Solitons in 1 dimension

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Abstract

Solitons are ubiquitous phenomena, appearing in sciences ranging from neuroscience to magnetism to oceanography. In this short exposition of solitons in 1 space and 1 time dimension, we begin by reviewing the basics of wave solutions leading up to the Korteweg-de Vries equation. Kink solitons are explored in the context of classical field theory, with examples of ϕ^4 theory and the sine-Gordon equation to illustrate key ideas. We end with a short and sweet discussion about solitons in more spatial dimensions, namely vortices and skyrmions. A webpage with animations accompanies this reading and can be found here.

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1 Introduction

Solitons are special solutions of nonlinear wave equations which have several interesting properties. One of the main distinguishing features of solitons, and of the related phenomena of solitary waves, is the lack of dispersion or dissipation observed as they propagate, meaning that their original shape and velocity are preserved. This particle-like nature explains the 'on' suffix. Solitons appear in numerous areas of physics, including hydrodynamics, optics, particle physics, and cosmology.

Let us first formulate a precise definition of a soliton, following that given in [1]. To do this, we must introduce the related concept of solitary waves. A *solitary wave* is a localised non-singular solution of a nonlinear field equation, whose energy density ε is localised and has a space-time dependence of the form

$$\varepsilon(\mathbf{x}, t) = \varepsilon(\mathbf{x} - \mathbf{u}t). \tag{1.1}$$

Note that there are several different definitions of solitary waves to be found in the literature; in particular some references require that the fields themselves have a travelling wave space-time dependence.
For the purposes of this text, *solitons* are solitary waves with the added constraint that their energy density profiles return asymptotically to their original shapes and velocities as $t \to \infty$. This essentially means that if we have two widely separated solitons at $t \to -\infty$, which then collide at some finite time, then as $t \to \infty$ we will again have two separate solitons with their original shapes and velocities. This stringent requirement means that there are far fewer equations known to admit soliton solutions than there are to admit solitary wave solutions.
However, in the literature the distinction between solitary waves and solitons is sometimes blurred, and the term 'soliton' is also used to refer to solitary waves. Sometimes solitons are defined in terms of integrability of partial differential equations. In the following, we will usually use the term 'soliton' in this broader sense [1].

In this text, we discuss the KdV equation as well as kink solutions in ϕ^4 theory and sine-Gordon theory, drawing primarily from [2–4]. To see interesting animations, see the associated webpage.

2 The Korteweg-de Vries Equation

In 1844, a young Scottish engineer observed a mass of water in moving in front of a ship. It "rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well defined heap of water, which continued its course along the channel without change of form or diminution of speed" [5]. He was compelled to follow the wave on horseback, watching the wave preserve its form. His name was John Scott Russell and he coined this phenomenon the "wave of translation". As we will see, this was nothing but a solution to the KdV equation. What follows is mainly taken from [2].

2.1 Wave equations

Adopting the notation $u_x = \partial u/\partial x$, the simplest wave equation in one (spatial) dimension is

$$u_{tt} - c^2 u_{xx}^2 = 0, \tag{2.1}$$

for some wave amplitude $u(x, t)$. This linear equation has general solution called the *d'Alembert solution*

$$u(x, t) = f(x - ct) + g(x + ct), \tag{2.2}$$

where f and g are arbitrary functions to be determined by initial values, e.g. $u(x,0)$, $u_t(x,0)$, ... One of these functions represents a *travelling wave* moving to the right and the other a wave moving to the left, both with speed c. For example, take the following *harmonic* wave ansatz;

$$u_0(x,t) = c_1 e^{i(x-ct)} + c_2 e^{-i(x+ct)}. \tag{2.3}$$

We immediately see that the wave equation (2.1) is satisfied since

$$u_{0,tt} = -c^2 u_0 \qquad u_{0,xx} = -u_0. \tag{2.4}$$

Physical systems often introduce specific constraints on $u(x,t)$ in addition to the wave equation (2.1). Examples include dispersive, dissipative and nonlinear wave solutions.

