The symbol alphabet of one-loop Feynman integrals

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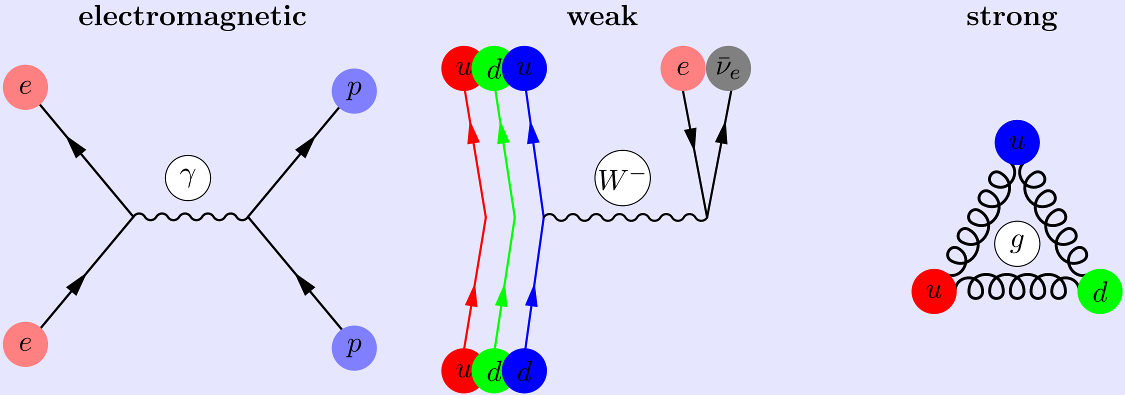
Alex	1	Recap & Symbol recursion
Mikey	2	Alphabets & Determinants

1 Recap & Symbol recursion

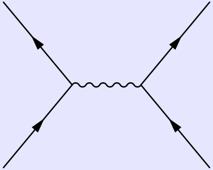
Recap (Motivation)

Motivation of particle physics: identify what makes up the world, explain physical phenomena from particle interactions (3/4 forces explained by quantum field theory).



Recap (Feynman diagrams)

Quantum field theory with Feynman diagrams:

$\mathcal{L} \to$  $\to \mathcal{M} \to \dfrac{d\sigma}{d\Omega}, \Gamma, \ldots$

Given Feynman rules, can interpret diagrams to get $\mathcal{M}$. Hooray!

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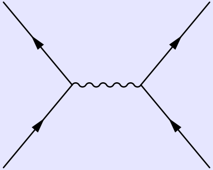
Quantum field theory with Feynman diagrams:

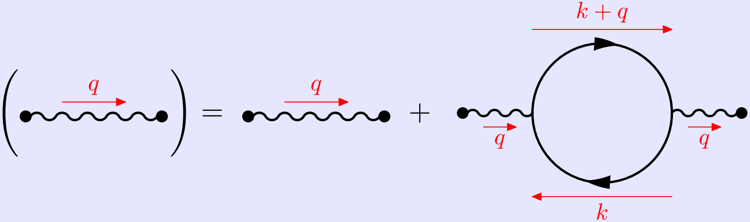
$$\mathcal{L} \rightarrow$$



$$\rightarrow \mathcal{M} \rightarrow \frac{d\sigma}{d\Omega}, \Gamma, \ldots$$

Given Feynman rules, can interpret diagrams to get $\mathcal{M}$. Hooray!
However, theoretical corrections involve loop diagrams which correspond to Feynman integrals over the undefined loop momentum. For example, here it's k where

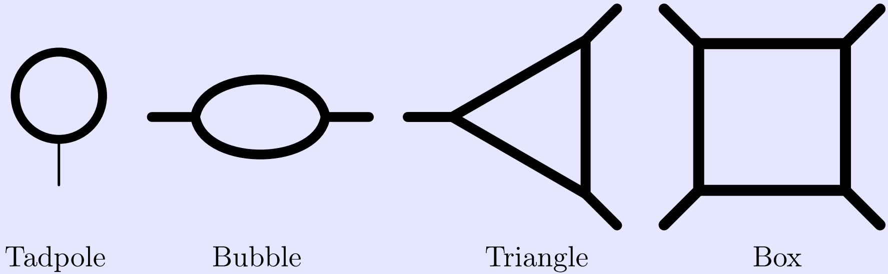
Exact



$+\,\mathcal{O}(2 \text{ loops})$.

