Quantising the $\mathrm{AdS}_5 \times S^5$ Superstring

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Abstract

Understanding the spectrum of quantised superstrings in $\mathrm{AdS}_5 \times S^5$ bosonic target space is the key to $\mathcal{N} = 4$ super Yang-Mills theory through the AdS/CFT correspondence. In this review of *Foundations of the $AdS_5 \times S^5$ Superstring, Part I* by Gleb Arutyunov and Sergey Frolov [1], results from the first two chapters are reproduced explicitly. The ultimate objective of this review is to understand the procedure of quantising the $\mathrm{AdS}_5 \times S^5$ superstring perturbatively. First, the Green-Schwarz Lagrangian is introduced in terms of the quotient Lie superalgebra $\mathfrak{psu}(2,2|4)$, and eventually fixed in the light-cone and κ-symmetry gauges. Once gauge-fixed, the model undergoes decompactification in preparation for perturbative quantisation in the large tension limit. The classical superstring in $\mathrm{AdS}_5 \times S^5$ is also shown to be integrable via the construction of a Lax pair which takes values in $\mathfrak{psu}(2,2|4)$.

Acknowledgements

Thank you, Sergey, for your advice and your patience in answering all of my questions about the paper, string theory and other (often unrelated) topics. What you taught me was rarely easy, but learning from you as both an advisor and lecturer has been a pleasure.

I should also thank my coursemates and friends for keeping me tied down to reality during this degree. Very few people have the chance to spend four years learning about the Universe in such good company. Finally, thanks to my family for their continuous support throughout the years which has allowed me to dedicate myself to learning. Unfortunately, Mum, Granda, John, Peter and Tina, this is probably as far as you will get but you have my word that it was an interesting project.

I have read and understood the plagiarism provisions in the General Regulations of the University Calendar for the current year, found at http://www.tcd.ie/calendar. I have also read and understood the guide, and completed the ‘Ready Steady Write’ Tutorial on avoiding plagiarism, located at https://libguides.tcd.ie/academic-integrity/ready-steady-write.

Alexander Farren, 7th April 2024.

Contents

Why $\mathrm{AdS}_5 \times S^5$?

String theory began in the 1960's as an attempt to explain the strong force felt by hadronic particles, which we now describe with quantum chromodynamics (QCD). Boasting several 'revolutions' – periods in physics when the theory took on a new life – string theory has proven itself to be a strong candidate for unifying our understanding of classical gravity with the three fundamental forces of the Standard Model. These are the strong, weak and electromagnetic forces. In the case of bosonic string theory, particles observed in the real world are identified with modes of a fundamental, one-dimensional object called a *string* which vibrates in $D = 26$ spacetime dimensions.

Supersymmetry, a symmetry relating bosonic force particles to fermionic matter particles, was scrutinised since its advent in the 1970/80's but has yet to be experimentally verified, remaining a speculative and controversial topic. However, supersymmetry did provide a remedy to crucial pitfalls of bosonic string theory. For example, particles with negative mass squared $m^2 < 0$ (called tachyons) no longer appeared in the theory and the critical dimension of spacetime (at which string theory can be quantised) went from $D = 26$ down to $D = 10$ [2, 3].

In 1997, the AdS/CFT correspondence [4] came to light and became a central focus for high-energy theorists. In certain limits, this correspondence provides a mathematical connection between a theory of gravity in D-dimensional anti-de Sitter (AdS) space with a $(D-1)$-dimensional conformal field theory (CFT). The most studied of these pairs is

$$\text{Type IIB } \mathrm{AdS}_5 \text{ superstring} \quad \leftrightarrow \quad \mathcal{N} = 4 \text{ Super-Yang-Mills.}$$

While neither side of the duality is directly observed in our world, and supersymmetry is yet a speculative and contended feature of our Universe, there is serious interest in deepening our understanding of what physical features facilitate this correspondence so that we may, for example, one day perform otherwise intractable calculations in the string dual of regular QCD Yang-Mills. The space AdS_5 is not only special by virtue of it being maximally symmetric in the context of general relativity, together with S^5 it is also a maximally *supersymmetric* background for supergravity. Pairing the space with any space other than S^5 does not preserve the full supersymmetry. In the hopes of using AdS/CFT to study $\mathcal{N} = 4$ SYM, we were thus cornered into considering $\mathrm{AdS}_5 \times S^5$.

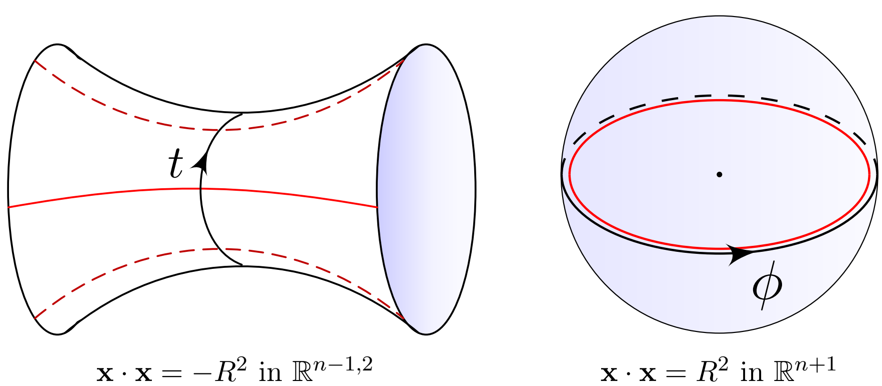


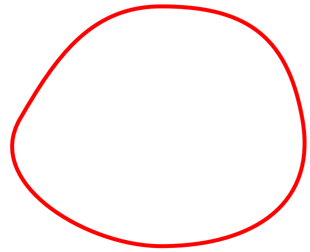
Figure 1. Classical strings (red) on hypersurfaces[1] AdS_n and S^n for $n = 2$.

[1]Because closed timelike curves are not desirable in physics, the AdS surface should be 'unwrapped'. See Figure 1 of [5].

This work is a review of [1] in which many results of the first two chapters are reproduced or corrected, with the end goal of describing the quantised superstring in an $\mathrm{AdS}_5 \times S^5$ target space. The focus of this work being to reproduce results explicitly, lengthy calculations from both chapters are included in full detail but are exiled to the appendices for the reader's convenience. Quite often results which are derived in these appendices are used in the main body.

Chapter 1 sets the scene by introducing superalgebra notation and the Green-Schwarz superstring. This superstring is described by a Lagrangian exhibiting a local fermionic symmetry known as κ-symmetry. We shall derive the symmetry and show its implications for integrability of the model. Various embeddings of the coset space for $\mathrm{AdS}_5 \times S^5$ into the supergroup $SU(2,2|4)$ are presented at the end, with a particular emphasis on the parametrisation which is suitable for the light-cone gauge fixing to follow.

In Chapter 2 the bosonic string is used to illustrate the light-cone gauge and first-order formalism which helps in the transition to Hamiltonian language, and eventually quantisation. The GS Lagrangian is then fixed in the light-cone and κ-symmetry gauges before proceeding in the planar limit to quantisation.



Chapter 1

String sigma model

A sigma model describes a particle or object living on a manifold. We will be studying in detail the model consisting of a single closed superstring propagating in the coset space

$$\frac{PSU(2,2|4)}{SO(4,1)\times SO(5)} = \mathrm{AdS}_5 \times S^5 + \text{ fermions} . \tag{1.1}$$

For this reason, we will sometimes refer to the $\mathrm{AdS}_5 \times S^5$ superstring as the coset sigma model. As previously mentioned, the superstring captures both bosonic and fermionic features. The bosonic modes of the string vibrate in $\mathrm{AdS}_5 \times S^5$ and one can think of their fermionic counterparts as a 'spin' at each point along the string. This is hard to visualise, but then again so is $\mathrm{AdS}_5 \times S^5$.

The qualities of 'bosonic' and 'fermionic' will be encoded into the group structure of matrix blocks entering in the Lagrangian. The slash in $PSU(2,2|4)$ is responsible for this distinction and turns the group into a *supergroup*. In the coming section, we will familiarise ourselves with the language of these superspaces so that we may understand the Lagrangian describing such a supersymmetric string, namely

$$\mathscr{L} = -\frac{T}{2}\Big[\gamma^{\alpha\beta}\,\mathrm{str}\big(A^{(2)}_\alpha A^{(2)}_\beta\big) + \kappa\varepsilon^{\alpha\beta}\,\mathrm{str}\big(A^{(1)}_\alpha A^{(3)}_\beta\big)\Big]. \tag{1.2}$$

This Lagrangian is due to Michael Green and John Schwarz and features a term with the factor κ. This second term is the benefactor of supersymmetry and also provides the model with a vital property known as κ-symmetry. A principal goal of this chapter is elucidating what exactly this symmetry entails and how we can use it to simplify the quantisation procedure.

In later sections the spacetime degrees of freedom of (1.1) will be embedded explicitly into supermatrices entering the Lagrangian which belong to the Lie superalgebra $\mathfrak{su}(2,2|4)$. This will make manifest the various symmetries of the model and will lay a clear path forward for canonical quantisation (or at least a version of it). Through the construction of a certain quantity called a Lax pair, we will end up showing that this model is classically integrable. An interesting connection will follow between this property of integrability, the equations of motion for $\mathscr{L}$ and κ-symmetry.

In the interest of space and time, facts and terminology from bosonic string theory are assumed. The reader is pointed to [2, 3, 6] for an introduction to the subject.

1.1 Super-duper algebra

Here the superalgebra $\mathfrak{su}(2,2|4)$, for which the considered matrix realisation admits a $\mathbb{Z}_4$-grading, and its quotient $\mathfrak{psu}(2,2|4)$ will be introduced. The generators of the bosonic subalgebra of $\mathfrak{su}(2,2|4)$ are constructed explicitly in terms of Dirac gamma matrices in preparation for the coset sigma model, which describes the dynamics of a string on the manifold (1.1).

Matrix realisation of $\mathfrak{su}(2,2|4)$

A Lie supergroup is a $\mathbb{Z}_2$-graded Lie group [7]. That is, the Lie group G is a Lie supergroup if we have $G = G_{\mathbf{0}} \oplus G_{\mathbf{1}}$, with even part $G_{\mathbf{0}}$ and odd part $G_{\mathbf{1}}$ such that any two homogeneous elements $a \in G_{\boldsymbol{\alpha}},\ b \in G_{\boldsymbol{\beta}}$ satisfy $ab \in G_{\boldsymbol{\alpha}+\boldsymbol{\beta}}$ where the degrees $|a| = \boldsymbol{\alpha}$ and $|b| = \boldsymbol{\beta}$ are in the abelian group $\mathbb{Z}_2 = \{\mathbf{0},\mathbf{1}\}$. This way, a product of two odd or two even elements is even, whereas the product of one odd and one even element is itself odd. The corresponding Lie super*algebra* $\mathscr{G} = \ln G = \mathscr{G}^{(0)} \oplus \mathscr{G}^{(1)}$ with homogeneous elements $\{\mathfrak{a},\mathfrak{b},\mathfrak{c},...\}$ is equipped with the Lie bracket $[\ ,\ ]$ satisfying

$$[\mathfrak{a},\mathfrak{b}] = -(-1)^{|\mathfrak{a}||\mathfrak{b}|}[\mathfrak{b},\mathfrak{a}], \tag{1.3}$$

$$[\mathfrak{a},[\mathfrak{b},\mathfrak{c}]] = [[\mathfrak{a},\mathfrak{b}],\mathfrak{c}] + (-1)^{|\mathfrak{a}||\mathfrak{b}|}[\mathfrak{b},[\mathfrak{a},\mathfrak{c}]]. \tag{1.4}$$

As you can see the bracket is antisymmetric unless *both* arguments are odd. For this reason, this bracket will play a very important role when we want to construct matrix commutation relations for bosons, and in the same framework, matrix anti-commutation relations for fermions. In analogy with the group product, the bracket satisfies $[\mathscr{G}^{(\alpha)},\mathscr{G}^{(\beta)}] \subseteq \mathscr{G}^{(\alpha+\beta)}$ modulo $\mathbb{Z}_2$.

The *special linear* Lie superalgebra $\mathfrak{sl}(N_1|N_2) = \mathscr{G}^{(0)} \oplus \mathscr{G}^{(1)}$ over the complex field, with dim $\mathscr{G}^{(0)} = N_1$ and dim $\mathscr{G}^{(1)} = N_2$, consists of square $(N_1+N_2)\times(N_1+N_2)$ matrices of the generic form

$$M = \begin{pmatrix} m & \theta \\ \eta & n \end{pmatrix} \tag{1.5}$$

with vanishing supertrace $\mathrm{str}\big(M\big) \equiv \mathrm{tr}(m) - \mathrm{tr}(n)$. Such matrices M which are diagonal are *even*, while those which are off-diagonal are *odd*. The Lie bracket for $\mathfrak{sl}(N_1|N_2)$ is always the standard matrix commutator. One might ask, what happened to the odd-odd anti-commutator? The matrix entries of θ and η are taken to be Grassmann variables such that, choosing some basis for off-diagonal matrices $\{E_i\}$,

$$[M,M'] \supset [\theta_i E_i, \theta'_j E_j] = \theta_i\theta'_j E_i E_j - \theta'_j\theta_i E_j E_i = \theta_i\theta'_j\{E_i,E_j\}.$$

We see that, indeed the Lie bracket is symmetric if we consider $\{E_i\}$ to be the basis of odd elements.

We will be considering subsuperalgebras of $\mathfrak{sl}(4|4)$ which is itself spanned by supertraceless 8×8 matrices as above, with m,n being even 4×4 matrices and θ,η being odd 4×4 matrices. In addition to being supertraceless, elements M of the *special pseudo-unitary* Lie superalgebra $\mathfrak{su}(2,2|4)$ also satisfy

$$MH + HM^\dagger = 0, \tag{1.6}$$

where

$$H = \begin{pmatrix} \Sigma & 0 \\ 0 & \mathbb{1}_4 \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \tag{1.7}$$

and $\mathbb{1}_n$ is the $n \times n$ identity matrix. Writing out the above conjugation explicitly, this implies

$$M = -HM^\dagger H^{-1} = -\begin{pmatrix} \Sigma & 0 \\ 0 & \mathbb{1}_4 \end{pmatrix} \begin{pmatrix} m^\dagger & \eta^\dagger \\ \theta^\dagger & n^\dagger \end{pmatrix} \begin{pmatrix} \Sigma & 0 \\ 0 & \mathbb{1}_4 \end{pmatrix} = -\begin{pmatrix} \Sigma m^\dagger \Sigma & \Sigma \eta^\dagger \\ \theta^\dagger \Sigma & n^\dagger \end{pmatrix}$$

such that

$$m = -\Sigma m^\dagger \Sigma, \qquad n = -n^\dagger, \qquad \eta = -\theta^\dagger \Sigma. \tag{1.8}$$

Clearly m and n span the unitary Lie algebras $\mathfrak{u}(2,2)$ and $\mathfrak{u}(4)$ respectively. The generator $\mathrm{i}\,\mathbb{1}_8$ of $\mathfrak{u}(1)$ is also an element of $\mathfrak{su}(2,2|4)$, which means the bosonic (even, diagonal) subalgebra of the latter is[2]

$$\mathfrak{su}(2,2|4)_{\text{even}} = \mathfrak{su}(2,2) \oplus \mathfrak{su}(4) \oplus \mathfrak{u}(1) \subset \mathfrak{u}(2,2) \oplus \mathfrak{u}(4) \oplus \mathfrak{u}(1). \tag{1.9}$$

The *quotient* algebra $\mathfrak{psu}(2,2|4)$ is defined as the quotient of $\mathfrak{su}(2,2|4)$ over the $\mathfrak{u}(1)$ generator, i.e.

$$\mathfrak{psu}(2,2|4) \equiv \frac{\mathfrak{su}(2,2|4)}{\mathrm{i}\,\mathbb{1}}. \tag{1.10}$$

Many times throughout the text a complex multiple of $\mathbb{1}_8$ in $\mathfrak{su}(2,2|4)$ will be taken to 0 in $\mathfrak{psu}(2,2|4)$. Elements of this quotient cannot be linearly represented as 8×8 matrices as the identity would be missing.

Writing elements of the bosonic subalgebra $\mathfrak{su}(2,2) \oplus \mathfrak{su}(4) \oplus \mathfrak{u}(1)$, and its complement in $\mathfrak{su}(2,2|4)$ will be crucial when deriving certain properties of the superstring. To this end, we should identify a suitable basis for both $\mathfrak{su}(4)$ and $\mathfrak{su}(2,2)$. We will be using the following representation of Dirac's matrices.

$$\begin{aligned} &\gamma^1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \qquad \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & \mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \\ 0 & -\mathrm{i} & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \end{pmatrix}, \qquad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \\ &\gamma^4 = \begin{pmatrix} 0 & 0 & -\mathrm{i} & 0 \\ 0 & 0 & 0 & \mathrm{i} \\ \mathrm{i} & 0 & 0 & 0 \\ 0 & -\mathrm{i} & 0 & 0 \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \Sigma. \end{aligned} \tag{1.11}$$

They obey the $SO(5)$ Clifford algebra relations

$$\gamma^i \gamma^j + \gamma^j \gamma^i = 2\delta^{ij}\,\mathbb{1}_4 \tag{1.12}$$

for $i,j = 1,\ldots,5$. All of these matrices are hermitian, meaning that $\mathrm{i}\gamma^i$ are elements of $\mathfrak{su}(4)$ since for any $i \leq 5$, with no summation, $(\mathrm{i}\gamma^i)^\dagger \mathrm{i}\gamma^i = \gamma^i\gamma^i = \mathbb{1}_4$ and the determinant of any γ^i or $\mathrm{i}\gamma^i$ is clearly 1. To show $\mathfrak{su}(4) \sim \mathfrak{so}(6)$, we will extend the spinor representation of $\mathfrak{so}(5)$, spanned by $n^{ij} = \frac{1}{4}[\gamma^i, \gamma^j]$ and satisifying

$$[n^{ij}, n^{kl}] = \delta^{jk} n^{il} - \delta^{ik} n^{jl} - \delta^{jl} n^{ik} + \delta^{il} n^{jk} \tag{1.13}$$

for $i, j... \leq 5$. In particular, we add the elements[3] $n^{i6} \equiv \frac{\mathrm{i}}{2}\gamma^i$ such that the above commutation relations are satisfied but this time for $i, j... \leq 6$. To see this, one performs the calculations in A.1 yielding

$$[n^{i6}, n^{kl}] = \left\{ \begin{matrix} -\delta^{6l} n^{ik} & l = 6, \\ +\delta^{6k} n^{il} & k = 6, \\ -\delta^{il} n^{k6} + \delta^{ik} n^{l6} & k, l \neq 6. \end{matrix} \right\} = \delta^{6k} n^{il} - \delta^{ik} n^{6l} - \delta^{6l} n^{ik} + \delta^{il} n^{6k}. \tag{1.14}$$

[2]To ensure elements of the subalgebra are supertraceless, m and n must be separately traceless. So $\mathfrak{u}$ becomes $\mathfrak{su}$.

[3]Looking at the calculations done in the appendix, it should be clear that $-\frac{\mathrm{i}}{2}\gamma^i$ are valid extensions too.

Thus the $\mathfrak{su}(4)$ matrices $\mathrm{i}\gamma^i$ provide a basis for the real vector space $\mathfrak{so}(6)$.

To describe $\mathfrak{su}(2,2)$, we should turn our attention to extending $\mathfrak{so}(4,1)$ instead and show $\mathfrak{so}(4,2) \sim \mathfrak{su}(2,2)$. We now set $m^{ij} = \frac{1}{4}[\gamma^i, \gamma^j]$ for $i, j = 0, ..., 4$ and distinguish $\gamma^0 = \mathrm{i}\gamma^5$. These matrices are taken from the generators of $\mathfrak{su}(2,2) = \mathrm{span}_{\mathbb{R}}\left\{\frac{1}{2}\gamma^i, \frac{\mathrm{i}}{2}\gamma^5\right\}$. The pseudo-orthogonal $\mathfrak{so}(4,1)$ relations are

$$[m^{ij}, m^{kl}] = \eta^{jk}m^{il} - \eta^{ik}m^{jl} - \eta^{jl}m^{ik} + \eta^{il}m^{jk} \tag{1.15}$$

with signature $\eta = \mathrm{diag}(-1,1,1,1,1)$. If we add in the elements $m^{i5} \equiv \frac{1}{2}\gamma^i$, a similar set of calculations in A.1 shows that the above relation is still satisfied for $i, j = 0, ..., 5$ if we set $\eta = \mathrm{diag}(-1,1,1,1,1,-1)$. To summarise, the upper and lower diagonal blocks are respectively spanned by

$$\begin{aligned} \mathfrak{su}(4) \sim \mathfrak{so}(6) &= \mathrm{span}_{\mathbb{R}}\left\{\frac{\mathrm{i}}{2}\gamma^i, \frac{1}{4}[\gamma^i, \gamma^j]\right\}, && i, j = 1, ..., 5, \\ \mathfrak{su}(2,2) \sim \mathfrak{so}(4,2) &= \mathrm{span}_{\mathbb{R}}\left\{\frac{1}{2}\gamma^i, \frac{\mathrm{i}}{2}\gamma^5, \frac{1}{4}[\gamma^i, \gamma^j], \frac{\mathrm{i}}{4}[\gamma^i, \gamma^5]\right\}, && i, j = 1, ..., 4. \end{aligned} \tag{1.16}$$

Finally, $\mathrm{i}\,\mathbb{1}_8$ spans $\mathfrak{u}(1)$ such that these generators together span the bosonic subalgebra $\mathfrak{su}(2,2|4)$.

$\mathbb{Z}_4$-grading

In addition to the $\mathbb{Z}_2$ grading we described above, it turns out that the automorphism group of $\mathfrak{sl}(4|4)$ is such that we can refine the grading to $\mathbb{Z}_4$. If we define the hypercharge Υ and take some generic matrix $M \in \mathfrak{sl}(4|4)$ as

$$\Upsilon = \begin{pmatrix} \mathbb{1}_4 & 0 \\ 0 & -\mathbb{1}_4 \end{pmatrix}, \qquad M = \begin{pmatrix} m & \theta \\ \eta & n \end{pmatrix}, \tag{1.17}$$

then there exists a continuous automorphism $\delta_\rho(M)$ acting as

$$\delta_\rho(M) \equiv \begin{pmatrix} m & \rho\theta \\ \frac{1}{\rho}\eta & n \end{pmatrix} = e^{\frac{1}{2}\Upsilon \ln \rho} M e^{-\frac{1}{2}\Upsilon \ln \rho}. \tag{1.18}$$

Moving on to the finite subgroup of automorphisms, if we define the supertranspose M^{st} of a matrix $M \in \mathfrak{sl}(4|4)$ as

$$M^{st} \equiv \begin{pmatrix} m^t & -\eta^t \\ \theta^t & n^t \end{pmatrix}, \tag{1.19}$$

then we see that $M \to -M^{st}$ is an automorphism of order four. Note that $(M^{st})^{st} = \delta_{-1}(M)$. Similar to this 'minus supertranspose', we will choose the automorphism

$$M \to \Omega(M) \equiv -\mathcal{K}M^{st}\mathcal{K}^{-1} \tag{1.20}$$

to refine the grading to $\mathbb{Z}_4$ where we have defined the matrices

$$\mathcal{K} \equiv \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}, \qquad K \equiv -\gamma^2\gamma^4 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{1.21}$$

Note the definition of $\Omega(M)$ immediately implies $\Omega(M_1M_2) = -\Omega(M_2)\Omega(M_1)$. We start by introducing the notation $\mathscr{G} \equiv \mathfrak{sl}(4|4)$ such that the graded subspaces of the vector space $\mathscr{G}$ are

$$\mathscr{G}^{(k)} \equiv \left\{ M \in \mathscr{G} \mid \Omega(M) = \mathrm{i}^k M \right\}. \tag{1.22}$$

The vector space $\mathscr{G}$ and some generic element $M \in \mathscr{G}$ can be decomposed uniquely with respect to $\Omega(M)$:

$$\begin{aligned} \mathscr{G} &= \mathscr{G}^{(0)} \oplus \mathscr{G}^{(1)} \oplus \mathscr{G}^{(2)} \oplus \mathscr{G}^{(3)}, \\ M &= M^{(0)} + M^{(1)} + M^{(2)} + M^{(3)}. \end{aligned} \tag{1.23}$$

To see that $[\mathscr{G}^{(a)}, \mathscr{G}^{(b)}] \subset \mathscr{G}^{(a+b)}$ modulo $\mathbb{Z}_4$, we can calculate

$$\Omega([M^{(a)}, M^{(b)}]) = -\mathrm{i}^{a+b} M^{(b)} M^{(a)} + \mathrm{i}^{a+b} M^{(a)} M^{(b)} = \mathrm{i}^{a+b} [M^{(a)}, M^{(b)}]. \tag{1.24}$$

According to the above, if we view $M^{(0)}$ and $M^{(2)}$ as even, then $M^{(1)}$ and $M^{(3)}$ would be odd. Given $M \in \mathscr{G} = \mathfrak{sl}(4|4)$, its projections $M^{(k)} \in \mathscr{G}^{(k)}$ can be expressed as

$$M^{(k)} = \frac{1}{4} \left(M + \mathrm{i}^{3k} \Omega(M) + \mathrm{i}^{2k} \Omega^2(M) + \mathrm{i}^k \Omega^3(M) \right) \tag{1.25}$$

since in this case $\Omega(M^{(k)}) = \mathrm{i}^k M^{(k)}$ as required[4]. In fact, the automorphism $\Omega(M)$ restricts to $\mathfrak{su}(2,2|4)$ such that we can relabel $\mathscr{G} = \mathfrak{su}(2,2|4)$ and think of the above decomposition as the $\mathbb{Z}_4$-grading of $\mathfrak{su}(2,2|4)$ with respect to the action of $\Omega(M)$. See A.2 for details of this restriction. Reassuringly, the explicit expressions are diagonal for even components and off-diagonal for odd components:

$$\begin{aligned} M^{(0)} &= \frac{1}{2} \begin{pmatrix} m - K m^t K^{-1} & 0 \\ 0 & n - K n^t K^{-1} \end{pmatrix}, \quad M^{(1)} = \frac{1}{2} \begin{pmatrix} 0 & \theta - \mathrm{i} K \eta^t K^{-1} \\ \theta + \mathrm{i} K \eta^t K^{-1} & 0 \end{pmatrix}, \\ M^{(2)} &= \frac{1}{2} \begin{pmatrix} m + K m^t K^{-1} & 0 \\ 0 & n + K n^t K^{-1} \end{pmatrix}, \quad M^{(3)} = \frac{1}{2} \begin{pmatrix} 0 & \theta + \mathrm{i} K \eta^t K^{-1} \\ \theta - \mathrm{i} K \eta^t K^{-1} & 0 \end{pmatrix}. \end{aligned} \tag{1.26}$$

We know the bosonic subalgebra $\mathfrak{su}(2,2) \oplus \mathfrak{su}(4) \oplus \mathfrak{u}(1) \subset \mathfrak{su}(2,2|4)$ coincides with the even-graded subspace $\mathscr{G}^{(0)} \oplus \mathscr{G}^{(2)} \subset \mathscr{G}$, so there should be a way to express the even components $M^{(k)}$ in terms of the generators of the bosonic algebra (1.16). It is argued in A.2 that the general forms of the even components are in fact linear combinations of the bosonic generators. For real coefficients m_a, n_a and $i, j = 1, ..., 4$, we have

$$M^{(0)} = \begin{pmatrix} m_1^{ij} [\gamma^i, \gamma^j] + \mathrm{i} m_2^i [\gamma^i, \gamma^5] & 0 \\ 0 & n_1^{ij} [\gamma^i, \gamma^j] + n_2^i [\gamma^i, \gamma^5] \end{pmatrix}, \tag{1.27}$$

$$M^{(2)} = \begin{pmatrix} m_3^i \gamma^i + \mathrm{i} m_4 \gamma^5 & 0 \\ 0 & \mathrm{i} n_3^i \gamma^i + \mathrm{i} n_4 \gamma^5 \end{pmatrix}. \tag{1.28}$$

The central element $\mathrm{i}\mathbb{1}_8 \in \mathfrak{u}(1) \subset \mathfrak{su}(2,2|4)$ also occurs in $\mathscr{G}^{(2)}$ since $\Omega(\mathbb{1}_8) = -\mathbb{1}_8$.

1.2 Green-Schwarz superstring

Following our discussion of how to decompose elements of $\mathfrak{psu}(2,2|4)$, we will now introduce the Green-Schwarz Lagrangian density (1.2) describing a closed supersymmetric string in an $\mathrm{AdS}_5 \times S^5$ background and derive its equations of motion. Kappa symmetry (κ-symmetry), a local fermionic symmetry stemming

[4] This follows from $\Omega^4(M) = M$ and $\mathrm{i}^{4k} = 1$.

from the Wess-Zumino term in the Lagrangian, is a property of the model vital for quantisation. We will spend some time deriving the symmetry and exploring its implication for the gauge transformations of fermionic degrees of freedom.

Consider a closed one-dimensional supersymmetric string propagating in an $\mathrm{AdS}_5 \times S^5$ background. Its worldsheet is a cylinder of circumference $2\pi r$ parametrised by the 'time' coordinate τ and spatial coordinate σ such that $-\pi r \leq \sigma \leq \pi r$. These are usually grouped as $(\sigma^\alpha) = (\tau, \sigma)$. The action for such a string is

$$S = \int d\tau \int_{-\pi r}^{\pi r} d\sigma \, \mathscr{L}, \tag{1.29}$$

where $\mathscr{L}$ is the Lagrangian density describing the dynamics. The Lagrangian will be accompanied by a constant prefactor which is the dimensionless string tension $T = R^2/2\pi\alpha'$, whereby R is the radius of AdS_5 and S^5 (see Figure 1) and α' is the Regge slope (e.g. [2, Ch. 2]). This tension T is related to the 't Hooft coupling constant λ of the dual Yang-Mills theory as $T = \sqrt{\lambda}/2\pi$.

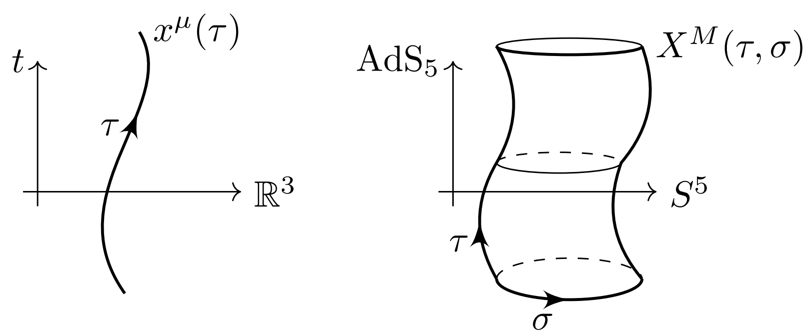


Figure 2. Worldline of a point particle in $\mathbb{R}^{3,1}$ and worldsheet of a closed string in $\mathrm{AdS}_5 \times S^5$.

Lagrangian

We need a few more ingredients to understand (1.2). Let $\mathfrak{g}$ be an element of the supergroup $SU(2,2|4)$. Introduce the matrix one-form taking values in $\mathfrak{su}(2,2|4)$

$$A \equiv -\mathfrak{g}^{-1} d\mathfrak{g} = A^{(0)} + A^{(1)} + A^{(2)} + A^{(3)}, \tag{1.30}$$

where the $\mathbb{Z}_4$-graded elements $A^{(k)} \in \mathscr{G}^{(k)}$ satisfy $\Omega(A^{(k)}) = \mathrm{i}^k A^{(k)}$ and

$$A^{(k)} = \frac{1}{4}\big[A + \mathrm{i}^{3k}\Omega(A) + \mathrm{i}^{2k}\Omega^2(A) + \mathrm{i}^k\Omega^3(A)\big]. \tag{1.31}$$

This one-form has zero-curvature ($dA - A \wedge A = 0$), which in component form translates to

$$\partial_\alpha A_\beta - \partial_\beta A_\alpha - [A_\alpha, A_\beta] = 0. \tag{1.32}$$

In terms of this $\mathfrak{su}(2,2|4)$ element A_α, Green and Schwarz proposed the following Lagrangian density to describe a superstring propagating in the coset (1.1):

$$\mathscr{L} = -\frac{T}{2}\Big[\gamma^{\alpha\beta}\,\mathrm{str}\big(A^{(2)}_\alpha A^{(2)}_\beta\big) + \kappa\varepsilon^{\alpha\beta}\,\mathrm{str}\big(A^{(1)}_\alpha A^{(3)}_\beta\big)\Big], \qquad \alpha, \beta \in \{\tau, \sigma\}. \tag{1.33}$$

The rescaled worldsheet metric $\gamma^{\alpha\beta} = h^{\alpha\beta}/\sqrt{-h} = \gamma^{\beta\alpha}$ is the Weyl-invariant combination of the worldsheet metric $h_{\alpha\beta}$. In general,

$$\gamma_{\alpha\beta} = h_{\alpha\beta}\sqrt{-h}, \qquad \det\big(\gamma_{\alpha\beta}\big) = \det(\gamma^{\alpha\beta}) = (\sqrt{-h})^2/h = -1, \tag{1.34}$$

and in conformal gauge we would set $(\gamma_{\alpha\beta}) = (\gamma^{\alpha\beta}) = \mathrm{diag}(1,-1)$, or equivalently $\gamma_{\tau\tau} = 1, \gamma_{\tau\sigma} = 0$. By convention we take the Levi-Civita symbol to satisfy $\varepsilon^{\tau\sigma} = 1$ where $\varepsilon^{\alpha\beta} = -\varepsilon^{\beta\alpha}$. The parameter κ will play a major role in quantising the model, wherein we will exploit a gauge symmetry called κ-symmetry which is a consequence of $\kappa^2 = 1$. This prefactor κ multiplies what is known as the Wess-Zumino term. Let us assume for now κ is a generic c-number. In that case,

$$\begin{aligned}\mathscr{L}^* &= -\frac{T}{2}\Big[\gamma^{\alpha\beta}\,\mathrm{str}\big(A^{(2)}_\alpha A^{(2)}_\beta\big)^* + \kappa^*\varepsilon^{\alpha\beta}\,\mathrm{str}\big(A^{(1)}_\alpha A^{(3)}_\beta\big)^*\Big] = -\frac{T}{2}\Big[\gamma^{\alpha\beta}\,\mathrm{str}\big(A^{(2)\dagger}_\beta A^{(2)\dagger}_\alpha\big) + \kappa^*\varepsilon^{\alpha\beta}\,\mathrm{str}\big(A^{(3)\dagger}_\beta A^{(1)\dagger}_\alpha\big)\Big]\\ &= -\frac{T}{2}\Big[\gamma^{\alpha\beta}\,\mathrm{str}\big(HA^{(2)}_\beta A^{(2)}_\alpha H^{-1}\big) + \kappa^*\varepsilon^{\alpha\beta}\,\mathrm{str}\big(HA^{(3)}_\beta A^{(1)}_\alpha H^{-1}\big)\Big]\end{aligned}$$

where we used the property $\mathrm{str}\big(M\big) = \mathrm{str}\big(M^t\big)$ and the reality condition for homogeneous elements $A^{(k)\dagger} = -HA^{(k)}H^{-1}$. Clearly for the Lagrangian to be real, we must have $\kappa = \kappa^*$ also.

As for the equations of motion for this Lagrangian, if we define

$$\Lambda^\alpha \equiv T\left[\gamma^{\alpha\beta}A^{(2)}_\beta - \frac{\kappa}{2}\varepsilon^{\alpha\beta}\big(A^{(1)}_\beta - A^{(3)}_\beta\big)\right], \tag{1.35}$$

then the variation of the action with respect to the element $\mathfrak{g}$ takes the form derived in A.3, namely

$$\delta S = -\iint d^2\sigma\ \mathrm{str}\big(\delta A_\alpha \Lambda^\alpha\big) = -\iint d^2\sigma\ \mathrm{str}[\mathfrak{g}^{-1}\delta\mathfrak{g}(\partial_\alpha\Lambda^\alpha - [A_\alpha,\Lambda^\alpha])]. \tag{1.36}$$

Then, $\delta S/\delta\mathfrak{g}$ can be set to zero to find the equations of motion as an element of $\mathfrak{su}(2,2|4)$:

$$\partial_\alpha\Lambda^\alpha - [A_\alpha,\Lambda^\alpha] = \rho\cdot\mathbb{1}_8. \tag{1.37}$$

The above obviously vanishes modulo $\mathrm{i}\,\mathbb{1}_8$ and we will be careful moving forward when working in $\mathfrak{psu}(2,2|4)$ since only the traceless part of the equation of motion will be under consideration. In turn this can be projected onto $\mathscr{G}^{(2)}$ and $\mathscr{G}^{(1,3)}$ to give

$$\gamma^{\alpha\beta}\partial_\alpha A^{(2)}_\beta - \gamma^{\alpha\beta}[A^{(0)}_\alpha, A^{(2)}_\beta] + \frac{\kappa}{2}\varepsilon^{\alpha\beta}\big([A^{(1)}_\alpha, A^{(1)}_\beta] - [A^{(3)}_\alpha, A^{(3)}_\beta]\big) = 0, \tag{1.38}$$

$$\gamma^{\alpha\beta}[A^{(3)}_\alpha, A^{(2)}_\beta] + \kappa\varepsilon^{\alpha\beta}[A^{(2)}_\alpha, A^{(3)}_\beta] = 0, \tag{1.39}$$

$$\gamma^{\alpha\beta}[A^{(1)}_\alpha, A^{(2)}_\beta] - \kappa\varepsilon^{\alpha\beta}[A^{(2)}_\alpha, A^{(1)}_\beta] = 0. \tag{1.40}$$

Treating the worldsheet metric $\gamma_{\alpha\beta}$ as an independent dynamic field and solving $\delta S/\delta\gamma_{\alpha\beta} = 0$ results in the equations of motion

$$\mathrm{str}\big(A^{(2)}_\alpha A^{(2)}_\beta\big) - \frac{1}{2}\gamma_{\alpha\beta}\gamma^{\rho\delta}\,\mathrm{str}\big(A^{(2)}_\rho A^{(2)}_\delta\big) = 0. \tag{1.41}$$

These are famously known as the *Virasoro constraints* and will play a recurring role in this work. Sometimes these are written as $T_{\alpha\beta} = 0$, since the stress-energy tensor is proportional to $\delta S/\delta h^{\alpha\beta}$ [2].

A common idea in the Lagrangian formalism is to identify the global symmetries of the system such that the associated Noether current's conservation simplifies the problem. This quotient factor $SO(4,1)\times SO(5)$

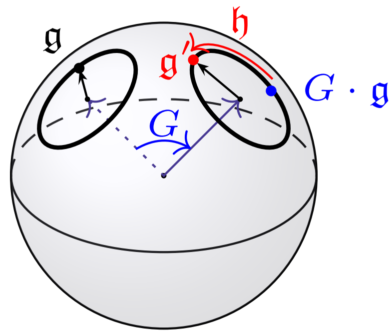


Figure 3. Subjected to the left action/multiplication of a $PSU(2,2|4)$ element G, a chosen coset representative $\mathfrak{g}$ in $PSU(2,2|4)/\big(SO(4,1)\times SO(5)\big)$ becomes some other element $G\mathfrak{g}$ in $PSU(2,2|4)$. To rewrite this element in terms of a coset representative (identified by the black arrow), one must introduce a compensating element $\mathfrak{h}$ in $SO(4,1)\times SO(5)$ such that $G\mathfrak{g} = \mathfrak{g}'\mathfrak{h}$. We interpret this as $G : \mathfrak{g} \to \mathfrak{g}'$.

is the isometry group of $\mathrm{AdS}_5 \times S^5$, whose coordinates will eventually be embedded into the coset element $\mathfrak{g}$. Because of this, we can think of $\mathfrak{g} \in PSU(2,2|4)$ as a coset representative modulo $SO(4,1)\times SO(5)$. Consider in Figure 3 an analogy to $SO(3)$ rotations of a point on a sphere $S^2 \cong SO(3)/SO(2)$, whereby the arrow from the center of the sphere to the point can be spun about itself, corresponding to a compensating $SO(2)$ transformation. As discussed in Chapter 1, the Lagrangian is invariant under $SO(4,1)\times SO(5)$ transformations such that global $PSU(2,2|4)$ transformations act on – and result in – coset representatives $\mathfrak{g}$. This gives rise to a Noether current $J^\alpha = \mathfrak{g}\Lambda_\alpha\mathfrak{g}^{-1}$ associated with the Lagrangian symmetry under left action $\mathfrak{g} \to G\mathfrak{g}$ by global element $G \in PSU(2,2|4)$ since $\mathscr{L}$ only depends on $A = -\mathfrak{g}^{-1}d\mathfrak{g}$. The current J^α is conserved (see (A.31)) thanks to the equations of motion (1.37). More precisely,

$$\partial_\alpha J^\alpha = \mathfrak{g}\left(\partial_\alpha \Lambda^\alpha - [A^\alpha, \Lambda_\alpha]\right)\mathfrak{g}^{-1} = \rho \mathbb{1}_8, \tag{1.42}$$

which tells us only the traceless part of $J^\alpha \in \mathfrak{su}(2,2|4)$ is conserved.

Kappa symmetry

We keep mentioning that the Green-Schwarz Lagrangian enjoys a local fermionic symmetry known as κ-symmetry. In this subsection, shadowed by explicit calculations in A.4, we will derive this symmetry by showing that $\delta_\epsilon \mathscr{L} = 0$ under the right action of a group element e^ϵ in $PSU(2,2|4)$,

$$\mathfrak{g} \to \mathfrak{g}e^\epsilon, \tag{1.43}$$

where $\epsilon = \epsilon(\tau,\sigma)$ is a local fermionic element in $\mathfrak{psu}(2,2|4)$. We start with $A \to A + \delta_\epsilon A$ where

$$\begin{aligned}\delta_\epsilon A &= -(e^{-\epsilon}\mathfrak{g}^{-1})d(\mathfrak{g}e^\epsilon) - A = -e^{-\epsilon}\mathfrak{g}^{-1}d\mathfrak{g}e^\epsilon - e^{-\epsilon}\mathfrak{g}^{-1}\mathfrak{g}de^\epsilon - A \\ &\approx (\mathbb{1}_8 - \epsilon)A(\mathbb{1}_8 + \epsilon) - e^{-\epsilon}e^\epsilon d\epsilon - A = [A,\epsilon] - d\epsilon.\end{aligned} \tag{1.44}$$

To find the change in Lagrangian $\delta_\epsilon \mathscr{L}$ following this group action, we must find the decomposition of $\delta_\epsilon A$. Noting that $[\mathscr{G}^{(a)}, \mathscr{G}^{(b)}] \subseteq \mathscr{G}^{(a+b)}$ and that ϵ is fermionic so that $\epsilon^{(0)} = \epsilon^{(2)} = 0$, we get by inspection

$$\begin{aligned}
\delta_\epsilon A^{(0)} &= [A^{(1)}, \epsilon^{(3)}] + [A^{(3)}, \epsilon^{(1)}], \\
\delta_\epsilon A^{(1)} &= [A^{(0)}, \epsilon^{(1)}] + [A^{(2)}, \epsilon^{(3)}] - d\epsilon^{(1)}, \\
\delta_\epsilon A^{(2)} &= [A^{(1)}, \epsilon^{(1)}] + [A^{(3)}, \epsilon^{(3)}], \\
\delta_\epsilon A^{(3)} &= [A^{(0)}, \epsilon^{(3)}] + [A^{(2)}, \epsilon^{(1)}] - d\epsilon^{(3)}.
\end{aligned} \tag{1.45}$$

In principle we have all the ingredients to determine how the Lagrangian transforms. As derived in A.4,

$$-\frac{2}{T}\delta_\epsilon \mathscr{L} = \delta_\epsilon \gamma^{\alpha\beta} \operatorname{str}\big(A^{(2)}_\alpha A^{(2)}_\beta\big) - 4 \operatorname{str}\left(\mathrm{P}^{\alpha\beta}_+ [A^{(1)}_\beta, A^{(2)}_\alpha]\epsilon^{(1)} + \mathrm{P}^{\alpha\beta}_- [A^{(3)}_\beta, A^{(2)}_\alpha]\epsilon^{(3)}\right), \tag{1.46}$$

where we introduced the projectors $P^{\alpha\beta}_\pm = \frac{1}{2}(\gamma^{\alpha\beta} \pm \kappa\varepsilon^{\alpha\beta})$. It follows that, for $\kappa^2 = 1$,

$$P^{\alpha\delta}_\pm P_{\mp\delta}{}^\beta = \frac{1}{4}(\gamma^{\alpha\delta}\delta^\beta_\delta - \kappa^2 \varepsilon^{\alpha\delta}\gamma_{\delta\mu}\varepsilon^{\mu\beta}) = \frac{1}{4}(\gamma^{\alpha\beta} - \kappa^2\gamma^{\alpha\beta}) = 0.$$

Similarly, $P^{\alpha\delta}_\pm P_{\pm\delta}{}^\beta = P^{\alpha\beta}_\pm$. Compiling identities, we see that the projection operators are orthogonal[5]:

$$P^{\alpha\beta}_\pm + P^{\alpha\beta}_\mp = \gamma^{\alpha\beta}, \qquad P^{\alpha\delta}_\pm P_{\pm\delta}{}^\beta = P^{\alpha\beta}_\pm, \qquad P^{\alpha\delta}_\pm P_{\mp\delta}{}^\beta = 0. \tag{1.47}$$

For any vector V^α we define the projections accordingly; $V^\alpha_\pm = P^{\alpha\beta}_\pm V_\beta = P^{\beta\alpha}_\mp V_\beta$. Returning to the change in the Lagrangian (1.46), the equations of motion (1.39) and (1.40) can be recast in the form

$$P^{\alpha\beta}_+ [A^{(3)}_\alpha, A^{(2)}_\beta] = -P^{\alpha\beta}_- [A^{(2)}_\alpha, A^{(3)}_\beta] = [A^{(3)}_{\delta,+}, A^{(2),\delta}_-] = 0, \tag{1.48}$$

$$P^{\alpha\beta}_+ [A^{(2)}_\alpha, A^{(1)}_\beta] = -P^{\alpha\beta}_- [A^{(1)}_\alpha, A^{(2)}_\beta] = [A^{(1)}_{\delta,+}, A^{(2),\delta}_-] = 0, \tag{1.49}$$

such that

$$-\frac{2}{T}\delta_\epsilon \mathscr{L} = \delta_\epsilon \gamma^{\alpha\beta} \operatorname{str}\big(A^{(2)}_\alpha A^{(2)}_\beta\big) - 4 \operatorname{str}\left([A^{(1)}_{\delta,+}, A^{(2),\delta}_-]\epsilon^{(1)} + [A^{(3)}_{\delta,+}, A^{(2),\delta}_-]\epsilon^{(3)}\right). \tag{1.50}$$

Our goal is now to check that $\delta_\epsilon \gamma^{\alpha\beta}$ can be brought to a suitable form such that $\delta_\epsilon \mathscr{L} = 0$, which would mean this local fermionic transformation constitutes a symmetry of the model. First, one consequence of (1.47) is that $A_{\tau,\pm}$ and $A_{\sigma,\pm}$ are in fact proportional. Indeed, we have

$$0 = P^{\alpha\beta}_\pm A_{\beta,\mp} = P^{\alpha\beta}_\pm P_{\mp\delta}{}^\beta A_\delta = \frac{1}{2}(\gamma^{\alpha\tau} \pm \kappa\varepsilon^{\alpha\tau})A_{\tau,\mp} + \frac{1}{2}(\gamma^{\alpha\sigma} \pm \kappa\varepsilon^{\alpha\sigma})A_{\sigma,\mp}$$

$$\alpha = \tau \implies 0 = \frac{1}{2}(\gamma^{\tau\tau} \pm \kappa\varepsilon^{\tau\tau})A_{\tau,\mp} + \frac{1}{2}(\gamma^{\tau\sigma} \pm \kappa\varepsilon^{\tau\sigma})A_{\sigma,\mp} = \frac{1}{2}\gamma^{\tau\tau}A_{\tau,\mp} + \frac{1}{2}(\gamma^{\tau\sigma} \pm \kappa)A_{\sigma,\mp},$$

and so we see that the different connection projected components $A_{\alpha,\pm}$ are proportional since

$$A_{\tau,\pm} = \frac{\gamma^{\tau\sigma} \pm \kappa}{\gamma^{\tau\tau}} A_{\sigma,\pm}. \tag{1.51}$$

To proceed any further, we will need to specify the forms of $\epsilon^{(1)}$ and $\epsilon^{(3)}$. We ansatz

$$\begin{aligned}
\epsilon^{(1)} &= A^{(2)}_{\alpha,-}\kappa^{(1),\alpha}_+ + \kappa^{(1),\alpha}_+ A^{(2)}_{\alpha,-}, \\
\epsilon^{(3)} &= A^{(2)}_{\alpha,+}\kappa^{(3),\alpha}_- + \kappa^{(3),\alpha}_- A^{(2)}_{\alpha,+}.
\end{aligned} \tag{1.52}$$

[5]They are orthogonal only if $\kappa^2 = 1$, which we will soon learn entails κ-symmetry.

