Deriving the central charges of the centrally extended $\mathfrak{su}(2|2)$ algebra

A review of

Foundations of the $\mathrm{AdS}_5 \times S^5$ Superstring [1]

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Abstract

In this review of *Foundations of the* $AdS_5 \times S^5$ *Superstring, Part I* [1] by Gleb Arutyunov and Sergey Frolov, results from the first two chapters are reproduced explicitly. The ultimate objective is to derive the central charges of the centrally extended $(\mathfrak{p})\mathfrak{su}(2|2)$ algebra, which is related to the symmetry algebra of the light-cone $\mathrm{AdS}_5 \times S^5$ string sigma model. First, the Green-Schwarz Lagrangian is introduced in terms of the superconformal algebra $\mathfrak{psu}(2,2|4)$, and eventually fixed by the light cone and κ-symmetry gauges. Once gauge-fixed, the model undergoes decompactification in preparation for quantisation in light cone coordinates. The superstring in $\mathrm{AdS}_5 \times S^5$ is shown to be integrable via the construction of a Lax pair which also takes values in $\mathfrak{psu}(2,2|4)$.

Acknowledgements

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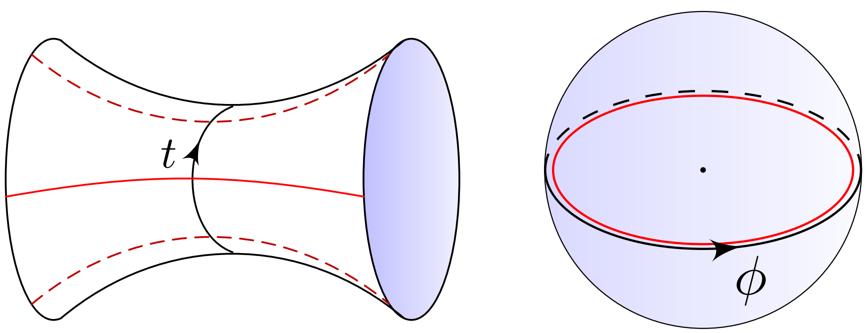
Why $\mathrm{AdS}_5 \times S^5$?

String theory began in the 1960's as an attempt to explain the strong force felt by hadrons, which we now describe with quantum chromodynamics (QCD). Boasting several 'revolutions' – periods in time when the theory took on a new life – string theory has proven itself to be a strong candidate for unifying gravity with the three fundamental forces of the Standard Model. These are the strong, weak and electromagnetic forces. While supersymmetry has not been experimentally verified, it provided a remedy to crucial pitfalls of bosonic string theory. For example, tachyons no longer appeared in the theory and the critical dimension of spacetime (at which string theory can be quantised) went from $D = 26$ to $D = 10$ instead [2, 3].

In 1997, the AdS/CFT correspondence [4] came to light and became a central focus for high-energy theorists. In certain limits, this correspondence provides a mathematical connection between a theory of gravity in D-dimensional anti-de Sitter (AdS) space with a lower $(D-1)$-dimensional conformal field theory (CFT). Type IIB superstring theory in an $\mathrm{AdS}_5 \times S^5$ background is a particular type of string theory which can be considered dual to $\mathcal{N} = 4$ super Yang-Mills theory, a staple of our understanding of non-abelian quantum field theories. The latter play an important role in particle physics, namely QCD.



$\mathbf{x} \cdot \mathbf{x} = -R^2$ in $\mathbb{R}^{n-1,2}$ $\qquad\qquad$ $\mathbf{x} \cdot \mathbf{x} = R^2$ in $\mathbb{R}^{n+1}$

Figure 1. Classical strings (red) on hypersurfaces[1] AdS_n and S^n for $n = 2$.

This work is a review of [1] in which most results of the first two chapters are reproduced or corrected, with the end goal of describing the symmetry algebra of the quantised superstring in an $\mathrm{AdS}_5 \times S^5$ target space.

Chapter 1 sets the scene by introducing superalgebra notation and the Green-Schwarz superstring. This superstring is described by a Lagrangian which exhibits a local fermionic symmetry known as κ-symmetry. We shall derive the symmetry and show its implications for integrability of the model. Various embeddings of the coset space for $\mathrm{AdS}_5 \times S^5$ into the supergroup $SU(2,2|4)$ are presented at the end, with a particular emphasis on the parametrisation which is suitable for the light cone gauge fixing to follow.

In Chapter 2 the bosonic string is used to illustrate the light cone gauge and first-order formalism which helps in the transition to Hamiltonian language, and eventually quantisation. The GS Lagrangian is then fixed in the light cone and κ-symmetry gauges before proceeding in the planar limit to quantisation.

Lengthy calculations from both chapters are exiled to the appendices for the reader's convenience.

[1] Because closed timelike curves are not allowed in physics, the AdS surface should be 'unwrapped'. See Figure 1 of [5].

① String sigma model

1.1 Super-duper algebra

The notion of 'bosonic' and 'fermionic' in supersymmetry can be encoded into the group structure of matrix blocks. In later sections, we will be embedding those particle degrees of freedom into supermatrices, which belong to the Lie superalgebra $\mathfrak{su}(2,2|4)$. Here we introduce the superconformal algebra $\mathfrak{psu}(2,2|4)$, for which the considered matrix realisation admits a $\mathbb{Z}_4$-grading. The generators of the bosonic subalgebra of $\mathfrak{su}(2,2|4)$ are constructed in preparation for the coset sigma-model, which describes the dynamics of a string on the manifold

$$\frac{PSU(2,2|4)}{SO(4,1)\times SO(5)} = \mathrm{AdS}_5 \times S^5. \tag{1.1}$$

1.1.1 Matrix realisation of $\mathfrak{su}(2,2|4)$

A superalgebra is a $\mathbb{Z}_2$-graded algebra. That is, the algebra G is a superalgebra if we have $G = G_0 \oplus G_1$, with even part G_0 and odd part G_1, and two homogeneous elements $a \in G_\alpha,\ b \in G_\beta$ satisfy $ab \in G_{\alpha+\beta}$ where the degrees $|a| = \alpha$ and $|b| = \beta$ are in the abelian group $\mathbb{Z}_2$. The corresponding Lie superalgebra $\mathscr{G} = \ln G = \mathscr{G}^{(0)} \oplus \mathscr{G}^{(1)}$ has elements $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ obeying the Lie bracket $[\ \ ,\ ]$ which satisfies

$$[\mathfrak{a},\mathfrak{b}] = -(-1)^{|\mathfrak{a}||\mathfrak{b}|}[\mathfrak{b},\mathfrak{a}], \tag{1.2}$$

$$[\mathfrak{a},[\mathfrak{b},\mathfrak{c}]] = [[\mathfrak{a},\mathfrak{b}],\mathfrak{c}] + (-1)^{|\mathfrak{a}||\mathfrak{b}|}[\mathfrak{b},[\mathfrak{a},\mathfrak{c}]]. \tag{1.3}$$

In analogy with the group product, the bracket satisfies $[\mathscr{G}^{(\alpha)}, \mathscr{G}^{(\beta)}] \subset \mathscr{G}^{(\alpha+\beta)}$ modulo $\mathbb{Z}_2$.

The *special linear* Lie superalgebra $\mathfrak{sl}(N_1|N_2) = \mathscr{G}^{(0)} \oplus \mathscr{G}^{(1)}$ over the complex field, with $\dim \mathscr{G}^{(0)} = N_1$ and $\dim \mathscr{G}^{(1)} = N_2$, consists of square $(N_1+N_2)\times(N_1+N_2)$ matrices of the generic form

$$M = \begin{pmatrix} m & \theta \\ \eta & n \end{pmatrix} \tag{1.4}$$

with vanishing supertrace $\mathrm{str}\big(M\big) \equiv \mathrm{tr}(m) - \mathrm{tr}(n)$. Such matrices M which are diagonal are *even*, while those which are off-diagonal are *odd*. The Lie bracket for $\mathfrak{sl}(N_1|N_2)$ is the standard matrix commutator.

We will be considering subsuperalgebras of $\mathfrak{sl}(4|4)$ which is itself spanned by supertraceless 8×8 matrices as above, with m, n being even 4×4 matrices and θ, η being odd 4×4 matrices. In addition to being supertraceless, elements M of the *special pseudo-unitary* Lie superalgebra $\mathfrak{su}(2,2|4)$ also satisfy

$$M = -H^{-1}M^\dagger H, \tag{1.5}$$

where

$$H = \begin{pmatrix} \Sigma & 0 \\ 0 & \mathbb{1}_4 \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \tag{1.6}$$

and $\mathbb{1}_n$ is the $n\times n$ identity matrix. If we write out the above conjugation explicitly, this implies

$$M = -H^{-1}M^\dagger H = -\begin{pmatrix} \Sigma & 0 \\ 0 & \mathbb{1}_4 \end{pmatrix}\begin{pmatrix} m^\dagger & \eta^\dagger \\ \theta^\dagger & n^\dagger \end{pmatrix}\begin{pmatrix} \Sigma & 0 \\ 0 & \mathbb{1}_4 \end{pmatrix} = -\begin{pmatrix} \Sigma m^\dagger \Sigma & \Sigma\eta^\dagger \\ \theta^\dagger\Sigma & n^\dagger \end{pmatrix}$$

such that

$$m = -\Sigma m^\dagger \Sigma, \qquad n = -n^\dagger, \qquad \eta = -\theta^\dagger \Sigma. \tag{1.7}$$

Clearly m and n span the unitary Lie algebras $\mathfrak{u}(2,2)$ and $\mathfrak{u}(4)$ respectively. The generator $\mathrm{i}\,\mathbb{1}_8$ of $\mathfrak{u}(1)$ is also an element of $\mathfrak{su}(2,2|4)$, which means the bosonic (even, diagonal) subalgebra of the latter is[2]

$$\mathfrak{su}(2,2|4)_{\text{even}} = \mathfrak{su}(2,2) \oplus \mathfrak{su}(4) \oplus \mathfrak{u}(1) \subset \mathfrak{u}(2,2) \oplus \mathfrak{u}(4) \oplus \mathfrak{u}(1). \tag{1.8}$$

The *quotient* algebra $\mathfrak{psu}(2,2|4)$ is defined as the quotient of $\mathfrak{su}(2,2|4)$ over the $\mathfrak{u}(1)$ generator, i.e.

$$\mathfrak{psu}(2,2|4) \equiv \frac{\mathfrak{su}(2,2|4)}{\mathrm{i}\,\mathbb{1}}. \tag{1.9}$$

Many times throughout the text a complex multiple of $\mathbb{1}_8$ in $\mathfrak{su}(2,2|4)$ will be taken to 0 in $\mathfrak{psu}(2,2|4)$. Elements of this quotient cannot be linearly represented as 8×8 matrices as the identity would be missing.

Writing elements of the bosonic subalgebra $\mathfrak{su}(2,2)\oplus\mathfrak{su}(4)\oplus\mathfrak{u}(1)$, and its complement in $\mathfrak{su}(2,2|4)$ will be crucial when deriving certain properties of the superstring. To this end, we should identify a suitable basis for both $\mathfrak{su}(4)$ and $\mathfrak{su}(2,2)$. We will be using the following representation of Dirac's matrices.

$$\begin{aligned}
&\gamma^1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \qquad \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & \mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \\ 0 & -\mathrm{i} & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \end{pmatrix}, \qquad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \\
&\gamma^4 = \begin{pmatrix} 0 & 0 & -\mathrm{i} & 0 \\ 0 & 0 & 0 & \mathrm{i} \\ \mathrm{i} & 0 & 0 & 0 \\ 0 & -\mathrm{i} & 0 & 0 \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \Sigma.
\end{aligned} \tag{1.10}$$

They obey the $SO(5)$ Clifford algebra relations

$$\gamma^i\gamma^j + \gamma^j\gamma^i = 2\delta^{ij}\,\mathbb{1}_4 \tag{1.11}$$

for $i,j = 1,\ldots,5$. All these matrices are hermitian, meaning that $\mathrm{i}\gamma^i$ are elements of $\mathfrak{su}(4)$ since for any $i \le 5$, with no summation, $(\mathrm{i}\gamma^i)^\dagger \mathrm{i}\gamma^i = \gamma^i\gamma^i = \mathbb{1}_4$ and the determinant of any γ^i or $\mathrm{i}\gamma^i$ is clearly 1. To show $\mathfrak{su}(4) \sim \mathfrak{so}(6)$, we will extend the spinor representation of $\mathfrak{so}(5)$, spanned by $n^{ij} = \frac{1}{4}[\gamma^i,\gamma^j]$ and satisifying

$$[n^{ij}, n^{kl}] = \delta^{jk}n^{il} - \delta^{ik}n^{jl} - \delta^{jl}n^{ik} + \delta^{il}n^{jk} \tag{1.12}$$

for $i,j... \le 5$. In particular, we add the elements[3] $n^{i6} \equiv \frac{\mathrm{i}}{2}\gamma^i$ such that the above commutation relations are satisfied but this time for $i,j... \le 6$. To see this, one performs the calculations in A.1 yielding

$$[n^{i6}, n^{kl}] = \left\{\begin{matrix} -\delta^{6l}n^{ik} & l = 6, \\ +\delta^{6k}n^{il} & k = 6, \\ -\delta^{il}n^{k6} + \delta^{ik}n^{l6} & k,l \neq 6. \end{matrix}\right\} = \delta^{6k}n^{il} - \delta^{ik}n^{6l} - \delta^{6l}n^{ik} + \delta^{il}n^{6k}. \tag{1.13}$$

Thus the $\mathfrak{su}(4)$ matrices $\mathrm{i}\gamma^i$ provide a basis for the real vector space $\mathfrak{so}(6)$.

[2]To ensure elements of the subalgebra are supertraceless, m and n must be separately traceless. So $\mathfrak{u}$ becomes $\mathfrak{su}$.

[3]Looking at the calculations done in the appendix, it should be clear that $-\frac{\mathrm{i}}{2}\gamma^i$ are valid extensions too.

To describe $\mathfrak{su}(2,2)$, we should turn our attention to extending $\mathfrak{so}(4,1)$ instead and show $\mathfrak{so}(4,2) \sim \mathfrak{su}(2,2)$. We now set $m^{ij} = \frac{1}{4}[\gamma^i, \gamma^j]$ for $i, j = 0, ..., 4$ and distinguish $\gamma^0 = \mathrm{i}\gamma^5$. These matrices are taken from the generators of $\mathfrak{su}(2,2) = \mathrm{span}_{\mathbb{R}} \left\{\frac{1}{2}\gamma^i, \frac{\mathrm{i}}{2}\gamma^5\right\}$. The pseudo-orthogonal $\mathfrak{so}(4,1)$ relations are

$$[m^{ij}, m^{kl}] = \eta^{jk}m^{il} - \eta^{ik}m^{jl} - \eta^{jl}m^{ik} + \eta^{il}m^{jk} \tag{1.14}$$

with signature $\eta = \mathrm{diag}(-1, 1, 1, 1, 1)$. If we add in the elements $m^{i5} \equiv \frac{1}{2}\gamma^i$, a similar set of calculations in A.1 shows that the above relation is still satisfied for $i, j = 0, ..., 5$ if we set $\eta = \mathrm{diag}(-1, 1, 1, 1, 1, -1)$. In summary,

$$\begin{aligned}
\mathfrak{su}(4) \sim \mathfrak{so}(6) &= \mathrm{span}_{\mathbb{R}} \left\{\frac{\mathrm{i}}{2}\gamma^i, \frac{1}{4}[\gamma^i, \gamma^j]\right\}, && i, j = 1, ..., 5, \\
\mathfrak{su}(2,2) \sim \mathfrak{so}(4,2) &= \mathrm{span}_{\mathbb{R}} \left\{\frac{1}{2}\gamma^i, \frac{\mathrm{i}}{2}\gamma^5, \frac{1}{4}[\gamma^i, \gamma^j], \frac{\mathrm{i}}{4}[\gamma^i, \gamma^5]\right\}, && i, j = 1, ..., 4.
\end{aligned} \tag{1.15}$$

Finally, $\mathrm{i}\mathbb{1}_8$ spans $\mathfrak{u}(1)$ such that these generators together span the bosonic subalgebra $\mathfrak{su}(2,2|4)$.

1.1.2 $\mathbb{Z}_4$-grading

In addition to the $\mathbb{Z}_2$ grading we described above, it turns out that the automorphism group of $\mathfrak{sl}(4|4)$ is such that we can refine the grading to $\mathbb{Z}_4$. If we define the hypercharge Υ and take some generic matrix $M \in \mathfrak{sl}(4|4)$ as

$$\Upsilon = \begin{pmatrix} \mathbb{1}_4 & 0 \\ 0 & -\mathbb{1}_4 \end{pmatrix}, \qquad M = \begin{pmatrix} m & \theta \\ \eta & n \end{pmatrix}, \tag{1.16}$$

then there exists a continuous automorphism $\delta_\rho(M)$ acting as

$$\delta_\rho(M) \equiv \begin{pmatrix} m & \rho\theta \\ \frac{1}{\rho}\eta & n \end{pmatrix} = e^{\frac{1}{2}\Upsilon \ln \rho} M e^{-\frac{1}{2}\Upsilon \ln \rho}. \tag{1.17}$$

Moving on to the finite subgroup of automorphisms, if we define the supertranspose M^{st} of a matrix $M \in \mathfrak{sl}(4|4)$ as

$$M^{st} \equiv \begin{pmatrix} m^t & -\eta^t \\ \theta^t & n^t \end{pmatrix}, \tag{1.18}$$

then we see that $M \to -M^{st}$ is an automorphism of order four. Note that $(M^{st})^{st} = \delta_{-1}(M)$. Equivalent to this 'minus supertranspose', we will choose the automorphism

$$M \to \Omega(M) \equiv -\mathcal{K}M^{st}\mathcal{K}^{-1} \tag{1.19}$$

to refine the grading to $\mathbb{Z}_4$ where we have defined the matrices

$$\mathcal{K} \equiv \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}, \qquad K \equiv -\gamma^2\gamma^4 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{1.20}$$

Note the definition of $\Omega(M)$ immediately implies $\Omega(M_1M_2) = -\Omega(M_2)\Omega(M_1)$. We start by introducing the notation $\mathscr{G} \equiv \mathfrak{sl}(4|4)$ such that the graded subspaces of the vector space $\mathscr{G}$ are

$$\mathscr{G}^{(k)} \equiv \left\{M \in \mathscr{G} \mid \Omega(M) = \mathrm{i}^k M\right\}. \tag{1.21}$$

The vector space $\mathscr{G}$ and some generic element $M \in \mathscr{G}$ can be decomposed uniquely with respect to $\Omega(M)$;

$$\begin{aligned} \mathscr{G} &= \mathscr{G}^{(0)} \oplus \mathscr{G}^{(1)} \oplus \mathscr{G}^{(2)} \oplus \mathscr{G}^{(3)}, \\ M &= M^{(0)} + M^{(1)} + M^{(2)} + M^{(3)}. \end{aligned} \tag{1.22}$$

To see that $[\mathscr{G}^{(a)}, \mathscr{G}^{(b)}] \subset \mathscr{G}^{(a+b)}$ modulo $\mathbb{Z}_4$, we can calculate

$$\Omega([M^{(a)}, M^{(b)}]) = -\mathrm{i}^{a+b} M^{(b)} M^{(a)} + \mathrm{i}^{a+b} M^{(a)} M^{(b)} = \mathrm{i}^{a+b} [M^{(a)}, M^{(b)}]. \tag{1.23}$$

According to the above, if we view $M^{(0)}$ and $M^{(2)}$ as even, then $M^{(1)}$ and $M^{(3)}$ would be odd. Given $M \in \mathscr{G} = \mathfrak{sl}(4|4)$, its projections $M^{(k)} \in \mathscr{G}^{(k)}$ can be expressed as

$$M^{(k)} = \frac{1}{4}\Big(M + \mathrm{i}^{3k}\Omega(M) + \mathrm{i}^{2k}\Omega^2(M) + \mathrm{i}^{k}\Omega^3(M)\Big) \tag{1.24}$$

since in this case $\Omega(M^{(k)}) = \mathrm{i}^k M^{(k)}$ as required[4]. In fact, the automorphisms $\Omega(M)$ restricts to $\mathfrak{su}(2,2|4)$ such that we can relabel $\mathscr{G} = \mathfrak{su}(2,2|4)$ and think of the above decomposition as the $\mathbb{Z}_4$-grading of $\mathfrak{su}(2,2|4)$ with respect to the action of $\Omega(M)$. See A.2 for details of this restriction. Reassuringly, the explicit expressions are diagonal for even components and off-diagonal for odd components.

$$\begin{aligned} M^{(0)} &= \frac{1}{2}\begin{pmatrix} m - K m^t K^{-1} & 0 \\ 0 & n - K n^t K^{-1} \end{pmatrix}, \quad M^{(1)} = \frac{1}{2}\begin{pmatrix} 0 & \theta - \mathrm{i} K \eta^t K^{-1} \\ \theta + \mathrm{i} K \eta^t K^{-1} & 0 \end{pmatrix}, \\ M^{(2)} &= \frac{1}{2}\begin{pmatrix} m + K m^t K^{-1} & 0 \\ 0 & n + K n^t K^{-1} \end{pmatrix}, \quad M^{(3)} = \frac{1}{2}\begin{pmatrix} 0 & \theta + \mathrm{i} K \eta^t K^{-1} \\ \theta - \mathrm{i} K \eta^t K^{-1} & 0 \end{pmatrix}. \end{aligned} \tag{1.25}$$

We know the bosonic subalgebra $\mathfrak{su}(2,2) \oplus \mathfrak{su}(4) \oplus \mathfrak{u}(1) \subset \mathfrak{su}(2,2|4)$ coincides with the even-graded subspace $\mathscr{G}^{(0)} \oplus \mathscr{G}^{(2)} \subset \mathscr{G}$. There should be a way to express the even components $M^{(k)}$ in terms of the generators of the bosonic algebra (1.15). It is argued in A.2 that the general forms of the even components are in fact linear combinations of the bosonic generators. For real coefficients m_a, n_a and $i, j = 1, ..., 4$, we have

$$M^{(0)} = \begin{pmatrix} m_1^{ij}[\gamma^i, \gamma^j] + \mathrm{i} m_2^i [\gamma^i, \gamma^5] & 0 \\ 0 & n_1^{ij}[\gamma^i, \gamma^j] + n_2^i [\gamma^i, \gamma^5] \end{pmatrix}, \tag{1.26}$$

$$M^{(2)} = \begin{pmatrix} m_3^i \gamma^i + \mathrm{i} m_4 \gamma^5 & 0 \\ 0 & \mathrm{i} n_3^i \gamma^i + \mathrm{i} n_4 \gamma^5 \end{pmatrix}. \tag{1.27}$$

The central element $\mathrm{i}\,\mathbb{1}_8 \in \mathfrak{u}(1) \subset \mathfrak{su}(2,2|4)$ also occurs in $\mathscr{G}^{(2)}$ since $\Omega(\mathbb{1}_8) = -\mathbb{1}_8$.

1.2 Green-Schwarz superstring

Following our discussion of how to decompose elements of $\mathfrak{psu}(2,2|4)$, we will now identify the symmetries of the Green-Schwarz Lagrangian density describing a closed supersymmetric string in an $\mathrm{AdS}_5 \times S^5$ background derive its equations of motion. We will briefly discuss the parity and time reversal symmetries of the Lagrangian. Kappa symmetry (κ-symmetry), a local fermionic symmetry stemming from the Wess-Zumino term in the Lagrangian, will be derived and finally the symmetry's implication for the gauge transformations of fermionic degrees of freedom will be highlighted. Consider a closed one-dimensional

[4] This follows from $\Omega^4(M) = M$ and $\mathrm{i}^{4k} = 1$.

supersymmetric string propagating in an $\mathrm{AdS}_5 \times S^5$ background. Its worldsheet is a cylinder of circumference $2r$ parametrised by the 'time' coordinate τ and spatial coordinate σ such that $-r \leq \sigma \leq r$. These are usually grouped as $(\sigma^\alpha) = (\tau, \sigma)$. The action describing such a string is

$$S = \int d\tau \int_{-r}^{r} d\sigma\, \mathscr{L} \tag{1.28}$$

where $\mathscr{L}$ is the Lagrangian density describing the dynamics. The constant prefactor $g = R^2/2\pi\alpha'$ is the dimensionless string tension where R is the radius of S^5 and α' is the Regge slope (e.g. [2, Ch. 2]). This tension g is related to the 't Hooft coupling constant λ of the dual Yang-Mills theory (cite) as $g = \sqrt{\lambda}/2\pi$.

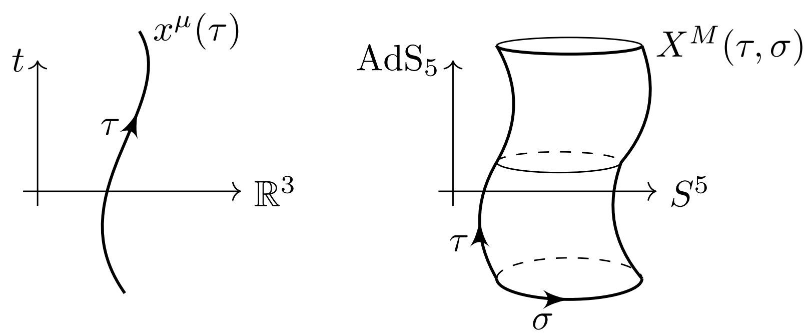


Figure 2. Worldline of a point particle in $\mathbb{R}^{3,1}$ and worldsheet of a closed string in $\mathrm{AdS}_5 \times S^5$.

1.2.1 Lagrangian

Let $\mathfrak{g}$ be an element of the supergroup $SU(2,2|4)$. Introduce the following one-form in $\mathfrak{su}(2,2|4)$

$$A \equiv -\mathfrak{g}^{-1} d\mathfrak{g} = A^{(0)} + A^{(1)} + A^{(2)} + A^{(3)}, \tag{1.29}$$

where the decomposed elements $A^{(k)} \in \mathscr{G}^{(k)}$ satisfy $\Omega(A^{(k)}) = \mathrm{i}^k A^{(k)}$ and

$$A^{(k)} = \frac{1}{4}\big[A + \mathrm{i}^{3k}\Omega(A) + \mathrm{i}^{2k}\Omega^2(A) + \mathrm{i}^k\Omega^3(A)\big]. \tag{1.30}$$

These one-forms also have zero curvature $dA - A \wedge A = 0$. In component form.

$$\partial_\alpha A_\beta - \partial_\beta A_\alpha - [A_\alpha, A_\beta] = 0. \tag{1.31}$$

We postulate the following Lagrangian density describing our superstring:

$$\mathscr{L} = -\frac{g}{2}\Big[\gamma^{\alpha\beta}\mathrm{str}\big(A_\alpha^{(2)} A_\beta^{(2)}\big) + \kappa\varepsilon^{\alpha\beta}\mathrm{str}\big(A_\alpha^{(1)} A_\beta^{(3)}\big)\Big] \qquad \alpha, \beta \in \{\tau, \sigma\}. \tag{1.32}$$

The rescaled worldsheet metric $\gamma^{\alpha\beta} = h^{\alpha\beta}/\sqrt{-h} = \gamma^{\beta\alpha}$ is the Weyl-invariant combination of the worldsheet metric $h_{\alpha\beta}$. In general,

$$\gamma^{\alpha\beta} = h_{\alpha\beta}\sqrt{-h}, \qquad \det(\gamma_{\alpha\beta}) = \det\Big(\gamma^{\alpha\beta}\Big) = (\sqrt{-h})^2/h = -1, \tag{1.33}$$

and in conformal gauge we set $(\gamma_{\alpha\beta}) = (\gamma^{\alpha\beta}) = \mathrm{diag}(1, -1)$. By convention we take the Levi-Civita symbol to satisfy $\varepsilon^{\tau\sigma} = 1$ where $\varepsilon^{\alpha\beta} = -\varepsilon^{\beta\alpha}$. To understand why this Lagrangian is justified, see A.3.

To derive the equations of motion, we will need to find $\delta\mathscr{L}$. If we define

$$\Lambda^\alpha \equiv g\left[\gamma^{\alpha\beta}A^{(2)}_\beta - \frac{\kappa}{2}\varepsilon^{\alpha\beta}\big(A^{(1)}_\beta - A^{(3)}_\beta\big)\right], \tag{1.34}$$

then the variation of the Lagrangian with respect to the element $\mathfrak{g}$ takes the form derived in A.4, namely

$$\delta\mathscr{L} = -\mathrm{str}\big(\delta A_\alpha \Lambda^\alpha\big). \tag{1.35}$$

$\delta\mathscr{L}/\delta\mathfrak{g}$ can be expanded and set to zero to find the equations of motion as an element of $\mathfrak{su}(2,2|4)$

$$\partial_\alpha\Lambda^\alpha - [A_\alpha, \Lambda^\alpha] = \rho\cdot \mathbb{1}_8. \tag{1.36}$$

The above obviously vanishes modulo $\mathrm{i}\mathbb{1}_8$ and we will be careful moving forward when working in $\mathfrak{psu}(2,2|4)$ since only the traceless part of the equation of motion will be under consideration. In turn this can be projected onto $\mathscr{G}^{(2)}$ and $\mathscr{G}^{(1,3)}$ to give

$$\gamma^{\alpha\beta}\partial_\alpha A^{(2)}_\beta - \gamma^{\alpha\beta}[A^{(0)}_\alpha, A^{(2)}_\beta] + \frac{\kappa}{2}\varepsilon^{\alpha\beta}\big([A^{(1)}_\alpha, A^{(1)}_\beta] - [A^{(3)}_\alpha, A^{(3)}_\beta]\big) = 0, \tag{1.37}$$

$$\gamma^{\alpha\beta}[A^{(3)}_\alpha, A^{(2)}_\beta] + \kappa\varepsilon^{\alpha\beta}[A^{(2)}_\alpha, A^{(3)}_\beta] = 0, \tag{1.38}$$

$$\gamma^{\alpha\beta}[A^{(1)}_\alpha, A^{(2)}_\beta] - \kappa\varepsilon^{\alpha\beta}[A^{(2)}_\alpha, A^{(1)}_\beta] = 0. \tag{1.39}$$

Treating the worldsheet metric $\gamma_{\alpha\beta}$ as an independent dynamic field results in the equations of motion

$$\mathrm{str}\big(A^{(2)}_\alpha A^{(2)}_\beta\big) - \frac{1}{2}\gamma_{\alpha\beta}\gamma^{\rho\delta}\mathrm{str}\big(A^{(2)}_\rho A^{(2)}_\delta\big) = 0, \tag{1.40}$$

which are famously known as the *Virasoro constraints* and will play a recurring role in this work.

A common idea in the Lagrangian formalism is to identify the global symmetries of the system such that the associated Noether current's conservation simplifies the problem. Consider in Figure 3 an analogy to $SO(3)$ rotations of a point on a sphere $S^2 \cong SO(3)/SO(2)$, whereby the arrow from the center of the sphere to the point can be spun about itself, corresponding to an $SO(2)$ symmetry.

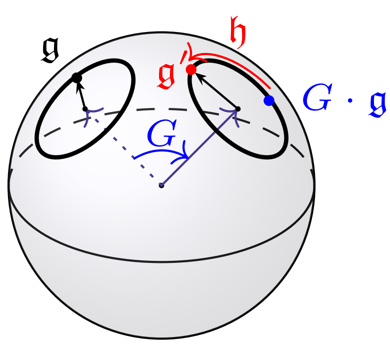


Figure 3. Subjected to the left action/multiplication of a $PSU(2,2|4)$ element G, a chosen coset representative $\mathfrak{g}$ in $PSU(2,2|4)/\big(SO(4,1)\times SO(5)\big)$ becomes some other element $G\mathfrak{g}$ in $PSU(2,2|4)$. To write this element in terms of a coset representative (identified by the black arrow), one must introduce a compensating element $\mathfrak{h}$ in $SO(4,1)\times SO(5)$ such that $G\mathfrak{g} = \mathfrak{g}'\mathfrak{h}$. We interpret this as $G : \mathfrak{g} \to \mathfrak{g}'$.

As discussed in A.3, the Lagrangian is invariant under $SO(4,1) \times SO(5)$ transformations such that global $PSU(2,2|4)$ transformations act on – and result in – coset representatives $\mathfrak{g}$. This gives rise to a Noether current $J^\alpha = \mathfrak{g}\Lambda_\alpha\mathfrak{g}^{-1}$ which is conserved (A.31) thanks to the equations of motion (1.36). More precisely,

$$\partial_\alpha J^\alpha = \mathfrak{g}\left(\partial_\alpha \Lambda^\alpha - [A^\alpha, \Lambda_\alpha]\right)\mathfrak{g}^{-1} = \rho \mathbb{1}_8, \tag{1.41}$$

which tells us only the traceless part of $J^\alpha \in \mathfrak{su}(2,2|4)$ is conserved.

1.2.2 Parity transformation and time reversal

We showed in 1.1.2 that the continuous automorphism $\delta_\rho(M)$ of $\mathfrak{sl}(4|4)$ restricted to $\mathfrak{psu}(2,2|4)$ provided the transformation parameter satisfied $|\rho|^2 = 1$. Here we will consider $\delta_{\pm \mathrm{i}}(M)$ which fall under that umbrella and seem to enact parity transformations, or equivalently time reversal, on the worldsheet (τ, σ).

We begin by considering the following transformation of $\mathfrak{g} \in PSU(2,2|4)$ enacted by some global bosonic element $U \in SU(2,2|4)$:

$$\mathfrak{g} \to U\mathfrak{g}U^{-1}. \tag{1.42}$$

In A.3, we pointed out that any $SO(4,1) \times SO(5)$ transformations of $\mathfrak{g}$ would leave the Lagrangian invariant. Assuming U is not such an element, the connection $A = -\mathfrak{g}^{-1}d\mathfrak{g}$ becomes

$$A \to A' = UAU^{-1}. \tag{1.43}$$

If we also impose that U commutes with $\mathcal{K}$, then a straightforward application of the $\mathbb{Z}_4$-grading (1.30) results in[5]

$$\begin{aligned} A'^{(2)} &= \frac{1}{2}\left[U(A^{(0)} + A^{(2)})U^{-1} - (U^{-1})^t(A^{(0)} - A^{(2)})U^t\right], \\ A'^{(1)} &= \frac{1}{2}\left[U(A^{(1)} + A^{(3)})U^{-1} + (U^{-1})^t(A^{(1)} - A^{(3)})U^t\right], \\ A'^{(3)} &= \frac{1}{2}\left[U(A^{(1)} + A^{(3)})U^{-1} - (U^{-1})^t(A^{(1)} - A^{(3)})U^t\right]. \end{aligned} \tag{1.44}$$

If we could factor the U from both sides of these expressions, resulting in a conjugation transformation, the situation would simplify dramatically. There are two obvious ways one could relate U^t to U^{-1}. The first is by letting

$$U^tU = \mathbb{1}_8, \qquad [U, \mathcal{K} = 0]. \tag{1.45}$$

The transformation (1.44) simplify to

$$A'^{(2)} = UA^{(2)}U^{-1}, \qquad A'^{(1)} = UA^{(1)}U^{-1}, \qquad A'^{(3)} = UA^{(3)}U^{-1},$$

and leave the supertrace terms in the Lagrangian (1.32) unchanged. This should not come as a surprise since such matrices form a subgroup of $SO(4) \times SO(4) \subset SO(4,1) \times SO(5)$ and can be removed from consideration. Another option is to let

$$U^tU = \Upsilon, \qquad [U, \mathcal{K} = 0]. \tag{1.46}$$

In this case $U^{-1} = \Upsilon U^t = U^t\Upsilon$. If we recall $\delta_{-1}(M) = \Upsilon M \Upsilon = \pm M$ for $(+)$ M even and $(-)$ for M odd, this implies

$$A'^{(2)} = UA^{(2)}U^{-1}, \qquad A'^{(1)} = UA^{(3)}U^{-1}, \qquad A'^{(3)} = UA^{(1)}U^{-1}. \tag{1.47}$$

Effectively this transformation switches $A^{(1)} \leftrightarrow A^{(3)}$. Since these two odd projections come with the antisymmetric $\varepsilon^{\alpha\beta}$ in the Lagrangian, this transformation changes the sign of the Wess-Zumino term in (1.32). The rest of the discussion is a little inconsistent so I will try to clear it up for the final thesis.

