Boundary Element Method (BEM) Approach for Torsional Problems

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I. Introduction

The boundary element method (BEM) is a commonly used approach in carrying out numerical simulations in a variety of engineering fields. The boundary element method uses classical integral equations in solving linear partial differential equations. The BEM in this study is developed in one of its most simple form; for the solution of the Poisson equation in an interior domain for evaluating the shear stresses in torsional problems. Within the BEM, discretization of the integrals allows its transformation into a system of linear equations to obtain solutions on its boundary and thus its stress distribution across the whole domain. In computing the shear stress, a spatial derivative is required to be computed. This study employs a smooth particle hydrodynamic-based gradient calculation using a cubic spline kernel function in approximating the derivative of the stress function. The validation will be done to to shows that the developed BEM subroutine is consistent with analytical solution.

This task will be divided into several section, the task structured as follows. Section 1 introduction. Section 2 discusses the stress formulation. Section 3 discusses the Boundary Element Method while Section 4 presents its formulation for torsional problems. Section 5 introduces the spatial gradient computation used. Section 6 deals with the analysis of results. Section 7 deals with conclusions. Section 8 source code and the last references.

II. Stress Formulation

In the case for torsional end loading, the stress formulation leads to the use of a stress function similar to that found in two-dimensional plane-stress problems. Consider a cross section of an arbitrary shape, under torsional loads as illustrated in Fig. 1, The in-plane or projected displacements of point P are determined as

$$\begin{aligned} u &= -r\beta sin\theta = -\beta y \\ v &= r\beta cos\theta = \beta x \end{aligned} \tag{1}$$

Using the assumption of linearity (small displacement), we can assume that the cylinder is fixed at $z = 0$ and take

$$\beta = \alpha z \tag{2}$$

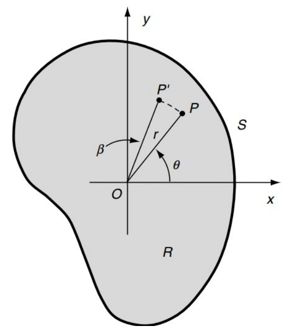

Fig. 1. In-plane displacement [1]

where α is the *angle of twist per unit length*. We can then group them as displacement vector $\mathbf{u}$ as

$$\mathbf{u} = \alpha \begin{pmatrix} -yz \\ xz \\ \varphi(x,y) \end{pmatrix} \tag{3}$$

where $\varphi(x,y)$ is known as the warping function. This then introduces a semi-inverse scheme for generating simplified problems for solutions of arbitrary cross-sectional shapes. The strain-displacement relations, obtained using Eq. 3, gives the following non-zero strain field

$$\begin{aligned} e_{xz} &= \frac{1}{2}\left(\frac{\partial\varphi}{\partial x} - \alpha y\right) \\ e_{yz} &= \frac{1}{2}\left(\frac{\partial\varphi}{\partial x} - \alpha x\right) \end{aligned} \tag{4}$$

Following from Hooke's law,

$$\begin{aligned} \tau_{xz} &= \mu\left(\frac{\partial\varphi}{\partial x} - \alpha y\right) \\ \tau_{xz} &= \mu\left(\frac{\partial\varphi}{\partial y} + \alpha x\right) \end{aligned} \tag{5}$$

where μ is the material property, namely the shear modulus.

For this case, where zero body forces are assumed, the equilibrium is reduced to

$$\frac{\partial\tau_{xz}}{\tau x} + \frac{\partial\tau_{yz}}{\tau y} = 0 \tag{6}$$

with the compatibility equation obtained by simply differentiating 5 with respect to y and x respectively and subtracting the results to obtain

$$\frac{\partial \tau_{xz}}{\tau y} - \frac{\partial \tau_{yz}}{\tau x} = -2\mu\alpha \tag{7}$$