To see that the homogeneity of $\epsilon^{(k)}$ is preserved for $\kappa_{\pm}^{(k),\alpha} \in \mathscr{G}^{(k)}$ and $k = 1, 3$, we calculate

$$\begin{aligned}\Omega(\epsilon^{(k)}) &= -\Omega(\kappa_{\pm}^{(k),\alpha})\Omega(A^{(2)}_{\alpha,\mp}) - \Omega(A^{(2)}_{\alpha,\mp})\Omega(\kappa_{\pm}^{(k),\alpha}) \\ &= -\mathrm{i}^{2+k}\left(\kappa_{\pm}^{(k),\alpha}A^{(2)}_{\alpha,\mp} + A^{(2)}_{\alpha,\mp}\kappa_{\pm}^{(k),\alpha}\right) = \mathrm{i}^k\epsilon^{(k)}.\end{aligned}$$

In addition, we can ask what requirements the matrices $\kappa^{(k)}$ should satisfy such that the fermionic elements $\epsilon^{(k)}$ belong to $\mathfrak{su}(2,2|4)$, i.e. $\epsilon^{(k)} = -H\epsilon^{(k)\dagger}H$. Using the fact that $A^{(2)} \in \mathfrak{su}(2,2|4)$, we find

$$H\epsilon^{(k)\dagger}H = -A^{(2)}_{\alpha,\mp}H\kappa_{\pm}^{(k),\alpha\dagger}H^{-1} - H\kappa_{\pm}^{(k),\alpha\dagger}H^{-1}A^{(2)}_{\alpha,\mp} \stackrel{?}{=} -A^{(2)}_{\alpha,\mp}\kappa_{\pm}^{(k),\alpha} - \kappa_{\pm}^{(k),\alpha}A^{(2)}_{\alpha,\mp}$$

which requires the reality condition $\kappa^{(k)} = H\kappa^{(k)\dagger}H^{-1}$.

The components $A^{(2)}$ can be taken as traceless since $\mathrm{i}\mathbb{1}_8 \in \mathscr{G}^{(2)}$, which does not contribute in the supertrace of the Lagrangian (1.33). Comparing with the generic form (1.28), this means we can assume

$$A^{(2)} = \begin{pmatrix} m^i\gamma^i & 0 \\ 0 & \mathrm{i}n^i\gamma^i \end{pmatrix} + \frac{1}{8}\operatorname{str}\big(\Upsilon A^{(2)}\big), \tag{1.53}$$

where m^i and n^i are real coefficients for $i = 1, ..., 5$ except m^5 which is imaginary. In this way,

$$A^{(2)}_{\alpha,\pm}A^{(2)}_{\beta,\pm} = \begin{pmatrix} m^i_{\alpha,\pm}m^j_{\beta,\pm}\gamma^i\gamma^j & 0 \\ 0 & -n^i_{\alpha,\pm}n^j_{\alpha,\pm}\gamma^i\gamma^j \end{pmatrix}.$$

We just showed that $A_{\tau,\pm}$ and $A_{\sigma,\pm}$ are proportional in (1.51), which means that $m^i_{\alpha,\pm}m^j_{\beta,\pm} = m^j_{\alpha,\pm}m^i_{\beta,\pm}$ no matter α, β and we can rewrite

$$\begin{aligned}A^{(2)}_{\alpha,\pm}A^{(2)}_{\beta,\pm} &= \begin{pmatrix} m^i_{\alpha,\pm}m^j_{\beta,\pm}\frac{1}{2}\{\gamma^i\gamma^j\} & 0 \\ 0 & -n^i_{\alpha,\pm}n^j_{\alpha,\pm}\frac{1}{2}\{\gamma^i\gamma^j\} \end{pmatrix} = \begin{pmatrix} m^i_{\alpha,\pm}m^i_{\beta,\pm}\mathbb{1}_4 & 0 \\ 0 & -n^i_{\alpha,\pm}n^i_{\alpha,\pm}\mathbb{1}_4 \end{pmatrix} \\ &= \frac{1}{2}(m^i_{\alpha,\pm}m^i_{\beta,\pm} + n^i_{\alpha,\pm}n^i_{\alpha,\pm})\Upsilon + \frac{1}{2}(m^i_{\alpha,\pm}m^i_{\beta,\pm} - n^i_{\alpha,\pm}n^i_{\alpha,\pm})\mathbb{1}_8 \\ &= \frac{1}{8}\Upsilon\operatorname{str}\big(A^{(2)}_{\alpha,\pm}A^{(2)}_{\beta,\pm}\big) + \frac{1}{2}(m^i_{\alpha,\pm}m^i_{\beta,\pm} - n^i_{\alpha,\pm}n^i_{\alpha,\pm})\mathbb{1}_8.\end{aligned} \tag{1.54}$$

Substituting our expressions for $\epsilon^{(k)}$ and these newly found properties of $A^{(2)}$, we find in A.4 that

$$\begin{aligned}-\frac{2}{T}\delta_\epsilon\mathscr{L} &= \delta_\epsilon\gamma^{\alpha\beta}\operatorname{str}\big(A^{(2)}_\alpha A^{(2)}_\beta\big) - \frac{1}{2}\operatorname{str}\big(A^{(2)}_{\alpha,-}A^{(2)}_{\beta,-}\big)\operatorname{str}\big(\Upsilon[\kappa_+^{(1),\beta}, A_+^{(1),\alpha}]\big) \\ &\qquad - \frac{1}{2}\operatorname{str}\big(A^{(2)}_{\alpha,+}A^{(2)}_{\beta,+}\big)\operatorname{str}\big(\Upsilon[\kappa_-^{(3),\beta}, A_-^{(3),\alpha}]\big) \\ &= \operatorname{str}\big(A^{(2)}_\alpha A^{(2)}_\beta\big)\left(\delta_\epsilon\gamma^{\alpha\beta} - \frac{1}{2}\operatorname{tr}\Big([\kappa_+^{(1),\beta}, A_+^{(1),\alpha}] - [\kappa_-^{(3),\beta}, A_-^{(3),\alpha}]\Big)\right).\end{aligned} \tag{1.55}$$

We used $\operatorname{str}\big(\Upsilon M\big) = \operatorname{tr}(M)$ for $M \in \mathfrak{psu}(2,2|4)$. Although it may seem like the trace whose argument is a commutator should vanish, the matrix commutator is symmetric since it is acting on odd matrices $\kappa^{(1,3)}$ and $A^{(1,3)}$. This explicitly shows the GS Lagrangian is invariant under this local fermionic transformation $\mathfrak{g} \to \mathfrak{g}\exp\epsilon(\tau,\sigma)$ provided

$$\delta_\epsilon\gamma^{\alpha\beta} = \frac{1}{2}\operatorname{tr}\Big([\kappa_+^{(1),\alpha}, A_+^{(1),\beta}] + [\kappa_-^{(3),\alpha}, A_-^{(3),\beta}]\Big). \tag{1.56}$$

This form of the variation shows explicitly that $\delta_\epsilon \gamma^{\alpha\beta}$ is a real tensor since, according to the reality conditions of κ and A, we have

$$(\delta_\epsilon \gamma^{\alpha\beta})^\dagger = -\frac{1}{2}\operatorname{tr}\left(-H[\kappa_+^{(1),\alpha}, A_+^{(1),\beta}]H^{-1} - H[\kappa_-^{(3),\alpha}, A_-^{(3),\beta}]H^{-1}\right) = \delta_\epsilon \gamma^{\alpha\beta}.$$

Crucially, such a form of $\delta_\epsilon \gamma^{\alpha\beta}$ is obtained if and only if $P_\pm^{\alpha\beta}$ are orthogonal, which tells us that κ-symmetry is obeyed if and only if $\kappa^2 = 1$. Thinking back to when we showed the parameter must be real, this means $\kappa = \pm 1$ is the condition for κ-symmetry.

Kappa symmetry gauge freedom

Now that we know the κ-symmetry transformations are in fact a symmetry of the Lagrangian for $\kappa = \pm 1$, we can ask if any fermionic degrees of freedom can be reduced as a result a corresponding gauge freedom.

Throughout Chapter 2, we will be employing the light-cone gauge in which we identify a time direction t along the longitudinal component of AdS_5 and an angle ϕ around the equator of S^5 (see Figure 1). The bosonic algebras $\mathfrak{so}(4,1)$, with distinguished element $\mathrm{i}\gamma^5$ corresponding to t, and $\mathfrak{so}(5)$ correspond to AdS_5 and S^5 respectively. For the moment, we can ignore the transversal dynamics (anything other than t and ϕ) such that the component $A^{(2)}$ has the generic form

$$A^{(2)} = \begin{pmatrix} \mathrm{i}x\gamma^5 & 0 \\ 0 & \mathrm{i}y\gamma^5 \end{pmatrix}$$

where x and y are linear combinations of t and ϕ. This is a valid assumption since any element in $\mathfrak{so}(5)$ can be brought to γ^5 by an $\mathfrak{su}(4)$ transformation, e.g. $\gamma^i \to (\mathrm{i}\gamma^5)(\mathrm{i}\gamma^i)\gamma^i = -\gamma^5$. If we work on-shell, i.e. when the equations of motion are satisfied, then the Virasoro constraints must be enforced. They are equivalent to

$$\operatorname{str}\big(A^{(2)}_{\alpha,-}A^{(2)}_{\beta,-}\big) = 0 \implies x^\alpha_\pm x^\beta_\pm = y^\alpha_\pm y^\beta_\pm.$$

In particular, this is satisfied by $y = x$. If we recall (1.51), the element $\epsilon^{(1)}$ (1.52) can be rewritten as

$$\epsilon^{(1)} = A^{(2)}_{\tau,-}\varkappa + \varkappa A^{(2)}_{\tau,-}, \qquad \varkappa = \kappa_+^{(1),\tau} - \frac{\gamma^{\tau\tau}}{\gamma^{\tau\sigma}+\kappa}\kappa_+^{(1),\sigma} = \begin{pmatrix} 0 & \varkappa_1 \\ \varkappa_2 & 0 \end{pmatrix}. \tag{1.57}$$

Substituting the above generic $A^{(2)}$ gives us

$$\epsilon^{(1)} = \mathrm{i}x_{\tau,-}\begin{pmatrix} 0 & \Sigma\varkappa_1 + \varkappa_1\Sigma \\ \Sigma\varkappa_2 + \varkappa_2\Sigma & 0 \end{pmatrix} \equiv 2\mathrm{i}x_{\tau,-}\begin{pmatrix} 0 & \varepsilon \\ -\varepsilon^\dagger\Sigma & 0 \end{pmatrix}$$

where we defined $2\varepsilon = \Sigma\varkappa_1 + \varkappa_1\Sigma$ and used the $\mathfrak{su}(2,2|4)$ fermionic reality condition $\varkappa_1^\dagger = -\varkappa_2\Sigma$ in

$$\Sigma\varkappa_2 + \varkappa_2\Sigma = \Sigma\varkappa_1^\dagger\Sigma - \varkappa_1^\dagger = -2\varepsilon^\dagger\Sigma.$$

If we let $(\varkappa_1)_{ij} = \varkappa_{ij}$ for entries $i,j = 1,...,4$, we find

$$\varepsilon = \begin{pmatrix} \varkappa_{11} & \varkappa_{12} & 0 & 0 \\ \varkappa_{21} & \varkappa_{22} & 0 & 0 \\ 0 & 0 & -\varkappa_{33} & -\varkappa_{34} \\ 0 & 0 & -\varkappa_{43} & -\varkappa_{43} \end{pmatrix}. \tag{1.58}$$

Thus, $\epsilon^{(1)}$ depends on 8 real fermionic parameters, namely the 8 entries $\varkappa_{ij}$ in the matrix ε . A similar discussion holds for $\epsilon^{(3)}$ which also depends on 8 fermionic parameters. All together, we can eliminate 16 fermionic degrees of freedom with $\epsilon^{(1)}$ and $\epsilon^{(3)}$ such that any odd element χ can be reduced to

$$\chi = \left(\begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & \bullet & \bullet \\ 0 & 0 & 0 & 0 & 0 & 0 & \bullet & \bullet \\ 0 & 0 & 0 & 0 & \bullet & \bullet & 0 & 0 \\ 0 & 0 & 0 & 0 & \bullet & \bullet & 0 & 0 \\ \hline 0 & 0 & \bullet & \bullet & 0 & 0 & 0 & 0 \\ 0 & 0 & \bullet & \bullet & 0 & 0 & 0 & 0 \\ \bullet & \bullet & 0 & 0 & 0 & 0 & 0 & 0 \\ \bullet & \bullet & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right). \tag{1.59}$$

Here the bullets indicate entries of the matrix realisation of χ which cannot be gauged away by κ-symmetry transformations. (This follows from the expressions for $\delta_\epsilon A^{(1)}$ and $\delta_\epsilon A^{(3)}$ in (1.45) which imply one can find ϵ such that the above form is fulfilled.) There are in fact 16 *real* degrees of freedom. It may seem like χ only has 8, since for any odd matrix the upper block θ and lower block η are related by the fermionic $\mathfrak{su}(2,2|4)$ reality condition $\eta = -\theta^\dagger \Sigma$ (1.8). However, the fact that we took the superalgebra $\mathfrak{sl}(4|4)$ over the complex field means each entry has two real parameters. This fermionic gauge freedom will prove to be *extremely* useful in Chapter 2 when we fix the above gauge before proceeding to quantisation of the light-cone Hamiltonian.

1.3 Integrability of classical superstrings

This section is a small detour from quantisation. A main gust in the sail of research into integrability of various $\mathrm{AdS}_5 \times S^5$ models was the presentation of a Lax representation of the equations of motion which is tantamount to showing the model is solvable, as we will see.

First, we will review the general concept of integrability and apply it to the principal chiral model to illustrate how one might construct a zero-curvature Lax representation from conserved currents. We will then show the Green-Schwarz string sigma model we just discussed is integrable by constructing such a flat Lax representation of the equations of motion. Interestingly, integrability of the model with this particular choice of Lax pair is in some way equivalent to the Virasoro constraints and necessitates κ-symmetry of the Lagrangian.

Classical integrability

In a physical context, integrability refers to the possibility of 'integrating' the equations of motion so as to find a solution to the problem at hand. Consider the following system of partial differential equations

$$\frac{\partial \Psi}{\partial \sigma} = L_\sigma(\sigma, \tau, z)\Psi, \tag{1.60}$$
$$\frac{\partial \Psi}{\partial \tau} = L_\tau(\sigma, \tau, z)\Psi, \tag{1.61}$$

where Ψ is a vector of dimension q and $L_\sigma,\ L_\tau$ are $q \times q$ matrices which all depend on a spectral parameter z taking values in $\mathbb{C}^2$. If we differentiate (1.60) with respect to τ and (1.61) with respect to σ, we get

$$\frac{\partial^2 \Psi}{\partial \tau \partial \sigma} = \partial_\tau L_\sigma(\sigma, \tau, z)\Psi + L_\sigma(\sigma, \tau, z)\partial_\tau \Psi,$$

$$\frac{\partial^2\Psi}{\partial\sigma\partial\tau} = \partial_\sigma L_\tau(\sigma,\tau,z)\Psi + L_\tau(\sigma,\tau,z)\partial_\sigma\Psi.$$

If we now substitute in the original equations for $\partial_\tau\Psi$ and $\partial_\sigma\Psi$, and equate the second order derivatives,

$$\partial_\tau L_\sigma - \partial_\sigma L_\tau - [L_\tau, L_\sigma] = 0. \tag{1.62}$$

This can be reformulated as the zero-curvature condition for connections L_α and $\alpha = \sigma,\tau$;

$$\partial_\alpha L_\beta - \partial_\beta L_\alpha - [L_\alpha, L_\beta] = 0. \tag{1.63}$$

If these connections satisfy (1.63) for all values of the spectral parameter z, then L_α are called *Lax connections* while (1.63) is the Lax representation of the integrable system of partial differential equations. We define the monodromy matrix $T(z)$ as the path-ordered exponential of the Lax connection L_σ

$$T(z) = \overleftarrow{\exp}\int_0^{2\pi} d\sigma\ L_\sigma(\tau,\sigma,z). \tag{1.64}$$

It can be shown (see A.5) that

$$\partial_\tau T(z) = [L_\tau(0,\tau,z), T(z)]. \tag{1.65}$$

This equation implies that the eigenvalues $\{\mu_i\}$ of the matrix $T(z)$ are constant in worldsheet time. It follows by considering the trace of $T^n(z)$, whose τ derivative is the trace of $[L_\tau(0,\tau,z), T^n(z)]$ and thus vanishes. Assuming $T(z)$ is diagonalisable, it follows that

$$\partial_\tau \operatorname{tr} T(z) = \partial_\tau \operatorname{tr} T^2(z) = \ldots = 0 \implies \sum \partial_\tau\mu_i = \sum \partial_\tau\mu_i^2 = \ldots = 0 \tag{1.66}$$

up to the dimension q of $T(z)$. The eigenvalues are integrals of motion, which means the conservation laws of the system are encoded in $T(z)$, thus motivating the identification of the Lax pair L_α of the model.

Explicit example – the principal chiral model

To gain some insight before trying our hand at the Green-Schwarz superstring, we will find a Lax pair for a simpler model. Remembering the one-form $A_\alpha = -\mathfrak{g}^{-1}\partial_\alpha\mathfrak{g}$, the action for the principal chiral model reads

$$S = -\frac{1}{2}\iint d^2\sigma\ \gamma^{\alpha\beta}\operatorname{tr}\bigl(\partial_\alpha\mathfrak{g}\mathfrak{g}^{-1}\partial_\beta\mathfrak{g}\mathfrak{g}^{-1}\bigr) = -\frac{1}{2}\iint d^2\sigma\ \gamma^{\alpha\beta}\operatorname{tr}\bigl(A_\alpha A_\beta\bigr). \tag{1.67}$$

In this case, $\mathfrak{g} = \mathfrak{g}(\tau,\sigma)$ is some generic local Lie group element. To find the variation of the action δS with respect to $\mathfrak{g}$ and thus the equations of motion, we need to find[6]

$$\frac{1}{2}\operatorname{tr}\delta(\gamma^{\alpha\beta}A_\alpha A_\beta) = \gamma^{\alpha\beta}\operatorname{tr}\bigl(\delta A_\alpha A_\beta\bigr). \tag{1.68}$$

Substituting the expression (A.20) for δA_α,

$$\begin{aligned}\frac{1}{2}\gamma^{\alpha\beta}\operatorname{tr}\delta(A_\alpha A_\beta) &= \gamma^{\alpha\beta}\operatorname{tr}\bigl[-\mathfrak{g}^{-1}\delta\mathfrak{g}A_\alpha A_\beta - \mathfrak{g}^{-1}\partial_\alpha\delta\mathfrak{g}A_\beta\bigr]\\ &= \gamma^{\alpha\beta}\operatorname{tr}\bigl[\delta\mathfrak{g}\partial_\alpha A_\beta\mathfrak{g}^{-1}\bigr] = -\gamma^{\alpha\beta}\operatorname{tr}\bigl[\mathfrak{g}^{-1}\delta\mathfrak{g}\partial_\alpha(\mathfrak{g}^{-1}\partial_\beta\mathfrak{g})\bigr]\end{aligned} \tag{1.69}$$

[6] In this step we treat $\gamma_{\alpha\beta}$ as a constant as we are implicitly finding the variation of S with respect to $\mathfrak{g}$ only.

so that $\delta S/\delta\mathfrak{g} = 0$ implies the equations of motion $\partial_\alpha(\gamma^{\alpha\beta}\mathfrak{g}^{-1}\partial_\beta\mathfrak{g}) = 0$. This is because the trace of this derivative does not necessarily vanish as the supertrace of elements of $\mathfrak{su}(2,2|4)$ would. We can manipulate (1.69) to find the same equations of motion in a different form,

$$\begin{aligned}\frac{1}{2}\gamma^{\alpha\beta}\,\mathrm{tr}\,\delta(A_\alpha A_\beta) &= \gamma^{\alpha\beta}\,\mathrm{tr}\left[\mathfrak{g}^{-1}\delta\mathfrak{g}\mathfrak{g}^{-1}\partial_\alpha\mathfrak{g}\mathfrak{g}^{-1}\partial_\beta\mathfrak{g} - \mathfrak{g}^{-1}\delta\mathfrak{g}\mathfrak{g}^{-1}\partial_\alpha\partial_\beta\mathfrak{g}\right]\\ &= \gamma^{\alpha\beta}\,\mathrm{tr}\left[\delta\mathfrak{g}\mathfrak{g}^{-1}\partial_\alpha(\partial_\beta\mathfrak{g}\mathfrak{g}^{-1})\right]\end{aligned} \tag{1.70}$$

which this time implies $\partial_\alpha(\gamma^{\alpha\beta}\partial_\beta\mathfrak{g}\mathfrak{g}^{-1}) = 0$. Putting this all together, the equations of motion are

$$\partial_\alpha(\gamma^{\alpha\beta}\mathfrak{g}^{-1}\partial_\beta\mathfrak{g}) = 0 = \partial_\alpha(\gamma^{\alpha\beta}\partial_\beta\mathfrak{g}\mathfrak{g}^{-1}), \tag{1.71}$$

and they can be conveniently written in terms of the (corrected) left and right currents

$$A_l^\alpha = \gamma^{\alpha\beta}\mathfrak{g}^{-1}\partial_\beta\mathfrak{g}, \qquad A_r^\alpha = -\gamma^{\alpha\beta}\partial_\beta\mathfrak{g}\mathfrak{g}^{-1} \tag{1.72}$$

as

$$\partial_\alpha A_l^\alpha = 0 = \partial_\alpha A_r^\alpha. \tag{1.73}$$

A_r is called the right (Noether) current, and A_l the left current because of their invariance under right and left action of $\mathfrak{g}$ by a constant group element G. In particular the left current coincides with our previous definition of A_α up to a sign.

The flatness of Lax pairs in this model must be invariant under σ and τ coordinate reparametrisation. This is because the Lax flatness represents the equations of motion which are themselves reparametrisation-invariant. (One can see this by looking at the action or simply (1.71).) It follows that the Lax connections L_α must then be one-forms. To see this, let L_α be a k-form, i.e. $L_\alpha \in \Omega^k$ and look at the zero-curvature condition (1.63)

$$\underbrace{\partial_\alpha L_\beta - \partial_\alpha L_\beta}_{\text{in }\Omega^{1+k}} - \underbrace{[L_\alpha, L_\beta]}_{\text{in }\Omega^{2k}} = 0. \tag{1.74}$$

For this equality to hold under reparametrisation, both terms must transform with the same overall prefactor, which means they are of the same tensor type. In other words, $\Omega^{1+k} = \Omega^{2k}$ so $k = 1$ and L_α is a co-vector or one-form. Consequently, we introduce the Lax connections

$$L_\alpha = \ell_1 A_\alpha + \ell_2\gamma_{\alpha\beta}\varepsilon^{\beta\rho}A_\rho \tag{1.75}$$

where ℓ_1, ℓ_2 are parameters to be determined and A is either A^r or A^l. It is natural to construct the Lax connections in terms of the currents, as the latter appear in the equations of motion and the Lax representation (1.63) would involve taking their derivative. In two dimensions we can recast the flatness condition (1.63) as

$$2\varepsilon^{\alpha\beta}\partial_\alpha L_\beta - \varepsilon^{\alpha\beta}[L_\alpha, L_\beta] = 0. \tag{1.76}$$

To show this we sum over contracted indices $\alpha = \tau, \sigma$ with convention $\varepsilon^{\tau\sigma} = +1$ and get

$$\begin{aligned}2\varepsilon^{\alpha\beta}\partial_\alpha L_\beta - \varepsilon^{\alpha\beta}[L_\alpha, L_\beta] &= 2(\partial_\tau L_\sigma - \partial_\sigma L_\tau) - [L_\tau, L_\sigma] + [L_\sigma, L_\tau]\\ &= 2(\partial_\tau L_\sigma - \partial_\sigma L_\tau - [L_\tau, L_\sigma]) = 0.\end{aligned}$$

Using the identity $\varepsilon^{\alpha\beta}\gamma_{\beta\rho}e^{\rho\delta} = \gamma^{\alpha\delta}$, we substitute (1.75) into (1.76) which reduces to

$$0 = 2\varepsilon^{\alpha\beta}\partial_\alpha(\ell_1 A_\beta + \ell_2\gamma_{\beta\mu}\varepsilon^{\mu\rho}A_\rho) - \varepsilon^{\alpha\beta}[\ell_1 A_\alpha + \ell_2\gamma_{\alpha\mu}\varepsilon^{\mu\rho}A_\rho, \ell_1 A_\beta + \ell_2\gamma_{\beta\nu}\varepsilon^{\nu\delta}A_\delta]$$

$$\begin{aligned}
&= 2\ell_1\varepsilon^{\alpha\beta}\partial_\alpha A_\beta + 2\ell_2\partial_\alpha(\varepsilon^{\alpha\beta}\gamma_{\beta\mu}\varepsilon^{\mu\rho}A_\rho) - \ell_1^2\varepsilon^{\alpha\beta}[A_\alpha, A_\beta] - \ell_1\ell_2\varepsilon^{\alpha\beta}\gamma_{\beta\nu}\varepsilon^{\nu\delta}[A_\alpha, A_\delta] \\
&\quad - \ell_1\ell_2\varepsilon^{\alpha\beta}\gamma_{\alpha\mu}\varepsilon^{\mu\rho}[A_\rho, A_\beta] - \ell_2^2\varepsilon^{\alpha\beta}\gamma_{\alpha\mu}\varepsilon^{\mu\rho}\gamma_{\beta\nu}\varepsilon^{\nu\delta}[A_\rho, A_\delta] \\
&= 2\ell_1\varepsilon^{\alpha\beta}\partial_\alpha A_\beta + 2\ell_2\partial_\alpha A^\alpha + \varepsilon^{\alpha\beta}(\ell_2^2 - \ell_1^2)[A_\alpha, A_\beta].
\end{aligned} \tag{1.77}$$

The second term vanishes because of the equations of motion $\partial_\alpha A^\alpha = 0$ (1.73). Let us now show the zero-curvature condition for the left and right currents (1.72) by using $\delta\mathfrak{g}^{-1} = -\mathfrak{g}^{-1}\delta\mathfrak{g}\mathfrak{g}^{-1}$. For the left,

$$\begin{aligned}
\partial_\alpha A^l_\beta = \partial_\alpha(\mathfrak{g}^{-1}\partial_\beta\mathfrak{g}) &= -\mathfrak{g}^{-1}\partial_\alpha\mathfrak{g}\mathfrak{g}^{-1}\partial_\beta\mathfrak{g} + \mathfrak{g}^{-1}\partial_\alpha\partial_\beta\mathfrak{g} \\
\implies \partial_\alpha A^l_\beta - \partial_\beta A^l_\alpha &= [\mathfrak{g}^{-1}\partial_\beta\mathfrak{g}, \mathfrak{g}^{-1}\partial_\alpha\mathfrak{g}] = [A^l_\beta, A^l_\alpha].
\end{aligned}$$

And for the right,

$$\begin{aligned}
\partial_\alpha A^r_\beta = -\partial_\alpha(\partial_\beta\mathfrak{g}\mathfrak{g}^{-1}) &= -\partial_\alpha\partial_\beta\mathfrak{g}\mathfrak{g}^{-1} + \partial_\beta\mathfrak{g}\mathfrak{g}^{-1}\partial_\alpha\mathfrak{g}\mathfrak{g}^{-1} \\
\implies \partial_\alpha A^r_\beta - \partial_\beta A^r_\alpha &= [\partial_\beta\mathfrak{g}\mathfrak{g}^{-1}, \partial_\alpha\mathfrak{g}\mathfrak{g}^{-1}] = [A^r_\beta, A^r_\alpha].
\end{aligned}$$

We can summarise these zero-curvature conditions into one;

$$\partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta] = 0. \tag{1.78}$$

Note that the sign of the commutator is determined by which current carries the minus sign (in this convention, it is the right current). Returning to (1.77), the newly-found flatness condition (1.78) implies

$$\begin{aligned}
2\ell_1\varepsilon^{\alpha\beta}\partial_\alpha A_\beta + \varepsilon^{\alpha\beta}(\ell_2^2 - \ell_1^2)[A_\alpha, A_\beta] &= 2\ell_1\varepsilon^{\alpha\beta}\partial_\alpha A_\beta - \varepsilon^{\alpha\beta}(\ell_2^2 - \ell_1^2)(\partial_\alpha A_\beta - \partial_\beta A_\alpha) \\
0 &= 2\varepsilon^{\alpha\beta}\big(\ell_1\partial_\alpha A_\beta - \frac{1}{2}(\ell_2^2 - \ell_1^2)(\partial_\alpha A_\beta - \partial_\beta A_\alpha)\big) \\
0 &= 2\varepsilon^{\alpha\beta}\big(\ell_1 - (\ell_2^2 - \ell_1^2)\big)\partial_\alpha A_\beta.
\end{aligned} \tag{1.79}$$

For (1.79) to vanish, we must then have $\ell_1^2 - \ell_2^2 + \ell_1 = 0$ for both $A^{l,r}$. Given ℓ_2, this equation has two solutions for ℓ_1. Introducing the spectral parameter z, these solutions are $\ell_2 = z/(1-z^2)$ and either

$$\ell_1 = +\frac{z^2}{1-z^2} \qquad \text{assigned to } A = A^l, \text{ or} \tag{1.80}$$

$$\ell_1 = -\frac{1}{1-z^2} \qquad \text{assigned to } A = A^r. \tag{1.81}$$

The explicit sign of ℓ_2 can always be attributed to the value of z, so is not fixed. Substituting in these values for ℓ_i, we obtain the left and right Lax connections

$$L^l_\alpha = +\frac{z^2}{1-z^2}A^l_\alpha + \frac{z}{1-z^2}\gamma_{\alpha\beta}\varepsilon^{\beta\rho}A^l_\rho, \tag{1.82}$$

$$L^r_\alpha = -\frac{1}{1-z^2}A^r_\alpha + \frac{z}{1-z^2}\gamma_{\alpha\beta}\varepsilon^{\beta\rho}A^r_\rho. \tag{1.83}$$

We finish by showing that the connections L^l and L^r are connected by the gauge transformation

$$L^r = hL^lh^{-1} + dhh^{-1}, \tag{1.84}$$

when $h = \mathfrak{g}$. We can show (1.84) component-wise by employing the expression (1.83), which becomes

$$\mathfrak{g}L^l_\alpha\mathfrak{g}^{-1} = \mathfrak{g}\big(\frac{z^2}{1-z^2}\mathfrak{g}^{-1}\partial_\alpha\mathfrak{g} - \frac{z}{1-z^2}\varepsilon_{\alpha\beta}\mathfrak{g}^{-1}\partial^\beta\mathfrak{g}\big)\mathfrak{g}^{-1}$$

$$
\begin{aligned}
&= +\frac{z^2}{1-z^2}\partial_\alpha \mathfrak{g}\mathfrak{g}^{-1} - \frac{z}{1-z^2}\varepsilon_{\alpha\beta}\partial^\beta \mathfrak{g}\mathfrak{g}^{-1} \\
\implies \mathfrak{g}L^l_\alpha \mathfrak{g}^{-1} + \partial_\alpha \mathfrak{g}\mathfrak{g}^{-1} &= (1+\frac{z^2}{1-z^2})\partial_\alpha \mathfrak{g}\mathfrak{g}^{-1} - \frac{z}{1-z^2}\varepsilon_{\alpha\beta}\partial^\beta \mathfrak{g}\mathfrak{g}^{-1} \\
&= -\frac{1}{1-z^2}A^l_\alpha + \frac{z}{1-z^2}\varepsilon_\alpha{}^\beta A^l_\beta = L^r_\alpha.
\end{aligned}
$$

In this sense, they represent the 'same' integrability. We have just shown how one would go about constructing a Lax pair for a simple action. In principle, there is no protocol for constructing such L_α other than guessing quantities (such as these one-forms A_α) for which the flatness condition is natural. The constituents A_α of the superstring sigma model Lagrangian indeed satisfy a zero-curvature of their own. We will now investigate this promising feature.

Lax pair

In the last example we found the Lax representation of the equations of motion for the principal chiral model action by writing the Lax connections in terms of conserved one-form currents. For our superstrings in $\mathrm{AdS}_5 \times S^5$ with Lagrangian density (1.33)

$$
\mathscr{L} = -\frac{T}{2}\Big[\gamma^{\alpha\beta}\,\mathrm{str}\big(A^{(2)}_\alpha A^{(2)}_\beta\big) + \kappa\varepsilon^{\alpha\beta}\,\mathrm{str}\big(A^{(1)}_\alpha A^{(3)}_\beta\big)\Big],
$$

the one-forms A_α are not quite conserved themselves but do satisfy a zero-curvature condition. To find a Lax representation of the string equations of motion, one should analogously ansatz a $\mathbb{Z}_4$-graded one-form

$$
L_\alpha = \ell_0 A^{(0)}_\alpha + \ell_1 A^{(2)}_\alpha + \ell_2 \gamma_{\alpha\beta}\varepsilon^{\beta\rho}A^{(2)}_\rho + \ell_3 A^{(1)}_\alpha + \ell_4 A^{(3)}_\alpha \tag{1.85}
$$

and then try to determine the parameters ℓ_i by imposing zero-curvature (1.76). The projections $\mathscr{G}^{(k)}$ of the zero-curvature condition are found and separately set to zero in A.6 to obtain the following requirements:

$$
\begin{aligned}
\mathscr{G}^{(0)} = 0 &\implies \ell_0 = 1, \qquad \ell_1^2 - \ell_2^2 = 1, \qquad \ell_3\ell_4 = 1, \\
\mathscr{G}^{(2)} = 0 &\implies \frac{\ell_3^2 - \ell_1}{\ell_2} = -\kappa, \qquad \frac{\ell_4^2 - \ell_1}{\ell_2} = \kappa, \\
\mathscr{G}^{(1)}, \mathscr{G}^{(3)} = 0 &\implies \frac{\ell_1\ell_4 - \ell_3}{\ell_2\ell_4} = \kappa, \qquad \frac{\ell_4 - \ell_1\ell_3}{\ell_2\ell_3} = \kappa.
\end{aligned} \tag{1.86}
$$

Some algebra shows that these requirements imply $\kappa^2 = 1$, which is not a big ask of the model whose local fermionic symmetry begs for the condition. In other words, κ-symmetry is required (by the Lax representation parameters ℓ_i) for our original Lax pair ansatz (1.85) to describe the string described by the Lagrangian density (1.33). This is not quite the statement that integrability requires κ-symmetry. In principle, if one found a Lax pair such that it can satisfy the zero-curvature condition by being a one-form but without $\kappa^2 = 1$, then integrability would hold independently of the symmetry. To date, no such Lax pair has been found.

Integrability and symmetries

We will see that certain gauge transformations leave the flatness of any Lax connections unchanged. Since the Lagrangian (and hence the physics) benefits from κ-symmetry, it should follow that integrability of the model is preserved under such transformations (1.43). In particular, the transformed Lax connections

$L'_\alpha = L_\alpha + \delta_\epsilon L_\alpha$ are shown to be gauge transformations, thus preserving flatness and integrability.

As shown in A.7, the Lax zero-curvature condition (1.63) is invariant under the gauge transformation

$$L_\alpha \to L'_\alpha = hL_\alpha h^{-1} + \partial_\alpha h h^{-1}. \tag{1.87}$$

In fact, if $h = \exp\Lambda \in G$ is a group element for $\Lambda \in \mathscr{G}$, then the transformation is equivalently defined by

$$\delta L_\alpha = [L_\alpha, \Lambda] - \partial_\alpha\Lambda. \tag{1.88}$$

The above variation is explicitly shown to preserve flatness after (A.67). Under κ-symmetry transformations with $\epsilon = \epsilon^{(1)}$ for example, the change in the ansatz Green-Schwarz Lax connections (1.85) is found in A.7 to be

$$\delta_\epsilon L_\alpha = [L_\alpha, \Lambda] - \partial_\alpha\Lambda - 2\ell_2\ell_3\varepsilon_{\alpha\beta}[A_-^{(2),\beta}, \epsilon^{(1)}] + \ell_2\varepsilon_{\alpha\beta}\Big(2[A_+^{(1),\beta}, \epsilon^{(1)}] + \delta_\epsilon\gamma^{\beta\delta}A_\delta^{(2)}\Big) \tag{1.89}$$

for $\Lambda = \ell_3\epsilon^{(1)} \in \mathfrak{su}(2,2|4)$. This is of the form $\delta_\epsilon L_\alpha = [L_\alpha, \Lambda] - \partial_\alpha\Lambda - c_\alpha$. If one can show that the extra term c_α vanishes, i.e.

$$c_\alpha = 2\ell_2\ell_3\varepsilon_{\alpha\beta}\underbrace{[A_-^{(2),\beta}, \epsilon^{(1)}]}_{I_1^\beta} - \ell_2\varepsilon_{\alpha\beta}\Big(2\underbrace{[A_+^{(1),\beta}, \epsilon^{(1)}]}_{I_2^\beta} + \delta_\epsilon\gamma^{\beta\delta}A_\delta^{(2)}\Big) = 0, \tag{1.90}$$

then that would equate the κ-symmetry transformation $\delta_\epsilon L_\alpha$ to the typical gauge transformation (1.89) for $\Lambda = \ell_3\epsilon^{(1)} \in \mathfrak{su}(2,2|4)$. That is precisely what is shown in A.7 by proving that the two terms containing $I_{1,\alpha}$ and $I_{2,\alpha}$ separately vanish. In particular, thanks to the proportionality of two projected components (1.51), the term $I_{1,\alpha}$ can be reduced to

$$I_{1,\alpha} = \frac{1}{8}\,\mathrm{str}\big(A_{\alpha,-}^{(2)}A_{\beta,-}^{(2)}\big)[\Upsilon, \kappa_+^{(1),\beta}]. \tag{1.91}$$

where $\kappa_+^{(1)}$ comes from the ansatz (1.52). Some careful manipulation shows an important relation between $I_{1,\alpha}$ and the Virasoro constraints (1.41) for $\kappa = \pm 1$:

$$\mathrm{str}\big(A_{\alpha,-}^{(2)}A_{\beta,-}^{(2)}\big) = 0 \iff \mathrm{str}\big(A_\alpha^{(2)}A_\beta^{(2)}\big) - \frac{1}{2}\gamma_{\alpha\beta}\gamma^{\mu\nu}\,\mathrm{str}\big(A_\mu^{(2)}A_\nu^{(2)}\big) = 0. \tag{1.92}$$

The second term $I_{2,\alpha}$ can be heavily simplified using arguments pertaining to the bosonic structure of $\mathfrak{su}(2,2|4)$ such that

$$\varepsilon_{\alpha\beta}\left(2I_2^\beta + \delta_\epsilon\gamma^{\beta\delta}A_\delta^{(2)}\right) = \varepsilon_{\alpha\beta}\left(-\frac{1}{2}\,\mathrm{str}\big(\Upsilon[\kappa_+^{(1),\beta}, A_+^{(1),\delta}]\big)A_{\delta,-}^{(2)} + \delta_\epsilon\gamma^{\beta\delta}A_\delta^{(2)}\right). \tag{1.93}$$

Looking at the expression (1.56) for $\delta_\epsilon\gamma^{\alpha\beta}$ when $\epsilon^{(3)} = 0$, the above vanishes.

In summary, $\kappa = \pm 1$ was shown to imply (1.56) in 1.2 and the equivalence (1.92) in A.7. In turn, the former implies the $I_{2,\alpha}$ term vanishes while the latter tells us $I_{1,\alpha} = 0$. Thus, κ-symmetry directly implies the extra term c_α drops out of the variation $\delta_\epsilon L_\alpha$ of the Lax connections (as long as the Virasoro constraints are satsfied), such that only a gauge transformation is leftover and the flatness is preserved. This is pictorially summarised in Figure 4.

$$\begin{tikzcd}
& & 2I_2^\beta + \delta_\epsilon \gamma^{\beta\delta} A_\delta^{(2)} = 0 \arrow[dr, Rightarrow] & \\
\text{Integrability} \arrow[r, dashed, "?"] & \kappa = \pm 1 \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & \begin{array}{l}\text{Integrability preserved under} \\ \kappa\text{-symmetry transformations}\end{array} \\
& & I_{1,\alpha} = 0 \iff \text{Virasoro} \arrow[ur, Rightarrow] &
\end{tikzcd}$$

Figure 4. A schema highlighting the relationship between the symmetries and integrability of the Green-Schwarz superstring on $\mathrm{AdS}_5 \times S^5$, which is a consequence of the zero-curvature of L_α given in (1.85).

We note that diffeomorphisms of the worldsheet coordinates σ^α of the type $\boldsymbol{\sigma} \to \boldsymbol{\sigma} = \tilde{\boldsymbol{\sigma}} + \mathbf{f}$ induce a change in the Lax pairs given by the expression derived in A.7:

$$\delta L_\alpha = f^\beta \partial_\beta L_\alpha + L_\beta \partial_\alpha f^\beta = [L_\beta f^\beta, L_\alpha] + \partial_\alpha (L_\beta f^\beta). \tag{1.94}$$

This is a gauge transformation of the form we saw before ($[L_\alpha, \Lambda] - \partial_\alpha \Lambda$) with parameter $\Lambda = -L_\beta f^\beta$, and is also the Lie derivative of L_α along the vector field $\mathbf{f}$. We have two integrability-preserving diffeomorphism freedoms $\tilde{\sigma}^\alpha$. These are not equivalent to the reparametrisation invariance of σ^α, but rather reflect the fact that integrability is highly dependent on the choice of coordinates (unlike the physics of the system). It is a weaker statement than actual Lagrangian invariance.

To conclude this section on integrability, we return to the gauge transformation (1.87). If we set $h = \mathfrak{g} \in PSU(2,2|4)$ and introduce the dual current $\tilde{A} = \mathfrak{g} A \mathfrak{g}^{-1} = -d\mathfrak{g}\mathfrak{g}^{-1}$ with homogeneous components $\tilde{A}^{(k)} \in \mathscr{G}^{(k)}$, then the new Lax connection takes the form

$$\begin{aligned}
L'_\alpha &= \ell_0 \tilde{A}_\alpha^{(0)} + \ell_1 \tilde{A}_\alpha^{(2)} + \ell_2 \gamma_{\alpha\beta} \varepsilon^{\beta\rho} \tilde{A}_\rho^{(2)} + \ell_3 \tilde{A}_\alpha^{(1)} + \ell_4 \tilde{A}_\alpha^{(3)} - \tilde{A}_\alpha \\
&= \ell'_0 \tilde{A}_\alpha^{(0)} + \ell'_1 \tilde{A}_\alpha^{(2)} + \ell'_2 \gamma_{\alpha\beta} \varepsilon^{\beta\rho} \tilde{A}_\rho^{(2)} + \ell'_3 \tilde{A}_\alpha^{(1)} + \ell'_4 \tilde{A}_\alpha^{(3)}
\end{aligned}$$

The shifted Lax parameters ℓ'_i can be expressed in terms of a spectral parameter while fulfilling (1.86) as

$$\ell'_0 = 0, \quad \ell'_1 = \frac{(1-z^2)^2}{2z^2}, \quad \ell'_2 = -\frac{1}{2\kappa}, \quad \ell'_3 = z - 1, \quad \ell'_4 = \frac{1}{z} - 1. \tag{1.95}$$

We can expand the new connection in around $w = 1 - z$ to leading order in w and we get

$$L_\alpha = \frac{2w}{\kappa} \mathcal{L}_\alpha + \mathcal{O}(w^2), \qquad \mathcal{L}_\alpha = \gamma_{\alpha\beta} \varepsilon^{\beta\delta} \tilde{A}_\delta^{(2)} + \frac{\kappa}{2} (\tilde{A}_\alpha^{(1)} - \tilde{A}_\alpha^{(3)}). \tag{1.96}$$

This expansion was confirmed using Mathematica (see Figure 7). Because we can vary w at will, the zero-curvature of $L_\alpha(w)$ should be fulfilled at each order in w. This implies

$$\partial_\alpha \mathcal{L}_\beta - \partial_\beta \mathcal{L}_\alpha = 0 \implies \partial_\alpha (\varepsilon^{\alpha\beta} \mathcal{L}_\beta) = 0. \tag{1.97}$$

This is no surprise since, recalling the expression (1.35) for Λ^α,

$$T \varepsilon^{\alpha\beta} \mathcal{L}_\alpha = \mathfrak{g} \Lambda^\alpha \mathfrak{g}^{-1} = J^\alpha \tag{1.98}$$

where J^α is the conserved Noether current (1.42) associated with $PSU(2,2|4)$ transformations.

1.4 Strings in coset space

In this section we will see how to include the spacetime coordinates of $\mathrm{AdS}_5 \times S^5$ in the Lagrangian, which so far has featured $\mathfrak{su}(2,2|4)$ matrices A_α with implicit dependence on the worldsheet coordinates.

Coset parametrisation

We first start by embedding the 5 unconstrained coordinates $\{\phi, y^i\}$ for $i = 1, ..., 4$ of S^5 into $\mathbb{R}^6$ by introducing 6 real coordinates Y^A for $A = 1, ..., 6$. Note $|y|^2 \equiv y^i y^i$ is *not* constant. These Y^A are

$$\begin{gathered} Y^1 + \mathrm{i}Y^2 = \frac{y^1 + \mathrm{i}y^2}{1 + |y|^2/4}, \qquad Y^3 + \mathrm{i}Y^4 = \frac{y^3 + \mathrm{i}y^4}{1 + |y|^2/4}, \\ Y_5 + \mathrm{i}Y_6 = \frac{1 - |y|^2/4}{1 + |y|^2/4} e^{\mathrm{i}\phi}. \end{gathered} \tag{1.99}$$

The metric induced on S^5 by this embedding into flat space is easily found (see A.8) by taking the modulus squared of the above expressions:

$$ds^2\big|_{S^5} = dY^A dY^A\big|_{S^5} = \Big(\frac{1 - |y|^2/4}{1 + |y|^2/4}\Big)^2 (d\phi)^2 + \frac{dy^i dy^i}{(1 + |y|^2/4)^2}. \tag{1.100}$$

Similarly, the embedding of AdS_5 with coordinates $\{t, z^i\}$ for $i = 1, ..., 4$ into $\mathbb{R}^6$ prescribed by

$$\begin{gathered} Z^1 + \mathrm{i}Z^2 = \frac{z^1 + \mathrm{i}z^2}{1 - |z|^2/4}, \qquad Z^3 + iZ^4 = \frac{z^3 + \mathrm{i}z^4}{1 - |z|^2/4}, \\ Z^0 + \mathrm{i}Z^5 = \frac{1 + |z|^2/4}{1 - |z|^2/4} e^{\mathrm{i}t}, \end{gathered} \tag{1.101}$$

and with the signature $\eta_{AB} = \mathrm{diag}(-1, 1, 1, 1, 1, -1)$, induces the metric

$$ds^2\big|_{\mathrm{AdS}_5} = \eta_{AB} dZ^A dZ^B\big|_{\mathrm{AdS}_5} = -\Big(\frac{1 + |z|^2/4}{1 - |z|^2/4}\Big)^2 (dt)^2 + \frac{1}{(1 - |z|^2/4)^2} dz^i dz^i. \tag{1.102}$$

If we group the coordinates z^i and y^i into one x^μ, for $\mu = 1, ..., 8$, then the $\mathrm{AdS}_5 \times S^5$ metric becomes diagonal:

$$ds^2\big|_{\mathrm{AdS}_5 \times S^5} \equiv -G_{tt}(dt)^2 + G_{\phi\phi}(d\phi)^2 + G_{\mu\mu} dx^\mu dx^\mu \tag{1.103}$$

where it is understood that $G_{\mu\mu} dx^\mu dx^\mu = \sum_{i=1}^4 G_{zz} dz^i dz^i + \sum_{i=1}^4 G_{yy} dy^i dy^i$ and the Lorentzian signature $\mathrm{AdS}_5 \times S^5$ metric entries are

$$G_{tt} = \Big(\frac{1 + |z|^2/4}{1 - |z|^2/4}\Big)^2, \qquad G_{\phi\phi} = \Big(\frac{1 - |y|^2/4}{1 + |y|^2/4}\Big)^2, \qquad G_{zz} = \frac{1}{(1 - |z|^2/4)^2}, \qquad G_{yy} = \frac{1}{(1 + |y|^2/4)^2}. \tag{1.104}$$

We group $G_{ii} = G_{zz}$ and $G_{(i+4)(i+4)} = G_{yy}$ for $i = 1, ...4$. To get rid of closed timelike curves, we extend the angle coordinate t to the real line. We can make this extension because nowhere else in the metric does t appear. In technical terms, this means we are considering the universal cover $\widetilde{\mathrm{AdS}_5 \times S^5}$.