[5] I might include the derivation of this in an appendix.

1.2.3 Kappa symmetry

It was mentioned that the Green-Schwarz Lagrangian enjoyed a local fermionic symmetry known as κ-symmetry. In this section, shadowed by explicit calculations in A.5, we will derive this symmetry by showing that $\delta_\epsilon \mathscr{L} = 0$ under the local action of a group element e^ϵ in $PSU(2,2|4)$

$$\mathfrak{g} \to \mathfrak{g} e^\epsilon. \tag{1.48}$$

where $\epsilon = \epsilon(\tau,\sigma)$ is a local fermionic element of $\mathfrak{psu}(2,2|4)$. We start with $A \to A + \delta_\epsilon A$ where

$$\begin{aligned} \delta_\epsilon A &= -(e^{-\epsilon}\mathfrak{g}^{-1})d(\mathfrak{g}e^\epsilon) - A = -e^{-\epsilon}\mathfrak{g}^{-1}d\mathfrak{g}e^\epsilon - e^{-\epsilon}\mathfrak{g}^{-1}\mathfrak{g}de^\epsilon - A \\ &\approx (\mathbb{1}_8 - \epsilon)A(\mathbb{1}_8 + \epsilon) - e^{-\epsilon}e^\epsilon d\epsilon - A = [A,\epsilon] - d\epsilon. \end{aligned} \tag{1.49}$$

To find the change in Lagrangian $\delta_\epsilon \mathscr{L}$ following this group action, we must find the decomposition of $\delta_\epsilon A$. Noting that $[\mathscr{G}^{(a)}, \mathscr{G}^{(b)}] \subseteq \mathscr{G}^{(a+b)}$ and ϵ is fermionic so that $\epsilon^{(0)} = \epsilon^{(2)} = 0$, we get by inspection

$$\begin{aligned} \delta_\epsilon A^{(0)} &= [A^{(1)}, \epsilon^{(3)}] + [A^{(3)}, \epsilon^{(1)}], \\ \delta_\epsilon A^{(1)} &= [A^{(0)}, \epsilon^{(1)}] + [A^{(2)}, \epsilon^{(3)}] - d\epsilon^{(1)}, \\ \delta_\epsilon A^{(2)} &= [A^{(1)}, \epsilon^{(1)}] + [A^{(3)}, \epsilon^{(3)}], \\ \delta_\epsilon A^{(3)} &= [A^{(0)}, \epsilon^{(3)}] + [A^{(2)}, \epsilon^{(1)}] - d\epsilon^{(3)}. \end{aligned} \tag{1.50}$$

We now have the ingredients to determine how the Lagrangian transforms. As derived in A.5,

$$-\frac{2}{g}\delta_\epsilon \mathscr{L} = \delta\gamma^{\alpha\beta}\mathrm{str}\big(A^{(2)}_\alpha A^{(2)}_\beta\big) - 4\mathrm{str}\Big\{\mathrm{P}^{\alpha\beta}_+[A^{(1)}_\beta, A^{(2)}_\alpha]\epsilon^{(1)} + \mathrm{P}^{\alpha\beta}_-[A^{(3)}_\beta, A^{(2)}_\alpha]\epsilon^{(3)}\Big\}. \tag{1.51}$$

where we introduced the projectors $P^{\alpha\beta}_\pm = \frac{1}{2}(\gamma^{\alpha\beta} \pm \kappa\varepsilon^{\alpha\beta})$. It follows that, for $\kappa = \pm 1$,

$$P^{\alpha\delta}_\pm P_{\mp\delta}{}^{\beta} = \frac{1}{4}(\gamma^{\alpha\delta}\delta^\beta_\delta - \kappa^2\varepsilon^{\alpha\delta}\gamma_{\delta\mu}\varepsilon^{\mu\beta}) = \frac{1}{4}(\gamma^{\alpha\beta} - \kappa^2\gamma^{\alpha\beta}) = 0.$$

Similarly, $P^{\alpha\delta}_\pm P_{\pm\delta}{}^{\beta} = P^{\alpha\beta}_\pm$. Compiling identities, we see that the projection operators are orthogonal[6]:

$$P^{\alpha\beta}_\pm + P^{\alpha\beta}_\mp = \gamma^{\alpha\beta}, \qquad P^{\alpha\delta}_\pm P_{\pm\delta}{}^{\beta} = P^{\alpha\beta}_\pm, \qquad P^{\alpha\delta}_\pm P_{\mp\delta}{}^{\beta} = 0. \tag{1.52}$$

For any vector V^α we define the projections accordingly; $V^\alpha_\pm = P^{\alpha\beta}_\pm V_\beta = P^{\beta\alpha}_\mp V_\beta$. Returning to the change in the Lagrangian (1.51), the equations of motion (1.38) and (1.39) can be recast in the form

$$P^{\alpha\beta}_+[A^{(3)}_\alpha, A^{(2)}_\beta] = -P^{\alpha\beta}_-[A^{(2)}_\alpha, A^{(3)}_\beta] = [A^{(3)}_{\delta,+}, A^{(2),\delta}_-] = 0, \tag{1.53}$$

$$P^{\alpha\beta}_+[A^{(2)}_\alpha, A^{(1)}_\beta] = -P^{\alpha\beta}_-[A^{(1)}_\alpha, A^{(2)}_\beta] = [A^{(1)}_{\delta,+}, A^{(2),\delta}_-] = 0, \tag{1.54}$$

such that

$$-\frac{2}{g}\delta_\epsilon \mathscr{L} = \delta\gamma^{\alpha\beta}\mathrm{str}\big(A^{(2)}_\alpha A^{(2)}_\beta\big) - 4\mathrm{str}\Big\{[A^{(1)}_{\delta,+}, A^{(2),\delta}_-]\epsilon^{(1)} + [A^{(3)}_{\delta,+}, A^{(2),\delta}_-]\epsilon^{(3)}\Big\}. \tag{1.55}$$

One consequence of (1.52) is that $A_{\tau,\pm}$ and $A_{\sigma,\pm}$ are in fact proportional. Indeed, we have

$$0 = P^{\alpha\beta}_\pm A_{\beta,\mp} = P^{\alpha\beta}_\pm P_{\mp\beta}{}^{\delta} A_\delta = \frac{1}{2}(\gamma^{\alpha\tau} \pm \kappa\varepsilon^{\alpha\tau})A_{\tau,\mp} + \frac{1}{2}(\gamma^{\alpha\sigma} \pm \kappa\varepsilon^{\alpha\sigma})A_{\sigma,\mp}$$

[6] They are orthogonal only if $\kappa = \pm 1$ such that there is κ-symmetry.

$$\alpha = \tau \implies 0 = \frac{1}{2}(\gamma^{\tau\tau} \pm \kappa\varepsilon^{\tau\tau})A_{\tau,\mp} + \frac{1}{2}(\gamma^{\tau\sigma} \pm \kappa\varepsilon^{\tau\sigma})A_{\sigma,\mp} = \frac{1}{2}\gamma^{\tau\tau}A_{\tau,\mp} + \frac{1}{2}(\gamma^{\tau\sigma} \pm \kappa)A_{\sigma,\mp},$$

and so we see that the different connection projected components $A_{\alpha,\pm}$ are proportional since

$$A_{\tau,\pm} = \frac{\gamma^{\tau\sigma} \pm \kappa}{\gamma^{\tau\tau}} A_{\sigma,\pm}. \tag{1.56}$$

To proceed any further, we will need to specify the forms of $\epsilon^{(1)}$ and $\epsilon^{(3)}$. We ansatz

$$\begin{aligned} \epsilon^{(1)} &= A^{(2)}_{\alpha,-}\kappa^{(1),\alpha}_{+} + \kappa^{(1),\alpha}_{+}A^{(2)}_{\alpha,-}, \\ \epsilon^{(3)} &= A^{(2)}_{\alpha,+}\kappa^{(3),\alpha}_{-} + \kappa^{(3),\alpha}_{-}A^{(2)}_{\alpha,+}. \end{aligned} \tag{1.57}$$

To see that the homogeneity of $\epsilon^{(k)}$ is preserved for $\kappa^{(k),\alpha}_{\pm} \in \mathscr{G}^{(k)}$ and $k = 1, 3$, we calculate

$$\begin{aligned} \Omega(\epsilon^{(k)}) &= -\Omega(\kappa^{(k),\alpha}_{\pm})\Omega(A^{(2)}_{\alpha,\mp}) - \Omega(A^{(2)}_{\alpha,\mp})\Omega(\kappa^{(k),\alpha}_{\pm}) \\ &= -\mathrm{i}^{2+k}\left(\kappa^{(k),\alpha}_{\pm}A^{(2)}_{\alpha,\mp} + A^{(2)}_{\alpha,\mp}\kappa^{(k),\alpha}_{\pm}\right) = \mathrm{i}^{k}\epsilon^{(k)}. \end{aligned}$$

In addition, we can ask what requirements the matrices $\kappa^{(k)}$ should satisfy such that the fermionic elements $\epsilon^{(k)}$ belong to $\mathfrak{su}(2,2|4)$, i.e. $\epsilon^{(k)} = -H\epsilon^{(k)\dagger}H$. Using the fact that $A^{(2)} \in \mathfrak{su}(2,2|4)$, we find

$$H\epsilon^{(k)\dagger}H = -A^{(2)}_{\alpha,\mp}H\kappa^{(k),\alpha\dagger}_{\pm}H^{-1} - H\kappa^{(k),\alpha\dagger}_{\pm}H^{-1}A^{(2)}_{\alpha,\mp} = -A^{(2)}_{\alpha,\mp}\kappa^{(k),\alpha}_{\pm} - \kappa^{(k),\alpha}_{\pm}A^{(2)}_{\alpha,\mp}$$

which implies the reality condition $\kappa^{(k)} = H\kappa^{(k)\dagger}H^{-1}$.

The components $A^{(2)}$ can be taken as traceless since $\mathrm{i}\mathbb{1}_8 \in \mathscr{G}^{(2)}$, which does not contribute in the supertrace of the Lagrangian (1.32). Comparing with the generic form (1.27), this means we can assume

$$A^{(2)} = \begin{pmatrix} m^i\gamma^i & 0 \\ 0 & \mathrm{i}n^i\gamma^i \end{pmatrix} + \frac{1}{8}\mathrm{str}\big(\Upsilon A^{(2)}\big), \tag{1.58}$$

where m^i and n^i are real coefficients for $i = 1, ..., 5$ except m^5 which is imaginary. In this way,

$$A^{(2)}_{\alpha,\pm}A^{(2)}_{\beta,\pm} = \begin{pmatrix} m^i_{\alpha,\pm}m^j_{\beta,\pm}\gamma^i\gamma^j & 0 \\ 0 & -n^i_{\alpha,\pm}n^j_{\alpha,\pm}\gamma^i\gamma^j \end{pmatrix}.$$

We just showed that $A_{\tau,\pm}$ and $A_{\sigma,\pm}$ are proportional in (1.56), which means that $m^i_{\alpha,\pm}m^j_{\beta,\pm} = m^j_{\alpha,\pm}m^i_{\beta,\pm}$ no matter α, β and we can rewrite

$$\begin{aligned} A^{(2)}_{\alpha,\pm}A^{(2)}_{\beta,\pm} &= \begin{pmatrix} m^i_{\alpha,\pm}m^j_{\beta,\pm}\frac{1}{2}\{\gamma^i\gamma^j\} & 0 \\ 0 & -n^i_{\alpha,\pm}n^j_{\alpha,\pm}\frac{1}{2}\{\gamma^i\gamma^j\} \end{pmatrix} = \begin{pmatrix} m^i_{\alpha,\pm}m^i_{\beta,\pm}\mathbb{1}_4 & 0 \\ 0 & -n^i_{\alpha,\pm}n^i_{\alpha,\pm}\mathbb{1}_4 \end{pmatrix} \\ &= \frac{1}{2}(m^i_{\alpha,\pm}m^i_{\beta,\pm} + n^i_{\alpha,\pm}n^i_{\alpha,\pm})\Upsilon + \frac{1}{2}(m^i_{\alpha,\pm}m^i_{\beta,\pm} - n^i_{\alpha,\pm}n^i_{\alpha,\pm})\mathbb{1}_8 \\ &= \frac{1}{8}\Upsilon\mathrm{str}\big(A^{(2)}_{\alpha,\pm}A^{(2)}_{\beta,\pm}\big) + \frac{1}{2}(m^i_{\alpha,\pm}m^i_{\beta,\pm} - n^i_{\alpha,\pm}n^i_{\alpha,\pm})\mathbb{1}_8. \end{aligned} \tag{1.59}$$

Substituting our expressions for $\epsilon^{(k)}$ and these newly found properties of $A^{(2)}$, we find in A.5 that

$$-\frac{2}{g}\delta_\epsilon\mathscr{L} = \delta_\epsilon\gamma^{\alpha\beta}\mathrm{str}\big(A^{(2)}_\alpha A^{(2)}_\beta\big) - \frac{1}{2}\mathrm{str}\big(A^{(2)}_{\alpha,-}A^{(2)}_{\beta,-}\big)\mathrm{str}\big(\Upsilon[\kappa^{(1),\beta}_+, A^{(1),\alpha}_+]\big) \\ - \frac{1}{2}\mathrm{str}\big(A^{(2)}_{\alpha,+}A^{(2)}_{\beta,+}\big)\mathrm{str}\big(\Upsilon[\kappa^{(3),\beta}_-, A^{(1),\alpha}_-]\big). \tag{1.60}$$

The second and third terms actually vanish because they have a factor which is the trace of a commutator. They also vanish because of (1.97). If we let the terms live, we find using the fact $\mathrm{str}\big(\Upsilon M\big) = \mathrm{tr}(M)$ that the Lagrangian is invariant under this local fermionic transformation $\mathfrak{g} \to \mathfrak{g}\exp\epsilon(\tau,\sigma)$ provided

$$\delta_\epsilon\gamma^{\alpha\beta} = \frac{1}{2}\,\mathrm{tr}\Big([\kappa^{(1),\alpha}_+, A^{(1),\beta}_+] + [\kappa^{(3),\alpha}_-, A^{(1),\beta}_-]\Big). \tag{1.61}$$

This vanishes as we expected from the fact that the rest of $\delta_\epsilon\mathscr{L}$ was zero. This form of the variation does however show that $\delta_\epsilon\gamma^{\alpha\beta}$ is a real tensor since, according to the reality conditions of κ and A, we have

$$(\delta_\epsilon\gamma^{\alpha\beta})^\dagger = -\frac{1}{2}\,\mathrm{tr}\left(-H[\kappa^{(1),\alpha}_+, A^{(1),\beta}_+]H^{-1} - H[\kappa^{(3),\alpha}_-, A^{(1),\beta}_-]H^{-1}\right) = \delta_\epsilon\gamma^{\alpha\beta}.$$

Crucially, such a form of the variation is obtained if and only if $P^{\alpha\beta}_\pm$ are orthogonal, which tells us that κ-symmetry is obeyed if and only if $\kappa = \pm 1$.

1.2.4 Kappa symmetry gauge freedom

Now that we know the κ-symmetry transformations are in fact a symmetry of the Lagrangian, we can ask if any fermionic degrees of freedom can be reduced as a result a corresponding gauge freedom. Throughout Chapter 2, we will be employing the light-cone gauge which assigns a time direction t along the longitudinal component of AdS_5 and an angle ϕ around the equator of S^5 (see Figure 1). The bosonic algebras $\mathfrak{so}(4,1)$, with distinguished element $\mathrm{i}\gamma^5$ corresponding to t, and $\mathfrak{so}(5)$ correspond to AdS_5 and S^5 respectively. For the moment, we can ignore the transversal dynamics (anything other than t and ϕ) such that the component $A^{(2)}$ has the generic form

$$A^{(2)} = \begin{pmatrix} \mathrm{i}x\gamma^5 & 0 \\ 0 & \mathrm{i}y\gamma^5 \end{pmatrix}.$$

where x and y are linear combinations of t and ϕ. This is a valid assumption since any element in $\mathfrak{so}(5)$ can be brought to γ^5 by an $\mathfrak{su}(4)$ transformation, e.g. $\gamma^i \to (\mathrm{i}\gamma^5)(\mathrm{i}\gamma^i)\gamma^i = -\gamma^5$.

If we work on-shell, i.e. when the equations of motion are satisfied, then the Virasoro constraints must be enforced. They are equivalent to

$$\mathrm{str}\big(A^{(2)}_{\alpha,-}A^{(2)}_{\beta,-}\big) = 0 \implies x^\alpha_\pm x^\beta_\pm = y^\alpha_\pm y^\beta_\pm.$$

In particular, this is satisfied by $y = x$. If we recall (1.56), the element $\epsilon^{(1)}$ (1.57) can be rewritten as

$$\epsilon^{(1)} = A^{(2)}_{\tau,-}\varkappa + \varkappa A^{(1)}_{\tau,-}, \qquad \varkappa = \kappa^{(1),\tau}_+ - \frac{\gamma^{\tau\tau}}{\gamma^{\tau\sigma}+\kappa}\kappa^{(1),\sigma}_+ = \begin{pmatrix} 0 & \varkappa_1 \\ \varkappa_2 & 0 \end{pmatrix}. \tag{1.62}$$

Substituting our generic $A^{(2)}$ into the above gives us

$$\epsilon^{(1)} = \mathrm{i}x_{\tau,-}\begin{pmatrix} 0 & \Sigma\varkappa_1 + \varkappa_1\Sigma \\ \Sigma\varkappa_2 + \varkappa_2\Sigma & 0 \end{pmatrix} \equiv 2\mathrm{i}x_{\tau,-}\begin{pmatrix} 0 & \varepsilon \\ -\varepsilon^\dagger\Sigma & 0 \end{pmatrix}$$

where we defined $2\varepsilon = \Sigma\varkappa_1 + \varkappa_1\Sigma$ and used the $\mathfrak{su}(2,2|4)$ fermionic reality condition $\varkappa_1^\dagger = -\varkappa_2\Sigma$ in

$$\Sigma\varkappa_2 + \varkappa_2\Sigma = \Sigma\varkappa_1^\dagger\Sigma - \varkappa_1^\dagger = -2\varepsilon^\dagger\Sigma.$$

If we let $(\varkappa_1)_{ij} = \varkappa_{ij}$ for entries $i, j = 1, ..., 4$, we find

$$\varepsilon = \begin{pmatrix} \varkappa_{11} & \varkappa_{12} & 0 & 0 \\ \varkappa_{21} & \varkappa_{22} & 0 & 0 \\ 0 & 0 & -\varkappa_{33} & -\varkappa_{34} \\ 0 & 0 & -\varkappa_{43} & -\varkappa_{43} \end{pmatrix}. \tag{1.63}$$

Nam dui ligula, fringilla a, euismod sodales, sollicitudin vel, wisi. Morbi auctor lorem non justo. Nam lacus libero, pretium at, lobortis vitae, ultricies et, tellus. Donec aliquet, tortor sed accumsan bibendum, erat ligula aliquet magna, vitae ornare odio metus a mi. Morbi ac orci et nisl hendrerit mollis. Suspendisse ut massa. Cras nec ante. Pellentesque a nulla. Cum sociis natoque penatibus et magnis dis parturient montes, nascetur ridiculus mus. Aliquam tincidunt urna. Nulla ullamcorper vestibulum turpis. Pellentesque cursus luctus mauris. Talk about $\epsilon^{(3)}$...

Thus, $\epsilon^{(1)}$ depends on 8 real fermionic parameters, namely the 8 entries $\varkappa_{ij}$ in the matrix ε[7]. A similar discussion holds for $\epsilon^{(3)}$ which also depends on 8 fermionic parameters. All together, we can eliminate 16 fermionic degrees of freedom with $\epsilon^{(1)}$ and $\epsilon^{(3)}$ such that any odd element χ can be reduced to

$$\chi = \left(\begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & \bullet & \bullet \\ 0 & 0 & 0 & 0 & 0 & 0 & \bullet & \bullet \\ 0 & 0 & 0 & 0 & \bullet & \bullet & 0 & 0 \\ 0 & 0 & 0 & 0 & \bullet & \bullet & 0 & 0 \\ \hline 0 & 0 & \bullet & \bullet & 0 & 0 & 0 & 0 \\ 0 & 0 & \bullet & \bullet & 0 & 0 & 0 & 0 \\ \bullet & \bullet & 0 & 0 & 0 & 0 & 0 & 0 \\ \bullet & \bullet & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right). \tag{1.64}$$

This follows from the expressions for $\delta_\epsilon A^{(1)}$ and $\delta_\epsilon A^{(3)}$ in (1.50). Here the bullets indicate entries of the matrix realisation of χ which cannot be gauged away by κ-symmetry transformations. There are in fact 16 *real* degrees of freedom. It may seem like χ only has 8, since for any odd matrix the upper block θ and lower block η are related by the fermionic $\mathfrak{su}(2,2|4)$ reality condition $\eta = -\theta^\dagger\Sigma$ (1.7). However, the fact that we took the superalgebra $\mathfrak{sl}(4|4)$ over the complex field means each entry has two real parameters. This fermionic gauge freedom will be exploited in Chapter 2 when we prepare to quantise the Lagrangian in the light cone gauge.

1.3 Integrability of classical superstrings

First, The general concept of integrability is introduced through the fundamental linear problem. An example (the principal chiral model) is then used to outline the construction of a zero-curvature Lax representation from conserved currents. The Green-Schwarz string sigma model we just discussed is shown to be integrable by constructing such a flat Lax representation of the equations of motion. Interestingly, integrability of the model with this choice of Lax pair is in some way equivalent to the Virasoro constraints and necessitates κ-symmetry of the Lagrangian.

[7] The fact that $\varkappa_{ij}$ are real follows from...Need to check $\epsilon^{(3)}$ and count independent parameters $\kappa_\pm^{(k),\alpha}$.

1.3.1 Classical integrability

In a physical context, integrability refers to the possibility of 'integrating' the equations of motion so as to find a solution to the problem at hand. Consider the following system of partial differential equations

$$\frac{\partial\Psi}{\partial\sigma} = L_\sigma(\sigma,\tau,z)\Psi, \tag{1.65}$$
$$\frac{\partial\Psi}{\partial\tau} = L_\tau(\sigma,\tau,z)\Psi, \tag{1.66}$$

where Ψ is a vector of dimension $\mathfrak{r}$ and $L_\sigma,\ L_\tau$ are $\mathfrak{r}\times\mathfrak{r}$ matrices which all depend on a spectral parameter z taking values in $\mathbb{C}^2$. If we differentiate (1.65) with respect to τ and (1.66) with respect to σ, we get

$$\frac{\partial^2\Psi}{\partial\tau\partial\sigma} = \partial_\tau L_\sigma(\sigma,\tau,z)\Psi + L_\sigma(\sigma,\tau,z)\partial_\tau\Psi,$$
$$\frac{\partial^2\Psi}{\partial\sigma\partial\tau} = \partial_\sigma L_\tau(\sigma,\tau,z)\Psi + L_\tau(\sigma,\tau,z)\partial_\sigma\Psi.$$

If we now substitute in the original equations for $\partial_\tau\Psi$ and $\partial_\sigma\Psi$, and equate the second order derivatives,

$$\partial_\tau L_\sigma - \partial_\sigma L_\tau - [L_\tau, L_\sigma] = 0. \tag{1.67}$$

This can be reformulated as the zero curvature condition for connections L_α and $\alpha=\sigma,\tau$;

$$\partial_\alpha L_\beta - \partial_\beta L_\alpha - [L_\alpha, L_\beta] = 0. \tag{1.68}$$

If these connections satisfy (1.68) for all values of the spectral parameter z, then L_α are called *Lax connections* while (1.68) is the Lax representation of the integrable system of partial differential equations. We define the monodromy matrix $T(z)$ as the path-ordered exponential of the Lax connection L_σ

$$T(z) = \overleftarrow{\exp}\int_0^{2\pi} d\sigma\ L_\sigma(\tau,\sigma,z). \tag{1.69}$$

It can be shown (see A.6) that

$$\partial_\tau T(z) = [L_\tau(0,\tau,z), T(z)]. \tag{1.70}$$

This equation implies that the eigenvalues $\{\mu_i\}$ of the matrix $T(z)$ are constant in worldsheet time. It follows by considering the trace of $T^n(z)$, whose τ derivative is the trace of $[L_\tau(0,\tau,z), T^n(z)]$ and thus vanishes. Assuming $T(z)$ is diagonalisable, it follows that

$$\partial_\tau \operatorname{tr} T(z) = \partial_\tau \operatorname{tr} T^2(z) = ... = 0 \implies \sum \partial_\tau\mu_i = \sum \partial_\tau\mu_i^2 = ... = 0 \tag{1.71}$$

up to the dimension $\mathfrak{r}$ of $T(z)$. The eigenvalues are integrals of motion, which means the conservation laws of the system are encoded in $T(z)$, thus motivating the identification of the Lax pair L_α of the model.

Explicit example - principal chiral model

Remembering the connection $A_\alpha = -\mathfrak{g}^{-1}\partial_\alpha\mathfrak{g}$, the action for the principal chiral model reads

$$S = -\frac{1}{2}\int d\tau d\sigma\ \gamma^{\alpha\beta}\operatorname{tr}\big(\partial_\alpha\mathfrak{g}\mathfrak{g}^{-1}\partial_\beta\mathfrak{g}\mathfrak{g}^{-1}\big) = -\frac{1}{2}\int d\tau d\sigma\ \gamma^{\alpha\beta}\operatorname{tr}(A_\alpha A_\beta). \tag{1.72}$$

To find the variation of the action δS with respect to $\mathfrak{g}$ and thus the equations of motion, we need to find[8]

$$\frac{1}{2}\operatorname{tr}\delta(\gamma^{\alpha\beta}A_\alpha A_\beta) = \gamma^{\alpha\beta}\operatorname{tr}(\delta A_\alpha A_\beta). \tag{1.73}$$

Substituting the expression (A.20) for δA_α,

$$\begin{aligned}\frac{1}{2}\gamma^{\alpha\beta}\operatorname{tr}\delta(A_\alpha A_\beta) &= \gamma^{\alpha\beta}\operatorname{tr}\left[-\mathfrak{g}^{-1}\delta\mathfrak{g}A_\alpha A_\beta - \mathfrak{g}^{-1}\partial_\alpha\delta\mathfrak{g}A_\beta\right]\\ &= \gamma^{\alpha\beta}\operatorname{tr}\left[\delta\mathfrak{g}\partial_\alpha A_\beta\mathfrak{g}^{-1}\right] = -\gamma^{\alpha\beta}\operatorname{tr}\left[\mathfrak{g}^{-1}\delta\mathfrak{g}\partial_\alpha(\mathfrak{g}^{-1}\partial_\beta\mathfrak{g})\right]\end{aligned} \tag{1.74}$$

so that $\delta S/\delta\mathfrak{g} = 0$ implies the equations of motion $\partial_\alpha(\gamma^{\alpha\beta}\mathfrak{g}^{-1}\partial_\beta\mathfrak{g}) = 0$. This is because the trace of this derivative does not necessarily vanish as the supertrace of elements of $\mathfrak{su}(2,2|4)$ would. We can manipulate (1.74) to find the same equations of motion in a different form,

$$\begin{aligned}\frac{1}{2}\gamma^{\alpha\beta}\operatorname{tr}\delta(A_\alpha A_\beta) &= \gamma^{\alpha\beta}\operatorname{tr}\left[\mathfrak{g}^{-1}\delta\mathfrak{g}\mathfrak{g}^{-1}\partial_\alpha\mathfrak{g}\mathfrak{g}^{-1}\partial_\beta\mathfrak{g} - \mathfrak{g}^{-1}\delta\mathfrak{g}\mathfrak{g}^{-1}\partial_\alpha\partial_\beta\mathfrak{g}\right]\\ &= \gamma^{\alpha\beta}\operatorname{tr}\left[\delta\mathfrak{g}\mathfrak{g}^{-1}\partial_\alpha(\partial_\beta\mathfrak{g}\mathfrak{g}^{-1})\right]\end{aligned} \tag{1.75}$$

which this time implies $\partial_\alpha(\gamma^{\alpha\beta}\partial_\beta\mathfrak{g}\mathfrak{g}^{-1}) = 0$. Putting this all together, the equations of motion are

$$\partial_\alpha(\gamma^{\alpha\beta}\partial_\beta\mathfrak{g}\mathfrak{g}^{-1}) = 0 = \partial_\alpha(\gamma^{\alpha\beta}\mathfrak{g}^{-1}\partial_\beta\mathfrak{g}), \tag{1.76}$$

and they can be conveniently written in terms of the (corrected) left and right currents

$$A_l^\alpha = -\gamma^{\alpha\beta}\partial_\beta\mathfrak{g}\mathfrak{g}^{-1}, \qquad A_r^\alpha = \gamma^{\alpha\beta}\mathfrak{g}^{-1}\partial_\beta\mathfrak{g} \tag{1.77}$$

as

$$\partial_\alpha A_l^\alpha = 0 = \partial_\alpha A_r^\alpha. \tag{1.78}$$

A_r is called the right (Noether) current, and A_l the left current because of their invariance under right and left action of $\mathfrak{g}$ by a constant group element $\mathfrak{h}$. In particular the right current coincides with our previous connections A_α up to a sign.