This coupled pair can be further reduce by introducing the *Prandtl stress function* $\Phi = \Phi(x, y)$ by

$$\tau_{xz} = \mu\alpha\frac{\partial \Phi}{\partial y}; \quad \tau_{yz} = -\mu\alpha\frac{\partial \Phi}{\partial x} \tag{8}$$

where

$$\begin{aligned} \frac{\partial \Phi}{\partial y} &= \frac{\partial \varphi}{\partial x} - y \\ \frac{\partial \Phi}{\partial x} &= \frac{\partial \varphi}{\partial y} - x \end{aligned} \tag{9}$$

Thus the equilibrium equation are then identically satisfied and the compatibility relations becomes

$$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = -2 \tag{10}$$

This relation is thus the governing equation for torsional problems referred to as the *Poisson equation*.

III. Boundary Element Method

There are many analytical solutions techniques which can be used to solve torsional problems. This study investigates the application of the Boundary Element Method (BEM). However, it should be noted that this method is still significantly inefficient when compared to other discretization methods such as finite element method, finite difference method and finite volume methods. the BEM works by solving linear partial differential equation formulated as boundary integral forms. A general Poisson equation using BEM is mathematically written as

$$\begin{aligned} \Phi = &\oint_{\partial\Omega} \left(\Phi\frac{\partial w_0}{\partial n} - w_0\frac{\partial \Phi}{\partial n} \right) ds \\ &+ \sum_{i=0}^{n} \oint_{\partial\Omega} \left(b_i\frac{\partial w_{i+1}}{\partial n} - w_{i+1}\frac{\partial b_i}{\partial n} \right) \end{aligned} \tag{11}$$

for (ξ, η) inside the domain Ω and

$$\begin{aligned} \frac{1}{2}\Phi = &\oint_{\partial\Omega} \left(\Phi\frac{\partial w_0}{\partial n} - w_0\frac{\partial \Phi}{\partial n} \right) ds \\ &+ \sum_{i=0}^{n} \oint_{\partial\Omega} \left(b_i\frac{\partial w_{i+1}}{\partial n} - w_{i+1}\frac{\partial b_i}{\partial n} \right) \end{aligned} \tag{12}$$

for (ξ, η) on the boundary $\partial\Omega$. Here, the weight function w_i is given as follows [2]:

$$\begin{aligned} w_i &= \frac{1}{4\pi} C_i r^{2i} \left(lnr^2 - D_i \right), \\ C_i &= \frac{1}{4^i . i!}, \\ D_i &= 2\left(1 + \frac{1}{2} + ... + \frac{1}{i} \right) \end{aligned} \tag{13}$$

for $i = 0, 1, 2, ..., n$ with the exception where $D_i = 0$ for $i = 0$.

IV. Torsional Problem using Boundary Element Method

For a torsional problem in terms of Φ and the Poisson equation in Eq. 10, the boundary condition are given as $\Phi = 0$ at $\partial\Omega$.

$$\nabla^2 \Phi = b_0 = -2\Phi = 0, \quad \text{at} \partial\Omega \tag{14}$$

Here we can see that $b_0 = -2$ and $b_j = 0$ for $j = 1, 2, ..., m$ therefore only the right summation terms exist in Eq. 11 and Eq. 12 at $i = 0$. In solving the torsional problem, we first solve for $\partial\Phi/\partial n$ by rewriting these equations, applying b_0 and the boundary condition,

$$\frac{1}{2}\Phi = \oint_{\partial\Omega} b_0\frac{\partial w_1}{\partial n} ds - \oint_{\partial\Omega} w_0\frac{\partial \Phi}{\partial n} ds \tag{15}$$

and

$$\Phi = \oint_{\partial\Omega} b_0\frac{\partial w_1}{\partial n} ds - \oint_{\partial\Omega} w_0\frac{\partial \Phi}{\partial n} ds \tag{16}$$

We then construct a system of linear equations in matrix form ***Ax=b*** from Eq. 15. As $\Phi = 0$ at the boundary, the equation is rewritten in terms of $\partial\Phi/\partial n$.