The time has come to put some life into these $\mathfrak{psu}(2,2|4)$ matrices. In 1.1 we understood that projections $M = M^{(2)} \in \mathscr{G}^{(2)}$ can take the arbitrary form (1.28). We can choose the coefficients of M such that

$$M = \frac{1}{2}\begin{pmatrix} z^i\gamma^i + \mathrm{i}t\gamma^5 & 0 \\ 0 & \mathrm{i}y^i\gamma^i + \mathrm{i}\phi\gamma^5 \end{pmatrix}. \tag{1.105}$$

for $i = 1, ..., 4$. The separation of the bosonic upper block, which corresponds to $\mathfrak{su}(2,2) \sim \mathfrak{so}(4,2) \sim \mathrm{AdS}_5$, and lower block $\mathfrak{su}(4) \sim \mathfrak{so}(6) \sim S^5$ is all too natural. The most obvious way to embed this bosonic element from $\mathfrak{su}(2,2|4)$ into $SU(2,2|4)$ is to exponentiate it. This leads us to define an embedding $\mathfrak{g}$ of the coset space $PSU(2,2|4)/\big(SO(4,1) \times SO(5)\big) \supset \mathrm{AdS}_5 \times S^5$ into $SU(2,2|4)$ as

$$\mathfrak{g} = \mathfrak{g}_{\mathfrak{f}}(\chi)\mathfrak{g}_{\mathfrak{b}}(t, \phi, x^\mu), \tag{1.106}$$

comprised of a fermionic element $\mathfrak{g}_{\mathfrak{f}}$ and a bosonic element $\mathfrak{g}_{\mathfrak{b}}$ of the form

$$\mathfrak{g}_{\mathfrak{b}} = \exp\frac{1}{2}\begin{pmatrix} \mathrm{i}t\gamma^5 + z^i\gamma^i & 0 \\ 0 & \mathrm{i}\phi\gamma^5 + \mathrm{i}y^i\gamma^i \end{pmatrix}, \qquad \mathfrak{g}_{\mathfrak{f}} = \exp\chi = \exp\begin{pmatrix} 0 & \Theta \\ -\Theta^\dagger\Sigma & 0 \end{pmatrix}. \tag{1.107}$$

We saw that the left action of a group element $G \in PSU(2,2|4)$ on a coset representative $\mathfrak{g}$ should result in $\mathfrak{g}'\mathfrak{h}$ for another coset representative $\mathfrak{g}'$ and a compensating element $\mathfrak{h} \in SO(4,1) \times SO(5)$. In the case of a purely bosonic global transformation $G \in SU(2,2) \times SU(4)$,

$$G\mathfrak{g} = G\mathfrak{g}_{\mathfrak{f}}G^{-1}G\mathfrak{g}_{\mathfrak{b}} \equiv G\mathfrak{g}_{\mathfrak{f}}G^{-1}\mathfrak{g}_{\mathfrak{b}}'\mathfrak{h}. \tag{1.108}$$

By using the power series representation of the exponential $\mathfrak{g}_{\mathfrak{f}}$, we see

$$G\mathfrak{g}_{\mathfrak{f}}G^{-1} = G(\mathbb{1}_8 + \chi + \frac{1}{2}\chi^2 + \frac{1}{6}\chi^3 + \ldots)G^{-1} = \exp G\chi G^{-1}. \tag{1.109}$$

This means the left action of G on $\mathfrak{g}$ induces the adjoint action of G on fermionic degrees of freedom found in χ. What are the consequences of this property for supersymmetry? Suppose $G : \chi \to \chi + \delta_\epsilon\chi = \chi + \epsilon$. To find $\delta_\epsilon\mathfrak{g}_{\mathfrak{b}}$, we substitute into (1.108) and $\mathfrak{g}$ becomes $G\mathfrak{g}$, or

$$e^\epsilon e^\chi \mathfrak{g}_{\mathfrak{b}} \overset{\mathrm{BCH}}{=} e^{\chi+\epsilon+\frac{1}{2}[\epsilon,\chi]+\mathcal{O}(\epsilon^2)}\mathfrak{g}_{\mathfrak{b}} = e^{\chi+\delta_\epsilon\chi}\mathfrak{g}_{\mathfrak{b}}'\mathfrak{h}.$$

Factoring out the $e^{\chi+\epsilon}$ from the left, for a 'small' compensating element $\mathfrak{h} \approx \mathbb{1}_8 + \delta\mathfrak{h}$ this means

$$\mathfrak{g}_{\mathfrak{b}} + \frac{1}{2}[\epsilon,\chi]\mathfrak{g}_{\mathfrak{b}} \approx (\mathfrak{g}_{\mathfrak{b}} + \delta_\epsilon\mathfrak{g}_{\mathfrak{b}})(\mathbb{1}_8 + \delta\mathfrak{h}) \approx \mathfrak{g}_{\mathfrak{b}} + \delta_\epsilon\mathfrak{g}_{\mathfrak{b}} + \mathfrak{g}_{\mathfrak{b}}\delta\mathfrak{h} \implies \delta_\epsilon\mathfrak{g}_{\mathfrak{b}} = \frac{1}{2}[\epsilon,\chi]\mathfrak{g}_{\mathfrak{b}} - \mathfrak{g}_{\mathfrak{b}}\delta\mathfrak{h}. \tag{1.110}$$

A better way to define the bosonic variation, instead of $\delta_\epsilon\mathfrak{g}_{\mathfrak{b}} = \mathfrak{g}_{\mathfrak{b}}' - \mathfrak{g}_{\mathfrak{b}}$ is $\delta_\epsilon\mathfrak{g}_{\mathfrak{b}} = (\mathfrak{g}_{\mathfrak{b}}' - \mathfrak{g}_{\mathfrak{b}})\mathfrak{g}_{\mathfrak{b}}^{-1}$ such that

$$\delta_\epsilon\mathfrak{g}_{\mathfrak{b}} = \frac{1}{2}[\epsilon,\chi] - \mathfrak{g}_{\mathfrak{b}}\delta\mathfrak{h}\mathfrak{g}_{\mathfrak{b}}^{-1}. \tag{1.111}$$

Here, $\mathfrak{g}_{\mathfrak{b}} \equiv e^{\chi_{\mathfrak{b}}} \to \mathfrak{g}_{\mathfrak{b}}' \equiv \mathfrak{g}_{\mathfrak{b}}e^{\chi_{\mathfrak{b}}+\delta_\epsilon\mathfrak{g}_{\mathfrak{b}}}$ which is better than the definition in [1] as the change $\delta_\epsilon\mathfrak{g}_{\mathfrak{b}}$ is now a bosonic $\mathfrak{psu}(2,2|4)$ element, so that it is comparable to $\delta_\epsilon\chi$. Note there is now an adjoint transformation on the element $\mathfrak{h} = e^{\delta\mathfrak{h}}$. Either way we see the consequence of the linear transformation of fermions: since χ is expressible in terms of bosonic degrees of freedom, it must be periodic in σ due to the intrinisc periodicity of the $\mathrm{AdS}_5 \times S^5$ spacetime coordinates. (Going all the way around the worldsheet should bring you back to the same point in target space.) In particular, we can quantify this with an integer *winding number* m for the S^5 equator angle such that

$$\phi(\pi r) - \phi(-\pi r) = 2\pi m.$$

Another valid parametrisation of the coset representative $\mathfrak{g}$ is of the form

$$\mathfrak{g} = \Lambda(t,\phi)\mathfrak{g}(\chi)\mathfrak{g}(\mathbb{X}) \tag{1.112}$$

where we split the bosonic degrees of freedom from $\mathfrak{g}_\mathfrak{b}$ into

$$\Lambda(t,\phi) = \exp\begin{pmatrix} \frac{\mathrm{i}}{2}t\gamma^5 & 0 \\ 0 & \frac{\mathrm{i}}{2}\phi\gamma^5 \end{pmatrix} \qquad \text{and} \qquad \mathbb{X} = \begin{pmatrix} \frac{1}{2}z^i\gamma^i & 0 \\ 0 & \frac{\mathrm{i}}{2}y^i\gamma^i \end{pmatrix} \tag{1.113}$$

for $i = 1, ..., 4$. We can go between (1.106) and (1.112) by the change $\chi \to \Lambda(t,\phi)\chi\Lambda(t,\phi)^{-1}$. Note that now the periodic boundary conditions of χ with respect to σ have changed. If we express the fermions as (1.107), then the redefinition entails

$$\Theta \to \Theta' = e^{\frac{\mathrm{i}}{2}(\phi - t)\gamma^5}\Theta$$

since $\Lambda^{-1}(t,\phi) = \Lambda(-t,-\phi)$. Therefore the new boundary condition is

$$\Theta'(\pi r) = e^{\mathrm{i}\pi m\gamma^5} e^{\frac{\mathrm{i}}{2}(\phi(-\pi r) - t(-\pi r))\gamma^5}\Theta = e^{\mathrm{i}\pi m\gamma^5}\Theta'(-\pi r).$$

Using $(\gamma^5)^2 = \mathbb{1}_4$, it is not difficult to show $e^{\mathrm{i}m\pi\gamma^5} = (-1)^m$ such that fermions have even or odd periodicity depending on the parity of the winding number m. With this expression for $\mathbb{X}$, it is also possible to construct an alternative embedding $\mathfrak{g}(\mathbb{X}) = \sqrt{\frac{1+\mathbb{X}}{1-\mathbb{X}}}$ such that the bilinear form $\mathrm{str}[(\mathfrak{g}_\mathfrak{b}^{-1}d\mathfrak{g}_\mathfrak{b})^2]$ reduces to the spacetime metric (1.103). This particular choice for $\mathfrak{g}(\mathbb{X})$ is expressed

$$\mathfrak{g}(\mathbb{X}) = \begin{pmatrix} \frac{1}{\sqrt{1-z^2/4}}[\mathbb{1}_4 + \frac{1}{2}z^i\gamma^i] & 0 \\ 0 & \frac{1}{\sqrt{1+y^2/4}}[\mathbb{1}_4 + \frac{\mathrm{i}}{2}y^i\gamma^i] \end{pmatrix}, \tag{1.114}$$

as derived in A.8. This will be the parameterisation we use moving forward into Chapter 2.

Linearly realised bosonic symmetries

With (1.112), time and angle shifts can be generated by left action of a group element $G = \Lambda(\delta t, \delta\phi)$ since

$$G \cdot \mathfrak{g} = \Lambda(\delta t, \delta\phi)\Lambda(t,\phi)\mathfrak{g}(\chi)\mathfrak{g}(\mathbb{X}) = \Lambda(t + \delta t, \phi + \delta\phi)\mathfrak{g}(\chi)\mathfrak{g}(\mathbb{X}). \tag{1.115}$$

Under such global transformations, the ordering of $\mathfrak{g}(\mathbb{X})$ and $\mathfrak{g}(\chi)$ after the time and angle components implies both the bosonic and fermionic degrees of freedom are neutrally charged, i.e. don't change. This makes the choice particularly suitable to the light-cone gauge where we will be redefining $(t,\phi) \to (x_+, x_-)$. However, this additive property of $\Lambda(t,\phi)$ only holds because G and Λ are both expressed in terms of γ^5. As we discussed previously, other bosonic elements are possible and in particular feature the generators $\frac{1}{4}[\gamma^i,\gamma^j]$. The question becomes, what is the most general bosonic group element G which acts in such an additive way on χ and $\mathbb{X}$ also? In other words, we are after the maximal bosonic subgroup with acts linearly on the latter.

It makes sense to consider the centraliser $\mathcal{C}$ in the bosonic subalgebra $\mathfrak{su}(2,2) \oplus \mathfrak{su}(4)$ of the $\mathfrak{u}(1)$ isometry group corresponding to t and ϕ shifts. Plainly put, this centraliser is made up of the bosonic elements of $\mathfrak{su}(2,2|4)$ which commute with γ^5. It can be expressed as

$$\mathcal{C} = \mathfrak{so}(4) \oplus \mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2) \tag{1.116}$$

where the first copy of $\mathfrak{so}(4)$ is part of $\mathfrak{so}(4,2) \sim \mathfrak{su}(2,2)$ and the second copy belongs to $\mathfrak{so}(6) \sim \mathfrak{su}(4)$. Both copies are spanned by generators $\frac{1}{4}[\gamma^i, \gamma^j]$ with $i, j = 1, ..., 4$ and these clearly commute with $\mathrm{i}\gamma^5$ which spans the time and angle bosonic subspace. Therefore if $G \in \exp \mathcal{C}$ then $G\Lambda(t,\phi)G^{-1} = \Lambda(t,\phi)$. The action of the centraliser on the full coset element is exactly what we are after, namely

$$G \cdot \mathfrak{g} = \Lambda(t,\phi) \cdot G\mathfrak{g}(\chi)G^{-1} \cdot G\mathfrak{g}(\mathbb{X})G^{-1} \cdot G \tag{1.117}$$

where the last element is recognised as a compensating element in $SO(4) \times SO(4) \subset SO(4,1) \times SO(5)$. To ensure that the transformation of $\mathfrak{g}(\mathbb{X})$ in (1.114) is linear, we should check whether conjugating it by G preserves the matrix structure in terms of γ^i. To this end, we want to calculate $[\gamma^i, \gamma^j]\gamma^k$. If $i = j$, then we get 0 identically. Suppose $i \neq j$, then there are two cases,

$$\begin{aligned} &\boldsymbol{i \neq k \neq j}: \qquad && [\gamma^i,\gamma^j]\gamma^k = \gamma^i\gamma^j\gamma^k - \gamma^j\gamma^i\gamma^k = \gamma^i(-\gamma^k\gamma^j) - \gamma^j(-\gamma^k\gamma^i) = \gamma^k[\gamma^i,\gamma^j], \\ &\boldsymbol{i \neq k = j}: \qquad && [\gamma^i,\gamma^j]\gamma^k = \gamma^i\gamma^j\gamma^k - \gamma^j\gamma^i\gamma^k = \gamma^i\gamma^j\gamma^k + \gamma^i\gamma^j\gamma^k = 2\gamma^i. \end{aligned}$$

Thus $\frac{1}{4}[\gamma^i, \gamma^j]$ commutes with γ^k if $i \neq k \neq j$, whereas their product gives $\frac{1}{2}\gamma^i$ if $i \neq k = j$. Either way, the form of $\mathfrak{g}(\mathbb{X})$ is preserved and we confirm that elements of the centraliser act on fermions and bosons as

$$G : \chi \to G\chi G^{-1}, \qquad G : \mathbb{X} \to G\mathbb{X}G^{-1}, \tag{1.118}$$

inducing a linear transformation of the dynamical degrees of freedom x^μ and χ.

To conclude Chapter 1, we will now introduce an extremely important notation which boils down to keeping track of the four copies of $\mathfrak{su}(2)$ comprising the centraliser (1.116). Any element $G \in \exp \mathcal{C} = SU(2)^4$ can be written as

$$G = \begin{pmatrix} \mathfrak{g}_1 & 0 & 0 & 0 \\ & \mathfrak{g}_2 & 0 & 0 \\ 0 & 0 & \mathfrak{g}_3 & 0 \\ 0 & 0 & 0 & \mathfrak{g}_4 \end{pmatrix} \tag{1.119}$$

with the 2×2 blocks $\mathfrak{g}_i$ representing an independent copy of $SU(2)$. Using the definition of γ^i (1.11), a straightforward calculation yields

$$\mathbb{X} = \begin{pmatrix} 0 & Z & 0 & 0 \\ Z^\dagger & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{i}Y \\ 0 & 0 & \mathrm{i}Y^\dagger & 0 \end{pmatrix} \tag{1.120}$$

with blocks

$$Z = \frac{1}{2}\begin{pmatrix} z_3 - \mathrm{i}z_4 & -z_1 + \mathrm{i}z_2 \\ z_1 + \mathrm{i}z_2 & z_3 + \mathrm{i}z_4 \end{pmatrix}, \qquad Y = \frac{1}{2}\begin{pmatrix} y_3 - \mathrm{i}y_4 & -y_1 + \mathrm{i}y_2 \\ y_1 + \mathrm{i}y_2 & y_3 + \mathrm{i}y_4 \end{pmatrix} \tag{1.121}$$

satisfying

$$Z^\dagger = \epsilon Z^t \epsilon^{-1}, \qquad Y^\dagger = \epsilon Y^t \epsilon^{-1}, \qquad \epsilon = \mathrm{i}\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{1.122}$$

The fermionic element χ can be taken to be of the κ-symmetry gauge-fixed form[7]

$$\chi = \left(\begin{array}{cc|cc} 0 & 0 & 0 & \eta \\ 0 & 0 & \theta^\dagger & 0 \\ \hline 0 & \theta & 0 & 0 \\ -\eta^\dagger & 0 & 0 & 0 \end{array}\right). \tag{1.123}$$

Therefore, the coset degrees of freedom transform under the centraliser as

$$\mathbb{X} \to G\mathbb{X}G^{-1} = \begin{pmatrix} 0 & \mathfrak{g}_1 Z \mathfrak{g}_2^{-1} & 0 & 0 \\ \mathfrak{g}_2 Z^\dagger \mathfrak{g}_1^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{i}\mathfrak{g}_3 Y \mathfrak{g}_4^{-1} \\ 0 & 0 & \mathrm{i}\mathfrak{g}_4 Y^\dagger \mathfrak{g}_3^{-1} & 0 \end{pmatrix},$$

$$\chi \to G\chi G^{-1} = \begin{pmatrix} 0 & 0 & 0 & \mathfrak{g}_1 \eta \mathfrak{g}_4^{-1} \\ 0 & 0 & \mathfrak{g}_2 \theta^\dagger \mathfrak{g}_3^{-1} & 0 \\ 0 & \mathfrak{g}_3 \theta \mathfrak{g}_2 - 1 & 0 & 0 \\ -\mathfrak{g}_4 \eta^\dagger \mathfrak{g}_1^{-1} & 0 & 0 & 0 \end{pmatrix}.$$

But because any $SU(2)$ element $\mathfrak{g}_i$ satisfies the condition $\mathfrak{g}_i^{-1} = \epsilon \mathfrak{g}_i^t \epsilon^{-1}$, taking for example the block Y, the quantity $Y\epsilon$ actually transforms under G as

$$Y\epsilon \to \mathfrak{g}_3 Y \mathfrak{g}_4^{-1} \epsilon = \mathfrak{g}_3 Y \epsilon \mathfrak{g}_4^t. \tag{1.124}$$

If we associate the indices $a = 1, 2$ with the fundamental representation of the $\mathfrak{g}_3$ copy of $SU(2)$ and $\dot{a} = \dot{1}, \dot{2}$ with that of the $\mathfrak{g}_4$ copy of $SU(2)$, we can think of this as a matrix with components

$$Y\epsilon = \left(Y^{a\dot{a}}\right) = \begin{pmatrix} Y^{1\dot{1}} & Y^{1\dot{2}} \\ Y^{2\dot{1}} & Y^{2\dot{2}} \end{pmatrix} \tag{1.125}$$

since it transforms as

$$Y^{a\dot{a}} \to \mathfrak{g}^a{}_b Y^{b\dot{b}} (\mathfrak{g}^t)_{\dot{b}}{}^{\dot{a}} = \mathfrak{g}^a{}_b \mathfrak{g}^{\dot{a}}{}_{\dot{b}} Y^{b\dot{b}}. \tag{1.126}$$

The subscript on $\mathfrak{g}_i$ is suppressed since the index style gives it away. We can easily find the components of $Y = (Y\epsilon)\epsilon$ in this *two-index* notation:

$$Y = \begin{pmatrix} Y^{1\dot{2}} & -Y^{1\dot{1}} \\ Y^{2\dot{2}} & -Y^{2\dot{1}} \end{pmatrix} \implies Y^\dagger = \epsilon Y^t \epsilon = \begin{pmatrix} -Y^{2\dot{1}} & Y^{1\dot{1}} \\ -Y^{2\dot{2}} & Y^{1\dot{2}} \end{pmatrix}. \tag{1.127}$$

In particular, if we view the skew-symmetric matrix ϵ as a Levi-Civita symbol then we can define a prescription for lowering the indices of the components such that

$$Y_{a\dot{a}} \equiv \left(Y^{a\dot{a}}\right)^\dagger = \epsilon_{ab} \epsilon_{\dot{a}\dot{b}} Y^{b\dot{b}} \tag{1.128}$$

with convention $\varepsilon_{12} = \varepsilon_{\dot{1}\dot{2}} = 1$. For example, if we weanted to find the conjugate of $Y^{1\dot{1}}$, we could compare the daggered components of Y with those of $Y^\dagger$ in (1.128) and read off $(Y^{1\dot{1}})^\dagger = Y^{2\dot{2}}$. Using the lowering prescription instead,

$$(Y^{1\dot{1}})^\dagger = Y_{1\dot{1}} = \epsilon_{1b} \epsilon_{\dot{1}\dot{b}} Y^{b\dot{b}} = Y^{2\dot{2}},$$

[7] This form of χ explicitly obeys the odd $\mathfrak{su}(2, 2|4)$ reality condition (1.8).

we get the same result. The same story holds with indices $\alpha = 3,4$ corresponding to g_1 and $\dot{\alpha} = \dot{3},\dot{4}$ corresponding to g_2 for the AdS_5 degrees of freedom. In terms of two-index components, we thus have

$$\mathbb{X} = \left(\begin{array}{cccc|cccc} 0 & 0 & Z^{3\dot{4}} & -Z^{3\dot{3}} & 0 & 0 & 0 & 0 \\ 0 & 0 & Z^{4\dot{4}} & -Z^{4\dot{3}} & 0 & 0 & 0 & 0 \\ -Z^{4\dot{3}} & Z^{3\dot{3}} & 0 & 0 & 0 & 0 & 0 & 0 \\ -Z^{4\dot{4}} & Z^{3\dot{4}} & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{i}Y^{1\dot{2}} & -\mathrm{i}Y^{1\dot{1}} \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{i}Y^{2\dot{2}} & -\mathrm{i}Y^{2\dot{1}} \\ 0 & 0 & 0 & 0 & -\mathrm{i}Y^{2\dot{1}} & \mathrm{i}Y^{1\dot{1}} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\mathrm{i}Y^{2\dot{2}} & \mathrm{i}Y^{1\dot{2}} & 0 & 0 \end{array}\right), \tag{1.129}$$

and similarly for fermions,

$$\chi = \left(\begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & \eta^{3\dot{2}} & -\eta^{3\dot{1}} \\ 0 & 0 & 0 & 0 & 0 & 0 & \eta^{4\dot{2}} & -\eta^{4\dot{1}} \\ 0 & 0 & 0 & 0 & \theta^{\dagger}_{1\dot{4}} & \theta^{\dagger}_{2\dot{4}} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\theta^{\dagger}_{1\dot{3}} & -\theta^{\dagger}_{2\dot{3}} & 0 & 0 \\ \hline 0 & 0 & \theta^{1\dot{4}} & -\theta^{1\dot{3}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \theta^{2\dot{4}} & -\theta^{2\dot{3}} & 0 & 0 & 0 & 0 \\ -\eta^{\dagger}_{3\dot{2}} & -\eta^{\dagger}_{4\dot{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ \eta^{\dagger}_{3\dot{1}} & \eta^{\dagger}_{4\dot{1}} & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right). \tag{1.130}$$

In the case of fermions the lowering corresponds in a different way to taking the conjugate:

$$\theta^{\dagger}_{a\dot{\alpha}} \equiv \left(\theta^{a\dot{\alpha}}\right)^*, \qquad \eta^{\dagger}_{\alpha\dot{a}} \equiv \left(\eta^{\alpha\dot{a}}\right)^*. \tag{1.131}$$

Now we understand how dynamical bosonic and fermionic degrees of freedom act under the bosonic symmetry group $\mathcal{C}$, which we can write as

$$\exp\mathcal{C} = SU(2)_{\alpha} \times SU(2)_{\dot{\alpha}} \times SU(2)_{a} \times SU(2)_{\dot{a}}. \tag{1.132}$$

In Chapter 2 we will see how to write the Lagrangian in terms of the two-index fields $(Z^{\alpha\dot{\alpha}}, Y^{a\dot{a}}, \theta^{a\dot{\alpha}}, \eta^{\alpha\dot{a}})$ such that the bosonic symmetry group of the model will be made manifest.

Chapter 2

Light-cone quantisation

Having accustomed ourselves with the language of superstrings in the coset (1.1), we will now take a step back. To quantise the $\mathrm{AdS}_5 \times S^5$ superstring, we will need to have an expression for the Hamiltonian of the Green-Schwarz action. Ideas taken from bosonic light-cone quantisation will prove useful in this endeavour.

Usually, one would quantise a model by translating the Lagrangian's generalised coordinates to Hamiltonian phase space whereupon the fields are promoted to operators. This is referred to as *canonical* quantisation as it involves promoting the Poisson bracket relating fields to their canonically conjugate momenta to matrix commutators. If the action presents constraints, one should also reduce the phase space to the *physical* phase space before making this quantum leap. We will see how to write the action of a string in $\mathrm{AdS}_5 \times S^5$ (first bosonic, then super-) in first-order form such that the canonical pairings and the model's constraints are made explicit by what is, and what isn't, the kinetic term in the Lagrangian. Working in light-cone coordinates, we will be invited to fix the light-cone gauge which will ultimately reveal the classical Hamiltonian density we are seeking.

In the decompactification limit, when the circumference of the cylindrical worldsheet goes to infinity, the model becomes a two-dimensional quantum field theory on the plane. We will see that the leading term in the large tension limit corresponds to free theory of 8 bosons and 8 fermions all having the same mass. It turns out that the $\mathcal{S}$-matrix factorises into two-particle scattering for the next-to-leading order theory, which would indicate that the quantised model is integrable.

2.1 How to fix a Lagrangian

At the end of this section, we will end up with a Lagrangian ready for decompactification. The Lagrangian will be fixed in two ways; in the light-cone and κ-symmetry gauges. Our first step will be to introduce the first-order formalism in light-cone coordinates through the bosonic case in order to prepare us for the superstring. We will then move forward with a special light-cone gauge known as the uniform light-cone gauge, which will be followed by fixing the κ-symmetry gauge and discussing the resulting gauge-fixed Lagrangian.

Bosonic first-order formalism

We start with the Green-Schwarz action with the fermions turned off,

$$S = -\frac{T}{2} \iint d^2\sigma \ \gamma^{\alpha\beta} \partial_\alpha X^M \partial_\beta X^N G_{MN}, \tag{2.1}$$

which described bosonic strings in 1.4 with target space metric G_{MN} and coordinates $X^M \in \{t, \phi, x^\mu\}$, and wordlsheet components $\alpha, \beta = \tau, \sigma$. We will use the shorthand $\partial_\tau X^M = \dot{X}^M$, $\partial_\sigma X^M = X'^M$. The conjugate momenta can be found as usual, remembering that $\gamma^{\alpha\beta} = \gamma^{\beta\alpha}$ so that

$$p_M = \frac{\partial \mathscr{L}}{\partial \dot{X}^M} = -T\gamma^{\tau\beta}\partial_\beta X^N G_{MN} = -T\gamma^{\tau\tau}\dot{X}_M - T\gamma^{\tau\sigma}X'_M. \tag{2.2}$$

We can rewrite the action in first-order form,

$$S = \iint d^2\sigma \ \left(p_M \dot{X}^M + \frac{\gamma^{\tau\sigma}}{\gamma^{\tau\tau}} C_1 + \frac{1}{2T\gamma^{\tau\tau}} C_2 \right), \tag{2.3}$$

with the following constraints determined in B.1:

$$C_1 = p_M X'^M, \qquad\qquad C_2 = p_M p^M + T^2 X'_M X'^M. \tag{2.4}$$

As derived in B.1, in flat space these constraints satisfy the equal-τ Poisson algebra

$$\begin{aligned}
\{C_1(\sigma), C_1(\sigma')\}_{\text{P.B.}} &= \partial_\sigma C_1(\sigma)\delta(\sigma - \sigma') + 2C_1(\sigma)\partial_\sigma\delta(\sigma - \sigma'), \\
\{C_1(\sigma), C_2(\sigma')\}_{\text{P.B.}} &= \partial_\sigma C_2(\sigma)\delta(\sigma - \sigma') + 2C_2(\sigma)\partial_\sigma\delta(\sigma - \sigma'), \\
\{C_2(\sigma), C_1(\sigma')\}_{\text{P.B.}} &= \partial_\sigma C_2(\sigma)\delta(\sigma - \sigma') + 2C_2(\sigma)\partial_\sigma\delta(\sigma - \sigma'), \\
\{C_2(\sigma), C_2(\sigma')\}_{\text{P.B.}} &= 4T^2\partial_\sigma C_1(\sigma)\delta(\sigma - \sigma') + 8T^2 C_1(\sigma)\partial_\sigma\delta(\sigma - \sigma').
\end{aligned} \tag{2.5}$$

In order to proceed with decompactification, which will ultimately facilitate quantisation, we want to express the action explicitly in *light-cone coordinates* parameterised by the constant a:

$$\begin{aligned}
t &= x_+ - a x_-, & x_+ &= a\phi + (1-a)t, \\
\phi &= x_+ + (1-a)x_-, & x_- &= \phi - t.
\end{aligned} \tag{2.6}$$

Equating the scalar $p_t\dot{t} + p_\phi\dot{\phi}$ with $p_+\dot{x}_- + p_-\dot{x}_+$, we get

$$\begin{aligned}
p_t &= (1-a)p_- - p_+, & p_+ &= (1-a)p_\phi - a p_t, \\
p_\phi &= p_+ + a p_-, & p_- &= p_\phi + p_t.
\end{aligned} \tag{2.7}$$

The transversal coordinates x^μ and their conjugate momenta p_μ are unchanged. The invariance of the action under shfits of t and ϕ has not changed, which leads to the conserved quantities

$$E = -\int_{-\pi r}^{-\pi r} d\sigma \ p_t, \qquad J = -\int_{-\pi r}^{-\pi r} d\sigma \ p_\phi \tag{2.8}$$

which evidently correspond to the target space energy and angular momentum of the string. We can now relate these quantities to the light-cone momenta

$$P_+ = \int_{-\pi r}^{-\pi r} d\sigma \ p_+ = J + a(E - J), \qquad P_- = \int_{-\pi r}^{-\pi r} d\sigma \ p_- = J + E. \tag{2.9}$$

A preference for the specific value of a depends on the context, but for us it will eventually be useful to set $a = 1/2$. Letting the parameter be free for now, in these coordinates the first term in the action is

$$p_M \dot{X}^M = p_+ \dot{x}_- + p_- \dot{x}_+ + p_\mu \dot{x}^\mu. \tag{2.10}$$

Meanwhile the constraints take the form

$$\begin{aligned} C_1 &= p_+ x'_- + p_- x'_+ + p_\mu x'^\mu, \qquad (2.11) \\ C_2 &= p_+^2 \left[G^{-1}_{\phi\phi} - G^{-1}_{tt}\right] + 2p_+p_- \left[aG^{-1}_{\phi\phi} + (1-a)G^{-1}_{tt}\right] + p_-^2 \left[a^2 G^{-1}_{\phi\phi} - (1-a)^2 G^{-1}_{tt}\right] \\ &\quad + T^2 x'^2_+ \left[G_{\phi\phi} - G_{tt}\right] + 2T^2 x'_+ x'_- \left[aG_{tt} + (1-a)G_{\phi\phi}\right] \qquad (2.12) \\ &\quad + T^2 x'^2_- \left[(1-a)^2 G_{\phi\phi} - a^2 G_{tt}\right] + 2\mathcal{H}_\perp, \end{aligned}$$

as derived in B.1. Here we defined a 'Hamiltonian' related to the transversal degrees of freedom (x^μ, p^μ),

$$\mathcal{H}_\perp = \frac{1}{2} p_\mu p_\nu G^{\mu\mu} + \frac{1}{2} T^2 x'^\mu x'^\nu G_{\mu\nu}. \tag{2.13}$$

In string theory, actions such as the Green-Schwarz action display two reparametrisation invariances in the coordinates (τ, σ). This provides two gauge freedoms which we will now exploit [6]. We fix the light-cone gauge by imposing the following conditions:

$$x_+ = \tau + am\frac{\sigma}{r}, \qquad p_+ = 1. \tag{2.14}$$

This is the *uniform* light-cone gauge because the total spacetime light-cone momentum $P_+ = \int d\sigma\ p_+ = 2\pi r$ is uniformly distributed around the string and is equal to its circumference. The integer m is the winding number from the periodicity condition at the equator,

$$\phi(\pi r) - \phi(-\pi r) = 2\pi m. \tag{2.15}$$

This same periodicity is the reason for the normalisation factor $1/r$ which is required by the consistency $x_+(\pi r) - x_+(-\pi r) = a(\phi(\pi r) - \phi(-\pi r))$.

Wanting to find the form of the action (2.3) in this gauge, we can rewrite the constraints as

$$\begin{aligned} C_1 &= x'_- + \frac{1}{r} amp_- + p_\mu x'^\mu, \qquad (2.16) \\ C_2 &= \left[G^{-1}_{\phi\phi} - G^{-1}_{tt}\right] + 2p_- \left[aG^{-1}_{\phi\phi} + (1-a)G^{-1}_{tt}\right] + p_-^2 \left[a^2 G^{-1}_{\phi\phi} - (1-a)^2 G^{-1}_{tt}\right] \\ &\quad + T^2 \left(\frac{1}{r}am\right)^2 \left[G_{\phi\phi} - G_{tt}\right] + 2T^2 \left(\frac{1}{r}am\right) x'_- \left[aG_{tt} + (1-a)G_{\phi\phi}\right] \qquad (2.17) \\ &\quad + T^2 x'^2_- \left[(1-a)^2 G_{\phi\phi} - a^2 G_{tt}\right] + 2\mathcal{H}_\perp. \end{aligned}$$

Solving the first constraint C_1, i.e. setting $C_1 = 0$, we find $x'_- = -\frac{1}{r}amp_- - p_\mu x'^\mu$, which implies that the second constraint C_2 can be solved to obtain a quadratic in $p_- = p_-(p_\mu, x^\mu, x'^\mu)$. Substituting (2.10) and solving the constraints, the light-cone gauge action becomes

$$S = \iint d^2\sigma \left(p_\mu \dot{x}^\mu + p_+ \dot{x}_- + p_- \dot{x}_+\right) = \iint d^2\sigma \left(p_\mu \dot{x}^\mu + \dot{x}_- + p_-\right).$$

Since $\dot{x}_- = \frac{\partial}{\partial\tau}x_-$ is a total derivative, it can be omitted from the action while preserving the correct physics. We are left with

$$S = \iint d^2\sigma \, (p_\mu \dot{x}^\mu - \mathcal{H}) \tag{2.18}$$

where $\mathcal{H} = -p_-(p_\mu, x^\mu, x'^\mu)$ is the light-cone Hamiltonian density since it's the only term not containing a time derivative. Note that in flat space (when $G_{tt} = G_{\phi\phi} = G_{\mu\mu} = 1$) and when $a = 1/2$, the above constraint C_2 (2.17) becomes

$$C_2 = 2p_- + T^2 \frac{1}{r} m x'_- + 2\mathcal{H}_\perp = 0.$$

We will soon set $m = 0$, resulting in $\mathcal{H}_\perp = -p_-$, which motivates the label of this transversal term. Physically $m = 0$ corresponds to the string not making it all the way around the equator of S^5.

In particular, now that we have identified $\mathcal{H} = -p_-$, the level-matching condition

$$\int_{-\pi r}^{\pi r} d\sigma \, x'_- = x_-(\pi r) - x_-(-\pi r) = \phi(\pi r) - \phi(-\pi r) - = 2\pi m \tag{2.19}$$

for physical states[8] can give us insight into the total worldsheet momentum which we will denote p_{ws}. This condition follows from (2.15) and $X^M(\tau, \sigma + 2\pi r) = X^M(\tau, \sigma)$ for non-angle coordinates t and x^μ. Combining this level-matching condition with $C_1 = 0$ we get

$$0 = \int_{-\pi r}^{\pi r} d\sigma \left(-\frac{1}{r} amp_- - p_\mu x'^\mu \right) = \frac{1}{r} am\mathbb{H} - \int_{-r}^{r} d\sigma \, p_\mu x'^\mu = 2\pi m \tag{2.20}$$

where $\mathbb{H} = \int d\sigma \, \mathcal{H}$ is the light-cone Hamiltonian of the superstring.[9] The second term is nothing but the total worldsheet momentum since it is the integral over the stress-energy tensor component $T^{\tau\sigma}$. So we in fact have shown that the worldsheet momentum for physical states satisfies

$$p_{\text{ws}} = -\int_{-r}^{r} d\sigma \, p_\mu x'^\mu = m \left(2\pi - \frac{a}{r}\mathbb{H} \right). \tag{2.21}$$

This component of the stress energy tensor corresponds to translations along σ, a symmetry of the action. Thus the worldsheet momentum p_{ws} is a conserved charge and, in particular, for $m = 0$ it is vanishing. Remembering that our original goal was to find a classical Hamiltonian to quantise, we update the light-cone gauge for $m = 0$:

$$x_+ = \tau, \qquad p_+ = 1. \tag{2.22}$$

The Virasoro constraint C_2, which is a quadratic in p_-, still depends on $x'_- = -p_\mu x'^\mu$ but takes the simpler form

$$\begin{aligned} C_2 = &\left[G_{\phi\phi}^{-1} - G_{tt}^{-1} \right] + 2p_- \left[aG_{\phi\phi}^{-1} + (1-a)G_{tt}^{-1} \right] + p_-^2 \left[a^2 G_{\phi\phi}^{-1} - (1-a)^2 G_{tt}^{-1} \right] \\ &+ T^2 x'^2_- \left[(1-a)^2 G_{\phi\phi} - a^2 G_{tt} \right] + 2\mathcal{H}_\perp. \end{aligned}$$

[8] These physical states must obey periodicity of the coordinates to be a closed string. If the level-matching is not observed, then we would be dealing with an open string whose endpoints are held at a fixed distance.

[9] It is worth noting that this can be expressed as $\mathbb{H} = \int d\sigma \mathcal{H} = E - J$, which means if one can solve the equation for $P_+(E, J)$ and substitute it into the bounds $\pm\frac{1}{2}P_+$, there will be a new equation for the target space energy E which may be then be related to the CFT scaling dimension.

Solving this quadratic for the light-cone Hamiltonian density $-p_-$ is straightfoward and we get

$$\mathcal{H} = \frac{\sqrt{G_{tt}G_{\phi\phi}\left[1+2\left((1-a)^2G_{\phi\phi}-a^2G_{tt}\right)\mathcal{H}_\perp + T^2\left((1-a)^2G_{\phi\phi}-a^2G_{tt}\right)^2 x_-'^2\right]}}{(1-a)^2G_{\phi\phi}-a^2G_{tt}} - \frac{aG_{tt}+(1-a)G_{\phi\phi}}{(1-a)^2G_{\phi\phi}-a^2G_{tt}}. \tag{2.23}$$

This highly complicated, non-polynomial expression for $\mathcal{H}(x^\mu, p_\mu, \chi)$ must have come as quite the disappointment to string theorists nearly two decades ago. We cannot simply promote fields to operators, even for the bosonic restriction which is not very inspiring. In the supersymmetric case, we will have to resort to a compromise as we will see.

Green-Schwarz first-order formalism

In analogy with the first-order form of the purely bosonic string (2.3), we can introduce an auxiliary field denoted $\mathcal{T} \in \mathfrak{psu}(2,2|4)$, such that the bosonic part of the Green-Schwarz superstring (1.33) changes to

$$\mathscr{L} = -\mathrm{str}\left[\mathcal{T} A_\tau^{(2)} + \frac{\gamma^{\tau\sigma}}{\gamma^{\tau\tau}}\mathcal{T} A_\sigma^{(2)} - \frac{1}{2T\gamma^{\tau\tau}}\left(\mathcal{T}^2 + T^2 A_\sigma^{(2)}A_\sigma^{(2)}\right)\right] - \frac{T}{2}\kappa\varepsilon^{\alpha\beta}\,\mathrm{str}\big(A_\alpha^{(1)}A_\beta^{(3)}\big). \tag{2.24}$$

The equation of motion for $\mathcal{T}$ is given by

$$0 = \frac{\partial\mathscr{L}}{\partial\mathcal{T}} = -\mathrm{str}\left[A_\tau^{(2)} + \frac{\gamma^{\tau\sigma}}{\gamma^{\tau\tau}}A_\sigma^{(2)} - \frac{1}{T\gamma^{\tau\tau}}\mathcal{T}\right]$$

which has an obvious solution, reminiscent of the bosonic momenta p_M in (2.2),

$$\mathcal{T} = T\gamma^{\tau\tau}A_\tau^{(2)} + T\gamma^{\tau\sigma}A_\sigma^{(2)}. \tag{2.25}$$

Substituting this expression for $\mathcal{T}$ into the Lagrangian minus the Wess-Zumino term, we recover in B.1 the Green-Schwarz Lagrangian kinetic term, as expected. Looking at (2.24), the constraints C_1 and C_2 of the superstring first-order formalism are

$$\begin{aligned} C_1 &= -\,\mathrm{str}\big(\mathcal{T} A_\sigma^{(2)}\big) = 0, \\ C_2 &= \,\mathrm{str}\big(\mathcal{T}^2 + T^2 A_\sigma^{(2)}A_\sigma^{(2)}\big) = 0, \end{aligned} \tag{2.26}$$

which we will solve after imposing light-cone gauge and fixing κ-symmetry. In general, the equation of motion shows that $\mathcal{T}$ can be viewed as an element of $\mathscr{G}^{(2)}$ without affecting the projections of A onto other graded subspaces. We can consequently write it as a generic element of $\mathscr{G}^{(2)}$,

$$\mathcal{T} = \mathcal{T}^{(2)} \equiv \frac{\mathrm{i}}{2}\mathcal{T}_+\Sigma_+ + \frac{\mathrm{i}}{4}\mathcal{T}_-\Sigma_- + \frac{1}{2}\mathcal{T}_\mu\Sigma^\mu + \mathcal{T}_{\mathbb{1}}\,\mathrm{i}\,\mathbb{1}_8, \tag{2.27}$$

which is a linear combination of 8×8 matrices of the form

$$\Sigma_+ = \begin{pmatrix}\Sigma & 0\\ 0 & \Sigma\end{pmatrix}, \quad \Sigma_- = \begin{pmatrix}-\Sigma & 0\\ 0 & \Sigma\end{pmatrix}, \quad \Sigma^k = \begin{pmatrix}\gamma^k & 0\\ 0 & 0\end{pmatrix}, \quad \Sigma^{4+k} = \begin{pmatrix}0 & 0\\ 0 & \mathrm{i}\gamma^k\end{pmatrix}. \tag{2.28}$$

These matrices are made of the Dirac matrices γ^k for $k = 1, ..., 4$ and $\gamma^5 = \Sigma$, and span the diagonal (bosonic) subspace of $\mathfrak{su}(2,2|4)$. Notice that Σ_{4+k} has a factor of i in front of γ^k; this is to ensure that $\Sigma_{4+k} \in \mathfrak{su}(2,2|4)$. The coefficient $\mathcal{T}_{\mathbb{1}}$ is extraneous to the Lagrangian as it always features alongside a $\mathrm{str}\big(A_\alpha^{(2)}\big) = 0$. The auxiliary field components $\mathcal{T}_\pm$ and $\mathcal{T}_\mu$ will be related to the momenta $p_\pm$ and p_μ by comparing the Lagrangian density to its light-cone gauge-fixed equivalent.

Kappa symmetry gauge fixing

As proven in 1.2, our Green-Schwarz Lagrangian enjoys a gauge freedom as a result of κ-symmetry, thanks to which we could gauge away 16 of the 32 fermionic entries in the matrix representation of the embedding element $\chi \in \mathfrak{su}(2,2|4)$. We saw that χ could be expressed as (1.123) for 2×2 matrices a and b, The following identities follow from – or can be considered a definition of – κ-symmetry gauge fixing:

$$\chi\Sigma_+ = -\Sigma_+\chi, \qquad \chi\Sigma_- = \Sigma_-\chi. \tag{2.29}$$

Explicit calculations in B.1 prove these identites for χ of the above form. Because $\mathfrak{g}(\chi)^{-1} = \mathfrak{g}(-\chi)$, we can Taylor expand the latter and apply the above identity for each copy of χ in the polynomials to get

$$\begin{aligned} \mathfrak{g}(\chi)^{-1}\Sigma_+ &= \Sigma_+\mathfrak{g}(\chi) &\implies\quad& \mathfrak{g}(\chi)^{-1}\Sigma_+\mathfrak{g}(\chi) = \Sigma_+\mathfrak{g}(\chi)^2, \\ \mathfrak{g}(\chi)^{-1}\Sigma_- &= \Sigma_-\mathfrak{g}(\chi)^{-1} &\implies\quad& \mathfrak{g}(\chi)^{-1}\Sigma_-\mathfrak{g}(\chi) = \Sigma_-. \end{aligned} \tag{2.30}$$

We will now find the explicit form of our current $A = -\mathfrak{g}^{-1}d\mathfrak{g}$ where $\mathfrak{g} = \Lambda(t,\phi)\mathfrak{g}(\chi)\mathfrak{g}(\mathbb{X})$ with the coset parametrisation given by

$$\Lambda(t,\phi) = \exp\frac{\mathrm{i}}{2}\begin{pmatrix} t\Sigma & 0 \\ 0 & \phi\Sigma \end{pmatrix}, \qquad \mathfrak{g}(\mathbb{X}) = \sqrt{\frac{\mathbb{1}+\mathbb{X}}{\mathbb{1}-\mathbb{X}}}, \qquad \mathfrak{g}(\chi) = \chi + \sqrt{\mathbb{1}+\chi^2}. \tag{2.31}$$

Because bosonic elements (1.113) are expressed in terms of γ^i for $i = 1,...,4$, we have

$$\mathfrak{g}(\mathbb{X})^{-1}\Sigma_\pm = \mathfrak{g}(-\mathbb{X})\Sigma_\pm = \Sigma_\pm\mathfrak{g}(\mathbb{X}). \tag{2.32}$$

One can revert back to the exponential definition of $\mathfrak{g}(\chi)$ in (1.107) with the substitution $\chi \to \sinh\chi$ since $\sinh\chi + \sqrt{\mathbb{1}+\sinh^2\chi} = \sinh\chi + \cosh\chi = \exp\chi$. In B.1 the even and odd components of A are derived to be

$$\begin{aligned} A_\mathrm{e} &= -\mathfrak{g}(\mathbb{X})^{-1}\left[\frac{\mathrm{i}}{2}\left(dx_+ + (\frac{1}{2}-a)dx_-\right)\Sigma_+(\mathbb{1}+2\chi^2) + \frac{\mathrm{i}}{4}dx_-\Sigma_- + B\right]\mathfrak{g}(\mathbb{X}) - \mathfrak{g}(\mathbb{X})^{-1}d\mathfrak{g}(\mathbb{X}), \\ A_\mathrm{o} &= -\mathfrak{g}(\mathbb{X})^{-1}\left[\mathrm{i}\left(dx_+ + (\frac{1}{2}-a)dx_-\right)\Sigma_+\chi\sqrt{\mathbb{1}+\chi^2} + F\right]\mathfrak{g}(\mathbb{X}). \end{aligned} \tag{2.33}$$

For simplicity, we decomposed $\mathfrak{g}(\chi)^{-1}\partial_\alpha\mathfrak{g}(\chi) = B_\alpha + F_\alpha$ into bosonic and fermionic parts

$$B_\alpha = \sqrt{\mathbb{1}+\chi^2}\partial_\alpha\sqrt{\mathbb{1}+\chi^2} - \chi\partial_\alpha\chi, \qquad F_\alpha = \sqrt{\mathbb{1}+\chi^2}\partial_\alpha\chi - \chi\partial_\alpha\sqrt{\mathbb{1}+\chi^2}. \tag{2.34}$$

One can show that these are respectively even and odd by rewriting the expressions as commutators of one or two odd elements. The formulae (2.33) were obtained using the identities (2.30) which depended on the commutation relations of χ and $\Sigma_\pm$. The latter relied on certain fermionic degrees of freedom in χ being gauged away by the κ-symmetry transformation as discussed in 1.2. In tandem with this κ-symmetry gauge fixing, we make the natural choice $a = 1/2$ such that the above odd current no longer depends on the coordinate x_-:

$$\begin{aligned} A_\mathrm{e} &= -\mathfrak{g}(\mathbb{X})^{-1}\left[\frac{\mathrm{i}}{2}\Sigma_+(\mathbb{1}+2\chi^2)dx_+ + \frac{\mathrm{i}}{4}\Sigma_-dx_- + B\right]\mathfrak{g}(\mathbb{X}) - \mathfrak{g}(\mathbb{X})^{-1}d\mathfrak{g}(\mathbb{X}), \\ A_\mathrm{o} &= -\mathfrak{g}(\mathbb{X})^{-1}\left[\mathrm{i}\Sigma_+\chi\sqrt{\mathbb{1}+\chi^2}dx_+ + F\right]\mathfrak{g}(\mathbb{X}). \end{aligned} \tag{2.35}$$

One gauge down, one more to go.