The flatness of Lax pairs in this model must be invariant under σ and τ coordinate reparametrisation. This is because the Lax flatness represents the equations of motion which are themselves reparametrisation-invariant. (One can see this by looking at the action or simply (1.76).) It follows that the Lax connections L_α must then be one-forms. To see this, let L_α be a k-form, i.e. $L_\alpha \in \Omega^k$ and look at the zero curvature condition (1.68)

$$\underbrace{\partial_\alpha L_\beta - \partial_\alpha L_\beta}_{\text{in }\Omega^{1+k}} - \underbrace{[L_\alpha, L_\beta]}_{\text{in }\Omega^{2k}} = 0. \tag{1.79}$$

For this equality to hold under reparametrisation, both terms must transform with the same overall prefactor, which means they are of the same tensor type. In other words, $\Omega^{1+k} = \Omega^{2k}$ so $k = 1$ and L_α is a co-vector or one-form. Consequently, we introduce the Lax connections

$$L_\alpha = \ell_1 A_\alpha + \ell_2\gamma_{\alpha\beta}\varepsilon^{\beta\rho}A_\rho \tag{1.80}$$

where ℓ_1, ℓ_2 are parameters to be determined and A is either A^r or A^l. It is natural to construct the Lax connections in terms of the currents, as the latter appear in the equations of motion and the Lax

[8] In this step we treat $\gamma_{\alpha\beta}$ as a constant as we are implicitly finding the variation of S with respect to $\mathfrak{g}$ only.

representation (1.68) would involve taking their derivative. In two dimensions we can recast the flatness condition (1.68) as

$$2\varepsilon^{\alpha\beta}\partial_\alpha L_\beta - \varepsilon^{\alpha\beta}[L_\alpha, L_\beta] = 0. \tag{1.81}$$

To show this we sum over contracted indices $\alpha = \tau, \sigma$ with convention $\varepsilon^{\tau\sigma} = +1$ and get

$$\begin{aligned}2\varepsilon^{\alpha\beta}\partial_\alpha L_\beta - \varepsilon^{\alpha\beta}[L_\alpha, L_\beta] &= 2(\partial_\tau L_\sigma - \partial_\sigma L_\tau) - [L_\tau, L_\sigma] + [L_\sigma, L_\tau] \\ &= 2(\partial_\tau L_\sigma - \partial_\sigma L_\tau - [L_\tau, L_\sigma]) = 0.\end{aligned}$$

Using the identity $\varepsilon^{\alpha\beta}\gamma_{\beta\rho}e^{\rho\delta} = \gamma^{\alpha\delta}$, we substitute (1.80) into (1.81) which reduces to

$$\begin{aligned}0 &= 2\varepsilon^{\alpha\beta}\partial_\alpha(\ell_1 A_\beta + \ell_2\gamma_{\beta\mu}\varepsilon^{\mu\rho}A_\rho) - \varepsilon^{\alpha\beta}[\ell_1 A_\alpha + \ell_2\gamma_{\alpha\mu}\varepsilon^{\mu\rho}A_\rho, \ell_1 A_\beta + \ell_2\gamma_{\beta\nu}\varepsilon^{\nu\delta}A_\delta] \\ &= 2\ell_1\varepsilon^{\alpha\beta}\partial_\alpha A_\beta + 2\ell_2\partial_\alpha(\varepsilon^{\alpha\beta}\gamma_{\beta\mu}\varepsilon^{\mu\rho}A_\rho) - \ell_1^2\varepsilon^{\alpha\beta}[A_\alpha, A_\beta] - \ell_1\ell_2\varepsilon^{\alpha\beta}\gamma_{\beta\nu}\varepsilon^{\nu\delta}[A_\alpha, A_\delta] \\ &\quad - \ell_1\ell_2\varepsilon^{\alpha\beta}\gamma_{\alpha\mu}\varepsilon^{\mu\rho}[A_\rho, A_\beta] - \ell_2^2\varepsilon^{\alpha\beta}\gamma_{\alpha\mu}\varepsilon^{\mu\rho}\gamma_{\beta\nu}\varepsilon^{\nu\delta}[A_\rho, A_\delta] \\ &= 2\ell_1\varepsilon^{\alpha\beta}\partial_\alpha A_\beta + 2\ell_2\partial_\alpha A^\alpha + \varepsilon^{\alpha\beta}(\ell_2^2 - \ell_1^2)[A_\alpha, A_\beta].\end{aligned} \tag{1.82}$$

The second term vanishes because of the equations of motion $\partial_\alpha A^\alpha = 0$ (1.78). Let us now show the zero-curvature condition for the left and right currents (1.77) by using $\delta\mathfrak{g}^{-1} = -\mathfrak{g}^{-1}\delta\mathfrak{g}\mathfrak{g}^{-1}$. For the left,

$$\begin{aligned}\partial_\alpha A^l_\beta &= -\partial_\alpha(\partial_\beta\mathfrak{g}\mathfrak{g}^{-1}) = -\partial_\alpha\partial_\beta\mathfrak{g}\mathfrak{g}^{-1} + \partial_\beta\mathfrak{g}\mathfrak{g}^{-1}\partial_\alpha\mathfrak{g}\mathfrak{g}^{-1} \\ \implies \partial_\alpha A^l_\beta - \partial_\beta A^l_\alpha &= [\partial_\beta\mathfrak{g}\mathfrak{g}^{-1}, \partial_\alpha\mathfrak{g}\mathfrak{g}^{-1}] = [A^l_\beta, A^l_\alpha].\end{aligned}$$

And for the right,

$$\begin{aligned}\partial_\alpha A^r_\beta &= \partial_\alpha(\mathfrak{g}^{-1}\partial_\beta\mathfrak{g}) = -\mathfrak{g}^{-1}\partial_\alpha\mathfrak{g}\mathfrak{g}^{-1}\partial_\beta\mathfrak{g} + \mathfrak{g}^{-1}\partial_\alpha\partial_\beta\mathfrak{g} \\ \implies \partial_\alpha A^r_\beta - \partial_\beta A^r_\alpha &= [\mathfrak{g}^{-1}\partial_\beta\mathfrak{g}, \mathfrak{g}^{-1}\partial_\alpha\mathfrak{g}] = [A^r_\beta, A^r_\alpha].\end{aligned}$$

We can summarise these zero curvature conditions into one;

$$\partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta] = 0. \tag{1.83}$$

Note that the sign of the commutator is determined by which current carries the minus sign (in this convention, it is the left current). Returning to (1.82), the newly-found flatness condition (1.83) implies

$$\begin{aligned}2\ell_1\varepsilon^{\alpha\beta}\partial_\alpha A_\beta + \varepsilon^{\alpha\beta}(\ell_2^2 - \ell_1^2)[A_\alpha, A_\beta] &= 2\ell_1\varepsilon^{\alpha\beta}\partial_\alpha A_\beta - \varepsilon^{\alpha\beta}(\ell_2^2 - \ell_1^2)(\partial_\alpha A_\beta - \partial_\beta A_\alpha) \\ 0 &= 2\varepsilon^{\alpha\beta}\big(\ell_1\partial_\alpha A_\beta - \frac{1}{2}(\ell_2^2 - \ell_1^2)(\partial_\alpha A_\beta - \partial_\beta A_\alpha)\big) \\ 0 &= 2\varepsilon^{\alpha\beta}\big(\ell_1 - (\ell_2^2 - \ell_1^2)\big)\partial_\alpha A_\beta.\end{aligned} \tag{1.84}$$

For (1.84) to vanish, we must then have $ell\ell_1^2 - \ell_2^2 + \ell_1 = 0$ for both $A^{l,r}$. Given ℓ_2, this equation has two solutions for ℓ_1. Introducing the spectral parameter z, these solutions take the form $\ell_2 = z/(1-z^2)$ and either

$$\ell_1 = +\frac{z^2}{1-z^2} \qquad \text{assigned to } A = A^l, \text{ or} \tag{1.85}$$

$$\ell_1 = -\frac{1}{1-z^2} \qquad \text{assigned to } A = A^r. \tag{1.86}$$

The explicit sign of ℓ_2 can always be attributed to the value of z, so is not fixed. Substituting in these values for ℓ_i, we obtain the left and right Lax connections

$$L^l_\alpha = +\frac{z^2}{1-z^2}A^l_\alpha + \frac{z}{1-z^2}\gamma_{\alpha\beta}\varepsilon^{\beta\rho}A^l_\rho, \tag{1.87}$$

$$L^r_\alpha = -\frac{1}{1-z^2}A^r_\alpha + \frac{z}{1-z^2}\gamma_{\alpha\beta}\varepsilon^{\beta\rho}A^r_\rho. \tag{1.88}$$

We finish by showing that the connections L^l and L^r are connected by the gauge transformation

$$L^l = hL^r h^{-1} + dhh^{-1}, \tag{1.89}$$

when $h = \mathfrak{g}$. We can show (1.89) component-wise by employing the expression (1.88), which becomes

$$\begin{aligned}
\mathfrak{g}L^r_\alpha\mathfrak{g}^{-1} &= \mathfrak{g}\big(\frac{z^2}{1-z^2}\mathfrak{g}^{-1}\partial_\alpha\mathfrak{g} - \frac{z}{1-z^2}\varepsilon_{\alpha\beta}\mathfrak{g}^{-1}\partial^\beta\mathfrak{g}\big)\mathfrak{g}^{-1} \\
&= +\frac{z^2}{1-z^2}\partial_\alpha\mathfrak{g}\mathfrak{g}^{-1} - \frac{z}{1-z^2}\varepsilon_{\alpha\beta}\partial^\beta\mathfrak{g}\mathfrak{g}^{-1} \\
\implies \mathfrak{g}L^r_\alpha\mathfrak{g}^{-1} + \partial_\alpha\mathfrak{g}\mathfrak{g}^{-1} &= (1 + \frac{z^2}{1-z^2})\partial_\alpha\mathfrak{g}\mathfrak{g}^{-1} - \frac{z}{1-z^2}\varepsilon_{\alpha\beta}\partial^\beta\mathfrak{g}\mathfrak{g}^{-1} \\
&= -\frac{1}{1-z^2}A^l_\alpha + \frac{z}{1-z^2}\varepsilon_\alpha{}^\beta A^l_\beta = L^l_\alpha.
\end{aligned}$$

1.3.2 Lax pair

In the last example we found the Lax representation of the equations of motion for the principal chiral model action. For our superstrings in $\mathrm{AdS}_5 \times S^5$ with Lagrangian density (1.32)

$$\mathscr{L} = -\frac{g}{2}\Big[\gamma^{\alpha\beta}\mathrm{str}\big(A^{(2)}_\alpha A^{(2)}_\beta\big) + \kappa\varepsilon^{\alpha\beta}\mathrm{str}\big(A^{(1)}_\alpha A^{(3)}_\beta\big)\Big],$$

the action is $S = \iint d\tau d\sigma\ \mathscr{L}$. To find a Lax representation of the string equations of motion, one should analogously ansatz a $\mathbb{Z}_4$-graded one-form

$$L_\alpha = \ell_0 A^{(0)}_\alpha + \ell_1 A^{(2)}_\alpha + \ell_2\gamma_{\alpha\beta}\varepsilon^{\beta\rho}A^{(2)}_\rho + \ell_3 A^{(1)}_\alpha + \ell_4 A^{(3)}_\alpha \tag{1.90}$$

and then try to determine the parameters ℓ_i by imposing zero curvature (1.81). The projections $\mathscr{G}^{(k)}$ of the zero curvature condition are found and separately set to zero in A.7 to obtain the following requirements:

$$\begin{aligned}
\mathscr{G}^{(0)} = 0 &\implies \ell_0 = 1, \qquad \ell_1^2 - \ell_2^2 = 1, \qquad \ell_3\ell_4 = 1, \\
\mathscr{G}^{(2)} = 0 &\implies \frac{\ell_3^2 - \ell_1}{\ell_2} = -\kappa, \qquad \frac{\ell_4^2 - \ell_1}{\ell_2} = \kappa, \\
\mathscr{G}^{(1)}, \mathscr{G}^{(3)} = 0 &\implies \frac{\ell_1\ell_4 - \ell_3}{\ell_2\ell_4} = \kappa, \qquad \frac{\ell_4 - \ell_1\ell_3}{\ell_2\ell_3} = \kappa.
\end{aligned} \tag{1.91}$$

Some algebra shows that these requirements imply $\kappa^2 = 1$. In other words, κ-symmetry is required (by the Lax representation parameters ℓ_i) for our original Lax pair ansatz (1.90) to describe the string described by the Lagrangian density (1.32). This is not quite the statement that integrability requires κ-symmetry. In principle, if one found a Lax pair such that it can satisfy the zero-curvature condition by being a one-form but without $\kappa^2 = 1$, then integrability would hold without the symmetry. But to date, no such Lax pair has been found (cite).

1.3.3 Integrability and symmetries

We will see that certain gauge transformations leave the flatness of any Lax connections unchanged. Since the Lagrangian (and hence the physics) benefits from κ-symmetry, it should follow that integrability of the model is preserved under transformations (1.48). In particular, the transformed Lax connections $L'_\alpha = L_\alpha + \delta_\epsilon L_\alpha$ are shown to be gauge transformations, thus preserving flatness and integrability.

As shown in A.8.1, the Lax zero curvature condition (1.68) is invariant under the gauge transformation

$$L_\alpha \to L'_\alpha = hL_\alpha h^{-1} + \partial_\alpha h h^{-1}. \tag{1.92}$$

In fact, if $h = \exp\Lambda \in G$ is a group element for $\Lambda \in \mathscr{G}$, then the transformation is equivalently defined by

$$\delta L_\alpha = [L_\alpha, \Lambda] - \partial_\alpha \Lambda. \tag{1.93}$$

The above variation is explicitly shown to preserve flatness after (A.67). Under κ-symmetry transformations with $\epsilon = \epsilon^{(1)}$, the change in the ansatz Green-Schwarz Lax connections (1.90) is found in A.8.2 to be

$$\delta_\epsilon L_\alpha = [L_\alpha, \Lambda] - \partial_\alpha\Lambda - 2\ell_2\ell_3\varepsilon_{\alpha\beta}[A_-^{(2),\beta}, \epsilon^{(1)}] + \ell_2\varepsilon_{\alpha\beta}\Big(2[A_+^{(1),\beta}, \epsilon^{(1)}] + \delta_\epsilon\gamma^{\beta\delta}A_\delta^{(2)}\Big) \tag{1.94}$$

for $\Lambda = \ell_3\epsilon^{(1)} \in \mathfrak{su}(2,2|4)$. This is of the form $\delta_\epsilon L_\alpha = [L_\alpha, \Lambda] - \partial_\alpha\Lambda - c_\alpha$. If one can show that the extra term c_α vanishes, i.e.

$$c_\alpha = 2\ell_2\ell_3\varepsilon_{\alpha\beta}\underbrace{[A_-^{(2),\beta}, \epsilon^{(1)}]}_{I_1^\beta} - \ell_2\varepsilon_{\alpha\beta}\Big(2\underbrace{[A_+^{(1),\beta}, \epsilon^{(1)}]}_{I_2^\beta} + \delta_\epsilon\gamma^{\beta\delta}A_\delta^{(2)}\Big) = 0, \tag{1.95}$$

then that would equate the κ-symmetry transformation $\delta_\epsilon L_\alpha$ to the typical gauge transformation (1.94) for $\Lambda = \ell_3\epsilon^{(1)} \in \mathfrak{su}(2,2|4)$. That is precisely what is shown in A.8.2 by proving that the two terms containing $I_{1,\alpha}$ and $I_{2,\alpha}$ separately vanish. In particular, thanks to the proportionality of two projected components (1.56), the term $I_{1,\alpha}$ can be reduced to

$$I_{1,\alpha} = \frac{1}{8}\mathrm{str}\big(A_{\alpha,-}^{(2)}A_{\beta,-}^{(2)}\big)[\Upsilon, \kappa_+^{(1),\beta}]. \tag{1.96}$$

where $\kappa_+^{(1)}$ comes from the ansatz (1.57). Some careful manipulation shows an important relation between $I_{1,\alpha}$ and the Virasoro constraints (1.40) for $\kappa = \pm 1$:

$$\mathrm{str}\big(A_{\alpha,-}^{(2)}A_{\beta,-}^{(2)}\big) = 0 \iff \mathrm{str}\big(A_\alpha^{(2)}A_\beta^{(2)}\big) - \frac{1}{2}\gamma_{\alpha\beta}\gamma^{\mu\nu}\mathrm{str}\big(A_\mu^{(2)}A_\nu^{(2)}\big) = 0. \tag{1.97}$$

The second term $I_{2,\alpha}$ can be heavily simplified using arguments pertaining to the bosonic structure of $\mathfrak{su}(2,2|4)$ such that

$$\varepsilon_{\alpha\beta}\left(2I_2^\beta + \delta_\epsilon\gamma^{\beta\delta}A_\delta^{(2)}\right) = \varepsilon_{\alpha\beta}\left(-\frac{1}{2}\mathrm{str}\big(\Upsilon[\kappa_+^{(1),\beta}, A_+^{(1),\delta}]\big)A_{\delta,-}^{(2)} + \delta_\epsilon\gamma^{\beta\delta}A_\delta^{(2)}\right). \tag{1.98}$$

Looking at the expression (1.61) for $\delta_\epsilon\gamma^{\alpha\beta}$ for $\epsilon^{(3)} = 0$, the above vanishes.

In summary, $\kappa = \pm 1$ was shown to imply (1.61) in 1.2.3 and the equivalence (1.97) in A.8.2. In turn, the former implies the $I_{2,\alpha}$ term vanishes while the latter tells us $I_{1,\alpha} = 0$. Thus, κ-symmetry directly implies the extra term c_α drops out of the variation $\delta_\epsilon L_\alpha$ of the Lax connections (as long as the Virasoro constraints are satsfied), such that only a gauge transformation is leftover and the flatness is preserved. This is pictorially summarised in Figure 4.

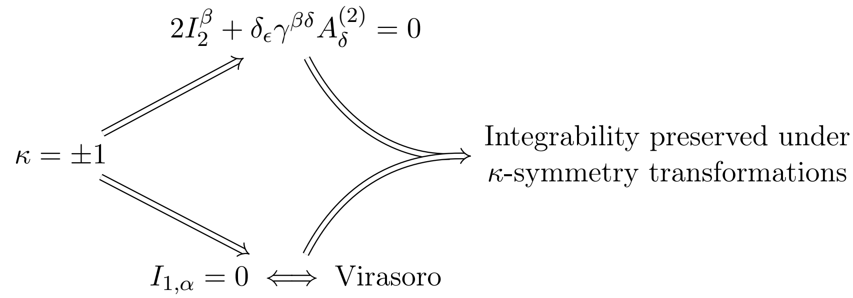


Figure 4. A schema highlighting the relationship between the symmetries and integrability of the Green-Schwarz superstring on $\mathrm{AdS}_5 \times S^5$, which is a consequence of the zero curvature of L_α given in (1.90).

We note that diffeomorphisms of the worldsheet coordinates σ^α of the type $\sigma^\alpha \to \sigma^\alpha = \tilde{\sigma}^\alpha + f^\alpha$ induce a change in the Lax pairs given by the expression derived in A.8.3:

$$\delta L_\alpha = f^\beta \partial_\beta L_\alpha + L_\beta \partial_\alpha f^\beta = [L_\beta f^\beta, L_\alpha] + \partial_\alpha (L_\beta f^\beta). \tag{1.99}$$

This is a gauge transformation of the form we saw before ($[L_\alpha, \Lambda] - \partial_\alpha \Lambda$) with parameter $\Lambda = -L_\beta f^\beta$, and is also the Lie derivative of L_α along the vector field $\mathbf{f}$. We have two integrability-preserving diffeomorphism freedoms $\tilde{\sigma}^\alpha$. These are not equivalent to the reparametrisation invariance of σ^α, but reflect the fact that integrability is highly dependent on the choice of coordinates (unlike the physics of the system). It is a weaker statement than actual Lagrangian invariance.

To conclude this section on integrability, we return to the gauge transformation (1.92). If we set $h = \mathfrak{g} \in \mathfrak{psu}(2,2|4)$ and introduce the dual current $\tilde{A} = \mathfrak{g} A \mathfrak{g}^{-1} = -d\mathfrak{g}\mathfrak{g}^{-1}$ with homogeneous components $\tilde{A}^{(k)}$, then the new Lax connection takes the form

$$\begin{aligned} L'_\alpha &= \ell_0 \tilde{A}_\alpha^{(0)} + \ell_1 \tilde{A}_\alpha^{(2)} + \ell_2 \gamma_{\alpha\beta} \varepsilon^{\beta\rho} \tilde{A}_\rho^{(2)} + \ell_3 \tilde{A}_\alpha^{(1)} + \ell_4 \tilde{A}_\alpha^{(3)} - \tilde{A}_\alpha \\ &= \ell'_0 \tilde{A}_\alpha^{(0)} + \ell'_1 \tilde{A}_\alpha^{(2)} + \ell'_2 \gamma_{\alpha\beta} \varepsilon^{\beta\rho} \tilde{A}_\rho^{(2)} + \ell'_3 \tilde{A}_\alpha^{(1)} + \ell'_4 \tilde{A}_\alpha^{(3)} \end{aligned}$$

The shifted Lax parameters ℓ'_i can be expressed in terms of a spectral parameter while fulfilling (1.91) as[9]

$$\ell'_0 = 0, \quad \ell'_1 = \frac{(1-z^2)^2}{2z^2}, \quad \ell'_2 = -\frac{1}{2\kappa}, \quad \ell'_3 = z - 1, \quad \ell'_4 = \frac{1}{z} - 1. \tag{1.100}$$

We can expand the new connection in around $w = 1 - z$ to leading order in w and we get

$$L_\alpha = \frac{2w}{\kappa} \mathcal{L}_\alpha + \mathcal{O}(w^2), \qquad \mathcal{L}_\alpha = \gamma_{\alpha\beta} \varepsilon^{\beta\delta} \tilde{A}_\delta^{(2)} + \frac{\kappa}{2} (\tilde{A}_\alpha^{(1)} - \tilde{A}_\alpha^{(3)}). \tag{1.101}$$

This expansion can be performed using Mathematica[10], as seen in Figure 5. Because we can vary w at will, the zero curvature of $L_\alpha(w)$ should be fulfilled at each order in w. This implies

$$\partial_\alpha \mathcal{L}_\beta - \partial_\beta \mathcal{L}_\alpha = 0 \implies \partial_\alpha (\varepsilon^{\alpha\beta} \mathcal{L}_\beta) = 0. \tag{1.102}$$

This is no surprise since, recalling the expression (1.34) for Λ^α,

$$g \varepsilon^{\alpha\beta} \mathcal{L}_\alpha = \mathfrak{g} \Lambda^\alpha \mathfrak{g}^{-1} = J^\alpha \tag{1.103}$$

where J^α is the conserved Noether current (1.41) associated with $PSU(2,2|4)$ transformations $\exp \mathfrak{g}$.

[9] I need to check these new ℓ'_2 satisfy the requirements.
[10] I am not sure whether I should include this snippet.

```
In[21]:= l₀[z_] := 0; (* Lax parameters *)
         l₁[z_] := (1 - z^2)^2 / 2 / z^2;
         l₂[z_] := - (z^2 - 1 / z^2) / 2 / κ;
         l₃[z_] := z - 1;
         l₄[z_] := 1 / z - 1;
         F[z_] := l₀[z] a₀ + l₁[z] a₂ + l₂[z] ∈a₂ + l₃[z] a₁ + l₄[z] a₃; (* Lα *)
         f[w_] := Series[F[1 - w], {w, 0, 1}]; (*Expanding Lα  in z around 1-z*)
         f[w]
Out[28]=
```

$$\left(-a_1 + a_3 + \frac{2 \in a_2}{\kappa}\right) w + O[w]^2$$

Figure 5. Temporary caption

1.4 Strings in coset space

In this section we will see how to include the spacetime coordinates of $\mathrm{AdS}_5 \times S^5$ in the Lagrangian, which so far has featured $\mathfrak{su}(2,2|4)$ matrices A_α with implicit dependence on the worldsheet coordinates.

1.4.1 Coset parametrisation

We first start by embedding the 5 unconstrained coordinates $\{\phi, y_i\}$ for $i = 1, ..., 4$ of S^5 into $\mathbb{R}^6$ by introducing 6 real coordinates Y_A for $A = 1, ..., 6$. Note $y^2(\tau, \sigma) \equiv y_i y_i$ is *not* constant. These Y_A are

$$\begin{gathered} Y_1 + \mathrm{i} Y_2 = \frac{y_1 + \mathrm{i} y_2}{1 + y^2/4}, \qquad Y_3 + \mathrm{i} Y_4 = \frac{y_3 + \mathrm{i} y_4}{1 + y^2/4}, \\ Y_5 + \mathrm{i} Y_6 = \frac{1 - y^2/4}{1 + y^2/4} e^{\mathrm{i}\phi}. \end{gathered} \tag{1.104}$$

The metric induced on S^5 by this embedding into flat space is easily found (see A.9) by taking the modulus squared of the above expressions:

$$ds^2\big|_{S^5} = dY_A dY_A\big|_{S^5} = \Big(\frac{1 - y^2/4}{1 + y^2/4}\Big)^2 (d\phi)^2 + \frac{dy_i dy_i}{(1 + y^2/4)^2}. \tag{1.105}$$

Similarly, the embedding of AdS_5 with coordinates $\{t, z^i\}$ for $i = 1, ..., 4$ into $\mathbb{R}^6$ prescribed by

$$\begin{gathered} Z_1 + \mathrm{i} Z_2 = \frac{z_1 + \mathrm{i} z_2}{1 - z^2/4}, \qquad Z_3 + iZ_4 = \frac{z_3 + \mathrm{i} z_4}{1 - z^2/4}, \\ Z_0 + \mathrm{i} Z_5 = \frac{1 + z^2/4}{1 - z^2/4} e^{\mathrm{i}t}, \end{gathered} \tag{1.106}$$

and with the signature $\eta_{AB} = \mathrm{diag}(-1, 1, 1, 1, 1, -1)$, induces the metric

$$ds^2\big|_{\mathrm{AdS}_5} = \eta^{AB} dZ_A dZ_B\big|_{\mathrm{AdS}_5} = -\Big(\frac{1 + z^2/4}{1 - z^2/4}\Big)^2 (dt)^2 + \frac{1}{(1 - z^2/4)^2} dz_i dz_i. \tag{1.107}$$

If we group the coordinates z_i and y_i into one x^μ, for $\mu = 1, ..., 8$, then the $\mathrm{AdS}_5 \times S^5$ metric becomes diagonal

$$ds^2\big|_{\mathrm{AdS}_5 \times S^5} \equiv -G_{tt}(dt)^2 + G_{\phi\phi}(d\phi)^2 + G_{\mu\mu} dx^\mu dx^\mu \tag{1.108}$$

where it is understood that $dx^\mu dx^\mu = dz_i dz_i + dy_i dy_i$ and the coefficients are

$$G_{tt} = \Big(\frac{1+z^2/4}{1-z^2/4}\Big)^2, \qquad G_{\phi\phi} = \Big(\frac{1-y^2/4}{1+y^2/4}\Big)^2, \qquad G_{zz} = \frac{1}{(1-z^2/4)^2}, \qquad G_{yy} = \frac{1}{(1+y^2/4)^2}. \quad (1.109)$$

We group $G_{ii} = G_{zz}$, $G_{(i+4)(i+4)} = G_{yy}$ for $i = 1, ...4$. Note that we do not consider closed timelike curves, since they are not physical. This means we are considering the universal cover $\widetilde{\mathrm{AdS}_5 \times S^5}$ [11].

The time has come to put some life into these $\mathfrak{psu}(2,2|4)$ matrices. In 1.1.2 we understood that projections $M = M^{(2)} \in \mathscr{G}^{(2)}$ can take the arbitrary form (1.27). We can choose the coefficients of M such that[12]

$$M = \frac{1}{2}\begin{pmatrix} z^i\gamma^i + \mathrm{i}t\gamma^5 & 0 \\ 0 & \mathrm{i}y^i\gamma^i + \mathrm{i}\phi\gamma^5 \end{pmatrix}. \quad (1.110)$$

for $i = 1, ..., 4$. The separation of the bosonic upper block, which corresponds to $\mathfrak{su}(2,2) \sim \mathfrak{so}(4,2) \sim \mathrm{AdS}_5$, and lower block $\mathfrak{su}(4) \sim \mathfrak{so}(6) \sim S^5$ is all too natural. The most obvious way to embed this bosonic element from $\mathfrak{su}(2,2|4)$ into $SU(2,2|4)$ is to exponentiate it. This leads us to define an embedding $\mathfrak{g}$ of the coset space $PSU(2,2|4)/\big(SO(4,1) \times SO(5)\big) \sim \mathrm{AdS}_5 \times S^5$ into $SU(2,2|4)$ as

$$\mathfrak{g} = \mathfrak{g}_\mathfrak{f}\mathfrak{g}_\mathfrak{b}, \quad (1.111)$$

made of a fermionic element $\mathfrak{g}_\mathfrak{f}$ and a bosonic element $\mathfrak{g}_\mathfrak{b}$ of the form

$$\mathfrak{g}_\mathfrak{b} = \exp\frac{1}{2}\begin{pmatrix} \mathrm{i}t\gamma^5 + z^i\gamma^i & 0 \\ 0 & \mathrm{i}\phi\gamma^5 + y^i\gamma^i \end{pmatrix} \qquad \mathfrak{g}_\mathfrak{f} = \exp\chi = \exp\begin{pmatrix} 0 & \Theta \\ -\Theta^\dagger\Sigma & 0 \end{pmatrix}. \quad (1.112)$$

We saw that the left action of a group element $G \in PSU(2,2|4)$ on a coset representative $\mathfrak{g}$ should result in $\mathfrak{g}'\mathfrak{h}$ for another coset representative $\mathfrak{g}'$ and a compensating element $\mathfrak{h} \in SO(4,1) \times SO(5)$. In this case,

$$G\mathfrak{g} = G\mathfrak{g}_\mathfrak{f}G^{-1}G\mathfrak{g}_\mathfrak{b} \equiv G\mathfrak{g}_\mathfrak{f}G^{-1}\mathfrak{g}'_\mathfrak{b}\mathfrak{h}. \quad (1.113)$$

By using the power series representation of the exponential $\mathfrak{g}_\mathfrak{f}$, we see

$$\begin{aligned} G\mathfrak{g}_\mathfrak{f}G^{-1} &= G(\mathbb{1}_8 + \chi + \frac{1}{2}\chi^2 + \frac{1}{6}\chi^3 + \ldots)G^{-1} \\ &= (\mathbb{1}_8 + G\chi G^{-1} + \frac{1}{2}G\chi G^{-1}G\chi G^{-1} + \frac{1}{6}G\chi G^{-1}G\chi G^{-1}G\chi G^{-1} + \ldots) \\ &= \exp G\chi G^{-1}. \end{aligned} \quad (1.114)$$

This means the left action of G on $\mathfrak{g}$ induces the adjoint action of G on fermionic degrees of freedom found in χ. Suppose $G : \chi \to \chi + \delta_\epsilon\chi = \chi + \epsilon$. To find $\delta_\epsilon\mathfrak{g}_\mathfrak{b}$, we substitute into (1.113) and $\mathfrak{g}$ becomes $G\mathfrak{g}$, or

$$\begin{aligned} e^\epsilon e^\chi \mathfrak{g}_\mathfrak{b} &\overset{\mathrm{BCH}}{=} e^{\chi+\epsilon+\frac{1}{2}[\varepsilon,\chi]+\mathcal{O}(\epsilon^2)}\mathfrak{g}_\mathfrak{b} = e^{\chi+\epsilon+\frac{1}{2}[\varepsilon,\chi]}\mathfrak{g}_\mathfrak{b} \\ &= e^{\chi+\delta_\epsilon\chi}e^{\frac{1}{2}[\varepsilon,\chi]}\mathfrak{g}_\mathfrak{b} = e^{\chi+\delta_\epsilon\chi}\mathfrak{g}'_\mathfrak{b}\mathfrak{h}. \end{aligned}$$

For a 'small' compensating element $\mathfrak{h} \approx \mathbb{1}_8 + \delta\mathfrak{h}$, this means

$$e^{\frac{1}{2}[\varepsilon,\chi]}\mathfrak{g}_\mathfrak{b} = (\mathfrak{g}_\mathfrak{b} + \delta_\epsilon\mathfrak{g}_\mathfrak{b})(\mathbb{1}_8 + \delta\mathfrak{h})$$

[11] I will read a bit more about this and what this implies for paths in the group.
[12] We still do not distinguish between y_i and y^i, i.e. $y_i \equiv y^i$ and $z_i \equiv z^i$.

$$\begin{aligned}\mathfrak{g_b} + \frac{1}{2}[\varepsilon,\chi]\mathfrak{g_b} &= \mathfrak{g_b} + \delta_\epsilon \mathfrak{g_b} + \mathfrak{g_b}\delta\mathfrak{h} \\ \implies \delta_\epsilon \mathfrak{g_b} &= \frac{1}{2}[\varepsilon,\chi]\mathfrak{g_b} - \mathfrak{g_b}\delta\mathfrak{h}. \end{aligned} \tag{1.115}$$

A better way to define the change $\delta\mathfrak{g_b}$, instead of $\mathfrak{g_b} -' \mathfrak{g_b}$ is $(\mathfrak{g}'_\mathfrak{b} - \mathfrak{g_b})\mathfrak{g}_\mathfrak{b}^{-1}$ such that

$$\delta_\epsilon \mathfrak{g_b} = \frac{1}{2}[\varepsilon,\chi] - \mathfrak{g_b}\delta\mathfrak{h}\mathfrak{g}_\mathfrak{b}^{-1} \tag{1.116}$$

Here, $\mathfrak{g_b} \equiv e^{\chi_\mathfrak{b}} \to \mathfrak{g}'_\mathfrak{b} \equiv \mathfrak{g_b} e^{\chi_\mathfrak{b} + \delta_\epsilon \mathfrak{g_b}}$ which is 'better' as the change $\delta_\epsilon \mathfrak{g_b}$ is a bosonic $\mathfrak{psu}(2,2|4)$ element, so that it is comparable to $\delta_\epsilon \chi$. Note there is now an adjoint transformation on the element $\mathfrak{h} = e^{\delta\mathfrak{h}}$. Another valid parametrisation of the coset representative $\mathfrak{g}$ is of the form

$$\mathfrak{g} = \Lambda(t,\phi)\mathfrak{g}(\chi)\mathfrak{g}(\mathbb{X}) \tag{1.117}$$

where we split the bosonic degrees of freedom from $\mathfrak{g_b}$ into

$$\Lambda(t,\phi) = \exp\begin{pmatrix} \frac{\mathrm{i}}{2}t\gamma^5 & 0 \\ 0 & \frac{\mathrm{i}}{2}\phi\gamma^5 \end{pmatrix} \qquad \text{and} \qquad \mathbb{X} = \begin{pmatrix} \frac{1}{2}z^i\gamma^i & 0 \\ 0 & \frac{\mathrm{i}}{2}y^i\gamma^i \end{pmatrix} \tag{1.118}$$

for $i = 1, ..., 4$. We can go between (1.111) and (1.117) by the change $\chi \to \Lambda(t,\phi)\chi\Lambda(t,\phi)^{-1}$.

It is possible to construct an alternative embedding $\mathfrak{g}(\mathbb{X}) = \sqrt{\frac{1+\mathbb{X}}{1-\mathbb{X}}}$ such that the bilinear form $\mathrm{str}[(\mathfrak{g}_\mathfrak{b}^{-1}d\mathfrak{g_b})^2]$ reduces to the spacetime metric (1.108). With the choice given above for $\mathbb{X}$, we get

$$\mathfrak{g}(\mathbb{X}) = \begin{pmatrix} \frac{1}{\sqrt{1-z^2/4}}[\mathbb{1}_4 + \frac{1}{2}z^i\gamma^i] & 0 \\ 0 & \frac{1}{\sqrt{1+y^2/4}}[\mathbb{1}_4 + \frac{\mathrm{i}}{2}y^i\gamma^i] \end{pmatrix}. \tag{1.119}$$

as derived in A.9. If we set $\mathfrak{g_b} = \Lambda(t,\phi)\mathfrak{g}(\mathbb{X})$, then we need to find

$$(\mathfrak{g}_\mathfrak{b}^{-1}d\mathfrak{g_b})^2 = \left(\mathfrak{g}(-\mathbb{X})\begin{pmatrix} \frac{\mathrm{i}}{2}dt\gamma^5 & 0 \\ 0 & \frac{\mathrm{i}}{2}d\phi\gamma^5 \end{pmatrix}\mathfrak{g}(\mathbb{X}) + \mathfrak{g}(-\mathbb{X})d\mathfrak{g}(\mathbb{X})\right)^2. \tag{1.120}$$

A long computation in A.9 yields (I have done much of this on paper)

$$\mathfrak{g}(-\mathbb{X})\begin{pmatrix} \frac{\mathrm{i}}{2}dt\gamma^5 & 0 \\ 0 & \frac{\mathrm{i}}{2}d\phi\gamma^5 \end{pmatrix}\mathfrak{g}(\mathbb{X}) = \dots, \tag{1.121}$$

$$\mathfrak{g}(-\mathbb{X})d\mathfrak{g}(\mathbb{X}) = \dots. \tag{1.122}$$

1.4.2 Linearly realised bosonic symmetries

I will reproduce this subsection while I am working on Chapter 2 as it will be necessary at the quantisation step.

② Light cone gauge and spectrum

2.1 How to fix a Lagrangian

At the end of this section, we will end up with a Lagrangian ready for decompactification. The Lagrangian will be fixed in two ways; in the light cone and κ-symmetry gauges. Our first step will be to introduce the first-order formalism in light cone coordinates through the bosonic case in order to prepare us for the Green-Schwarz superstring. We will then move forward with a special light cone gauge known as the uniform light-cone gauge, which will be followed by fixing the κ-symmetry gauge and discussing the resulting gauge-fixed Lagrangian.