Recap (One-loop Feynman integrals)

We can express any one-loop Feynman integral in terms of the first four scalar integrals



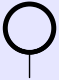
which correspond to a basis of integrals called master integrals

$$I_n^D\left(\{p_i \cdot p_j\}; m_i^2; \epsilon\right) \equiv e^{\gamma_E \epsilon} \int \frac{d^D k}{\mathrm{i}\pi^{D/2}} \frac{1}{(k-p_1)^2 - m_1^2} \frac{1}{(k-p_1-p_2)^2 - m_2^2} \cdots$$

and ϵ is the dimensional regularisation parameter such that $D = 4 - 2\epsilon$.

Recap (One-loop Feynman integrals)

If $D = 2\left\lceil \frac{n}{2} \right\rceil - 2\epsilon$, it turns out that all one-loop Feynman integrals are basically logarithms!



$$= -e^{\gamma_E \epsilon} \left(m^2\right)^{-\epsilon} \Gamma\left(\epsilon\right)$$

$$= -\frac{1}{\epsilon} + \log\left(m^2\right) - \frac{1}{12}\epsilon\left(6\log^2(m^2) + \pi^2\right) + \frac{1}{12}\epsilon^2\left(2\log^3(m^2) + \pi^2\log\left(m^2\right) + 4\zeta(3)\right)$$
$$+ \frac{1}{480}\epsilon^3\left(-160\zeta(3)\log\left(m^2\right) - 20\log^4(m^2) - 20\pi^2\log^2(m^2) - 3\pi^4\right) + O\left(\epsilon^4\right),$$

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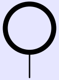

$$
\begin{aligned}
&= -e^{\gamma_E \epsilon} \left(m^2\right)^{-\epsilon} \Gamma\left(\epsilon\right) \\
&= -\frac{1}{\epsilon} + \log\left(m^2\right) - \frac{1}{12}\epsilon \left(6 \log^2(m^2) + \pi^2\right) + \frac{1}{12}\epsilon^2 \left(2 \log^3(m^2) + \pi^2 \log\left(m^2\right) + 4\zeta(3)\right) \\
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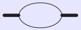
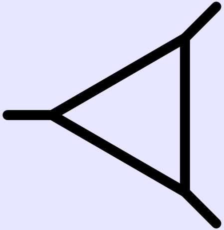


$$
\begin{aligned}
&= -\frac{2}{\epsilon} \frac{e^{\gamma_E \epsilon} \Gamma\left(1-\epsilon\right)^2 \Gamma\left(1+\epsilon\right)}{\Gamma\left(1-2\epsilon\right)} (-p^2)^{-1-\epsilon} \\
&= -\frac{1}{\epsilon} + \log\left(-p^2\right) - \frac{1}{12}\epsilon \left(6 \log^2(-p^2) - \pi^2\right) + \frac{1}{12}\epsilon^2 \left(2 \log^3(-p^2) - \pi^2 \log\left(-p^2\right) + 28\zeta(3)\right) \\
&\quad + \frac{1}{1440}\epsilon^3 \left(-3360\zeta(3) \log\left(-p^2\right) - 60 \log^4(-p^2) + 60\pi^2 \log^2(-p^2) + 47\pi^4\right) + O\left(\epsilon^4\right).
\end{aligned}
$$

Recap (Physics)

Physics $\sim$  $\sim \log, \mathrm{Li}_2, \mathrm{Li}_{m_1 \dots m_k}$

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All one-loop integrals can be expressed in terms of multiple polylogarithms (MPLs).

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$$\mathrm{Li}_{n_1 \ldots n_k}(z_1, \ldots, z_k) = \sum_{m_1 > \ldots > m_k}^{\infty} \frac{z_1^{m_1}}{m_1^{n_1}} \cdots \frac{z_k^{m_k}}{m_k^{n_k}}.$$

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Some familiar special cases are

$$\begin{aligned} &\text{the ordinary logarithm} && \mathrm{Li}_1(z) = -\log(1-z), \\ &\text{classical polylogarithms} && \mathrm{Li}_n(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^n}. \end{aligned}$$

Recap (Coproduct on MPLs)

If we let $\mathcal{A}$ denote the $\mathbb{Q}$-vector space of MPLs, then $\mathcal{H} = \mathcal{A}/(\mathrm{i}\pi\mathcal{A})$ can be endowed with a coproduct $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ which is a coassociative homomorphism.