Dispersive waves

Consider the simplest dispersive wave equation which further limits solutions to (2.1)

$$u_t + u_x + u_{xxx} = 0. \tag{2.5}$$

To find a harmonic solution of the form $u(x,t) \propto e^{i(kx-\omega t)}$, the frequency ω is related to the wave number k through the dispersion relation imposed by (2.5)

$$\omega = k - k^3. \tag{2.6}$$

This means that $u(x,t) = f(x-ct)$ if $c = \omega/k = 1-k^2$, which describes a wave whose velocity is proportional to its wave number, i.e. a dispersive wave. This characteristic can be formulated in terms of unequal group velocity (rate at which the wave packet/envelope moves) and phase velocity (rate at which a wave propagates in a medium). The superposition of two such waves with different wave numbers would lose its shape over time, since the two waves propogate at different rates.

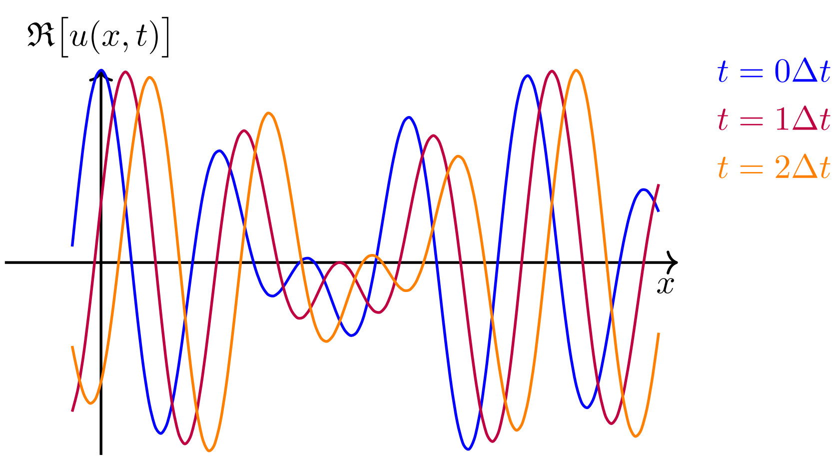

Figure 1. A superposition of two different $\Re\big[u_i(x,t)\big] \propto \cos\big(k_i x - k_i t + k_i^3 t\big)$ for different wave numbers $k_{1,2}$. These are the real parts of the harmonic solution to (2.5). Over two time steps Δt, we see that $u(x,t)$ does not preserve its shape due to different propagation speeds dependent on k_i.

Dissipative waves

If instead of adding an odd spatial derivative of $u(x,t)$ to $u_t + u_x = 0$ we added an even derivative, for example

$$u_t + u_x - u_{xx} = 0, \tag{2.7}$$

then the dispersion relation would become $\omega = k - ik^2$. This time, the harmonic solution would have the form

$$u(x,t) \propto e^{-k^2 t} e^{ik(x-t)}, \tag{2.8}$$

which is a travelling wave decaying exponentially over time, i.e. *dissipating*. The decay comes from the minus sign in fron of u_{xx}. Had it been a plus, then the wave would grow exponentially as time went on.

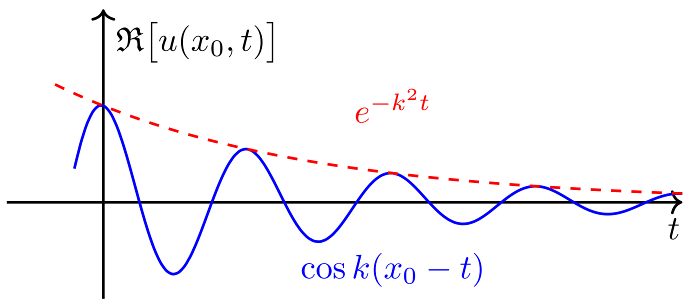

Figure 2. At some point x_0, the real part of the harmonic wave solution to the dissipative equation (2.7) decays with time as $e^{-k^2 t}$.