2.1.1 Bosonic first-order formalism

We start with the Polyakov action for bosonic strings in the space described in 1.4.1 with metric G_{MN},

$$S = -\frac{g}{2}\int d\tau d\sigma\ \gamma^{\alpha\beta}\partial_\alpha X^M \partial_\beta X^N G_{MN} \tag{2.1}$$

where $X^M \in \{t, \phi, x^\mu\}$ and $\alpha, \beta = \tau, \sigma$. We will use the shorthand $\partial_\tau X^M = \dot{X}^M,\ \partial_\sigma X^M = X'^M$. The conjugate momenta can be found as usual, remembering that $\gamma^{\alpha\beta} = \gamma^{\beta\alpha}$ so that

$$\begin{aligned} p_M &= \frac{\delta S}{\delta \dot{X}^M} = -g\gamma^{\tau\beta}\partial_\beta X^N G_{MN}, \\ \implies p^N &= -g\gamma^{\tau\tau}\dot{X}^N - g\gamma^{\tau\sigma}X'^N. \end{aligned} \tag{2.2}$$

We can rewrite the action in first-order form,

$$S = \int d\tau d\sigma\ \left(p_N \dot{X}^M + \frac{\gamma^{\tau\sigma}}{\gamma^{\tau\tau}}C_1 + \frac{1}{2g\gamma^{\tau\tau}}C_2\right), \tag{2.3}$$

with the following constraints determined in B.1.1

$$C_1 = p_M X'^M, \qquad\qquad C_2 = p_M p^M + g^2 X'_M X'^M. \tag{2.4}$$

We can express the action explicitly in light cone coordinates

$$\begin{aligned} t &= x_+ - a x_-, \qquad & p_t &= (1-a)p_- - p_+, \\ \phi &= x_+ + (1-a)x_-, \qquad & p_\phi &= p_+ + a p_-, \end{aligned} \tag{2.5}$$

which are parameterised by the constant a. The transversal coordinates x^μ and canonical momenta p_μ are unchanged. The first term in the action becomes

$$p_M \dot{X}^M = p_+ \dot{x}_- + p_- \dot{x}_+ + p_\mu \dot{x}^\mu, \tag{2.6}$$

while the constraints can be written (see B.1.1) as

$$C_1 = p_+ x'_- + p_- x'_+ + p_\mu x'^\mu, \tag{2.7}$$

$$\begin{aligned} C_2 = {} & p_+^2\left[G_{\phi\phi}^{-1} - G_{tt}^{-1}\right] + 2p_+p_-\left[aG_{\phi\phi}^{-1} + (1-a)G_{tt}^{-1}\right] + p_-^2\left[a^2 G_{\phi\phi}^{-1} - (1-a)^2 G_{tt}^{-1}\right] \\ & + g^2 x_+'^2\left[G_{\phi\phi} - G_{tt}\right] + 2g^2 x'_+ x'_-\left[aG_{tt} + (1-a)G_{\phi\phi}\right] + g^2 x_-'^2\left[(1-a)^2 G_{\phi\phi} - a^2 G_{tt}\right] \\ & + 2\mathcal{H}_\perp. \end{aligned} \tag{2.8}$$

Here we defined the 'Hamiltonian' governing the transversal degrees of freedom (x^μ, p^μ)

$$\mathcal{H}_\perp = \frac{1}{2} p_\mu p_\nu G^{\mu\mu} + \frac{1}{2} g^2 x'_\mu x'_\nu G^{\mu\nu}. \tag{2.9}$$

In string theory, actions such as the Polyakov or Green-Schwarz actions display two reparametrisation invariances in the coordinates (τ, σ). This provides two gauge freedoms which we will now exploit. We fix the light cone gauge by imposing the following conditions.

$$x_+ = \tau + \frac{\pi}{r} am\sigma, \qquad p_+ = 1. \tag{2.10}$$

This is the *uniform* light cone gauge because the total momentum $P_+ = \int d\sigma\, p_+$ is uniformly distributed around the string. The integer m is the winding number from the periodicity condition at the equator,

$$\phi(r) - \phi(-r) = 2\pi m. \tag{2.11}$$

This same periodicity is the reason for the normalisation factor π/r which is required by the consistency $x_+(r) - x_+(-r) = a(\phi(r) - \phi(r))$. Note that in flat space (when $G_{tt} = G_{\phi\phi}$) and when $a = 1/2$, the above constraint C_2 becomes

$$C_2 = p_- G_{tt}^{-1} + \frac{1}{2} m g^2 x'_- G_{tt} + \mathcal{H}_\perp.$$

We will soon set $m = 0$, resulting in $\mathcal{H}_\perp \sim -p_-$, which motivates the label of the transversal term since we will soon find that the Hamiltonian of the system $\mathcal{H}$ is in fact equal to $-p_-$ in general. I have quite a lot to fill in here. Namely I need to explain how 'solving' the constraints and resubstituting variables results in the light cone gauge-fixed Lagrangian, which implies $\mathcal{H} = -p_-$. I also need to highlight the importance of a non-vanishing angular momentum J caused by ϕ-shift invariance. The rest of the section is more discussion with one computation I already did on Mathematica:

$$\mathrm{H} == -\frac{(-1+a)\,G_\phi + G_t\left(-a + G_\phi\sqrt{\frac{1+a^4 C^2 g^2 G_t^2 + a^2 G_t\left(-2\,(C - a\,C)^2 g^2 G_\phi - H_x\right) + (-1+a)^2 G_\phi\left((-1+a)^2 C^2 g^2 G_\phi + H_x\right)}{G_t\, G_\phi}}\right)}{a^2 G_t - (-1+a)^2 G_\phi} \;||$$

$$\mathrm{H} - \frac{-((-1+a)\,G_\phi) + G_t\left(a + G_\phi\sqrt{\frac{1+a^4 C^2 g^2 G_t^2 + a^2 G_t\left(-2\,(C - a\,C)^2 g^2 G_\phi - H_x\right) + (-1+a)^2 G_\phi\left((-1+a)^2 C^2 g^2 G_\phi + H_x\right)}{G_t\, G_\phi}}\right)}{a^2 G_t - (-1+a)^2 G_\phi} == 0$$

2.1.2 Green-Schwarz first-order formalism

In analogy with the first-order form of the Polyakov string (2.3), we can introduce an auxiliary field denoted $\pi \in \mathfrak{psu}(2,2|4)$, such that the Green-Schwarz superstring (1.32) becomes

$$\mathscr{L} = -\mathrm{str}\left[\pi A_\tau^{(2)} + \kappa \frac{g}{2} \varepsilon^{\alpha\beta} A_\alpha^{(\sigma)} A_\beta^{(3)} + \frac{\gamma^{\tau\sigma}}{\gamma^{\tau\tau}} \pi A_\sigma^{(2)} - \frac{1}{2g\gamma^{\tau\tau}} \left(\pi^2 + g^2 A_\sigma^{(2)} A_\sigma^{(2)}\right)\right]. \tag{2.12}$$

The equation of motion for π is given by

$$0 = \frac{\delta \mathscr{L}}{\delta \pi} = -\mathrm{str}\left[A_\tau^{(2)} + \frac{\gamma^{\tau\sigma}}{\gamma^{\tau\tau}} A_\sigma^{(2)} - \frac{1}{g\gamma^{\tau\tau}} \pi\right]$$

which has an obvious solution, reminiscent of the bosonic momenta p^M in (2.2),

$$\pi = g\gamma^{\tau\tau} A_\tau^{(2)} + g\gamma^{\tau\sigma} A_\sigma^{(2)}. \tag{2.13}$$

Substituting this expression for π into the Lagrangian minus the Wess-Zumino term, we recover in B.1.2 the Green-Schwarz Lagrangian kinetic term, as expected. Looking at (2.12), the constraints C_1 and C_2 of the bosonic first-order formalism are

$$C_1 = -\mathrm{str}\big(\pi A_\sigma^{(2)}\big) = 0, \tag{2.14}$$

$$C_2 = \mathrm{str}\big(\pi^2 + g^2 A_\sigma^{(2)} A_\sigma^{(2)}\big) = 0, \tag{2.15}$$

which we will solve after imposing light cone gauge and fixing κ-symmetry. In general, the equation of motion shows that π can be viewed as an element of $\mathscr{G}^{(2)}$ without affecting the projections of A onto other graded subspaces. We can write it as a generic element of $\mathscr{G}^{(2)}$,

$$\pi = \pi^{(2)} \equiv \frac{\mathrm{i}}{2}\pi_+\Sigma_+ + \frac{\mathrm{i}}{2}\pi_-\Sigma_- + \frac{1}{2}\pi_\mu\Sigma^\mu + \pi_{\mathbb{1}}\mathrm{i}\mathbb{1}_8, \tag{2.16}$$

which is a linear combination of 8×8 matrices of the form

$$\Sigma_+ = \begin{pmatrix} \Sigma & 0 \\ 0 & \Sigma \end{pmatrix}, \quad \Sigma_- = \begin{pmatrix} -\Sigma & 0 \\ 0 & \Sigma \end{pmatrix}, \quad \Sigma^k = \begin{pmatrix} \gamma^k & 0 \\ 0 & 0 \end{pmatrix}, \quad \Sigma^{4+k} = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{i}\gamma^k \end{pmatrix}. \tag{2.17}$$

These matrices are made of the Dirac matrices γ^i for $i = 1, ..., 4$ and the matrix $\Sigma = \mathrm{diag}(1, 1, -1, -1)$, and span the diagonal (bosonic) subspace of $\mathfrak{su}(2, 2|4)$. The coefficient $\pi_{\mathbb{1}}$ is extraneous to the Lagrangian as it always features alongside a $\mathrm{str}\big(A_\alpha^{(2)}\big) = 0$.

2.1.3 Kappa symmetry gauge fixing

As proven in 1.2.4, the Green-Schwarz Lagrangian enjoys a local fermionic symmetry; κ-symmetry. Using such transformations, we saw that we could gauge away 16 of the 32 fermionic entries in the matrix representation of the embedding element $\chi \in \mathfrak{su}(2, 2|4)$. Schematically, for 2×2 matrices a and b,[13]

$$\chi = \left(\begin{array}{cc|cc} 0 & 0 & 0 & a \\ 0 & 0 & b & 0 \\ \hline 0 & b^\dagger & 0 & 0 \\ -a^\dagger & 0 & 0 & 0 \end{array}\right). \tag{2.18}$$

The following identities follow from – or can be considered a definition of – κ-symmetry:

$$\chi\Sigma_+ = -\Sigma_+\chi, \qquad \chi\Sigma_- = \Sigma_-\chi. \tag{2.19}$$

Explicit calculations in B.1.3 prove these identites for χ of the above form. Because $\mathfrak{g}(\chi)^{-1} = \mathfrak{g}(-\chi)$, we can Taylor expand the latter and apply the above identity for each copy of χ in the polynomials to get

$$\begin{aligned} \mathfrak{g}(\chi)^{-1}\Sigma_+ &= \Sigma_+\mathfrak{g}(\chi) &&\Longrightarrow\quad \mathfrak{g}(\chi)^{-1}\Sigma_+\mathfrak{g}(\chi) = \Sigma_+\mathfrak{g}(\chi)^2, \\ \mathfrak{g}(\chi)^{-1}\Sigma_- &= \Sigma_-\mathfrak{g}(\chi)^{-1} &&\Longrightarrow\quad \mathfrak{g}(\chi)^{-1}\Sigma_-\mathfrak{g}(\chi) = \Sigma_-. \end{aligned} \tag{2.20}$$

[13] This form of χ explicitly obeys the odd $\mathfrak{su}(2, 2|4)$ reality condition (1.7).

We will now find the explicit form of our current $A = -\mathfrak{g}^{-1}d\mathfrak{g}$ where $\mathfrak{g} = \Lambda(t,\phi)\mathfrak{g}(\chi)\mathfrak{g}(\mathbb{X})$ with the coset parametrisation given by

$$\Lambda(t,\phi) = \exp \frac{\mathrm{i}}{2}\begin{pmatrix} t\Sigma & 0 \\ 0 & \phi\Sigma \end{pmatrix}, \qquad \mathfrak{g}(\mathbb{X}) = \sqrt{\frac{\mathbb{1}+\mathbb{X}}{\mathbb{1}-\mathbb{X}}}, \qquad \mathfrak{g}(\chi) = \chi + \sqrt{\mathbb{1}+\chi^2}. \tag{2.21}$$

One can revert back to the exponential definition of $\mathfrak{g}(\chi)$ in (1.112) with the substitution $\chi \to \sinh\chi$ since $\sinh\chi + \sqrt{\mathbb{1}+\sinh^2\chi} = \sinh\chi + \cosh\chi = \exp\chi$. In B.1.3 we derive that the even and odd components of A are

$$\begin{aligned} A_{\mathrm{e}} &= -\mathfrak{g}(\mathbb{X})^{-1}\left[\frac{\mathrm{i}}{2}\left(dx_+ + (\frac{1}{2}-a)dx_-\right)\Sigma_+(\mathbb{1}+2\chi^2) + \frac{\mathrm{i}}{4}dx_-\Sigma_- - d\mathfrak{g}(\mathbb{X})\mathfrak{g}(\mathbb{X})^{-1}\right]\mathfrak{g}(\mathbb{X}) \\ &\quad - \mathfrak{g}(\mathbb{X})^{-1}\left[\sqrt{\mathbb{1}+\chi^2}d\sqrt{\mathbb{1}+\chi^2} - \chi d\chi\right]\mathfrak{g}(\mathbb{X}), \\ A_{\mathrm{o}} &= -\mathfrak{g}(\mathbb{X})^{-1}\left[\mathrm{i}\left(dx_+ + (\frac{1}{2}-a)dx_-\right)\Sigma_+\chi\sqrt{\mathbb{1}+\chi^2} + \sqrt{\mathbb{1}+\chi^2}d\chi - \chi d\sqrt{\mathbb{1}+\chi^2}\right]\mathfrak{g}(\mathbb{X}). \end{aligned} \tag{2.22}$$

These formulae were obtained using the identities (2.20) which depended on the commutation relations of χ and $\Sigma_\pm$. The latter relied on certain fermionic degrees of freedom in χ being gauged away by the κ-symmetry transformation as discussed in 1.2.3. In tandem with this κ-symmetry gauge fixing, we make the natural choice $a = 1/2$ such that the above odd current no longer depends on the coordinate x_-. The next two sections I have not fully done on paper yet but will probably not deviate too much from your paper's structure.

2.1.4 Light cone gauge fixing

2.1.5 Gauge-fixed Lagrangian

2.2 Decompactification limit

Properties of the light-cone string theory in the decompactification limit are explored. In this limit, the total light-cone momentum goes to infinity. The resulting massive two-dimensional model enjoys multi-soliton solutions. A single soliton solution, namely the giant magnon, is constructed and its dispersion relation found. The light-cone $\mathrm{AdS}_5 \times S^5$ superstring is described in the large tension expansion and then perturbatively quantised. Particles with special scattering properties form closed sectors, some of which are discussed. Finally, a perturbative world-sheet S-matrix satisfying the classical Yang-Baxter equation is presented.

2.3 Perturbative world-sheet S-matrix

The basics are covered for perturbative S-matrix scattering theory in the Heisenberg picture. In particular, incident and scattered states are created or destroyed by field operators $a^\dagger$, a as in second quantisation. An example ($Y^{a\dot{a}}$ bosons from S^5) is used to present the computation of a perturbative S-matrix. The procedure of factorisating an S-matrix is explained.

2.4 Symmetry algebra

I suppose I will show as much as I can of this.

Ⓐ Appendix – Chapter 1

A.1 Extended spinor algebras

We aim to show that adding $n^{i6} = \frac{\mathrm{i}}{2}\gamma^i$ to the five gamma matrices preserves the relations (1.12) by computing $[n^{i6}, n^{kl}]$. First, we note that

$$n^{kl} = \frac{1}{4}[\gamma^k, \gamma^l] = \frac{1}{4}\left(\gamma^k\gamma^l - (2\delta^{kl}\mathbb{1} - \gamma^k\gamma^l)\right) = \frac{1}{2}\left(\gamma^k\gamma^l - \delta^{kl}\mathbb{1}\right). \tag{A.1}$$

Clearly if either $l = 6$ or $k = 6$, we get a result of the form

$$[n^{i6}, n^{kl}] \overset{l=6}{=} -\delta^{l6}\frac{1}{4}[\gamma^i, \gamma^k] = -\delta^{l6}n^{ik}, \qquad [n^{i6}, n^{kl}] \overset{k=6}{=} \delta^{k6}n^{il}. \tag{A.2}$$

If however $k \neq 6$ and $l \neq 6$, we get

$$\begin{aligned}
[n^{i6}, n^{kl}] &= \frac{\mathrm{i}}{4}[\gamma^i, \gamma^k\gamma^l - \delta^{kl}\mathbb{1}] = \frac{\mathrm{i}}{4}[\gamma^i, \gamma^k\gamma^l] = \frac{\mathrm{i}}{4}\gamma^k[\gamma^i, \gamma^l] + \frac{\mathrm{i}}{4}[\gamma^i, \gamma^k]\gamma^l \\
&= \mathrm{i}\gamma^k n^{il} + \mathrm{i}n^{ik}\gamma^l = \frac{\mathrm{i}}{2}\gamma^k\left(\gamma^i\gamma^l - \delta^{il}\mathbb{1}\right) + \frac{\mathrm{i}}{2}\left(\gamma^i\gamma^k - \delta^{ik}\mathbb{1}\right)\gamma^l \\
&= \frac{\mathrm{i}}{2}\left(\gamma^k\gamma^i\gamma^l + \gamma^i\gamma^k\gamma^l\right) - \delta^{il}n^{k6} - \delta^{ik}n^{l6} = \frac{\mathrm{i}}{2}\left(2\delta^{ik}\gamma^l - \gamma^i\gamma^k\gamma^l + \gamma^i\gamma^k\gamma^l\right) - \delta^{il}n^{k6} - \delta^{ik}n^{l6} \\
&= -\delta^{il}n^{k6} + \delta^{ik}n^{l6}.
\end{aligned}$$

Adding all these cases, which do not contribute whenever their conditions are not met, we find (1.13) which extends the $\mathfrak{so}(5)$ spinor relations (1.12) to $\mathfrak{so}(6)$.

Similarly, let us define $\gamma^0 \equiv \mathrm{i}\gamma^5$ and look at the extended generators of $\mathfrak{so}(4,1)$ satisfying (1.14)

$$m^{ij} = \frac{1}{4}[\gamma^i, \gamma^j], \qquad m^{i5} \equiv \frac{1}{2}\gamma^i, \qquad i, j = 0, ..., 4. \tag{A.3}$$

The addition of m^{i5} should preserve the relations $[m^{ij}, m^{kl}]$ for $i, j... = 0, ..., 5$. As above we start with

$$[m^{i5}, m^{kl}] \overset{l=5}{=} \delta^{l5}\frac{1}{4}[\gamma^i, \gamma^k] = \delta^{l5}m^{ik}, \qquad [m^{i5}, m^{kl}] \overset{k=5}{=} -\delta^{k5}m^{il}. \tag{A.4}$$

If however $k \neq 5$ and $l \neq 5$, we get

$$\begin{aligned}
[m^{i5}, m^{kl}] &= \frac{1}{4}[\gamma^i, \gamma^k\gamma^l - \delta^{kl}\mathbb{1}] = \frac{1}{4}[\gamma^i, \gamma^k\gamma^l] = \frac{1}{4}\gamma^k[\gamma^i, \gamma^l] + \frac{1}{4}[\gamma^i, \gamma^k]\gamma^l \\
&= \gamma^k m^{il} + m^{ik}\gamma^l = \frac{1}{2}\gamma^k\left(\gamma^i\gamma^l - \delta^{il}\mathbb{1}\right) + \frac{1}{2}\left(\gamma^i\gamma^k - \delta^{ik}\mathbb{1}\right)\gamma^l \\
&= \frac{1}{2}\left(\gamma^k\gamma^i\gamma^l + \gamma^i\gamma^k\gamma^l\right) - \delta^{il}m^{k5} - \delta^{ik}m^{l5} = \frac{1}{2}\left(2\delta^{ik}\gamma^l - \gamma^i\gamma^k\gamma^l + \gamma^i\gamma^k\gamma^l\right) - \delta^{il}m^{k5} - \delta^{ik}m^{l5} \\
&= -\delta^{il}m^{k5} + \delta^{ik}m^{l5}.
\end{aligned}$$

All that is left to do is recognise that if $\eta^{i5} = -\delta^{i5}$, then the relations are satisfied by the generators m^{ij} for $i, j = 0, ..., 5$ and become those of $\mathfrak{so}(4,2)$ instead.

A.2 Endowing $\mathfrak{su}(2,2|4)$ with a $\mathbb{Z}_4$-grading

Let us first discuss the continuous subgroup of $\mathrm{Aut}(\mathfrak{sl}(4|4))$. We want to show that the continuous dilatation transformations $\delta_\rho(M)$ (1.17) can be written as $\delta_\rho(M) = e^{\frac{1}{2}\Upsilon \ln\rho} M e^{-\frac{1}{2}\Upsilon\ln\rho}$. We start with by noticing $\Upsilon^2 = \mathbb{1}_8$ and so e^Υ becomes

$$\begin{aligned} e^{\Upsilon} &= \mathbb{1}_8 + \Upsilon + \frac{1}{2!}\mathbb{1}_8 + \frac{1}{3!}\Upsilon + \ldots = \sum_{n=0}^{\infty}\frac{1}{(2n)!}\mathbb{1}_8 + \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\Upsilon \\ &= \cosh(1)\mathbb{1}_8 + \sinh(1)\Upsilon = \begin{pmatrix} e\mathbb{1}_4 & 0 \\ 0 & \frac{1}{e}\mathbb{1}_4\end{pmatrix}. \end{aligned} \tag{A.5}$$

By raising both sides of the equation to the power of $\frac{1}{2}\ln\rho = \ln\rho^{1/2}$, we get

$$e^{\frac{1}{2}\Upsilon\ln\rho} = \begin{pmatrix} \rho^{\frac{1}{2}}\mathbb{1}_4 & 0 \\ 0 & \rho^{-\frac{1}{2}}\mathbb{1}_4 \end{pmatrix} \implies e^{-\frac{1}{2}\Upsilon\ln\rho} = \begin{pmatrix} \rho^{-\frac{1}{2}}\mathbb{1}_4 & 0 \\ 0 & \rho^{\frac{1}{2}}\mathbb{1}_4 \end{pmatrix}, \tag{A.6}$$

which clearly shows (1.17). This transformation is an automorphism on $\mathfrak{su}(2,2|4)$ if it preserves the fermionic reality condition $\eta = -\theta^\dagger\Sigma$. To this end, the transformation parameter must satisfy $|\rho|^2 = 1$. It is clear that $\delta_{-1}(M) = \Upsilon M \Upsilon^{-1}$ and we note $\delta_{-1}(M) = M$ if M is even whereas $\delta_{-1}(M) = -M$ if M is odd.

Next we want to show that the fourth-order automorphism $\Omega(M)$ restricts to the subalgebra $\mathfrak{su}(2,2|4) \subset \mathfrak{sl}(4|4)$. To do this, we should show that $\Omega(M)^\dagger = -H\Omega(M)H^{-1}$. Since $[K,\Sigma] = [\gamma^5, \gamma^2\gamma^4] = 0$, we know that $[\mathcal{K}, H] = 0$ which will be useful since we can use the reality condition (1.5) for $M \in \mathfrak{su}(2,2|4)$. However, the issue is that $(M^{st})^\dagger \neq (M^\dagger)^{st}$ in general. In particular, for M even

$$M = \begin{pmatrix} m & 0 \\ 0 & n\end{pmatrix} \implies (M^{st})^\dagger = (M^\dagger)^{st} = \begin{pmatrix} m^* & 0 \\ 0 & n^*\end{pmatrix},$$

while for M odd

$$M = \begin{pmatrix} 0 & \theta \\ \eta & 0\end{pmatrix} \implies (M^{st})^\dagger = -(M^\dagger)^{st} = \begin{pmatrix} 0 & \bar\theta \\ -\bar\eta & 0\end{pmatrix}.$$

One can find $\Omega(M)^\dagger$ using the identities

$$\begin{gathered} \mathcal{K}^\dagger = \mathcal{K}^{-1} = \mathcal{K}^{st} = -\mathcal{K}, \qquad \Upsilon^\dagger = \Upsilon^{-1} = \Upsilon^{st} = \Upsilon, \qquad H^\dagger = H^{-1} = H^{st} = H, \\ [\mathcal{K}, H] = [\mathcal{K}, \Upsilon] = [\Upsilon, H] = 0, \end{gathered} \tag{A.7}$$

and the fact $\delta_{-1}(M) = \pm M = \Upsilon M \Upsilon^{-1}$ for M even $(+)$ or odd $(-)$. We get $(M^{st})^\dagger = \Upsilon (M^\dagger)^{st}\Upsilon^{-1}$ so

$$\begin{aligned} \Omega(M)^\dagger &= \mathcal{K}^\dagger (M^{st})^\dagger \mathcal{K}^\dagger = -\mathcal{K}(M^{st})^\dagger\mathcal{K}^{-1} \\ &= -\mathcal{K}\Upsilon(M^\dagger)^{st}\Upsilon^{-1}\mathcal{K}^{-1} = -\mathcal{K}\Upsilon(-HMH^{-1})^{st}\Upsilon^{-1}\mathcal{K}^{-1} \\ &= \mathcal{K}\Upsilon H M^{st} H^{-1}\Upsilon^{-1}\mathcal{K}^{-1} = -(\Upsilon H)(-\mathcal{K}M^{st}\mathcal{K}^{-1})(\Upsilon H)^{-1} \\ &= -(\Upsilon H)\Omega(M)(\Upsilon H)^{-1}. \end{aligned}$$

For M even, so $M = M^{(0)} + M^{(2)}$, the hypercharge Υ can be ignored in the above expression since $\delta_{-1}(M) = M$. This means $\Omega(M)$ restricts to the bosonic subalgebra of $\mathfrak{su}(2,2|4)$. To see that it also restricts to the entire subalgebra, we should look at $M^{(k)\dagger}$. In fact, if we use the properties

$$\Omega^2(M) = (M^{st})^{st} = \delta_{-1}(M) = \Upsilon M \Upsilon \implies \Omega^3(M) = \Upsilon\Omega(M)\Upsilon,$$

we get the strong result for any $k = 0, 1, 2, 3$

$$
\begin{aligned}
M^{(k)\dagger} &= \frac{1}{4}\left[-HMH^{-1} + (-1)^{3k}\mathrm{i}^{3k}\Omega(M)^\dagger + \mathrm{i}^{2k}\Omega^2(M)^\dagger + (-1)^k\mathrm{i}^k\Omega^3(M)^\dagger\right] \\
&= \frac{1}{4}\left[-HMH^{-1} + \mathrm{i}^k\left(-(\Upsilon H)\Omega(M)(\Upsilon H)^{-1}\right) + \mathrm{i}^{2k}\left(\Upsilon M\Upsilon\right)^\dagger + \mathrm{i}^{3k}\left(\Upsilon\Omega(M)\Upsilon\right)^\dagger\right] \\
&= \frac{1}{4}\left[-HMH^{-1} + \mathrm{i}^k\left(-H\Omega^3(M)H^{-1}\right) + \mathrm{i}^{2k}\left(-H\Upsilon M\Upsilon H^{-1}\right) + \mathrm{i}^{3k}\left(-H\Omega(M)H^{-1}\right)\right] \\
&= \frac{1}{4}\left[-HMH^{-1} + \mathrm{i}^k\left(-H\Omega^3(M)H^{-1}\right) + \mathrm{i}^{2k}\left(-H\Omega^2H^{-1}\right) + \mathrm{i}^{3k}\left(-H\Omega(M)H^{-1}\right)\right] \\
&= -HM^{(k)}H^{-1}.
\end{aligned}
$$

Because any $M \in \mathfrak{sl}(4|4)$ can be uniquely decomposed by the $\mathbb{Z}_4$-grading (1.24), and since we just showed each component $M^{(k)}$ is independently an element of $(\mathfrak{p})\mathfrak{su}(2,2|4)$, it must be that the subalgebra $\mathfrak{su}(2,2|4)$ can itself be endowed with the $\mathbb{Z}_4$-grading $\Omega(M)$. From now on, we relabel $\mathscr{G} = \mathfrak{su}(2,2|4)$ and the $\mathbb{Z}_4$-graded decomposition of $\mathscr{G}$ is given with respect to the automorphism $\Omega(M)$ by (1.24).

If we start with a matrix M of the generic form (1.4), then the explicit components $M^{(k)}$ can be found by computing $\Omega^k(M)$ and evaluating (1.24). Using the usual identities,

$$
\begin{aligned}
\Omega(M) &= -\begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}\begin{pmatrix} m^t & -\eta^t \\ \theta^t & n^t \end{pmatrix}\begin{pmatrix} K^{-1} & 0 \\ 0 & K^{-1} \end{pmatrix} = \begin{pmatrix} -Km^tK^{-1} & K\eta^tK^{-1} \\ -K\theta^tK^{-1} & -Kn^tK^{-1} \end{pmatrix}, \\
\Omega^2(M) &= \delta_{-1}(M) = \begin{pmatrix} m & -\theta \\ -\eta & n \end{pmatrix}, \qquad \Omega^3(M) = \Upsilon\Omega(M)\Upsilon = \begin{pmatrix} -Km^tK^{-1} & -K\eta^tK^{-1} \\ K\theta^tK^{-1} & -Kn^tK^{-1} \end{pmatrix}.
\end{aligned}
$$

we find the decomposition $M^{(k)}$ (1.25). To find explicit expressions for the even components $M^{(0)}, M^{(2)}$ in terms of bosonic generators (1.15), we notice that the matrix $K = -\gamma^2\gamma^4$ was constructed such that

$$
\begin{aligned}
\left(K(\gamma^i)^tK^{-1}\right)^* &= \gamma^2\gamma^4(\gamma^i)^\dagger\gamma^4\gamma^2 = \gamma^2\gamma^4\gamma^i\gamma^4\gamma^2 = \gamma^2(2\delta^{i4} - \gamma^i\gamma^4)\gamma^4\gamma^2 \\
&= 2\delta^{i4}\gamma^2\gamma^4\gamma^2 - \gamma^2\gamma^i\gamma^2 = -2\delta^{i4}\gamma^4 - (2\delta^{i2} - \gamma^i\gamma^2)\gamma^2 \\
&= \gamma^i - 2(\delta^{i4}\gamma^4 + \delta^{i2}\gamma^2) \overset{\dagger}{=} (\gamma^i)^*
\end{aligned}
$$

or equivalently $K(\gamma^i)^tK^{-1} = \gamma^i$. In turn this means

$$
K[\gamma^i, \gamma^j]^tK^{-1} = K[(\gamma^j)^t, (\gamma^i)^t]^tK^{-1} = [K(\gamma^j)^tK^{-1}, K(\gamma^i)^tK^{-1}] = -[\gamma^i, \gamma^j].
$$

Looking at the expressions for $M^{(k)}$ (1.25), it is apparent that one can span $\mathscr{G}^{(0)}$ by expressing the even 'upper block' elements m in terms of the $\mathfrak{so}(4,1) \subset \mathfrak{su}(2,2)$ generators $\left\{\frac{1}{4}[\gamma^i,\gamma^j], \frac{\mathrm{i}}{4}[\gamma^i,\gamma^5]\right\}$ for $i,j = 1,...,4$ and the 'lower block' elements n in terms of the $\mathfrak{so}(5)$ generators $\left\{\frac{1}{4}[\gamma^i,\gamma^j]\right\}$ for $i,j = 1,...,5$. Similarly, the elements m of the projection $\mathscr{G}^{(2)}$ can be spanned by the remaining bosonic generators $\left\{\frac{1}{2}\gamma^i, \frac{\mathrm{i}}{2}\gamma^5\right\} \in \mathfrak{su}(2,2)$ for $i = 1,...,4$ and the elements n by $\left\{\frac{\mathrm{i}}{2}\gamma^i\right\} \in \mathfrak{su}(4)$ for $i = 1,...,5$. Explicit matrices are given in (1.26) and (1.27) respectively.

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† Only γ^2 and γ^4 are imaginary such that $(\gamma^i)^* = -\gamma^i$ for $i = 2,4$ and $(\gamma^i)^* = \gamma^i$ otherwise.

A.3 Justifying Green-Schwarz

In this appendix I will show the following in favour of the GS Lagrangian:

1. $\mathscr{L}_{\rm GS}$ reduces to the Polyakov action in the absence of fermionic degrees of freedom (1.5.2 in the paper),
2. $\mathscr{L}_{\rm GS}$ is invariant under $SO(4,1)\times SO(5)$ transformations, reflecting the isometries of $\mathrm{AdS}_5\times S^5$.

