Statistical Analysis of Monte Carlo Simulation for Borehole and Cantilever Models

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I. Introduction

Borehole models are commonly used in geotechnical engineering to analyze the behavior of foundations and the surrounding soil [1]. The model considers a cylindrical hole drilled into the soil and evaluates the bearing capacity and settlement of the foundation based on the soil properties and the applied load. In borehole modeling, input parameters such as soil strength, soil modulus, and foundation geometry can significantly affect the output responses.

On the other hand, cantilever analytical models are frequently used in structural engineering to analyze the behavior of structures subjected to bending forces [2]. The model considers a beam that is fixed at one end and subjected to a load at the other end, resulting in a bending moment that causes stresses and deflection along the length of the beam. In cantilever modeling, input parameters such as beam length, beam thickness, and applied load can significantly affect the output responses.

Monte Carlo simulation can be used as a powerful tool to analyze the effects of uncertainty of the input parameter on the distribution of outputs responses [3]. The output responses then evaluated by calculating its statistical properties, such as mean, variance, skewness and kurtosis.

II. Methodology

The methodology used in this work is straightforward. Initially, based on the given mathematical model of borehole and cantilever, and the uncertainty quantification of each input of the model, the Monte Carlo simulation is conducted. The Monte Carlo simulation used random sampling of the input variables to obtained the outcomes of the systems.

Statistical analysis is used to evaluate the results of the Monte Carlo simulation. The probability distribution function of the output variables will be estimated. Besides that, the statistical parameters such as mean, skewness and kurtosis will be calculated.

III. Mathematical Models

A. Borehole Function

The Borehole function models water flow through a borehole. Its simplicity and quick evaluation makes it a commonly used function for testing a wide variety of methods in computer experiments. Mathematically, the borehole function is written as follows.

$$f(\mathbf{x}) = \frac{2\pi T_u (H_u - H_l)}{\ln(r/r_w)\left(1 + \frac{2LT_u}{\ln(r/r_w) r_w^2 K_w} + \frac{T_u}{T_l}\right)} \quad (1)$$

The response $f(\mathbf{x})$ is water flow rate, in m3/yr.

The input variables and their usual input ranges are explained on table I. For the purposes of uncertainty quantification, the distributions of the input random variables are given on table II. The $\mathcal{N}(\mu,\sigma)$ is the normal distribution with mean μ and variance σ^2. Lognormal(μ,σ) is the lognormal distribution of a variable, such that the natural logarithm of the variable has a $\mathcal{N}(\mu,\sigma)$ distribution [4].

TABLE I
Borehole Input Variables

$r_w \in [0.05, 0.15]$	radius of borehole (m)
$r \in [100, 50000]$	radius of influence (m)
$T_u \in [63070, 115600]$	transmissivity of upper aquifer (m2/yr)
$H_u \in [990, 1110]$	potentiometric head of upper aquifer (m)
$T_l \in [63.1, 116]$	transmissivity of lower aquifer (m2/yr)
$H_l \in [700, 820]$	potentiometric head of lower aquifer (m)
$L \in [1120, 1680]$	length of borehole (m)
$K_W \in [9855, 12045]$	hydraulic conductivity of borehole (m/yr)

TABLE II
Uncertainty Quantification for Borehole Input

$r_w \sim \mathcal{N}(\mu = 0.10, \sigma = 0.0161812)$
$r \sim$ Lognormal$(\mu = 7.71, \sigma = 1.0056)$
$T_u \sim$ Uniform$[63070, 115600]$
$H_u \sim$ Uniform$[990, 1110]$
$T_l \sim$ Uniform$[63.1, 116]$
$H_l \sim$ Uniform$[700, 820]$
$L \sim$ Uniform$[1120, 1680]$
$K_W \sim$ Uniform$[9855, 12045]$

B. Cantilever Beam Function

The cantilever beam functions, used for uncertainty quantification, model a simple uniform cantilever beam with horizontal and vertical loads. The mathematical equation for the cantilever beam functions are shown on equation 2 and 3. The beam length L and displacement tolerance D_0 at the free end of the beam are problem constants, with values $L = 100$ inches, and $D_0 = 2.2535$ inches. The parameters