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Explicitly, the coproduct acts as follows for

$$\begin{aligned} \text{the ordinary logarithm} \quad & \Delta\left(\log(z)\right) = 1 \otimes \log(z) + \log(z) \otimes 1, \\ \text{classical polylogarithms} \quad & \Delta\left(\mathrm{Li}_n(z)\right) = 1 \otimes \mathrm{Li}_n(z) + \sum_{k=0}^{n-1} \frac{1}{k!} \mathrm{Li}_{n-k}(z) \otimes \log^k(z). \end{aligned}$$

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Examples of coproducts

$$\Delta\left(\mathrm{Li}_2(z)\right) = 1 \otimes \mathrm{Li}_2(z) + \mathrm{Li}_2(z) \otimes 1 - \log(1-z) \otimes \log(z)$$

$$\Delta\left(\mathrm{Li}_3(z)\right) = 1 \otimes \mathrm{Li}_3(z) + \mathrm{Li}_3(z) \otimes 1 + \mathrm{Li}_2(z) \otimes \log(z) - \frac{1}{2} \log(1-z) \otimes \log^2(z)$$

Recap (Symbol)

The symbol map: A linear map, built from the coproduct, which captures the main combinatorial and analytical properties of certain transcendental functions.

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$$\Big\downarrow \mathcal{S}$$

$$-z \otimes (1-z) + \bar{z} \otimes (1-\bar{z}) - (1-z) \otimes \bar{z} + z \otimes (1-\bar{z}) + (1-\bar{z}) \otimes z - \bar{z} \otimes (1-z)$$

Recap (Symbol)

Coassociativity $\Rightarrow \Delta$ can be uniquely iterated.

The maximal iteration of the coproduct is called the symbol $\mathcal{S}$:

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The symbol map

To find the symbol of an MPL of weight n:

1. Iteratively apply the coproduct to the MPL $n-1$ times.
2. Extract the terms in which all entries have weight one (i.e. the ordinary logarithms).

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Some simple examples:

$$\Delta\left(\mathrm{Li}_2(z)\right) = 1 \otimes \mathrm{Li}_2(z) + \mathrm{Li}_2(z) \otimes 1 - \log(1-z) \otimes \log z$$

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$$\begin{aligned}(\mathrm{id} \otimes \Delta)\Delta\left(\mathrm{Li}_3(z)\right) &= 1 \otimes 1 \otimes \mathrm{Li}_3(z) + 1 \otimes \mathrm{Li}_3(z) \otimes 1 \\ &+ 1 \otimes \mathrm{Li}_2(z) \otimes \log(z) + \mathrm{Li}_2(z) \otimes 1 \otimes \log(z) + \mathrm{Li}_2(z) \otimes \log(z) \otimes 1 \\ &- \frac{1}{2} 1 \otimes \log(1-z) \otimes \log^2 z - \frac{1}{2}\log(1-z) \otimes 1 \otimes \log^2 z \\ &- \frac{1}{2}\log(1-z) \otimes \log^2 z \otimes 1 - \log(1-z) \otimes \log z \otimes \log z\end{aligned}$$

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Recap (Symbol)

Since all entries in the symbol are ordinary logarithms, it is conventional to show only their arguments. For example,

$$\mathcal{S}\left(\mathrm{Li}_2(z)\right) = -(1-z)\otimes z,$$
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General form of the symbol

In general, the symbol of any MPL f of weight n has the form

$$\mathcal{S}(f) = \sum_{i_1,\ldots,i_n} c_{i_1,\ldots,i_j} f_{i_1} \otimes \cdots \otimes f_{i_n},$$

where the $c_{i_1,\ldots,i_n}$ are coefficients, and the values f_i are known as the symbol letters.

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For Feynman integrals: Letters are functions of the external momenta and propagator masses:

$$f_i = f_i(p_1^2, \ldots, p_n^2; m_1^2, \ldots, m_k^2).$$

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Our focus: one-loop integrals.