Light-cone gauge fixing

We should aim to write $\mathscr{L}$ in the form (2.18) in order to read off the Hamiltonian density $\mathcal{H}$ which will be required at the quantisation step in 2.2. To proceed, we need to identify to which graded subspaces each term in $A_{\mathrm{e,o}}$ belongs such that we can evaluate the Lagrangian explicitly and identify the canonical momenta $p_\pm$ and the worldsheet momentum p_{ws}. Because the component $A^{(0)}$ is proportional to the identity, we can replace $A^{(2)}$ with the whole even component A_{e} and the Lagrangian will be unchanged. Explicitly, sending $A^{(2)} \to A^{(0)} + A^{(2)}$ adds the terms $\mathrm{str}\big(A^{(0)}A^{(0)}\big) = 0$ and $\mathrm{str}\big(A^{(2)}A^{(0)}\big) \propto \mathrm{str}\big(A^{(2)}\big) = 0$. Making this choice and substituting the even current into (2.24), we can write

$$\mathscr{L} = p_+\dot{x}_- + \mathbf{p}_-\dot{x}_+ - \mathrm{str}\big(\mathit{\Pi}\, A^{\perp}_{\mathrm{e},\tau} + \frac{T}{2}\kappa\varepsilon^{\alpha\beta}A^{(1)}_{\alpha}A^{(3)}_{\beta}\big), \tag{2.36}$$

where we can read off the conjugate momentum p_+ and the factor $\mathbf{p}_-$:

$$p_+ = \frac{\mathrm{i}}{4}\,\mathrm{str}\big(\mathit{\Pi}\,\Sigma_-\mathfrak{g}(\mathbb{X})^2\big), \qquad \mathbf{p}_- = \frac{\mathrm{i}}{2}\,\mathrm{str}\big(\mathit{\Pi}\,\Sigma_+\mathfrak{g}(\mathbb{X})(\mathbb{1}+2\chi^2)\mathfrak{g}(\mathbb{X})\big). \tag{2.37}$$

The part of the even current which depends only on the transversal degrees of freedom

$$A^{\perp}_{\mathrm{e}} = -\mathfrak{g}(\mathbb{X})^{-1}\left[\sqrt{\mathbb{1}+\chi^2}\,d\sqrt{\mathbb{1}+\chi^2} - \chi d\chi\right]\mathfrak{g}(\mathbb{X}) - \mathfrak{g}(\mathbb{X})^{-1}d\mathfrak{g}(\mathbb{X}) \tag{2.38}$$

is isolated for brevity. To obtain these expressions (2.37), we used (2.32) to carry the matrices $\Sigma_\pm$ past $\mathfrak{g}(\mathbb{X})^{-1}$. The factor $\mathbf{p}_-$ is bold because it is boldly pretending to be p_-, which contains a second term proportional to $\dot{x}_+$ coming from the odd current (2.35) through the Wess-Zumino term. However, specifically because we eliminated the x_- dependence from the odd part of A_α by choosing $a = 1/2$, the light-cone momentum p_+ is no pretender so we can go ahead and impose the uniform light-cone gauge

$$x_+ = \tau + m\frac{\sigma}{2r}, \qquad p_+ = 1. \tag{2.39}$$

Gauge-fixed Lagrangian

In preparation for the decompactification limit when the circumference of the worldsheet (or in other words the length of the string) goes to infinity, we must set $m = 0$ to keep the light-cone Hamiltonian $\mathbb{H} = E - J$ finite. As $P_+ \to \infty$, the angular momentum J grows rapidly which necessitates that the string move fast in $\mathrm{AdS}_5 \times S^5$ such that E is comparably large. Importantly, setting $m = 0$ results in $x_+ = \tau$ as we saw for the bosonic string. Having fixed the κ-symmetry gauge such that the Lagrangian is given by (2.36), before imposing the light-cone gauge we start with

$$\mathscr{L} = p_+\dot{x}_- + \mathbf{p}_-\dot{x}_+ - \mathrm{str}\big(\mathit{\Pi}\, A^{\perp}_{\mathrm{e},\tau}\big) + \mathscr{L}_{\mathrm{WZ}}. \tag{2.40}$$

Both p_+ and $\dot{x}_+$ will ultimately be set to 1. The $\dot{x}_-$ term can consequently be dropped as it is a total time derivative. The constraints C_1 and C_2 offer a way to solve for $\mathit{\Pi}_\pm$ and $\mathit{\Pi}_\mu$ in terms of the worldsheet momenta $p_\pm$ and p_μ. Looking at C_1, it features $A_{\mathrm{e},\sigma}$ which means it is also possible to isolate x'_- which is crucial to impose the level-matching condition. Once we have $\{\mathit{\Pi}\}$ in terms of $\{p\}$, we can simplify $\mathbf{p}_-$ and the Wess-Zumino term to determine the Hamiltonian density $\mathcal{H} = -p_-$.[10] In the appendix B.3 the details of the derivation of $\mathscr{L}_{\mathrm{GF}}$ explicitly reproduce – and agree with – the results found in [8].

[10] The equality $\mathcal{H} = -p_-$ still holds in the case of this superstring because the only term in (2.40) without a time derivative, i.e. the only term which is not a kinetic term, is the one associated to $\dot{x}_+ = 1$.

To begin, the definition (2.37) of p_+ is used in B.3 to show that the momentum p_+ reduces to

$$p_+ = \pi_+ G_+ - \frac{1}{2}\pi_- G_- \quad \text{for} \quad G_\pm = \frac{1}{2}(\sqrt{G_{tt}} \pm \sqrt{G_{\phi\phi}}). \tag{2.41}$$

This expression is then used in B.3 to find

$$-\operatorname{str}\big(\pi A^{\perp}_{\mathrm{e},\tau}\big) = p_\mu \dot{x}^\mu - \frac{\mathrm{i}}{2} p_+ \operatorname{str}\big(\Sigma_+ \chi \dot{\chi}\big) + \frac{1}{2}\mathfrak{g}_\nu \pi_\mu \operatorname{str}\big([\Sigma_\nu, \Sigma_\mu] B_\tau\big) \tag{2.42}$$

provided p_μ are equal to

$$p_i = \sqrt{G_{zz}}\pi_i, \qquad p_{4+i} = \sqrt{G_{yy}}\pi_{4+i}. \tag{2.43}$$

Replacing ∂_τ with ∂_σ in (2.42) one can solve $C_1 = 0$ (B.29) yielding

$$x'_- = -\frac{1}{p_+}\left[p_\mu x'^\mu - \frac{\mathrm{i}}{2} p_+ \operatorname{str}\big(\Sigma_+ \chi \partial_\sigma \chi\big) + \frac{1}{2}\mathfrak{g}_\nu \pi_\mu \operatorname{str}\big([\Sigma_\nu, \Sigma_\mu] B_\sigma\big)\right], \tag{2.44}$$

Integrating x'_- over σ and setting the result to zero is the level-matching condition we saw was necessary for physical states. Note we see already that in flat space, when $G_+ = G_{\mu\mu} = 1$ and $G_- = 0$, the momenta satisfy $p_+ = \pi_+$ and $p_\mu = \pi_\mu$. The factor $\mathfrak{g}_\nu$ comes from the decomposition (B.24)

$$\mathfrak{g}(\mathbb{X}) = \mathfrak{g}_+ \mathbb{1}_8 + \mathfrak{g}_- \Upsilon + \mathfrak{g}_\mu \Sigma_\mu, \qquad \mathfrak{g}(\mathbb{X})^2 = G_+ \mathbb{1}_8 + G_- \Upsilon + G_\mu \Sigma_\mu,$$

and explicit forms of the coefficients are presented in (B.25) and (B.26). $C_2 = 0$ is solved in B.3 to find

$$\pi_- = -\frac{G_+(\pi_\mu^2 + T^2 \mathcal{A}^2)}{p_+ + \sqrt{p_+^2 - G_- G_+(\pi_\mu^2 + T^2\mathcal{A}^2)}} \tag{2.45}$$

where $\mathcal{A}^2 \equiv \operatorname{str}\big(A^{(2)}_\sigma A^{(2)}_\sigma\big)$ is found in B.3 to be (B.36). Let us catch our breath and see where the Lagrangian stands. Taking stock of each term other than $p_+ \dot{x}_- \to 0$,

$$\mathbf{p}_- \dot{x}_+ = \Big(-2\frac{G_-}{G_+} p_+ + \frac{G_+^2 - G_-^2}{G_+}\pi_- - \frac{1}{2} p_+ \operatorname{str}\big(\chi^2\big) \tag{2.46}$$

$$+ \frac{\mathrm{i}}{2}\pi_\mu \mathfrak{g}_\nu \operatorname{str}\big([\Sigma_\nu, \Sigma_\mu]\chi^2(\mathfrak{g}_+\Sigma_+ - \mathfrak{g}_-\Sigma_-)\big)\Big)\dot{x}_+, \tag{2.47}$$

$$-\operatorname{str}\big(\pi A^{\perp}_{\mathrm{e},\tau}\big) = p_\mu \dot{x}^\mu - \frac{\mathrm{i}}{2} p_+ \operatorname{str}\big(\Sigma_+ \chi \dot{\chi}\big) + \frac{1}{2}\mathfrak{g}_\nu \pi_\mu \operatorname{str}\big([\Sigma_\nu, \Sigma_\mu] B_\tau\big), \tag{2.48}$$

$$\mathscr{L}_{\mathrm{WZ}} = \kappa\frac{T}{2}(G_+^2 - G_-^2)\operatorname{str}\big([\mathrm{i}F_\tau - \dot{x}_+\Sigma_+\chi\sqrt{\mathbb{1}+\chi^2}]\mathcal{K}F^{st}_\sigma\mathcal{K}^{-1}\big) \tag{2.49}$$

$$- \kappa\frac{T}{2}G_\mu G_\nu \operatorname{str}\big(\Sigma_\nu[\mathrm{i}F_\tau - \dot{x}_+\Sigma_+\chi\sqrt{\mathbb{1}+\chi^2}]\Sigma_\mu\mathcal{K}F^{st}_\sigma\mathcal{K}^{-1}\big).$$

The gauge-fixed Lagrangian is thus the sum of the above with $x_+ = \tau$ and $p_+ = 1$, and can be written

$$\mathscr{L}_{\mathrm{GF}} = \mathscr{L}_{\mathrm{Kin}} - \mathcal{H} \tag{2.50}$$

where the kinetic part below houses all the transversal τ derivatives $\dot{x}^\mu$:

$$\begin{aligned} \mathscr{L}_{\mathrm{Kin}} &= p_\mu \dot{x}^\mu - \frac{\mathrm{i}}{2}\operatorname{str}\big(\Sigma_+\chi\dot{\chi}\big) + \frac{1}{2}\mathfrak{g}_\nu\pi_\mu\operatorname{str}\big([\Sigma_\nu,\Sigma_\mu]B_\tau\big) \\ &+ \mathrm{i}\kappa\frac{T}{2}(G_+^2 - G_-^2)\operatorname{str}\big(F_\tau\mathcal{K}F^{st}_\sigma\mathcal{K}^{-1}\big) - \mathrm{i}\kappa\frac{T}{2}G_\mu G_\nu\operatorname{str}\big(\Sigma_\nu F_\tau\Sigma_\mu\mathcal{K}F^{st}_\sigma\mathcal{K}^{-1}\big). \end{aligned} \tag{2.51}$$

The gauge-fixed light-cone Hamiltonian density can be read off as whatever multiplies $\dot{x}_+$ in $\mathscr{L}$:

$$\mathcal{H} = -p_- = -\mathbf{p}_- + \mathcal{H}_{\mathrm{WZ}} \tag{2.52}$$

and the aforementioned Wess-Zumino contribution is

$$\begin{aligned}\mathcal{H}_{\mathrm{WZ}} = \kappa\frac{T}{2}(G_+^2 - G_-^2)\,\mathrm{str}\big(\Sigma_+\chi\sqrt{\mathbb{1}+\chi^2}\mathcal{K}F_\sigma^{st}\mathcal{K}^{-1}\big)\\ -\kappa\frac{T}{2}G_\mu G_\nu\,\mathrm{str}\big(\Sigma_\nu\Sigma_+\chi\sqrt{\mathbb{1}+\chi^2}\Sigma_\mu\mathcal{K}F_\sigma^{st}\mathcal{K}^{-1}\big).\end{aligned} \tag{2.53}$$

Of course the explicit expressions for π_-, $G_\pm$, G_μ, ... in terms of the fields should be substituted in to find $\mathcal{H}(p_\mu, x^\mu, x'^\mu)$. Still the functional form of the Hamiltonian is too complicated to quantise so we must resort to another approach. Before moving on, let us remind ourselves that the level matching condition would require the integral over the worldsheet circumference of (2.44) to vanish.

2.2 Perturbative Quantisation

Now we will try our hand at simplifying the gauge-fixed model a little bit more before actually quantising it. In particular, we will consider string states with infinite light-cone momentum P_+ such the worldsheet gets 'decompactified' into a plane. This will allow us to obtain a two-dimensional quantum field theory which will be much more accessible and give us insight into the scattering properties of the model.

Decompactification

We just gauge fixed the action such that it takes the form

$$S = \int_{-\infty}^{\infty} d\tau \int_{-P_+/2}^{+P_+/2} d\sigma\, \mathscr{L}_{\mathrm{GF}}. \tag{2.54}$$

In the limit as $P_+ \to \infty$, the circumference of the worldsheet becomes so large that the geometry becomes that of a plane. A specific one-field solution of this limit is presented in B.2 whereby the solution is a soliton with dispersion relation

$$E - J = 2T\big|\sin\frac{p_{\mathrm{ws}}}{2}\big|. \tag{2.55}$$

This specific solution is of interest as it resembles the plane-wave dispersion which characterises a similar, integrable model. Our interest, however, will lie in the properties of the $\mathrm{AdS}_5\times S^5$ superstring under this decompactification *and* the large tension regime, $T \gg 1$.

The idea is to resort to rescaling $\sigma \to \sigma T$ such that the worldsheet circumference becomes $2\pi r T$ while we inversely rescale $(x^\mu, p_\mu, \chi) \to (x^\mu, p_\mu, \chi)/\sqrt{T}$. As a result, the action takes the form

$$S = \iint d^\sigma\left(\mathscr{L}_2 + \frac{1}{T}\mathscr{L}_4 + \frac{1}{T^2}\mathscr{L}_6 + \dots\right) \tag{2.56}$$

whereby the Lagrangian can be calculated at each order in the fields (or equivalently the inverse tension). This calculation is rather involved but was done in [1] from [8].

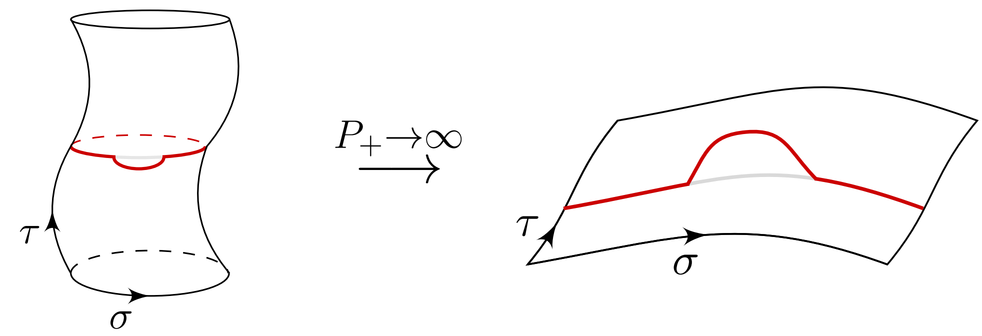


Figure 5. A string excitation in the decompactification limit

Another harmonic oscillator?

The next step is to rewrite the leading order quadratic Lagrangian in the two-index notation described at the end of 1.4. A derivation of some of the terms below is presented in B.4 to give the reader a taste of the calculation. The result is

$$\mathscr{L}_2 = P_{a\dot{a}}\dot{Y}^{a\dot{a}} + P_{\alpha\dot{\alpha}}\dot{Z}^{\alpha\dot{\alpha}} + \mathrm{i}\eta^{\dagger}_{\alpha\dot{a}}\dot{\eta}^{\alpha\dot{a}} + \mathrm{i}\theta^{\dagger}_{a\dot{\alpha}}\dot{\theta}^{a\dot{\alpha}} - \mathcal{H}_2 \tag{2.57}$$

with quadratic Hamiltonian

$$\begin{aligned}\mathcal{H}_2 = &\frac{1}{4}P_{a\dot{a}}P^{a\dot{a}} + Y_{a\dot{a}}Y^{a\dot{a}} + Y'_{a\dot{a}}Y'^{a\dot{a}} + \frac{1}{4}P_{\alpha\dot{\alpha}}P^{\alpha\dot{\alpha}} + Z_{\alpha\dot{\alpha}}Z^{\alpha\dot{\alpha}} + Z'_{\alpha\dot{\alpha}}Z'^{\alpha\dot{\alpha}} \\ &+ \eta^{\dagger}_{\alpha\dot{a}}\eta^{\alpha\dot{a}} + \frac{\kappa}{2}\eta^{\alpha\dot{a}}\eta'_{\alpha\dot{a}} - \frac{\kappa}{2}\eta^{\dagger\alpha\dot{a}}\eta'^{\dagger}_{\alpha\dot{a}} + \theta^{\dagger}_{a\dot{\alpha}}\theta^{a\dot{\alpha}} + \frac{\kappa}{2}\theta^{a\dot{\alpha}}\theta'_{a\dot{\alpha}} - \frac{\kappa}{2}\theta^{\dagger a\dot{\alpha}}\theta'^{\dagger}_{a\dot{\alpha}}.\end{aligned} \tag{2.58}$$

Looking at the kinetic term of the Lagrangian, one can read off the canonical pairs such that

$$[Y^{a\dot{a}}(\tau,\sigma), P_{b\dot{b}}(\tau,\sigma')] = \mathrm{i}\delta^a_b\delta^{\dot{a}}_{\dot{b}}\delta(\sigma-\sigma')\mathbb{1}, \qquad [Z^{\alpha\dot{\alpha}}(\tau,\sigma), P_{\beta\dot{\beta}}(\tau,\sigma')] = \mathrm{i}\delta^{\alpha}_{\beta}\delta^{\dot{\alpha}}_{\dot{\beta}}\delta(\sigma-\sigma')\mathbb{1}, \tag{2.59}$$

$$\{\theta^{a\dot{\alpha}}(\tau,\sigma), \theta^{\dagger}_{b\dot{\beta}}(\tau,\sigma')\} = \delta^a_b\delta^{\dot{\alpha}}_{\dot{\beta}}\delta(\sigma-\sigma'), \qquad \{\eta^{\alpha\dot{a}}(\tau,\sigma), \eta^{\dagger}_{\beta\dot{b}}(\tau,\sigma')\} = \delta^{\alpha}_{\beta}\delta^{\dot{a}}_{\dot{b}}\delta(\sigma-\sigma'). \tag{2.60}$$

The spacetime in which this field theory lives is the worldsheet, and there are 8 of bosons and 8 fermions because $a, \dot{a}, \alpha$ and $\dot{\alpha}$ each can take two values. In the decompactification limit the worldsheet coordinate σ is unbounded. We are thus dealing with a quantum field theory in $\mathbb{R}^{1,1}$ which describes 8 bosons and 8 fermions all with unit mass so that $\omega_p = \sqrt{1+p^2}$. To analyse this quantum field theory, we will choose the following mode decomposition. For bosons, we use the standard harmonic oscillator ladder formalism:

$$\begin{aligned}Y^{a\dot{a}}(\tau,\sigma) &= \int \frac{dp}{\sqrt{2\pi}}\frac{1}{2\sqrt{\omega_p}}\left(a^{a\dot{a}}(\tau,p)e^{\mathrm{i}p\sigma} + \epsilon^{ab}\epsilon^{\dot{a}\dot{b}}a^{\dagger}_{b\dot{b}}(\tau,p)e^{-\mathrm{i}p\sigma}\right), \\ P_{a\dot{a}}(\tau,\sigma) &= \int \frac{dp}{\sqrt{2\pi}}\mathrm{i}\sqrt{\omega_p}\left(a^{\dagger}_{a\dot{a}}(\tau,p)e^{-\mathrm{i}p\sigma} - \epsilon_{ab}\epsilon_{\dot{a}\dot{b}}a^{b\dot{b}}(\tau,p)e^{\mathrm{i}p\sigma}\right), \\ Z^{\alpha\dot{\alpha}}(\tau,\sigma) &= \int \frac{dp}{\sqrt{2\pi}}\frac{1}{2\sqrt{\omega_p}}\left(a^{\alpha\dot{\alpha}}(\tau,p)e^{\mathrm{i}p\sigma} + \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}a^{\dagger}_{\beta\dot{\beta}}(\tau,p)e^{-\mathrm{i}p\sigma}\right), \\ P_{\alpha\dot{\alpha}}(\tau,\sigma) &= \int \frac{dp}{\sqrt{2\pi}}\mathrm{i}\sqrt{\omega_p}\left(a^{\dagger}_{\alpha\dot{\alpha}}(\tau,p)e^{-\mathrm{i}p\sigma} - \epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}a^{\beta\dot{\beta}}(\tau,p)e^{\mathrm{i}p\sigma}\right).\end{aligned} \tag{2.61}$$

For fermions, we must also include functions f_p, h_p which play the 1-dimensional role of spinors:

$$\begin{aligned}
\theta^{a\dot{\alpha}}(\tau,\sigma) &= \int \frac{dp}{\sqrt{2\pi}} \frac{e^{-\mathrm{i}\pi/4}}{\sqrt{\omega_p}} \left(f_p\, a^{a\dot{\alpha}}(\tau,p) e^{\mathrm{i}p\sigma} + h_p\, a^{\dagger a\dot{\alpha}}(\tau,p) e^{-\mathrm{i}p\sigma} \right), \\
\eta^{\alpha\dot{a}}(\tau,\sigma) &= \int \frac{dp}{\sqrt{2\pi}} \frac{e^{-\mathrm{i}\pi/4}}{\sqrt{\omega_p}} \left(f_p\, a^{\alpha\dot{a}}(\tau,p) e^{\mathrm{i}p\sigma} + h_p\, a^{\dagger \alpha\dot{a}}(\tau,p) e^{-\mathrm{i}p\sigma} \right).
\end{aligned} \tag{2.62}$$

The specific form of the fermionic wavefunctions can be specified when it comes time to diagonalise the Hamiltonian, in order to easily find the spectrum of this theory. One can derive the commutation relations for the ladder operators themselves by inverting the Fourier transforms above. For real f_p and h_p, a calculation in B.5 shows

$$[a^{a\dot{a}}(\tau,p), a^\dagger_{b\dot{b}}(\tau,p')] = \delta^a_b \delta^{\dot{a}}_{\dot{b}} \delta(p-p') \mathbb{1}, \tag{2.63}$$

$$\{a^{a\dot{\alpha}}(p), a^\dagger_{b\dot{\beta}}(p')\} = \omega_p \frac{(f^2_{-p} + h^2_{-p})}{(f_{-p} f_p - h_{-p} h_p)^2} \delta^a_b \delta^{\dot{\alpha}}_{\dot{\beta}} \delta(p-p') \mathbb{1} \tag{2.64}$$

It will prove useful to group the two-index notation since, in terms of creation and annihilation operators $a^{M\dot{M}}(\tau,p)$ and $a^\dagger_{M\dot{M}}(\tau,p)$, the Lagrangian is found in B.5 to be of the diagonal form

$$\mathbb{L}_2 = \int dp \left(\mathrm{i} a^\dagger_{M\dot{M}}(p) \dot{a}^{M\dot{M}}(p) - \omega_p\, a^\dagger_{M\dot{M}}(p) a^{M\dot{M}}(p) \right) \tag{2.65}$$

provided the fermionic wavefunctions satisfy

$$f_p = \sqrt{\frac{1+\omega_p}{2}}, \qquad h_p = \frac{\kappa p}{2 f_p} \implies f_p^2 = 1 + h_p^2 = \omega_p - h_p^2, \tag{2.66}$$

and uppercase Latin indices take values $M = 1, ..., 4$ and $\dot{M} = \dot{1}, ..., \dot{4}$. We can take $\kappa = 1$ for definiteness. The lowercase Latin indices $a, \dot{a}$ corresponding to S^5 are taken to be even, while the lowercase Greek indices $\alpha, \dot{\alpha}$ corresponding to AdS_5 are odd (i.e. $|a| = |\dot{a}| = 0$ while $|\alpha| = |\dot{\alpha}| = 1$). In this notation, we can consider the mixed bracket

$$\left[a^{M\dot{M}}(\tau,p),\, a^\dagger_{N\dot{N}}(\tau,p') \right\} = -(-1)^{|M|+|\dot{M}|} \left\{ a^\dagger_{N\dot{N}}(\tau,p')\, a^{M\dot{M}}(\tau,p) \right] = \delta^M_N \delta^{\dot{M}}_{\dot{N}} \delta(p-p') \tag{2.67}$$

such that bosonic operators $a_{a\dot{a}}$ and $a_{\alpha,\dot{\alpha}}$ satisfy equal-τ commutation relations whereas we get anti-commutation relations fermionic modes $a_{a\dot{\alpha}}$ and $a_{\alpha,\dot{a}}$. This form (2.65) of the Lagrangian is not strictly correct as we should have kept the real form of the non-zero kinetic term. Bringing the term to the present form involved integrating by parts and using the Heisenberg evolution result

$$\dot{a}^{M\dot{M}}(\tau,p) = \mathrm{i}[\mathbb{H}_2, a^{M\dot{M}}(\tau,p)] = -\mathrm{i}\omega_p\, a^{M\dot{M}}(\tau,p) \tag{2.68}$$

to find commutation relations between the ladder operators and their τ derivatives.[11] Note that all of this*follows* from imposing the canonical relations (2.59) and (2.60) with a generic mode decomposition.

[11] One could also point out that substituting $\dot{a}^{a\dot{a}}(p)$ into the above formula returns in $\mathbb{L}_{\mathrm{AdS}_5} = 0$, a puzzling outcome. Had we been careful, we would have retained non-zero boundary terms when integrating by parts. These constant shifts of the Lagrangian, which are equivalent to canonical transformations, would have saved us from this zero.

We have made our lives much easier with regard to analysing the spectrum of this decompactified model. This is because we see that the Hamiltonian is that of a standard harmonic oscillator, since

$$\mathbb{H}_2 = \int dp\ \omega_p\, a^\dagger_{M\dot{M}}(p) a^{M\dot{M}}(p) \tag{2.69}$$

which has eigenstates in the Q-particle Fock space

$$\left\{ |\Psi\rangle = a^\dagger_{M_1\dot{M}_1}(p_1) a^\dagger_{M_2\dot{M}_2}(p_2) \ldots a^\dagger_{M_Q\dot{M}_Q}(p_Q)|0\rangle \right\}$$

spanned by creation operators $a^\dagger_{M\dot{M}}(p)$ and stemming from a vacuum $|0\rangle$ defined such that any annihilation operator destroys it, i.e. $a^{M\dot{M}}(p)|0\rangle = 0$ for any $M, \dot{M}$. Thus the ground state is $\mathbb{H}_2|0\rangle = E_0|0\rangle = 0$ and the excited states have the usual spectrum

$$\mathbb{H}_2|\Psi\rangle = E_\Psi|\Psi\rangle = \left(\sum_{i=1}^{Q} \omega_{p_i} \right) |\Psi\rangle. \tag{2.70}$$

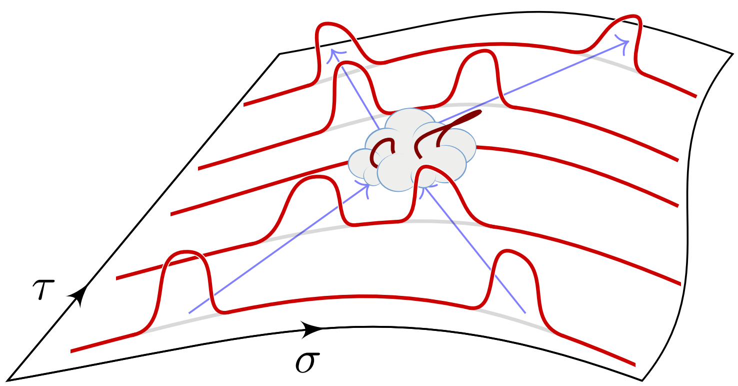


Figure 6. Scattering of two superstring worldsheet excitations.

The field theory's total momentum operator $\mathbb{P}$ originates from the classical worldsheet momentum

$$\mathbb{P} = p_{\text{ws}} = -\frac{1}{T}\int d\sigma\ p_\mu x'^\mu$$

since the particles' 'spacetime' *is* the worldsheet. A straightforward calculation in B.5 shows

$$\mathbb{P} = -\frac{1}{T}\int d\sigma \left(P_{a\dot{a}} Y'^{a\dot{a}} + P_{\alpha\dot{\alpha}} Z'^{\alpha\dot{\alpha}} + \mathrm{i}\theta^\dagger_{a\dot{\alpha}}\theta'^{a\dot{\alpha}} + \mathrm{i}\eta^\dagger_{\alpha\dot{a}}\eta'^{\alpha\dot{a}} \right) = \frac{1}{T}\int dp\ p\ a^\dagger_{M\dot{M}}(p) a^{M\dot{M}}(p), \tag{2.71}$$

which implies the energy eigenstates are also eigenstates of $\mathbb{P}$ with eigenvalue

$$\mathbb{P}|\Psi\rangle = P_\Psi|\Psi\rangle = \left(\frac{1}{T}\sum_{i=1}^{Q} p_i \right) |\Psi\rangle. \tag{2.72}$$

We now have a different interpretation for the level matching condition: the momenta of any number of particles must add to zero for an energy eigenstate $|\Psi\rangle$ corresponding to a physical state on the string.

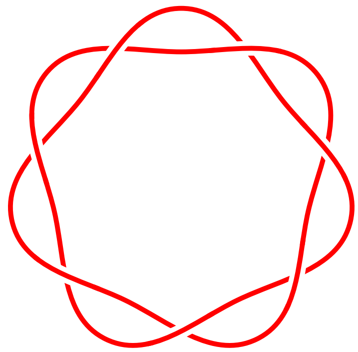
The way in which decompactification entered the mix was subtle: we were able to think of σ as an unbounded real number which in turn allowed us to use Fourier transforms to decompose the fields into specific modes. These modes are found to correspond to harmonic oscillators with a $\mathbb{Z}_2$-grading, but what we have discussed so far is a *free* theory, Interactions between different species of bosons and fermions are first found in the quartic Hamiltonian (see [1]) of the perturbative expansion for $T \gg 1$.

Closed sectors

Just as in regular quantum field theory, only certain decay modes are possible because of selection rules. In the case of our decompactified model, the charges which are to be conserved in a scattering process are those associated with the $SU(2)^4$ transformations (1.132). For example, the Q-particle states spanned uniquely by creators $a^\dagger_{1\dot{1}}$ have maximal charge $(Q/2, Q/2)$ such that they can only scatter between themselves. This is because each creation operator is charged under the $SU(2)_a$ and $SU(2)_{\dot{a}}$, each giving a 'spin' of $1/2$. Explicitly, using for example σ^3 as the spin projector for $SU(2)_a$, we would have

$$\sigma^3_a \begin{pmatrix} a^\dagger_{1\dot{a}} \\ a^\dagger_{2\dot{a}} \end{pmatrix} = \begin{pmatrix} +\frac{1}{2} a^\dagger_{1\dot{a}} \\ -\frac{1}{2} a^\dagger_{2\dot{a}} \end{pmatrix}. \tag{2.73}$$

A host of other properties of the model follows from this analysis, namely the factoring of the $\mathcal{S}$-matrix into two-body scattering.



Chapter 3
Conclusion

Albeit perturbatively, we have succeeded to some extent in quantising the $\mathrm{AdS}_5 \times S^5$ superstring. To get to that point the model had to be simplified using some tricks, namely fixing the light-cone and κ-symmetry gauge in a favourable manner as well as decompactifying the worldsheet cylinder to a plane. This resulted in a perturbative expansion of the action in which we kept the leading order, quadratic, free Lagrangian which described 8 massive bosons and 8 massive fermions, all with the same mass. However, the original goal was to understand the *full spectrum* of the superstring so that one may relate, for example, the target space energy of string states to the scaling dimension of operators through AdS/CFT. Naively, the way to approach full canonical quantisation would involve including the full range of interactions from the light-cone Hamiltonian density up to higher orders in the fields. Researchers found a way around this. The review [1] on which this work was based was supposed to preceed another review, *Part II*, in which the plane would be 'recompactified' to a cylinder in preparation for use of the thermodynamic Bethe ansatz. In the end, it is rather surprising that by analysing the scattering of vibrational modes of a single string on its worldsheet, one can retrieve the full spectrum and begin to use the AdS/CFT duality.

Appendix A
Chapter 1

A.1 Extended spinor algebras

We aim to show that adding $n^{i6} = \frac{\mathrm{i}}{2}\gamma^i$ to the five gamma matrices preserves the relations (1.13) by computing $[n^{i6}, n^{kl}]$. First, we note that

$$n^{kl} = \frac{1}{4}[\gamma^k, \gamma^l] = \frac{1}{4}\left(\gamma^k\gamma^l - (2\delta^{kl}\mathbb{1} - \gamma^k\gamma^l)\right) = \frac{1}{2}\left(\gamma^k\gamma^l - \delta^{kl}\mathbb{1}\right). \tag{A.1}$$

Clearly if either $l = 6$ or $k = 6$, we get a result of the form

$$[n^{i6}, n^{kl}] \overset{l=6}{=} -\delta^{l6}\frac{1}{4}[\gamma^i, \gamma^k] = -\delta^{l6}n^{ik}, \qquad [n^{i6}, n^{kl}] \overset{k=6}{=} \delta^{k6}n^{il}. \tag{A.2}$$

If however $k \neq 6$ and $l \neq 6$, we get

$$\begin{aligned}
[n^{i6}, n^{kl}] &= \frac{\mathrm{i}}{4}[\gamma^i, \gamma^k\gamma^l - \delta^{kl}\mathbb{1}] = \frac{\mathrm{i}}{4}[\gamma^i, \gamma^k\gamma^l] = \frac{\mathrm{i}}{4}\gamma^k[\gamma^i, \gamma^l] + \frac{\mathrm{i}}{4}[\gamma^i, \gamma^k]\gamma^l \\
&= \mathrm{i}\gamma^k n^{il} + \mathrm{i}n^{ik}\gamma^l = \frac{\mathrm{i}}{2}\gamma^k\left(\gamma^i\gamma^l - \delta^{il}\mathbb{1}\right) + \frac{\mathrm{i}}{2}\left(\gamma^i\gamma^k - \delta^{ik}\mathbb{1}\right)\gamma^l \\
&= \frac{\mathrm{i}}{2}\left(\gamma^k\gamma^i\gamma^l + \gamma^i\gamma^k\gamma^l\right) - \delta^{il}n^{k6} - \delta^{ik}n^{l6} = \frac{\mathrm{i}}{2}\left(2\delta^{ik}\gamma^l - \gamma^i\gamma^k\gamma^l + \gamma^i\gamma^k\gamma^l\right) - \delta^{il}n^{k6} - \delta^{ik}n^{l6} \\
&= -\delta^{il}n^{k6} + \delta^{ik}n^{l6}.
\end{aligned}$$

Adding all these cases, which do not contribute whenever their conditions are not met, we find (1.14) which extends the $\mathfrak{so}(5)$ spinor relations (1.13) to $\mathfrak{so}(6)$.

Similarly, let us define $\gamma^0 \equiv \mathrm{i}\gamma^5$ and look at the extended generators of $\mathfrak{so}(4,1)$ satisfying (1.15)

$$m^{ij} = \frac{1}{4}[\gamma^i, \gamma^j], \qquad m^{i5} \equiv \frac{1}{2}\gamma^i, \qquad i, j = 0, ..., 4. \tag{A.3}$$

The addition of m^{i5} should preserve the relations $[m^{ij}, m^{kl}]$ for $i, j... = 0, ..., 5$. As above we start with

$$[m^{i5}, m^{kl}] \overset{l=5}{=} \delta^{l5}\frac{1}{4}[\gamma^i, \gamma^k] = \delta^{l5}m^{ik}, \qquad [m^{i5}, m^{kl}] \overset{k=5}{=} -\delta^{k5}m^{il}. \tag{A.4}$$

If however $k \neq 5$ and $l \neq 5$, we get

$$\begin{aligned}
[m^{i5}, m^{kl}] &= \frac{1}{4}[\gamma^i, \gamma^k\gamma^l - \delta^{kl}\mathbb{1}] = \frac{1}{4}[\gamma^i, \gamma^k\gamma^l] = \frac{1}{4}\gamma^k[\gamma^i, \gamma^l] + \frac{1}{4}[\gamma^i, \gamma^k]\gamma^l \\
&= \gamma^k m^{il} + m^{ik}\gamma^l = \frac{1}{2}\gamma^k\left(\gamma^i\gamma^l - \delta^{il}\mathbb{1}\right) + \frac{1}{2}\left(\gamma^i\gamma^k - \delta^{ik}\mathbb{1}\right)\gamma^l \\
&= \frac{1}{2}\left(\gamma^k\gamma^i\gamma^l + \gamma^i\gamma^k\gamma^l\right) - \delta^{il}m^{k5} - \delta^{ik}m^{l5} = \frac{1}{2}\left(2\delta^{ik}\gamma^l - \gamma^i\gamma^k\gamma^l + \gamma^i\gamma^k\gamma^l\right) - \delta^{il}m^{k5} - \delta^{ik}m^{l5} \\
&= -\delta^{il}m^{k5} + \delta^{ik}m^{l5}.
\end{aligned}$$

All that is left to do is recognise that if $\eta^{i5} = -\delta^{i5}$, then the relations are satisfied by the generators m^{ij} for $i, j = 0, ..., 5$ and become those of $\mathfrak{so}(4,2)$ instead.

A.2 Endowing $\mathfrak{su}(2,2|4)$ with a $\mathbb{Z}_4$-grading

Let us first discuss the continuous subgroup of $\mathrm{Aut}(\mathfrak{sl}(4|4))$. We want to show that the continuous dilatation transformations $\delta_\rho(M)$ (1.18) can be written as $\delta_\rho(M) = e^{\frac{1}{2}\Upsilon \ln \rho} M e^{-\frac{1}{2}\Upsilon \ln\rho}$. We start with by noticing $\Upsilon^2 = \mathbb{1}_8$ and so e^Υ becomes

$$\begin{aligned} e^\Upsilon &= \mathbb{1}_8 + \Upsilon + \frac{1}{2!}\mathbb{1}_8 + \frac{1}{3!}\Upsilon + \ldots = \sum_{n=0}^\infty \frac{1}{(2n)!}\mathbb{1}_8 + \sum_{n=0}^\infty \frac{1}{(2n+1)!}\Upsilon \\ &= \cosh(1)\mathbb{1}_8 + \sinh(1)\Upsilon = \begin{pmatrix} e\mathbb{1}_4 & 0 \\ 0 & \frac{1}{e}\mathbb{1}_4 \end{pmatrix}. \end{aligned} \tag{A.5}$$

By raising both sides of the equation to the power of $\frac{1}{2}\ln\rho = \ln\rho^{1/2}$, we get

$$e^{\frac{1}{2}\Upsilon\ln\rho} = \begin{pmatrix} \rho^{\frac{1}{2}}\mathbb{1}_4 & 0 \\ 0 & \rho^{-\frac{1}{2}}\mathbb{1}_4 \end{pmatrix} \implies e^{-\frac{1}{2}\Upsilon\ln\rho} = \begin{pmatrix} \rho^{-\frac{1}{2}}\mathbb{1}_4 & 0 \\ 0 & \rho^{\frac{1}{2}}\mathbb{1}_4 \end{pmatrix}, \tag{A.6}$$

which clearly shows (1.18). This transformation is an automorphism on $\mathfrak{su}(2,2|4)$ if it preserves the fermionic reality condition $\eta = -\theta^\dagger\Sigma$. To this end, the transformation parameter must satisfy $|\rho|^2 = 1$. It is clear that $\delta_{-1}(M) = \Upsilon M \Upsilon^{-1}$ and we note $\delta_{-1}(M) = M$ if M is even whereas $\delta_{-1}(M) = -M$ if M is odd.

Next we want to show that the fourth-order automorphism $\Omega(M)$ restricts to the subalgebra $\mathfrak{su}(2,2|4) \subset \mathfrak{sl}(4|4)$. To do this, we should show that $\Omega(M)^\dagger = -H\Omega(M)H^{-1}$. Since $[K,\Sigma] = [\gamma^5, \gamma^2\gamma^4] = 0$, we know that $[\mathcal{K}, H] = 0$ which will be useful since we can use the reality condition (1.6) for $M \in \mathfrak{su}(2,2|4)$. However, the issue is that $(M^{st})^\dagger \neq (M^\dagger)^{st}$ in general. In particular, for M even

$$M = \begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix} \implies (M^{st})^\dagger = (M^\dagger)^{st} = \begin{pmatrix} m^* & 0 \\ 0 & n^* \end{pmatrix},$$

while for M odd

$$M = \begin{pmatrix} 0 & \theta \\ \eta & 0 \end{pmatrix} \implies (M^{st})^\dagger = -(M^\dagger)^{st} = \begin{pmatrix} 0 & \theta^* \\ -\eta^* & 0 \end{pmatrix}.$$

One can find $\Omega(M)^\dagger$ using the identities

$$\begin{gathered} \mathcal{K}^\dagger = \mathcal{K}^{-1} = \mathcal{K}^{st} = -\mathcal{K}, \qquad \Upsilon^\dagger = \Upsilon^{-1} = \Upsilon^{st} = \Upsilon, \qquad H^\dagger = H^{-1} = H^{st} = H, \\ [\mathcal{K}, H] = [\mathcal{K}, \Upsilon] = [\Upsilon, H] = 0, \end{gathered} \tag{A.7}$$

and the fact $\delta_{-1}(M) = \pm M = \Upsilon M \Upsilon^{-1}$ for M even $(+)$ or odd $(-)$. We get $(M^{st})^\dagger = \Upsilon(M^\dagger)^{st}\Upsilon^{-1}$ so

$$\begin{aligned} \Omega(M)^\dagger &= \mathcal{K}^\dagger (M^{st})^\dagger \mathcal{K}^\dagger = -\mathcal{K}(M^{st})^\dagger\mathcal{K}^{-1} \\ &= -\mathcal{K}\Upsilon(M^\dagger)^{st}\Upsilon^{-1}\mathcal{K}^{-1} = -\mathcal{K}\Upsilon(-HMH^{-1})^{st}\Upsilon^{-1}\mathcal{K}^{-1} \\ &= \mathcal{K}\Upsilon H M^{st} H^{-1}\Upsilon^{-1}\mathcal{K}^{-1} = -(\Upsilon H)(-\mathcal{K}M^{st}\mathcal{K}^{-1})(\Upsilon H)^{-1} \\ &= -(\Upsilon H)\Omega(M)(\Upsilon H)^{-1}. \end{aligned}$$

For M even, so $M = M^{(0)} + M^{(2)}$, the hypercharge Υ can be ignored in the above expression since $\delta_{-1}(M) = M$. This means $\Omega(M)$ restricts to the bosonic subalgebra of $\mathfrak{su}(2,2|4)$. To see that it also restricts to the entire subalgebra, we should look at $M^{(k)\dagger}$. In fact, if we use the properties

$$\Omega^2(M) = (M^{st})^{st} = \delta_{-1}(M) = \Upsilon M \Upsilon \implies \Omega^3(M) = \Upsilon\Omega(M)\Upsilon,$$

we get the strong result for any $k = 0, 1, 2, 3$

$$\begin{aligned} M^{(k)\dagger} &= \frac{1}{4}\left[-HMH^{-1} + (-1)^{3k}\mathrm{i}^{3k}\Omega(M)^\dagger + \mathrm{i}^{2k}\Omega^2(M)^\dagger + (-1)^k\mathrm{i}^k\Omega^3(M)^\dagger\right] \\ &= \frac{1}{4}\left[-HMH^{-1} + \mathrm{i}^k\left(-(\Upsilon H)\Omega(M)(\Upsilon H)^{-1}\right) + \mathrm{i}^{2k}\left(\Upsilon M\Upsilon\right)^\dagger + \mathrm{i}^{3k}\left(\Upsilon\Omega(M)\Upsilon\right)^\dagger\right] \\ &= \frac{1}{4}\left[-HMH^{-1} + \mathrm{i}^k\left(-H\Omega^3(M)H^{-1}\right) + \mathrm{i}^{2k}\left(-H\Upsilon M\Upsilon H^{-1}\right) + \mathrm{i}^{3k}\left(-H\Omega(M)H^{-1}\right)\right] \\ &= \frac{1}{4}\left[-HMH^{-1} + \mathrm{i}^k\left(-H\Omega^3(M)H^{-1}\right) + \mathrm{i}^{2k}\left(-H\Omega^2H^{-1}\right) + \mathrm{i}^{3k}\left(-H\Omega(M)H^{-1}\right)\right] \\ &= -HM^{(k)}H^{-1}. \end{aligned}$$

Because any $M \in \mathfrak{sl}(4|4)$ can be uniquely decomposed by the $\mathbb{Z}_4$-grading (1.25), and since we just showed each component $M^{(k)}$ is independently an element of $(\mathfrak{p})\mathfrak{su}(2,2|4)$, it must be that the subalgebra $\mathfrak{su}(2,2|4)$ can itself be endowed with the $\mathbb{Z}_4$-grading $\Omega(M)$. From now on, we relabel $\mathscr{G} = \mathfrak{su}(2,2|4)$ and the $\mathbb{Z}_4$-graded decomposition of $\mathscr{G}$ is given with respect to the automorphism $\Omega(M)$ by (1.25).

If we start with a matrix M of the generic form (1.5), then the explicit components $M^{(k)}$ can be found by computing $\Omega^k(M)$ and evaluating (1.25). Using the usual identities,

$$\Omega(M) = -\begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}\begin{pmatrix} m^t & -\eta^t \\ \theta^t & n^t \end{pmatrix}\begin{pmatrix} K^{-1} & 0 \\ 0 & K^{-1} \end{pmatrix} = \begin{pmatrix} -Km^tK^{-1} & K\eta^tK^{-1} \\ -K\theta^tK^{-1} & -Kn^tK^{-1} \end{pmatrix},$$

$$\Omega^2(M) = \delta_{-1}(M) = \begin{pmatrix} m & -\theta \\ -\eta & n \end{pmatrix}, \qquad \Omega^3(M) = \Upsilon\Omega(M)\Upsilon = \begin{pmatrix} -Km^tK^{-1} & -K\eta^tK^{-1} \\ K\theta^tK^{-1} & -Kn^tK^{-1} \end{pmatrix}.$$

we find the decomposition $M^{(k)}$ (1.26). To find explicit expressions for the even components $M^{(0)}, M^{(2)}$ in terms of bosonic generators (1.16), we notice that the matrix $K = -\gamma^2\gamma^4$ was constructed such that

$$\begin{aligned} \left(K(\gamma^i)^tK^{-1}\right)^* &= \gamma^2\gamma^4(\gamma^i)^\dagger\gamma^4\gamma^2 = \gamma^2\gamma^4\gamma^i\gamma^4\gamma^2 = \gamma^2(2\delta^{i4} - \gamma^i\gamma^4)\gamma^4\gamma^2 \\ &= 2\delta^{i4}\gamma^2\gamma^4\gamma^2 - \gamma^2\gamma^i\gamma^2 = -2\delta^{i4}\gamma^4 - (2\delta^{i2} - \gamma^i\gamma^2)\gamma^2 \\ &= \gamma^i - 2(\delta^{i4}\gamma^4 + \delta^{i2}\gamma^2) \stackrel{\dagger}{=} (\gamma^i)^* \end{aligned}$$

or equivalently $K(\gamma^i)^tK^{-1} = \gamma^i$. In turn this means

$$K[\gamma^i, \gamma^j]^tK^{-1} = K[(\gamma^j)^t, (\gamma^i)^t]^tK^{-1} = [K(\gamma^j)^tK^{-1}, K(\gamma^i)^tK^{-1}] = -[\gamma^i, \gamma^j].$$

Looking at the expressions for $M^{(k)}$ (1.26), it is apparent that one can span $\mathscr{G}^{(0)}$ by expressing the even 'upper block' elements m in terms of the $\mathfrak{so}(4,1) \subset \mathfrak{su}(2,2)$ generators $\left\{\frac{1}{4}[\gamma^i, \gamma^j], \frac{\mathrm{i}}{4}[\gamma^i, \gamma^5]\right\}$ for $i, j = 1, ..., 4$ and the 'lower block' elements n in terms of the $\mathfrak{so}(5)$ generators $\left\{\frac{1}{4}[\gamma^i, \gamma^j]\right\}$ for $i, j = 1, ..., 5$. Similarly, the elements m of the projection $\mathscr{G}^{(2)}$ can be spanned by the remaining bosonic generators $\left\{\frac{1}{2}\gamma^i, \frac{\mathrm{i}}{2}\gamma^5\right\} \in \mathfrak{su}(2,2)$ for $i = 1, ..., 4$ and the elements n by $\left\{\frac{\mathrm{i}}{2}\gamma^i\right\} \in \mathfrak{su}(4)$ for $i = 1, ..., 5$. Explicit matrices are given in (1.27) and (1.28) respectively.