A.4 Green-Schwarz equations of motion

To begin a long derivation of the equations of motion, we will show that for $M_1,\ M_2\in\mathfrak{su}(2,2|4)$

$$\mathrm{str}\big(\Omega^k(M_1)M_2\big)=\mathrm{str}\big(M_1\Omega^{4-k}(M_2)\big),\qquad k=1,2,3. \tag{A.8}$$

By definition $\Omega(M)=-\mathcal{K}M^{st}\mathcal{K}^{-1}$. Using the supertrace identities

$$\mathrm{str}\big(AB\big)=\mathrm{str}\big(BA\big),\qquad \mathrm{str}\big(A^{st}\big)=str A, \tag{A.9}$$

and the fact that $\Omega^4(M)=M$,

$$\begin{aligned}\mathrm{str}\big(\Omega^k(M_1)M_2\big)&=\mathrm{str}\big(\Omega^k(M_1)\Omega^4(M_2)\big)=\mathrm{str}\big(\mathcal{K}(\Omega^{k-1}(M_1))^{st}\mathcal{K}^{-1}\mathcal{K}(\Omega^{4-1}(M_2))^{st}\mathcal{K}^{-1}\big)\\&=\mathrm{str}\big(\Omega^{k-1}(M_1)\Omega^{4-1}(M_2)\big)=...=\mathrm{str}\big(\Omega^{k-k}(M_1)\Omega^{4-k}(M_2)\big)\\ \implies \mathrm{str}\big(\Omega^k(M_1)M_2\big)&=\mathrm{str}\big(M_1\Omega^{4-k}(M_2)\big).\end{aligned} \tag{A.10}$$

Armed with (A.10), we can show that $\mathrm{str}\big(A^{(j)}_\alpha A^{(4-j)}_\beta\big)=\mathrm{str}\big(A_\alpha A^{(4-j)}_\beta\big)=\mathrm{str}\big(A^{(j)}_\alpha A_\beta\big)$ when $j=1,2,3$. For convenience, define $\Omega^k(A_\alpha)\equiv\Omega^k_\alpha$. Then according to (1.30), and remembering that $\mathrm{i}^4=1$ so $\mathrm{i}^{-j}=\mathrm{i}^{3j}$,

$$\begin{aligned}\mathrm{str}\big(A^{(j)}_\alpha A^{(4-j)}_\beta\big)&=\frac{1}{16}\,\mathrm{str}\Big[\big(A_\alpha+\mathrm{i}^{3j}\Omega_\alpha+\mathrm{i}^{2j}\Omega^2_\alpha+\mathrm{i}^{j}\Omega^3_\alpha\big)\big(A_\beta+\mathrm{i}^{3(4-j)}\Omega_\beta+\mathrm{i}^{2(4-j)}\Omega^2_\beta+\mathrm{i}^{(4-j)}\Omega^3_\beta\big)\Big]\\&=\frac{1}{16}\,\mathrm{str}\Big[\big(A_\alpha+\mathrm{i}^{3j}\Omega_\alpha+\mathrm{i}^{2j}\Omega^2_\alpha+\mathrm{i}^{j}\Omega^3_\alpha\big)\big(A_\beta+\mathrm{i}^{-3j}\Omega_\beta+\mathrm{i}^{-2j}\Omega^2_\beta+\mathrm{i}^{-j}\Omega^3_\beta\big)\Big]\\&=\frac{1}{16}\,\mathrm{str}\Big[\big(A_\alpha+\mathrm{i}^{3j}\Omega_\alpha+\mathrm{i}^{2j}\Omega^2_\alpha+\mathrm{i}^{j}\Omega^3_\alpha\big)\big(A_\beta+\mathrm{i}^{j}\Omega_\beta+\mathrm{i}^{2j}\Omega^2_\beta+\mathrm{i}^{3j}\Omega^3_\beta\big)\Big]\\&=\frac{1}{16}\Big[\mathrm{str}\big(A_\alpha A_\beta\big)+\mathrm{i}^{j}\mathrm{str}\big(A_\alpha\Omega_\beta\big)+\mathrm{i}^{2j}\mathrm{str}\big(A_\alpha\Omega^2_\beta\big)+\mathrm{i}^{3j}\mathrm{str}\big(A_\alpha\Omega^3_\beta\big)\\&\qquad+\mathrm{i}^{3j}\mathrm{str}\big(\Omega_\alpha A_\beta\big)+\mathrm{str}\big(\Omega_\alpha\Omega_\beta\big)+\mathrm{i}^{j}\mathrm{str}\big(\Omega_\alpha\Omega^2_\beta\big)+\mathrm{i}^{2j}\mathrm{str}\big(\Omega_\alpha\Omega^3_\beta\big)\\&\qquad+\mathrm{i}^{2j}\mathrm{str}\big(\Omega^2_\alpha A_\beta\big)+\mathrm{i}^{3j}\mathrm{str}\big(\Omega^2_\alpha\Omega_\beta\big)+\mathrm{str}\big(\Omega^2_\alpha\Omega^2_\beta\big)+\mathrm{i}^{j}\mathrm{str}\big(\Omega^2_\alpha\Omega^3_\beta\big)\\&\qquad+\mathrm{i}^{j}\mathrm{str}\big(\Omega^3_\alpha A_\beta\big)+\mathrm{i}^{2j}\mathrm{str}\big(\Omega^3_\alpha\Omega_\beta\big)+\mathrm{i}^{3j}\mathrm{str}\big(\Omega^3_\alpha\Omega^2_\beta\big)+\mathrm{str}\big(\Omega^3_\alpha\Omega^3_\beta\big)\Big].\end{aligned} \tag{A.11}$$

The terms with the same color are related by (A.10) so that on one hand

$$\begin{aligned}\mathrm{str}\big(A^{(j)}_\alpha A^{(4-j)}_\beta\big)&=\frac{4}{16}\Big[\mathrm{str}\big(A_\alpha A_\beta\big)+\mathrm{i}^{3j}\mathrm{str}\big(\Omega_\alpha A_\beta\big)+\mathrm{i}^{2j}\mathrm{str}\big(\Omega^2_\alpha A_\beta\big)+\mathrm{i}^{j}\mathrm{str}\big(\Omega^3_\alpha A_\beta\big)\Big]\\&=\mathrm{str}\Big[\frac{1}{4}\big(A_\alpha+\mathrm{i}^{3j}\Omega_\alpha+\mathrm{i}^{2j}\Omega^2_\alpha+\mathrm{i}^{j}\Omega^3_\alpha\big)A_\beta\Big]\end{aligned} \tag{A.12}$$

$$= \mathrm{str}\big(A_\alpha^{(j)}A_\beta\big),$$

and on the other hand, again using $\mathrm{i}^{3j} = \mathrm{i}^{-j} = \mathrm{i}^4\mathrm{i}^{-j} = \mathrm{i}^{4-j}$,

$$\begin{aligned}\mathrm{str}\big(A_\alpha^{(j)}A_\beta^{(4-j)}\big) &= \frac{4}{16}\big[\mathrm{str}\big(A_\alpha A_\beta\big)+\mathrm{i}^{3(4-j)}\mathrm{str}\big(A_\alpha\Omega_\beta\big)+\mathrm{i}^{2(4-j)}\mathrm{str}\big(A_\alpha\Omega_\beta^2\big)+\mathrm{i}^{(4-j)}\mathrm{str}\big(A_\alpha\Omega_\beta^3\big)\big] \qquad (A.13)\\ &= \mathrm{str}\left[A_\alpha\frac{1}{4}(A_\beta+\mathrm{i}^{3(4-j)}\Omega_\beta+\mathrm{i}^{2(4-j)}\Omega_\beta^2+\mathrm{i}^{(4-j)}\Omega_\beta^3)\right]\\ &= \mathrm{str}\big(A_\alpha A_\beta^{(4-j)}\big).\end{aligned}$$

In particular this means that

$$\mathrm{str}\big(A_\alpha^{(1)}A_\beta^{(3)}\big) = \mathrm{str}\big(A_\alpha A_\beta^{(3)}\big) = \mathrm{str}\big(A_\alpha^{(1)}A_\beta\big), \qquad (A.14)$$

$$\mathrm{str}\big(A_\alpha^{(2)}A_\beta^{(2)}\big) = \mathrm{str}\big(A_\alpha A_\beta^{(2)}\big) = \mathrm{str}\big(A_\alpha^{(2)}A_\beta\big), \qquad (A.15)$$

which further implies, using the product rule and the cyclicity of the supertrace,

$$\begin{aligned}\delta\mathrm{str}\big(A_\alpha^{(1)}A_\beta^{(3)}\big) &= \mathrm{str}\big(\delta A_\alpha A_\beta^{(3)}\big)+\mathrm{str}\big(A_\alpha^{(1)}\delta A_\beta\big)\\ &= \mathrm{str}\big(\delta A_\alpha A_\beta^{(3)}+\delta A_\beta A_\alpha^{(1)}\big), \qquad (A.16)\\ \delta\mathrm{str}\big(A_\alpha^{(2)}A_\beta^{(2)}\big) &= \mathrm{str}\big(\delta A_\alpha A_\beta^{(2)}\big)+\mathrm{str}\big(A_\alpha^{(2)}\delta A_\beta\big)\\ &= \mathrm{str}\big(\delta A_\alpha A_\beta^{(2)}+\delta A_\beta A_\alpha^{(2)}\big). \qquad (A.17)\end{aligned}$$

Substituting (A.16) and (A.17) into $\delta\mathscr{L}$ gives

$$\begin{aligned}\delta\mathscr{L} &= -\frac{g}{2}\Big[\gamma^{\alpha\beta}\delta\mathrm{str}\big(A_\alpha^{(2)}A_\beta^{(2)}\big)+\kappa\varepsilon^{\alpha\beta}\delta\mathrm{str}\big(A_\alpha^{(1)}A_\beta^{(3)}\big)\Big]\\ &= -\frac{g}{2}\Big[\gamma^{\alpha\beta}\mathrm{str}\big(\delta A_\alpha A_\beta^{(2)}+\delta A_\beta A_\alpha^{(2)}\big)+\kappa\varepsilon^{\alpha\beta}\mathrm{str}\big(\delta A_\alpha A_\beta^{(3)}+\delta A_\beta A_\alpha^{(1)}\big)\Big]\\ &\overset{\dagger}{=} -\frac{g}{2}\,\mathrm{str}\Big[\gamma^{\alpha\beta}\big(\delta A_\alpha A_\beta^{(2)}+\delta A_\alpha A_\beta^{(2)}\big)+\kappa\varepsilon^{\alpha\beta}\big(\delta A_\alpha A_\beta^{(3)}-\delta A_\alpha A_\beta^{(1)}\big)\Big]\\ &= -\,\mathrm{str}\Big[\delta A_\alpha g\big(\gamma^{\alpha\beta}A_\beta^{(2)}-\frac{\kappa}{2}\varepsilon^{\alpha\beta}\big(A_\beta^{(1)}-A_\beta^{(3)}\big)\big)\Big]\\ \implies \delta\mathscr{L} &= = -\mathrm{str}\big(\delta A_\alpha\Lambda^\alpha\big), \qquad (A.18)\end{aligned}$$

For a matrix $\mathfrak{g}\in SU(2,2|4)$, the variation $\delta\mathfrak{g}^{-1}$ or the derivative $\partial_\alpha\mathfrak{g}^{-1}$ can be found by looking at

$$0 = \delta(\mathfrak{g}\mathfrak{g}^{-1}) = \delta\mathfrak{g}\mathfrak{g}^{-1}+\mathfrak{g}\delta\mathfrak{g}^{-1} \implies \delta\mathfrak{g}^{-1} = -\mathfrak{g}^{-1}\delta\mathfrak{g}\mathfrak{g}^{-1}. \qquad (A.19)$$

In particular $\partial_\alpha\mathfrak{g}^{-1} = A_\alpha\mathfrak{g}^{-1}$. The variation δA_α is then

$$\begin{aligned}\delta A_\alpha &= \delta(-\mathfrak{g}^{-1}\partial_\alpha\mathfrak{g}) = -\delta\mathfrak{g}^{-1}\partial_\alpha\mathfrak{g}-\mathfrak{g}^{-1}\partial_\alpha\delta\mathfrak{g}\\ &= -(-\mathfrak{g}^{-1}\delta\mathfrak{g}\mathfrak{g}^{-1})\partial_\alpha\mathfrak{g}-\mathfrak{g}^{-1}\partial_\alpha\delta\mathfrak{g} = -\mathfrak{g}^{-1}\delta\mathfrak{g}A_\alpha-\mathfrak{g}^{-1}\partial_\alpha\delta\mathfrak{g}. \qquad (A.20)\end{aligned}$$

Substituting into (A.18),

$$\delta\mathscr{L} = \mathrm{str}\big(\mathfrak{g}^{-1}\delta\mathfrak{g}A_\alpha\Lambda^\alpha+\mathfrak{g}^{-1}\partial_\alpha\delta(\mathfrak{g})\Lambda^\alpha\big). \qquad (A.21)$$

†Using $\gamma^{\alpha\beta} = \gamma^{\beta\alpha}$ and $\varepsilon^{\alpha\beta} = -\varepsilon^{\beta\alpha}$ to change indices in the second terms.

The second term can be rewritten as (assuming cyclicity due to its presence in a supertrace)

$$\begin{aligned}\mathfrak{g}^{-1}\partial_\alpha(\delta\mathfrak{g})\Lambda^\alpha &= \cancel{\partial_\alpha(\mathfrak{g}^{-1}\delta\mathfrak{g}\Lambda^\alpha)} - A_\alpha\mathfrak{g}^{-1}\delta\mathfrak{g}\Lambda^\alpha - \mathfrak{g}^{-1}\delta\mathfrak{g}\partial_\alpha\Lambda^\alpha \\ &= -\mathfrak{g}^{-1}\delta\mathfrak{g}\Lambda^\alpha A_\alpha - \mathfrak{g}^{-1}\delta\mathfrak{g}\partial_\alpha\Lambda^\alpha\end{aligned}$$

where we drop the total derivative as the variation $\delta\mathfrak{g}$ vanishes at the bounds of integration in δS. Finally we can write the variation in the Lagrangian as

$$\delta\mathscr{L} = -\operatorname{str}\left[\mathfrak{g}^{-1}\delta\mathfrak{g}(\partial_\alpha\Lambda^\alpha - [A_\alpha, \Lambda^\alpha])\right], \tag{A.22}$$

which holds for arbitrary variations $\delta\mathfrak{g}$. If we view the term in the supertrace as an element of $\mathfrak{su}(2,2|4)$, this must mean that

$$\operatorname{str}\big(\partial_\alpha\Lambda^\alpha - [A_\alpha, \Lambda^\alpha]\big) = 0 \implies \partial_\alpha\Lambda^\alpha - [A_\alpha, \Lambda^\alpha] = \rho \cdot \mathbb{1}_8, \tag{A.23}$$

where ρ is determined by taking the supertrace of both sides. If instead the LHS was an element of $\mathfrak{psu}(2,2|4)$, then it would be equal to 0 modulo $\mathrm{i}\mathbb{1}$ in $\mathfrak{psu}(2,2|4)$, i.e.

$$\partial_\alpha\Lambda^\alpha - [A_\alpha, \Lambda^\alpha] = 0. \tag{A.24}$$

This single equation (A.24) can be projected onto different $\mathbb{Z}_4$-components. Let us first rewrite

$$\begin{aligned}[A_\alpha, \Lambda^\alpha] &= [A_\alpha, g\gamma^{\alpha\beta}A^{(2)}_\beta] - [A_\alpha, g\frac{\kappa}{2}\varepsilon^{\alpha\beta}(A^{(3)}_\beta - A^{(1)}_\beta)] \\ &= g\gamma^{\alpha\beta}[A^{(0)}_\alpha + A^{(1)}_\alpha + A^{(2)}_\alpha + A^{(3)}_\alpha, A^{(2)}_\beta] - g\frac{\kappa}{2}\varepsilon^{\alpha\beta}[A^{(0)}_\alpha + A^{(1)}_\alpha + A^{(2)}_\alpha + A^{(3)}_\alpha, (A^{(1)}_\beta - A^{(3)}_\beta)] \\ &= g\gamma^{\alpha\beta}\Big\{[A^{(0)}_\alpha, A^{(2)}_\beta] + [A^{(1)}_\alpha, A^{(2)}_\beta] + [A^{(2)}_\alpha, A^{(2)}_\beta] + [A^{(3)}_\alpha, A^{(2)}_\beta]\Big\} \\ &\quad - g\frac{\kappa}{2}\varepsilon^{\alpha\beta}\Big\{[A^{(0)}_\alpha, A^{(1)}_\beta] + [A^{(1)}_\alpha, A^{(1)}_\beta] + [A^{(2)}_\alpha, A^{(1)}_\beta] + [A^{(3)}_\alpha, A^{(1)}_\beta] \\ &\qquad - [A^{(0)}_\alpha, A^{(3)}_\beta] - [A^{(1)}_\alpha, A^{(3)}_\beta] - [A^{(2)}_\alpha, A^{(3)}_\beta] - [A^{(3)}_\alpha, A^{(3)}_\beta]\Big\}.\end{aligned}$$

The red term vanishes due to the symmetry of $\gamma^{\alpha\beta}$ under exchange of indices and the blue terms cancel due to the asymmetry of $\varepsilon^{\alpha\beta}$. We can now decompose each term in (A.24) where the colours indicate whether the term belongs to $\mathscr{G}^{(1)}$, $\mathscr{G}^{(2)}$ or $\mathscr{G}^{(3)}$;

$$\partial_\alpha\Lambda^\alpha = g\gamma^{\alpha\beta}\partial_\alpha A^{(2)}_\beta - g\frac{\kappa}{2}\varepsilon^{\alpha\beta}\partial_\alpha A^{(1)}_\beta - g\frac{\kappa}{2}\varepsilon^{\alpha\beta}\partial_\alpha A^{(3)}_\beta, \tag{A.25}$$

$$\begin{aligned}[A_\alpha, \Lambda^\alpha] &= g\gamma^{\alpha\beta}\Big\{[A^{(0)}_\alpha, A^{(2)}_\beta] + [A^{(1)}_\alpha, A^{(2)}_\beta] + [A^{(3)}_\alpha, A^{(2)}_\beta]\Big\} \\ &\quad - g\frac{\kappa}{2}\varepsilon^{\alpha\beta}\Big\{[A^{(0)}_\alpha, A^{(1)}_\beta] + [A^{(1)}_\alpha, A^{(1)}_\beta] + [A^{(2)}_\alpha, A^{(1)}_\beta] \\ &\qquad - [A^{(0)}_\alpha, A^{(3)}_\beta] - [A^{(2)}_\alpha, A^{(3)}_\beta] - [A^{(3)}_\alpha, A^{(3)}_\beta]\Big\}.\end{aligned} \tag{A.26}$$

Projecting the equations of motion (A.24) onto $\mathscr{G}^{(2)}$ gives

$$\gamma^{\alpha\beta}\partial_\alpha A^{(2)}_\beta - \gamma^{\alpha\beta}[A^{(0)}_\alpha, A^{(2)}_\beta] + \frac{\kappa}{2}\varepsilon^{\alpha\beta}\big([A^{(1)}_\alpha, A^{(1)}_\beta] - [A^{(3)}_\alpha, A^{(3)}_\beta]\big) = 0, \tag{A.27}$$

In order to proceed, we use the zero curvature condition for A (1.31) (recast in the form of (1.81)) to find

$$\varepsilon^{\alpha\beta}\partial_\alpha A^{(1)}_\beta = \varepsilon^{\alpha\beta}\Big\{[A^{(0)}_\alpha, A^{(1)}_\beta] + [A^{(2)}_\alpha, A^{(3)}_\beta]\Big\},$$

$$\varepsilon^{\alpha\beta}\partial_\alpha A^{(3)}_\beta = \varepsilon^{\alpha\beta}\Big\{[A^{(0)}_\alpha, A^{(3)}_\beta] + [A^{(2)}_\alpha, A^{(1)}_\beta]\Big\}.$$

which tell us the $\mathscr{G}^{(1,3)}$ projections are, respectively,

$$\gamma^{\alpha\beta}[A^{(3)}_\alpha, A^{(2)}_\beta] + \kappa\varepsilon^{\alpha\beta}[A^{(2)}_\alpha, A^{(3)}_\beta] = 0, \tag{A.28}$$

$$\gamma^{\alpha\beta}[A^{(1)}_\alpha, A^{(2)}_\beta] - \kappa\varepsilon^{\alpha\beta}[A^{(2)}_\alpha, A^{(1)}_\beta] = 0. \tag{A.29}$$

We next find the equations of motion for the worldsheet metric $\gamma^{\alpha\beta}$ by finding $\delta\mathscr{L}/\delta\gamma^{\alpha\beta}$. First, we calculate $\delta\gamma^{\alpha\beta} = \delta(h^{\alpha\beta}\sqrt{-h}) = \sqrt{-h}\delta h^{\alpha\beta} + h^{\alpha\beta}\delta\sqrt{-h}$. A standard computation yields

$$\begin{aligned}
\delta\sqrt{-h} &= \delta\sqrt{-\det(h_{\alpha\beta})} = -\frac{1}{2\sqrt{-\det(h_{\alpha\beta})}}\delta\det(h_{\alpha\beta}) \\
&= -\frac{1}{2\sqrt{-h}}\delta e^{\operatorname{tr}\ln\left(h_{\alpha\beta}\right)} = -\frac{1}{2\sqrt{-h}}e^{\operatorname{tr}\ln\left(h_{\alpha\beta}\right)}\delta\operatorname{tr}\ln(h_{\alpha\beta}) \\
&= \frac{-h}{2\sqrt{-h}}\operatorname{tr}\delta\ln(h_{\alpha\beta}) = \frac{\sqrt{-h}}{2}\operatorname{tr}\left[(h_{\alpha\beta})^{-1}\delta(h_{\alpha\beta})\right] \\
\implies \delta\sqrt{-h} &= \frac{1}{2}\sqrt{-h}h^{\alpha\beta}\delta h_{\alpha\beta} \overset{\dagger}{=} -\frac{1}{2}\sqrt{-h}h_{\alpha\beta}\delta h^{\alpha\beta} \overset{\ddagger}{=} \frac{h}{2}\gamma_{\alpha\beta}\delta h^{\alpha\beta}.
\end{aligned} \tag{A.30}$$

Substituting $\delta\gamma^{\alpha\beta}$ into $\delta\mathscr{L}$ (varying only $\gamma^{\alpha\beta}$ in (1.32)) we find

$$\begin{aligned}
\delta\mathscr{L} &= -\frac{g}{2}\Big[\delta\gamma^{\alpha\beta}\operatorname{str}\big(A^{(2)}_\alpha A^{(2)}_\beta\big)\Big] = -\frac{g}{2}\Big[\sqrt{-h}\delta h^{\alpha\beta}\operatorname{str}\big(A^{(2)}_\alpha A^{(2)}_\beta\big) + \frac{h}{2}\gamma_{\alpha\beta}\delta h^{\alpha\beta}h^{\rho\delta}\operatorname{str}\big(A^{(2)}_\rho A^{(2)}_\delta\big)\Big] \\
&= -\frac{g}{2}\sqrt{-h}\delta h^{\alpha\beta}\Big[\operatorname{str}\big(A^{(2)}_\alpha A^{(2)}_\beta\big) - \frac{1}{2}\gamma_{\alpha\beta}\sqrt{-h}h^{\rho\delta}\operatorname{str}\big(A^{(2)}_\rho A^{(2)}_\delta\big)\Big] \\
&= -\frac{g}{2}\sqrt{-h}\delta h^{\alpha\beta}\Big[\operatorname{str}\big(A^{(2)}_\alpha A^{(2)}_\beta\big) - \frac{1}{2}\gamma_{\alpha\beta}\gamma^{\rho\delta}\operatorname{str}\big(A^{(2)}_\rho A^{(2)}_\delta\big)\Big].
\end{aligned}$$

Finally, the equations of motion are found by imposing $\delta\mathscr{L}/\delta h^{\alpha\beta} = 0$, which results in the Virasoro constraint (1.40).

To show the Noether current $J^\alpha = \mathfrak{g}\Lambda^\alpha\mathfrak{g}^{-1}$ (associated with the global $PSU(2,2|4)$ symmetry of the Lagrangian) is conserved, we use (A.24) by going to $\mathfrak{psu}(2,2|4)$ such that

$$\begin{aligned}
\partial_\alpha J^\alpha &= \partial_\alpha\mathfrak{g}\Lambda^\alpha\mathfrak{g}^{-1} + \mathfrak{g}\partial_\alpha\Lambda^\alpha\mathfrak{g}^{-1} + \mathfrak{g}\Lambda^\alpha\partial_\alpha\mathfrak{g}^{-1} \\
&= -\mathfrak{g}(-\mathfrak{g}^{-1}\partial_\alpha\mathfrak{g}\Lambda^\alpha\mathfrak{g}^{-1}) + \mathfrak{g}\partial_\alpha\Lambda^\alpha\mathfrak{g}^{-1} + \mathfrak{g}\Lambda^\alpha(-\mathfrak{g}^{-1}\partial_\alpha\mathfrak{g}\mathfrak{g}^{-1}) \\
&= -\mathfrak{g}A_\alpha\Lambda^\alpha\mathfrak{g}^{-1} + \mathfrak{g}\partial_\alpha\Lambda^\alpha\mathfrak{g}^{-1} + \mathfrak{g}\Lambda^\alpha A_\alpha\mathfrak{g}^{-1} = \mathfrak{g}(\partial_\alpha\Lambda^\alpha - [A_\alpha,\Lambda^\alpha])\mathfrak{g}^{-1} \\
\implies \partial_\alpha J^\alpha &= 0 \text{ in } \mathfrak{psu}(2,2|4).
\end{aligned} \tag{A.31}$$

A.5 Kappa symmetry transformation

Here we derive the κ-symmetry transformation $\delta_\epsilon\mathscr{L}$ of the Green-Schwarz Lagrangian

$$\mathscr{L} = -\frac{g}{2}\Big[\gamma^{\alpha\beta}\operatorname{str}\big(A^{(2)}_\alpha A^{(2)}_\beta\big) + \kappa\varepsilon^{\alpha\beta}\operatorname{str}\big(A^{(1)}_\alpha A^{(3)}_\beta\big)\Big].$$

† Because $\delta(h_{\alpha\beta}h^{\alpha\beta}) = 0 = h^{\alpha\beta}\delta h_{\alpha\beta} + h_{\alpha\beta}\delta h^{\alpha\beta}$.

‡ Because $\gamma^{\alpha\beta}$ is defined as $\gamma^{\alpha\beta} = \sqrt{-h}h^{\alpha\beta}$, we have $\gamma_{\alpha\beta}\gamma^{\beta\rho} = h_{\alpha\beta}h^{\beta\rho} = \delta^\rho_\alpha \implies \gamma_{\alpha\beta} = h_{\alpha\beta}/\sqrt{-h}$.

Under the transformation (1.48) where $A^{(k)}_\xi \to A^{(k)}_\xi + \delta_\epsilon A^{(k)}_\xi$,

$$-\frac{2}{g}\mathscr{L} \to \overbrace{\gamma^{\alpha\beta}\,\mathrm{str}\left[(A^{(2)}_\alpha + \delta_\varepsilon A^{(2)}_\alpha)(A^{(2)}_\beta + \delta_\varepsilon A^{(2)}_\beta)\right]}^{①} + \overbrace{\kappa\varepsilon^{\alpha\beta}\,\mathrm{str}\left[(A^{(1)}_\alpha + \delta_\varepsilon A^{(1)}_\alpha)(A^{(3)}_\beta + \delta_\varepsilon A^{(3)}_\beta)\right]}^{②}$$
$$+ \delta_\epsilon\gamma^{\alpha\beta}\mathrm{str}\big(A^{(2)}_\alpha A^{(2)}_\beta\big). \tag{A.32}$$

Our job is now to evaluate ① and ②, add them to the $\delta\gamma^{\alpha\beta}$ term, subtract $-2\mathscr{L}/g$ and finally get $\delta_\epsilon\mathscr{L}$. Let us start by using the transformations of A (1.50) to find

$$\begin{aligned}
① &= \gamma^{\alpha\beta}\,\mathrm{str}\left[(A^{(2)}_\alpha + \delta_\varepsilon A^{(2)}_\alpha)(A^{(2)}_\beta + \delta_\varepsilon A^{(2)}_\beta)\right]\\
&= \gamma^{\alpha\beta}\,\mathrm{str}\Big\{(A^{(2)}_\alpha + [A^{(1)}_\alpha,\epsilon^{(1)}] + [A^{(3)}_\alpha,\epsilon^{(3)}])(A^{(2)}_\beta + [A^{(1)}_\beta,\epsilon^{(1)}] + [A^{(3)}_\beta,\epsilon^{(3)}])\Big\}\\
&= \gamma^{\alpha\beta}\,\mathrm{str}\Big\{A^{(2)}_\alpha A^{(2)}_\beta + A^{(2)}_\alpha[A^{(1)}_\beta,\epsilon^{(1)}] + A^{(2)}_\alpha[A^{(3)}_\beta,\epsilon^{(3)}] + [A^{(1)}_\alpha,\epsilon^{(1)}]A^{(2)}_\beta + [A^{(3)}_\alpha,\epsilon^{(3)}]A^{(2)}_\beta + \mathcal{O}(\epsilon^2)\Big\}.
\end{aligned}$$

Dropping sub-leading $\mathcal{O}(\epsilon^2)$ contributions, and using the fact that

$$\begin{aligned}
\gamma^{\alpha\beta}\,\mathrm{str}\Big\{A^{(2)}_\alpha[A^{(3)}_\beta,\epsilon^{(3)}]+[A^{(3)}_\alpha,\epsilon^{(3)}]A^{(2)}_\beta\Big\} &= \gamma^{\alpha\beta}\,\mathrm{str}\Big\{A^{(2)}_\alpha A^{(3)}_\beta\epsilon^{(3)} - A^{(2)}_\alpha\epsilon^{(3)}A^{(3)}_\beta + A^{(3)}_\alpha\epsilon^{(3)}A^{(2)}_\beta - \epsilon^{(3)}A^{(3)}_\alpha A^{(2)}_\beta\Big\}\\
\text{(cyclicity)}\quad &= \gamma^{\alpha\beta}\,\mathrm{str}\Big\{A^{(2)}_\alpha A^{(3)}_\beta\epsilon^{(3)} - A^{(3)}_\beta A^{(2)}_\alpha\epsilon^{(3)} + A^{(2)}_\beta A^{(3)}_\alpha\epsilon^{(3)} - A^{(3)}_\alpha A^{(2)}_\beta\epsilon^{(3)}\Big\}\\
(\alpha\leftrightarrow\beta)\quad &= \gamma^{\alpha\beta}\,\mathrm{str}\Big\{A^{(2)}_\alpha A^{(3)}_\beta\epsilon^{(3)} - A^{(3)}_\beta A^{(2)}_\alpha\epsilon^{(3)} + A^{(2)}_\alpha A^{(3)}_\beta\epsilon^{(3)} - A^{(3)}_\beta A^{(2)}_\alpha\epsilon^{(3)}\Big\}\\
&= 2\gamma^{\alpha\beta}\,\mathrm{str}\Big\{[A^{(2)}_\alpha, A^{(3)}_\beta]\epsilon^{(3)}\Big\} = -2\gamma^{\alpha\beta}\,\mathrm{str}\Big\{[A^{(3)}_\beta, A^{(2)}_\alpha]\epsilon^{(3)}\Big\},
\end{aligned}\tag{A.33}$$

along with an analog for the $\epsilon^{(1)}$ terms, we get

$$① = \gamma^{\alpha\beta}\mathrm{str}\big(A^{(2)}_\alpha A^{(2)}_\beta\big) - 2\gamma^{\alpha\beta}\,\mathrm{str}\Big\{[A^{(3)}_\beta, A^{(2)}_\alpha]\epsilon^{(3)} + [A^{(1)}_\beta, A^{(2)}_\alpha]\epsilon^{(1)}\Big\}. \tag{A.34}$$

In calculating ②, it will be useful to derive the following identity implied by the flatness condition (1.31)

$$\begin{aligned}
\varepsilon^{\alpha\beta}\partial_\alpha A^{(1)}_\beta &= \frac{1}{2}\varepsilon^{\alpha\beta}(\partial_\alpha A^{(1)}_\beta - \partial_\beta A^{(1)}_\alpha)\\
&= \frac{1}{2}\varepsilon^{\alpha\beta}\Big\{[A^{(0)}_\alpha, A^{(1)}_\beta] + [A^{(1)}_\alpha, A^{(0)}_\beta] + [A^{(2)}_\alpha, A^{(3)}_\beta] + [A^{(3)}_\alpha, A^{(2)}_\beta]\Big\}\\
&= \varepsilon^{\alpha\beta}\Big\{[A^{(0)}_\alpha, A^{(1)}_\beta] + [A^{(2)}_\alpha, A^{(3)}_\beta]\Big\} \quad \text{and similarly,}
\end{aligned}\tag{A.35}$$

$$\varepsilon^{\alpha\beta}\partial_\alpha A^{(3)}_\beta = \varepsilon^{\alpha\beta}\Big\{[A^{(0)}_\alpha, A^{(3)}_\beta] + [A^{(1)}_\alpha, A^{(2)}_\beta]\Big\}. \tag{A.36}$$

Once again referring to (1.50), we find

$$\begin{aligned}
② &= \kappa\varepsilon^{\alpha\beta}\,\mathrm{str}\left[(A^{(1)}_\alpha + \delta_\varepsilon A^{(1)}_\alpha)(A^{(3)}_\beta + \delta_\varepsilon A^{(3)}_\beta)\right]\\
&= \kappa\varepsilon^{\alpha\beta}\,\mathrm{str}\Big\{\Big(A^{(1)}_\alpha - \partial_\alpha\epsilon^{(1)} + [A^{(0)}_\alpha,\epsilon^{(1)}] + [A^{(2)}_\alpha,\epsilon^{(3)}]\Big)\Big(A^{(3)}_\beta - \partial_\beta\epsilon^{(3)} + [A^{(0)}_\beta,\epsilon^{(3)}] + [A^{(2)}_\beta,\epsilon^{(1)}]\Big)\Big\}\\
&= \kappa\varepsilon^{\alpha\beta}\,\mathrm{str}\Big\{A^{(1)}_\alpha A^{(3)}_\beta - A^{(1)}_\alpha\partial_\beta\epsilon^{(3)} + A^{(1)}_\alpha[A^{(0)}_\beta,\epsilon^{(3)}] + A^{(1)}_\alpha[A^{(2)}_\beta,\epsilon^{(1)}]\\
&\qquad - \partial_\alpha\epsilon^{(1)}A^{(3)}_\beta + [A^{(0)}_\alpha,\epsilon^{(1)}]A^{(3)}_\beta + [A^{(2)}_\alpha,\epsilon^{(3)}]A^{(3)}_\beta\Big\}
\end{aligned}$$

$$= \kappa\varepsilon^{\alpha\beta}\mathrm{str}\big(A^{(1)}_\alpha A^{(3)}_\beta\big) + \kappa\varepsilon^{\alpha\beta}\,\mathrm{str}\Big\{A^{(1)}_\beta\partial_\alpha\epsilon^{(3)} - A^{(3)}_\beta\partial_\alpha\epsilon^{(1)} + A^{(1)}_\alpha[A^{(0)}_\beta,\epsilon^{(3)}] + A^{(1)}_\alpha[A^{(2)}_\beta,\epsilon^{(1)}] + [A^{(0)}_\alpha,\epsilon^{(1)}]A^{(3)}_\beta + [A^{(2)}_\alpha,\epsilon^{(3)}]A^{(3)}_\beta\Big\}. \tag{A.37}$$

We can write

$$A^{(1)}_\beta\partial_\alpha\epsilon^{(3)} = \partial_\alpha(A^{(1)}_\beta\epsilon^{(3)}) - \partial_\alpha A^{(1)}_\beta\epsilon^{(3)}, \tag{A.38}$$

(and similarly for $A^{(3)}_\beta\partial_\alpha\epsilon^{(1)}$) whereby the total derivatives vanish in $\delta_\varepsilon\mathscr{L}$. This leaves

$$2 - \kappa\varepsilon^{\alpha\beta}\mathrm{str}\big(A^{(1)}_\alpha A^{(3)}_\beta\big) = \kappa\varepsilon^{\alpha\beta}\,\mathrm{str}\Big\{\partial_\alpha A^{(3)}_\beta\epsilon^{(1)} - \partial_\alpha A^{(1)}_\beta\epsilon^{(3)} + A^{(1)}_\alpha[A^{(0)}_\beta,\epsilon^{(3)}] + A^{(1)}_\alpha[A^{(2)}_\beta,\epsilon^{(1)}] + [A^{(0)}_\alpha,\epsilon^{(1)}]A^{(3)}_\beta + [A^{(2)}_\alpha,\epsilon^{(3)}]A^{(3)}_\beta\Big\}.$$

We are now ready to use our identity and substitute in (A.35) and (A.36), giving

$$2 - \kappa\varepsilon^{\alpha\beta}\mathrm{str}\big(A^{(1)}_\alpha A^{(3)}_\beta\big) = \kappa\varepsilon^{\alpha\beta}\,\mathrm{str}\Big\{[A^{(0)}_\alpha,A^{(1)}_\beta]\epsilon^{(1)} + [A^{(2)}_\alpha,A^{(3)}_\beta]\epsilon^{(1)} - [A^{(0)}_\alpha,A^{(3)}_\beta]\epsilon^{(3)} + [A^{(1)}_\alpha,A^{(2)}_\beta]\epsilon^{(3)} + A^{(1)}_\alpha[A^{(0)}_\beta,\epsilon^{(3)}] + A^{(1)}_\alpha[A^{(2)}_\beta,\epsilon^{(1)}] + [A^{(0)}_\alpha,\epsilon^{(1)}]A^{(3)}_\beta + [A^{(2)}_\alpha,\epsilon^{(3)}]A^{(3)}_\beta\Big\}.$$

If we expand the commutators, employ cyclicity of the supertrace, and gather like-terms in $\epsilon^{(1)}$ and $\epsilon^{(3)}$, this simplifies greatly to

$$2 = \kappa\varepsilon^{\alpha\beta}\mathrm{str}\big(A^{(1)}_\alpha A^{(3)}_\beta\big) + 2\kappa\varepsilon^{\alpha\beta}\,\mathrm{str}\Big\{[A^{(1)}_\alpha,A^{(2)}_\beta]\epsilon^{(1)} + [A^{(3)}_\beta,A^{(2)}_\alpha]\epsilon^{(3)}\Big\}. \tag{A.39}$$