Nonlinear waves

Certain wave phenomena cannot be decomposed into a superposition of other wave solutions. Such waves are called *nonlinear* as we cannot think of them as linear combinations of component wave forms. The simplest type of nonlinearity is of the form uu_x such that

$$u_t + (1+u)u_x = 0, \tag{2.9}$$

The above equation has general solution of the form

$$u(x,t) = f\big(x - (1+u)t\big) \tag{2.10}$$

where f is an arbitrary function. After a finite time, solutions of this form admit multi-valued functions f so that a discontinuity appears. Physically, this means the wave shape changes over time, and eventually breaks as seen in Fig. 3 .

2.2 Generalised KdV equation

Certain physical system constrain wave solutions with a combination of the above characteristics. Fore example, one might consider systems with solutions $u(x,t)$ which present nonlinearity *and* dissipation such that

$$u_t + (1+u)u_x - u_{xx} = 0 \tag{2.11}$$

which is the *Burgers equation*, or nonlinearity and dispersion

$$u_t + (1+u)u_x + u_{xxx} = 0. \tag{2.12}$$

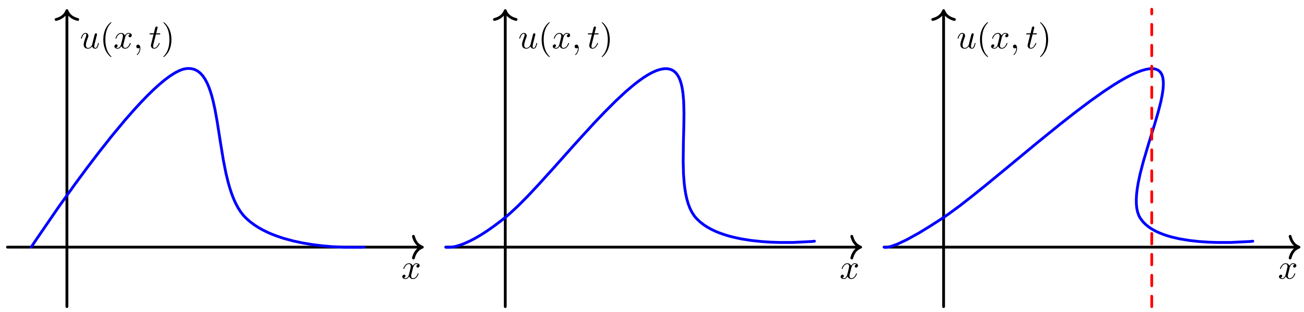

Figure 3. A schematic nonlinear wave losing its shape over time (from left to right), until it breaks due to a three-value discontinuity indicated by the red line. This is actually how water waves on the surface break down.

The latter is a variant of the *Korteweg-de Vries (KdV) equation.* If we take the general form

$$u_t + \frac{\alpha}{\gamma} u u_x + \frac{\beta}{\gamma^3} u_{xxx} = 0 \tag{2.13}$$

for non-zero coefficients α, β, γ, we obtain the generalised KdV equation. The most common choice is

$$u_t - 6uu_x + u_{xxx} = 0. \tag{2.14}$$

Many variations of the KdV equation have been explored, and it is of course possible to extend the above discussion to more than one spatial dimension. The 1+1 dimensional KdV equation is *integrable* which means we can find a solution and, in particular, a Lax pair L_α (with $\alpha = x, t$) can be found satisfying the zero curvature condition [6]

$$\partial_x L_t - \partial_t L_x = [L_x, L_t]. \tag{2.15}$$

2.3 Solution

Eventually Russell dismounted his horse and, after performing laboratory experiments by dropping weights on water baths, found an empirical formula for the propagation velocity c of wave packets across the surface of the baths;

$$c^2 = g(h + a) \tag{2.16}$$

where $g = 9.81$ ms^{-2} is the acceleration due to gravity at the Earth's surface, h is the height of water in the bath, and a is the amplitude of the wave crest. Because of this formula, waves propagating in a nonlinear fashion due to gravity (e.g. tidal waves) are called surface *gravity waves.* In 1895, Korteweg and de Vries showed that the solitary wave solution to (2.14) is of the form

$$\zeta(x,t) = a\,\mathrm{sech}^2\left\{\frac{1}{2}\sqrt{\frac{3a}{h^3}}(x - ct)\right\}, \tag{2.17}$$

where c is given by Russell's empirical formula in the limit of relatively small waves, $a \ll h$,

$$c = \sqrt{gh}(1 + \frac{1}{2}\frac{a}{h}). \tag{2.18}$$

So we see that the solitary wave Russell saw was in fact a solution to the KdV equation. From this expression for c, we see that waves with taller crests travel faster than shorter ones. (See website.)