† Only γ^2 and γ^4 are imaginary such that $(\gamma^i)^* = -\gamma^i$ for $i = 2, 4$ and $(\gamma^i)^* = \gamma^i$ otherwise.

A.3 Green-Schwarz equations of motion

To begin a long derivation of the equations of motion, we will show that for $M_1,\ M_2 \in \mathfrak{su}(2,2|4)$

$$\mathrm{str}\big(\Omega^k(M_1)M_2\big) = \mathrm{str}\big(M_1\Omega^{4-k}(M_2)\big), \qquad k = 1,2,3. \tag{A.8}$$

By definition $\Omega(M) = -\mathcal{K}M^{st}\mathcal{K}^{-1}$. Using the supertrace identities

$$\mathrm{str}\big(AB\big) = \mathrm{str}\big(BA\big), \qquad \mathrm{str}\big(A^{st}\big) = \mathrm{str}\big(A\big) \tag{A.9}$$

and the fact that $\Omega^4(M) = M$,

$$\begin{aligned}
\mathrm{str}\big(\Omega^k(M_1)M_2\big) &= \mathrm{str}\big(\Omega^k(M_1)\Omega^4(M_2)\big) = \mathrm{str}\big(\mathcal{K}(\Omega^{k-1}(M_1))^{st}\mathcal{K}^{-1}\mathcal{K}(\Omega^{4-1}(M_2))^{st}\mathcal{K}^{-1}\big)\\
&= \mathrm{str}\big(\Omega^{k-1}(M_1)\Omega^{4-1}(M_2)\big) = \ldots = \mathrm{str}\big(\Omega^{k-k}(M_1)\Omega^{4-k}(M_2)\big)\\
\implies \mathrm{str}\big(\Omega^k(M_1)M_2\big) &= \mathrm{str}\big(M_1\Omega^{4-k}(M_2)\big).
\end{aligned} \tag{A.10}$$

Armed with (A.10), we can show that $\mathrm{str}\big(A_\alpha^{(j)}A_\beta^{(4-j)}\big) = \mathrm{str}\big(A_\alpha A_\beta^{(4-j)}\big) = \mathrm{str}\big(A_\alpha^{(j)}A_\beta\big)$ when $j = 1,2,3$. For convenience, define $\Omega^k(A_\alpha) \equiv \Omega_\alpha^k$. Then according to (1.31), and remembering that $\mathrm{i}^4 = 1$ so $\mathrm{i}^{-j} = \mathrm{i}^{3j}$,

$$\begin{aligned}
\mathrm{str}\big(A_\alpha^{(j)}A_\beta^{(4-j)}\big) &= \frac{1}{16}\,\mathrm{str}\Big[(A_\alpha + \mathrm{i}^{3j}\Omega_\alpha + \mathrm{i}^{2j}\Omega_\alpha^2 + \mathrm{i}^{j}\Omega_\alpha^3)(A_\beta + \mathrm{i}^{3(4-j)}\Omega_\beta + \mathrm{i}^{2(4-j)}\Omega_\beta^2 + \mathrm{i}^{(4-j)}\Omega_\beta^3)\Big]\\
&= \frac{1}{16}\,\mathrm{str}\Big[(A_\alpha + \mathrm{i}^{3j}\Omega_\alpha + \mathrm{i}^{2j}\Omega_\alpha^2 + \mathrm{i}^{j}\Omega_\alpha^3)(A_\beta + \mathrm{i}^{-3j}\Omega_\beta + \mathrm{i}^{-2j}\Omega_\beta^2 + \mathrm{i}^{-j}\Omega_\beta^3)\Big]\\
&= \frac{1}{16}\,\mathrm{str}\Big[(A_\alpha + \mathrm{i}^{3j}\Omega_\alpha + \mathrm{i}^{2j}\Omega_\alpha^2 + \mathrm{i}^{j}\Omega_\alpha^3)(A_\beta + \mathrm{i}^{j}\Omega_\beta + \mathrm{i}^{2j}\Omega_\beta^2 + \mathrm{i}^{3j}\Omega_\beta^3)\Big]\\
&= \frac{1}{16}\big[\mathrm{str}\big(A_\alpha A_\beta\big) + \mathrm{i}^{j}\,\mathrm{str}\big(A_\alpha\Omega_\beta\big) + \mathrm{i}^{2j}\,\mathrm{str}\big(A_\alpha\Omega_\beta^2\big) + \mathrm{i}^{3j}\,\mathrm{str}\big(A_\alpha\Omega_\beta^3\big)\\
&\quad + \mathrm{i}^{3j}\,\mathrm{str}\big(\Omega_\alpha A_\beta\big) + \mathrm{str}\big(\Omega_\alpha\Omega_\beta\big) + \mathrm{i}^{j}\,\mathrm{str}\big(\Omega_\alpha\Omega_\beta^2\big) + \mathrm{i}^{2j}\,\mathrm{str}\big(\Omega_\alpha\Omega_\beta^3\big)\\
&\quad + \mathrm{i}^{2j}\,\mathrm{str}\big(\Omega_\alpha^2 A_\beta\big) + \mathrm{i}^{3j}\,\mathrm{str}\big(\Omega_\alpha^2\Omega_\beta\big) + \mathrm{str}\big(\Omega_\alpha^2\Omega_\beta^2\big) + \mathrm{i}^{j}\,\mathrm{str}\big(\Omega_\alpha^2\Omega_\beta^3\big)\\
&\quad + \mathrm{i}^{j}\,\mathrm{str}\big(\Omega_\alpha^3 A_\beta\big) + \mathrm{i}^{2j}\,\mathrm{str}\big(\Omega_\alpha^3\Omega_\beta\big) + \mathrm{i}^{3j}\,\mathrm{str}\big(\Omega_\alpha^3\Omega_\beta^2\big) + \mathrm{str}\big(\Omega_\alpha^3\Omega_\beta^3\big)\big].
\end{aligned} \tag{A.11}$$

The terms with the same color are related by (A.10) so that on one hand

$$\begin{aligned}
\mathrm{str}\big(A_\alpha^{(j)}A_\beta^{(4-j)}\big) &= \frac{4}{16}\big[\mathrm{str}\big(A_\alpha A_\beta\big) + \mathrm{i}^{3j}\,\mathrm{str}\big(\Omega_\alpha A_\beta\big) + \mathrm{i}^{2j}\,\mathrm{str}\big(\Omega_\alpha^2 A_\beta\big) + \mathrm{i}^{j}\,\mathrm{str}\big(\Omega_\alpha^3 A_\beta\big)\big]\\
&= \mathrm{str}\,\big[\frac{1}{4}(A_\alpha + \mathrm{i}^{3j}\Omega_\alpha + \mathrm{i}^{2j}\Omega_\alpha^2 + \mathrm{i}^{j}\Omega_\alpha^3)A_\beta\big]\\
&= \mathrm{str}\big(A_\alpha^{(j)}A_\beta\big),
\end{aligned} \tag{A.12}$$

and on the other hand, again using $\mathrm{i}^{3j} = \mathrm{i}^{-j} = \mathrm{i}^4\mathrm{i}^{-j} = \mathrm{i}^{4-j}$,

$$\begin{aligned}
\mathrm{str}\big(A_\alpha^{(j)}A_\beta^{(4-j)}\big) &= \frac{4}{16}\big[\mathrm{str}\big(A_\alpha A_\beta\big) + \mathrm{i}^{3(4-j)}\,\mathrm{str}\big(A_\alpha\Omega_\beta\big) + \mathrm{i}^{2(4-j)}\,\mathrm{str}\big(A_\alpha\Omega_\beta^2\big) + \mathrm{i}^{(4-j)}\,\mathrm{str}\big(A_\alpha\Omega_\beta^3\big)\big]\\
&= \mathrm{str}\,\big[A_\alpha\frac{1}{4}(A_\beta + \mathrm{i}^{3(4-j)}\Omega_\beta + \mathrm{i}^{2(4-j)}\Omega_\beta^2 + \mathrm{i}^{(4-j)}\Omega_\beta^3)\big]\\
&= \mathrm{str}\big(A_\alpha A_\beta^{(4-j)}\big).
\end{aligned} \tag{A.13}$$

In particular this means that

$$\mathrm{str}\big(A_\alpha^{(1)}A_\beta^{(3)}\big) = \mathrm{str}\big(A_\alpha A_\beta^{(3)}\big) = \mathrm{str}\big(A_\alpha^{(1)}A_\beta\big), \tag{A.14}$$

$$\mathrm{str}\big(A_\alpha^{(2)}A_\beta^{(2)}\big) = \mathrm{str}\big(A_\alpha A_\beta^{(2)}\big) = \mathrm{str}\big(A_\alpha^{(2)}A_\beta\big), \tag{A.15}$$

which further implies, using the product rule and the cyclicity of the supertrace,

$$\begin{aligned}\delta\,\mathrm{str}\big(A_\alpha^{(1)}A_\beta^{(3)}\big) &= \mathrm{str}\big(\delta A_\alpha A_\beta^{(3)}\big) + \mathrm{str}\big(A_\alpha^{(1)}\delta A_\beta\big)\\ &= \mathrm{str}\big(\delta A_\alpha A_\beta^{(3)} + \delta A_\beta A_\alpha^{(1)}\big),\end{aligned} \tag{A.16}$$

$$\begin{aligned}\delta\,\mathrm{str}\big(A_\alpha^{(2)}A_\beta^{(2)}\big) &= \mathrm{str}\big(\delta A_\alpha A_\beta^{(2)}\big) + \mathrm{str}\big(A_\alpha^{(2)}\delta A_\beta\big)\\ &= \mathrm{str}\big(\delta A_\alpha A_\beta^{(2)} + \delta A_\beta A_\alpha^{(2)}\big).\end{aligned} \tag{A.17}$$

Substituting (A.16) and (A.17) into $\delta\mathscr{L}$ gives[12]

$$\begin{aligned}\delta\mathscr{L} &= -\frac{T}{2}\Big[\gamma^{\alpha\beta}\delta\,\mathrm{str}\big(A_\alpha^{(2)}A_\beta^{(2)}\big) + \kappa\varepsilon^{\alpha\beta}\delta\,\mathrm{str}\big(A_\alpha^{(1)}A_\beta^{(3)}\big)\Big]\\ &= -\frac{T}{2}\Big[\gamma^{\alpha\beta}\,\mathrm{str}\big(\delta A_\alpha A_\beta^{(2)} + \delta A_\beta A_\alpha^{(2)}\big) + \kappa\varepsilon^{\alpha\beta}\,\mathrm{str}\big(\delta A_\alpha A_\beta^{(3)} + \delta A_\beta A_\alpha^{(1)}\big)\Big]\\ &\overset{\dagger}{=} -\frac{T}{2}\,\mathrm{str}\Big[\gamma^{\alpha\beta}\big(\delta A_\alpha A_\beta^{(2)} + \delta A_\alpha A_\beta^{(2)}\big) + \kappa\varepsilon^{\alpha\beta}\big(\delta A_\alpha A_\beta^{(3)} - \delta A_\alpha A_\beta^{(1)}\big)\Big]\\ &= -\,\mathrm{str}\Big[\delta A_\alpha T\big(\gamma^{\alpha\beta}A_\beta^{(2)} - \frac{\kappa}{2}\varepsilon^{\alpha\beta}\big(A_\beta^{(1)} - A_\beta^{(3)}\big)\big)\Big]\\ \implies \delta\mathscr{L} &= = -\,\mathrm{str}\big(\delta A_\alpha\Lambda^\alpha\big),\end{aligned} \tag{A.18}$$

For a matrix $\mathfrak{g} \in SU(2,2|4)$, the variation $\delta\mathfrak{g}^{-1}$ or the derivative $\partial_\alpha\mathfrak{g}^{-1}$ can be found by looking at

$$0 = \delta(\mathfrak{g}\mathfrak{g}^{-1}) = \delta\mathfrak{g}\mathfrak{g}^{-1} + \mathfrak{g}\delta\mathfrak{g}^{-1} \implies \delta\mathfrak{g}^{-1} = -\mathfrak{g}^{-1}\delta\mathfrak{g}\mathfrak{g}^{-1}. \tag{A.19}$$

In particular $\partial_\alpha\mathfrak{g}^{-1} = A_\alpha\mathfrak{g}^{-1}$. The variation δA_α is then

$$\begin{aligned}\delta A_\alpha &= \delta(-\mathfrak{g}^{-1}\partial_\alpha\mathfrak{g}) = -\delta\mathfrak{g}^{-1}\partial_\alpha\mathfrak{g} - \mathfrak{g}^{-1}\partial_\alpha\delta\mathfrak{g}\\ &= -(-\mathfrak{g}^{-1}\delta\mathfrak{g}\mathfrak{g}^{-1})\partial_\alpha\mathfrak{g} - \mathfrak{g}^{-1}\partial_\alpha\delta\mathfrak{g} = -\mathfrak{g}^{-1}\delta\mathfrak{g}A_\alpha - \mathfrak{g}^{-1}\partial_\alpha\delta\mathfrak{g}.\end{aligned} \tag{A.20}$$

Substituting into (A.18),

$$\delta\mathscr{L} = \mathrm{str}\big(\mathfrak{g}^{-1}\delta\mathfrak{g}A_\alpha\Lambda^\alpha + \mathfrak{g}^{-1}\partial_\alpha\delta(\mathfrak{g})\Lambda^\alpha\big). \tag{A.21}$$

The second term can be rewritten as (assuming cyclicity due to its presence in a supertrace)

$$\begin{aligned}\mathfrak{g}^{-1}\partial_\alpha(\delta\mathfrak{g})\Lambda^\alpha &= \cancel{\partial_\alpha(\mathfrak{g}^{-1}\delta\mathfrak{g}\Lambda^\alpha)} - A_\alpha\mathfrak{g}^{-1}\delta\mathfrak{g}\Lambda^\alpha - \mathfrak{g}^{-1}\delta\mathfrak{g}\partial_\alpha\Lambda^\alpha\\ &= -\mathfrak{g}^{-1}\delta\mathfrak{g}\Lambda^\alpha A_\alpha - \mathfrak{g}^{-1}\delta\mathfrak{g}\partial_\alpha\Lambda^\alpha\end{aligned}$$

where we drop the total derivative as the variation $\delta\mathfrak{g}$ vanishes at the bounds of integration in δS. Finally we can write the variation in the Lagrangian as

$$\delta\mathscr{L} = -\,\mathrm{str}\big[\mathfrak{g}^{-1}\delta\mathfrak{g}(\partial_\alpha\Lambda^\alpha - [A_\alpha, \Lambda^\alpha])\big], \tag{A.22}$$

[12] Here $\delta\mathscr{L}$ is shorthand for the the variation *inside* the action integral.

† Using $\gamma^{\alpha\beta} = \gamma^{\beta\alpha}$ and $\varepsilon^{\alpha\beta} = -\varepsilon^{\beta\alpha}$ to change indices in the second terms.

which holds for arbitrary variations $\delta\mathfrak{g}$. If we view the term in the supertrace as an element of $\mathfrak{su}(2,2|4)$, this must mean that

$$\operatorname{str}\big(\partial_\alpha\Lambda^\alpha - [A_\alpha,\Lambda^\alpha]\big) = 0 \implies \partial_\alpha\Lambda^\alpha - [A_\alpha,\Lambda^\alpha] = \rho\cdot\mathbb{1}_8, \tag{A.23}$$

where ρ is determined by taking the supertrace of both sides. If instead the LHS was an element of $\mathfrak{psu}(2,2|4)$, then it would be equal to 0 modulo $\mathrm{i}\mathbb{1}$ in $\mathfrak{psu}(2,2|4)$, i.e.

$$\partial_\alpha\Lambda^\alpha - [A_\alpha,\Lambda^\alpha] = 0. \tag{A.24}$$

This single equation (A.24) can be projected onto different $\mathbb{Z}_4$-components. Let us first rewrite

$$\begin{aligned}
[A_\alpha,\Lambda^\alpha] &= [A_\alpha, T\gamma^{\alpha\beta}A^{(2)}_\beta] - [A_\alpha, T\frac{\kappa}{2}\varepsilon^{\alpha\beta}(A^{(3)}_\beta - A^{(1)}_\beta)]\\
&= T\gamma^{\alpha\beta}[A^{(0)}_\alpha + A^{(1)}_\alpha + A^{(2)}_\alpha + A^{(3)}_\alpha, A^{(2)}_\beta] - T\frac{\kappa}{2}\varepsilon^{\alpha\beta}[A^{(0)}_\alpha + A^{(1)}_\alpha + A^{(2)}_\alpha + A^{(3)}_\alpha, (A^{(1)}_\beta - A^{(3)}_\beta)]\\
&= T\gamma^{\alpha\beta}\Big\{[A^{(0)}_\alpha, A^{(2)}_\beta] + [A^{(1)}_\alpha, A^{(2)}_\beta] + [A^{(2)}_\alpha, A^{(2)}_\beta] + [A^{(3)}_\alpha, A^{(2)}_\beta]\Big\}\\
&\quad - T\frac{\kappa}{2}\varepsilon^{\alpha\beta}\Big\{[A^{(0)}_\alpha, A^{(1)}_\beta] + [A^{(1)}_\alpha, A^{(1)}_\beta] + [A^{(2)}_\alpha, A^{(1)}_\beta] + [A^{(3)}_\alpha, A^{(1)}_\beta]\\
&\qquad - [A^{(0)}_\alpha, A^{(3)}_\beta] - [A^{(1)}_\alpha, A^{(3)}_\beta] - [A^{(2)}_\alpha, A^{(3)}_\beta] - [A^{(3)}_\alpha, A^{(3)}_\beta]\Big\}.
\end{aligned}$$

The red term vanishes due to the symmetry of $\gamma^{\alpha\beta}$ under exchange of indices and the blue terms cancel due to the asymmetry of $\varepsilon^{\alpha\beta}$. We can now decompose each term in (A.24) where the colours indicate whether the term belongs to $\mathscr{G}^{(1)}$, $\mathscr{G}^{(2)}$ or $\mathscr{G}^{(3)}$;

$$\partial_\alpha\Lambda^\alpha = T\gamma^{\alpha\beta}\partial_\alpha A^{(2)}_\beta - T\frac{\kappa}{2}\varepsilon^{\alpha\beta}\partial_\alpha A^{(1)}_\beta - T\frac{\kappa}{2}\varepsilon^{\alpha\beta}\partial_\alpha A^{(3)}_\beta, \tag{A.25}$$

$$\begin{aligned}
[A_\alpha,\Lambda^\alpha] &= T\gamma^{\alpha\beta}\Big\{[A^{(0)}_\alpha, A^{(2)}_\beta] + [A^{(1)}_\alpha, A^{(2)}_\beta] + [A^{(3)}_\alpha, A^{(2)}_\beta]\Big\}\\
&\quad - T\frac{\kappa}{2}\varepsilon^{\alpha\beta}\Big\{[A^{(0)}_\alpha, A^{(1)}_\beta] + [A^{(1)}_\alpha, A^{(1)}_\beta] + [A^{(2)}_\alpha, A^{(1)}_\beta]\\
&\qquad - [A^{(0)}_\alpha, A^{(3)}_\beta] - [A^{(2)}_\alpha, A^{(3)}_\beta] - [A^{(3)}_\alpha, A^{(3)}_\beta]\Big\}.
\end{aligned} \tag{A.26}$$

Projecting the equations of motion (A.24) onto $\mathscr{G}^{(2)}$ gives

$$\gamma^{\alpha\beta}\partial_\alpha A^{(2)}_\beta - \gamma^{\alpha\beta}[A^{(0)}_\alpha, A^{(2)}_\beta] + \frac{\kappa}{2}\varepsilon^{\alpha\beta}\big([A^{(1)}_\alpha, A^{(1)}_\beta] - [A^{(3)}_\alpha, A^{(3)}_\beta]\big) = 0, \tag{A.27}$$

In order to proceed, we use the zero-curvature condition for A (1.32) (recast in the form of (1.76)) to find

$$\begin{aligned}
\varepsilon^{\alpha\beta}\partial_\alpha A^{(1)}_\beta &= \varepsilon^{\alpha\beta}\Big\{[A^{(0)}_\alpha, A^{(1)}_\beta] + [A^{(2)}_\alpha, A^{(3)}_\beta]\Big\},\\
\varepsilon^{\alpha\beta}\partial_\alpha A^{(3)}_\beta &= \varepsilon^{\alpha\beta}\Big\{[A^{(0)}_\alpha, A^{(3)}_\beta] + [A^{(2)}_\alpha, A^{(1)}_\beta]\Big\}.
\end{aligned}$$

which tell us the $\mathscr{G}^{(1,3)}$ projections are, respectively,

$$\gamma^{\alpha\beta}[A^{(3)}_\alpha, A^{(2)}_\beta] + \kappa\varepsilon^{\alpha\beta}[A^{(2)}_\alpha, A^{(3)}_\beta] = 0, \tag{A.28}$$

$$\gamma^{\alpha\beta}[A^{(1)}_\alpha, A^{(2)}_\beta] - \kappa\varepsilon^{\alpha\beta}[A^{(2)}_\alpha, A^{(1)}_\beta] = 0. \tag{A.29}$$

We next find the equations of motion for the worldsheet metric $\gamma^{\alpha\beta}$ by finding $\delta S/\delta\gamma^{\alpha\beta}$. First, we calculate $\delta\gamma^{\alpha\beta} = \delta(h^{\alpha\beta}\sqrt{-h}) = \sqrt{-h}\delta h^{\alpha\beta} + h^{\alpha\beta}\delta\sqrt{-h}$. A standard computation yields

$$\begin{aligned}
\delta\sqrt{-h} &= \delta\sqrt{-\det\big(h_{\alpha\beta}\big)} = -\frac{1}{2\sqrt{-\det\big(h_{\alpha\beta}\big)}}\delta\det\big(h_{\alpha\beta}\big) \\
&= -\frac{1}{2\sqrt{-h}}\delta e^{\operatorname{tr}\ln(h_{\alpha\beta})} = -\frac{1}{2\sqrt{-h}}e^{\operatorname{tr}\ln(h_{\alpha\beta})}\delta\operatorname{tr}\ln\big(h_{\alpha\beta}\big) \\
&= \frac{-h}{2\sqrt{-h}}\operatorname{tr}\delta\ln\big(h_{\alpha\beta}\big) = \frac{\sqrt{-h}}{2}\operatorname{tr}\left[(h_{\alpha\beta})^{-1}\delta(h_{\alpha\beta})\right] \\
\implies \delta\sqrt{-h} &= \frac{1}{2}\sqrt{-h}h^{\alpha\beta}\delta h_{\alpha\beta} = -\frac{1}{2}\sqrt{-h}h_{\alpha\beta}\delta h^{\alpha\beta} = \frac{h}{2}\gamma_{\alpha\beta}\delta h^{\alpha\beta}.
\end{aligned} \tag{A.30}$$

Substituting $\delta\gamma^{\alpha\beta}$ into $\delta\mathscr{L}$ (varying only $\gamma^{\alpha\beta}$ in (1.33)) we find

$$\begin{aligned}
\delta\mathscr{L} &= -\frac{T}{2}\Big[\delta\gamma^{\alpha\beta}\operatorname{str}\big(A^{(2)}_\alpha A^{(2)}_\beta\big)\Big] = -\frac{T}{2}\Big[\sqrt{-h}\delta h^{\alpha\beta}\operatorname{str}\big(A^{(2)}_\alpha A^{(2)}_\beta\big) + \frac{h}{2}\gamma_{\alpha\beta}\delta h^{\alpha\beta}h^{\rho\delta}\operatorname{str}\big(A^{(2)}_\rho A^{(2)}_\delta\big)\Big] \\
&= -\frac{T}{2}\sqrt{-h}\delta h^{\alpha\beta}\Big[\operatorname{str}\big(A^{(2)}_\alpha A^{(2)}_\beta\big) - \frac{1}{2}\gamma_{\alpha\beta}\sqrt{-h}h^{\rho\delta}\operatorname{str}\big(A^{(2)}_\rho A^{(2)}_\delta\big)\Big] \\
&= -\frac{T}{2}\sqrt{-h}\delta h^{\alpha\beta}\Big[\operatorname{str}\big(A^{(2)}_\alpha A^{(2)}_\beta\big) - \frac{1}{2}\gamma_{\alpha\beta}\gamma^{\rho\delta}\operatorname{str}\big(A^{(2)}_\rho A^{(2)}_\delta\big)\Big].
\end{aligned}$$

The Virasoro constraints (1.41) are finally found by imposing $\delta S/\delta h^{\alpha\beta} = 0$.

To show the Noether current $J^\alpha = \mathfrak{g}\Lambda^\alpha\mathfrak{g}^{-1}$ (associated with the global $PSU(2,2|4)$ symmetry of the Lagrangian) is conserved, we use (A.24) by going to $\mathfrak{psu}(2,2|4)$ such that

$$\begin{aligned}
\partial_\alpha J^\alpha &= \partial_\alpha\mathfrak{g}\Lambda^\alpha\mathfrak{g}^{-1} + \mathfrak{g}\partial_\alpha\Lambda^\alpha\mathfrak{g}^{-1} + \mathfrak{g}\Lambda^\alpha\partial_\alpha\mathfrak{g}^{-1} \\
&= -\mathfrak{g}(-\mathfrak{g}^{-1}\partial_\alpha\mathfrak{g}\Lambda^\alpha\mathfrak{g}^{-1}) + \mathfrak{g}\partial_\alpha\Lambda^\alpha\mathfrak{g}^{-1} + \mathfrak{g}\Lambda^\alpha(-\mathfrak{g}^{-1}\partial_\alpha\mathfrak{g}\mathfrak{g}^{-1}) \\
&= -\mathfrak{g}A_\alpha\Lambda^\alpha\mathfrak{g}^{-1} + \mathfrak{g}\partial_\alpha\Lambda^\alpha\mathfrak{g}^{-1} + \mathfrak{g}\Lambda^\alpha A_\alpha\mathfrak{g}^{-1} = \mathfrak{g}(\partial_\alpha\Lambda^\alpha - [A_\alpha,\Lambda^\alpha])\mathfrak{g}^{-1}
\end{aligned} \tag{A.31}$$

which manifestly vanishes according to the equations of motion when working in $\mathfrak{psu}(2,2|4)$.

A.4 Kappa symmetry transformation

Here we derive the κ-symmetry transformation $\delta_\epsilon\mathscr{L}$ of the Green-Schwarz Lagrangian

$$\mathscr{L} = -\frac{T}{2}\Big[\gamma^{\alpha\beta}\operatorname{str}\big(A^{(2)}_\alpha A^{(2)}_\beta\big) + \kappa\varepsilon^{\alpha\beta}\operatorname{str}\big(A^{(1)}_\alpha A^{(3)}_\beta\big)\Big].$$

Under the transformation (1.43) where $A^{(k)}_\xi \to A^{(k)}_\xi + \delta_\epsilon A^{(k)}_\xi$,

$$\begin{aligned}
-\frac{2}{T}\mathscr{L} \to\ & \overbrace{\gamma^{\alpha\beta}\operatorname{str}\left[(A^{(2)}_\alpha + \delta_\varepsilon A^{(2)}_\alpha)(A^{(2)}_\beta + \delta_\varepsilon A^{(2)}_\beta)\right]}^{①} + \overbrace{\kappa\varepsilon^{\alpha\beta}\operatorname{str}\left[(A^{(1)}_\alpha + \delta_\varepsilon A^{(1)}_\alpha)(A^{(3)}_\beta + \delta_\varepsilon A^{(3)}_\beta)\right]}^{②} \\
&+ \delta_\epsilon\gamma^{\alpha\beta}\operatorname{str}\big(A^{(2)}_\alpha A^{(2)}_\beta\big).
\end{aligned} \tag{A.32}$$

Our job is now to evaluate ① and ②, add them to the $\delta\gamma^{\alpha\beta}$ term, subtract $-2\mathscr{L}/T$ and finally get $\delta_\epsilon\mathscr{L}$. Let us start by using the transformations of A (1.45) to find

$$① = \gamma^{\alpha\beta}\operatorname{str}\left[(A^{(2)}_\alpha + \delta_\varepsilon A^{(2)}_\alpha)(A^{(2)}_\beta + \delta_\varepsilon A^{(2)}_\beta)\right]$$

$$
\begin{aligned}
&= \gamma^{\alpha\beta}\,\mathrm{str}\left\{(A_\alpha^{(2)} + [A_\alpha^{(1)},\epsilon^{(1)}] + [A_\alpha^{(3)},\epsilon^{(3)}])(A_\beta^{(2)} + [A_\beta^{(1)},\epsilon^{(1)}] + [A_\beta^{(3)},\epsilon^{(3)}])\right\} \\
&= \gamma^{\alpha\beta}\,\mathrm{str}\left\{A_\alpha^{(2)}A_\beta^{(2)} + A_\alpha^{(2)}[A_\beta^{(1)},\epsilon^{(1)}] + A_\alpha^{(2)}[A_\beta^{(3)},\epsilon^{(3)}] + [A_\alpha^{(1)},\epsilon^{(1)}]A_\beta^{(2)} + [A_\alpha^{(3)},\epsilon^{(3)}]A_\beta^{(2)} + \mathcal{O}(\epsilon^2)\right\}.
\end{aligned}
$$

Dropping sub-leading $\mathcal{O}(\epsilon^2)$ contributions, and using the fact that

$$
\begin{aligned}
\gamma^{\alpha\beta}\,\mathrm{str}\left\{A_\alpha^{(2)}[A_\beta^{(3)},\epsilon^{(3)}]+[A_\alpha^{(3)},\epsilon^{(3)}]A_\beta^{(2)}\right\} &= \gamma^{\alpha\beta}\,\mathrm{str}\left\{A_\alpha^{(2)}A_\beta^{(3)}\epsilon^{(3)} - A_\alpha^{(2)}\epsilon^{(3)}A_\beta^{(3)} + A_\alpha^{(3)}\epsilon^{(3)}A_\beta^{(2)} - \epsilon^{(3)}A_\alpha^{(3)}A_\beta^{(2)}\right\} \\
(\text{cyclicity}) \quad &= \gamma^{\alpha\beta}\,\mathrm{str}\left\{A_\alpha^{(2)}A_\beta^{(3)}\epsilon^{(3)} - A_\beta^{(3)}A_\alpha^{(2)}\epsilon^{(3)} + A_\beta^{(2)}A_\alpha^{(3)}\epsilon^{(3)} - A_\alpha^{(3)}A_\beta^{(2)}\epsilon^{(3)}\right\} \\
(\alpha\leftrightarrow\beta) \quad &= \gamma^{\alpha\beta}\,\mathrm{str}\left\{A_\alpha^{(2)}A_\beta^{(3)}\epsilon^{(3)} - A_\beta^{(3)}A_\alpha^{(2)}\epsilon^{(3)} + A_\alpha^{(2)}A_\beta^{(3)}\epsilon^{(3)} - A_\beta^{(3)}A_\alpha^{(2)}\epsilon^{(3)}\right\} \\
&= 2\gamma^{\alpha\beta}\,\mathrm{str}\left\{[A_\alpha^{(2)},A_\beta^{(3)}]\epsilon^{(3)}\right\} = -2\gamma^{\alpha\beta}\,\mathrm{str}\left\{[A_\beta^{(3)},A_\alpha^{(2)}]\epsilon^{(3)}\right\},
\end{aligned}
\tag{A.33}
$$

along with an analog for the $\epsilon^{(1)}$ terms, we get

$$
1 = \gamma^{\alpha\beta}\,\mathrm{str}\big(A_\alpha^{(2)}A_\beta^{(2)}\big) - 2\gamma^{\alpha\beta}\,\mathrm{str}\left\{[A_\beta^{(3)},A_\alpha^{(2)}]\epsilon^{(3)} + [A_\beta^{(1)},A_\alpha^{(2)}]\epsilon^{(1)}\right\}.
\tag{A.34}
$$

In calculating 2, it will be useful to derive the following identity implied by the flatness condition (1.32)

$$
\begin{aligned}
\varepsilon^{\alpha\beta}\partial_\alpha A_\beta^{(1)} &= \frac{1}{2}\varepsilon^{\alpha\beta}(\partial_\alpha A_\beta^{(1)} - \partial_\beta A_\alpha^{(1)}) \\
&= \frac{1}{2}\varepsilon^{\alpha\beta}\Big\{[A_\alpha^{(0)},A_\beta^{(1)}] + [A_\alpha^{(1)},A_\beta^{(0)}] + [A_\alpha^{(2)},A_\beta^{(3)}] + [A_\alpha^{(3)},A_\beta^{(2)}]\Big\} \\
&= \varepsilon^{\alpha\beta}\Big\{[A_\alpha^{(0)},A_\beta^{(1)}] + [A_\alpha^{(2)},A_\beta^{(3)}]\Big\} \quad \text{and similarly,}
\end{aligned}
\tag{A.35}
$$

$$
\varepsilon^{\alpha\beta}\partial_\alpha A_\beta^{(3)} = \varepsilon^{\alpha\beta}\Big\{[A_\alpha^{(0)},A_\beta^{(3)}] + [A_\alpha^{(1)},A_\beta^{(2)}]\Big\}.
\tag{A.36}
$$

Once again referring to (1.45), we find

$$
\begin{aligned}
2 &= \kappa\varepsilon^{\alpha\beta}\,\mathrm{str}\left[(A_\alpha^{(1)} + \delta_\varepsilon A_\alpha^{(1)})(A_\beta^{(3)} + \delta_\varepsilon A_\beta^{(3)})\right] \\
&= \kappa\varepsilon^{\alpha\beta}\,\mathrm{str}\left\{\Big(A_\alpha^{(1)} - \partial_\alpha\epsilon^{(1)} + [A_\alpha^{(0)},\epsilon^{(1)}] + [A_\alpha^{(2)},\epsilon^{(3)}]\Big)\Big(A_\beta^{(3)} - \partial_\beta\epsilon^{(3)} + [A_\beta^{(0)},\epsilon^{(3)}] + [A_\beta^{(2)},\epsilon^{(1)}]\Big)\right\} \\
&= \kappa\varepsilon^{\alpha\beta}\,\mathrm{str}\Big\{A_\alpha^{(1)}A_\beta^{(3)} - A_\alpha^{(1)}\partial_\beta\epsilon^{(3)} + A_\alpha^{(1)}[A_\beta^{(0)},\epsilon^{(3)}] + A_\alpha^{(1)}[A_\beta^{(2)},\epsilon^{(1)}] \\
&\qquad - \partial_\alpha\epsilon^{(1)}A_\beta^{(3)} + [A_\alpha^{(0)},\epsilon^{(1)}]A_\beta^{(3)} + [A_\alpha^{(2)},\epsilon^{(3)}]A_\beta^{(3)}\Big\} \\
&= \kappa\varepsilon^{\alpha\beta}\,\mathrm{str}\big(A_\alpha^{(1)}A_\beta^{(3)}\big) + \kappa\varepsilon^{\alpha\beta}\,\mathrm{str}\Big\{A_\beta^{(1)}\partial_\alpha\epsilon^{(3)} - A_\beta^{(3)}\partial_\alpha\epsilon^{(1)} + A_\alpha^{(1)}[A_\beta^{(0)},\epsilon^{(3)}] + A_\alpha^{(1)}[A_\beta^{(2)},\epsilon^{(1)}] \\
&\qquad + [A_\alpha^{(0)},\epsilon^{(1)}]A_\beta^{(3)} + [A_\alpha^{(2)},\epsilon^{(3)}]A_\beta^{(3)}\Big\}.
\end{aligned}
\tag{A.37}
$$

We can write

$$
A_\beta^{(1)}\partial_\alpha\epsilon^{(3)} = \partial_\alpha(A_\beta^{(1)}\epsilon^{(3)}) - \partial_\alpha A_\beta^{(1)}\epsilon^{(3)},
\tag{A.38}
$$

(and similarly for $A_\beta^{(3)}\partial_\alpha\epsilon^{(1)}$) whereby the total derivatives vanish in $\delta_\varepsilon\mathscr{L}$. This leaves

$$
\begin{aligned}
2 - \kappa\varepsilon^{\alpha\beta}\,\mathrm{str}\big(A_\alpha^{(1)}A_\beta^{(3)}\big) = \kappa\varepsilon^{\alpha\beta}\,\mathrm{str}\{&\partial_\alpha A_\beta^{(3)}\epsilon^{(1)} - \partial_\alpha A_\beta^{(1)}\epsilon^{(3)} + A_\alpha^{(1)}[A_\beta^{(0)},\epsilon^{(3)}] + A_\alpha^{(1)}[A_\beta^{(2)},\epsilon^{(1)}] \\
&+ [A_\alpha^{(0)},\epsilon^{(1)}]A_\beta^{(3)} + [A_\alpha^{(2)},\epsilon^{(3)}]A_\beta^{(3)}\}.
\end{aligned}
$$

We are now ready to use our identity and substitute in (A.35) and (A.36), giving

$$
② - \kappa\varepsilon^{\alpha\beta}\,\mathrm{str}\big(A^{(1)}_\alpha A^{(3)}_\beta\big) = \kappa\varepsilon^{\alpha\beta}\,\mathrm{str}\Big\{[A^{(0)}_\alpha, A^{(1)}_\beta]\epsilon^{(1)} + [A^{(2)}_\alpha, A^{(3)}_\beta]\epsilon^{(1)} - [A^{(0)}_\alpha, A^{(3)}_\beta]\epsilon^{(3)} + [A^{(1)}_\alpha, A^{(2)}_\beta]\epsilon^{(3)}
+ A^{(1)}_\alpha[A^{(0)}_\beta, \epsilon^{(3)}] + A^{(1)}_\alpha[A^{(2)}_\beta, \epsilon^{(1)}] + [A^{(0)}_\alpha, \epsilon^{(1)}]A^{(3)}_\beta + [A^{(2)}_\alpha, \epsilon^{(3)}]A^{(3)}_\beta\Big\}.
$$

If we expand the commutators, employ cyclicity of the supertrace, and gather like-terms in $\epsilon^{(1)}$ and $\epsilon^{(3)}$, this simplifies greatly to

$$
② = \kappa\varepsilon^{\alpha\beta}\,\mathrm{str}\big(A^{(1)}_\alpha A^{(3)}_\beta\big) + 2\kappa\varepsilon^{\alpha\beta}\,\mathrm{str}\Big\{[A^{(1)}_\alpha, A^{(2)}_\beta]\epsilon^{(1)} + [A^{(3)}_\beta, A^{(2)}_\alpha]\epsilon^{(3)}\Big\}. \tag{A.39}
$$

Adding our equations (A.34) for ① and (A.39) for ② (with a little index manipulation) gives

$$
\begin{aligned}
① + ② &= \gamma^{\alpha\beta}\,\mathrm{str}\big(A^{(2)}_\alpha A^{(2)}_\beta\big) + \kappa\varepsilon^{\alpha\beta}\,\mathrm{str}\big(A^{(1)}_\alpha A^{(3)}_\beta\big) \\
&\quad - 2\gamma^{\alpha\beta}\,\mathrm{str}\Big\{[A^{(1)}_\beta, A^{(2)}_\alpha]\epsilon^{(1)} + [A^{(3)}_\beta, A^{(2)}_\alpha]\epsilon^{(3)}\Big\} - 2\kappa\varepsilon^{\alpha\beta}\,\mathrm{str}\Big\{[A^{(1)}_\beta, A^{(2)}_\alpha]\epsilon^{(1)} - [A^{(3)}_\beta, A^{(2)}_\alpha]\epsilon^{(3)}\Big\} \\
&= -\frac{T}{2}\mathscr{L} - 4\,\mathrm{str}\Big\{\mathrm{P}^{\alpha\beta}_+[A^{(1)}_\beta, A^{(2)}_\alpha]\epsilon^{(1)} + \mathrm{P}^{\alpha\beta}_-[A^{(3)}_\beta, A^{(2)}_\alpha]\epsilon^{(3)}\Big\},
\end{aligned} \tag{A.40}
$$

where we defined the projectors $\mathrm{P}^{\alpha\beta}_\pm = \frac{1}{2}(\gamma^{\alpha\beta} \pm \kappa\varepsilon^{\alpha\beta})$. The change in the Lagrangian density is

$$
\begin{aligned}
-\frac{2}{T}\delta_\varepsilon\mathscr{L} &= ① + ② + \delta_\epsilon\gamma^{\alpha\beta}\,\mathrm{str}\big(A^{(2)}_\alpha A^{(2)}_\beta\big) + \frac{2}{T}\mathscr{L} \\
&= \delta_\epsilon\gamma^{\alpha\beta}\,\mathrm{str}\big(A^{(2)}_\alpha A^{(2)}_\beta\big) - 4\,\mathrm{str}\Big\{\mathrm{P}^{\alpha\beta}_+[A^{(1)}_\beta, A^{(2)}_\alpha]\epsilon^{(1)} + \mathrm{P}^{\alpha\beta}_-[A^{(3)}_\beta, A^{(2)}_\alpha]\epsilon^{(3)}\Big\}.
\end{aligned} \tag{A.41}
$$

Looking ahead at (1.54), it would be useful to know how expressions of the form $X^\alpha_\pm Y^\beta_\pm$ can be manipulated. We will actually prove

$$
P^{\alpha\gamma}_\pm P^{\beta\delta}_\pm = P^{\alpha\delta}_\pm P^{\beta\gamma}_\pm. \tag{A.42}
$$

Expanding the left-hand side will result in terms of the form $\gamma^{\alpha\gamma}\varepsilon^{\beta\delta}$. This can be rewritten as

$$
\gamma^{\alpha\gamma}\varepsilon^{\beta\delta} = \varepsilon^{\alpha\mu}\gamma_{\mu\nu}\varepsilon^{\nu\gamma}\varepsilon^{\beta\delta} = -\varepsilon^{\alpha\mu}\gamma_{\mu\nu}(\gamma^{\nu\beta}\gamma^{\gamma\delta} - \gamma^{\nu\delta}\gamma^{\gamma\beta}) = \varepsilon^{\alpha\delta}\gamma^{\gamma\beta} - \varepsilon^{\alpha\beta}\gamma^{\gamma\delta},
$$

such that

$$
\begin{aligned}
4P^{\alpha\gamma}_\pm P^{\beta\delta}_\pm &= (\gamma^{\alpha\gamma} \pm \kappa\varepsilon^{\alpha\gamma})(\gamma^{\beta\delta} \pm \kappa\varepsilon^{\beta\delta}) = \gamma^{\alpha\gamma}\gamma^{\beta\delta} \pm \kappa(\gamma^{\alpha\gamma}\varepsilon^{\beta\delta} + \gamma^{\beta\delta}\gamma^{\alpha\gamma}) + \varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta} \\
&= \gamma^{\alpha\gamma}\gamma^{\beta\delta} \pm \kappa\big[(\varepsilon^{\alpha\delta}\gamma^{\gamma\beta} - \cancel{\varepsilon^{\alpha\beta}\gamma^{\gamma\delta}}) + (\varepsilon^{\beta\gamma}\gamma^{\delta\alpha} - \cancel{\varepsilon^{\beta\alpha}\gamma^{\delta\gamma}})\big] - (\gamma^{\alpha\beta}\gamma^{\gamma\delta} - \gamma^{\alpha\delta}\gamma^{\gamma\beta}) \\
&= \gamma^{\alpha\delta}\gamma^{\beta\gamma} \pm \kappa(\gamma^{\alpha\delta}\varepsilon^{\beta\gamma} + \gamma^{\beta\gamma}\gamma^{\alpha\delta}) - (\gamma^{\alpha\beta}\gamma^{\gamma\delta} - \gamma^{\alpha\gamma}\gamma^{\delta\beta}) \\
&= \gamma^{\alpha\delta}\gamma^{\beta\gamma} \pm \kappa(\gamma^{\alpha\delta}\varepsilon^{\beta\gamma} + \gamma^{\beta\gamma}\gamma^{\alpha\delta}) + \varepsilon^{\alpha\delta}\varepsilon^{\beta\gamma} = 4P^{\alpha\delta}_\pm P^{\beta\gamma}_\pm.
\end{aligned}
$$

This identity (A.42) relating projections tells us that, no matter the circumstance,

$$
X^\alpha_\pm \ldots Y^\beta_\pm = X^\beta_\pm \ldots Y^\alpha_\pm. \tag{A.43}
$$

Recalling (1.48) and (1.52), the first half of the second term in (A.41) becomes

$$
\mathrm{str}\left([A^{(1)}_{\delta,+}, A^{(2),\delta}_-]\epsilon^{(1)}\right) = \mathrm{str}\left(A^{(1),\delta}_+ A^{(2)}_{\delta,-} A^{(2)}_{\alpha,-}\kappa^{(1),\alpha}_+ + A^{(1),\delta}_+ A^{(2)}_{\delta,-}\kappa^{(1),\alpha}_+ A^{(2)}_{\alpha,-}\right.
$$

$$- A^{(2)}_{\delta,-}A^{(1),\delta}_{+}A^{(2)}_{\alpha,-}\kappa^{(1),\alpha}_{+} - A^{(2)}_{\delta,-}A^{(1),\delta}_{+}\kappa^{(1),\alpha}_{+}A^{(2)}_{\alpha,-}\Big).$$

Notice the identity (A.43) actually equates the second and therm term, cancelling them. We are left with

$$\begin{aligned}\operatorname{str}\Big([A^{(1)}_{\delta,+}, A^{(2),\delta}_{-}]\epsilon^{(1)}\Big) &= \operatorname{str}\Big(A^{(1),\delta}_{+}A^{(2)}_{\delta,-}A^{(2)}_{\alpha,-}\kappa^{(1),\alpha}_{+} - A^{(2)}_{\delta,-}A^{(1),\delta}_{+}\kappa^{(1),\alpha}_{+}A^{(2)}_{\alpha,-}\Big)\\ &= \operatorname{str}\Big(A^{(2)}_{\delta,-}A^{(2)}_{\alpha,-}[\kappa^{(1),\alpha}_{+}, A^{(1),\delta}_{+}]\Big) = \frac{1}{8}\operatorname{str}\big(A^{(2)}_{\delta,-}A^{(2)}_{\alpha,-}\big)\operatorname{str}\big(\Upsilon[\kappa^{(1),\alpha}_{+}, A^{(1),\delta}_{+}]\big)\end{aligned}$$

since the term proportional to the identity in (1.54) vanishes in the supertrace. Similarly,

$$\operatorname{str}\Big([A^{(3)}_{\delta,-}, A^{(2),\delta}_{+}]\epsilon^{(3)}\Big) = \frac{1}{8}\operatorname{str}\big(A^{(2)}_{\delta,+}A^{(2)}_{\alpha,+}\big)\operatorname{str}\big(\Upsilon[\kappa^{(3),\alpha}_{-}, A^{(3),\delta}_{-}]\big).$$

Putting the two halves of the second term together, the change in the Lagrangian becomes (1.55). To determine what variation in the worldsheet metric does a local fermionic transformation leave the Lagrangian invariant, i.e. what $\delta_\epsilon\gamma^{\alpha\beta}$ would kill $\delta_\epsilon\mathscr{L}$, we need to factor out $\operatorname{str}\big(A^{(2)}_{\alpha}A^{(2)}_{\beta}\big)$. Using

$$P^{\alpha\beta}_{\pm} = P^{\beta\alpha}_{\mp} \qquad \text{and} \qquad P^{\alpha\delta}_{\pm}P_{\pm\delta}{}^{\beta} = P^{\alpha\beta}_{\pm},$$

the terms involving $\operatorname{str}\big(A^{(2)}_{\alpha\pm}A^{(2)}_{\beta,\pm}\big)$ can be manipulated to get

$$\begin{aligned}\operatorname{str}\big(A^{(2)}_{\alpha,-}A^{(2)}_{\beta,-}\big)\operatorname{tr}\Big([\kappa^{(1),\beta}_{+}, A^{(1),\alpha}_{+}]\Big) &= \operatorname{str}\big(A^{(2)}_{\mu}A^{(2)}_{\nu}\big)\operatorname{tr}\Big(P_{-\alpha}{}^{\mu}P_{-\beta}{}^{\nu}P^{\beta}_{+\rho}P^{\alpha}_{+\zeta}[\kappa^{(1),\rho}, A^{(1),\zeta}]\Big)\\ &= \operatorname{str}\big(A^{(2)}_{\mu}A^{(2)}_{\nu}\big)\operatorname{tr}\Big(P_{+}{}^{\mu}{}_{\zeta}P_{+}{}^{\nu}{}_{\rho}[\kappa^{(1),\rho}, A^{(1),\zeta}]\Big)\\ &= \operatorname{str}\big(A^{(2)}_{\alpha}A^{(2)}_{\beta}\big)\operatorname{tr}\Big([\kappa^{(1),\beta}_{+}, A^{(1),\alpha}_{+}]\Big),\end{aligned}$$

effectively removing the '+' or '-' in the prefactors. Factoring out $\operatorname{str}\big(A^{(2)}_{\alpha}A^{(2)}_{\beta}\big)$ gives (1.55).