$$\oint_{\partial\Omega} b_0\frac{\partial w_1}{\partial n} ds = \oint_{\partial\Omega} w_0\frac{\partial \Phi}{\partial n} ds \tag{17}$$

discretizing the integral into N-number of panels and taking the constant b_0 out from the integral,

$$b_0 \sum_{k=1}^{N} \oint_{\partial\Omega_{ik}} \frac{\partial w_1}{\partial n} = \sum_{k=1}^{N} \left(\frac{\partial \Phi}{\partial n} \right) \oint_{\partial\Omega_{ik}} w_0 ds \tag{18}$$

which can be written in matrix form as follows

$$\begin{aligned} &\begin{bmatrix} \oint_{\partial\Omega_{1,1}} w_0 ds & ... & \oint_{\partial\Omega_{1,N}} w_0 ds \\ \vdots & \ddots & \vdots \\ \oint_{\partial\Omega_{N,1}} w_0 ds & ... & \oint_{\partial\Omega_{N,N}} w_0 ds \end{bmatrix} \begin{bmatrix} \left(\frac{\partial\Phi}{\partial n}\right)_1 \\ \vdots \\ \left(\frac{\partial\Phi}{\partial n}\right)_N \end{bmatrix} \\ &= b_0 \begin{bmatrix} \sum_{k=1}^{N} \oint_{\partial\Omega_{1,k}} \frac{\partial w_1}{\partial n} ds \\ \vdots \\ \sum_{k=1}^{N} \oint_{\partial\Omega_{N,k}} \frac{\partial w_1}{\partial n} ds \end{bmatrix} \end{aligned} \tag{19}$$

In avoiding improper integral when $i = k$ in Eq. 19, we evaluate the integrals as

$$\oint_{\partial\Omega_{kk}} w_0 ds = \frac{S_k}{2\pi} \left(ln\left(\frac{S_k}{2} \right) - 1 \right) \tag{20}$$

$$\oint_{\partial\Omega_{kk}} \frac{\partial w_1}{\partial n} ds = 0 \tag{21}$$

Therefore, the linear equations in Eq. 19 can be easily solved using LU decomposition or gauss elimination where $\partial\Phi/\partial n$ at every panel in boundary $\partial\Omega$ are computed for. From the following computation, we can then proceed to compute for

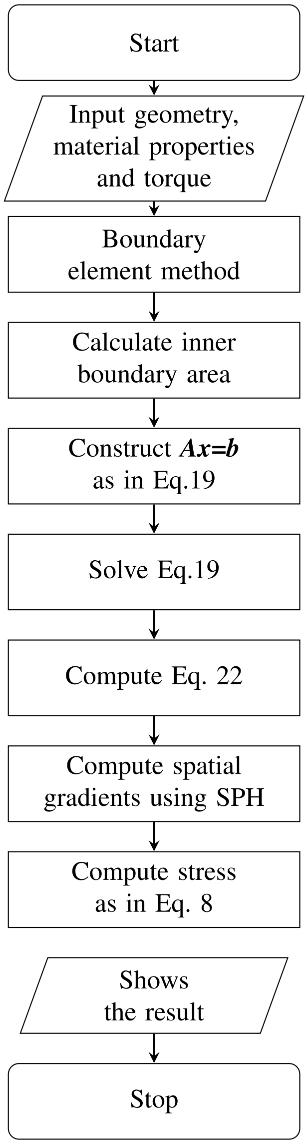

Fig. 2. Boundary Element Method Algorithm

Φ at any point inside Ω as $\partial\Omega$ is now known. Rewriting Eq. 16 in summation obtains:

$$\Phi = b_0 \sum_{k=1}^{N} \oint_{\partial\Omega_k} \frac{\partial w_1}{\partial n} ds - \sum_{k=1}^{N} \left(\frac{\partial \Phi}{\partial n}\right)_k \oint_{\partial\Omega_k} w_0 ds \tag{22}$$

The flowchart of the BEM algorithm developed for solving torsional problem is presented in Fig. 2.