3 Kink solitons

Classical field theories with multiple vacua often contain soliton solutions which connect these vacua. In $1+1$ dimensions, such solutions are known as *kink solutions*.

We will begin by considering the general case of a single scalar field $\phi(x,t)$ in $1+1$ dimensions. The Lagrangian of such a scalar field theory is of the form

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - U(\phi), \tag{3.1}$$

where $U(\phi)$ is some potential which is a polynomial in ϕ, and we have chosen the Minkowski metric to have $(+,-)$ signature. The equation of motion obtained from this Lagrangian is

$$0 = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = \partial_\mu\partial^\mu\phi + \frac{dU}{d\phi}. \tag{3.2}$$

This is known as the *nonlinear Klein-Gordon equation*, since it is nonlinear unless U is quadratic in ϕ. Invoking Noether's theorem, we also find a conserved energy

$$E = \int\left(\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\phi'^2 + U(\phi)\right)dx\,. \tag{3.3}$$

For solitons that are *static* solutions, the time derivatives vanish and (3.2) becomes

$$\frac{d^2\phi}{dx^2} = \frac{dU}{d\phi}, \tag{3.4}$$

while the energy is

$$E = \int\left(\frac{1}{2}\phi'^2 + U(\phi)\right)dx\,. \tag{3.5}$$

We will first consider a general potential U with minimum value 0. Invoking Noether's theorem, we find a conserved energy

$$E = \int\left(\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\phi'^2 + U(\phi)\right)dx\,. \tag{3.6}$$

We will find the kink solution by finding a minimum of the energy. However, for any fixed t, the field $\phi(x,t)$ must satisfy $\phi(x,t) \to$ vacuum as $x \to \pm\infty$, to prevent the field having infinite energy. We hence fix boundary conditions $\phi(\infty)$ and $\phi(-\infty)$, and require the kinks to minimise the energy subject to these boundary conditions.

To do this, we write the potential in the form

$$U(\phi) = \frac{1}{2}\left(\frac{dW(\phi)}{d\phi}\right)^2, \tag{3.7}$$

in which case the energy is

$$\begin{aligned} E &= \frac{1}{2}\int \left(\phi'^2 + \left(\frac{dW}{d\phi}\right)^2\right) dx \\ &= \frac{1}{2}\int \left(\phi' \mp \frac{dW}{d\phi}\right)^2 dx \pm \int \frac{dW}{d\phi}\frac{d\phi}{dx}\, dx \\ &= \frac{1}{2}\int \left(\phi' \mp \frac{dW}{d\phi}\right)^2 dx \pm \int dW \\ &= \frac{1}{2}\int \left(\phi' \mp \frac{dW}{d\phi}\right)^2 dx \pm\ \left(W(\phi(\infty)) - W(\phi(-\infty))\right). \end{aligned} \tag{3.8}$$

Since the second term depends only on the fixed boundary conditions, the energy will be minimised when the first term vanishes. Note that we shall have to choose the sign that makes the second term non-negative, since energy is non-negative. The energy is thus minimised when

$$\phi' = \pm\frac{dW}{d\phi}. \tag{3.9}$$

This is the *Bogomolny equation*. The corresponding energy is

$$E = \pm\left(W(\infty) - W(-\infty)\right), \tag{3.10}$$

which is the *Bogomolny energy bound*.