Adding our equations (A.34) for 1 and (A.39) for 2 (with a little index manipulation) gives

$$\begin{aligned} 1 + 2 &= \gamma^{\alpha\beta}\mathrm{str}\big(A^{(2)}_\alpha A^{(2)}_\beta\big) + \kappa\varepsilon^{\alpha\beta}\mathrm{str}\big(A^{(1)}_\alpha A^{(3)}_\beta\big) \\ &\quad - 2\gamma^{\alpha\beta}\,\mathrm{str}\Big\{[A^{(1)}_\beta,A^{(2)}_\alpha]\epsilon^{(1)} + [A^{(3)}_\beta,A^{(2)}_\alpha]\epsilon^{(3)}\Big\} - 2\kappa\varepsilon^{\alpha\beta}\,\mathrm{str}\Big\{[A^{(1)}_\beta,A^{(2)}_\alpha]\epsilon^{(1)} - [A^{(3)}_\beta,A^{(2)}_\alpha]\epsilon^{(3)}\Big\} \\ &= -\frac{g}{2}\mathscr{L} - 4\,\mathrm{str}\Big\{\mathrm{P}^{\alpha\beta}_+[A^{(1)}_\beta,A^{(2)}_\alpha]\epsilon^{(1)} + \mathrm{P}^{\alpha\beta}_-[A^{(3)}_\beta,A^{(2)}_\alpha]\epsilon^{(3)}\Big\}, \end{aligned} \tag{A.40}$$

where we defined the projectors $\mathrm{P}^{\alpha\beta}_\pm = \frac{1}{2}(\gamma^{\alpha\beta} \pm \kappa\varepsilon^{\alpha\beta})$. We have found the change in the Lagrangian density,

$$\begin{aligned} -\frac{2}{g}\delta_\varepsilon\mathscr{L} &= 1 + 2 + \delta_\epsilon\gamma^{\alpha\beta}\mathrm{str}\big(A^{(2)}_\alpha A^{(2)}_\beta\big) + \frac{2}{g}\mathscr{L} \\ \implies -\frac{2}{g}\delta_\varepsilon\mathscr{L} &= \delta_\epsilon\gamma^{\alpha\beta}\mathrm{str}\big(A^{(2)}_\alpha A^{(2)}_\beta\big) - 4\,\mathrm{str}\Big\{\mathrm{P}^{\alpha\beta}_+[A^{(1)}_\beta,A^{(2)}_\alpha]\epsilon^{(1)} + \mathrm{P}^{\alpha\beta}_-[A^{(3)}_\beta,A^{(2)}_\alpha]\epsilon^{(3)}\Big\}. \end{aligned} \tag{A.41}$$

Looking ahead at (1.59), it would be useful to know how expressions of the form $X^\alpha_\pm Y^\beta_\pm$ can be manipulated. We will actually prove

$$P^{\alpha\gamma}_\pm P^{\beta\delta}_\pm = P^{\alpha\delta}_\pm P^{\beta\gamma}_\pm. \tag{A.42}$$

Expanding the left-hand side will result in terms of the form $\gamma^{\alpha\gamma}\varepsilon^{\beta\delta}$. This can be rewritten as

$$\begin{aligned} \gamma^{\alpha\gamma}\varepsilon^{\beta\delta} &= \varepsilon^{\alpha\mu}\gamma_{\mu\nu}\varepsilon^{\nu\gamma}\varepsilon^{\beta\delta} = -\varepsilon^{\alpha\mu}\gamma_{\mu\nu}(\gamma^{\nu\beta}\gamma^{\gamma\delta} - \gamma^{\nu\delta}\gamma^{\gamma\beta}) \\ &= \varepsilon^{\alpha\delta}\gamma^{\gamma\beta} - \varepsilon^{\alpha\beta}\gamma^{\gamma\delta}, \end{aligned}$$

such that

$$\begin{aligned}
4P_\pm^{\alpha\gamma}P_\pm^{\beta\delta} &= (\gamma^{\alpha\gamma} \pm \kappa\varepsilon^{\alpha\gamma})(\gamma^{\beta\delta} \pm \kappa\varepsilon^{\beta\delta}) = \gamma^{\alpha\gamma}\gamma^{\beta\delta} \pm \kappa(\gamma^{\alpha\gamma}\varepsilon^{\beta\delta} + \gamma^{\beta\delta}\gamma^{\alpha\gamma}) + \varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta} \\
&= \gamma^{\alpha\gamma}\gamma^{\beta\delta} \pm \kappa\big[(\varepsilon^{\alpha\delta}\gamma^{\gamma\beta} - \cancel{\varepsilon^{\alpha\beta}\gamma^{\gamma\delta}}) + (\varepsilon^{\beta\gamma}\gamma^{\delta\alpha} - \cancel{\varepsilon^{\beta\alpha}\gamma^{\delta\gamma}})\big] - (\gamma^{\alpha\beta}\gamma^{\gamma\delta} - \gamma^{\alpha\delta}\gamma^{\gamma\beta}) \\
&= \gamma^{\alpha\delta}\gamma^{\beta\gamma} \pm \kappa(\gamma^{\alpha\delta}\varepsilon^{\beta\gamma} + \gamma^{\beta\gamma}\gamma^{\alpha\delta}) - (\gamma^{\alpha\beta}\gamma^{\gamma\delta} - \gamma^{\alpha\gamma}\gamma^{\delta\beta}) \\
&= \gamma^{\alpha\delta}\gamma^{\beta\gamma} \pm \kappa(\gamma^{\alpha\delta}\varepsilon^{\beta\gamma} + \gamma^{\beta\gamma}\gamma^{\alpha\delta}) + \varepsilon^{\alpha\delta}\varepsilon^{\beta\gamma} = 4P_\pm^{\alpha\delta}P_\pm^{\beta\gamma}.
\end{aligned}$$

This identity (A.42) relating projections tells us that, no matter the circumstance,

$$X_\pm^\alpha \ldots Y_\pm^\beta = X_\pm^\beta \ldots Y_\pm^\alpha. \tag{A.43}$$

Recalling (1.53) and (1.57), the first half of the second term in (A.41) becomes

$$\begin{aligned}
\operatorname{str}\Big\{[A^{(1)}_{\delta,+}, A_-^{(2),\delta}]\epsilon^{(1)}\Big\} = \operatorname{str}\Big\{ & A_+^{(1),\delta}A^{(2)}_{\delta,-}A^{(2)}_{\alpha,-}\kappa_+^{(1),\alpha} + A_+^{(1),\delta}A^{(2)}_{\delta,-}\kappa_+^{(1),\alpha}A^{(2)}_{\alpha,-} \\
& - A^{(2)}_{\delta,-}A_+^{(1),\delta}A^{(2)}_{\alpha,-}\kappa_+^{(1),\alpha} - A^{(2)}_{\delta,-}A_+^{(1),\delta}\kappa_+^{(1),\alpha}A^{(2)}_{\alpha,-}\Big\}.
\end{aligned}$$

Notice the identity (A.43) actually equates the second and therm term, cancelling them. We are left with

$$\begin{aligned}
\operatorname{str}\Big\{[A^{(1)}_{\delta,+}, A_-^{(2),\delta}]\epsilon^{(1)}\Big\} &= \operatorname{str}\Big\{A_+^{(1),\delta}A^{(2)}_{\delta,-}A^{(2)}_{\alpha,-}\kappa_+^{(1),\alpha} - A^{(2)}_{\delta,-}A_+^{(1),\delta}\kappa_+^{(1),\alpha}A^{(2)}_{\alpha,-}\Big\} \\
&= \operatorname{str}\Big\{A^{(2)}_{\delta,-}A^{(2)}_{\alpha,-}[\kappa_+^{(1),\alpha}, A_+^{(1),\delta}]\Big\} = \frac{1}{8}\operatorname{str}\big(A^{(2)}_{\delta,-}A^{(2)}_{\alpha,-}\big)\operatorname{str}\big(\Upsilon[\kappa_+^{(1),\alpha}, A_+^{(1),\delta}]\big)
\end{aligned}$$

since the term proportional to the identity in (1.59) vanishes in the supertrace and similarly

$$\operatorname{str}\Big\{[A^{(3)}_{\delta,-}, A_+^{(2),\delta}]\epsilon^{(3)}\Big\} = \frac{1}{8}\operatorname{str}\big(A^{(2)}_{\delta,+}A^{(2)}_{\alpha,+}\big)\operatorname{str}\big(\Upsilon[\kappa_-^{(3),\alpha}, A_-^{(1),\delta}]\big).$$

Putting the two halves of the second term together, the change in the Lagrangian becomes (1.60). To determine what variation in the worldsheet metric does a local fermionic transformation leave the Lagrangian invariant, i.e. what $\delta_\epsilon\gamma^{\alpha\beta}$ would kill $\delta_\epsilon\mathscr{L}$, we need to factor out $\operatorname{str}\big(A^{(2)}_{\delta,+}A^{(2)}_{\alpha,+}\big)$ and get (1.61)

$$\delta_\epsilon\gamma^{\alpha\beta} = \frac{1}{2}\operatorname{tr}\Big([\kappa_+^{(1),\alpha}, A_+^{(1),\beta}] + [\kappa_-^{(3),\alpha}, A_-^{(1),\beta}]\Big).$$

This is a valid assumption as $\delta(\gamma^{\alpha\beta}\gamma_{\alpha\beta}) = 0$ is satisfied and the components of the variation are real.

A.6 Monodromy matrix evolution

We want to compute $\partial_\tau T(z)$ where $T(z)$ is given by (1.69). It will be useful to introduce the notation

$$T(z,a,b) = \overleftarrow{\exp}\int_b^a d\sigma\; L_\sigma(\tau,\sigma,z), \qquad T(z,2\pi,0) = T(z). \tag{A.44}$$

Path-ordered exponentials of this type satisfy $T(z,a,c) = T(z,a,b)T(z,b,c)$. In particular, we can break up any interval $[s_1, s_n]$ into smaller sub-intervals such that

$$T(z,s_n,s_1) = T(z,s_n,s_{n-1})T(z,s_{n-1},s_{n-2})\cdots T(z,s_2,s_1). \tag{A.45}$$

This becomes useful when computing $\partial_\tau T(z)$. Our strategy will be to apply the product rule to (A.45) and shrink to 0 the sub-interval size $\Delta s = s_2 - s_1 = \ldots = s_{n-1} - s_n$ such that $\Delta s ||L_\sigma||_{\text{HS}} \ll 1$. So,

$$\begin{aligned}\partial_\tau T(z, s_n, s_1) &= \sum_{k=1}^{n} T(z, s_n, s_{k+1}) \partial_\tau T(z, s_{k+1}, s_k) T(z, s_k, s_1)\\ (\Delta s \text{ small}) \quad &\approx \sum_{k=1}^{n} T(z, s_n, s_{k+1}) \partial_\tau e^{\Delta s L_\sigma} T(z, s_k, s_1) = \sum_{k=1}^{n} \Delta s T(z, s_n, s_{k+1}) \partial_\tau L_\sigma T(z, s_{k+1}, s_1)\\ &\approx \int_{s_1}^{s_n} d\sigma\ T(z, s_n, \sigma) \partial_\tau L_\sigma T(z, \sigma, s_1)\end{aligned}$$

by approximating the integral as a Riemann sum. In particular, for $s_n = 2\pi$ and $s_1 = 0$ we retrieve

$$\begin{aligned}\partial_\tau T(z) &= \int_0^{2\pi} d\sigma\ T(z, 2\pi, \sigma) \partial_\tau L_\sigma T(z, \sigma, 0)\\ &= \int_0^{2\pi} d\sigma \big[\overleftarrow{\exp} \int_\sigma^{2\pi} L_\sigma \big] \partial_\tau L_\sigma \big[\overleftarrow{\exp} \int_0^{\sigma} L_\sigma \big]\\ &\overset{(1.68)}{=} \int_0^{2\pi} d\sigma \big[\overleftarrow{\exp} \int_\sigma^{2\pi} L_\sigma \big] \big(\partial_\sigma L_\tau + [L_\tau, L_\sigma]\big) \big[\overleftarrow{\exp} \int_0^{\sigma} L_\sigma \big].\end{aligned} \tag{A.46}$$

The Leibniz rule for the derivative of an integral states

$$\partial_x \int_{a(x)}^{b(x)} dt\ f(x,t) = f(b(x),t) b'(x) - f(a(x),t) a'(x) + \int_{a(x)}^{b(x)} dt\ f'(x,t). \tag{A.47}$$

If we identify $x \sim \sigma$ and $f(x,t) \sim L_\sigma(\tau, \sigma, z)$ then (A.46) is equal to

$$\partial_\tau T(z) = \int_0^{2\pi} d\sigma\ \partial_\sigma \Big[(\overleftarrow{\exp} \int_\sigma^{2\pi} L_\sigma) L_\tau (\overleftarrow{\exp} \int_0^{\sigma} L_\sigma) \Big]. \tag{A.48}$$

Taking the anti-derivative and evaluating at the bounds yields the evolution equation (1.70) for $T(z)$

$$\begin{aligned}\partial_\tau T(z) &= \Big[(\overleftarrow{\exp} \int_{2\pi}^{2\pi} L_\sigma) L_\tau(2\pi, \tau, z) (\overleftarrow{\exp} \int_0^{2\pi} L_\sigma) \Big] - \Big[(\overleftarrow{\exp} \int_0^{2\pi} L_\sigma) L_\tau(0, \tau, z) (\overleftarrow{\exp} \int_0^{0} L_\sigma) \Big]\\ &= L_\tau(2\pi, \tau, z) T(z) - T(z) L_\tau(0, \tau, z) = [L_\tau(0, \tau, z), T(z)]\end{aligned}$$

by the effective periodicity $\sigma + 2\pi = \sigma$ of any function of the worldsheet spatial coordinate.

A.7 Lax pair parameters

The projections of the zero curvature condition (1.81) for the ansatz (1.90) are

$$\begin{aligned}0 &= 2\varepsilon^{\alpha\beta} \partial_\alpha L_\beta - \varepsilon^{\alpha\beta} [L_\alpha, L_\beta]\\ &= 2\varepsilon^{\alpha\beta} \big\{ \ell_0 \partial_\alpha A_\beta^{(0)} + \ell_1 \partial_\alpha A_\beta^{(2)} + \ell_2 \varepsilon^{\mu\nu} \partial_\alpha (\gamma_{\beta\mu} A_\nu^{(2)}) + \ell_3 \partial_\alpha A_\beta^{(1)} + \ell_4 \partial_\alpha A_\beta^{(3)} \big\}\\ &\quad - \varepsilon^{\alpha\beta} \Big[\ell_0 A_\alpha^{(0)} + \ell_1 A_\alpha^{(2)} + \ell_2 \gamma_{\alpha\delta} \varepsilon^{\delta\rho} A_\rho^{(2)} + \ell_3 A_\alpha^{(1)} + \ell_4 A_\alpha^{(3)},\\ &\qquad \ell_0 A_\beta^{(0)} + \ell_1 A_\beta^{(2)} + \ell_2 \gamma_{\beta\mu} \varepsilon^{\mu\nu} A_\nu^{(2)} + \ell_3 A_\beta^{(1)} + \ell_4 A_\beta^{(3)} \Big]\end{aligned}$$

$$
\begin{aligned}
&= 2\varepsilon^{\alpha\beta}\ell_0\partial_\alpha A^{(0)}_\beta - \varepsilon^{\alpha\beta}\Big\{[\ell_0 A^{(0)}_\alpha, \ell_0 A^{(0)}_\beta] + [\ell_1 A^{(2)}_\alpha, \ell_1 A^{(2)}_\beta] + [\ell_1 A^{(2)}_\alpha, \ell_2\gamma_{\beta\mu}\varepsilon^{\mu\nu}A^{(2)}_\nu] \\
&\qquad + [\ell_2\gamma_{\alpha\delta}\varepsilon^{\delta\rho}A^{(2)}_\rho, \ell_1 A^{(2)}_\beta] + [\ell_2\gamma_{\alpha\delta}\varepsilon^{\delta\rho}A^{(2)}_\rho, \ell_2\gamma_{\beta\mu}\varepsilon^{\mu\nu}A^{(2)}_\nu] + 2[\ell_3 A^{(1)}_\alpha, \ell_4 A^{(3)}_\alpha]\Big\} \\
&+ 2\varepsilon^{\alpha\beta}\ell_1\partial_\alpha A^{(2)}_\beta + 2\varepsilon^{\alpha\beta}\ell_2\varepsilon^{\mu\nu}\partial_\alpha(\gamma_{\beta\mu}A^{(2)}_\nu) - \varepsilon^{\alpha\beta}\Big\{[\ell_0 A^{(0)}_\alpha, \ell_1 A^{(2)}_\beta] + [\ell_1 A^{(2)}_\alpha, \ell_0 A^{(0)}_\beta] \\
&\qquad + [\ell_0 A^{(0)}_\alpha, \ell_2\gamma_{\beta\mu}\varepsilon^{\mu\nu}A^{(2)}_\nu] + [\ell_2\gamma_{\alpha\delta}\varepsilon^{\delta\rho}A^{(2)}_\rho, \ell_0 A^{(0)}_\beta] + [\ell_3 A^{(1)}_\alpha, \ell_3 A^{(1)}_\beta] + [\ell_4 A^{(3)}_\alpha, \ell_4 A^{(3)}_\beta]\Big\} \\
&+ 2\varepsilon^{\alpha\beta}\ell_3\partial_\alpha A^{(1)}_\beta - \varepsilon^{\alpha\beta}\Big\{[\ell_0 A^{(0)}_\alpha, \ell_3 A^{(1)}_\beta] + [\ell_3 A^{(1)}_\alpha, \ell_0 A^{(0)}_\beta] + [\ell_1 A^{(2)}_\alpha, \ell_4 A^{(3)}_\beta] \\
&\qquad + [\ell_2\gamma_{\alpha\delta}\varepsilon^{\delta\rho}A^{(2)}_\rho, \ell_4 A^{(3)}_\beta] + [\ell_4 A^{(3)}_\alpha, \ell_1 A^{(2)}_\beta] + [\ell_4 A^{(3)}_\alpha, \ell_2\gamma_{\beta\mu}\varepsilon^{\mu\nu}A^{(2)}_\nu]\Big\} \\
&+ 2\varepsilon^{\alpha\beta}\ell_4\partial_\alpha A^{(3)}_\beta - \varepsilon^{\alpha\beta}\Big\{[\ell_0 A^{(0)}_\alpha, \ell_4 A^{(3)}_\beta] + [\ell_4 A^{(3)}_\alpha, \ell_0 A^{(0)}_\beta] + [\ell_1 A^{(2)}_\alpha, \ell_3 A^{(1)}_\beta] \\
&\qquad + [\ell_2\gamma_{\alpha\delta}\varepsilon^{\delta\rho}A^{(2)}_\rho, \ell_3 A^{(1)}_\beta] + [\ell_3 A^{(1)}_\alpha, \ell_1 A^{(2)}_\beta] + [\ell_3 A^{(1)}_\alpha, \ell_2\gamma_{\beta\mu}\varepsilon^{\mu\nu}A^{(2)}_\nu]\Big\} \\
&= \mathscr{G}^{(0)} + \mathscr{G}^{(2)} + \mathscr{G}^{(1)} + \mathscr{G}^{(3)}.
\end{aligned}
$$

Starting with $\mathscr{G}^{(0)} = 0$, we find

$$
\begin{aligned}
\mathscr{G}^{(0)} &= 2\varepsilon^{\alpha\beta}\ell_0\partial_\alpha A^{(0)}_\beta - \varepsilon^{\alpha\beta}\Big\{[\ell_0 A^{(0)}_\alpha, \ell_0 A^{(0)}_\beta] + [\ell_1 A^{(2)}_\alpha, \ell_1 A^{(2)}_\beta] + [\ell_1 A^{(2)}_\alpha, \ell_2\gamma_{\beta\mu}\varepsilon^{\mu\nu}A^{(2)}_\nu] \\
&\qquad + [\ell_2\gamma_{\alpha\delta}\varepsilon^{\delta\rho}A^{(2)}_\rho, \ell_1 A^{(2)}_\beta] + [\ell_2\gamma_{\alpha\delta}\varepsilon^{\delta\rho}A^{(2)}_\rho, \ell_2\gamma_{\beta\mu}\varepsilon^{\mu\nu}A^{(2)}_\nu] + 2[\ell_3 A^{(1)}_\alpha, \ell_4 A^{(3)}_\alpha]\Big\} \\
0 &= 2\varepsilon^{\alpha\beta}\ell_0\partial_\alpha A^{(0)}_\beta - \varepsilon^{\alpha\beta}\Big\{\ell_0^2[A^{(0)}_\alpha, A^{(0)}_\beta] + \ell_1^2[A^{(2)}_\alpha, A^{(2)}_\beta] + \ell_1\ell_2\gamma_{\beta\mu}\varepsilon^{\mu\nu}[A^{(2)}_\alpha, A^{(2)}_\nu] \\
&\qquad + \ell_1\ell_2\gamma_{\alpha\delta}\varepsilon^{\delta\rho}[A^{(2)}_\rho, A^{(2)}_\beta] + \ell_2^2\gamma_{\alpha\delta}\varepsilon^{\delta\rho}\gamma_{\beta\mu}\varepsilon^{\mu\nu}[A^{(2)}_\rho, A^{(2)}_\nu] + 2\ell_3\ell_4[A^{(1)}_\alpha, A^{(3)}_\alpha]\Big\} \\
0 &= 2\varepsilon^{\alpha\beta}\ell_0\partial_\alpha A^{(0)}_\beta - \varepsilon^{\alpha\beta}\Big\{\ell_0^2[A^{(0)}_\alpha, A^{(0)}_\beta] + \ell_1^2[A^{(2)}_\alpha, A^{(2)}_\beta] + 2\ell_3\ell_4[A^{(1)}_\alpha, A^{(3)}_\alpha]\Big\} \\
&\qquad - \ell_1\ell_2\varepsilon^{\alpha\beta}\gamma_{\beta\mu}\varepsilon^{\mu\nu}[A^{(2)}_\alpha, A^{(2)}_\nu] - \ell_1\ell_2\varepsilon^{\alpha\beta}\gamma_{\alpha\delta}\varepsilon^{\delta\rho}[A^{(2)}_\rho, A^{(2)}_\beta] - \ell_2^2\varepsilon^{\alpha\beta}\gamma_{\alpha\delta}\varepsilon^{\delta\rho}\gamma_{\beta\mu}\varepsilon^{\mu\nu}[A^{(2)}_\rho, A^{(2)}_\nu] \\
0 &\stackrel{\dagger}{=} 2\varepsilon^{\alpha\beta}\ell_0\partial_\alpha A^{(0)}_\beta - \varepsilon^{\alpha\beta}\Big\{\ell_0^2[A^{(0)}_\alpha, A^{(0)}_\beta] + \ell_1^2[A^{(2)}_\alpha, A^{(2)}_\beta] + 2\ell_3\ell_4[A^{(1)}_\alpha, A^{(3)}_\alpha]\Big\} \\
&\qquad - \cancel{\ell_1\ell_2\gamma^{\alpha\nu}[A^{(2)}_\alpha, A^{(2)}_\nu]} + \cancel{\ell_1\ell_2\gamma^{\beta\rho}[A^{(2)}_\rho, A^{(2)}_\beta]} + \ell_2^2\varepsilon^{\mu\nu}\gamma^{\beta\rho}\gamma_{\beta\mu}[A^{(2)}_\rho, A^{(2)}_\nu] \\
0 &\stackrel{\ddagger}{=} 2\varepsilon^{\alpha\beta}\ell_0\partial_\alpha A^{(0)}_\beta - \varepsilon^{\alpha\beta}\Big\{\ell_0^2[A^{(0)}_\alpha, A^{(0)}_\beta] + (\ell_1^2 - \ell_2^2)[A^{(2)}_\alpha, A^{(2)}_\beta] + 2\ell_3\ell_4[A^{(1)}_\alpha, A^{(3)}_\alpha]\Big\}.
\end{aligned} \tag{A.49}
$$

To simplify this expression further, we will need to use the projection onto $\mathscr{G}^{(0)}$ of the flatness condition (1.81) for A_α,

$$
2\varepsilon^{\alpha\beta}\partial_\alpha A^{(0)}_\beta = \varepsilon^{\alpha\beta}\Big\{[A^{(0)}_\alpha, A^{(0)}_\beta] + [A^{(2)}_\alpha, A^{(2)}_\beta] + 2[A^{(1)}_\alpha, A^{(3)}_\beta]\Big\}. \tag{A.50}
$$

Substituting (A.50) into (A.49) we get

$$
\mathscr{G}^{(0)} = \varepsilon^{\alpha\beta}\Big\{(\ell_0 - \ell_0^2)[A^{(0)}_\alpha, A^{(0)}_\beta] + (\ell_0 + \ell_2^2 - \ell_1^2)[A^{(2)}_\alpha, A^{(2)}_\beta] + 2(\ell_0 - \ell_3\ell_4)[A^{(1)}_\alpha, A^{(3)}_\beta]\Big\} = 0 \tag{A.51}
$$

which tells us, assuming each commutator vanishes independently[14],

$$
\ell_0 = 1, \qquad \ell_1^2 - \ell_2^2 = 1, \qquad \ell_3\ell_4 = 1. \tag{A.52}
$$

† Using the identity $\varepsilon^{ij}\gamma_{jk}\varepsilon^{kl} = \gamma^{il}$.

‡ Since $\gamma^{\beta\rho}\gamma_{\beta\mu} = \delta^\rho_\mu$ and we can relabel summation indices $\mu, \nu \to \alpha, \beta$.

[14] The connection components $A^{(k)}$ are independent of one another.

We can assume these commutators vanish independently, since if their prefactors were not always vanishing, we would be imposing an additional constraint which did not follow from the equations of motion. In addition, the prospect $\ell_0 = 0$ is not valid as it would imply $\ell_3\ell_4 = 0$ which would mean either the $\mathscr{G}^{(1)}$ or $\mathscr{G}^{(3)}$ projection of L_α is always zero. Moving to $\mathscr{G}^{(2)} = 0$, we get

$$\begin{aligned}
\mathscr{G}^{(2)} &= 2\varepsilon^{\alpha\beta}\ell_1\partial_\alpha A_\beta^{(2)} + 2\varepsilon^{\alpha\beta}\ell_2\varepsilon^{\mu\nu}\partial_\alpha(\gamma_{\beta\mu}A_\nu^{(2)}) - \varepsilon^{\alpha\beta}\Big\{[\ell_0 A_\alpha^{(0)}, \ell_1 A_\beta^{(2)}] + [\ell_1 A_\alpha^{(2)}, \ell_0 A_\beta^{(0)}] \\
&\quad + [\ell_0 A_\alpha^{(0)}, \ell_2\gamma_{\beta\mu}\varepsilon^{\mu\nu}A_\nu^{(2)}] + [\ell_2\gamma_{\alpha\delta}\varepsilon^{\delta\rho}A_\rho^{(2)}, \ell_0 A_\beta^{(0)}] + [\ell_3 A_\alpha^{(1)}, \ell_3 A_\beta^{(1)}] + [\ell_4 A_\alpha^{(3)}, \ell_4 A_\beta^{(3)}]\Big\} \\
0 &= 2\varepsilon^{\alpha\beta}\ell_1\partial_\alpha A_\beta^{(2)} + 2\ell_2\partial_\alpha(\gamma^{\alpha\beta}A_\beta^{(2)}) - \varepsilon^{\alpha\beta}\Big\{2\ell_0\ell_1[A_\alpha^{(0)}, A_\beta^{(2)}] + \ell_3^2[A_\alpha^{(1)}, A_\beta^{(1)}] + \ell_4^2[A_\alpha^{(3)}, A_\beta^{(3)}]\Big\} \\
&\quad - \ell_0\ell_2\varepsilon^{\alpha\beta}\gamma_{\beta\mu}\varepsilon^{\mu\nu}[A_\alpha^{(0)}, A_\nu^{(2)}] - \ell_0\ell_2\varepsilon^{\alpha\beta}\gamma_{\alpha\delta}\varepsilon^{\delta\rho}[A_\rho^{(2)}, A_\beta^{(2)}] \qquad \text{(A.53)} \\
0 &= 2\varepsilon^{\alpha\beta}\ell_1\partial_\alpha A_\beta^{(2)} + 2\ell_2\partial_\alpha(\gamma^{\alpha\beta}A_\beta^{(2)}) - 2\ell_0(\varepsilon^{\alpha\beta}\ell_1 + \gamma^{\alpha\beta}\ell_2)[A_\alpha^{(0)}, A_\beta^{(2)}] - \varepsilon^{\alpha\beta}\Big\{\ell_3^2[A_\alpha^{(1)}, A_\beta^{(1)}] + \ell_4^2[A_\alpha^{(3)}, A_\beta^{(3)}]\Big\}.
\end{aligned}$$

In this case, the projection onto $\mathscr{G}^{(2)}$ of (1.81) is

$$2\varepsilon^{\alpha\beta}\partial_\alpha A_\beta^{(2)} = \varepsilon^{\alpha\beta}\Big\{2[A_\alpha^{(0)}, A_\beta^{(2)}] + [A_\alpha^{(1)}, A_\beta^{(1)}] + [A_\alpha^{(3)}, A_\beta^{(3)}]\Big\}. \tag{A.54}$$

Substituting (A.54) into (A.53) and recalling $l_0 = 1$, we now get in the conformal gauge s.t. $\partial_\alpha\gamma_{\beta\mu} = 0$

$$\mathscr{G}^{(2)} = 2\ell_2\gamma^{\alpha\beta}\partial_\alpha A_\beta^{(2)} - 2\ell_2\gamma^{\alpha\beta}[A_\alpha^{(0)}, A_\beta^{(2)}] - \varepsilon^{\alpha\beta}\Big\{(\ell_3^2 - \ell_1)[A_\alpha^{(1)}, A_\beta^{(1)}] + (\ell_4^2 - \ell_1)[A_\alpha^{(3)}, A_\beta^{(3)}]\Big\} = 0 \tag{A.55}$$

which agrees with the string equations of motion (1.37) provided the parameters ℓ_i satisfy

$$\frac{\ell^3 - \ell_1}{\ell_2} = -\kappa, \qquad \frac{\ell^4 - \ell_1}{\ell_2} = \kappa. \tag{A.56}$$

For $\mathscr{G}^{(1)}$ and $\mathscr{G}^{(3)}$ the equations will look identical up to exchange of $\ell_3 \leftrightarrow \ell_4$. Starting with

$$\begin{aligned}
\mathscr{G}^{(1)} &= 2\varepsilon^{\alpha\beta}\ell_3\partial_\alpha A_\beta^{(1)} - \varepsilon^{\alpha\beta}\Big\{[\ell_0 A_\alpha^{(0)}, \ell_3 A_\beta^{(1)}] + [\ell_3 A_\alpha^{(1)}, \ell_0 A_\beta^{(0)}] + [\ell_1 A_\alpha^{(2)}, \ell_4 A_\beta^{(3)}] \\
&\quad + [\ell_2\gamma_{\alpha\delta}\varepsilon^{\delta\rho}A_\rho^{(2)}, \ell_4 A_\beta^{(3)}] + [\ell_4 A_\alpha^{(3)}, \ell_1 A_\beta^{(2)}] + [\ell_4 A_\alpha^{(3)}, \ell_2\gamma_{\beta\mu}\varepsilon^{\mu\nu}A_\nu^{(2)}]\Big\} \\
0 &= 2\varepsilon^{\alpha\beta}\ell_3\partial_\alpha A_\beta^{(1)} - 2\varepsilon^{\alpha\beta}\Big\{\ell_0\ell_3[A_\alpha^{(0)}, A_\beta^{(1)}] + \ell_1\ell_4[A_\alpha^{(2)}, A_\beta^{(3)}]\Big\} \\
&\quad + \ell_2\ell_4\varepsilon^{\alpha\beta}\gamma_{\alpha\delta}\varepsilon^{\delta\rho}[A_\rho^{(2)}, A_\beta^{(3)}] + \ell_2\ell_4\varepsilon^{\alpha\beta}\gamma_{\beta\mu}\varepsilon^{\mu\nu}[A_\alpha^{(3)}, A_\nu^{(2)}] \\
0 &= 2\varepsilon^{\alpha\beta}\ell_3\partial_\alpha A_\beta^{(1)} - 2\varepsilon^{\alpha\beta}\Big\{\ell_0\ell_3[A_\alpha^{(0)}, A_\beta^{(1)}] + \ell_1\ell_4[A_\alpha^{(2)}, A_\beta^{(3)}]\Big\} + 2\ell_2\ell_4\gamma^{\alpha\beta}[A_\alpha^{(2)}, A_\beta^{(3)}],
\end{aligned} \tag{A.57}$$

and similarly

$$\mathscr{G}^{(3)} = 2\varepsilon^{\alpha\beta}\ell_4\partial_\alpha A_\beta^{(3)} - 2\varepsilon^{\alpha\beta}\Big\{\ell_0\ell_4[A_\alpha^{(0)}, A_\beta^{(3)}] + \ell_1\ell_3[A_\alpha^{(2)}, A_\beta^{(1)}]\Big\} + 2\ell_2\ell_3\gamma^{\alpha\beta}[A_\rho^{(2)}, A_\beta^{(1)}]. \tag{A.58}$$

The projections onto $\mathscr{G}^{(1)}$ and $\mathscr{G}^{(3)}$ of the flatness condition for A_α are

$$2\varepsilon^{\alpha\beta}\partial_\alpha A_\beta^{(1)} = \varepsilon^{\alpha\beta}\Big\{2[A_\alpha^{(0)}, A_\beta^{(1)}] + 2[A_\alpha^{(2)}, A_\beta^{(3)}]\Big\}, \tag{A.59}$$

$$2\varepsilon^{\alpha\beta}\partial_\alpha A_\beta^{(3)} = \varepsilon^{\alpha\beta}\Big\{2[A_\alpha^{(0)}, A_\beta^{(3)}] + 2[A_\alpha^{(2)}, A_\beta^{(1)}]\Big\}. \tag{A.60}$$

Substituting (A.59) into (A.57) and (A.60) into (A.58), we obtain

$$\mathscr{G}^{(1)} = \big(\ell_2\ell_4\gamma^{\alpha\beta} - (\ell_1\ell_4 - \ell_3)\big)[A_\alpha^{(2)}, A_\beta^{(3)}] = 0, \tag{A.61}$$

$$\mathscr{G}^{(3)} = \big(\ell_2\ell_3\gamma^{\alpha\beta} - (\ell_1\ell_3 - \ell_4)\big)[A_\alpha^{(2)}, A_\beta^{(1)}] = 0. \tag{A.62}$$

Comparing with the string equations (1.38), the parameters ℓ_i would need to satisfy

$$\frac{\ell_1\ell_4 - \ell_3}{\ell_2\ell_4} = \kappa, \qquad \frac{\ell_4 - \ell_1\ell_3}{\ell_2\ell_3} = \kappa. \tag{A.63}$$

These requirements are summarised in (1.91). Summing the second row of equations gives us

$$0 = \frac{\ell_3^2 - \ell_1}{\ell_2} + \frac{\ell_4^2 - \ell_1}{\ell_2} = \frac{\ell_3^2 + \ell_4^2 - 2\ell_1}{\ell_2} \implies \ell_3^2 + \ell_4^2 = 2\ell_1. \tag{A.64}$$

We next multiply the bottom row to give

$$\kappa^2 = \frac{\ell_1\ell_4 - \ell_3}{\ell_2\ell_4}\frac{\ell_4 - \ell_1\ell_3}{\ell_2\ell_3} = \frac{\ell_1\ell_4^2 - \ell_1^2\ell_3\ell_4 - \ell_3\ell_4 + \ell_1\ell_3^2}{\ell_2^2\ell_3\ell_4} \underset{\ell_3\ell_4=1}{=} \frac{\ell_1(\ell_3^2 + \ell_4^2) - \ell_1^2 - 1}{\ell_2^2},$$

which we further simplify using (A.64), yielding

$$\kappa^2 = \frac{2\ell_1^2 - \ell_1^2 - 1}{\ell_2^2} = \frac{\ell_1^2 - 1}{\ell_2^2}.$$

Comparing with $\ell_1^2 - \ell_2^2 = 1$ (A.52), this immediately tells us that $\kappa^2 = 1$.