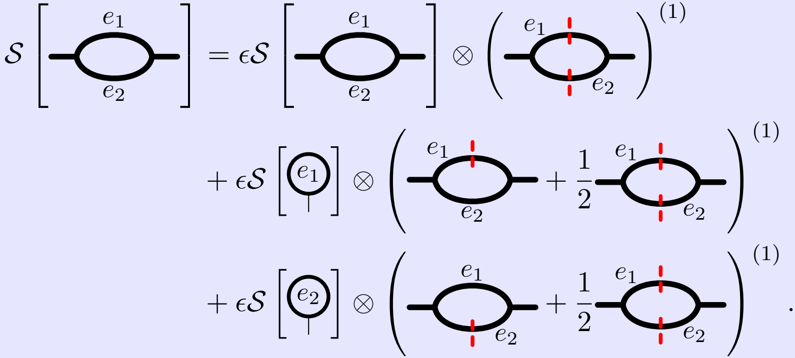
☑ **Generic case:** All propagators and external momenta are unspecified.

☐ **Non-generic case:** For example, setting $p_1^2 \rightarrow 0, m_2^2 \rightarrow 0, m_1^2 = m_2^2, \ldots$

New stuff!

Symbol recursion

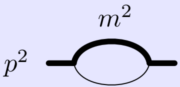
In [Abr+17], a recursive formula for the symbol entries, which relates words with n letters to words with $n+1$ letters is presented. For example, the bubble has



If we know the cut integrals and pinched symbols, we have all of the information!

Symbol recursion

Examining the recursion allows us to predict the alphabet and dictionary. Let us look at the bubble with one massive propagator whose symbol has the three-letter words

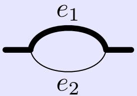


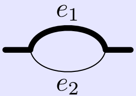
1st letter	2nd letter	3rd letter
m^2	m^2	m^2
m^2	m^2	p^2
m^2	m^2	$(m^2 - p^2)$
m^2	p^2	p^2
m^2	p^2	$(m^2 - p^2)$
m^2	$(m^2 - p^2)$	p^2
m^2	$(m^2 - p^2)$	$(m^2 - p^2)$
$(m^2 - p^2)$	p^2	p^2
$(m^2 - p^2)$	p^2	$(m^2 - p^2)$
$(m^2 - p^2)$	$(m^2 - p^2)$	p^2
$(m^2 - p^2)$	$(m^2 - p^2)$	$(m^2 - p^2)$.

Why ?

Symbol recursion

For the bubble with $m_2^2 = 0$, the recursion reads

$S\Big[$ 

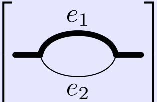
$\Big] = \epsilon S\Big[$ 

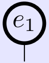
$\Big]\left(\otimes p^2 - \otimes(m^2 - p^2)^2\right)$

$+\,\epsilon S\Big[$ 

$\Big]\left(\frac{1}{2}\otimes m^2 - \frac{1}{2}\otimes p^2\right).$

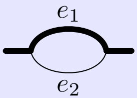
The base of the recursion involves the coefficients of ϵ^{-1} in the respective Laurent series of

$S\Big[$ 

$\Big]^{(-1)} = -\frac{1}{2},\qquad S\Big[$ 

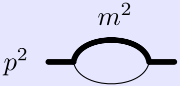
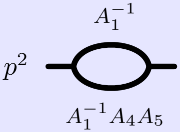
$\Big]^{(-1)} = -1$

such that the order ϵ^0 symbol words of the bubble with one massive propagator are

$S\Big[$ 

$\Big]^{(-1)} = \otimes(m^2 - p^2) - \frac{1}{2}\otimes m^2 + \frac{1}{2}\otimes p^2 - \frac{1}{2}\otimes p^2.$

Symbol recursion

Diagram	Alphabet	Dictionary
	$\{m^2, p^2, m^2 - p^2\}$	▶ Only the letter m^2 can precede m^2. ▶ The letter p^2 cannot come first.
	$\{A_1, \ldots, A_5\}$ $p^2 = \frac{A_3(1+A_4)^2}{A_1A_4A_5}$	▶ $A_1 \otimes \ldots \otimes A_1 \otimes A_2$ not allowed.
$\vdots$	$\vdots$	$\vdots$

❷ Alphabets & Determinants

Genealogy

Part of our investigation is to report which letters may appear after other letters in a symbol word.
In [Han+24], the *genealogical* method is showcased which offers a sufficient condition for a letter being forbidden from other occurring after another in a word. This method only requires the graph polynomial $\mathcal{F}$.