3.1 ϕ^4 potential

In ϕ^4 theory, we choose the potential

$$U(\phi) = \frac{1}{2}(1-\phi^2)^2. \tag{3.11}$$

This potential has two minima, located at

$$\phi(x) = \pm 1, \tag{3.12}$$

which means that there are two vacuum field configurations, both of zero energy. We want to consider kink solutions that connect the two vacua. From (3.7), we see that

$$\frac{dW}{d\phi} = 1 - \phi^2, \tag{3.13}$$

so we choose $W(\phi) = \phi - \phi^3/3$. When $\phi = \pm 1$, we have $W = \pm 2/3$, so we need to choose the upper signs in (3.8), in which case the energy of the kink is $E = 4/3$. Finally, we can actually solve for ϕ using the Bogomolny equation (3.9), which for ϕ^4 theory is

$$\phi' = 1 - \phi^2. \tag{3.14}$$

Upon rearranging this, we obtain

$$\frac{1}{1-\phi^2}d\phi = dx, \tag{3.15}$$

which can easily be integrated to get

$$\phi(x) = \tanh(x-a), \tag{3.16}$$

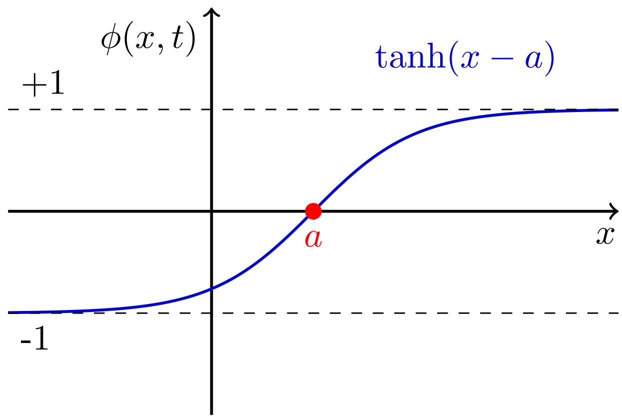

Figure 4. Kink soliton $\phi(x) = \tanh(x-a)$ centred at a.

where a is a constant of integration that labels the x-intercept of the kink. It can be thought of as the location of the 'centre' of the kink. Hence the solutions are labelled by the parameter a, which is known as the *modulus* of the solution. In the case of kinks, the moduli space of all possible modulus values is simply $\mathbb{R}$. Since the solutions are determined by the chosen combination of the boundary conditions $\phi = \pm 1$, there is also an *anti-kink* solution, which is given by

$$\phi(x) = -\tanh(x-b) \tag{3.17}$$

for some constant b, and also has energy $E = 4/3$.
So far, we have been considering static solutions. We now want to consider kinks that move in time. Since our theory is Lorentz invariant, we can simply boost our static solution (3.16) to obtain

$$\phi(x,t) = \tanh\gamma(x-vt), \tag{3.18}$$

where velocities $v \in (-1,1)$ define the Lorentz factor

$$\gamma = (1-v^2)^{-1/2}. \tag{3.19}$$

For small velocities v such that $\gamma \approx 1$, this solution can be approximated by

$$\phi(x,t) = \tanh(x-vt). \tag{3.20}$$

More generally, we can consider a field

$$\phi(x,t) = \tanh(x-a(t)) \tag{3.21}$$

for an arbitrary function $a(t)$. This is an approximate solution of the field equation for small $\dot{a}$.

Soliton interactions occur when we have kink-anti-kink configurations. Since the centre a of a soliton is only well-defined in the case of a pure kink or anti-kink, there is no longer a clear way to discuss how the centre of the particle moves. Instead, we will consider the momentum of the field obtained from Noether's theorem using the translation invariance of the Lagrangian (3.1). Applying Noether's theorem to this Lagrangian, we obtain the energy-momentum tensor

$$T^\mu_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi - \delta^\mu_\nu\mathcal{L} = \partial^\mu\phi\partial_\nu\phi - \delta^\mu_\nu\mathcal{L}. \tag{3.22}$$