A.8 Lax pair transformations

A.8.1 Gauge transformation

Here we will show that the zero curvature condition of Lax pairs is invariant under gauge transformations (1.92). Recall $\partial_\alpha h^{-1} = -h^{-1}\partial_\alpha h h^{-1}$ for matrices h. Using this and (1.92), we find by the product rule

$$\partial_\alpha L'_\beta = \partial_\alpha h L_\beta h^{-1} + h\partial_\alpha L_\beta h^{-1} - hL_\beta h^{-1}\partial_\alpha h h^{-1} + \partial_\alpha\partial_\beta h h^{-1} - \partial_\beta h h^{-1}\partial_\alpha h h^{-1}. \tag{A.65}$$

Being careful with indices, this means that

$$\begin{aligned}
\partial_\alpha L'_\beta - \partial_\beta L'_\alpha &= \partial_\alpha h L_\beta h^{-1} + h\partial_\alpha L_\beta h^{-1} - hL_\beta h^{-1}\partial_\alpha h h^{-1} + \partial_\alpha\partial_\beta h h^{-1} - \partial_\beta h h^{-1}\partial_\alpha h h^{-1} \\
&\quad - \partial_\beta h L_\alpha h^{-1} - h\partial_\beta L_\alpha h^{-1} + hL_\alpha h^{-1}\partial_\beta h h^{-1} - \partial_\beta\partial_\alpha h h^{-1} + \partial_\alpha h h^{-1}\partial_\beta h h^{-1} \\
&= hL_\alpha h^{-1}\partial_\beta h h^{-1} - \partial_\beta h h^{-1} hL_\alpha h^{-1} + \partial_\alpha h h^{-1} hL_\beta h^{-1} - hL_\beta h^{-1}\partial_\alpha h h^{-1} \\
&\quad + \partial_\alpha h h^{-1}\partial_\beta h h^{-1} - \partial_\beta h h^{-1}\partial_\alpha h h^{-1} + h\big(\partial_\alpha L_\beta - \partial_\beta L_\alpha\big)h^{-1} \\
&\overset{\dagger}{=} [hL_\alpha h^{-1}, \partial_\beta h h^{-1}] + [\partial_\alpha h h^{-1}, hL_\beta h^{-1}] + [\partial_\alpha h h^{-1}, \partial_\beta h h^{-1}] + [hL_\alpha h^{-1}, hL_\beta h^{-1}] \\
&= [hL_\alpha h^{-1} + \partial_\alpha h h^{-1}, hL_\beta h^{-1} + \partial_\beta h h^{-1}] = [L'_\alpha, L'_\beta]
\end{aligned}$$

where we used the fact that $[A + B, C] = [A, C] + [B, C]$ as the commutator is bilinear.

† Using flatness (1.68) and the fact that $h[A, B]h^{-1} = [hAh^{-1}, hBh^{-1}]$ since $AB = Ah^{-1}hB$.

A.8.2 Kappa symmetry transformation

To find how the Lax pair (1.90) described in 1.3.2 transform under κ-symmetry transformations, i.e. to find

$$\delta_\epsilon L_\alpha = \Big(\ell_0\delta_\epsilon A_\alpha^{(0)} + \ell_1\delta_\epsilon A_\alpha^{(2)} + \ell_2\gamma_{\alpha\beta}\varepsilon^{\beta\rho}\delta_\epsilon A_\rho^{(2)} + \ell_3\delta_\epsilon A_\alpha^{(1)} + \ell_4\delta_\epsilon A_\alpha^{(3)}\Big) + \ell_2\delta_\epsilon\gamma_{\alpha\beta}\varepsilon^{\beta\rho}A_\rho^{(2)}$$

we should start by recalling $\delta_\epsilon A^{(k)}$ (1.50). If we restrict our discussion to transformations of type $\epsilon = \epsilon^{(1)}$,

$$\begin{aligned} \delta_\epsilon A^{(0)} &= [A^{(3)}, \epsilon^{(1)}], \qquad & \delta_\epsilon A^{(1)} &= [A^{(0)}, \epsilon^{(1)}] - d\epsilon^{(1)}, \\ \delta_\epsilon A^{(2)} &= [A^{(1)}, \epsilon^{(1)}], \qquad & \delta_\epsilon A^{(3)} &= [A^{(2)}, \epsilon^{(1)}]. \end{aligned} \tag{A.66}$$

Substituting these variations into $\delta_\epsilon L_\alpha$ above, remembering the conditions imposed on ℓ_i (1.91), and setting $\Lambda = \ell_3\epsilon^{(1)}$, we get

$$\begin{aligned} \delta_\epsilon L_\alpha &= [A_\alpha^{(3)}, \epsilon^{(1)}] + \ell_1[A_\alpha^{(1)}, \epsilon^{(1)}] + \ell_2\gamma_{\alpha\beta}\varepsilon^{\beta\rho}[A_\rho^{(1)}, \epsilon^{(1)}] + \ell_3[A_\alpha^{(0)}, \epsilon^{(1)}] - \ell_3\partial_\alpha\epsilon^{(1)} + \ell_4[A_\alpha^{(2)}, \epsilon^{(1)}] \\ &\quad + \ell_2\delta_\epsilon\gamma_{\alpha\beta}\varepsilon^{\beta\rho}A_\rho^{(2)} \\ &= [A_\alpha^{(0)} + \ell_4 A_\alpha^{(3)}, \Lambda] + \ell_1[A_\alpha^{(1)}, \epsilon^{(1)}] + \ell_2\gamma_{\alpha\beta}\varepsilon^{\beta\rho}[A_\rho^{(1)}, \epsilon^{(1)}] - \partial_\alpha\Lambda + \ell_4[A_\alpha^{(2)}, \epsilon^{(1)}] + \ell_1\ell_3[A_\alpha^{(2)}, \epsilon^{(1)}] \\ &\quad - \ell_1\ell_3[A_\alpha^{(2)}, \epsilon^{(1)}] + \ell_2\ell_3[\gamma_{\alpha\beta}\varepsilon^{\beta\rho}A_\rho^{(2)}, \epsilon^{(1)}] - \ell_2\ell_3[\gamma_{\alpha\beta}\varepsilon^{\beta\rho}A_\rho^{(2)}, \epsilon^{(1)}] + \ell_3^2[A_\alpha^{(1)}, \epsilon^{(1)}] - \ell_3^2[A_\alpha^{(1)}, \epsilon^{(1)}] \\ &\quad + \ell_2\delta_\epsilon\gamma_{\alpha\beta}\varepsilon^{\beta\rho}A_\rho^{(2)} \\ &= [A_\alpha^{(0)} + \ell_1 A_\alpha^{(2)} + \ell_2\gamma_{\alpha\beta}\varepsilon^{\beta\rho}A_\rho^{(2)} + \ell_3 A_\alpha^{(1)} + \ell_4 A_\alpha^{(3)}, \Lambda] - \partial_\alpha\Lambda + \ell_1[A_\alpha^{(1)}, \epsilon^{(1)}] + \ell_2\gamma_{\alpha\beta}\varepsilon^{\beta\rho}[A_\rho^{(1)}, \epsilon^{(1)}] \\ &\quad + \ell_4[A_\alpha^{(2)}, \epsilon^{(1)}] - \ell_1\ell_3[A_\alpha^{(2)}, \epsilon^{(1)}] - \ell_2\ell_3[\gamma_{\alpha\beta}\varepsilon^{\beta\rho}A_\rho^{(2)}, \epsilon^{(1)}] - \ell_3^2[A_\alpha^{(1)}, \epsilon^{(1)}] + \ell_2\delta_\epsilon\gamma_{\alpha\beta}\varepsilon^{\beta\rho}A_\rho^{(2)} \\ &= [L_\alpha, \Lambda] - \partial_\alpha\Lambda + (\ell_4 - \ell_1\ell_3)[A_\alpha^{(2)}, \epsilon^{(1)}] - \ell_2\ell_3[\gamma_{\alpha\beta}\varepsilon^{\beta\rho}A_\rho^{(2)}, \epsilon^{(1)}] \\ &\quad + [(\ell_1 - \ell_3^2)A_\alpha^{(1)} + \ell_2\gamma_{\alpha\beta}\varepsilon^{\beta\rho}A_\rho^{(1)}, \epsilon^{(1)}] + \ell_2\delta_\epsilon\gamma_{\alpha\beta}\varepsilon^{\beta\rho}A_\rho^{(2)} \\ &= [L_\alpha, \Lambda] - \partial_\alpha\Lambda + \ell_2\ell_3\kappa[A_\alpha^{(2)}, \epsilon^{(1)}] - \ell_2\ell_3\gamma_{\alpha\beta}\varepsilon^{\beta\rho}[A_\rho^{(2)}, \epsilon^{(1)}] \\ &\quad + [\ell_2\kappa A_\alpha^{(1)} + \ell_2\gamma_{\alpha\beta}\varepsilon^{\beta\rho}A_\rho^{(1)}, \epsilon^{(1)}] + \ell_2\delta_\epsilon\gamma_{\alpha\beta}\varepsilon^{\beta\rho}A_\rho^{(2)} \\ &= [L_\alpha, \Lambda] - \partial_\alpha\Lambda + \ell_2\ell_3[\kappa A_\alpha^{(2)} - \gamma_{\alpha\beta}\varepsilon^{\beta\rho}A_\rho^{(2)}, \epsilon^{(1)}] + \ell_2[\kappa A_\alpha^{(1)} + \gamma_{\alpha\beta}\varepsilon^{\beta\rho}A_\rho^{(1)}, \epsilon^{(1)}] + \ell_2\delta_\epsilon\gamma_{\alpha\beta}\varepsilon^{\beta\rho}A_\rho^{(2)}. \end{aligned}$$

We have almost manipulated the expression into a form using $P_\pm^{\alpha\beta}$. All we need to see is the relation

$$\begin{aligned} \kappa A_\alpha^{(2)} - \gamma_{\alpha\beta}\varepsilon^{\beta\rho}A_\rho^{(2)} &= \kappa\gamma_{\alpha\mu}A^{(2),\mu} - \gamma_{\alpha\beta}\varepsilon^{\beta\rho}A_\rho^{(2)} \overset{\dagger}{=} \kappa\varepsilon_{\alpha\sigma}\gamma^{\sigma\nu}\varepsilon_{\nu\mu}A^{(2),\mu} - \gamma_{\alpha\beta}\varepsilon^{\beta\rho}A_\rho^{(2)} \\ &= \varepsilon_{\alpha\beta}\big[\kappa\gamma^{\beta\nu}\varepsilon_{\nu\mu}A^{(2),\mu} - A^{(2),\beta}\big] = -\varepsilon_{\alpha\beta}\big[\gamma^{\beta\delta} - \kappa\varepsilon^{\beta\delta}\big]A_\delta^{(2)} \\ &= -2\varepsilon_{\alpha\beta}P_-^{\beta\delta}A_\delta^{(2)} = -2\varepsilon_{\alpha\beta}A_-^{(2),\beta} \end{aligned}$$

which ultimately results in equation (1.94)

$$\delta_\epsilon L_\alpha = [L_\alpha, \Lambda] - \partial_\alpha\Lambda - 2\ell_2\ell_3\varepsilon_{\alpha\beta}[A_-^{(2),\beta}, \epsilon^{(1)}] + \ell_2\varepsilon_{\alpha\beta}\Big(2[A_+^{(1),\beta}, \epsilon^{(1)}] + \delta_\epsilon\gamma^{\beta\delta}A_\delta^{(2)}\Big). \tag{A.67}$$

Now, suppose for some arbitrary one-form c_α the infinitesimal transformation resulted in

$$\delta_\epsilon L_\alpha = [L_\alpha, \Lambda] - \partial_\alpha\Lambda + c_\alpha.$$

†Using $\varepsilon^{\alpha\beta}\gamma_{\beta\delta}\varepsilon^{\delta\rho} = \gamma^{\alpha\rho}$.

Then, we would get the new Lax connections $L'_\alpha = L_\alpha + \delta_\epsilon L_\alpha$. To check the new zero-curvature condition, let us calculate its ingredients first. Namely,

$$\partial_\alpha L'_\beta = \partial_\alpha L_\beta + [\partial_\alpha L_\beta, \Lambda] + [L_\beta, \partial_\alpha \Lambda] - \partial_\alpha \partial_\beta \Lambda + \partial_\alpha c_\beta$$

which means

$$\partial_\alpha L'_\beta - \partial_\beta L'_\alpha = \partial_\alpha L_\beta - \partial_\beta L_\alpha + [\partial_\alpha L_\beta - \partial_\beta L_\alpha, \Lambda] + [L_\beta, \partial_\alpha \Lambda] - [L_\alpha, \partial_\beta \Lambda] + \partial_\alpha c_\beta - \partial_\beta c_\alpha.$$

We will now find the new $[L'_\alpha, L'_\beta]$ and compare with the above expression to see the conditions imposed on c_α such that the zero-curvature of L_α is preserved. Ignoring terms quadratic in the infinitesimal transformation parameter $\epsilon^{(1)}$ (or equivalently Λ),

$$\begin{aligned}
[L'_\alpha, L'_\beta] = [L_\alpha, L_\beta] &+ [L_\alpha, [L_\beta, \Lambda]] - [L_\alpha, \partial_\beta \Lambda] + [L_\alpha, c_\beta] \\
&+ [[L_\alpha, \Lambda], L_\beta] + [[L_\alpha, \Lambda], c_\beta] + \mathcal{O}(\Lambda^2) \\
&- [\partial_\alpha \Lambda, L_\beta] - [\partial_\alpha \Lambda, c_\beta] + \mathcal{O}(\Lambda^2) \\
&+ [c_\alpha, L_\beta] + [c_\alpha, [L_\beta, \Lambda]] - [c_\alpha, \partial_\beta \Lambda] + [c_\alpha, c_\beta].
\end{aligned}$$

We now use the Jacobi identity[15] to write

$$\begin{aligned}
[L'_\alpha, L'_\beta] &= [L_\alpha, L_\beta] + \Big([L_\alpha, [L_\beta, \Lambda]] + [[L_\alpha, \Lambda], L_\beta]\Big) + [L_\beta, \partial_\alpha \Lambda] - [L_\alpha, \partial_\beta \Lambda] \\
&\quad + [L_\alpha, c_\beta] + [c_\alpha, L_\beta] + [[L_\alpha, \Lambda], c_\beta] + [c_\alpha, [L_\beta, \Lambda]] - [\partial_\alpha \Lambda, c_\beta] - [c_\alpha, \partial_\beta \Lambda] + [c_\alpha, c_\beta] \\
&\quad + \mathcal{O}(\Lambda^2) \\
&= [L_\alpha, L_\beta] + [[L_\alpha, L_\beta], \Lambda] + [L_\beta, \partial_\alpha \Lambda] - [L_\alpha, \partial_\beta \Lambda] \\
&\quad + [L_\alpha, c_\beta] + [c_\alpha, L_\beta] + [[L_\alpha, \Lambda], c_\beta] + [c_\alpha, [L_\beta, \Lambda]] - [\partial_\alpha \Lambda, c_\beta] - [c_\alpha, \partial_\beta \Lambda] + [c_\alpha, c_\beta] \\
&\quad + \mathcal{O}(\Lambda^2).
\end{aligned}$$

Comparing with what we previously found, i.e.

$$\partial_\alpha L'_\beta - \partial_\beta L'_\alpha = \partial_\alpha L_\beta - \partial_\beta L_\alpha + [\partial_\alpha L_\beta - \partial_\beta L_\alpha, \Lambda] + [L_\beta, \partial_\alpha \Lambda] - [L_\alpha, \partial_\beta \Lambda] + \partial_\alpha c_\beta - \partial_\beta c_\alpha,$$

and substituting the old zero-curvature condition (1.68), the new zero curvature-condition

$$\partial_\alpha L'_\beta - \partial_\beta L'_\alpha = [L'_\alpha, L'_\beta]$$

is satisfied, provided the extra term c_α obeys the following condition

$$\partial_\alpha c_\beta - \partial_\beta c_\alpha = [L_\alpha, c_\beta] + [c_\alpha, L_\beta] + [[L_\alpha, \Lambda], c_\beta] + [c_\alpha, [L_\beta, \Lambda]] - [\partial_\alpha \Lambda, c_\beta] - [c_\alpha, \partial_\beta \Lambda] + [c_\alpha, c_\beta].$$

Obviously if $c_\alpha = 0$ then the above is satisfied[16]. We will now prove that

$$c_\alpha = 2\ell_2\ell_3\varepsilon_{\alpha\beta} \underbrace{[A_-^{(2),\beta}, \epsilon^{(1)}]}_{I_1^\beta} - \ell_2\varepsilon_{\alpha\beta}\Big(2\underbrace{[A_+^{(1),\beta}, \epsilon^{(1)}]}_{I_2^\beta} + \delta_\epsilon \gamma^{\beta\delta} A_\delta^{(2)}\Big) = 0 \tag{A.68}$$

[15] $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$.

[16] I tried substituting non-trivial forms of c_α, for example $\partial_\alpha \Lambda$, but was not able to find one which satisfied the condition.

by reducing $I_{1,2}$ and showing that the two terms vanish separately, hence the transformation is a gauge transformation of the Lax connection, i.e it preserves flatness. Beginning with I_1, we remember that $A_{\alpha,-}$ and $A_{\beta,-}$ are proportional to each other; when $\alpha = \beta$ they are just equal, but whenever $\alpha \neq \beta$ they are related by (1.56). Either way, $[A_{\alpha,\pm}, A_{\beta,\pm}] = 0$. In particular, taking the $\mathscr{G}^{(0)}$ projection of this equality, we get that $[A^{(k)}_{\alpha,\pm}, A^{(k)}_{\beta,\pm}] = 0$ for $k = 0, 1, 2, 3$ since the different grading elements $A^{(k)}$ are independent of one another. All of this to say that $[A^{(2)}_{\alpha,-}, A^{(2)}_{\beta,-}] = 0$ such that, substituting the expression (1.57) for $\epsilon^{(1)}$,

$$\begin{aligned}
I_{1,\alpha} &= [A^{(2)}_{\alpha,-}, \epsilon^{(1)}] = [A^{(2)}_{\alpha,-}, A^{(2)}_{\beta,-}\kappa^{(1),\beta}_{+} + \kappa^{(1),\beta}_{+} A^{(2)}_{\beta,-}] \\
&= A^{(2)}_{\beta,-}[A^{(2)}_{\alpha,-}, \kappa^{(1),\beta}_{+}] + [A^{(2)}_{\alpha,-}, \kappa^{(1),\beta}_{+}]A^{(2)}_{\beta,-} \\
&= A^{(2)}_{\beta,-}A^{(2)}_{\alpha,-}\kappa^{(1),\beta}_{+} - \cancel{A^{(2)}_{\beta,-}\kappa^{(1),\beta}_{+}A^{(2)}_{\alpha,-}} + \cancel{A^{(2)}_{\alpha,-}\kappa^{(1),\beta}_{+}A^{(2)}_{\beta,-}} - \kappa^{(1),\beta}_{+}A^{(2)}_{\alpha,-}A^{(2)}_{\beta,-} \\
&= [A^{(2)}_{\alpha,-}A^{(2)}_{\beta,-}, \kappa^{(1),\beta}_{+}].
\end{aligned}$$

In the last line we used again the fact that the projected components $A^{(2)}_{\alpha,-}$ and $A^{(2)}_{\beta,-}$ are proportional to commute them and to cancel the equal and opposite terms. Lastly we recall (1.59) and notice that the term proportional to the identity will commute with $\kappa^{(1),\beta}_{+}$ such that we are left with

$$I_{1,\alpha} = \frac{1}{8}\mathrm{str}\big(A^{(2)}_{\alpha,-}A^{(2)}_{\beta,-}\big)[\Upsilon, \kappa^{(1),\beta}_{+}]. \tag{A.69}$$

To proceed, we will show that the Virasoro constraints are satisfied if and only if $\mathrm{str}\big(A^{(2)}_{\alpha,-}A^{(2)}_{\beta,-}\big) = 0$. It will be crucial to use the following identities relating $\varepsilon_{\alpha\beta}$ and $\gamma_{\alpha\beta}$.

(i) We note that $\varepsilon_{\alpha\mu}\varepsilon_{\beta\nu}$ and $(\gamma_{\alpha\beta}\gamma_{\mu\nu} - \gamma_{\alpha\nu}\gamma_{\beta\mu})$ share the same symmetry under exchange of pairs of indices $(\alpha\mu) \leftrightarrow (\beta\nu)$, and anti-symmetry under exchange of $\alpha \leftrightarrow \mu$ and $\beta \leftrightarrow \nu$. So they must be proportional and, by looking at $\varepsilon_{\tau\sigma}\varepsilon_{\tau\sigma} = 1 \propto \det\gamma = -1$ for example, we see that we in fact have

$$\varepsilon_{\alpha\mu}\varepsilon_{\beta\nu} = -(\gamma_{\alpha\beta}\gamma_{\mu\nu} - \gamma_{\alpha\nu}\gamma_{\beta\mu}) = \gamma_{\alpha\nu}\gamma_{\beta\mu} - \gamma_{\alpha\beta}\gamma_{\mu\nu}.$$

One could also use $\varepsilon^{\alpha\beta}\varepsilon^{\gamma\delta} = \delta^{\alpha\gamma}\delta^{\beta\delta} - \delta^{\alpha\delta}\delta^{\beta\gamma} = -(\gamma^{\alpha\gamma}\gamma^{\beta\delta} - \gamma^{\alpha\delta}\gamma^{\beta\gamma})$. Note the overall minus sign appears because each $\gamma^{\alpha\beta}$ factor is associated to a different index of $\varepsilon^{\alpha\beta}$. Both dimensions' sign appears exactly once in each term, and since $\det\gamma = -1$, an extra minus is needed to keep the Kronecker delta terms positive when non-zero.

(ii) We use identity (i) to derive

$$\begin{aligned}
\varepsilon_{\nu}{}^{\lambda}\varepsilon^{\rho}{}_{\mu} &= \gamma^{\lambda\alpha}\gamma^{\rho\beta}\varepsilon_{\nu\alpha}\varepsilon_{\beta\mu} = \gamma^{\lambda\alpha}\gamma^{\rho\beta}(\gamma_{\nu\mu}\gamma_{\beta\alpha} - \gamma_{\nu\beta}\gamma_{\alpha\mu}) \\
&= \gamma_{\mu\nu}\gamma^{\lambda\rho} - \delta^{\rho}_{\nu}\delta^{\lambda}_{\mu}.
\end{aligned}$$

With these two identities (i) and (ii) in mind we calculate the following, with cyclicity in $\mu \leftrightarrow \nu$,

$$\begin{aligned}
\mathrm{str}\big(A^{(2)}_{\alpha,-}A^{(2)}_{\beta,-}\big) &= \mathrm{str}\big(P_{-\alpha}{}^{\mu}A^{(2)}_{\mu}P_{-\beta}{}^{\nu}A^{(2)}_{\nu}\big) = P_{-\alpha\mu}P_{-\beta\nu}\mathrm{str}\big(A^{(2),\mu}A^{(2),\nu}\big) \\
0 &= \frac{1}{4}\big[\gamma_{\alpha\mu}\gamma_{\beta\nu} - \kappa\gamma_{\alpha\mu}\varepsilon_{\beta\nu} - \kappa\gamma_{\beta\nu}\varepsilon_{\alpha\mu} + \kappa^2\varepsilon_{\alpha\mu}\varepsilon_{\beta\nu}\big]\mathrm{str}\big(A^{(2),\mu}A^{(2),\nu}\big) \\
0 &\overset{\text{(i)}}{=} \frac{1}{4}\big[\gamma_{\alpha\mu}\gamma_{\beta\nu} - \kappa\gamma_{\alpha\mu}\varepsilon_{\beta\nu} - \kappa\gamma_{\beta\nu}\varepsilon_{\alpha\mu} + \gamma_{\alpha\nu}\gamma_{\beta\mu} - \gamma_{\alpha\beta}\gamma_{\mu\nu}\big]\mathrm{str}\big(A^{(2),\mu}A^{(2),\nu}\big)
\end{aligned}$$

$$
\begin{aligned}
0 &= \frac{1}{2}\gamma_{\alpha\mu}\gamma_{\beta\nu}\mathrm{str}\big(A^{(2),\mu}A^{(2),\nu}\big) - \frac{1}{4}\gamma_{\alpha\beta}\gamma_{\mu\nu}\mathrm{str}\big(A^{(2),\mu}A^{(2),\nu}\big) - \frac{\kappa}{4}\big[\gamma_{\alpha\mu}\varepsilon_{\beta\nu} + \gamma_{\beta\nu}\varepsilon_{\alpha\mu}\big]\mathrm{str}\big(A^{(2),\mu}A^{(2),\nu}\big)\\
0 &= \frac{1}{2}\big[\mathrm{str}\big(A^{(2)}_{\alpha}A^{(2)}_{\beta}\big) - \frac{1}{2}\gamma_{\alpha\beta}\gamma^{\mu\nu}\mathrm{str}\big(A^{(2)}_{\mu}A^{(2)}_{\nu}\big)\big] - \frac{\kappa}{4}\big[\gamma_{\alpha\mu}\varepsilon_{\beta\nu} + \gamma_{\beta\nu}\varepsilon_{\alpha\mu}\big]\mathrm{str}\big(A^{(2),\mu}A^{(2),\nu}\big)\\
0 &= \frac{1}{2}(\text{Virasoro})_{\alpha\beta} - \frac{\kappa}{4}\big[\gamma_{\alpha\mu}\varepsilon_{\beta\nu} + \gamma_{\beta\nu}\varepsilon_{\alpha\mu}\big]\mathrm{str}\big(A^{(2),\mu}A^{(2),\nu}\big)\\
\gamma^{\alpha\rho}\varepsilon^{\beta\lambda}0 &= \gamma^{\alpha\rho}\varepsilon^{\beta\lambda}\frac{1}{2}(\text{Virasoro})_{\alpha\beta} - \gamma^{\alpha\rho}\varepsilon^{\beta\lambda}\frac{\kappa}{4}\big[\gamma_{\alpha\mu}\varepsilon_{\beta\nu} + \gamma_{\beta\nu}\varepsilon_{\alpha\mu}\big]\mathrm{str}\big(A^{(2),\mu}A^{(2),\nu}\big)\\
0 &= \gamma^{\alpha\rho}\varepsilon^{\beta\lambda}\frac{1}{2}(\text{Virasoro})_{\alpha\beta} - \frac{\kappa}{4}\big[\delta^{\rho}_{\mu}\varepsilon^{\beta\lambda}\varepsilon_{\beta\nu} + \varepsilon_{\nu}{}^{\lambda}\varepsilon^{\rho}{}_{\mu}\big]\mathrm{str}\big(A^{(2),\mu}A^{(2),\nu}\big)\\
0 &\overset{\text{(ii)}}{=} \gamma^{\alpha\rho}\varepsilon^{\beta\lambda}\frac{1}{2}(\text{Virasoro})_{\alpha\beta} - \frac{\kappa}{4}\big[-2\delta^{\rho}_{\mu}\delta^{\lambda}_{\nu} + \gamma^{\lambda\rho}\gamma_{\mu\nu}\big]\mathrm{str}\big(A^{(2),\mu}A^{(2),\nu}\big)\\
0 &= \gamma^{\alpha\rho}\varepsilon^{\beta\lambda}\frac{1}{2}(\text{Virasoro})_{\alpha\beta} + \frac{\kappa}{2}\big[\mathrm{str}\big(A^{(2),\rho}A^{(2),\lambda}\big) - \frac{1}{2}\gamma^{\lambda\rho}\gamma_{\mu\nu}\mathrm{str}\big(A^{(2),\mu}A^{(2),\nu}\big)\big]\\
0 &= \gamma^{\alpha\rho}\varepsilon^{\beta\lambda}\frac{1}{2}(\text{Virasoro})_{\alpha\beta} + \frac{\kappa}{2}(\text{Virasoro})^{\lambda\rho} = \frac{1}{2}\gamma^{\alpha\rho}\big[\varepsilon^{\beta\lambda} + \kappa\gamma^{\beta\nu}\big](\text{Virasoro})_{\alpha\beta}.
\end{aligned}
$$

This proves equivalence with the Virasoro constraints:

$$
\mathrm{str}\big(A^{(2)}_{\alpha,-}A^{(2)}_{\beta,-}\big) = 0 \iff \mathrm{str}\big(A^{(2)}_{\alpha}A^{(2)}_{\beta}\big) - \frac{1}{2}\gamma_{\alpha\beta}\gamma^{\mu\nu}\mathrm{str}\big(A^{(2)}_{\mu}A^{(2)}_{\nu}\big) = 0. \tag{A.70}
$$

Thus, $I_{1,\alpha} = 0$ and we only need to show $I_2^{\beta} + \delta_{\epsilon}\gamma^{\beta\delta}A^{(2)}_{\delta} = 0$. Looking at

$$
I_2^{\alpha} = [A_+^{(1),\alpha}, \varepsilon^{(1)}] = [A_+^{(1),\alpha}, A^{(2)}_{\beta,-}\kappa_+^{(1),\beta} + \kappa_+^{(1),\beta}A^{(2)}_{\beta,-}], \tag{A.71}
$$

we use (A.43) which helps simplify I_2^{α} down to

$$
\begin{aligned}
I_2^{\alpha} &= [A_+^{(1),\alpha}, A^{(2)}_{\beta,-}\kappa_+^{(1),\beta} + \kappa_+^{(1),\beta}A^{(2)}_{\beta,-}] = [A_+^{(1),\beta}, A^{(2)}_{\beta,-}\kappa_+^{(1),\alpha} + \kappa_+^{(1),\alpha}A^{(2)}_{\beta,-}]\\
&\overset{\dagger}{=} A^{(2)}_{\beta,-}[A_+^{(1),\beta}, \kappa_+^{(1),\alpha}] + \cancel{[A_+^{(1),\beta}, A^{(2)}_{\beta,-}]}\kappa_+^{(1),\alpha} + \kappa_+^{(1),\alpha}\cancel{[A_+^{(1),\beta}, A^{(2)}_{\beta,-}]} + [A_+^{(1),\beta}, \kappa_+^{(1),\alpha}]A^{(2)}_{\beta,-}\\
&= A^{(2)}_{\beta,-}[A_+^{(1),\beta}, \kappa_+^{(1),\alpha}] + [A_+^{(1),\beta}, \kappa_+^{(1),\alpha}]A^{(2)}_{\beta,-}.
\end{aligned}
$$

Since $[A_+^{(1),\beta}, \kappa_+^{(1),\alpha}] \in \mathscr{G}^{(2)}$, this commutator is traceless and can be expressed generically using (1.58) as

$$
[A_+^{(1),\beta}, \kappa_+^{(1),\alpha}] = \begin{pmatrix} m_a^{\alpha\beta}\gamma^a & 0 \\ 0 & n_a^{\alpha\beta}\gamma^a \end{pmatrix} + \frac{1}{8}\mathrm{str}\big(\Upsilon[A_+^{(1),\beta}, \kappa_+^{(1),\alpha}]\big)\mathbb{1}_8,
$$

which clearly implies

$$
I_2^{\alpha} = \{A^{(2)}_{\beta,-}, \begin{pmatrix} m_a^{\alpha\beta}\gamma^a & 0 \\ 0 & n_a^{\alpha\beta}\gamma^a \end{pmatrix}\} + \frac{1}{4}\mathrm{str}\big(\Upsilon[A_+^{(1),\beta}, \kappa_+^{(1),\alpha}]\big)A^{(2)}_{\beta,-}. \tag{A.72}
$$

Again, as discussed in 1.1.2 and used in (1.58), elements $A^{(2)}_{\beta,-}$ can be expressed as

$$
A^{(2)}_{\beta,-} = \begin{pmatrix} p^i_{\beta,-}\gamma^i & 0 \\ 0 & q^i_{\beta,-}\gamma^i \end{pmatrix},
$$

† The fermionic equations of motion (1.39) are equivalent to $[A_+^{(1),\beta}, A^{(2)}_{\alpha}] = 0$ and thus $[A_+^{(1),\beta}, A^{(2)}_{\beta,-}] = 0$

‡ Any diagonal matrix can be expressed as a linear combination of $\mathbb{1}_8$ and Υ.

which can be used to simplify the anti-commutator

$$\{A^{(2)}_{\beta,-},\begin{pmatrix} m_a^{\alpha\beta}\gamma^a & 0 \\ 0 & n_a^{\alpha\beta}\gamma^a\end{pmatrix}\} = \begin{pmatrix} m_a^{\alpha\beta}p^i_{\beta,-}\{\gamma^i,\gamma^a\} & 0 \\ 0 & n_a^{\alpha\beta}q^i_{\beta,-}\{\gamma^i,\gamma^a\}\end{pmatrix} = \begin{pmatrix} m_a^{\alpha\beta}p^i_{\beta,-}\delta^{ia}\mathbb{1}_4 & 0 \\ 0 & n_a^{\alpha\beta}q^i_{\beta,-}\delta^{ia}\mathbb{1}_4\end{pmatrix}$$
$$= \begin{pmatrix} m_a^{\alpha\beta}p^a_{\beta,-}\mathbb{1}_4 & 0 \\ 0 & n_a^{\alpha\beta}q^a_{\beta,-}\mathbb{1}_4\end{pmatrix} \overset{\ddagger}{\equiv} \frac{1}{2}\rho_1^\alpha\mathbb{1}_8 + \frac{1}{2}\rho_2^\alpha\Upsilon.$$

This means

$$2I_2^\alpha = \rho_1^\alpha\mathbb{1}_8 + \rho_2^\alpha\Upsilon - \frac{1}{2}\mathrm{str}\big(\Upsilon[\kappa_+^{(1),\alpha},A_+^{(1),\beta}]\big)A^{(2)}_{\beta,-}. \tag{A.73}$$

Because of its original definition as a commutator (A.71), I_2^α must be supertraceless which means that $\rho_2^\alpha = 0$ since $\mathrm{str}\big(\mathbb{1}_8\big) = \mathrm{str}\big(A\big) = 0$. The first term will not contribute as we are working modulo $i\mathbb{1}_8$ in $\mathfrak{psu}(2,2|4)$. Finally, the last term will cancel with the $\delta_\epsilon\gamma^{\alpha\beta}$ (1.61) in (A.68):

$$\varepsilon_{\alpha\beta}\left(2I_2^\beta + \delta_\epsilon\gamma^{\beta\delta}A_\delta^{(2)}\right) = \varepsilon_{\alpha\beta}\left(-\frac{1}{2}\mathrm{str}\big(\Upsilon[\kappa_+^{(1),\beta},A_+^{(1),\delta}]\big)A^{(2)}_{\delta,-} + \delta_\epsilon\gamma^{\beta\delta}A_\delta^{(2)}\right) = 0.$$

A.8.3 Diffeomorphisms

To show diffeomorphisms $\sigma^\alpha \to \sigma^\alpha = \tilde{\sigma}^\alpha + f^\alpha(\sigma,\tau)$ induce a gauge transformation of the Lax connections, we first calculate $\tilde{L}_\alpha(\tilde{\boldsymbol{\sigma}})$ in two different ways. On one hand, a one-form transforms as

$$\begin{aligned}\tilde{L}_\alpha(\tilde{\boldsymbol{\sigma}}) &= L_\beta(\boldsymbol{\sigma})\frac{\partial\sigma^\beta}{\partial\tilde{\sigma}^\alpha} = L_\beta(\boldsymbol{\sigma})\frac{\partial(\tilde{\sigma}^\beta+f^\beta)}{\partial\tilde{\sigma}^\alpha} = L_\beta(\boldsymbol{\sigma})(\delta_\alpha^\beta + \frac{\partial f^\beta}{\partial\sigma^\delta}\frac{\partial\sigma^\delta}{\partial\tilde{\sigma}^\alpha})\\ &= L_\alpha(\boldsymbol{\sigma}) + L_\beta(\boldsymbol{\sigma})\partial_\delta f^\beta\delta_\alpha^\delta + \mathcal{O}(f^2) = L_\alpha(\boldsymbol{\sigma}) + (L_\beta\partial_\alpha f^\beta)(\boldsymbol{\sigma}) + \mathcal{O}(f^2).\end{aligned}$$

On the other hand, using the Taylor expansion of $\tilde{L}_\alpha(\boldsymbol{\sigma}-\mathbf{f})$ in f^α around $\mathbf{f}=0$, and substituting what we just found above,

$$\begin{aligned}\tilde{L}_\alpha(\tilde{\boldsymbol{\sigma}}) &= \tilde{L}_\alpha(\boldsymbol{\sigma}-\mathbf{f}) = \tilde{L}_\alpha(\boldsymbol{\sigma}) - f^\beta(\frac{\partial}{\partial f^\beta}\tilde{L}_\alpha)(\boldsymbol{\sigma}) + \mathcal{O}(f^2)\\ &= \tilde{L}_\alpha(\boldsymbol{\sigma}) - f^\beta\frac{\partial\sigma^\rho}{\partial f^\beta}\partial_\rho\big[L_\alpha + L_\beta\partial_\alpha f^\beta + \mathcal{O}(f^2)\big)\big](\boldsymbol{\sigma}) + \mathcal{O}(f^2)\\ &= \tilde{L}_\alpha(\boldsymbol{\sigma}) - (f^\beta\delta_\beta^\rho\partial_\rho L_\alpha)(\boldsymbol{\sigma}) + \mathcal{O}(f^2) = \tilde{L}_\alpha(\boldsymbol{\sigma}) - (f^\beta\partial_\beta L_\alpha)(\boldsymbol{\sigma}) + \mathcal{O}(f^2).\end{aligned}$$

Equating the two expressions for $\tilde{L}_\alpha(\tilde{\boldsymbol{\sigma}})$ up to linear order in f^α, we get

$$\delta L_\alpha = \tilde{L}_\alpha(\boldsymbol{\sigma}) - L_\alpha(\boldsymbol{\sigma}) = f^\beta\partial_\beta L_\alpha + L_\beta\partial_\alpha f^\beta. \tag{A.74}$$

Using the zero-curvature condition for L_α, we find (1.99).