Genealogy

A Feynman parameter integral has a simplex as its integration bounds.

$$\mathcal{I} = e^{l\epsilon\gamma_E}\Gamma(\omega)\int_{a_j\geq 0} d^n a\ \delta\left(1-\sum_{j=1}^{n} a_j\right)\frac{\mathcal{U}^{\omega-\frac{D}{2}}}{\mathcal{F}^{\omega}}$$

Taking a discontinuity of a Feynman parameter integral will introduce a heaviside function $\Theta(\mathcal{F})$. This sets $\mathcal{F}=0$ as an extra bound of integration. This restriction may cause previous $\alpha_i = 0$ or $\sum \alpha_i = 1$ integration boundaries to become obsolete, limiting which singularities can occur.

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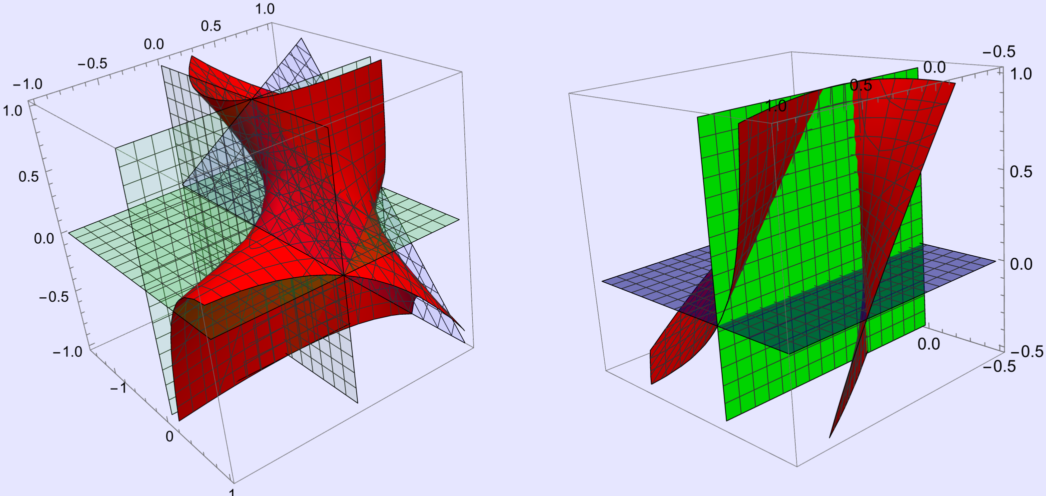
Taking discontinuities restricts access to singularities, and it will stay that way after any further discontinuities. We must verify which singularities are still relevant after restriction the integration region.

Contour plots

By plotting $\mathcal{F} = 0$ and setting a given letter to zero, we see which α_i's are no longer involved in the integration.



Genealogy results

Let

$$Y = \mathbb{C}^{E-1} \Bigg\backslash \left(\mathcal{F} = 0 \cup \mathcal{U} = 0 \bigcup_{i=1}^{E} \alpha_i = 0 \right) \tag{1}$$

be the Feynman parameter space without the singular loci and integration bounds. It was show in [FMT24] that if $|\chi(Y)|$ decreases when a letter is set to zero, the integral becomes singular.

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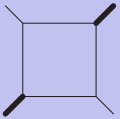
be the Feynman parameter space without the singular loci and integration bounds. It was show in [FMT24] that if $|\chi(Y)|$ decreases when a letter is set to zero, the integral becomes singular. This *Euler characteristic test* is equivalent to counting the number of solutions to

$$\frac{\mu_1}{\mathcal{U}} + \frac{\mu_2}{\mathcal{F}} \frac{\partial \mathcal{F}}{\partial \alpha_i} + \frac{\nu_i}{\alpha_i} = 0$$

before and after setting a letter to zero. The generic ν_i is set to zero when α_i becomes obsolete. Then we set another letter to zero. If the number of critical points does not change, then this letter is forbidden from occurring after the first.