This corresponds to the conserved energy

$$E = \int T_0^0 \, dx = \int \left(\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\phi'^2 + U(\phi) \right) dx, \tag{3.23}$$

and the conserved momentum

$$P = -\int T_1^0 dx = -\int \dot{\phi}\phi' \, dx \, . \tag{3.24}$$

In ϕ^4 theory, $U(\phi)$ is given by (3.11), and the solution ϕ is given by (3.16), so that the momentum is

$$P = \int \dot{a} \,\mathrm{sech}^4(x-a) \, dx \stackrel{\dagger}{=} \frac{4}{3\dot{a}} = M\dot{a}, \tag{3.25}$$

which is exactly the result we would expect for a particle of mass M and position a. Now consider a

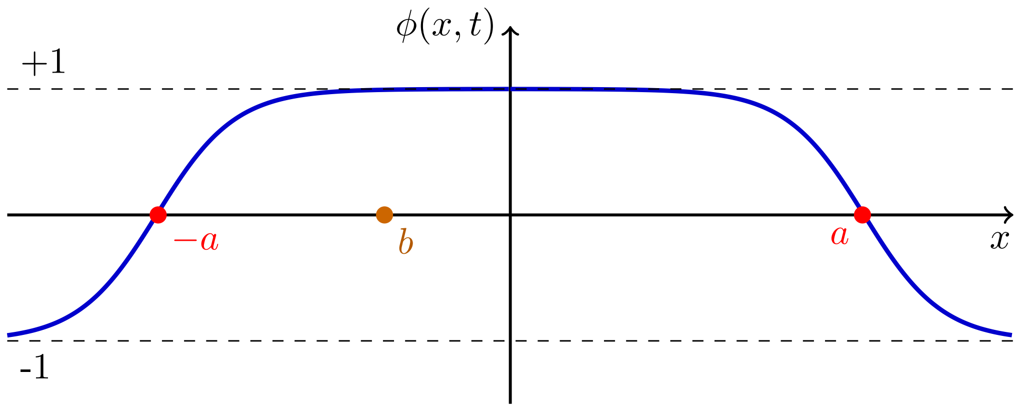

Figure 5. Kink-anti-kink solution to the ϕ^4 potential, defined for $-a \ll b \ll a$ (3.26).

configuration with a kink centred at $-a$ and an anti-kink centred at a, where we assume that a is large enough so that the kink and anti-kink are well-separated. We choose a point b lying between the kink and anti-kink such that $-a \ll b \ll a$, and consider the region to the left of b to correspond to the kink, and the region to the right of b to correspond to the anti-kink.

Since the equation of motion is nonlinear, we cannot merely add the kink and anti-kink solutions to obtain a static solution for this configuration. However, when the kink and anti-kink are well separated, we have an approximate static solution

$$\phi(x) = \tanh(x+a) - \tanh(x-a) - 1, \tag{3.26}$$

where the -1 term is included to impose the correct boundary conditions for $x \to \pm\infty$. We choose this as our initial configuration. The configuration will subsequently evolve dynamically due to the interaction between the kink and anti-kink.

Assuming that the fields are initially at rest, we have $\dot{\phi} = 0$ at $t = 0$. For values of x close to b, we have $\phi(x) = 1 + \eta(x)$ for small η. To find an expression for η, we note that near $x = b$ the kink and anti-kink can be approximated by

$$\tanh(x+a) \approx 1 - 2e^{-2(x+a)}, \quad \tanh(x-a) \approx -1 + 2e^{2(x-a)}, \tag{3.27}$$

†Using the identities $\mathrm{sech}^4\, x = \mathrm{sech}^2\, x(1-\tanh^2 x)$ and $\frac{d}{dx}\tanh x = \mathrm{sech}^2\, x$.

so that our initial configuration becomes

$$\phi(x) = 1 - 2e^{-2(x+a)} - 2e^{2(x-a)}. \tag{3.28}$$

We can thus read off

$$\eta = -2e^{-2(x+a)} - 2e^{2(x-a)}. \tag{3.29}$$

We now want to consider the dynamics of this configuration. Focusing on the kink alone, the force F is given by the rate of change of the momentum:

$$F = \frac{dP}{dt} = -\int_{-\infty}^{b} \frac{\partial}{\partial t} T_1^0 \, dx \tag{3.30}$$