A.9 Details of embedding

Given the S^5 coordinates (1.104), we can find the differentials and their moduli squared:

$$(dY_1)^2 + (dY_2)^2 = \frac{(dy_1)^2+(dy_2)^2}{(1+y^2/4)^2} + \frac{1}{4}\frac{1}{(1+y^2/4)^4}\left\{(y_idy_i)^2[(y_1)^2+(y_2)^2] - (4+y^2)(y_idy_i)(y_1dy_1+y_2dy_2)\right\},$$
$$(dY_3)^2 + (dY_4)^2 = \frac{(dy_3)^2+(dy_4)^2}{(1+y^2/4)^2} + \frac{1}{4}\frac{1}{(1+y^2/4)^4}\left\{(y_idy_i)^2[(y_3)^2+(y_4)^2] - (4+y^2)(y_idy_i)(y_3dy_3+y_4dy_4)\right\},$$

$$(dY_5)^2+(dY_6)^2=\Big(\frac{1-y^2/4}{1+y^2/4}\Big)^2(d\phi)^2+\frac{(y_idy_i)^2}{(1+y^2/4)^4}.$$

Their sum gives the induced metric $ds^2|_{S^5}$ (1.105) since adding the two first equations results in

$$(dY_1)^2+(dY_2)^2+(dY_3)^2+(dY_4)^2=\frac{dy_idy_i}{(1+y^2/4)^2}-\frac{(y_idy_i)^2}{(1+y^2/4)^4}=ds^2|_{S^5}-(dY_5)^2+(dY_6)^2.$$

For the AdS$_5$ coordinates (1.106), we can simply replace y^2 with $-z^2$ to find

$$(dZ_1)^2+(dZ_2)^2=\frac{(dz_1)^2+(dz_2)^2}{(1-z^2/4)^2}-\frac{1}{4}\frac{1}{(1-z^2/4)^4}\left\{(z_idz_i)^2[(z_1)^2+(z_2)^2]+(4-z^2)(z_idz_i)(z_1dz_1+z_2dz_2)\right\},$$
$$(dZ_3)^2+(dZ_4)^2=\frac{(dz_3)^2+(dz_4)^2}{(1-z^2/4)^2}-\frac{1}{4}\frac{1}{(1-z^2/4)^4}\left\{(z_idz_i)^2[(z_3)^2+(z_4)^2]+(4-z^2)(z_idz_i)(z_3dz_3+z_4dz_4)\right\},$$
$$(dZ_0)^2+(dZ_5)^2=\Big(\frac{1+z^2/4}{1-z^2/4}\Big)^2(dt)^2-\frac{(z_idz_i)^2}{(1-z^2/4)^4}.$$

This time their sum has signature $(\eta^{AB})=\mathrm{diag}(-1,1,1,1,1,-1)$ which results in the induced metric $ds^2|_{\mathrm{AdS}_5}$ (1.107) since

$$(dZ_1)^2+(dZ_2)^2+(dZ_3)^2+(dZ_4)^2=\frac{dz_idz_i}{(1-z^2/4)^2}-\frac{(z_idz_i)^2}{(1-z^2/4)^4}=ds^2|_{\mathrm{AdS}_5}+(dZ_0)^2+(dZ_5)^2.$$

We shall now find the representation of the bosonic element $\mathfrak{g}_\mathfrak{b}$, whose bilinear form $\mathrm{str}[(\mathfrak{g}_\mathfrak{b}^{-1}d\mathfrak{g}_\mathfrak{b})^2]$ reproduces the metric (1.108) as described in 1.4.1. First, we introduce the matrices

$$\mathfrak{g}_\mathfrak{b}=\Lambda(t,\phi)\mathfrak{g}(\mathbb{X}),\qquad \mathfrak{g}(\mathbb{X})=\sqrt{\frac{\mathbb{1}+\mathbb{X}}{\mathbb{1}-\mathbb{X}}}=\big((\mathbb{1}_8-\mathbb{X})^{-1}(\mathbb{1}_8+\mathbb{X})\big)^{\frac{1}{2}}. \tag{A.75}$$

where $\mathbb{X}$ is given by (1.118). To compute $\mathfrak{g}(\mathbb{X})$, we will need to find the inverse of

$$\mathbb{1}_8-\mathbb{X}=\begin{pmatrix}\mathbb{1}_4-\frac{1}{2}z^i\gamma^i & 0\\ 0 & \mathbb{1}_4-\frac{\mathrm{i}}{2}y^i\gamma^i\end{pmatrix}. \tag{A.76}$$

We know that $(\gamma^i)^2=\mathbb{1}_4$. Looking at a simpler case, for example

$$\begin{aligned}(\mathbb{1}_4-a\gamma^1-b\gamma^2)(\mathbb{1}_4+a\gamma^1+b\gamma^2)&=\mathbb{1}_4-a^2\mathbb{1}_4-b^2\mathbb{1}_4-ab\cancel{(\gamma^1\gamma^2+\gamma^2\gamma^1)}\\&=(1-a^2-b^2)\mathbb{1}_4,\end{aligned}$$

it becomes clear that the inverse of (A.76) should be[17]

$$(\mathbb{1}_8-\mathbb{X})^{-1}=\begin{pmatrix}\frac{1}{1-z^2/4}[\mathbb{1}_4+\frac{1}{2}z^i\gamma^i] & 0\\ 0 & \frac{1}{1+y^2/4}[\mathbb{1}_4+\frac{\mathrm{i}}{2}y^i\gamma^i]\end{pmatrix}. \tag{A.77}$$

Substituting into (A.75), we easily get (1.119)

$$\mathfrak{g}(\mathbb{X})=\begin{pmatrix}\frac{1}{\sqrt{1-z^2/4}}[\mathbb{1}_4+\frac{1}{2}z^i\gamma^i] & 0\\ 0 & \frac{1}{\sqrt{1+y^2/4}}[\mathbb{1}_4+\frac{\mathrm{i}}{2}y^i\gamma^i]\end{pmatrix}. \tag{A.78}$$

[17] In the lower right block, because $z^2\to -y^2$, the normalising factor becomes $1-z^2/4\to 1+y^2/4$ instead.

Next, we aim to find $\mathfrak{g}_{\mathfrak{b}}^{-1} d\mathfrak{g}_{\mathfrak{b}}$ and start with

$$\begin{aligned}
\mathfrak{g}(\mathbb{X})^{-1} d\mathfrak{g}(\mathbb{X}) &= \begin{pmatrix} \frac{1}{\sqrt{1-z^2/4}}[\mathbb{1}_4 - \frac{1}{2} z^i \gamma^i] & 0 \\ 0 & \frac{1}{\sqrt{1+y^2/4}}[\mathbb{1}_4 - \frac{\mathrm{i}}{2} y^i \gamma^i] \end{pmatrix} \\
&\times \begin{pmatrix} \frac{z^i dz^i/4}{(1-z^2/4)^{\frac{3}{2}}}[\mathbb{1}_4 + \frac{1}{2} z^i \gamma^i] + \frac{1}{\sqrt{1-z^2/4}}[\frac{1}{2} dz^i \gamma^i] & 0 \\ 0 & -\frac{y^i dy^i/4}{(1+y^2/4)^{\frac{3}{2}}}[\mathbb{1}_4 + \frac{\mathrm{i}}{2} y^i \gamma^i] + \frac{1}{\sqrt{1+y^2/4}}[\frac{\mathrm{i}}{2} dy^i \gamma^i] \end{pmatrix} \\
&= \begin{pmatrix} \frac{z^i dz^i/4}{(1-z^2/4)^2}[\mathbb{1}_4 - \frac{1}{4} z^i z^j \gamma^j \gamma^i] + \frac{dz^i/2}{1-z^2/4}[\gamma^i - \frac{1}{2} z^j \gamma^j \gamma^i] & 0 \\ 0 & -\frac{y^i dy^i/4}{(1+y^2/4)^2}[\mathbb{1}_4 + \frac{1}{4} y^i y^j \gamma^j \gamma^i] + \frac{\mathrm{i} dy^i/2}{1+y^2/4}[\gamma^i - \frac{\mathrm{i}}{2} y^j \gamma^j \gamma^i] \end{pmatrix} \\
&\equiv \begin{pmatrix} Z & 0 \\ 0 & Y \end{pmatrix}.
\end{aligned}$$

We can separately compute the bite-sized 4×4 matrices

$$\begin{aligned}
X &= \frac{z^i dz^i/4}{(1-z^2/4)^2}[\mathbb{1}_4 - \frac{1}{4} z^i z^j \gamma^j \gamma^i] + \frac{dz^i/2}{1-z^2/4}[\gamma^i - \frac{1}{2} z^j \gamma^j \gamma^i] \\
&\overset{\dagger}{=} \frac{z^i dz^i/4}{(1-z^2/4)^2}[\mathbb{1}_4 - z^2 \mathbb{1}_4/4] + \frac{1}{1-z^2/4}[dz^i \gamma^i - \frac{1}{2} dz^i z^j \gamma^j \gamma^i] \\
&= \ldots \\
Y &= \ldots
\end{aligned}$$

[†] Using $\gamma^i \gamma^j (A_i B_j + A_j B_i) = 2 A_i B_i \mathbb{1}_4$.

Ⓑ Appendix – Chapter 2

B.1 First-order formalism

B.1.1 Bosonic string

Here we derive 2.4. If we start by summing (2.1) explicitly[18], we find

$$\begin{aligned}
S &= -\frac{g}{2}\int d\tau d\sigma\; G_{MN}\left(\gamma^{\tau\tau}\dot{X}^M\dot{X}^N + 2\gamma^{\tau\sigma}\dot{X}^M X'^N + \gamma^{\sigma\sigma}X'^M X'^N\right)\\
&= \int d\tau d\sigma\; G_{MN}\left(-\frac{g}{2}\gamma^{\tau\tau}\dot{X}^M\dot{X}^N - g\gamma^{\tau\sigma}\dot{X}^M X'^N - \frac{g}{2}\gamma^{\sigma\sigma}X'^M X'^N\right)\\
&= \int d\tau d\sigma\; G_{MN}\left(-\frac{g}{2}\gamma^{\tau\tau}\dot{X}^M\dot{X}^N - \frac{g}{2}\gamma^{\tau\tau}\dot{X}^M\dot{X}^N + \frac{g}{2}\gamma^{\tau\tau}\dot{X}^M\dot{X}^N - g\gamma^{\tau\sigma}\dot{X}^M X'^N - \frac{g}{2}\gamma^{\sigma\sigma}X'^M X'^N\right)\\
&\overset{(2.2)}{=} \int d\tau d\sigma\; G_{MN}\big(p^N\dot{X}^M + \underbrace{\frac{g}{2}\gamma^{\tau\tau}\dot{X}^M\dot{X}^N - \frac{g}{2}\gamma^{\sigma\sigma}X'^M X'^N}_{\star^{MN}}\big), \quad \text{where}
\end{aligned}$$

$$\begin{aligned}
\star^{MN} &= \frac{g}{2}\gamma^{\tau\tau}\dot{X}^M\dot{X}^N - \frac{g}{2}\gamma^{\sigma\sigma}X'^M X'^N - g\frac{\gamma^{\tau\sigma}\gamma^{\tau\sigma}}{\gamma^{\tau\tau}}X'^M X'^N + g\frac{\gamma^{\tau\sigma}\gamma^{\tau\sigma}}{\gamma^{\tau\tau}}X'^M X'^N - g\gamma^{\tau\sigma}\dot{X}^M X'^N + g\gamma^{\tau\sigma}\dot{X}^M X'^N\\
&= \frac{\gamma^{\tau\sigma}}{\gamma^{\tau\tau}}\left(-g\gamma^{\tau\tau}\dot{X}^M X'^N - g\gamma^{\tau\sigma}X'^M X'^N\right) + \frac{1}{2g\gamma^{\tau\tau}}\Big(g^2\gamma^{\tau\tau}\gamma^{\tau\tau}\dot{X}^M\dot{X}^N - g^2\gamma^{\tau\tau}\gamma^{\sigma\sigma}X'^M X'^N\\
&\qquad + 2g^2\gamma^{\tau\sigma}\gamma^{\tau\sigma}X'^M X'^N + 2g^2\gamma^{\tau\tau}\gamma^{\tau\sigma}\dot{X}^M X'^N\Big)\\
&= \frac{\gamma^{\tau\sigma}}{\gamma^{\tau\tau}}p^M X'^N + \frac{1}{2g\gamma^{\tau\tau}}\left(p^M p^N - g^2\gamma^{\tau\tau}\gamma^{\sigma\sigma}X'^M X'^N + g^2\gamma^{\tau\sigma}\gamma^{\tau\sigma}X'^M X'^N\right)\\
&= \frac{\gamma^{\tau\sigma}}{\gamma^{\tau\tau}}p^M X'^N + \frac{1}{2g\gamma^{\tau\tau}}\left(p^M p^N - g^2\det(\gamma^{\alpha\beta})X'^M X'^N\right) = \frac{\gamma^{\tau\sigma}}{\gamma^{\tau\tau}}p^M X'^N + \frac{1}{2g\gamma^{\tau\tau}}\left(p^M p^N + g^2 X'^M X'^N\right).
\end{aligned}$$

Looking at (2.3), we can identify the constraints (2.4). In light cone coordinates (1.64). the first term in the first-order form action becomes

$$\begin{aligned}
p_M\dot{X}^M &= p_t\dot{t} + p_\phi\dot{\phi} + p_\mu\dot{x}^\mu\\
&= [(1-a)p_- - p_+]\,[\dot{x}_+ - a\dot{x}_-] + [p_+ + ap_-]\,[\dot{x}_+ + (1-a)\dot{x}_-] + p_\mu\dot{x}^\mu\\
&= (1-a)p_-\dot{x}_+ - a(1-a)p_-\dot{x}_- - p_+\dot{x}_+ + ap_+\dot{x}_- + p_+\dot{x}_+\\
&\qquad + (1-a)p_+\dot{x}_- + ap_-\dot{x}_+ + a(1-a)p_-\dot{x}_- + p_\mu\dot{x}^\mu\\
&= p_+\dot{x}_- + p_-\dot{x}_+ + p_\mu\dot{x}^\mu,
\end{aligned}$$

Similarly, the first constraint turns into

$$\begin{aligned}
C_1 = p_M X'^M &= p_t t' + p_\phi\phi' + p_\mu x'^\mu\\
&= p_+ x'_- + p_- x'_+ + p_\mu x'^\mu,
\end{aligned}$$

and the second becomes

$$C_2 = p_\mu p_\nu G^{\mu\mu} - p_t^2 G_{tt}^{-1} + p_\phi^2 G_{\phi\phi}^{-1} + g^2 x'^\mu x'^\nu G_{\mu\nu} - g^2 t'^2 G_{tt} + g^2\phi'^2 G_{\phi\phi}$$

[18] Note that the spacetime metric G_{MN} is diagonal and thus symmetric.

$$= 2\mathcal{H}_x + \textcircled{p} + \textcircled{x'}.$$

We have isolated the term $\mathcal{H}_\perp$ (2.9) involving the transversal degrees of freedom. Computing the two others terms in C_2, we get

$$\begin{aligned}
\textcircled{p} &= -p_t^2 G_{tt}^{-1} + p_\phi^2 G_{\phi\phi}^{-1} = -\left[(1-a)p_- - p_+\right]^2 G_{tt}^{-1} + \left[p_+ + ap_-\right]^2 G_{\phi\phi}^{-1} \\
&= -(1-a)^2 p_-^2 + 2(1-a)G_{tt}^{-1}p_+p_- - G_{tt}^{-1}p_+^2 + G_{\phi\phi}^{-1}p_+^2 + 2aG_{\phi\phi}^{-1}p_+p_- + a^2G_{\phi\phi}^{-1}p_-^2 \\
&= p_+^2\left[G_{\phi\phi}^{-1} - G_{tt}^{-1}\right] + 2p_+p_-\left[aG_{\phi\phi}^{-1} + (1-a)G_{tt}^{-1}\right] + p_-^2\left[a^2G_{\phi\phi}^{-1} - (1-a)^2G_{tt}^{-1}\right],
\end{aligned}$$

and

$$\begin{aligned}
\textcircled{x'} &= -g^2t'^2G_{tt} + g^2\phi'^2G_{tt} = -g^2\left[x'_+ - ax'_-\right]^2 G_{tt} + g^2\left[x'_+ + (1-a)x'_-\right]^2 G_{\phi\phi} \\
&= -g^2G_{tt}x'^2_+ + 2g^2aG_{tt}x'_+x'_- - g^2a^2G_{tt}x'^2_- + g^2G_{\phi\phi}x'^2_+ + 2g^2(1-a)G_{\phi\phi}x'_+x'_- + g^2(1-a)^2G_{\phi\phi}x'^2_- \\
&= g^2x'^2_+\left[G_{\phi\phi} - G_{tt}\right] + 2g^2x'_+x'_-\left[aG_{tt} + (1-a)G_{\phi\phi}\right] + g^2x'^2_-\left[(1-a)^2G_{\phi\phi} - a^2G_{tt}\right].
\end{aligned}$$

Putting these three terms together, we retrieve (2.8).

B.1.2 Superstring

Substituting this expression for π into the Lagrangian minus the Wess-Zumino term,

$$\begin{aligned}
\mathscr{L} - \mathscr{L}_{\text{WZ}} &= -\text{str}\Big[\pi A_\tau^{(2)} + \frac{\gamma^{\tau\sigma}}{\gamma^{\tau\tau}}\pi A_\sigma^{(2)} - \frac{1}{2g\gamma^{\tau\tau}}\left(\pi^2 + g^2A_\sigma^{(2)}A_\sigma^{(2)}\right)\Big] \\
&= -\text{str}\Big[g\gamma^{\tau\tau}A_\tau^{(2)}A_\tau^{(2)} + g\gamma^{\tau\sigma}A_\sigma^{(2)}A_\tau^{(2)} + g\gamma^{\tau\sigma}A_\tau^{(2)}A_\sigma^{(2)} + g\frac{\gamma^{\tau\sigma}\gamma^{\tau\sigma}}{\gamma^{\tau\tau}}A_\sigma^{(2)}A_\sigma^{(2)} \\
&\qquad - \frac{1}{2g\gamma^{\tau\tau}}\left(g^2\gamma^{\tau\tau}A_\tau^{(2)}A_\tau^{(2)} + 2g^2\gamma^{\tau\tau}\gamma^{\tau\sigma}A_\tau^{(2)}A_\sigma^{(2)} + g^2\gamma^{\tau\sigma}\gamma^{\tau\sigma}A_\sigma^{(2)}A_\sigma^{(2)} + g^2A_\sigma^{(2)}A_\sigma^{(2)}\right)\Big] \\
&= -\text{str}\Big[g\gamma^{\alpha\beta}A_\alpha^{(2)}A_\beta^{(2)} - g\gamma^{\sigma\sigma}A_\sigma^{(2)}A_\sigma^{(2)} + g\frac{\gamma^{\tau\sigma}\gamma^{\tau\sigma}}{\gamma^{\tau\tau}}A_\sigma^{(2)}A_\sigma^{(2)} \\
&\qquad - \frac{g}{2}\gamma^{\tau\tau}A_\tau^{(2)}A_\tau^{(2)} - g\gamma^{\tau\sigma}A_\tau^{(2)}A_\sigma^{(2)} - \frac{g}{2}\frac{\gamma^{\tau\sigma}\gamma^{\tau\sigma}}{\gamma^{\tau\tau}}A_\sigma^{(2)}A_\sigma^{(2)} + \frac{g}{2\gamma^{\tau\tau}}\det(\gamma^{\alpha\beta})A_\sigma^{(2)}A_\sigma^{(2)}\Big] \\
&= -\text{str}\Big[g\gamma^{\alpha\beta}A_\alpha^{(2)}A_\beta^{(2)} - g\gamma^{\sigma\sigma}A_\sigma^{(2)}A_\sigma^{(2)} + g\frac{\gamma^{\tau\sigma}\gamma^{\tau\sigma}}{\gamma^{\tau\tau}}A_\sigma^{(2)}A_\sigma^{(2)} \\
&\qquad - \frac{g}{2}\gamma^{\tau\tau}A_\tau^{(2)}A_\tau^{(2)} - g\gamma^{\tau\sigma}A_\tau^{(2)}A_\sigma^{(2)} - \frac{g}{2}\frac{\gamma^{\tau\sigma}\gamma^{\tau\sigma}}{\gamma^{\tau\tau}}A_\sigma^{(2)}A_\sigma^{(2)} + \frac{g}{2}\gamma^{\sigma\sigma}A_\sigma^{(2)}A_\sigma^{(2)} - \frac{g}{2}\frac{\gamma^{\tau\sigma}\gamma^{\tau\sigma}}{\gamma^{\tau\tau}}A_\sigma^{(2)}A_\sigma^{(2)}\Big] \\
&= -\text{str}\Big[g\gamma^{\alpha\beta}A_\alpha^{(2)}A_\beta^{(2)} - \frac{g}{2}\gamma^{\tau\tau}A_\tau^{(2)}A_\tau^{(2)} - g\gamma^{\tau\sigma}A_\tau^{(2)}A_\sigma^{(2)} - \frac{g}{2}\gamma^{\sigma\sigma}A_\sigma^{(2)}A_\sigma^{(2)}\Big] = -\frac{g}{2}\gamma^{\alpha\beta}\text{str}\big(A_\alpha^{(2)}A_\beta^{(2)}\big),
\end{aligned}$$

which is indeed the kinetic term of the Green-Schwarz Lagrangian (1.32).

B.1.3 Kappa symmetry

To begin we have two identites to prove. Namely,

$$\Sigma_+^{-1}\chi\Sigma_+ = \left(\begin{array}{cc|cc} \mathbb{1}_2 & 0 & 0 & 0 \\ 0 & -\mathbb{1}_2 & 0 & 0 \\ \hline 0 & 0 & \mathbb{1}_2 & 0 \\ 0 & 0 & 0 & -\mathbb{1}_2 \end{array}\right)\left(\begin{array}{cc|cc} 0 & 0 & 0 & a \\ 0 & 0 & b & 0 \\ \hline 0 & b^\dagger & 0 & 0 \\ -a^\dagger & 0 & 0 & 0 \end{array}\right)\left(\begin{array}{cc|cc} \mathbb{1}_2 & 0 & 0 & 0 \\ 0 & -\mathbb{1}_2 & 0 & 0 \\ \hline 0 & 0 & \mathbb{1}_2 & 0 \\ 0 & 0 & 0 & -\mathbb{1}_2 \end{array}\right)$$

$$
= \left(\begin{array}{cc|cc} \mathbb{1}_2 & 0 & 0 & 0 \\ 0 & -\mathbb{1}_2 & 0 & 0 \\ \hline 0 & 0 & \mathbb{1}_2 & 0 \\ 0 & 0 & 0 & -\mathbb{1}_2 \end{array}\right) \left(\begin{array}{cc|cc} 0 & 0 & 0 & -a \\ 0 & 0 & b & 0 \\ \hline 0 & -b^\dagger & 0 & 0 \\ -a^\dagger & 0 & 0 & 0 \end{array}\right) = \left(\begin{array}{cc|cc} 0 & 0 & 0 & -a \\ 0 & 0 & -b & 0 \\ \hline 0 & -b^\dagger & 0 & 0 \\ a^\dagger & 0 & 0 & 0 \end{array}\right) = -\chi,
$$

$$
\Sigma_-^{-1}\chi\Sigma_- = \left(\begin{array}{cc|cc} -\mathbb{1}_2 & 0 & 0 & 0 \\ 0 & \mathbb{1}_2 & 0 & 0 \\ \hline 0 & 0 & \mathbb{1}_2 & 0 \\ 0 & 0 & 0 & -\mathbb{1}_2 \end{array}\right) \left(\begin{array}{cc|cc} 0 & 0 & 0 & a \\ 0 & 0 & b & 0 \\ \hline 0 & b^\dagger & 0 & 0 \\ -a^\dagger & 0 & 0 & 0 \end{array}\right) \left(\begin{array}{cc|cc} -\mathbb{1}_2 & 0 & 0 & 0 \\ 0 & \mathbb{1}_2 & 0 & 0 \\ \hline 0 & 0 & \mathbb{1}_2 & 0 \\ 0 & 0 & 0 & -\mathbb{1}_2 \end{array}\right)
$$

$$
= \left(\begin{array}{cc|cc} -\mathbb{1}_2 & 0 & 0 & 0 \\ 0 & \mathbb{1}_2 & 0 & 0 \\ \hline 0 & 0 & \mathbb{1}_2 & 0 \\ 0 & 0 & 0 & -\mathbb{1}_2 \end{array}\right) \left(\begin{array}{cc|cc} 0 & 0 & 0 & -a \\ 0 & 0 & b & 0 \\ \hline 0 & b^\dagger & 0 & 0 \\ a^\dagger & 0 & 0 & 0 \end{array}\right) = \left(\begin{array}{cc|cc} 0 & 0 & 0 & a \\ 0 & 0 & b & 0 \\ \hline 0 & b^\dagger & 0 & 0 \\ -a^\dagger & 0 & 0 & 0 \end{array}\right) = \chi.
$$

which equivalently imply (2.19). Our next task is to find $A = -\mathfrak{g}^{-1}d\mathfrak{g}$ given by (2.21) and sort it into even and odd elements A_e and A_o such that $A = A_\mathrm{e} + A_\mathrm{o}$. By the product rule,

$$
\begin{aligned}
A &= -\mathfrak{g}^{-1}d\mathfrak{g} = -\mathfrak{g}(\mathbb{X})^{-1}\mathfrak{g}(\chi)^{-1}\Lambda^{-1}d\left(\Lambda\mathfrak{g}(\chi)\mathfrak{g}(\mathbb{X})\right) \\
&= \underbrace{-\mathfrak{g}(\mathbb{X})^{-1}\mathfrak{g}(\chi)^{-1}\Lambda^{-1}d\Lambda\mathfrak{g}(\chi)\mathfrak{g}(\mathbb{X})}_{①} \underbrace{-\mathfrak{g}(\mathbb{X})^{-1}\mathfrak{g}(\chi)^{-1}d\mathfrak{g}(\chi)\mathfrak{g}(\mathbb{X})}_{②} \underbrace{-\mathfrak{g}(\mathbb{X})^{-1}d\mathfrak{g}(\mathbb{X})}_{③}.
\end{aligned}
$$

Let us take care of ① first. In light cone coordinates, the longitudinal matrix $\Lambda(t,\phi)$ is given by

$$
\begin{aligned}
\Lambda(t,\phi) &= \exp\frac{\mathrm{i}}{2}\begin{pmatrix} \Sigma(x_+ - ax_-) & 0 \\ 0 & \Sigma\left(x_+ + (1-a)x_-\right) \end{pmatrix} \\
&= \exp\frac{\mathrm{i}}{2}\begin{pmatrix} \Sigma\left(x_+ + (\frac{1}{2}-a)x_-\right) - \Sigma\frac{1}{2}x_- & 0 \\ 0 & \Sigma\left(x_+ + (\frac{1}{2}-a)x_-\right) + \Sigma\frac{1}{2}x_- \end{pmatrix} \\
&= \exp\frac{\mathrm{i}}{2}\left[\Sigma_+\left(x_+ + (\frac{1}{2}-a)x_-\right) + \frac{1}{2}\Sigma_-x_-\right].
\end{aligned}
$$

In particular this means[19]

$$
\Lambda^{-1}d\Lambda = \frac{\mathrm{i}}{2}\left[\Sigma_+\left(dx_+ + (\frac{1}{2}-a)dx_-\right) + \frac{1}{2}\Sigma_-dx_-\right]
$$

such that we can use the identites (2.20) to find

$$
\begin{aligned}
① &= -\mathfrak{g}(\mathbb{X})^{-1}\mathfrak{g}(\chi)^{-1}\frac{\mathrm{i}}{2}\left[\Sigma_+\left(dx_+ + (\frac{1}{2}-a)dx_-\right) + \frac{1}{2}\Sigma_-dx_-\right]\mathfrak{g}(\chi)\mathfrak{g}(\mathbb{X}) \\
&= -\mathfrak{g}(\mathbb{X})^{-1}\frac{\mathrm{i}}{2}\left[\Sigma_+\mathfrak{g}(\chi)^2\left(dx_+ + (\frac{1}{2}-a)dx_-\right) + \frac{1}{2}\Sigma_-dx_-\right]\mathfrak{g}(\mathbb{X}) \\
&= -\mathfrak{g}(\mathbb{X})^{-1}\left[\frac{\mathrm{i}}{2}\left(dx_+ + (\frac{1}{2}-a)dx_-\right)\Sigma_+(\mathbb{1} + 2\chi^2 + 2\chi\sqrt{\mathbb{1}+\chi^2}) + \frac{\mathrm{i}}{4}dx_-\Sigma_-\right]\mathfrak{g}(\mathbb{X}).
\end{aligned}
$$

The only odd term in ① is clearly the one with the factor $\Sigma_+\chi\sqrt{\mathbb{1}+\chi^2}$ which is the product of one odd element (χ) and even elements ($\Sigma_+, \mathfrak{g}(\mathbb{X}), \mathfrak{g}(\mathbb{X})^{-1}$ and $\sqrt{\mathbb{1}+\chi^2}$). To find ②, we should calculate

$$
\mathfrak{g}(\chi)^{-1}d\mathfrak{g}(\chi) = (-\chi + \sqrt{\mathbb{1}+\chi^2})d(\chi + \sqrt{\mathbb{1}+\chi^2})
$$

[19] The argument in the exponential Λ commutes with its derivative, which is still in terms of Σ matrices.

$$= (\sqrt{\mathbb{1}+\chi^2}d\sqrt{\mathbb{1}+\chi^2} - \chi d\chi)_{\text{even}} + (\sqrt{\mathbb{1}+\chi^2}d\chi - \chi d\sqrt{\mathbb{1}+\chi^2})_{\text{odd}}$$

This is consistent since, under the substitution $\chi \to \sinh\chi$, the above becomes $d\chi$ when $\mathfrak{g}(\chi) = \exp\chi$. Thus

$$\begin{aligned}
② &= -\mathfrak{g}(\mathbb{X})^{-1}\mathfrak{g}(\chi)^{-1}d\mathfrak{g}(\chi)\mathfrak{g}(\mathbb{X}) \\
&= -\mathfrak{g}(\mathbb{X})^{-1}(\sqrt{\mathbb{1}+\chi^2}d\sqrt{\mathbb{1}+\chi^2} - \chi d\chi)_{\text{even}}\mathfrak{g}(\mathbb{X}) - \mathfrak{g}(\mathbb{X})^{-1}(\sqrt{\mathbb{1}+\chi^2}d\chi - \chi d\sqrt{\mathbb{1}+\chi^2})_{\text{odd}}\mathfrak{g}(\mathbb{X})
\end{aligned}$$

Finally, ③ is an even term by definition. Adding the three yields (2.22).

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