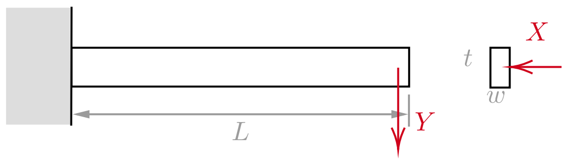

Fig. 1. Load Condition of Cantilever Beam

TABLE III
CANTILEVER INPUT VARIABLES

$R \sim \mathcal{N}(\sigma = 40000, \sigma = 2000)$	yield stress
$E \sim \mathcal{N}(\nu = 2.9 \times 10^7, \sigma = 1.4 \times 10^6)$	Youngs's modulus
$X \sim \mathcal{N}(\nu = 500, \sigma = 100)$	horizontal load
$Y \sim \mathcal{N}(\nu = 1000, \sigma = 100)$	vertical load

w and t are width and thickness of the cross-section. The responses are displacement (D) and stress (S). They have the following constraints: $S \leq R$ and $D \leq D_0$ [4].

$$D(\mathbf{x}) = \frac{4L^3}{Ewt}\sqrt{\left(\frac{Y}{t^2}\right)^2 + \left(\frac{X}{w^2}\right)^2} \tag{2}$$

$$S(\mathbf{x}) = \frac{600Y}{wt^2} + \frac{600X}{w^2 t} \tag{3}$$

Or if scaled,

$$g_D = \frac{displacement}{D_0} - 1 \leq 0 \tag{4}$$

$$g_S = \frac{stress}{R} - 1 \leq 0 \tag{5}$$

The negative value of g_D and g_S indicate safe regions. The input random variables and their distributions are given on table III.

IV. Monte Carlo Simulation

Monte Carlo simulation is a powerful computational method that utilises random sampling to generate possible outcomes of a system or process [5]. The idea of the Monte Carlo simulation is to run multiple times of a mathematical model of a system, with random value of the input variables for each run, resulting ranges of outputs. And after that, the statistical analysis is used to evaluate the result of the outputs.

V. Results and Discussions

The Monte Carlo simulation is conducted with 1×10^5 number of repetitions. This value is chosen since the resulting data from 1×10^4 and 1×10^5 repetitions give insignificant changes.

A. Borehole

Based on the Monte Carlo simulation, the distribution of the water flow rate of borehole is shown by the figure 2.

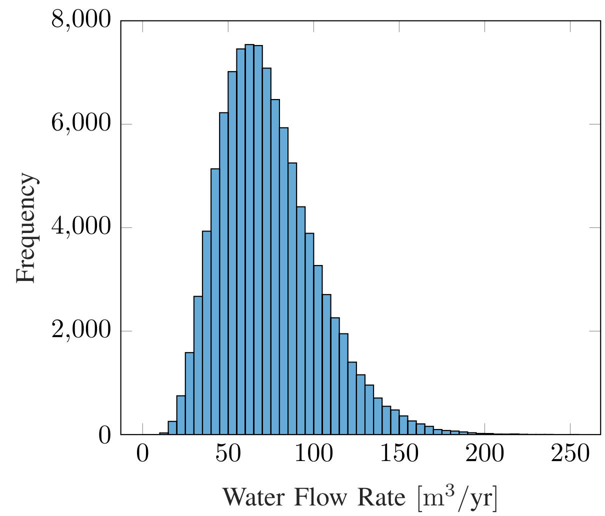

Fig. 2. Distribution of Mass Flow Rate Value

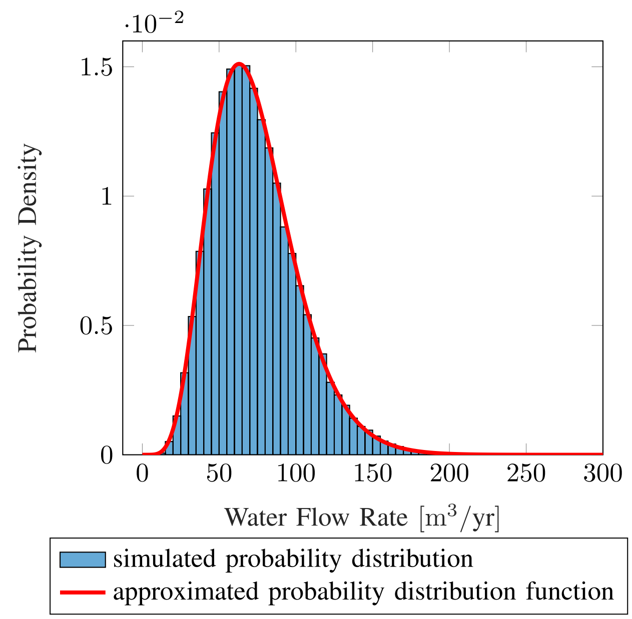

Fig. 3. Approximated Probability Density Function of Borehole Flow Rate

Curve fitting is conducted to approximate the Probability Density Function (PDF) of the flow rate of borehole. It is found that the Gamma distribution function with shape parameter $k = 6.865$ and scale parameter $\theta = 10.746$ fits the distribution. The approximated gamma distribution function is depicted on figure 3.