V. Corrected Smooth Particle Hydrodynamics

In computing the shear stresses present in our geometry under torque loading, we employ the stress relation in Eq. 8. Since the distribution of the *Prandtl stress function* Φ has already been computed via Eq. 22, it is then only necessary to evaluate the gradient of Φ with respect to the y-axis and x-axis, namely $\partial\Phi/\partial y$ and $\partial\Phi/\partial x$ respectively. There are many methods which can be used to compute spatial gradients in the interior points, amongst them the most popular method is the forward-time centered-space (FTCS). However, in this particular study, an implementation of a simple smooth particle hydrodynamics (SPH) based gradient calculation method is used in computing the spatial gradient of the cloud point.

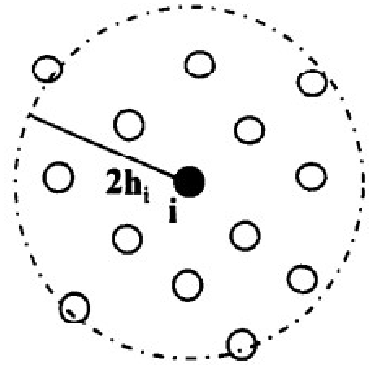

Fig. 3. SPH particle neighborhood [3]

SPH relies on interpolation theory as its foundation. The partial differential equation of the conservation laws of continuum dynamics are transformed into integral equation through a smoothing kernel function W to 'estimate' physical quantities between neighbouring particles. Within the SPH framework, the information is only known at discrete points thus the integrals are evaluated as sums of neighbouring particles. Consider a function f represented by a kernel function determined by parameter h, the kernel estimate is written as

$$\langle f(x) \rangle = \int f(x') W(|x - x'|/h) dx' \tag{23}$$

If we set $\rho(x')dx'$ as the differential mass dm and transforming into a sum, then the discrete kernel estimate becomes

$$\langle f(x) \rangle = \sum_{j=1}^{N} m_j f_j W(|x - x'|/h)/\rho_j \tag{24}$$

Fig. 3 illustrates a particle neighborhood, in our case a particular point inside the domain Ω, centered at a particle of interest within radius $2h$. Using integration by parts and recalling that W is 0 at infinity, a particle equation for the gradient can be obtained as

$$\langle \nabla f(x) \rangle \approx - \sum_{j=1}^{N} m_j f_j \nabla W(|x_j - x|/h)|_{x=x_j}/\rho_j \tag{25}$$

Here we can see that the spatial derivatives moves from operating on the physical quantities and onto the interpolation kernel. Approximations of Eq. 24 and Eq. 25 to functions, in our case the stress function Φ, and their derivatives allows estimates of its gradient.

In this study, we employ a cubic spline kernel [4],

$$W(s) = \begin{cases} 1 - 3s^2/2 + 3s^3/4, & \text{if } 0 \le s < 1; \\ (2-s)^3/4, & \text{if } 1 \le s < 3; \\ 0, & \text{if } s \ge 2 \end{cases} \tag{26}$$

where $s = r/h$. Note that from the above kernel function, its derivative ∇W can be easily computed to fulfill Eq. 25 thus obtaining $\nabla\Phi$ for Eq. 8. Where its resultant stress is computed as follows,

$$q = \sqrt{\tau_{yz}^2 + \tau_{xz}^2} \tag{27}$$

VI. Results Validation

The validation of the developed BEM is done by comparing the calculation result from BEM with the analytical solution. The different between BEM result and analytical solution is quantified by using L2 error, which mathematically written as follows.

$$e = \sqrt{\frac{\sum_{i=1}^{n}(\tau_{BEM}^i - \tau_a^i)^2}{\sum_{i=1}^{n}(\tau_a^i)^2}} \tag{28}$$

where n is the number of spatial discretization in the domain, τ_{bem} is the shear stress value calculated using BEM, and τ_a is the shear stress value calculated using analytical solution.

A. Circular Cross-Section

For the first case for validation, the circular section beam is used. This circular beam is subjected to a torsion of 1000 N.m. The beam itself is made of material that has shear modulus μ of 1×10^6.