Genealogy results

Tabulating the constraints, we can categorise the 10 letters of the *two mass easy* box based on the which ones may occur sequentially after others.



$$\left\{p_1^2, p_3^2, s, t, \quad p_1^2 - s,\ p_1^2 - t,\ p_3^2 - s,\ p_3^2 - t, \quad st - p_1^2 p_3^2, \quad s + t - p_1^2 - p_3^2\right\}$$

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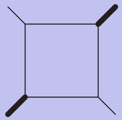


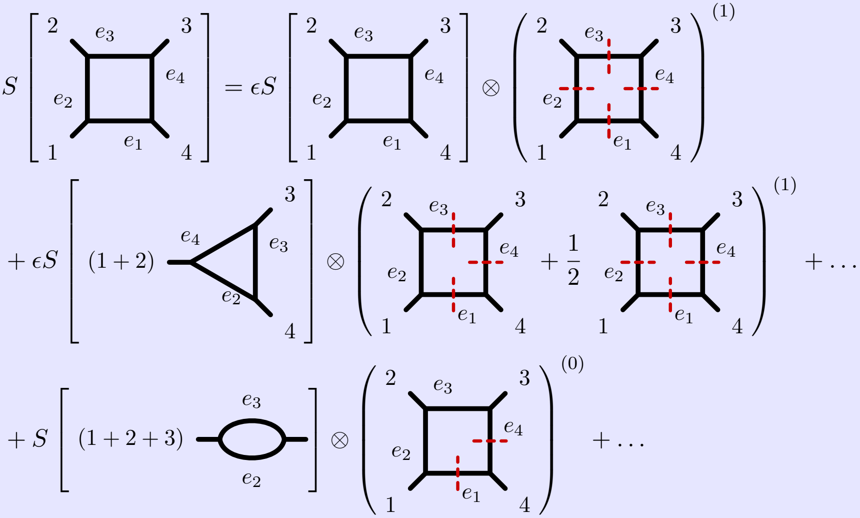
$$\Big\{p_1^2, p_3^2, s, t, \quad p_1^2 - s,\ p_1^2 - t,\ p_3^2 - s,\ p_3^2 - t, \quad st - p_1^2 p_3^2, \quad s + t - p_1^2 - p_3^2\Big\}$$

A valid word might look like

$$s \otimes s \otimes \cdots \otimes s \ \otimes\ (p_1^2 - s) \otimes \cdots \otimes (p_1^2 - s)\ \otimes\ (st - p_1^2 p_3^2) \otimes \cdots \otimes (s + t - p_1^2 - p_3^2).$$

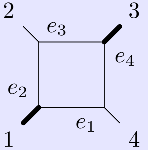
Explicit results from recursion

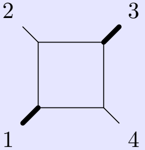
The generic box recursion formally reads diagrammatically



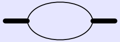
Explicit results from recursion

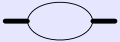
After entering the cut data into the formula and reducing triangles to bubbles,

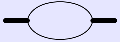
$S\Bigg[$ 

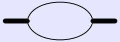
$\Bigg] = \epsilon S\Bigg[$ 

$\Bigg]\left(\otimes(s+t-p_1^2-p_3^2)-\otimes(st-p_1^2p_3^2)\right)$

$+\,S\Big[\; p_3^2$  $\Big]\left(-\otimes p_3^2+\otimes(p_3^2-s)(p_3^2-t)-\otimes(s+t-p_1^2-p_3^2)\right)$

$+\,S\Big[\; t$  $\Big]\left(+\otimes t-\otimes(t-p_1^2)(t-p_3^2)+\otimes(s+t-p_1^2-p_3^2)\right)$

$+\,S\Big[\; p_1^2$  $\Big]\left(-\otimes p_1^2+\otimes(p_1^2-s)(p_1^2-t)-\otimes(s+t-p_1^2-p_3^2)\right)$

$+\,S\Big[\; s$  $\Big]\left(+\otimes s-\otimes(s-p_1^2)(s-p_3^2)+\otimes(s+t-p_1^2-p_3^2)\right).$