The calculation of mean, variance, skewness and kurtosis of the water flow rate is done by using the following formula, respectively.

$$< U > = \int_{-\infty}^{\infty} V f(V) dV \tag{6}$$

$$\text{Var}(U) =< u^2 >= \int_{-\infty}^{\infty} u^2 f(V) dV \tag{7}$$

$$\hat{\mu}_3 = \frac{< u^3 >}{\sigma_u^3} \tag{8}$$

$$\hat{\mu}_4 = \frac{< u^4 >}{\sigma_u^4} \tag{9}$$

The $< U >$ denotes the mean of random variable U, in this case water flow rate, V denotes the sample-space variable corresponding to random variable U, $f(V)$ denotes the probability density function, Var(U) denotes the variance of random variable, $u \equiv U- < U >$ is the fluctuation, $\hat{\mu}_3$ denotes the skewness, $\hat{\mu}_4$ denotes the kurtosis and $\sigma_u \equiv \sqrt{\text{Var}(U)}$ is the standard deviation. By utilizing these equation to the known probability distribution, the mean, variance, skewness and kurtosis of water flow rate is summarized on table IV. The table shows that the skewness of the probability density function is positive, which indicates that the probability distribution graph has large tail to the right. It is consitent with the graph on figure 3.

TABLE IV
STATISTICAL PROPERTIES OF BOREHOLE'S WATER FLOW RATE

mean	73.773 $[\text{m}^3/\text{yr}]$
variance	803.372 $[\text{m}^3/\text{yr}]^2$
skewness	0.809
kurtosis	3.945

B. Cantilever

In this work, the cantilever width (w) and thickness (t) is 4 inches and 2 inches, respectively. The distribution of scaled displacement and stress value obtained from the Monte Carlo simulation are given on figure 4 and 5.

After that, curve fitting is conducted on both distribution of displacement and stress values to approximate their probability density function. It is obtained that normal distribution function fits both of them. For the scaled displacement g_D, the normal distribution function has $\mu = 0.934$ and $\sigma = 0.214$. For the scaled stress g_S, the normal distribution function has $\mu = 0.175$ and $\sigma = 0.121$. The probability distribution function graph of both scaled displacement and stress are shown on figure 6 and 7.

As mentioned in section III, there are several constraints in the cantilever, which are $g_D \leq 0$ and $g_S \leq 0$. From the probability density function of g_D, the probability of $g_D \leq 0$ can be calculates as follows.

$$P(g_D \leq 0) = \int_{-\infty}^{0} f(G) dG \tag{10}$$

where $f(G)$ denotes the probability density function of g_D and G is the sample-space on g_D. Hence, by substituting the known normal distribution function of g_D to equation above,