By using BEM, the Prandtl stress function Φ can be calculate in the domain. Visually the Prandtl stress function Φ for the circular section is depicted on the figure 4. The shear stress in x-direction τ_{zx}, shear stress in y-direction τ_{yx}, and the total shear stress τ_s are shown by the figure 5 ,6, 7, respectively.

The analytical solution of resultant stress (τ_s) for torsional problem with circular cross-section beam is

$$\tau_s(x, y) = \frac{2T}{\pi r^2}\sqrt{\frac{x^2}{r^4} + \frac{y^2}{r^4}} \tag{29}$$

where T is the torsional load and radius r is $\sqrt{x^2 + y^2}$. The $L2$ error for this case is 0.00307. This $L2$ error value is low, which means that the BEM is consistent result with the analytical solution.

B. Triangular Cross-Section

The next case is the torsional problem in triangular cross-section beam. The torsion load and material properties of the beam is similar with the previous case. The Prandtl stress function Φ for the given case can be seen on figure 8. Besides that, the shear stress contour of the cross-section are depicted on figure 9, 10, and 11.

The analytical solution of the resultant shear stress τ_s for torsional problem in triangular cross-section beam is as follows.

$$\tau_s = \sqrt{\tau_{xz}^2 + \tau_{xz}^2} \tag{30}$$

$$\tau_{xz} = \frac{\mu\alpha}{a}(x - a)y \tag{31}$$

$$\tau_{yz} = \frac{\mu\alpha}{2a}(x^2 + 2ax - y^2) \tag{32}$$

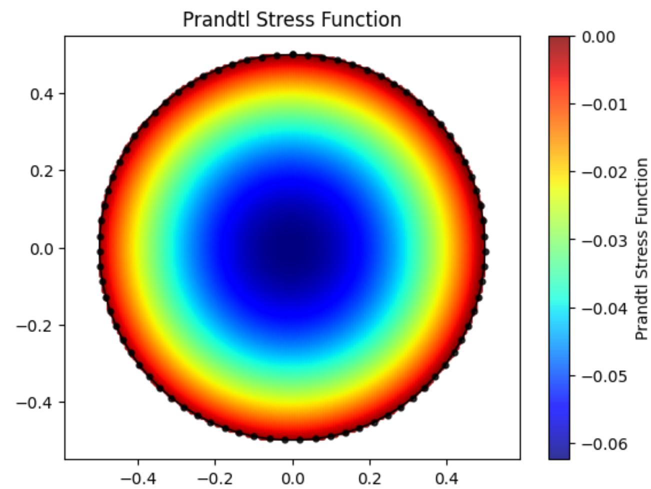

Fig. 4. Prandtl stress function contour of circular section beam

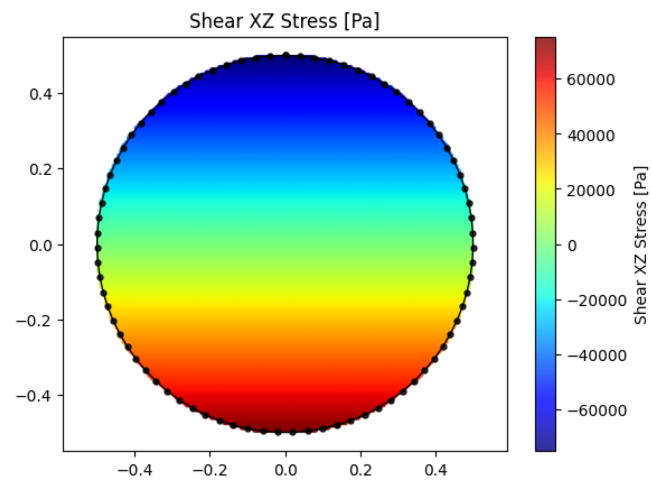

Fig. 5. τ_{zx} contour of circular section beam

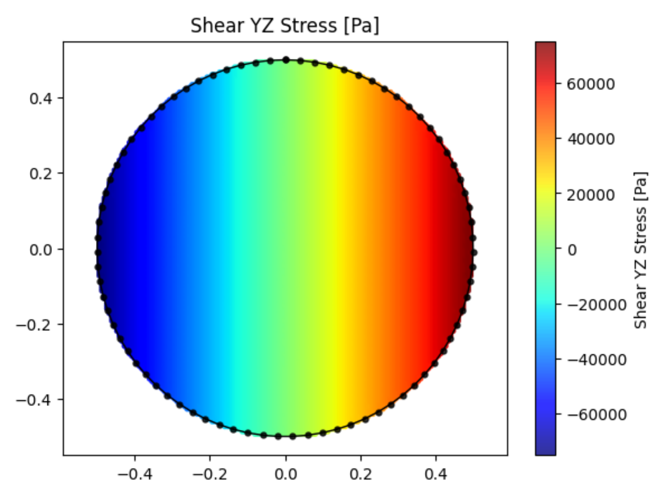

Fig. 6. τ_{zy} contour of circular section beam

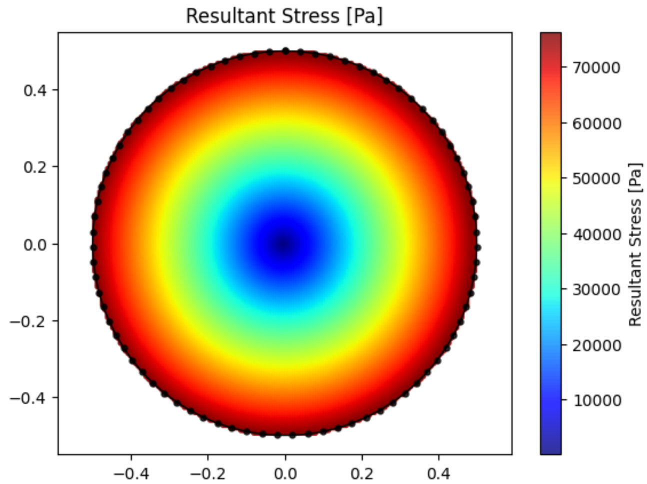

Fig. 7. τ_s contour of circular section beam

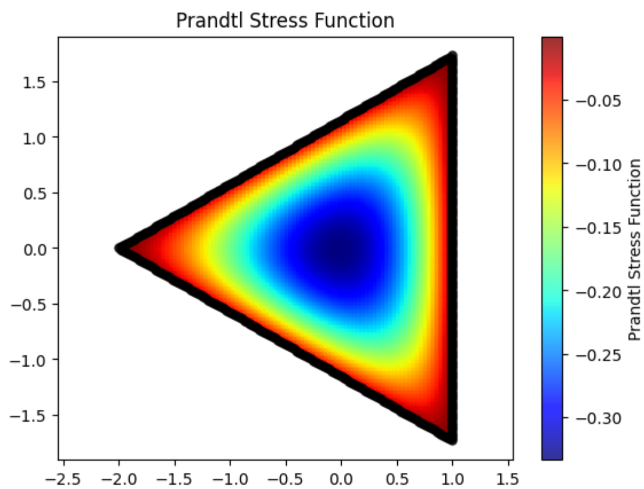

Fig. 8. Prandtl stress function contour of circular section beam

where a is one third of the height of the triangle. Besides, the analytical solution assumed that the triangle centroid is located at $(0, 0)$. Upon comparing the obtained results with the analytical solution, the $L2$ error of 0.01043 was observed, indicating a commendably low level of deviation. This suggests that the Boundary Element Method (BEM) developed in this project aligns well with the analytical solution, affirming its consistency and accuracy.