A.5 Monodromy matrix evolution

We want to compute $\partial_\tau T(z)$ where $T(z)$ is given by (1.64). It will be useful to introduce the notation

$$T(z,a,b) = \overleftarrow{\exp}\int_b^a d\sigma\ L_\sigma(\tau,\sigma,z), \qquad T(z,2\pi,0) = T(z). \tag{A.44}$$

Path-ordered exponentials of this type satisfy $T(z,a,c) = T(z,a,b)T(z,b,c)$. In particular, we can break up any interval $[s_1, s_n]$ into smaller sub-intervals such that

$$T(z,s_n,s_1) = T(z,s_n,s_{n-1})T(z,s_{n-1},s_{n-2})\cdots T(z,s_2,s_1). \tag{A.45}$$

This becomes useful when computing $\partial_\tau T(z)$. Our strategy will be to apply the product rule to (A.45) and shrink to 0 the sub-interval size $\Delta s = s_2 - s_1 = \ldots = s_{n-1} - s_n$ such that $\Delta s||L_\sigma||_{\text{HS}} \ll 1$. So,

$$\begin{aligned}\partial_\tau T(z,s_n,s_1) &= \sum_{k=1}^{n} T(z,s_n,s_{k+1})\partial_\tau T(z,s_{k+1},s_k)T(z,s_k,s_1)\\ (\Delta s \text{ small}) \quad &\approx \sum_{k=1}^{n} T(z,s_n,s_{k+1})\partial_\tau e^{\Delta s L_\sigma}T(z,s_k,s_1) = \sum_{k=1}^{n}\Delta s T(z,s_n,s_{k+1})\partial_\tau L_\sigma T(z,s_{k+1},s_1)\end{aligned}$$

$$\approx \int_{s_1}^{s_n} d\sigma\ T(z, s_n, \sigma)\partial_\tau L_\sigma T(z, \sigma, s_1)$$

by approximating the integral as a Riemann sum. In particular, for $s_n = 2\pi$ and $s_1 = 0$ we retrieve

$$\begin{aligned}\partial_\tau T(z) &= \int_0^{2\pi} d\sigma\ T(z, 2\pi, \sigma)\partial_\tau L_\sigma T(z, \sigma, 0)\\ &= \int_0^{2\pi} d\sigma\big[\ \overleftarrow{\exp}\int_\sigma^{2\pi} L_\sigma\big]\partial_\tau L_\sigma\big[\ \overleftarrow{\exp}\int_0^{\sigma} L_\sigma\big]\\ &\overset{(1.63)}{=} \int_0^{2\pi} d\sigma\big[\ \overleftarrow{\exp}\int_\sigma^{2\pi} L_\sigma\big]\big(\partial_\sigma L_\tau + [L_\tau, L_\sigma]\big)\big[\ \overleftarrow{\exp}\int_0^{\sigma} L_\sigma\big].\end{aligned} \tag{A.46}$$

The Leibniz rule for the derivative of an integral states

$$\partial_x \int_{a(x)}^{b(x)} dt\ f(x,t) = f(b(x),t)b'(x) - f(a(x),t)a'(x) + \int_{a(x)}^{b(x)} dt\ f'(x,t). \tag{A.47}$$

If we identify $x \sim \sigma$ and $f(x,t) \sim L_\sigma(\tau, \sigma, z)$ then (A.46) is equal to

$$\partial_\tau T(z) = \int_0^{2\pi} d\sigma\ \partial_\sigma\Big[(\ \overleftarrow{\exp}\int_\sigma^{2\pi} L_\sigma)L_\tau(\ \overleftarrow{\exp}\int_0^{\sigma} L_\sigma)\Big]. \tag{A.48}$$

Taking the anti-derivative and evaluating at the bounds yields the evolution equation (1.65) for $T(z)$

$$\begin{aligned}\partial_\tau T(z) &= \Big[(\ \overleftarrow{\exp}\int_{2\pi}^{2\pi} L_\sigma)L_\tau(2\pi,\tau,z)(\ \overleftarrow{\exp}\int_0^{2\pi} L_\sigma)\Big] - \Big[(\ \overleftarrow{\exp}\int_0^{2\pi} L_\sigma)L_\tau(0,\tau,z)(\ \overleftarrow{\exp}\int_0^{0} L_\sigma)\Big]\\ &= L_\tau(2\pi,\tau,z)T(z) - T(z)L_\tau(0,\tau,z) = [L_\tau(0,\tau,z), T(z)]\end{aligned}$$

by the effective periodicity $\sigma + 2\pi = \sigma$ of any function of the worldsheet spatial coordinate.

A.6 Lax pair parameters

The projections of the zero-curvature condition (1.76) for the ansatz (1.85) are

$$\begin{aligned}0 &= 2\varepsilon^{\alpha\beta}\partial_\alpha L_\beta - \varepsilon^{\alpha\beta}[L_\alpha, L_\beta]\\ &= 2\varepsilon^{\alpha\beta}\big\{\ell_0\partial_\alpha A_\beta^{(0)} + \ell_1\partial_\alpha A_\beta^{(2)} + \ell_2\varepsilon^{\mu\nu}\partial_\alpha(\gamma_{\beta\mu}A_\nu^{(2)}) + \ell_3\partial_\alpha A_\beta^{(1)} + \ell_4\partial_\alpha A_\beta^{(3)}\big\}\\ &\quad - \varepsilon^{\alpha\beta}\Big[\ell_0 A_\alpha^{(0)} + \ell_1 A_\alpha^{(2)} + \ell_2\gamma_{\alpha\delta}\varepsilon^{\delta\rho}A_\rho^{(2)} + \ell_3 A_\alpha^{(1)} + \ell_4 A_\alpha^{(3)},\\ &\qquad\quad \ell_0 A_\beta^{(0)} + \ell_1 A_\beta^{(2)} + \ell_2\gamma_{\beta\mu}\varepsilon^{\mu\nu}A_\nu^{(2)} + \ell_3 A_\beta^{(1)} + \ell_4 A_\beta^{(3)}\Big]\\ &= 2\varepsilon^{\alpha\beta}\ell_0\partial_\alpha A_\beta^{(0)} - \varepsilon^{\alpha\beta}\Big\{[\ell_0 A_\alpha^{(0)}, \ell_0 A_\beta^{(0)}] + [\ell_1 A_\alpha^{(2)}, \ell_1 A_\beta^{(2)}] + [\ell_1 A_\alpha^{(2)}, \ell_2\gamma_{\beta\mu}\varepsilon^{\mu\nu}A_\nu^{(2)}]\\ &\qquad + [\ell_2\gamma_{\alpha\delta}\varepsilon^{\delta\rho}A_\rho^{(2)}, \ell_1 A_\beta^{(2)}] + [\ell_2\gamma_{\alpha\delta}\varepsilon^{\delta\rho}A_\rho^{(2)}, \ell_2\gamma_{\beta\mu}\varepsilon^{\mu\nu}A_\nu^{(2)}] + 2[\ell_3 A_\alpha^{(1)}, \ell_4 A_\alpha^{(3)}]\Big\}\\ &\quad + 2\varepsilon^{\alpha\beta}\ell_1\partial_\alpha A_\beta^{(2)} + 2\varepsilon^{\alpha\beta}\ell_2\varepsilon^{\mu\nu}\partial_\alpha(\gamma_{\beta\mu}A_\nu^{(2)}) - \varepsilon^{\alpha\beta}\Big\{[\ell_0 A_\alpha^{(0)}, \ell_1 A_\beta^{(2)}] + [\ell_1 A_\alpha^{(2)}, \ell_0 A_\beta^{(0)}]\\ &\qquad + [\ell_0 A_\alpha^{(0)}, \ell_2\gamma_{\beta\mu}\varepsilon^{\mu\nu}A_\nu^{(2)}] + [\ell_2\gamma_{\alpha\delta}\varepsilon^{\delta\rho}A_\rho^{(2)}, \ell_0 A_\beta^{(0)}] + [\ell_3 A_\alpha^{(1)}, \ell_3 A_\beta^{(1)}] + [\ell_4 A_\alpha^{(3)}, \ell_4 A_\beta^{(3)}]\Big\}\\ &\quad + 2\varepsilon^{\alpha\beta}\ell_3\partial_\alpha A_\beta^{(1)} - \varepsilon^{\alpha\beta}\Big\{[\ell_0 A_\alpha^{(0)}, \ell_3 A_\beta^{(1)}] + [\ell_3 A_\alpha^{(1)}, \ell_0 A_\beta^{(0)}] + [\ell_1 A_\alpha^{(2)}, \ell_4 A_\beta^{(3)}]\end{aligned}$$

$$
\begin{aligned}
&\qquad +[\ell_2\gamma_{\alpha\delta}\varepsilon^{\delta\rho}A^{(2)}_\rho,\ell_4A^{(3)}_\beta]+[\ell_4A^{(3)}_\alpha,\ell_1A^{(2)}_\beta]+[\ell_4A^{(3)}_\alpha,\ell_2\gamma_{\beta\mu}\varepsilon^{\mu\nu}A^{(2)}_\nu]\Big\}\\
&+2\varepsilon^{\alpha\beta}\ell_4\partial_\alpha A^{(3)}_\beta-\varepsilon^{\alpha\beta}\Big\{[\ell_0A^{(0)}_\alpha,\ell_4A^{(3)}_\beta]+[\ell_4A^{(3)}_\alpha,\ell_0A^{(0)}_\beta]+[\ell_1A^{(2)}_\alpha,\ell_3A^{(1)}_\beta]\\
&\qquad +[\ell_2\gamma_{\alpha\delta}\varepsilon^{\delta\rho}A^{(2)}_\rho,\ell_3A^{(1)}_\beta]+[\ell_3A^{(1)}_\alpha,\ell_1A^{(2)}_\beta]+[\ell_3A^{(1)}_\alpha,\ell_2\gamma_{\beta\mu}\varepsilon^{\mu\nu}A^{(2)}_\nu]\Big\}\\
&=\mathscr{G}^{(0)}+\mathscr{G}^{(2)}+\mathscr{G}^{(1)}+\mathscr{G}^{(3)}.
\end{aligned}
$$

Starting with $\mathscr{G}^{(0)}=0$, we find

$$
\begin{aligned}
\mathscr{G}^{(0)}&=2\varepsilon^{\alpha\beta}\ell_0\partial_\alpha A^{(0)}_\beta-\varepsilon^{\alpha\beta}\Big\{[\ell_0A^{(0)}_\alpha,\ell_0A^{(0)}_\beta]+[\ell_1A^{(2)}_\alpha,\ell_1A^{(2)}_\beta]+[\ell_1A^{(2)}_\alpha,\ell_2\gamma_{\beta\mu}\varepsilon^{\mu\nu}A^{(2)}_\nu]\\
&\qquad +[\ell_2\gamma_{\alpha\delta}\varepsilon^{\delta\rho}A^{(2)}_\rho,\ell_1A^{(2)}_\beta]+[\ell_2\gamma_{\alpha\delta}\varepsilon^{\delta\rho}A^{(2)}_\rho,\ell_2\gamma_{\beta\mu}\varepsilon^{\mu\nu}A^{(2)}_\nu]+2[\ell_3A^{(1)}_\alpha,\ell_4A^{(3)}_\alpha]\Big\}\\
0&=2\varepsilon^{\alpha\beta}\ell_0\partial_\alpha A^{(0)}_\beta-\varepsilon^{\alpha\beta}\Big\{\ell_0^2[A^{(0)}_\alpha,A^{(0)}_\beta]+\ell_1^2[A^{(2)}_\alpha,A^{(2)}_\beta]+\ell_1\ell_2\gamma_{\beta\mu}\varepsilon^{\mu\nu}[A^{(2)}_\alpha,A^{(2)}_\nu]\\
&\qquad +\ell_1\ell_2\gamma_{\alpha\delta}\varepsilon^{\delta\rho}[A^{(2)}_\rho,A^{(2)}_\beta]+\ell_2^2\gamma_{\alpha\delta}\varepsilon^{\delta\rho}\gamma_{\beta\mu}\varepsilon^{\mu\nu}[A^{(2)}_\rho,A^{(2)}_\nu]+2\ell_3\ell_4[A^{(1)}_\alpha,A^{(3)}_\alpha]\Big\}\\
0&=2\varepsilon^{\alpha\beta}\ell_0\partial_\alpha A^{(0)}_\beta-\varepsilon^{\alpha\beta}\Big\{\ell_0^2[A^{(0)}_\alpha,A^{(0)}_\beta]+\ell_1^2[A^{(2)}_\alpha,A^{(2)}_\beta]+2\ell_3\ell_4[A^{(1)}_\alpha,A^{(3)}_\alpha]\Big\}\\
&\qquad -\ell_1\ell_2\varepsilon^{\alpha\beta}\gamma_{\beta\mu}\varepsilon^{\mu\nu}[A^{(2)}_\alpha,A^{(2)}_\nu]-\ell_1\ell_2\varepsilon^{\alpha\beta}\gamma_{\alpha\delta}\varepsilon^{\delta\rho}[A^{(2)}_\rho,A^{(2)}_\beta]-\ell_2^2\varepsilon^{\alpha\beta}\gamma_{\alpha\delta}\varepsilon^{\delta\rho}\gamma_{\beta\mu}\varepsilon^{\mu\nu}[A^{(2)}_\rho,A^{(2)}_\nu]\\
0&\overset{\dagger}{=}2\varepsilon^{\alpha\beta}\ell_0\partial_\alpha A^{(0)}_\beta-\varepsilon^{\alpha\beta}\Big\{\ell_0^2[A^{(0)}_\alpha,A^{(0)}_\beta]+\ell_1^2[A^{(2)}_\alpha,A^{(2)}_\beta]+2\ell_3\ell_4[A^{(1)}_\alpha,A^{(3)}_\alpha]\Big\}\\
&\qquad -\cancel{\ell_1\ell_2\gamma^{\alpha\nu}[A^{(2)}_\alpha,A^{(2)}_\nu]}+\cancel{\ell_1\ell_2\gamma^{\beta\rho}[A^{(2)}_\rho,A^{(2)}_\beta]}+\ell_2^2\varepsilon^{\mu\nu}\gamma^{\beta\rho}\gamma_{\beta\mu}[A^{(2)}_\rho,A^{(2)}_\nu]\\
0&\overset{\ddagger}{=}2\varepsilon^{\alpha\beta}\ell_0\partial_\alpha A^{(0)}_\beta-\varepsilon^{\alpha\beta}\Big\{\ell_0^2[A^{(0)}_\alpha,A^{(0)}_\beta]+(\ell_1^2-\ell_2^2)[A^{(2)}_\alpha,A^{(2)}_\beta]+2\ell_3\ell_4[A^{(1)}_\alpha,A^{(3)}_\alpha]\Big\}.
\end{aligned}
\tag{A.49}
$$

To simplify this expression further, we will need to use the projection onto $\mathscr{G}^{(0)}$ of the flatness condition (1.76) for A_α,

$$
2\varepsilon^{\alpha\beta}\partial_\alpha A^{(0)}_\beta=\varepsilon^{\alpha\beta}\Big\{[A^{(0)}_\alpha,A^{(0)}_\beta]+[A^{(2)}_\alpha,A^{(2)}_\beta]+2[A^{(1)}_\alpha,A^{(3)}_\beta]\Big\}.
\tag{A.50}
$$

Substituting (A.50) into (A.49) we get

$$
\mathscr{G}^{(0)}=\varepsilon^{\alpha\beta}\Big\{(\ell_0-\ell_0^2)[A^{(0)}_\alpha,A^{(0)}_\beta]+(\ell_0+\ell_2^2-\ell_1^2)[A^{(2)}_\alpha,A^{(2)}_\beta]+2(\ell_0-\ell_3\ell_4)[A^{(1)}_\alpha,A^{(3)}_\beta]\Big\}=0
\tag{A.51}
$$

which tells us, assuming each commutator vanishes independently[13],

$$
\ell_0=1,\qquad \ell_1^2-\ell_2^2=1,\qquad \ell_3\ell_4=1.
\tag{A.52}
$$

We can assume these commutators vanish independently, since if their prefactors were not always vanishing, we would be imposing an additional constraint which did not follow from the equations of motion. In addition, the prospect $\ell_0=0$ is not valid as it would imply $\ell_3\ell_4=0$ which would mean either the $\mathscr{G}^{(1)}$ or $\mathscr{G}^{(3)}$ projection of L_α is always zero. Moving to $\mathscr{G}^{(2)}=0$, we get

$$
\mathscr{G}^{(2)}=2\varepsilon^{\alpha\beta}\ell_1\partial_\alpha A^{(2)}_\beta+2\varepsilon^{\alpha\beta}\ell_2\varepsilon^{\mu\nu}\partial_\alpha(\gamma_{\beta\mu}A^{(2)}_\nu)-\varepsilon^{\alpha\beta}\Big\{[\ell_0A^{(0)}_\alpha,\ell_1A^{(2)}_\beta]+[\ell_1A^{(2)}_\alpha,\ell_0A^{(0)}_\beta]
$$

[†] Using the identity $\varepsilon^{ij}\gamma_{jk}\varepsilon^{kl}=\gamma^{il}$.

[‡] Since $\gamma^{\beta\rho}\gamma_{\beta\mu}=\delta^\rho_\mu$ and we can relabel summation indices $\mu,\nu\to\alpha,\beta$.

[13] The connection components $A^{(k)}$ are independent of one another.

$$
\begin{aligned}
&+[\ell_0 A_\alpha^{(0)}, \ell_2\gamma_{\beta\mu}\varepsilon^{\mu\nu}A_\nu^{(2)}] + [\ell_2\gamma_{\alpha\delta}\varepsilon^{\delta\rho}A_\rho^{(2)}, \ell_0 A_\beta^{(0)}] + [\ell_3 A_\alpha^{(1)}, \ell_3 A_\beta^{(1)}] + [\ell_4 A_\alpha^{(3)}, \ell_4 A_\beta^{(3)}]\Big\} \\
0 &= 2\varepsilon^{\alpha\beta}\ell_1\partial_\alpha A_\beta^{(2)} + 2\ell_2\partial_\alpha(\gamma^{\alpha\beta}A_\beta^{(2)}) - \varepsilon^{\alpha\beta}\Big\{2\ell_0\ell_1[A_\alpha^{(0)}, A_\beta^{(2)}] + \ell_3^2[A_\alpha^{(1)}, A_\beta^{(1)}] + \ell_4^2[A_\alpha^{(3)}, A_\beta^{(3)}]\Big\} \\
&\quad - \ell_0\ell_2\varepsilon^{\alpha\beta}\gamma_{\beta\mu}\varepsilon^{\mu\nu}[A_\alpha^{(0)}, A_\nu^{(2)}] - \ell_0\ell_2\varepsilon^{\alpha\beta}\gamma_{\alpha\delta}\varepsilon^{\delta\rho}[A_\rho^{(2)}, A_\beta^{(2)}] \qquad \text{(A.53)} \\
0 &= 2\varepsilon^{\alpha\beta}\ell_1\partial_\alpha A_\beta^{(2)} + 2\ell_2\partial_\alpha(\gamma^{\alpha\beta}A_\beta^{(2)}) - 2\ell_0(\varepsilon^{\alpha\beta}\ell_1 + \gamma^{\alpha\beta}\ell_2)[A_\alpha^{(0)}, A_\beta^{(2)}] - \varepsilon^{\alpha\beta}\Big\{\ell_3^2[A_\alpha^{(1)}, A_\beta^{(1)}] + \ell_4^2[A_\alpha^{(3)}, A_\beta^{(3)}]\Big\}.
\end{aligned}
$$

In this case, the projection onto $\mathscr{G}^{(2)}$ of (1.76) is

$$
2\varepsilon^{\alpha\beta}\partial_\alpha A_\beta^{(2)} = \varepsilon^{\alpha\beta}\Big\{2[A_\alpha^{(0)}, A_\beta^{(2)}] + [A_\alpha^{(1)}, A_\beta^{(1)}] + [A_\alpha^{(3)}, A_\beta^{(3)}]\Big\}. \tag{A.54}
$$

Substituting (A.54) into (A.53) and recalling $l_0 = 1$, we now get

$$
\mathscr{G}^{(2)} = 2\ell_2\partial_\alpha(\gamma^{\alpha\beta}A_\beta^{(2)}) - 2\ell_2\gamma^{\alpha\beta}[A_\alpha^{(0)}, A_\beta^{(2)}] - \varepsilon^{\alpha\beta}\Big\{(\ell_3^2 - \ell_1)[A_\alpha^{(1)}, A_\beta^{(1)}] + (\ell_4^2 - \ell_1)[A_\alpha^{(3)}, A_\beta^{(3)}]\Big\} = 0 \tag{A.55}
$$

which agrees with the string equations of motion (1.38) provided the parameters ℓ_i satisfy

$$
\frac{\ell^3 - \ell_1}{\ell_2} = -\kappa, \qquad \frac{\ell^4 - \ell_1}{\ell_2} = \kappa. \tag{A.56}
$$

For $\mathscr{G}^{(1)}$ and $\mathscr{G}^{(3)}$ the equations will look identical up to exchange of $\ell_3 \leftrightarrow \ell_4$. Starting with

$$
\begin{aligned}
\mathscr{G}^{(1)} &= 2\varepsilon^{\alpha\beta}\ell_3\partial_\alpha A_\beta^{(1)} - \varepsilon^{\alpha\beta}\Big\{[\ell_0 A_\alpha^{(0)}, \ell_3 A_\beta^{(1)}] + [\ell_3 A_\alpha^{(1)}, \ell_0 A_\beta^{(0)}] + [\ell_1 A_\alpha^{(2)}, \ell_4 A_\beta^{(3)}] \\
&\quad + [\ell_2\gamma_{\alpha\delta}\varepsilon^{\delta\rho}A_\rho^{(2)}, \ell_4 A_\beta^{(3)}] + [\ell_4 A_\alpha^{(3)}, \ell_1 A_\beta^{(2)}] + [\ell_4 A_\alpha^{(3)}, \ell_2\gamma_{\beta\mu}\varepsilon^{\mu\nu}A_\nu^{(2)}]\Big\} \\
0 &= 2\varepsilon^{\alpha\beta}\ell_3\partial_\alpha A_\beta^{(1)} - 2\varepsilon^{\alpha\beta}\Big\{\ell_0\ell_3[A_\alpha^{(0)}, A_\beta^{(1)}] + \ell_1\ell_4[A_\alpha^{(2)}, A_\beta^{(3)}]\Big\} \\
&\quad + \ell_2\ell_4\varepsilon^{\alpha\beta}\gamma_{\alpha\delta}\varepsilon^{\delta\rho}[A_\rho^{(2)}, A_\beta^{(3)}] + \ell_2\ell_4\varepsilon^{\alpha\beta}\gamma_{\beta\mu}\varepsilon^{\mu\nu}[A_\alpha^{(3)}, A_\nu^{(2)}] \\
0 &= 2\varepsilon^{\alpha\beta}\ell_3\partial_\alpha A_\beta^{(1)} - 2\varepsilon^{\alpha\beta}\Big\{\ell_0\ell_3[A_\alpha^{(0)}, A_\beta^{(1)}] + \ell_1\ell_4[A_\alpha^{(2)}, A_\beta^{(3)}]\Big\} + 2\ell_2\ell_4\gamma^{\alpha\beta}[A_\alpha^{(2)}, A_\beta^{(3)}], \qquad \text{(A.57)}
\end{aligned}
$$

and similarly

$$
\mathscr{G}^{(3)} = 2\varepsilon^{\alpha\beta}\ell_4\partial_\alpha A_\beta^{(3)} - 2\varepsilon^{\alpha\beta}\Big\{\ell_0\ell_4[A_\alpha^{(0)}, A_\beta^{(3)}] + \ell_1\ell_3[A_\alpha^{(2)}, A_\beta^{(1)}]\Big\} + 2\ell_2\ell_3\gamma^{\alpha\beta}[A_\rho^{(2)}, A_\beta^{(1)}]. \tag{A.58}
$$

The projections onto $\mathscr{G}^{(1)}$ and $\mathscr{G}^{(3)}$ of the flatness condition for A_α are

$$
2\varepsilon^{\alpha\beta}\partial_\alpha A_\beta^{(1)} = \varepsilon^{\alpha\beta}\Big\{2[A_\alpha^{(0)}, A_\beta^{(1)}] + 2[A_\alpha^{(2)}, A_\beta^{(3)}]\Big\}, \tag{A.59}
$$

$$
2\varepsilon^{\alpha\beta}\partial_\alpha A_\beta^{(3)} = \varepsilon^{\alpha\beta}\Big\{2[A_\alpha^{(0)}, A_\beta^{(3)}] + 2[A_\alpha^{(2)}, A_\beta^{(1)}]\Big\}. \tag{A.60}
$$

Substituting (A.59) into (A.57) and (A.60) into (A.58), we obtain

$$
\mathscr{G}^{(1)} = \big(\ell_2\ell_4\gamma^{\alpha\beta} - (\ell_1\ell_4 - \ell_3)\big)[A_\alpha^{(2)}, A_\beta^{(3)}] = 0, \tag{A.61}
$$

$$
\mathscr{G}^{(3)} = \big(\ell_2\ell_3\gamma^{\alpha\beta} - (\ell_1\ell_3 - \ell_4)\big)[A_\alpha^{(2)}, A_\beta^{(1)}] = 0. \tag{A.62}
$$

Comparing with the string equations (1.39), the parameters ℓ_i would need to satisfy

$$\frac{\ell_1\ell_4-\ell_3}{\ell_2\ell_4}=\kappa, \qquad \frac{\ell_4-\ell_1\ell_3}{\ell_2\ell_3}=\kappa. \tag{A.63}$$

These requirements are summarised in (1.86). Summing the second row of equations gives us

$$0=\frac{\ell_3^2-\ell_1}{\ell_2}+\frac{\ell_4^2-\ell_1}{\ell_2}=\frac{\ell_3^2+\ell_4^2-2\ell_1}{\ell_2} \implies \ell_3^2+\ell_4^2=2\ell_1. \tag{A.64}$$

We next multiply the bottom row to give

$$\kappa^2=\frac{\ell_1\ell_4-\ell_3}{\ell_2\ell_4}\frac{\ell_4-\ell_1\ell_3}{\ell_2\ell_3}=\frac{\ell_1\ell_4^2-\ell_1^2\ell_3\ell_4-\ell_3\ell_4+\ell_1\ell_3^2}{\ell_2^2\ell_3\ell_4}\underset{\ell_3\ell_4=1}{=}\frac{\ell_1(\ell_3^2+\ell_4^2)-\ell_1^2-1}{\ell_2^2},$$

which we further simplify using (A.64), yielding

$$\kappa^2=\frac{2\ell_1^2-\ell_1^2-1}{\ell_2^2}=\frac{\ell_1^2-1}{\ell_2^2}.$$

Comparing with $\ell_1^2-\ell_2^2=1$ (A.52), this immediately tells us that $\kappa^2=1$.

A.7 Lax pair transformations

Gauge transformation

Here we will show that the zero-curvature condition of Lax pairs is invariant under gauge transformations (1.87). Recall $\partial_\alpha h^{-1}=-h^{-1}\partial_\alpha h h^{-1}$ for matrices h. Using this and (1.87), we find by the product rule

$$\partial_\alpha L'_\beta=\partial_\alpha h L_\beta h^{-1}+h\partial_\alpha L_\beta h^{-1}-hL_\beta h^{-1}\partial_\alpha h h^{-1}+\partial_\alpha\partial_\beta h h^{-1}-\partial_\beta h h^{-1}\partial_\alpha h h^{-1}. \tag{A.65}$$

Being careful with indices, this means that

$$\begin{aligned}
\partial_\alpha L'_\beta-\partial_\beta L'_\alpha &= \partial_\alpha h L_\beta h^{-1}+h\partial_\alpha L_\beta h^{-1}-hL_\beta h^{-1}\partial_\alpha h h^{-1}+\partial_\alpha\partial_\beta h h^{-1}-\partial_\beta h h^{-1}\partial_\alpha h h^{-1}\\
&\quad -\partial_\beta h L_\alpha h^{-1}-h\partial_\beta L_\alpha h^{-1}+hL_\alpha h^{-1}\partial_\beta h h^{-1}-\partial_\beta\partial_\alpha h h^{-1}+\partial_\alpha h h^{-1}\partial_\beta h h^{-1}\\
&= hL_\alpha h^{-1}\partial_\beta h h^{-1}-\partial_\beta h h^{-1}hL_\alpha h^{-1}+\partial_\alpha h h^{-1}hL_\beta h^{-1}-hL_\beta h^{-1}\partial_\alpha h h^{-1}\\
&\qquad +\partial_\alpha h h^{-1}\partial_\beta h h^{-1}-\partial_\beta h h^{-1}\partial_\alpha h h^{-1}+h\big(\partial_\alpha L_\beta-\partial_\beta L_\alpha\big)h^{-1}\\
&\overset{\dagger}{=}[hL_\alpha h^{-1},\partial_\beta h h^{-1}]+[\partial_\alpha h h^{-1},hL_\beta h^{-1}]+[\partial_\alpha h h^{-1},\partial_\beta h h^{-1}]+[hL_\alpha h^{-1},hL_\beta h^{-1}]\\
&=[hL_\alpha h^{-1}+\partial_\alpha h h^{-1},hL_\beta h^{-1}+\partial_\beta h h^{-1}]=[L'_\alpha,L'_\beta]
\end{aligned}$$

where we used the fact that $[A+B,C]=[A,C]+[B,C]$ as the commutator is bilinear.

Kappa symmetry transformation

To find how the Lax pair (1.85) described in 1.3 transform under κ-symmetry transformations, i.e. to find

$$\delta_\epsilon L_\alpha=\left(\ell_0\delta_\epsilon A_\alpha^{(0)}+\ell_1\delta_\epsilon A_\alpha^{(2)}+\ell_2\gamma_{\alpha\beta}\varepsilon^{\beta\rho}\delta_\epsilon A_\rho^{(2)}+\ell_3\delta_\epsilon A_\alpha^{(1)}+\ell_4\delta_\epsilon A_\alpha^{(3)}\right)+\ell_2\delta_\epsilon\gamma_{\alpha\beta}\varepsilon^{\beta\rho}A_\rho^{(2)}$$

[†]Using flatness (1.63) and the fact that $h[A,B]h^{-1}=[hAh^{-1},hBh^{-1}]$ since $AB=Ah^{-1}hB$.

we should start by recalling $\delta_\epsilon A^{(k)}$ (1.45). If we restrict our discussion to transformations of type $\epsilon = \epsilon^{(1)}$,

$$\begin{aligned} \delta_\epsilon A^{(0)} &= [A^{(3)}, \epsilon^{(1)}], & \delta_\epsilon A^{(1)} &= [A^{(0)}, \epsilon^{(1)}] - d\epsilon^{(1)}, \\ \delta_\epsilon A^{(2)} &= [A^{(1)}, \epsilon^{(1)}], & \delta_\epsilon A^{(3)} &= [A^{(2)}, \epsilon^{(1)}]. \end{aligned} \tag{A.66}$$

Substituting these variations into $\delta_\epsilon L_\alpha$ above, remembering the conditions imposed on ℓ_i (1.86), and setting $\Lambda = \ell_3 \epsilon^{(1)}$, we get

$$\begin{aligned} \delta_\epsilon L_\alpha &= [A^{(3)}_\alpha, \epsilon^{(1)}] + \ell_1 [A^{(1)}_\alpha, \epsilon^{(1)}] + \ell_2 \gamma_{\alpha\beta}\varepsilon^{\beta\rho}[A^{(1)}_\rho, \epsilon^{(1)}] + \ell_3 [A^{(0)}_\alpha, \epsilon^{(1)}] - \ell_3 \partial_\alpha \epsilon^{(1)} + \ell_4 [A^{(2)}_\alpha, \epsilon^{(1)}] \\ &\quad + \ell_2 \delta_\epsilon \gamma_{\alpha\beta}\varepsilon^{\beta\rho} A^{(2)}_\rho \\ &= [A^{(0)}_\alpha + \ell_4 A^{(3)}_\alpha, \Lambda] + \ell_1 [A^{(1)}_\alpha, \epsilon^{(1)}] + \ell_2 \gamma_{\alpha\beta}\varepsilon^{\beta\rho}[A^{(1)}_\rho, \epsilon^{(1)}] - \partial_\alpha \Lambda + \ell_4 [A^{(2)}_\alpha, \epsilon^{(1)}] + \ell_1 \ell_3 [A^{(2)}_\alpha, \epsilon^{(1)}] \\ &\quad - \ell_1 \ell_3 [A^{(2)}_\alpha, \epsilon^{(1)}] + \ell_2 \ell_3 [\gamma_{\alpha\beta}\varepsilon^{\beta\rho} A^{(2)}_\rho, \epsilon^{(1)}] - \ell_2 \ell_3 [\gamma_{\alpha\beta}\varepsilon^{\beta\rho} A^{(2)}_\rho, \epsilon^{(1)}] + \ell_3^2 [A^{(1)}_\alpha, \epsilon^{(1)}] - \ell_3^2 [A^{(1)}_\alpha, \epsilon^{(1)}] \\ &\quad + \ell_2 \delta_\epsilon \gamma_{\alpha\beta}\varepsilon^{\beta\rho} A^{(2)}_\rho \\ &= [A^{(0)}_\alpha + \ell_1 A^{(2)}_\alpha + \ell_2 \gamma_{\alpha\beta}\varepsilon^{\beta\rho} A^{(2)}_\rho + \ell_3 A^{(1)}_\alpha + \ell_4 A^{(3)}_\alpha, \Lambda] - \partial_\alpha \Lambda + \ell_1 [A^{(1)}_\alpha, \epsilon^{(1)}] + \ell_2 \gamma_{\alpha\beta}\varepsilon^{\beta\rho}[A^{(1)}_\rho, \epsilon^{(1)}] \\ &\quad + \ell_4 [A^{(2)}_\alpha, \epsilon^{(1)}] - \ell_1 \ell_3 [A^{(2)}_\alpha, \epsilon^{(1)}] - \ell_2 \ell_3 [\gamma_{\alpha\beta}\varepsilon^{\beta\rho} A^{(2)}_\rho, \epsilon^{(1)}] - \ell_3^2 [A^{(1)}_\alpha, \epsilon^{(1)}] + \ell_2 \delta_\epsilon \gamma_{\alpha\beta}\varepsilon^{\beta\rho} A^{(2)}_\rho \\ &= [L_\alpha, \Lambda] - \partial_\alpha \Lambda + (\ell_4 - \ell_1 \ell_3)[A^{(2)}_\alpha, \epsilon^{(1)}] - \ell_2 \ell_3 [\gamma_{\alpha\beta}\varepsilon^{\beta\rho} A^{(2)}_\rho, \epsilon^{(1)}] \\ &\quad + [(\ell_1 - \ell_3^2) A^{(1)}_\alpha + \ell_2 \gamma_{\alpha\beta}\varepsilon^{\beta\rho} A^{(1)}_\rho, \epsilon^{(1)}] + \ell_2 \delta_\epsilon \gamma_{\alpha\beta}\varepsilon^{\beta\rho} A^{(2)}_\rho \\ &= [L_\alpha, \Lambda] - \partial_\alpha \Lambda + \ell_2 \ell_3 \kappa [A^{(2)}_\alpha, \epsilon^{(1)}] - \ell_2 \ell_3 \gamma_{\alpha\beta}\varepsilon^{\beta\rho}[A^{(2)}_\rho, \epsilon^{(1)}] \\ &\quad + [\ell_2 \kappa A^{(1)}_\alpha + \ell_2 \gamma_{\alpha\beta}\varepsilon^{\beta\rho} A^{(1)}_\rho, \epsilon^{(1)}] + \ell_2 \delta_\epsilon \gamma_{\alpha\beta}\varepsilon^{\beta\rho} A^{(2)}_\rho \\ &= [L_\alpha, \Lambda] - \partial_\alpha \Lambda + \ell_2 \ell_3 [\kappa A^{(2)}_\alpha - \gamma_{\alpha\beta}\varepsilon^{\beta\rho} A^{(2)}_\rho, \epsilon^{(1)}] + \ell_2 [\kappa A^{(1)}_\alpha + \gamma_{\alpha\beta}\varepsilon^{\beta\rho} A^{(1)}_\rho, \epsilon^{(1)}] + \ell_2 \delta_\epsilon \gamma_{\alpha\beta}\varepsilon^{\beta\rho} A^{(2)}_\rho. \end{aligned}$$

We have almost manipulated the expression into a form using $P^{\alpha\beta}_\pm$. All we need to see is the relation

$$\begin{aligned} \kappa A^{(2)}_\alpha - \gamma_{\alpha\beta}\varepsilon^{\beta\rho} A^{(2)}_\rho &= \kappa \gamma_{\alpha\mu} A^{(2),\mu} - \gamma_{\alpha\beta}\varepsilon^{\beta\rho} A^{(2)}_\rho \overset{\dagger}{=} \kappa \varepsilon_{\alpha\sigma} \gamma^{\sigma\nu} \varepsilon_{\nu\mu} A^{(2),\mu} - \gamma_{\alpha\beta}\varepsilon^{\beta\rho} A^{(2)}_\rho \\ &= \varepsilon_{\alpha\beta} \left[\kappa \gamma^{\beta\nu} \varepsilon_{\nu\mu} A^{(2),\mu} - A^{(2),\beta}\right] = -\varepsilon_{\alpha\beta} \left[\gamma^{\beta\delta} - \kappa \varepsilon^{\beta\delta}\right] A^{(2)}_\delta \\ &= -2\varepsilon_{\alpha\beta} P^{\beta\delta}_- A^{(2)}_\delta = -2\varepsilon_{\alpha\beta} A^{(2),\beta}_- \end{aligned}$$

which ultimately results in equation (1.89)

$$\delta_\epsilon L_\alpha = [L_\alpha, \Lambda] - \partial_\alpha \Lambda - 2\ell_2 \ell_3 \varepsilon_{\alpha\beta} [A^{(2),\beta}_-, \epsilon^{(1)}] + \ell_2 \varepsilon_{\alpha\beta} \Big(2[A^{(1),\beta}_+, \epsilon^{(1)}] + \delta_\epsilon \gamma^{\beta\delta} A^{(2)}_\delta \Big). \tag{A.67}$$

Now, suppose for some arbitrary one-form c_α the infinitesimal transformation resulted in

$$\delta_\epsilon L_\alpha = [L_\alpha, \Lambda] - \partial_\alpha \Lambda + c_\alpha.$$

Then, we would get the new Lax connections $L'_\alpha = L_\alpha + \delta_\epsilon L_\alpha$. To check the new zero-curvature condition, let us calculate its ingredients first. Namely,

$$\partial_\alpha L'_\beta = \partial_\alpha L_\beta + [\partial_\alpha L_\beta, \Lambda] + [L_\beta, \partial_\alpha \Lambda] - \partial_\alpha \partial_\beta \Lambda + \partial_\alpha c_\beta$$

---

†Using $\varepsilon^{\alpha\beta}\gamma_{\beta\delta}\varepsilon^{\delta\rho} = \gamma^{\alpha\rho}$.

which means

$$\partial_\alpha L'_\beta - \partial_\beta L'_\alpha = \partial_\alpha L_\beta - \partial_\beta L_\alpha + [\partial_\alpha L_\beta - \partial_\beta L_\alpha, \Lambda] + [L_\beta, \partial_\alpha \Lambda] - [L_\alpha, \partial_\beta \Lambda] + \partial_\alpha c_\beta - \partial_\beta c_\alpha .$$

We will now find the new $[L'_\alpha, L'_\beta]$ and compare with the above expression to see the conditions imposed on c_α such that the zero-curvature of L_α is preserved. Ignoring terms quadratic in the infinitesimal transformation parameter $\epsilon^{(1)}$ (or equivalently Λ),

$$\begin{aligned}
[L'_\alpha, L'_\beta] = [L_\alpha, L_\beta] &+ [L_\alpha, [L_\beta, \Lambda]] - [L_\alpha, \partial_\beta \Lambda] + [L_\alpha, c_\beta] \\
&+ [[L_\alpha, \Lambda], L_\beta] + [[L_\alpha, \Lambda], c_\beta] + \mathcal{O}(\Lambda^2) \\
&- [\partial_\alpha \Lambda, L_\beta] - [\partial_\alpha \Lambda, c_\beta] + \mathcal{O}(\Lambda^2) \\
&+ [c_\alpha, L_\beta] + [c_\alpha, [L_\beta, \Lambda]] - [c_\alpha, \partial_\beta \Lambda] + [c_\alpha, c_\beta].
\end{aligned}$$

We now use the Jacobi identity[14] to write

$$\begin{aligned}
[L'_\alpha, L'_\beta] &= [L_\alpha, L_\beta] + \Big([L_\alpha, [L_\beta, \Lambda]] + [[L_\alpha, \Lambda], L_\beta]\Big) + [L_\beta, \partial_\alpha \Lambda] - [L_\alpha, \partial_\beta \Lambda] \\
&\qquad + [L_\alpha, c_\beta] + [c_\alpha, L_\beta] + [[L_\alpha, \Lambda], c_\beta] + [c_\alpha, [L_\beta, \Lambda]] - [\partial_\alpha \Lambda, c_\beta] - [c_\alpha, \partial_\beta \Lambda] + [c_\alpha, c_\beta] \\
&\qquad + \mathcal{O}(\Lambda^2) \\
&= [L_\alpha, L_\beta] + [[L_\alpha, L_\beta], \Lambda] + [L_\beta, \partial_\alpha \Lambda] - [L_\alpha, \partial_\beta \Lambda] \\
&\qquad + [L_\alpha, c_\beta] + [c_\alpha, L_\beta] + [[L_\alpha, \Lambda], c_\beta] + [c_\alpha, [L_\beta, \Lambda]] - [\partial_\alpha \Lambda, c_\beta] - [c_\alpha, \partial_\beta \Lambda] + [c_\alpha, c_\beta] \\
&\qquad + \mathcal{O}(\Lambda^2).
\end{aligned}$$

Comparing with what we previously found, i.e.

$$\partial_\alpha L'_\beta - \partial_\beta L'_\alpha = \partial_\alpha L_\beta - \partial_\beta L_\alpha + [\partial_\alpha L_\beta - \partial_\beta L_\alpha, \Lambda] + [L_\beta, \partial_\alpha \Lambda] - [L_\alpha, \partial_\beta \Lambda] + \partial_\alpha c_\beta - \partial_\beta c_\alpha ,$$

and substituting the old zero-curvature condition (1.63), the new zero-curvature-condition

$$\partial_\alpha L'_\beta - \partial_\beta L'_\alpha = [L'_\alpha, L'_\beta]$$

is satisfied, provided the extra term c_α obeys the following condition

$$\partial_\alpha c_\beta - \partial_\beta c_\alpha = [L_\alpha, c_\beta] + [c_\alpha, L_\beta] + [[L_\alpha, \Lambda], c_\beta] + [c_\alpha, [L_\beta, \Lambda]] - [\partial_\alpha \Lambda, c_\beta] - [c_\alpha, \partial_\beta \Lambda] + [c_\alpha, c_\beta].$$

Obviously if $c_\alpha = 0$ then the above is satisfied[15]. We will now prove that

$$c_\alpha = 2\ell_2 \ell_3 \varepsilon_{\alpha\beta} \underbrace{[A_-^{(2),\beta}, \epsilon^{(1)}]}_{I_1^\beta} - \ell_2 \varepsilon_{\alpha\beta} \Big(2 \underbrace{[A_+^{(1),\beta}, \epsilon^{(1)}]}_{I_2^\beta} + \delta_\epsilon \gamma^{\beta\delta} A_\delta^{(2)} \Big) = 0 \tag{A.68}$$

by reducing $I_{1,2}$ and showing that the two terms vanish separately, hence the transformation is a gauge transformation of the Lax connection, i.e it preserves flatness. Beginning with I_1, we remember that $A_{\alpha,-}$ and $A_{\beta,-}$ are proportional to each other; when $\alpha = \beta$ they are just equal, but whenever $\alpha \neq \beta$ they are related by (1.51). Either way, $[A_{\alpha,\pm}, A_{\beta,\pm}] = 0$. In particular, taking the $\mathscr{G}^{(0)}$ projection of this equality,

[14] $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$.

[15] I tried substituting non-trivial forms of c_α, for example $\partial_\alpha \Lambda$, but was not able to find one which satisfied the condition.

we get that $[A^{(k)}_{\alpha,\pm}, A^{(k)}_{\beta,\pm}] = 0$ for $k = 0, 1, 2, 3$ since the different grading elements $A^{(k)}$ are independent of one another. All of this to say that $[A^{(2)}_{\alpha,-}, A^{(2)}_{\beta,-}] = 0$ such that, substituting the expression (1.52) for $\epsilon^{(1)}$,

$$\begin{aligned}
I_{1,\alpha} &= [A^{(2)}_{\alpha,-}, \epsilon^{(1)}] = [A^{(2)}_{\alpha,-}, A^{(2)}_{\beta,-}\kappa^{(1),\beta}_{+} + \kappa^{(1),\beta}_{+} A^{(2)}_{\beta,-}] \\
&= A^{(2)}_{\beta,-}[A^{(2)}_{\alpha,-}, \kappa^{(1),\beta}_{+}] + [A^{(2)}_{\alpha,-}, \kappa^{(1),\beta}_{+}]A^{(2)}_{\beta,-} \\
&= A^{(2)}_{\beta,-}A^{(2)}_{\alpha,-}\kappa^{(1),\beta}_{+} - \cancel{A^{(2)}_{\beta,-}\kappa^{(1),\beta}_{+}A^{(2)}_{\alpha,-}} + \cancel{A^{(2)}_{\alpha,-}\kappa^{(1),\beta}_{+}A^{(2)}_{\beta,-}} - \kappa^{(1),\beta}_{+}A^{(2)}_{\alpha,-}A^{(2)}_{\beta,-} \\
&= [A^{(2)}_{\alpha,-}A^{(2)}_{\beta,-}, \kappa^{(1),\beta}_{+}].
\end{aligned}$$

In the last line we used again the fact that the projected components $A^{(2)}_{\alpha,-}$ and $A^{(2)}_{\beta,-}$ are proportional to commute them and to cancel the equal and opposite terms. Lastly we recall (1.54) and notice that the term proportional to the identity will commute with $\kappa^{(1),\beta}_{+}$ such that we are left with

$$I_{1,\alpha} = \frac{1}{8}\,\mathrm{str}\big(A^{(2)}_{\alpha,-}A^{(2)}_{\beta,-}\big)[\Upsilon, \kappa^{(1),\beta}_{+}]. \tag{A.69}$$

To proceed, we will show that the Virasoro constraints are satisfied if and only if $\mathrm{str}\big(A^{(2)}_{\alpha,-}A^{(2)}_{\beta,-}\big) = 0$. It will be crucial to use the following identities relating $\varepsilon_{\alpha\beta}$ and $\gamma_{\alpha\beta}$.

(i) We note that $\varepsilon_{\alpha\mu}\varepsilon_{\beta\nu}$ and $(\gamma_{\alpha\beta}\gamma_{\mu\nu} - \gamma_{\alpha\nu}\gamma_{\beta\mu})$ share the same symmetry under exchange of pairs of indices $(\alpha\mu) \leftrightarrow (\beta\nu)$, and anti-symmetry under exchange of $\alpha \leftrightarrow \mu$ and $\beta \leftrightarrow \nu$. So they must be proportional and, by looking at $\varepsilon_{\tau\sigma}\varepsilon_{\tau\sigma} = 1 \propto \det\gamma = -1$ for example, we see that we in fact have

$$\varepsilon_{\alpha\mu}\varepsilon_{\beta\nu} = -(\gamma_{\alpha\beta}\gamma_{\mu\nu} - \gamma_{\alpha\nu}\gamma_{\beta\mu}) = \gamma_{\alpha\nu}\gamma_{\beta\mu} - \gamma_{\alpha\beta}\gamma_{\mu\nu}.$$

One could also use $\varepsilon^{\alpha\beta}\varepsilon^{\gamma\delta} = \delta^{\alpha\gamma}\delta^{\beta\delta} - \delta^{\alpha\delta}\delta^{\beta\gamma} = -(\gamma^{\alpha\gamma}\gamma^{\beta\delta} - \gamma^{\alpha\delta}\gamma^{\beta\gamma})$. Note the overall minus sign appears because each $\gamma^{\alpha\beta}$ factor is associated to a different index of $\varepsilon^{\alpha\beta}$. Both dimensions' sign appears exactly once in each term, and since $\det\gamma = -1$, an extra minus is needed to keep the Kronecker delta terms positive when non-zero.

(ii) We use identity (i) to derive $\varepsilon_{\nu}{}^{\lambda}\varepsilon^{\rho}{}_{\mu} = \gamma^{\lambda\alpha}\gamma^{\rho\beta}\varepsilon_{\nu\alpha}\varepsilon_{\beta\mu} = \gamma^{\lambda\alpha}\gamma^{\rho\beta}(\gamma_{\nu\mu}\gamma_{\beta\alpha} - \gamma_{\nu\beta}\gamma_{\alpha\mu}) = \gamma_{\mu\nu}\gamma^{\lambda\rho} - \delta^{\rho}_{\nu}\delta^{\lambda}_{\mu}$.