Since the energy-momentum tensor is conserved, we have $\partial_\mu T^\mu_\nu = 0$. Setting $\nu = 0$ in this equation, we obtain

$$\frac{\partial}{\partial t} T_1^0 + \frac{\partial}{\partial x} T_1^1 = 0. \tag{3.31}$$

This allows us to write the force as

$$F = \int_{-\infty}^{b} \frac{\partial}{\partial x} T_1^1 \, dx = T_1^1|_b = \left(-\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\phi'^2 + \frac{1}{2}(1-\phi^2) \right)_b, \tag{3.32}$$

where we used the fact that T_1^1 vanishes for $x \to -\infty$. Expanding this expression up to $\mathcal{O}(\eta^2)$, we get

$$\begin{aligned}
F &= \left(-\frac{1}{2}\eta'^2 + 2\eta^2 \right)_b \\
&= \left(-\frac{1}{2}(4e^{-2(x+a)} - 4e^{2(x-a)})^2 + 2(-2e^{-2(x+a)} - 2e^{2(x-a)})^2 \right)_b \\
&= 2\left(-\frac{1}{2}(4e^{-2(x+a)})(-4e^{2(x-a)}) + 2(-2e^{-2(x+a)})(-2e^{2(x-a)}) \right)_b \\
&= 32e^{-4a}.
\end{aligned} \tag{3.33}$$

We note that this is independent of our particular choice of the point b. The force is positive, meaning that the kink and anti-kink attract each other. As discussed in [1], an attraction results in a negative time delay, i.e. the kinks accelerate past each other.

3.2 Sine-Gordon potential

We now move on from ϕ^4 theory to briefly consider solitons in sine-Gordon theory. In this case, we have the potential

$$U(\phi) = 1 - \cos\phi, \tag{3.34}$$

which has infinitely many distinct vacua, located at $\phi = 2n\pi$ for integer n. Referring to (3.7), we see that we must choose W such that

$$\frac{dW}{d\phi} = 2\sin\frac{\phi}{2}. \tag{3.35}$$

Substituting this into the Bogomolny equation (3.9) yields

$$\frac{d\phi}{dx} = 2\sin\frac{\phi}{2}, \tag{3.36}$$

which has the solution

$$\phi(x) = 4\tan^{-1} e^{x-a}. \tag{3.37}$$

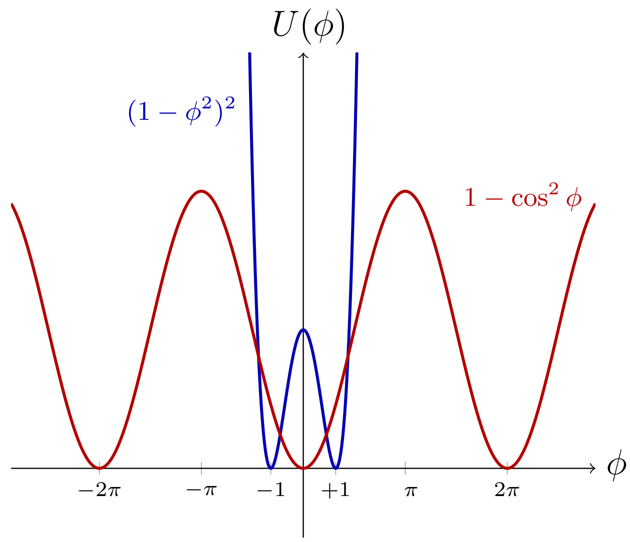

Figure 6. The ϕ^4 and sine-Gordon potentials connect different vacua, but both admit kink solitons.