Explicit results from recursion

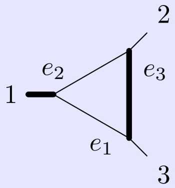
Letters: $\left\{p_1^2, p_3^2, s, t, \qquad p_1^2 - s,\ p_1^2 - t,\ p_3^2 - s,\ p_3^2 - t, \qquad st - p_1^2 p_3^2, \quad s + t - p_1^2 - p_3^2\right\}$

- Only one of p_1^2, p_3^2, s, t appears in a word, and it appears at least once.
- A word consists first of a sequence of blue letters, then a single red letter may appear, then the rest of the letters are black.
- A word consisting of a sequence of blue followed by $s + t - p_1^2 - p_3^2$ can be constructed in 2 ways which cancel such that this word never occurs in the symbol. Since a blue letter always comes first, $s + t - p_1^2 - p_3^2$ can only appear in the third position onward.
- Each word that can appear does so uniquely and has a coefficient of ± 1.

Hence we have fully characterised the symbol of the 2 mass easy box, beyond what is possible with other approaches.

Spurious cancellations

In some cases, symbol letters cancel out during the first few steps of the recursion and become forbidden from appearing at the starting positions of words. In the case of the one mass opposite triangle,



the weight-1 order cancels out entirely. The first non-trivial symbol order is

$$m^2 \otimes m^2 + m^2 \otimes (m^2 + p^2) + p^2 \otimes m^2 - p^2 \otimes (m^2 + p^2).$$

Since the recursive term in the recursion formula is at the start of the word, such cancellations will propagate throughout higher weights, also restricting how they may start.

Gram and Cayley determinants

Kinematic data enters into the expression for cut Feynman integrals in terms of the determinants Gram_C and Y_C defined as

$$\mathrm{Gram}_C = \det((q_i - q_*) \cdot (q_j - q_*))_{i,j \in C/*}$$

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In an expansion they appear individually inside logs. For instance,

$$\mathcal{C}_{[n-1]} \tilde{J}_n = -\frac{2^{-\frac{n}{2}}\, i^{\frac{n}{2}}}{\sqrt{Y_{[n]}}} \left(1 + 2\epsilon \ln \left(1 + \sqrt{\frac{Y_{[n]} G_{[n-1]} - G_{[n]} Y_{[n-1]}}{Y_{[n]} G_{[n-1]}}} \right) + \epsilon \ln \left(\frac{G_{[n-1]}}{4\, Y_{[n-1]}} \right) \right) + \mathcal{O}\left(\epsilon^2\right) .$$

Only the zeroth or first orders in ϵ for the cuts are needed in the recursion.

Using determinants to explain cancellations

One-loop diagrams can be expressed in terms of their cuts via

$$\sum_{i\in[n]} \mathcal{C}_i \tilde{J}_n + \sum_{\substack{i,j\in[n]\\ i<j}} \mathcal{C}_{ij} \tilde{J}_n = -\epsilon\, \tilde{J}_n \mod i\pi\,.$$

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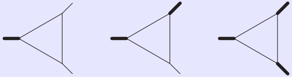
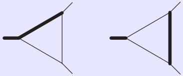
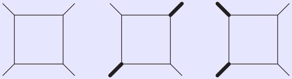
Using the ϵ expansions for different types of cuts, we wish to find some conditions on Gram and Y_C that would explain cancellations of letters during the first few steps of the recursion. There are many subtleties to iron out, including different expressions for odd/even n, only having expressions for maximal, NM, NNM cuts, different limiting procedures, divergences...

Summary of diagrams

Diagram	Alphabet	Dictionary
	✓ ✓ ✓	✓ ✓ ✓
	✓ ✓ ⚙	✓ ✓ ✓
	⚙ ⚙	✓ ✓
	✓ ✓ ⚙	✓ ✓ ⚙

Open questions

- What is the symbol alphabet?
- At which order ϵ do letters appear?
- Where do letters appear in words relative to each other?
- When can we take limits to recover more diagrams from the general case?

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  There is a well-defined way to do this from cut formulas.

More open questions

- What is the easiest way to determine the minimum alphabet?
- How can we explain the alphabets directly from the polytope geometry?
- Can this be generalised to more than 1 loop?
- ...

Thank you