With these two identities (i) and (ii) in mind we calculate the following with cyclicity in $\mu \leftrightarrow \nu$,

$$\begin{aligned}
\mathrm{str}\big(A^{(2)}_{\alpha,-}A^{(2)}_{\beta,-}\big) &= \mathrm{str}\big(P_{-\alpha}{}^{\mu}A^{(2)}_{\mu}P_{-\beta}{}^{\nu}A^{(2)}_{\nu}\big) = P_{-\alpha\mu}P_{-\beta\nu}\,\mathrm{str}\big(A^{(2),\mu}A^{(2),\nu}\big) \\
0 &= \frac{1}{4}\big[\gamma_{\alpha\mu}\gamma_{\beta\nu} - \kappa\gamma_{\alpha\mu}\varepsilon_{\beta\nu} - \kappa\gamma_{\beta\nu}\varepsilon_{\alpha\mu} + \kappa^2\varepsilon_{\alpha\mu}\varepsilon_{\beta\nu}\big]\,\mathrm{str}\big(A^{(2),\mu}A^{(2),\nu}\big) \\
0 &\overset{\text{(i)}}{=} \frac{1}{4}\big[\gamma_{\alpha\mu}\gamma_{\beta\nu} - \kappa\gamma_{\alpha\mu}\varepsilon_{\beta\nu} - \kappa\gamma_{\beta\nu}\varepsilon_{\alpha\mu} + \gamma_{\alpha\nu}\gamma_{\beta\mu} - \gamma_{\alpha\beta}\gamma_{\mu\nu}\big]\,\mathrm{str}\big(A^{(2),\mu}A^{(2),\nu}\big) \\
0 &= \frac{1}{2}\gamma_{\alpha\mu}\gamma_{\beta\nu}\,\mathrm{str}\big(A^{(2),\mu}A^{(2),\nu}\big) - \frac{1}{4}\gamma_{\alpha\beta}\gamma_{\mu\nu}\,\mathrm{str}\big(A^{(2),\mu}A^{(2),\nu}\big) - \frac{\kappa}{4}\big[\gamma_{\alpha\mu}\varepsilon_{\beta\nu} + \gamma_{\beta\nu}\varepsilon_{\alpha\mu}\big]\,\mathrm{str}\big(A^{(2),\mu}A^{(2),\nu}\big) \\
0 &= \frac{1}{2}\Big[\mathrm{str}\big(A^{(2)}_{\alpha}A^{(2)}_{\beta}\big) - \frac{1}{2}\gamma_{\alpha\beta}\gamma^{\mu\nu}\,\mathrm{str}\big(A^{(2)}_{\mu}A^{(2)}_{\nu}\big)\Big] - \frac{\kappa}{4}\big[\gamma_{\alpha\mu}\varepsilon_{\beta\nu} + \gamma_{\beta\nu}\varepsilon_{\alpha\mu}\big]\,\mathrm{str}\big(A^{(2),\mu}A^{(2),\nu}\big) \\
0 &= \frac{1}{2}(\text{Virasoro})_{\alpha\beta} - \frac{\kappa}{4}\big[\gamma_{\alpha\mu}\varepsilon_{\beta\nu} + \gamma_{\beta\nu}\varepsilon_{\alpha\mu}\big]\,\mathrm{str}\big(A^{(2),\mu}A^{(2),\nu}\big) \\
\gamma^{\alpha\rho}\varepsilon^{\beta\lambda}0 &= \gamma^{\alpha\rho}\varepsilon^{\beta\lambda}\frac{1}{2}(\text{Virasoro})_{\alpha\beta} - \gamma^{\alpha\rho}\varepsilon^{\beta\lambda}\frac{\kappa}{4}\big[\gamma_{\alpha\mu}\varepsilon_{\beta\nu} + \gamma_{\beta\nu}\varepsilon_{\alpha\mu}\big]\,\mathrm{str}\big(A^{(2),\mu}A^{(2),\nu}\big)
\end{aligned}$$

$$
\begin{aligned}
0 &= \gamma^{\alpha\rho}\varepsilon^{\beta\lambda}\frac{1}{2}(\text{Virasoro})_{\alpha\beta} - \frac{\kappa}{4}\big[\delta^{\rho}_{\mu}\varepsilon^{\beta\lambda}\varepsilon_{\beta\nu} + \varepsilon_{\nu}{}^{\lambda}\varepsilon^{\rho}{}_{\mu}\big]\operatorname{str}\big(A^{(2),\mu}A^{(2),\nu}\big)\\
0 &\overset{\text{(ii)}}{=} \gamma^{\alpha\rho}\varepsilon^{\beta\lambda}\frac{1}{2}(\text{Virasoro})_{\alpha\beta} - \frac{\kappa}{4}\big[-2\delta^{\rho}_{\mu}\delta^{\lambda}_{\nu} + \gamma^{\lambda\rho}\gamma_{\mu\nu}\big]\operatorname{str}\big(A^{(2),\mu}A^{(2),\nu}\big)\\
0 &= \gamma^{\alpha\rho}\varepsilon^{\beta\lambda}\frac{1}{2}(\text{Virasoro})_{\alpha\beta} + \frac{\kappa}{2}\big[\operatorname{str}\big(A^{(2),\rho}A^{(2),\lambda}\big) - \frac{1}{2}\gamma^{\lambda\rho}\gamma_{\mu\nu}\operatorname{str}\big(A^{(2),\mu}A^{(2),\nu}\big)\big]\\
0 &= \gamma^{\alpha\rho}\varepsilon^{\beta\lambda}\frac{1}{2}(\text{Virasoro})_{\alpha\beta} + \frac{\kappa}{2}(\text{Virasoro})^{\lambda\rho} = \frac{1}{2}\gamma^{\alpha\rho}\big[\varepsilon^{\beta\lambda} + \kappa\gamma^{\beta\nu}\big](\text{Virasoro})_{\alpha\beta}.
\end{aligned}
$$

This proves equivalence with the Virasoro constraints:

$$
\operatorname{str}\big(A^{(2)}_{\alpha,-}A^{(2)}_{\beta,-}\big) = 0 \iff \operatorname{str}\big(A^{(2)}_{\alpha}A^{(2)}_{\beta}\big) - \frac{1}{2}\gamma_{\alpha\beta}\gamma^{\mu\nu}\operatorname{str}\big(A^{(2)}_{\mu}A^{(2)}_{\nu}\big) = 0. \tag{A.70}
$$

Thus, $I_{1,\alpha} = 0$ and we only need to show $I_2^{\beta} + \delta_{\epsilon}\gamma^{\beta\delta}A^{(2)}_{\delta} = 0$. Looking at

$$
I_2^{\alpha} = [A_+^{(1),\alpha}, \varepsilon^{(1)}] = [A_+^{(1),\alpha}, A^{(2)}_{\beta,-}\kappa_+^{(1),\beta} + \kappa_+^{(1),\beta}A^{(2)}_{\beta,-}], \tag{A.71}
$$

we use (A.43) which helps simplify I_2^{α} down to

$$
\begin{aligned}
I_2^{\alpha} &= [A_+^{(1),\alpha}, A^{(2)}_{\beta,-}\kappa_+^{(1),\beta} + \kappa_+^{(1),\beta}A^{(2)}_{\beta,-}] = [A_+^{(1),\beta}, A^{(2)}_{\beta,-}\kappa_+^{(1),\alpha} + \kappa_+^{(1),\alpha}A^{(2)}_{\beta,-}]\\
&\overset{\dagger}{=} A^{(2)}_{\beta,-}[A_+^{(1),\beta}, \kappa_+^{(1),\alpha}] + \cancel{[A_+^{(1),\beta}, A^{(2)}_{\beta,-}]}\kappa_+^{(1),\alpha} + \kappa_+^{(1),\alpha}\cancel{[A_+^{(1),\beta}, A^{(2)}_{\beta,-}]} + [A_+^{(1),\beta}, \kappa_+^{(1),\alpha}]A^{(2)}_{\beta,-}\\
&= A^{(2)}_{\beta,-}[A_+^{(1),\beta}, \kappa_+^{(1),\alpha}] + [A_+^{(1),\beta}, \kappa_+^{(1),\alpha}]A^{(2)}_{\beta,-}.
\end{aligned}
$$

Since $[A_+^{(1),\beta}, \kappa_+^{(1),\alpha}] \in \mathscr{G}^{(2)}$, this commutator is traceless and can be expressed generically using (1.53) as

$$
[A_+^{(1),\beta}, \kappa_+^{(1),\alpha}] = \begin{pmatrix} m_a^{\alpha\beta}\gamma^a & 0 \\ 0 & n_a^{\alpha\beta}\gamma^a \end{pmatrix} + \frac{1}{8}\operatorname{str}\big(\Upsilon[A_+^{(1),\beta}, \kappa_+^{(1),\alpha}]\big)\mathbb{1}_8,
$$

which clearly implies

$$
I_2^{\alpha} = \{A^{(2)}_{\beta,-}, \begin{pmatrix} m_a^{\alpha\beta}\gamma^a & 0 \\ 0 & n_a^{\alpha\beta}\gamma^a \end{pmatrix}\} + \frac{1}{4}\operatorname{str}\big(\Upsilon[A_+^{(1),\beta}, \kappa_+^{(1),\alpha}]\big)A^{(2)}_{\beta,-}. \tag{A.72}
$$

Again, as discussed in 1.1 and used in (1.53), elements $A^{(2)}_{\beta,-}$ can be expressed as

$$
A^{(2)}_{\beta,-} = \begin{pmatrix} p^i_{\beta,-}\gamma^i & 0 \\ 0 & q^i_{\beta,-}\gamma^i \end{pmatrix},
$$

which can be used to simplify the anti-commutator

$$
\begin{aligned}
\{A^{(2)}_{\beta,-}, \begin{pmatrix} m_a^{\alpha\beta}\gamma^a & 0 \\ 0 & n_a^{\alpha\beta}\gamma^a \end{pmatrix}\} &= \begin{pmatrix} m_a^{\alpha\beta}p^i_{\beta,-}\{\gamma^i,\gamma^a\} & 0 \\ 0 & n_a^{\alpha\beta}q^i_{\beta,-}\{\gamma^i,\gamma^a\} \end{pmatrix} = \begin{pmatrix} m_a^{\alpha\beta}p^i_{\beta,-}\delta^{ia}\mathbb{1}_4 & 0 \\ 0 & n_a^{\alpha\beta}q^i_{\beta,-}\delta^{ia}\mathbb{1}_4 \end{pmatrix}\\
&= \begin{pmatrix} m_a^{\alpha\beta}p^a_{\beta,-}\mathbb{1}_4 & 0 \\ 0 & n_a^{\alpha\beta}q^a_{\beta,-}\mathbb{1}_4 \end{pmatrix} \equiv \frac{1}{2}\rho_1^{\alpha}\mathbb{1}_8 + \frac{1}{2}\rho_2^{\alpha}\Upsilon.
\end{aligned}
$$

† The fermionic equations of motion (1.40) are equivalent to $[A_+^{(1),\beta}, A^{(2)}_{\alpha}] = 0$ and thus $[A_+^{(1),\beta}, A^{(2)}_{\beta,-}] = 0$

This means

$$2I_2^\alpha = \rho_1^\alpha \mathbb{1}_8 + \rho_2^\alpha \Upsilon - \frac{1}{2}\,\mathrm{str}\big(\Upsilon[\kappa_+^{(1),\alpha}, A_+^{(1),\beta}]\big) A^{(2)}_{\beta,-}. \tag{A.73}$$

Because of its original definition as a commutator (A.71), I_2^α must be supertraceless which means that $\rho_2^\alpha = 0$ since $\mathrm{str}\big(\mathbb{1}_8\big) = \mathrm{str}\big(A\big) = 0$. The first term will not contribute as we are working modulo $i\mathbb{1}_8$ in $\mathfrak{psu}(2,2|4)$. Finally, the last term will cancel with the $\delta_\epsilon \gamma^{\alpha\beta}$ (1.56) in (A.68):

$$\varepsilon_{\alpha\beta}\left(2I_2^\beta + \delta_\epsilon \gamma^{\beta\delta} A^{(2)}_\delta\right) = \varepsilon_{\alpha\beta}\left(-\frac{1}{2}\,\mathrm{str}\big(\Upsilon[\kappa_+^{(1),\beta}, A_+^{(1),\delta}]\big) A^{(2)}_{\delta,-} + \delta_\epsilon \gamma^{\beta\delta} A^{(2)}_\delta\right) = 0.$$

Diffeomorphisms

To show diffeomorphisms $\sigma^\alpha \to \sigma^\alpha = \tilde{\sigma}^\alpha + f^\alpha(\sigma,\tau)$ induce a gauge transformation of the Lax connections, we first calculate $\tilde{L}_\alpha(\tilde{\boldsymbol{\sigma}})$ in two different ways. On one hand, a one-form transforms as

$$\begin{aligned}\tilde{L}_\alpha(\tilde{\boldsymbol{\sigma}}) &= L_\beta(\boldsymbol{\sigma})\frac{\partial \sigma^\beta}{\partial \tilde{\sigma}^\alpha} = L_\beta(\boldsymbol{\sigma})\frac{\partial(\tilde{\sigma}^\beta + f^\beta)}{\partial \tilde{\sigma}^\alpha} = L_\beta(\boldsymbol{\sigma})(\delta_\alpha^\beta + \frac{\partial f^\beta}{\partial \sigma^\delta}\frac{\partial \sigma^\delta}{\partial \tilde{\sigma}^\alpha}) \\ &= L_\alpha(\boldsymbol{\sigma}) + L_\beta(\boldsymbol{\sigma})\partial_\delta f^\beta \delta_\alpha^\delta + \mathcal{O}(f^2) = L_\alpha(\boldsymbol{\sigma}) + (L_\beta \partial_\alpha f^\beta)(\boldsymbol{\sigma}) + \mathcal{O}(f^2).\end{aligned}$$

On the other hand, using the Taylor expansion of $\tilde{L}_\alpha(\boldsymbol{\sigma} - \mathbf{f})$ in f^α around $\mathbf{f} = 0$, and substituting what we just found above,

$$\begin{aligned}\tilde{L}_\alpha(\tilde{\boldsymbol{\sigma}}) = \tilde{L}_\alpha(\boldsymbol{\sigma} - \mathbf{f}) &= \tilde{L}_\alpha(\boldsymbol{\sigma}) - f^\beta(\frac{\partial}{\partial f^\beta}\tilde{L}_\alpha)(\boldsymbol{\sigma}) + \mathcal{O}(f^2) \\ &= \tilde{L}_\alpha(\boldsymbol{\sigma}) - f^\beta \frac{\partial \sigma^\rho}{\partial f^\beta}\partial_\rho\big[L_\alpha + L_\beta \partial_\alpha f^\beta + \mathcal{O}(f^2))\big](\boldsymbol{\sigma}) + \mathcal{O}(f^2) \\ &= \tilde{L}_\alpha(\boldsymbol{\sigma}) - (f^\beta \delta_\beta^\rho \partial_\rho L_\alpha)(\boldsymbol{\sigma}) + \mathcal{O}(f^2) = \tilde{L}_\alpha(\boldsymbol{\sigma}) - (f^\beta \partial_\beta L_\alpha)(\boldsymbol{\sigma}) + \mathcal{O}(f^2).\end{aligned}$$

Equating the two expressions for $\tilde{L}_\alpha(\tilde{\boldsymbol{\sigma}})$ up to linear order in f^α, we get

$$\delta L_\alpha = \tilde{L}_\alpha(\boldsymbol{\sigma}) - L_\alpha(\boldsymbol{\sigma}) = f^\beta \partial_\beta L_\alpha + L_\beta \partial_\alpha f^\beta. \tag{A.74}$$

Using the zero-curvature condition for L_α, we find (1.94).

```
In[21]:= l0[z_] := 0; (* Lax parameters *)
l1[z_] := (1 - z^2)^2 / 2 / z^2;
l2[z_] := - (z^2 - 1 / z^2) / 2 / κ;
l3[z_] := z - 1;
l4[z_] := 1 / z - 1;
F[z_] := l0[z] a0 + l1[z] a2 + l2[z] ϵa2 + l3[z] a1 + l4[z] a3; (* Lα *)
f[w_] := Series[F[1 - w], {w, 0, 1}]; (*Expanding Lα  in z around 1-z*)
f[w]
Out[28]=
```

$$\left(-a_1 + a_3 + \frac{2\,\epsilon a_2}{\kappa}\right) w + O[w]^2$$

Figure 7. Expanding the shifted Lax connection to get (1.96).

A.8 Details of embedding

Given the S^5 coordinates (1.99), we can find the differentials and their moduli squared:

$$(dY^1)^2+(dY^2)^2=\frac{(dy^1)^2+(dy^2)^2}{(1+|y|^2/4)^2}+\frac{1}{4}\frac{1}{(1+|y|^2/4)^4}\left\{(y^idy^i)^2[(y^1)^2+(y^2)^2]-(4+|y|^2)(y^idy^i)(y^1dy^1+y^2dy^2)\right\},$$
$$(dY^3)^2+(dY^4)^2=\frac{(dy^3)^2+(dy^4)^2}{(1+|y|^2/4)^2}+\frac{1}{4}\frac{1}{(1+|y|^2/4)^4}\left\{(y^idy^i)^2[(y^3)^2+(y^4)^2]-(4+|y|^2)(y^idy^i)(y^3dy^3+y^4dy^4)\right\},$$
$$(dY^5)^2+(dY^6)^2=\Big(\frac{1-|y|^2/4}{1+|y|^2/4}\Big)^2(d\phi)^2+\frac{(y^idy^i)^2}{(1+|y|^2/4)^4}.$$

Their sum gives the induced metric $ds^2|_{S^5}$ (1.100) since adding the two first equations results in

$$(dY^1)^2+(dY^2)^2+(dY^3)^2+(dY^4)^2=\frac{dy^idy^i}{(1+|y|^2/4)^2}-\frac{(y^idy^i)^2}{(1+|y|^2/4)^4}=ds^2|_{S^5}-(dY^5)^2+(dY^6)^2.$$

For the AdS_5 coordinates (1.101), we can simply replace $|y|^2$ with $-|z|^2$ to find

$$(dZ^1)^2+(dZ^2)^2=\frac{(dz^1)^2+(dz^2)^2}{(1-|z|^2/4)^2}-\frac{1}{4}\frac{1}{(1-|z|^2/4)^4}\left\{(z^idz^i)^2[(z^1)^2+(z^2)^2]+(4-|z|^2)(z^idz^i)(z^1dz^1+z^2dz^2)\right\},$$
$$(dZ^3)^2+(dZ^4)^2=\frac{(dz^3)^2+(dz^4)^2}{(1-|z|^2/4)^2}-\frac{1}{4}\frac{1}{(1-|z|^2/4)^4}\left\{(z^idz^i)^2[(z^3)^2+(z^4)^2]+(4-|z|^2)(z^idz^i)(z^3dz^3+z^4dz^4)\right\},$$
$$(dZ^0)^2+(dZ^5)^2=\Big(\frac{1+|z|^2/4}{1-|z|^2/4}\Big)^2(dt)^2-\frac{(z^idz^i)^2}{(1-|z|^2/4)^4}.$$

This time their sum has signature $(\eta_{AB})=\mathrm{diag}(-1,1,1,1,1,-1)$ which results in the induced metric $ds^2|_{\mathrm{AdS}_5}$ (1.102) since with this signature

$$(dZ^1)^2+(dZ^2)^2+(dZ^3)^2+(dZ^4)^2=\frac{dz^idz^i}{(1-|z|^2/4)^2}-\frac{(z^idz^i)^2}{(1-|z|^2/4)^4}=ds^2|_{\mathrm{AdS}_5}+(dZ^0)^2+(dZ^5)^2.$$

We shall now find the representation of the bosonic element $\mathfrak{g}_\mathfrak{b}$, whose bilinear form $\mathrm{str}[(\mathfrak{g}_\mathfrak{b}^{-1}d\mathfrak{g}_\mathfrak{b})^2]$ reproduces the metric (1.103) as described in 1.4. First, we introduce the matrices

$$\mathfrak{g}_\mathfrak{b}=\Lambda(t,\phi)\mathfrak{g}(\mathbb{X}),\qquad \mathfrak{g}(\mathbb{X})=\sqrt{\frac{\mathbb{1}+\mathbb{X}}{\mathbb{1}-\mathbb{X}}}=\big((\mathbb{1}_8-\mathbb{X})^{-1}(\mathbb{1}_8+\mathbb{X})\big)^{\frac{1}{2}}. \tag{A.75}$$

where $\mathbb{X}$ is given by (1.113). To compute $\mathfrak{g}(\mathbb{X})$, we will need to find the inverse of

$$\mathbb{1}_8-\mathbb{X}=\begin{pmatrix}\mathbb{1}_4-\frac{1}{2}z^i\gamma^i & 0\\ 0 & \mathbb{1}_4-\frac{\mathrm{i}}{2}y^i\gamma^i\end{pmatrix}. \tag{A.76}$$

We know that $(\gamma^i)^2=\mathbb{1}_4$. Looking at a simpler case, for example

$$\begin{aligned}(\mathbb{1}_4-a\gamma^1-b\gamma^2)(\mathbb{1}_4+a\gamma^1+b\gamma^2)&=\mathbb{1}_4-a^2\mathbb{1}_4-b^2\mathbb{1}_4-ab\cancel{(\gamma^1\gamma^2+\gamma^2\gamma^1)}\\&=(1-a^2-b^2)\mathbb{1}_4,\end{aligned}$$

it becomes clear that the inverse of (A.76) should be

$$(\mathbb{1}_8-\mathbb{X})^{-1}=\begin{pmatrix}\frac{1}{1-|z|^2/4}[\mathbb{1}_4+\frac{1}{2}z^i\gamma^i] & 0\\ 0 & \frac{1}{1+|y|^2/4}[\mathbb{1}_4+\frac{\mathrm{i}}{2}y^i\gamma^i]\end{pmatrix}. \tag{A.77}$$

Substituting into (A.75), we easily get (1.114)

$$\mathfrak{g}(\mathbb{X}) = \begin{pmatrix} \frac{1}{\sqrt{1-|z|^2/4}}[\mathbb{1}_4 + \frac{1}{2}z^i\gamma^i] & 0 \\ 0 & \frac{1}{\sqrt{1+|y|^2/4}}[\mathbb{1}_4 + \frac{\mathrm{i}}{2}y^i\gamma^i] \end{pmatrix}. \tag{A.78}$$

Appendix B
Chapter 2

B.1 First-order formalism

Bosonic string

Here we derive (2.4). If we start by summing (2.1) explicitly[16], we find

$$
\begin{aligned}
S &= -\frac{T}{2}\iint d^2\sigma\; G_{MN}\left(\gamma^{\tau\tau}\dot{X}^M\dot{X}^N + 2\gamma^{\tau\sigma}\dot{X}^M X'^N + \gamma^{\sigma\sigma}X'^M X'^N\right)\\
&= \iint d^2\sigma\; G_{MN}\left(-\frac{T}{2}\gamma^{\tau\tau}\dot{X}^M\dot{X}^N - T\gamma^{\tau\sigma}\dot{X}^M X'^N - \frac{T}{2}\gamma^{\sigma\sigma}X'^M X'^N\right)\\
&= \iint d^2\sigma\; G_{MN}\left(-\frac{T}{2}\gamma^{\tau\tau}\dot{X}^M\dot{X}^N - \frac{T}{2}\gamma^{\tau\tau}\dot{X}^M\dot{X}^N + \frac{T}{2}\gamma^{\tau\tau}\dot{X}^M\dot{X}^N - T\gamma^{\tau\sigma}\dot{X}^M X'^N - \frac{T}{2}\gamma^{\sigma\sigma}X'^M X'^N\right)\\
&\overset{(2.2)}{=} \iint d^2\sigma\; G_{MN}\big(p^N\dot{X}^M + \underbrace{\frac{T}{2}\gamma^{\tau\tau}\dot{X}^M\dot{X}^N - \frac{T}{2}\gamma^{\sigma\sigma}X'^M X'^N}_{\star^{MN}}\big), \quad \text{where}
\end{aligned}
$$

$$
\begin{aligned}
\star^{MN} &= \frac{T}{2}\gamma^{\tau\tau}\dot{X}^M\dot{X}^N - \frac{T}{2}\gamma^{\sigma\sigma}X'^M X'^N - T\frac{\gamma^{\tau\sigma}\gamma^{\tau\sigma}}{\gamma^{\tau\tau}}X'^M X'^N + T\frac{\gamma^{\tau\sigma}\gamma^{\tau\sigma}}{\gamma^{\tau\tau}}X'^M X'^N - T\gamma^{\tau\sigma}\dot{X}^M X'^N + T\gamma^{\tau\sigma}\dot{X}^M X'^N\\
&= \frac{\gamma^{\tau\sigma}}{\gamma^{\tau\tau}}\left(-T\gamma^{\tau\tau}\dot{X}^M X'^N - T\gamma^{\tau\sigma}X'^M X'^N\right) + \frac{1}{2T\gamma^{\tau\tau}}\Big(T^2\gamma^{\tau\tau}\gamma^{\tau\tau}\dot{X}^M\dot{X}^N - T^2\gamma^{\tau\tau}\gamma^{\sigma\sigma}X'^M X'^N\\
&\qquad + 2T^2\gamma^{\tau\sigma}\gamma^{\tau\sigma}X'^M X'^N + 2T^2\gamma^{\tau\tau}\gamma^{\tau\sigma}\dot{X}^M X'^N\Big)\\
&= \frac{\gamma^{\tau\sigma}}{\gamma^{\tau\tau}}p^M X'^N + \frac{1}{2T\gamma^{\tau\tau}}\left(p^M p^N - T^2\gamma^{\tau\tau}\gamma^{\sigma\sigma}X'^M X'^N + T^2\gamma^{\tau\sigma}\gamma^{\tau\sigma}X'^M X'^N\right)\\
&= \frac{\gamma^{\tau\sigma}}{\gamma^{\tau\tau}}p^M X'^N + \frac{1}{2T\gamma^{\tau\tau}}\left(p^M p^N - T^2\det(\gamma^{\alpha\beta})X'^M X'^N\right) = \frac{\gamma^{\tau\sigma}}{\gamma^{\tau\tau}}p^M X'^N + \frac{1}{2T\gamma^{\tau\tau}}\left(p^M p^N + T^2 X'^M X'^N\right).
\end{aligned}
$$

Looking at (2.3), we can identify the constraints (2.4). In light cone coordinates (2.6), the first term in the first-order form action becomes

$$
\begin{aligned}
p_M\dot{X}^M &= p_t\dot{t} + p_\phi\dot{\phi} + p_\mu\dot{x}^\mu\\
&= \left[(1-a)p_- - p_+\right]\left[\dot{x}_+ - a\dot{x}_-\right] + \left[p_+ + ap_-\right]\left[\dot{x}_+ + (1-a)\dot{x}_-\right] + p_\mu\dot{x}^\mu\\
&= (1-a)p_-\dot{x}_+ - a(1-a)p_-\dot{x}_- - p_+\dot{x}_+ + ap_+\dot{x}_- + p_+\dot{x}_+\\
&\qquad + (1-a)p_+\dot{x}_- + ap_-\dot{x}_+ + a(1-a)p_-\dot{x}_- + p_\mu\dot{x}^\mu\\
&= p_+\dot{x}_- + p_-\dot{x}_+ + p_\mu\dot{x}^\mu,
\end{aligned}
$$

Similarly, the two constraints turn into

$$
\begin{aligned}
C_1 &= p_M X'^M = p_t t' + p_\phi\phi' + p_\mu x'^\mu = p_+x'_- + p_-x'_+ + p_\mu x'^\mu,\\
C_2 &= p_\mu p_\nu G^{\mu\mu} - p_t^2 G_{tt}^{-1} + p_\phi^2 G_{\phi\phi}^{-1} + g^2 x'^\mu x'^\nu G_{\mu\nu} - g^2 t'^2 G_{tt} + g^2\phi'^2 G_{\phi\phi}\\
&= 2\mathcal{H}_x + p + x'.
\end{aligned}
$$

[16] Note that the spacetime metric G_{MN} is diagonal and thus symmetric.

We have isolated the term $\mathcal{H}_\perp$ (2.13) involving the transversal degrees of freedom. Computing the two others terms in C_2, we get

$$\begin{aligned}
p &= -p_t^2 G_{tt}^{-1} + p_\phi^2 G_{\phi\phi}^{-1} = -\left[(1-a)p_- - p_+\right]^2 G_{tt}^{-1} + \left[p_+ + ap_-\right]^2 G_{\phi\phi}^{-1} \\
&= -(1-a)^2 p_-^2 G_{tt}^{-1} + 2(1-a) G_{tt}^{-1} p_+ p_- - G_{tt}^{-1} p_+^2 + G_{\phi\phi}^{-1} p_+^2 + 2a G_{\phi\phi}^{-1} p_+ p_- + a^2 G_{\phi\phi}^{-1} p_-^2 \\
&= p_+^2 \left[G_{\phi\phi}^{-1} - G_{tt}^{-1}\right] + 2p_+ p_- \left[a G_{\phi\phi}^{-1} - (1-a) G_{tt}^{-1}\right] + p_-^2 \left[a^2 G_{\phi\phi}^{-1} - (1-a)^2 G_{tt}^{-1}\right],
\end{aligned}$$

and

$$\begin{aligned}
x' &= -g^2 t'^2 G_{tt} + g^2 \phi'^2 G_{tt} = -g^2 \left[x'_+ - a x'_-\right]^2 G_{tt} + g^2 \left[x'_+ + (1-a) x'_-\right]^2 G_{\phi\phi} \\
&= -g^2 G_{tt} x'^2_+ + 2g^2 a G_{tt} x'_+ x'_- - g^2 a^2 G_{tt} x'^2_- + g^2 G_{\phi\phi} x'^2_+ + 2g^2 (1-a) G_{\phi\phi} x'_+ x'_- + g^2 (1-a)^2 G_{\phi\phi} x'^2_- \\
&= g^2 x'^2_+ \left[G_{\phi\phi} - G_{tt}\right] + 2g^2 x'_+ x'_- \left[a G_{tt} + (1-a) G_{\phi\phi}\right] + g^2 x'^2_- \left[(1-a)^2 G_{\phi\phi} - a^2 G_{tt}\right].
\end{aligned}$$

Putting these three terms together, we retrieve (2.12).

Virasoro algebra

Here we derive the Virasoro Poisson algebra (2.5). The constraints at fixed τ are

$$C_1(\sigma) = p_M(\sigma)\partial_\sigma X^M(\sigma), \qquad C_2 = p_M(\sigma)p^M(\sigma) + T^2 \partial_\sigma X_M(\sigma)\partial_\sigma X^M(\sigma).$$

Here, $X'(\sigma)$ will always mean $\partial_\sigma X(\sigma)$. The Poisson bracket satisfies

$$\begin{aligned}
\{X^M(\sigma), p_N(\sigma')\}_{\text{P.B.}} = \frac{\partial X^M(\sigma)}{\partial X^L(\sigma'')} \frac{\partial p_N(\sigma')}{\partial p_L(\sigma'')} - 0 = \delta^M_L \delta^L_N \delta(\sigma - \sigma'')\delta(\sigma' - \sigma'') = \delta^M_N \delta(\sigma - \sigma'), \\
\{X^M(\sigma), X^N(\sigma)\}_{\text{P.B.}} = \{p_M(\sigma), p_N(\sigma')\}_{\text{P.B.}} = 0.
\end{aligned} \tag{B.1}$$

For the rest of this appendix the P.B. subscript will be suppressed. This in turn implies

$$\begin{aligned}
\{X'^M(\sigma), p_N(\sigma')\} = \{p_M(\sigma), X'^N(\sigma')\} = \delta^M_N \partial_\sigma \delta(\sigma - \sigma'), \\
\{X'^M(\sigma), X^N(\sigma)\} = \{p'_M(\sigma), p_N(\sigma')\} = 0
\end{aligned} \tag{B.2}$$

since $\partial_\sigma \delta(\sigma - \sigma') = -\partial_{\sigma'} \delta(\sigma - \sigma')$.[17] If we know the key relations

$$\{X'^N(\sigma), C_{1,2}(\sigma')\} \quad \text{and } \{p_N(\sigma), C_{1,2}(\sigma')\},$$

we can rather easily find the Poisson algebra. Starting with

$$\begin{aligned}
\{X'^N(\sigma), C_1(\sigma')\} &= \{X'^N(\sigma), p_M(\sigma') X'^M(\sigma')\} \\
&= p_M(\sigma')\{X'^N(\sigma), X'^M(\sigma')\} + X'^M(\sigma')\{X'^N(\sigma), p_M(\sigma')\} \\
&= X'^M(\sigma')\delta^N_M \partial_\sigma \delta(\sigma - \sigma') = \partial_\sigma \left(X'^N(\sigma')\delta(\sigma - \sigma')\right),
\end{aligned}$$

because of the useful identity

$$\int d\sigma' \, \delta(\sigma - \sigma') \frac{\partial X(\sigma')}{\partial \sigma'} = \cancel{\delta(\sigma - \sigma')} X(\sigma')\big| - \int d\sigma' \, X(\sigma') \partial_{\sigma'} \delta(\sigma - \sigma')$$

[17] This can easily be checked by comparing the effect on a test function with some integration by parts.

$$= \partial_\sigma \int d\sigma' \, X(\sigma')\delta(\sigma - \sigma') = X'(\sigma) = \int d\sigma' \, \delta(\sigma - \sigma')X'(\sigma),$$

we can effectively equate $\delta(\sigma - \sigma')X'^N(\sigma') = \delta(\sigma - \sigma')X'^N(\sigma)$ such that

$$\{X'^N(\sigma), C_1(\sigma')\} = \partial_\sigma \left(X'^N(\sigma)\delta(\sigma - \sigma') \right) = X''^N(\sigma)\delta(\sigma - \sigma') + X'^N(\sigma)\partial_\sigma\delta(\sigma - \sigma').$$

Similarly, using (B.2)

$$\{p_N(\sigma), C_1(\sigma')\} = \partial_\sigma \left(p_N(\sigma)\delta(\sigma - \sigma') \right) = p'_N(\sigma)\delta(\sigma - \sigma') + p_N(\sigma)\partial_\sigma\delta(\sigma - \sigma').$$

In flat space the metric can be taken out of Poisson brackets, so a similar set of calculations yield

$$\begin{aligned} \{X'^N(\sigma), C_2(\sigma')\} &= 2p'^N(\sigma)\delta(\sigma - \sigma') + 2p^N(\sigma)\partial_\sigma\delta(\sigma - \sigma'), \\ \{p_N(\sigma), C_2(\sigma')\} &= 2T^2 X''_N(\sigma)\delta(\sigma - \sigma') + 2T^2 X'_N(\sigma)\partial_\sigma\delta(\sigma - \sigma'). \end{aligned}$$

As promised, it has now become simple to compute

$$\begin{aligned} \{C_1(\sigma), C_1(\sigma')\} &= p_N(\sigma)\{X'^N(\sigma), C_1(\sigma')\} + X'^N(\sigma)\{p_N(\sigma), C_1(\sigma')\} \\ &= p_N(\sigma)X''^N(\sigma)\delta(\sigma - \sigma') + p_N(\sigma)X'^N(\sigma)\partial_\sigma\delta(\sigma - \sigma') \\ &\qquad + X'^N(\sigma)p'_N(\sigma)\delta(\sigma - \sigma') + X'^N(\sigma)p_N(\sigma)\partial_\sigma\delta(\sigma - \sigma') \\ &= \partial_\sigma C_1(\sigma)\delta(\sigma - \sigma') + 2C_1(\sigma)\partial_\sigma\delta(\sigma - \sigma') \end{aligned}$$

and

$$\begin{aligned} \{C_1(\sigma), C_2(\sigma')\} &= p_N(\sigma)\{X'^N(\sigma), C_2(\sigma')\} + X'^N(\sigma)\{p_N(\sigma), C_2(\sigma')\} \\ &= 2p_N(\sigma)p'^N(\sigma)\delta(\sigma - \sigma') + 2p_N(\sigma)p^N(\sigma)\partial_\sigma\delta(\sigma - \sigma') \\ &\qquad + 2X'^N(\sigma)X''_N(\sigma)\delta(\sigma - \sigma') + 2X'^N(\sigma)X'_N(\sigma)\partial_\sigma\delta(\sigma - \sigma') \\ &= \partial_\sigma C_2(\sigma)\delta(\sigma - \sigma') + 2C_2(\sigma)\partial_\sigma\delta(\sigma - \sigma'). \end{aligned}$$

One could explicitly compute $\{C_2(\sigma), C_1(\sigma')\}$ to get the same expression, or just use the anti-symmetry of the Poisson bracket combined with the extra minus sign which comes from

$$f(\sigma)\partial_\sigma\delta(\sigma - \sigma') = -f(\sigma')\partial_{\sigma'}\delta(\sigma - \sigma').$$

The final relation in (2.5) is

$$\begin{aligned} \{C_2(\sigma), C_2(\sigma')\} &= 2p_N(\sigma)\{p^N(\sigma), C_2(\sigma')\} + 2T^2 X'^N(\sigma)\{X'_N(\sigma), C_2(\sigma')\} \\ &= 4T^2 p^N(\sigma)X''^N(\sigma)\delta(\sigma - \sigma') + 4T^2 p_N(\sigma)X'^N(\sigma)\partial_\sigma\delta(\sigma - \sigma') \\ &\qquad + 4T^2 X'^N(\sigma)p'_N(\sigma)\delta(\sigma - \sigma') + 4T^2 X'^N(\sigma)p_N(\sigma)\partial_\sigma\delta(\sigma - \sigma') \\ &= 4T^2\partial_\sigma C_1(\sigma)\delta(\sigma - \sigma') + 8T^2 C_1(\sigma)\partial_\sigma\delta(\sigma - \sigma'). \end{aligned}$$

Superstring

Substituting this expression for Π into the Lagrangian minus the Wess-Zumino term,

$$\mathscr{L} - \mathscr{L}_{\mathrm{WZ}} = -\mathrm{str}\Big[\Pi A^{(2)}_\tau + \frac{\gamma^{\tau\sigma}}{\gamma^{\tau\tau}}\Pi A^{(2)}_\sigma - \frac{1}{2T\gamma^{\tau\tau}}\left(\Pi^2 + T^2 A^{(2)}_\sigma A^{(2)}_\sigma\right)\Big]$$

$$
\begin{aligned}
&= -\mathrm{str}\Big[T\gamma^{\tau\tau}A^{(2)}_\tau A^{(2)}_\tau + T\gamma^{\tau\sigma}A^{(2)}_\sigma A^{(2)}_\tau + T\gamma^{\tau\sigma}A^{(2)}_\tau A^{(2)}_\sigma + T\frac{\gamma^{\tau\sigma}\gamma^{\tau\sigma}}{\gamma^{\tau\tau}}A^{(2)}_\sigma A^{(2)}_\sigma \\
&\qquad - \frac{1}{2T\gamma^{\tau\tau}}\left(T^2\gamma^{\tau\tau}A^{(2)}_\tau A^{(2)}_\tau + 2T^2\gamma^{\tau\tau}\gamma^{\tau\sigma}A^{(2)}_\tau A^{(2)}_\sigma + T^2\gamma^{\tau\sigma}\gamma^{\tau\sigma}A^{(2)}_\sigma A^{(2)}_\sigma + T^2 A^{(2)}_\sigma A^{(2)}_\sigma\right)\Big] \\
&= -\mathrm{str}\Big[T\gamma^{\alpha\beta}A^{(2)}_\alpha A^{(2)}_\beta - T\gamma^{\sigma\sigma}A^{(2)}_\sigma A^{(2)}_\sigma + T\frac{\gamma^{\tau\sigma}\gamma^{\tau\sigma}}{\gamma^{\tau\tau}}A^{(2)}_\sigma A^{(2)}_\sigma \\
&\qquad - \frac{T}{2}\gamma^{\tau\tau}A^{(2)}_\tau A^{(2)}_\tau - T\gamma^{\tau\sigma}A^{(2)}_\tau A^{(2)}_\sigma - \frac{T}{2}\frac{\gamma^{\tau\sigma}\gamma^{\tau\sigma}}{\gamma^{\tau\tau}}A^{(2)}_\sigma A^{(2)}_\sigma + \frac{T}{2\gamma^{\tau\tau}}\det(\gamma^{\alpha\beta})A^{(2)}_\sigma A^{(2)}_\sigma\Big] \\
&= -\mathrm{str}\Big[T\gamma^{\alpha\beta}A^{(2)}_\alpha A^{(2)}_\beta - T\gamma^{\sigma\sigma}A^{(2)}_\sigma A^{(2)}_\sigma + T\frac{\gamma^{\tau\sigma}\gamma^{\tau\sigma}}{\gamma^{\tau\tau}}A^{(2)}_\sigma A^{(2)}_\sigma \\
&\qquad - \frac{T}{2}\gamma^{\tau\tau}A^{(2)}_\tau A^{(2)}_\tau - T\gamma^{\tau\sigma}A^{(2)}_\tau A^{(2)}_\sigma - \frac{T}{2}\frac{\gamma^{\tau\sigma}\gamma^{\tau\sigma}}{\gamma^{\tau\tau}}A^{(2)}_\sigma A^{(2)}_\sigma + \frac{T}{2}\gamma^{\sigma\sigma}A^{(2)}_\sigma A^{(2)}_\sigma - \frac{T}{2}\frac{\gamma^{\tau\sigma}\gamma^{\tau\sigma}}{\gamma^{\tau\tau}}A^{(2)}_\sigma A^{(2)}_\sigma\Big] \\
&= -\mathrm{str}\Big[T\gamma^{\alpha\beta}A^{(2)}_\alpha A^{(2)}_\beta - \frac{T}{2}\gamma^{\tau\tau}A^{(2)}_\tau A^{(2)}_\tau - T\gamma^{\tau\sigma}A^{(2)}_\tau A^{(2)}_\sigma - \frac{T}{2}\gamma^{\sigma\sigma}A^{(2)}_\sigma A^{(2)}_\sigma\Big] = -\frac{T}{2}\gamma^{\alpha\beta}\,\mathrm{str}\big(A^{(2)}_\alpha A^{(2)}_\beta\big),
\end{aligned}
$$

which is indeed the kinetic term of the Green-Schwarz Lagrangian (1.33).

Kappa symmetry

To begin we have two easy identities to prove. Namely,

$$
\Sigma_+^{-1}\chi\Sigma_+ = \left(\begin{array}{cc|cc} \mathbb{1}_2 & 0 & 0 & 0 \\ 0 & -\mathbb{1}_2 & 0 & 0 \\ \hline 0 & 0 & \mathbb{1}_2 & 0 \\ 0 & 0 & 0 & -\mathbb{1}_2 \end{array}\right)\left(\begin{array}{cc|cc} 0 & 0 & 0 & a \\ 0 & 0 & b & 0 \\ \hline 0 & b^\dagger & 0 & 0 \\ -a^\dagger & 0 & 0 & 0 \end{array}\right)\left(\begin{array}{cc|cc} \mathbb{1}_2 & 0 & 0 & 0 \\ 0 & -\mathbb{1}_2 & 0 & 0 \\ \hline 0 & 0 & \mathbb{1}_2 & 0 \\ 0 & 0 & 0 & -\mathbb{1}_2 \end{array}\right) = -\chi,
$$

$$
\Sigma_-^{-1}\chi\Sigma_- = \left(\begin{array}{cc|cc} -\mathbb{1}_2 & 0 & 0 & 0 \\ 0 & \mathbb{1}_2 & 0 & 0 \\ \hline 0 & 0 & \mathbb{1}_2 & 0 \\ 0 & 0 & 0 & -\mathbb{1}_2 \end{array}\right)\left(\begin{array}{cc|cc} 0 & 0 & 0 & a \\ 0 & 0 & b & 0 \\ \hline 0 & b^\dagger & 0 & 0 \\ -a^\dagger & 0 & 0 & 0 \end{array}\right)\left(\begin{array}{cc|cc} -\mathbb{1}_2 & 0 & 0 & 0 \\ 0 & \mathbb{1}_2 & 0 & 0 \\ \hline 0 & 0 & \mathbb{1}_2 & 0 \\ 0 & 0 & 0 & -\mathbb{1}_2 \end{array}\right) = \chi.
$$

These equivalently imply (2.29). Our next task is to find $A = -\mathfrak{g}^{-1}d\mathfrak{g}$ given by (2.31) and sort it into even elements A_e and odd elements A_o such that $A = A_\mathrm{e} + A_\mathrm{o}$. By the product rule,

$$
\begin{aligned}
A = -\mathfrak{g}^{-1}d\mathfrak{g} &= -\mathfrak{g}(\mathbb{X})^{-1}\mathfrak{g}(\chi)^{-1}\Lambda^{-1}d\left(\Lambda\mathfrak{g}(\chi)\mathfrak{g}(\mathbb{X})\right) \\
&= \underbrace{-\mathfrak{g}(\mathbb{X})^{-1}\mathfrak{g}(\chi)^{-1}\Lambda^{-1}d\Lambda\mathfrak{g}(\chi)\mathfrak{g}(\mathbb{X})}_{①}\underbrace{-\mathfrak{g}(\mathbb{X})^{-1}\mathfrak{g}(\chi)^{-1}d\mathfrak{g}(\chi)\mathfrak{g}(\mathbb{X})}_{②}\underbrace{-\mathfrak{g}(\mathbb{X})^{-1}d\mathfrak{g}(\mathbb{X})}_{③}.
\end{aligned}
$$

Let us take care of ① first. In light cone coordinates, the longitudinal matrix $\Lambda(t,\phi)$ is given by

$$
\begin{aligned}
\Lambda(t,\phi) &= \exp\frac{\mathrm{i}}{2}\begin{pmatrix}\Sigma(x_+ - ax_-) & 0 \\ 0 & \Sigma\left(x_+ + (1-a)x_-\right)\end{pmatrix} \\
&= \exp\frac{\mathrm{i}}{2}\begin{pmatrix}\Sigma\left(x_+ + (\frac{1}{2}-a)x_-\right) - \Sigma\frac{1}{2}x_- & 0 \\ 0 & \Sigma\left(x_+ + (\frac{1}{2}-a)x_-\right) + \Sigma\frac{1}{2}x_-\end{pmatrix} \\
&= \exp\frac{\mathrm{i}}{2}\left[\Sigma_+\left(x_+ + (\frac{1}{2}-a)x_-\right) + \frac{1}{2}\Sigma_- x_-\right].
\end{aligned}
$$

The argument in the exponential Λ commutes with its derivative, which is still in terms of Σ matrices. This means

$$\Lambda^{-1}d\Lambda = \frac{\mathrm{i}}{2}\left[\Sigma_+\left(dx_+ + (\frac{1}{2}-a)dx_-\right) + \frac{1}{2}\Sigma_- dx_-\right]$$

such that we can use the identites (2.30) to find

$$\begin{aligned}
① &= -\mathfrak{g}(\mathbb{X})^{-1}\mathfrak{g}(\chi)^{-1}\frac{\mathrm{i}}{2}\left[\Sigma_+\left(dx_+ + (\frac{1}{2}-a)dx_-\right) + \frac{1}{2}\Sigma_- dx_-\right]\mathfrak{g}(\chi)\mathfrak{g}(\mathbb{X}) \\
&= -\mathfrak{g}(\mathbb{X})^{-1}\frac{\mathrm{i}}{2}\left[\Sigma_+\mathfrak{g}(\chi)^2\left(dx_+ + (\frac{1}{2}-a)dx_-\right) + \frac{1}{2}\Sigma_- dx_-\right]\mathfrak{g}(\mathbb{X}) \\
&= -\mathfrak{g}(\mathbb{X})^{-1}\left[\frac{\mathrm{i}}{2}\left(dx_+ + (\frac{1}{2}-a)dx_-\right)\Sigma_+(\mathbb{1}+2\chi^2+2\chi\sqrt{\mathbb{1}+\chi^2}) + \frac{\mathrm{i}}{4}dx_-\Sigma_-\right]\mathfrak{g}(\mathbb{X}).
\end{aligned}$$

The only odd term in ① is clearly the one with the factor $\Sigma_+\chi\sqrt{\mathbb{1}+\chi^2}$ which is the product of one odd element (χ) and even elements ($\Sigma_+, \mathfrak{g}(\mathbb{X}), \mathfrak{g}(\mathbb{X})^{-1}$ and $\sqrt{\mathbb{1}+\chi^2}$). To find ②, we should calculate

$$\begin{aligned}
\mathfrak{g}(\chi)^{-1}d\mathfrak{g}(\chi) &= (-\chi+\sqrt{\mathbb{1}+\chi^2})d(\chi+\sqrt{\mathbb{1}+\chi^2}) \\
&= (\sqrt{\mathbb{1}+\chi^2}d\sqrt{\mathbb{1}+\chi^2} - \chi d\chi)_{\text{even}} + (\sqrt{\mathbb{1}+\chi^2}d\chi - \chi d\sqrt{\mathbb{1}+\chi^2})_{\text{odd}}.
\end{aligned}$$

This is consistent since, under the substitution $\chi \to \sinh\chi$, the above becomes $d\chi$ when $\mathfrak{g}(\chi) = \exp\chi$. If we define the even and odd parts of $\mathfrak{g}(\chi)^{-1}d\mathfrak{g}(\chi)$ as B and F respectively[18],

$$② = -\mathfrak{g}(\mathbb{X})^{-1}\mathfrak{g}(\chi)^{-1}d\mathfrak{g}(\chi)\mathfrak{g}(\mathbb{X}) \equiv -\mathfrak{g}(\mathbb{X})^{-1}B\mathfrak{g}(\mathbb{X}) - \mathfrak{g}(\mathbb{X})^{-1}F\mathfrak{g}(\mathbb{X}).$$

Finally, ③ is an even term as it only depends on $\mathbb{X}$. Adding the three yields (2.33).