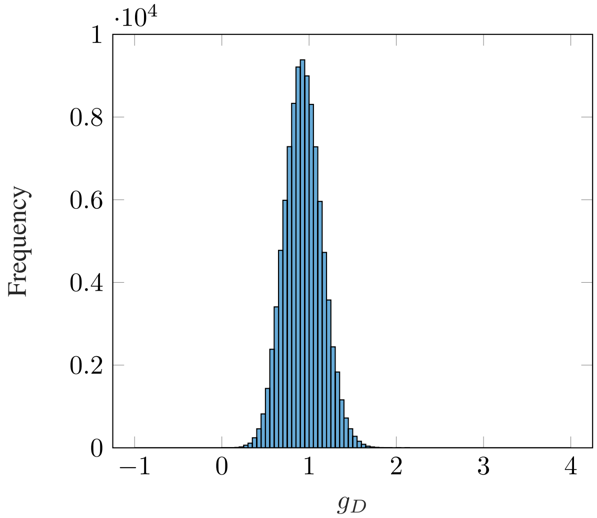

Fig. 4. g_D Distribution

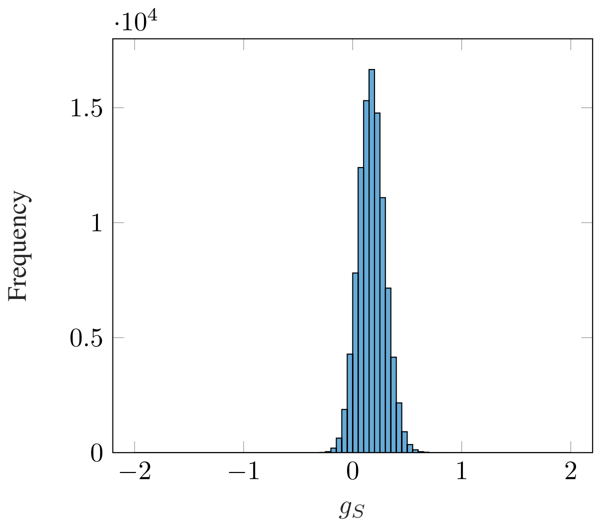

Fig. 5. g_S Distribution

it can be known that $P(g_D \leq 0) = 0$ which indicates that the displacement of the beam inevitably exceed the displacement tolerance. By using the same procedure, the probability of stress less than yield stress can be calculated. Hence, the probability of stress is less than yield stress $P(g_S \leq 0) = 0.070$, which tells there is only 7% chance that stress is in safe region (less than yield stress).

The statistical properties of scaled displacement and stress distribution are calculated by using the equation 6 to 9 as already explained. The summary of the statistical properties are given on table V and VI.

VI. CONCLUSION

The analysis of random outputs of borehole and cantilever utilizing Monte Carlo simulation has conducted. The probability density function can be approximated from the random

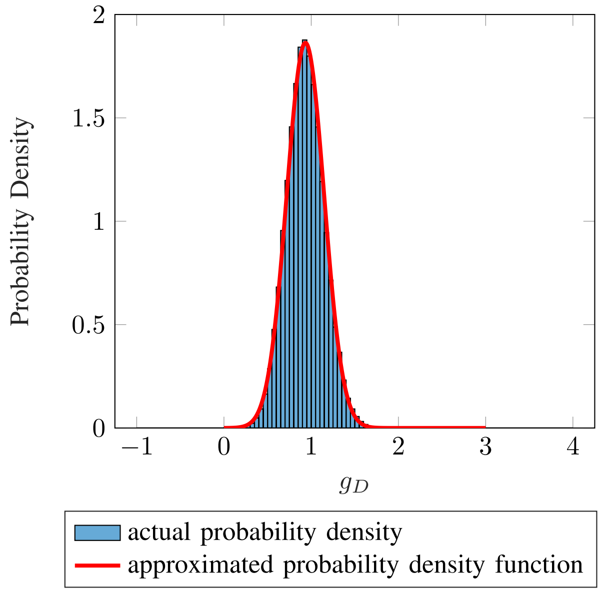

Fig. 6. Approximated Probability Density Function of g_D

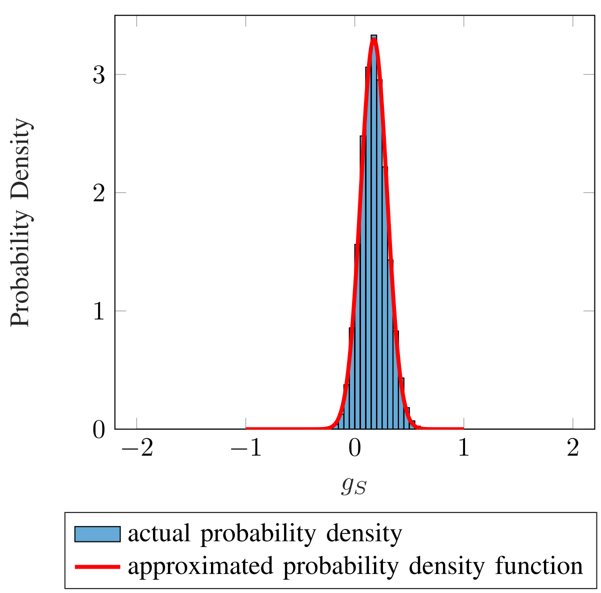

Fig. 7. Approximated Probability Density Function of g_S

TABLE V
STATISTICAL PROPERTIES OF SCALED DISPLACEMENT g_D

mean	0.934
variance	0.046
skewness	0.138
kurtosis	3.070

TABLE VI
STATISTICAL PROPERTIES OF SCALED STRESS g_S

mean	0.175
variance	0.015
skewness	0.148
kurtosis	3.095

variable distribution by using curve fitting. Besides that, the statistical properties, such as mean, variance, skewness and kurtosis are also calculated.

VII. SOURCE CODE

The source code used for this work can be accessed by clicking the *href* below.

>>> **CODE** <<<

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