This solution takes values in the range $[0, 2\pi]$, with inflection point at π. Since the sine-Gordon equation

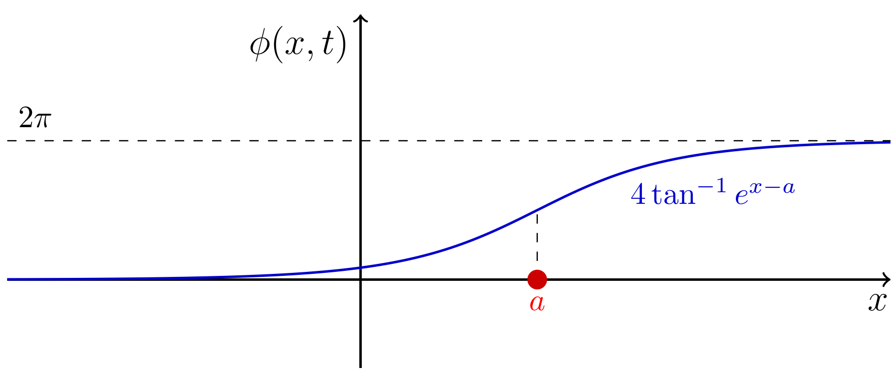

Figure 7. The sine-Gordon kink, which connects the vacua $\phi(x) = 0$ and $\phi(x) = 2\pi$.

is integrable, dynamical multi-kink solutions can be derived exactly. For example, it can be shown that there is a two-kink /soliton-soliton solution given by

$$\phi_{\rm SS}(x,t) = 4\tan^{-1}\left(\frac{v\sinh(\gamma x)}{\cosh(\gamma t)}\right), \tag{3.38}$$

where γ is again given by (3.19). Since $\phi(x,t) = \phi(x,-t)$, this solution corresponds to two solitons initially moving towards each other, each with speed v, and then subsequently bouncing off each other and moving apart. This indicates that two kinks repel each other, in contrast to the attraction we previously observed between kinks and anti-kinks in ϕ^4 theory.

4 Vortices and Skyrmions

To conclude, we will provide a short qualitative description of solitons in more than one spatial dimension. The logical next step is to consider topological solitons in $2+1$ spacetime dimensions. These are known as *vortices*; an example is the two-dimensional hedgehog, which can be visualised as a vector field on $\mathbb{C}$.

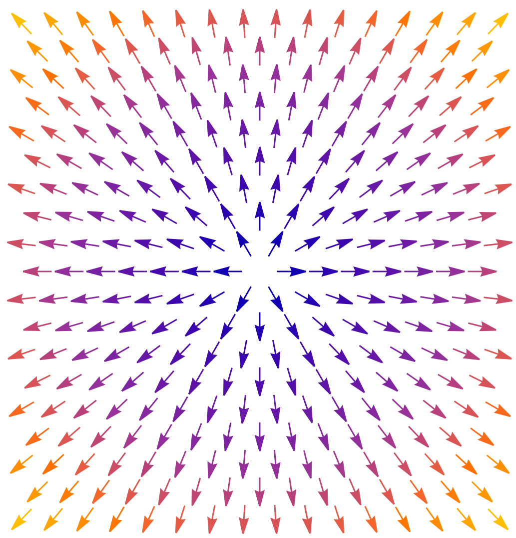

Figure 8. The 2+1 dimensional hedgehog, which is a special case of the Ginzburg- Landau vortex $\Phi(r,\theta) \sim r^N e^{iN\theta}$ in (complex) polar coordinates for $N = 1$.

In $3+1$ spacetime dimensions, solitons arise in the Skyrme model [7], which describes nucleons as solitons in an effective field theory. The classical field equations have soliton solutions, known as *skyrmions*, with an integer topological charge. This topological charge is identified with the baryon number, providing an explanation for baryon number conservation.

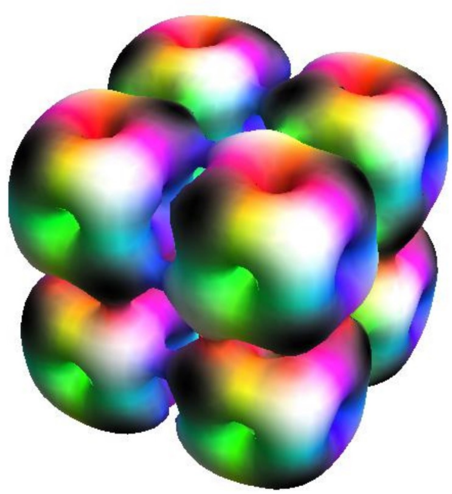

Figure 9. $B = 31$ skyrmion by D. Feist [8].

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