VII. Conclusion

In this project, the Boundary Element Method (BEM) has been explored as a potential numerical tool for solving the Poisson equation and evaluating shear stresses in torsional problems. The BEM, relying on classical integral equations, efficiently addresses linear partial differential equations by discretizing integrals and transforming them into a system of linear equations. The reliability of the developed BEM subroutine is affirmed through validation against analytical

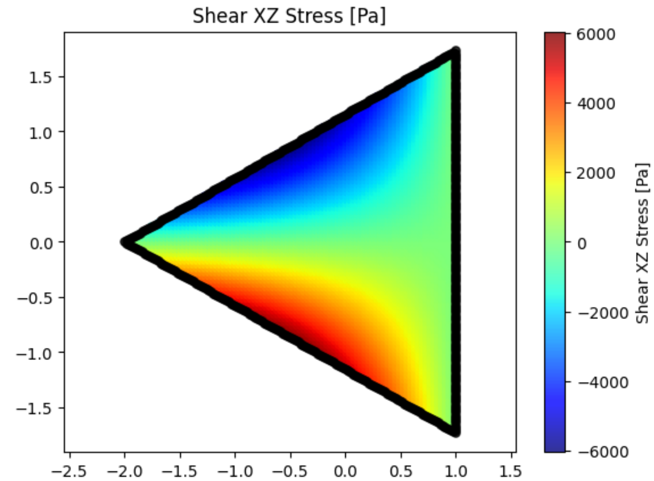

Fig. 9. τ_{zx} contour of circular section beam

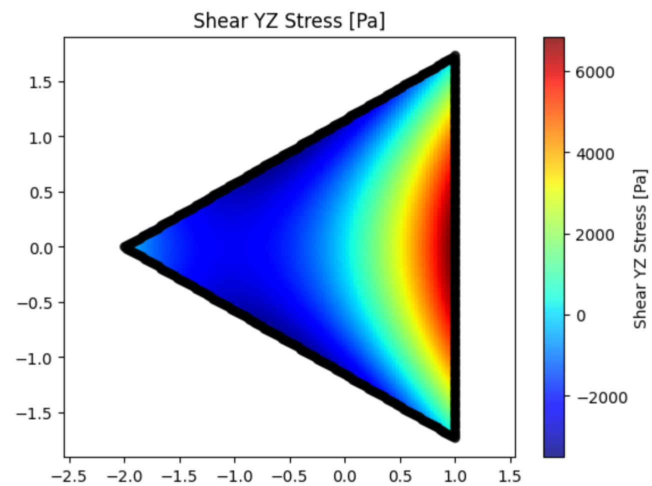

Fig. 10. τ_{zy} contour of circular section beam

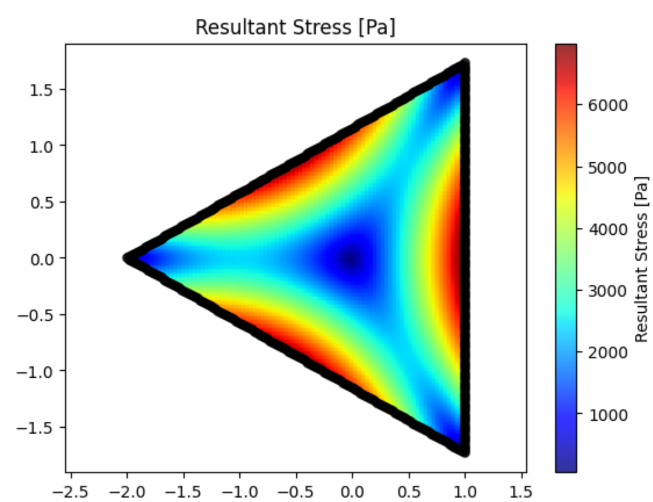

Fig. 11. τ_s contour of circular section beam

solutions, demonstrating a low L2 norm. Despite its success, a notable limitation is acknowledged: the current code cannot handle multiply connected regions. Future enhancements may target this limitation to broaden the applicability of the BEM subroutine.

VIII. Source Code

The source code thas havs been developed in this project can be accessed in https://github.com/farrasarira/BL-Approximation-using-Thwaites-and-Head-s-Entrainment-Method.

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