B.2 Giant magnon

In this section we will discuss a solution to a heavily simplified version of the classical superstring in $\mathrm{AdS}_5\times S^5$. Namely, excitations of the string will be confined to a single transverse field of the sphere. For example, setting all transversal fields to zero except y^1 reduces $\mathrm{AdS}_5\times S^5$ to $\mathbb{R}\times S^2$ spanned by $t\times(\phi, y^1)$. It will be convenient to deal with the new variable

$$z = \frac{y^1}{1+|y|^2/4} \tag{B.3}$$

where this time $|y|^2 = (y^1)^2$. Had we instead considered an excitation in AdS_5, the denominator would not be well-defined at $|z|^2 = 4$, hence the restriction to S^5 in particular. It follows from (1.104) that $G_{tt} = G_{zz} = 1$ and the other components of the target space metric are

$$G_{\phi\phi} = \left(\frac{1-|y|^2/4}{1+|y|^2/4}\right)^2, \qquad G_{yy} = \frac{1}{(1+|y|^2/4)^2}.$$

We are interested in finding the metric induced on $\mathbb{R}\times S^2$ by this reduction. So far we have

$$ds^2|_{\mathrm{AdS}_5\times S^5} = -dt^2 + dz^i dz^i + G_{\phi\phi}d\phi^2 + G_{yy}dy^i dy^i$$

[18] These terms can be rewritten in terms of commutators of homogeneous elements, which in turns gives away their degree. This is shown for B_σ in (B.32).

which means

$$ds^2|_{\mathbb{R}\times S^2} = -dt^2 + G_{\phi\phi}d\phi^2 + G_{yy}(dy^1)^2. \tag{B.4}$$

Wanting to express these metric components and the differential dy^1 in terms of z, we look at

$$dz = \left[\frac{1}{1+|y|^2/4} - \frac{|y|^2/2}{(1+|y|^2/4)^2}\right]dy^1 = \left[\frac{1-|y|^2/4}{(1+|y|^2/4)^2}\right]dy^1$$

which directly implies

$$G_{yy}(dy^1)^2 = \left(\frac{1+|y|^2/4}{1-|y|^2/4}\right)^2 dz^2 = \frac{1}{G_{\phi\phi}}dz^2.$$

Looking at the form of $G_{\phi\phi}$, it is natural to look at

$$1-z^2 = \frac{(1+|y|^2/4)^2-|y|^2}{(1+|y|^2/4)^2} = \frac{1-|y|^2/2+(|y|^2/4)^2}{(1+|y|^2/4)^2} = \left(\frac{1-|y|^2/4}{1+|y|^2/4}\right)^2 = G_{\phi\phi}.$$

Thus the metric induced on $\mathbb{R}\times S^2$ by eliminating all but one transversal degrees of freedom is

$$ds^2|_{\mathbb{R}\times S^2} = -dt^2 + (1-z^2)^{-1}dz^2 + (1-z^2)d\phi^2. \tag{B.5}$$

We can now proceed by analysing the restricted first-order formalism. We already found the Hamiltonian $\mathcal{H}$ (2.23) by solving the constraint $C_2 = 0$. Notice that it features the string tension T, both explicitly and implicitly through $\mathcal{H}_\perp$. In 2.2 we will be taking the large tension limit. Also notice that the tension comes along a σ-derivative; x'_-. To retain a finite light cone Hamiltonian $\mathcal{H}$, it consequently jumps out to us that we should make use of the reparametrisation invariance of the spatial coordinate to redefine $\sigma \to T\sigma$ such that $Tx'_- \to x'_-$. We have successfully removed all tension dependence of $\mathcal{H}$ and are left with the action

$$S = T\int_{-\infty}^{\infty} d\tau \int_{-\pi rT}^{\pi rT} d\sigma\, \left(p_z\dot{z} - \mathcal{H}\right). \tag{B.6}$$

The Hamiltonian can be calculated by evaluating (2.23) when $G_{tt} = 1$, $G_{\phi\phi} = 1-z^2$ and

$$x'^2_- = -p_\mu x'^\mu = -p_z z', \qquad 2\mathcal{H}_\perp = (1-z^2)p_z^2 + (1-z^2)^{-1}z'^2.$$

Using the shorthand

$$\mathcal{Z}_a = (1-a)^2 G_{\phi\phi} - a^2 G_{tt} = (1-a)^2(1-z^2)^2 - a^2 = 1-2a-(1-a)^2z^2, \tag{B.7}$$

the Hamiltonian (2.23) becomes

$$\mathcal{H} = -\frac{1-(1-a)z^2}{\mathcal{Z}_a} + \frac{1}{\mathcal{Z}_a}\sqrt{(1-z^2)[1+\mathcal{Z}_a 2\mathcal{H}_\perp + \mathcal{Z}_a^2 x'^2_-]}$$

which we can rewrite as

$$\mathcal{H}(z,z',p_z) = -\frac{1-(a-1)z^2}{\mathcal{Z}_a} + \frac{1}{\mathcal{Z}_a}\sqrt{1+(1-z^2)\mathcal{Z}_a p_z^2}\sqrt{1-z^2+\mathcal{Z}_a z'^2}. \tag{B.8}$$

Since we are interested in soliton dynamics, it would be convenient to have an equation relating z, z' and $\dot{z}$, for example. These variables are naturally related by wave-like differential equations and could prove useful in writing down a solution to the former. To this end, we will now switch to the Lagrangian formalism. Given the Hamiltonian (B.8), one can find the momentum $p_z(z, z', \dot{z})$ conjugate to z by solving the equation of motion $\delta S/\delta p_z = 0$. The latter implies

$$0 = \dot{z} - \frac{\partial \mathcal{H}}{\partial p_z} = \dot{z} - p_z(1-z)^2 \sqrt{\frac{1 - z^2 + \mathcal{Z}_a z'^2}{1 + (1-z^2)\mathcal{Z}_a p_z^2}}.$$

Some simple algebraic manipulation yields the expression

$$p_z(z, z', \dot{z}) = \frac{\dot{z}}{\sqrt{(1-z^2)}\sqrt{(1-z^2)^2 - [\dot{z}^2 - (1-z^2)z'^2]\mathcal{Z}_a}}. \tag{B.9}$$

Substituting this expression into the restricted action gives us the Lagrangian

$$\mathscr{L}(z, z', \dot{z}) = \frac{1-(1-a)z^2}{\mathcal{Z}_a} - \frac{1}{\mathcal{Z}_a\sqrt{(1-z^2)}}\sqrt{(1-z^2)^2 - [\dot{z}^2 - (1-z^2)z'^2]\mathcal{Z}_a}. \tag{B.10}$$

Note that $\mathscr{L}$ has a term of the form $\sqrt{\dot{X}^2 - X'^2}$ which is reminiscent of the Nambu-Goto action. Having jumped the Legendre gap, we ansatz a general solution to the wave equation;

$$z = z(\sigma - v\tau), \qquad \dot{z} = vz' \tag{B.11}$$

where v is anticipated to be the speed with which the soliton travels in the σ direction. It is important to stress that this solution would describe a solitonic vibration of the *worldsheet*, i.e. a localised excitation which propagates in the (τ, σ) space in contrast to a wave propagating in $\mathrm{AdS}_5 \times S^5$ spacetime. This description is useful because we can eliminate the $\dot{z}$ degree of freedom by substituting (B.11) to find the reduced Lagrangian

$$\mathscr{L}_\mathrm{R}(z, z') = \frac{1-(1-a)z^2}{\mathcal{Z}_a} - \frac{1}{\mathcal{Z}_a\sqrt{(1-z^2)}}\sqrt{(1-z^2)^2 + (1-v^2-z^2)z'^2\mathcal{Z}_a}. \tag{B.12}$$

The new equivalent to a conjugate momentum π_z is

$$\pi_z = \frac{\partial \mathscr{L}_\mathrm{R}}{\partial z'} = -\frac{(1-v^2-z^2)z'}{\sqrt{(1-z^2)}\sqrt{(1-z^2)^2 + (1-v^2-z^2)z'^2\mathcal{Z}_a}} \tag{B.13}$$

such that the reduced Hamiltonian is

$$\mathcal{H}_\mathrm{R} = \pi_z z' - \mathscr{L}_\mathrm{R} = -\frac{1-(1-a)z^2}{\mathcal{Z}_a} + \frac{(1-z^2)\sqrt{(1-z^2)}}{\mathcal{Z}_a\sqrt{(1-z^2)^2 + (1-v^2-z^2)z'^2\mathcal{Z}_a}} \tag{B.14}$$

and one should in principle invert (B.13) to find $\mathcal{H}_\mathrm{R}(z, \pi_z)$. However, σ is clearly cyclic so we have a Hamiltonian where the 'time' coordinate is cyclic, which means $\mathcal{H}_\mathrm{R}$ is constant in σ. Solitonic solutions are localised so they must satisfy the sensible boundary conditions $z(\pm\infty) = 0 = z'(\pm\infty)$. In this regime,

$$\mathcal{H}_\mathrm{R}(\pm\infty, \pm\infty) = -\frac{1}{\mathcal{Z}_a} + \frac{1}{\mathcal{Z}_a} = 0$$

which tells us that the constant $\mathcal{H}_\mathrm{R}$ vanishes for all σ. Solving (B.14) for z',

$$(1-z^2)^2 + (1-v^2-z^2)z'^2\mathcal{Z}_a = \frac{(1-z^2)^3}{(1-(1-a)z^2)^2}$$

$$(1-v^2-z^2)z'^2\mathcal{Z}_a = \left(\frac{1-z^2}{1-(1-a)z^2}\right)^2 [(1-z^2)-(1-(1-a)z^2)^2]$$

where

$$\begin{aligned}(1-z^2)-(1-(1-a)z^2)^2 &= (1-z^2)-(1-2(1-a)z^2+(1-a)^2z^4)\\ &= -z^2+2(1-a)z^2-(1-a)^2z^4 = z^2\left(1-2a-(a-1)^2z^2\right) = z^2\mathcal{Z}_a\end{aligned}$$

so that finally

$$z'^2 = \left(\frac{1-z^2}{1-(a-1)z^2}\right)^2 \frac{z^2}{1-v^2-z^2}. \tag{B.15}$$

This non-linear differential equation can in fact be solved for $z(\sigma - v\tau)$ for various values of a. Since $0 \le |z| \le 1$ as discussed, the energy $T\int d\sigma\,\mathcal{H}$ is only finite for

$$0 \le a \le 1, \qquad 0 \le |v| \le 1. \tag{B.16}$$

We can of course assume $v > 0$ by choosing a direction of propagation. The solution is the inverse of

$$(a-1)\sqrt{1-v^2-z^2} - av\arctan\left(\frac{\sqrt{1-v^2-z^2}}{v}\right) + \sqrt{v^2-1}\arctan\left(\frac{\sqrt{1-v^2-z^2}}{\sqrt{v^2-1}}\right). \tag{B.17}$$

which looks like In particular, we can take $z > 0$ by always going to $y^1 > 0$ so that $0 \le z \le z_0 \equiv \sqrt{1-v^2}$.

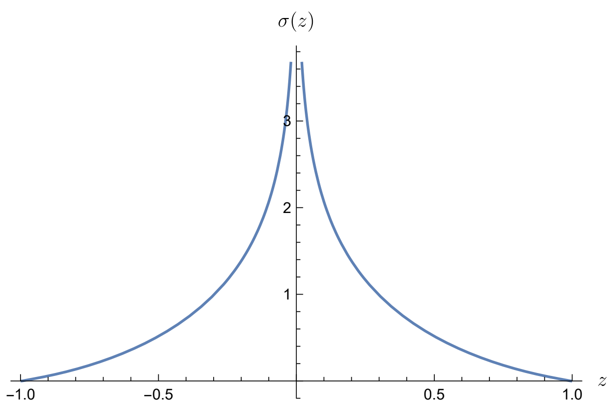


Figure 8. Giant magnon solution $\sigma(z)$ for $a = v = 0.1$

As mentioned, we are interested in the dispersion relation for this solution which relates its target space energy to its worldsheet momentum. Since σ is rescaled, the worldsheet bounds go to infinity in the large tension limit

$$E - J = T\int_{-\pi rT}^{\pi rT} d\sigma\,\mathcal{H} \overset{r\to\infty}{=} T\int_{-\infty}^{\infty} d\sigma\,\mathcal{H} = 2T\int_0^{z_0} dz\,\frac{\mathcal{H}}{|z'|}.$$

Thus we need to calculate the integral of

$$\frac{\mathcal{H}}{|z'|} = \frac{z}{\sqrt{z_0^2-z^2}},$$

which has a simple anti-derivative such that

$$E - J = -2T\sqrt{z_0^2 - z^2}\Big|_0^{z_0} = 2Tz_0 = 2T\sqrt{1-v^2}. \tag{B.18}$$

Turning to the momentum, we are looking to evaluate

$$p_{\text{ws}} = -\int_{-\infty}^{\infty} d\sigma\, p_z z' = 2\int_0^{z_0} dz\, |p_z| \tag{B.19}$$

where the conjugate momentum (B.9) reduces to

$$p_z = \frac{vz}{(1-z^2)\sqrt{z_0^2 - z^2}}. \tag{B.20}$$

The indefinite integral is easy to evaluate:

$$\begin{aligned} I &= \int dz\, \frac{z}{(1-z^2)\sqrt{z_0^2-z^2}} = \int dx\, \frac{z_0 \sin x}{1 - z_0^2 \sin^2 x} = \int dx\, \frac{z_0 \sin x}{1 - z_0^2 + z_0^2 \cos^2 x} \\ &= -\int du\, \frac{1}{1 - z_0^2 + u^2} = -\frac{1}{\sqrt{1-z_0^2}} \arctan \frac{u}{\sqrt{1-z_0^2}}. \end{aligned}$$

Retrieving $u = z_0 \cos x = z_0 \cos \arcsin z/z_0 = \sqrt{z_0^2 - z^2}$ and evaluating at the desired bounds,

$$I\Big|_{z=0}^{z=z_0} = 0 + \frac{1}{\sqrt{1-z_0^2}} \arctan \frac{z_0}{\sqrt{1-z_0^2}} = \frac{1}{\sqrt{1-z_0^2}} \arccos \sqrt{1-z_0^2} = \frac{1}{v} \arccos v,$$

which ultimately yields $p_{\text{ws}} = 2 \arccos v$. Thus, the dispersion relation is (2.55).

B.3 Gauge-fixed Lagrangian

This is an appendix reserved to the computation of (2.50) and uses intermediate results from [8] as guidance (some conventions differ). For brevity, we will give $\mathbb{X}$ a holiday and temporarily write $\mathfrak{g}(\mathbb{X}) = \mathfrak{g}$.

Finding p_+

To evaluate p_+ in the expression (2.37), we will need to find

$$\begin{aligned} p_+ = &-\frac{1}{8}\pi_+ \operatorname{str}\big(\Sigma_+\Sigma_-\mathfrak{g}^2\big) - \frac{1}{16}\pi_- \operatorname{str}\big(\Sigma_-^2\mathfrak{g}^2\big) \\ &+ \frac{\mathrm{i}}{8}\pi_\mu \operatorname{str}\big(\Sigma_\mu\Sigma_-\mathfrak{g}^2\big) - \frac{1}{4}\pi_{\mathbb{1}} \operatorname{str}\big(\Sigma_-\mathfrak{g}^2\big) \end{aligned} \tag{B.21}$$

where $\pi_\pm$ and π_μ are the coefficients of (2.27). Because of their definitions in terms of $\Sigma = \gamma^5$, the matrices $\Sigma_\pm$ satisfy

$$\Sigma_\pm\Sigma_\mp = -\Upsilon, \qquad \Sigma_\pm^2 = \mathbb{1}_8. \tag{B.22}$$

In turn we can simplify the first two terms in (B.21) to

$$p_+ = \frac{1}{8}\pi_+ \operatorname{tr}\big(\mathfrak{g}^2\big) - \frac{1}{16}\pi_- \operatorname{str}\big(\mathfrak{g}^2\big) + \dots\, .$$

Given the square-root bosonic parametrisation $\mathfrak{g}$ (1.114), the square $\mathfrak{g}^2$ is

$$\mathfrak{g}^2 = \begin{pmatrix} \frac{1+|z|^2/4}{1-|z|^2/4}\mathbb{1}_4 + \frac{1}{1-|z|^2/4}z^i\gamma^i & 0 \\ 0 & \frac{1-|y|^2/4}{1+|y|^2/4}\mathbb{1}_4 + \frac{1}{1+|y|^2/4}\mathrm{i}y^i\gamma^i \end{pmatrix} = \begin{pmatrix} \sqrt{G_{tt}}\mathbb{1}_4 + \sqrt{G_{zz}}z^i\gamma^i & 0 \\ 0 & \sqrt{G_{\phi\phi}}\mathbb{1}_4 + \sqrt{G_{yy}}\mathrm{i}y^i\gamma^i \end{pmatrix}. \tag{B.23}$$

Since the matrices γ^i are traceless by definition, this means we can use (1.104) to get

$$\mathrm{tr}\big(\mathfrak{g}^2\big) = 4\sqrt{G_{tt}} + 4\sqrt{G_{\phi\phi}}, \qquad \mathrm{str}\big(\mathfrak{g}^2\big) = 4\sqrt{G_{tt}} - 4\sqrt{G_{\phi\phi}}.$$

Defining $G_\pm = (\sqrt{G_{tt}} \pm \sqrt{G_{\phi\phi}})/2$, we can concisely write the first two terms as

$$p_+ = G_+\pi_+ - \frac{1}{2}G_-\pi_- + \dots .$$

All that is left is to show that the third and fourth terms in (B.21) indeed vanish. Starting with the third and assuming $\mu \le 4$ for example,

$$\begin{aligned} \mathrm{str}\big(\Sigma_\mu\Sigma_-\mathfrak{g}^2\big) &\propto \mathrm{str}\left\{ \begin{pmatrix} \gamma^\mu & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -\gamma^5 & 0 \\ 0 & \gamma^5 \end{pmatrix} \begin{pmatrix} \mathbb{1}_4 + az^i\gamma^i & 0 \\ 0 & b\mathbb{1}_4 + cy^i\gamma^i \end{pmatrix} \right\} \\ &\propto \mathrm{str}\left\{ \begin{pmatrix} -\gamma^\mu\gamma^5 - az^i\gamma^\mu\gamma^5\gamma^i & 0 \\ 0 & 0 \end{pmatrix} \right\} = -\,\mathrm{tr}\big(\gamma^\mu\gamma^5\big) - az^i\,\mathrm{tr}\Big(\gamma^\mu\gamma^5\gamma^i\Big). \end{aligned}$$

The cyclicity of the trace combined with the anticommutation relations of the gamma matrices imply both traces vanish, and hence so does $\mathrm{str}\big(\Sigma_\mu\Sigma_-\mathfrak{g}^2\big)$ for any μ. The same steps applied to $\mathrm{str}\big(\Sigma_-\mathfrak{g}^2\big)$ reveal that it is proportional to two traces, both of which similarly vanish. Thus p_+ is given by (2.41).

Solving $C_1 = 0$

From the discussion above and the expression (1.114), it is clear that we can write

$$\mathfrak{g} = \mathfrak{g}_+\mathbb{1}_8 + \mathfrak{g}_-\Upsilon + \mathfrak{g}_\mu\Sigma_\mu, \qquad \mathfrak{g}^2 = G_+\mathbb{1}_8 + G_-\Upsilon + G_\mu\Sigma_\mu, \tag{B.24}$$

where we define the coefficients for $i = 1,\dots,4$ by

$$\begin{aligned} \mathfrak{g}_+ &= \frac{1}{8}\,\mathrm{tr}(\mathfrak{g}) = \frac{1}{2}\frac{1}{\sqrt{1-|z|^2}} + \frac{1}{2}\frac{1}{\sqrt{1+|y|^2}}, &\qquad \mathfrak{g}_i &= \frac{1}{2}\frac{z^i}{\sqrt{1-|z|^2}}, \\ \mathfrak{g}_- &= \frac{1}{8}\,\mathrm{str}(\mathfrak{g}) = \frac{1}{2}\frac{1}{\sqrt{1-|z|^2}} - \frac{1}{2}\frac{1}{\sqrt{1+|y|^2}}, &\qquad \mathfrak{g}_{4+i} &= \frac{\mathrm{i}}{2}\frac{y^i}{\sqrt{1+|y|^2}} \end{aligned} \tag{B.25}$$

and similarly

$$\begin{aligned} G_+ &= \frac{1}{8}\,\mathrm{tr}\big(\mathfrak{g}^2\big) = \frac{1}{2}\frac{1+|z|^2/4}{1-|z|^2/4} + \frac{1}{2}\frac{1-|y|^2/4}{1+|y|^2/4}, &\qquad G_i &= \frac{z^i}{1-|z|^2/4}, \\ G_- &= \frac{1}{8}\,\mathrm{str}\big(\mathfrak{g}^2\big) = \frac{1}{2}\frac{1+|z|^2/4}{1-|z|^2/4} - \frac{1}{2}\frac{1-|y|^2/4}{1+|y|^2/4}, &\qquad G_{4+i} &= \frac{\mathrm{i}y^i}{1+|y|^2/4}. \end{aligned} \tag{B.26}$$

These explicit expressions for the coefficients are not so important. What matters is that we can express $\mathfrak{g}$ and $\mathfrak{g}^2$ in terms of the $\Sigma_\pm$ and Σ_μ matrices which will prove useful moving forward. Also recall the definitions of the bosonic and fermionic parts of $\mathfrak{g}(\chi)^{-1}\partial_\alpha\mathfrak{g}(\chi)$:

$$B_\alpha = \sqrt{\mathbb{1}+\chi^2}\partial_\alpha\sqrt{\mathbb{1}+\chi^2} - \chi\partial_\alpha\chi, \qquad F_\alpha = \sqrt{\mathbb{1}+\chi^2}\partial_\alpha\chi - \chi\partial_\alpha\sqrt{\mathbb{1}+\chi^2}. \tag{B.27}$$

As explained in 2.1, we want to solve the constraints C_1 and C_2 (2.26) in order to find x'_- in terms of the other spacetime coordinates. Using (2.41), we start by finding

$$\pi_+ = G_+^{-1}(p_+ + \frac{1}{2}\pi_- G_-). \tag{B.28}$$

We should solve C_1 in the light cone gauge with $x_+ = \tau$ but leaving p_+ free for now,

$$\begin{aligned} -\operatorname{str}(\pi A_\sigma^{(2)}) &= -\operatorname{str}(\pi A_{\mathrm{e},\sigma}) = \operatorname{str}\Big\{\pi\mathfrak{g}^{-1}\frac{\mathrm{i}}{4}x'_-\Sigma_-\mathfrak{g} + \underbrace{\pi\mathfrak{g}^{-1}B_\sigma\mathfrak{g} + \pi\mathfrak{g}^{-1}\partial_\sigma\mathfrak{g}}_{-\pi A^\perp_{\mathrm{e},\sigma}}\Big\} \\ &= p_+x'_- - \operatorname{str}\big(\pi A^\perp_{\mathrm{e},\sigma}\big) = 0. \end{aligned} \tag{B.29}$$

We will now evaluate $-\operatorname{str}\big(\pi A^\perp_{\mathrm{e},\alpha}\big)$ which is doubly useful; for $\alpha = \tau$ the term appears in the to-be-gauge-fixed Lagrangian (2.40) and for $\alpha = \sigma$ as we just saw the term appears in C_1. Using the expression for $\mathfrak{g}^{-1}d\mathfrak{g}$ in terms of Z and Y in A.8 and $\Sigma_\pm\Sigma_\mp = -\Upsilon$,

$$\begin{aligned} \operatorname{str}\big(\pi\mathfrak{g}^{-1}\partial_\alpha\mathfrak{g}\big) &= \operatorname{str}\Big\{\Big(\frac{\mathrm{i}}{2}\pi_+\Sigma_+ + \frac{\mathrm{i}}{2}\pi_-\Sigma_- + \frac{1}{2}\pi_\mu\Sigma^\mu + \pi_{\mathbb{1}}\mathrm{i}\mathbb{1}_8\Big)\mathfrak{g}^{-1}\partial_\alpha\mathfrak{g}\Big\} \\ &= \frac{1}{2}\pi_j\frac{1}{2}\frac{1}{1-|z|^2/4}\partial_\alpha z^i\operatorname{tr}\Big(\gamma^j\gamma^i\Big) - \frac{1}{2}\pi_{4+j}\frac{\mathrm{i}}{2}\frac{1}{1+|y|^2/4}\partial_\alpha y^i\operatorname{tr}\Big(\mathrm{i}\gamma^j\gamma^i\Big) \\ &= \frac{\pi_i}{1-|z|^2/4}\partial_\alpha z^i + \frac{\pi_{4+i}}{1+|y|^2/4}\partial_\alpha y^i \equiv p_\mu\partial_\alpha x^\mu \end{aligned} \tag{B.30}$$

where we found the transversal momenta for $i = 1, \ldots, 4$ to be

$$p_i = \frac{\pi_i}{1-|z|^2/4} = \sqrt{G_{zz}}\pi_i, \qquad p_{4+i} = \frac{\pi_{4+i}}{1+|y|^2/4} = \sqrt{G_{yy}}\pi_{4+i}. \tag{B.31}$$

To make use of the commutation relations (2.29) of $\Sigma_\pm$ and χ in evaluating the term with B_α in $-\operatorname{str}\big(\pi A^\perp_{\mathrm{o},\alpha}\big)$, we rewrite B_α in the following way:

$$\begin{aligned} B_\alpha &= \frac{1}{2}\sqrt{\mathbb{1}+\chi^2}\partial_\alpha\sqrt{\mathbb{1}+\chi^2} + \frac{1}{2}\sqrt{\mathbb{1}+\chi^2}\partial_\alpha\sqrt{\mathbb{1}+\chi^2} - \frac{1}{2}\chi\partial_\alpha\chi - \frac{1}{2}\chi\partial_\alpha\chi \\ &= \cancel{\frac{1}{2}\partial_\alpha\left(\mathbb{1}+\chi^2\right)} - \frac{1}{2}\partial_\alpha\sqrt{\mathbb{1}+\chi^2}\sqrt{\mathbb{1}+\chi^2} + \frac{1}{2}\sqrt{\mathbb{1}+\chi^2}\partial_\alpha\sqrt{\mathbb{1}+\chi^2} \\ &\quad - \cancel{\frac{1}{2}\partial_\alpha\left(\chi^2\right)} + \frac{1}{2}\partial_\alpha\chi\chi - \frac{1}{2}\chi\partial_\alpha\chi \\ &= \frac{1}{2}\partial_\alpha\chi\chi - \frac{1}{2}\chi\partial_\alpha\chi + \frac{1}{2}\sqrt{\mathbb{1}+\chi^2}\partial_\alpha\sqrt{\mathbb{1}+\chi^2} - \frac{1}{2}\partial_\alpha\sqrt{\mathbb{1}+\chi^2}\sqrt{\mathbb{1}+\chi^2}. \end{aligned} \tag{B.32}$$

Using cyclicity of the supertrace and $\Sigma_+\chi = -\chi\Sigma_+$,

$$\begin{aligned} \operatorname{str}\big(\Sigma_+B_\alpha\big) &= \frac{1}{2}\operatorname{str}\big(\Sigma_+\partial_\alpha\chi\chi - \Sigma_+\chi\partial_\alpha\chi\big) + \frac{1}{2}\operatorname{str}\big(\Sigma_+\sqrt{\mathbb{1}+\chi^2}\partial_\alpha\sqrt{\mathbb{1}+\chi^2} - \Sigma_+\partial_\alpha\sqrt{\mathbb{1}+\chi^2}\sqrt{\mathbb{1}+\chi^2}\big) \\ &= \frac{1}{2}\operatorname{str}\big(-\Sigma_+\chi\partial_\alpha\chi - \Sigma_+\chi\partial_\alpha\chi\big) + \frac{1}{2}\operatorname{str}\big(\Sigma_+\sqrt{\mathbb{1}+\chi^2}\partial_\alpha\sqrt{\mathbb{1}+\chi^2} - \Sigma_+\sqrt{\mathbb{1}+\chi^2}\partial_\alpha\sqrt{\mathbb{1}+\chi^2}\big) \\ &= -\operatorname{str}\big(\Sigma_+\chi\partial_\alpha\chi\big). \end{aligned} \tag{B.33}$$

In contrast, because χ and Σ_- commute, $\operatorname{str}\big(\Sigma_-B_\alpha\big) = 0$. Together with the decomposition of $\mathfrak{g}^2$ (B.24), these two identities imply

$$\operatorname{str}\big(\pi\mathfrak{g}^{-1}B_\alpha\mathfrak{g}\big) = \frac{\mathrm{i}}{2}\pi_+\operatorname{str}\big(\Sigma_+B_\alpha\mathfrak{g}^2\big) + \frac{\mathrm{i}}{4}\pi_-\operatorname{str}\big(\Sigma_-B_\alpha\mathfrak{g}^2\big) + \frac{1}{2}\pi_\mu\operatorname{str}\big(\Sigma_\mu\mathfrak{g}^{-1}B_\alpha\mathfrak{g}\big)$$

$$
\begin{aligned}
&= \frac{\mathrm{i}}{2}\mathit{\Pi}_+ \left\{ G_+ \operatorname{str}\big(\Sigma_+ B_\alpha\big) - G_- \cancel{\operatorname{str}\big(\Sigma_- B_\alpha\big)} + G_\mu \cancel{\operatorname{str}\big(\Sigma_+ B_\alpha \Sigma_\mu\big)} \right\} \\
&\quad + \frac{\mathrm{i}}{4}\mathit{\Pi}_- \left\{ G_+ \cancel{\operatorname{str}\big(\Sigma_- B_\alpha\big)} - G_- \operatorname{str}\big(\Sigma_+ B_\alpha\big) + G_\mu \cancel{\operatorname{str}\big(\Sigma_- B_\alpha \Sigma_\mu\big)} \right\} \\
&\quad + \frac{1}{2}\mathit{\Pi}_\mu \operatorname{str}\big(\Sigma_\mu \mathfrak{g}^{-1} B_\alpha \mathfrak{g}\big) \\
&= \frac{\mathrm{i}}{2}\big(\mathit{\Pi}_+ G_+ - \frac{1}{2}\mathit{\Pi}_- G_-\big) \operatorname{str}\big(\Sigma_+ B_\alpha\big) + \frac{1}{2}\mathit{\Pi}_\mu \operatorname{str}\big(\Sigma_\mu \mathfrak{g}^{-1} B_\alpha \mathfrak{g}\big) \\
&= -\frac{\mathrm{i}}{2} p_+ \operatorname{str}\big(\Sigma_+ \chi \partial_\alpha \chi\big) + \frac{1}{2}\mathit{\Pi}_\mu \operatorname{str}\big(\Sigma_\mu \mathfrak{g}^{-1} B_\alpha \mathfrak{g}\big).
\end{aligned}
$$

The $\Sigma_\pm B_\alpha \Sigma_\mu$ terms vanish because $\Sigma_\pm B_\alpha = B_\alpha \Sigma_\pm$ while $\Sigma_\pm \Sigma_\mu = -\Sigma_\mu \Sigma_\pm$. The last term can be simplified using (B.24) and (2.32) as follows:

$$
\begin{aligned}
\operatorname{str}\big(\mathfrak{g}\Sigma_\mu \mathfrak{g}^{-1} B_\alpha\big) &= \mathfrak{g}_+ \operatorname{str}\big(\Sigma_\mu \mathfrak{g}^{-1} B_\alpha\big) - \mathfrak{g}_- \operatorname{str}\big(\Sigma_+ \Sigma_- \Sigma_\mu \mathfrak{g}^{-1} B_\alpha\big) + \mathfrak{g}_\nu \operatorname{str}\big(\Sigma_\nu \Sigma_\mu \mathfrak{g}^{-1} B_\alpha\big) \\
&= \mathfrak{g}_+ \operatorname{str}\big(\Sigma_\mu \mathfrak{g}^{-1} B_\alpha\big) - \mathfrak{g}_- \operatorname{str}\big(\Sigma_\mu \Sigma_+ \Sigma_- \mathfrak{g}^{-1} B_\alpha\big) + \mathfrak{g}_\nu \operatorname{str}\big(\Sigma_\mu \Sigma_\nu \mathfrak{g}^{-1} B_\alpha\big) \\
&\quad + \mathfrak{g}_\nu \operatorname{str}\big([\Sigma_\nu, \Sigma_\mu] \mathfrak{g}^{-1} B_\alpha\big) \\
&= \operatorname{str}\big(\Sigma_\mu \mathfrak{g}\mathfrak{g}^{-1} B_\alpha\big) + \mathfrak{g}_\nu \operatorname{str}\big([\Sigma_\nu, \Sigma_\mu] \mathfrak{g}^{-1} B_\alpha\big) \\
&= \operatorname{str}\big(\Sigma_\mu B_\alpha\big) + \mathfrak{g}_\nu \operatorname{str}\big([\Sigma_\nu, \Sigma_\mu] \mathfrak{g}^{-1} B_\alpha\big).
\end{aligned}
$$

Importantly, any trace involving an odd number of Σ_μ's and B_α is zero using the trick

$$
\operatorname{str}\big(\Sigma_\mu^{2n+1} B_\alpha\big) = \operatorname{str}\big(\Sigma_\mu^{2n+1} \Sigma_+ \Sigma_+ B_\alpha\big) = -\operatorname{str}\big(\Sigma_+ \Sigma_\mu^{2n+1} \Sigma_+ B_\alpha\big) = -\operatorname{str}\big(\Sigma_\mu^{2n+1} B_\alpha\big) = 0.
$$

Similarly, by the commutativity of B_α and $\Sigma_\pm$ and the property (2.32),

$$
\mathfrak{g}_\nu \operatorname{str}\big([\Sigma_\nu, \Sigma_\mu] \mathfrak{g}^{-1} B_\alpha\big) = \mathfrak{g}_\nu \operatorname{str}\big([\Sigma_\nu, \Sigma_\mu] \mathfrak{g}^{-1} \Sigma_+ \Sigma_+ B_\alpha\big) = \mathfrak{g}_\nu \operatorname{str}\big([\Sigma_\nu, \Sigma_\mu] \mathfrak{g} B_\alpha\big).
$$

Thus the last term of $\operatorname{str}\big(\mathit{\Pi}\,\mathfrak{g}^{-1} B_\alpha \mathfrak{g}\big)$ can be rewritten such that

$$
\operatorname{str}\big(\mathit{\Pi}\,\mathfrak{g}^{-1} B_\alpha \mathfrak{g}\big) = -\frac{\mathrm{i}}{2} p_+ \operatorname{str}\big(\Sigma_+ \chi \partial_\alpha \chi\big) + \frac{1}{2}\mathfrak{g}_\nu \mathit{\Pi}_\mu \operatorname{str}\big([\Sigma_\nu, \Sigma_\mu] \mathfrak{g} B_\alpha\big). \tag{B.34}
$$

Combining expressions (B.30) and (B.34) we finally get,

$$
-\operatorname{str}\big(\mathit{\Pi} A^\perp_{\mathrm{o},\alpha}\big) = p_\mu \partial_\alpha x^\mu - \frac{\mathrm{i}}{2} p_+ \operatorname{str}\big(\Sigma_+ \chi \partial_\alpha \chi\big) + \frac{1}{2}\mathfrak{g}_\nu \mathit{\Pi}_\mu \operatorname{str}\big([\Sigma_\nu, \Sigma_\mu] \mathfrak{g} B_\alpha\big) = 0
$$

Excluding the $\mathfrak{g}$ factor, the supertrace term has the factors

$$
\mathfrak{g}_\nu \sim \mathcal{O}(\,\text{fields}^1), \qquad \mathit{\Pi}_\mu \sim \mathcal{O}(\,\text{fields}^1), \qquad B_\sigma \sim \mathcal{O}(\,\text{fields}^2)
$$

meaning it is already quartic in the fields $\mathbb{X}, \chi$. Because in the decompacitifaction limit we rescale the fields such that terms of order six are neglected and, looking at $\mathfrak{g}_\pm, \mathfrak{g}_\nu$, we find to leading order

$$
-\operatorname{str}\big(\mathit{\Pi} A^\perp_{\mathrm{o},\alpha}\big) = p_\mu \partial_\alpha x^\mu - \frac{\mathrm{i}}{2} p_+ \operatorname{str}\big(\Sigma_+ \chi \partial_\alpha \chi\big) + \frac{1}{2}\mathfrak{g}_\nu \mathit{\Pi}_\mu \operatorname{str}\big([\Sigma_\nu, \Sigma_\mu] B_\alpha\big) = 0
$$

which in the first instance implies (2.42) for $\alpha = \tau$ and for $\alpha = \sigma$ we solve the constraint $C_1 = p_+ x'_- - \operatorname{str}\big(\mathit{\Pi} A_{\mathrm{e},\sigma}\big) = 0$ to get

$$
x'_- = -\frac{1}{p_+}\left[p_\mu x'^\mu - \frac{\mathrm{i}}{2} p_+ \operatorname{str}\big(\Sigma_+ \chi \chi'\big) + \frac{1}{2}\mathfrak{g}_\nu \mathit{\Pi}_\mu \operatorname{str}\big([\Sigma_\nu, \Sigma_\mu] B_\sigma\big) \right].
$$

This expression (2.44) agrees with [8], from which it appears in [1]. (Note this p_+ is half of the p_+ in [8] which makes this x'_- twice the x'_- in [8].)

Wess-Zumino term

Here we will evaluate the Wess-Zumino term

$$\mathscr{L}_{\mathrm{WZ}} = -\frac{T}{2}\kappa\varepsilon^{\alpha\beta}\,\mathrm{str}\big(A^{(1)}_\alpha A^{(3)}_\beta\big) = -\frac{T}{2}\kappa\,\mathrm{str}\left(A^{(1)}_\tau A^{(3)}_\sigma - A^{(1)}_\sigma A^{(3)}_\tau\right).$$

Substituting the definition $F = (\mathfrak{g}(\chi)^{-1}d\mathfrak{g}(\chi))_{\mathrm{odd}}$ in the odd current $A_{\mathrm{o},\alpha}$ (2.33),

$$A_{\mathrm{o},\tau} = -\mathfrak{g}^{-1}\left[\mathrm{i}\dot{x}_+\Sigma_+\chi\sqrt{\mathbb{1}+\chi^2} + F_\tau\right]\mathfrak{g}, \qquad A_{\mathrm{o},\sigma} = -\mathfrak{g}^{-1}F_\sigma\mathfrak{g}.$$

To find the grading projections $A^{(1)}$ and $A^{(3)}$, we can use the decomposition formula (1.31) with $A \to A_{\mathrm{o}}$ as the bosonic projections are not relevant. Noting that $\Omega^2(A_{\mathrm{o}}) = -A_{\mathrm{o}}$ since A_{o} is odd,

$$\begin{aligned} A^{(1)} &= \frac{1}{2}\left[A_{\mathrm{o}} - \mathrm{i}\Omega(A_{\mathrm{o}})\right] = \frac{1}{2}\left[A_{\mathrm{o}} + \mathrm{i}\mathcal{K}A_{\mathrm{o}}^{st}\mathcal{K}^{-1}\right], \\ A^{(3)} &= \frac{1}{2}\left[A_{\mathrm{o}} + \mathrm{i}\Omega(A_{\mathrm{o}})\right] = \frac{1}{2}\left[A_{\mathrm{o}} - \mathrm{i}\mathcal{K}A_{\mathrm{o}}^{st}\mathcal{K}^{-1}\right]. \end{aligned}$$

We will clearly need to deal with terms of the type $\mathcal{K}\mathfrak{g}^{st}...(\mathfrak{g}^{-1})^{st}\mathcal{K}^{-1}$. It turns out $\mathcal{K}\mathfrak{g}^{st} = \mathfrak{g}\mathcal{K}$. To see this, we look at the expression (1.114) for $\mathfrak{g}$, the definition of $\mathcal{K} = \mathrm{diag}(-\gamma^2\gamma^4, -\gamma^2\gamma^4)$, and use the fact that γ^i are Hermitian to rewrite the two relevant diagonal entries as

$$\begin{aligned} \mathcal{K}\mathfrak{g}^{st} \sim z^i\gamma^2\gamma^4(\gamma^i)^t &= z^i\gamma^2\gamma^4(\gamma^i)^* = z^1\gamma^2\gamma^4\gamma^1 - z^2\gamma^2\gamma^4\gamma^2 + z^3\gamma^2\gamma^4\gamma^3 - z^4\gamma^2\gamma^4\gamma^4 \\ &= z^1\gamma^1\gamma^2\gamma^4 + z^2\gamma^2\gamma^2\gamma^4 + z^3\gamma^3\gamma^2\gamma^4 + z^4\gamma^4\gamma^2\gamma^4 = z^i\gamma^i\gamma^2\gamma^4 \sim \mathfrak{g}\mathcal{K}. \end{aligned} \tag{B.35}$$

Similarly, because they are both inverses through a sign change, $(\mathfrak{g}^{-1})^{st}\mathcal{K}^{-1} = \mathcal{K}^{-1}\mathfrak{g}^{-1}$. In particular,

$$\mathcal{K}A_{\mathrm{o},\tau}^{st}\mathcal{K}^{-1} = -\mathfrak{g}\mathcal{K}\left[\mathrm{i}\dot{x}_+(\chi\sqrt{\mathbb{1}+\chi^2})^{st}\Sigma_+ + F_\tau^{st}\right]\mathcal{K}^{-1}\mathfrak{g}^{-1}, \qquad \mathcal{K}A_{\mathrm{o},\sigma}^{st}\mathcal{K}^{-1} = -\mathfrak{g}\mathcal{K}F_\sigma^{st}\mathcal{K}^{-1}\mathfrak{g}^{-1}.$$

The Wess-Zumino term is proportional to

$$\begin{aligned} \mathrm{str}\left(A^{(1)}_\tau A^{(3)}_\sigma - A^{(1)}_\sigma A^{(3)}_\tau\right) &= \frac{1}{4}\,\mathrm{str}\left(\left[A_{\mathrm{o},\tau} + \mathrm{i}\mathcal{K}A_{\mathrm{o},\tau}^{st}\mathcal{K}^{-1}\right]\left[A_{\mathrm{o},\sigma} - \mathrm{i}\mathcal{K}A_{\mathrm{o},\sigma}^{st}\mathcal{K}^{-1}\right] - (\tau\leftrightarrow\sigma)\right) \\ &= \frac{1}{4}\,\mathrm{str}\left(\cancel{A_{\mathrm{o},\tau}A_{\mathrm{o},\sigma}} + \mathrm{i}\mathcal{K}A_{\mathrm{o},\tau}^{st}\mathcal{K}^{-1}A_{\mathrm{o},\sigma} - A_{\mathrm{o},\tau}\mathrm{i}\mathcal{K}A_{\mathrm{o},\sigma}^{st}\mathcal{K}^{-1} + \cancel{\mathcal{K}A_{\mathrm{o},\tau}^{st}A_{\mathrm{o},\sigma}^{st}\mathcal{K}^{-1}} - (\tau\leftrightarrow\sigma)\right) \\ &= \frac{\mathrm{i}}{4}\,\mathrm{str}\left(\mathcal{K}A_{\mathrm{o},\tau}^{st}\mathcal{K}^{-1}A_{\mathrm{o},\sigma} - A_{\mathrm{o},\tau}\mathcal{K}A_{\mathrm{o},\sigma}^{st}\mathcal{K}^{-1} - \mathcal{K}A_{\mathrm{o},\sigma}^{st}\mathcal{K}^{-1}A_{\mathrm{o},\tau} + A_{\mathrm{o},\sigma}\mathcal{K}A_{\mathrm{o},\tau}^{st}\mathcal{K}^{-1}\right) \\ &= \frac{\mathrm{i}}{2}\,\mathrm{str}\left(A_{\mathrm{o},\sigma}\mathcal{K}A_{\mathrm{o},\tau}^{st}\mathcal{K}^{-1} - A_{\mathrm{o},\tau}\mathcal{K}A_{\mathrm{o},\sigma}^{st}\mathcal{K}^{-1}\right). \end{aligned}$$

Using the expressions we derived above for $A_{\mathrm{o},\alpha}$ and $\mathcal{K}A_{\mathrm{o},\alpha}\mathcal{K}^{-1}$, this becomes

$$\begin{aligned} \mathrm{str}\left(A^{(1)}_\tau A^{(3)}_\sigma - A^{(1)}_\sigma A^{(3)}_\tau\right) &= \frac{\mathrm{i}}{2}\,\mathrm{str}\left(F_\sigma\mathfrak{g}^2\mathcal{K}\left[\mathrm{i}\dot{x}_+(\Sigma_+\chi\sqrt{\mathbb{1}+\chi^2})^{st} + F_\tau^{st}\right]\mathcal{K}^{-1}\mathfrak{g}^{-2}\right) \\ &\quad - \frac{\mathrm{i}}{2}\,\mathrm{str}\left(\left[\mathrm{i}\dot{x}_+\Sigma_+\chi\sqrt{\mathbb{1}+\chi^2} + F_\tau\right]\mathfrak{g}^2\mathcal{K}F_\sigma^{st}\mathcal{K}^{-1}\mathfrak{g}^{-2}\right) \\ &= -\frac{1}{2}\,\mathrm{str}\big(F_\sigma\mathfrak{g}^2\mathcal{K}\dot{x}_+(\Sigma_+\chi\sqrt{\mathbb{1}+\chi^2})^{st}\mathcal{K}^{-1}\mathfrak{g}^{-2}\big) + \frac{\mathrm{i}}{2}\,\mathrm{str}\big(F_\sigma\mathfrak{g}^2\mathcal{K}F_\tau^{st}\mathcal{K}^{-1}\mathfrak{g}^{-2}\big) \\ &\quad + \frac{1}{2}\,\mathrm{str}\big(\dot{x}_+\Sigma_+\chi\sqrt{\mathbb{1}+\chi^2}\mathfrak{g}^2\mathcal{K}F_\sigma^{st}\mathcal{K}^{-1}\mathfrak{g}^{-2}\big) - \frac{\mathrm{i}}{2}\,\mathrm{str}\big(F_\tau\mathfrak{g}^2\mathcal{K}F_\sigma^{st}\mathcal{K}^{-1}\mathfrak{g}^{-2}\big). \end{aligned}$$

But the top line line is in fact equal to the bottom, since $\operatorname{str} A = \operatorname{str} A^{st}$ and

$$\begin{aligned}\operatorname{str}\big(F_\sigma \mathfrak{g}^2 \mathcal{K}(\Sigma_+\chi\sqrt{\mathbb{1}+\chi^2})^{st}\mathcal{K}^{-1}\mathfrak{g}^{-2}\big)^{st} &= \operatorname{str}\big((\mathfrak{g}^{-2})^{st}\mathcal{K}\Sigma_+\chi\sqrt{\mathbb{1}+\chi^2}\mathcal{K}^{-1}(\mathfrak{g}^2)^{st}F_\sigma^{st}\big)\\ &= \operatorname{str}\big(\Sigma_+\chi\sqrt{\mathbb{1}+\chi^2}\mathfrak{g}^2\mathcal{K}F_\sigma^{st}\mathcal{K}^{-1}\mathfrak{g}^{-2}\big)\end{aligned}$$

with a similar step involving the identities

$$\mathcal{K}^{st} = \mathcal{K}^{-1} = -\mathcal{K}, \qquad (\mathfrak{g}^{\pm 1})^{st}\mathcal{K} = \mathcal{K}\mathfrak{g}^{\pm 1},$$

for the other term. Thus we double the bottom line yielding a preliminary form of the Wess-Zumino term

$$\mathscr{L}_{\mathrm{WZ}} = \mathrm{i}\kappa\frac{T}{2}\operatorname{str}\big(F_\tau\mathfrak{g}^2\mathcal{K}F_\sigma^{st}\mathcal{K}^{-1}\mathfrak{g}^{-2}\big) - \kappa\dot{x}_+\frac{T}{2}\operatorname{str}\big(\Sigma_+\chi\sqrt{\mathbb{1}+\chi^2}\mathfrak{g}^2\mathcal{K}F_\sigma^{st}\mathcal{K}^{-1}\mathfrak{g}^{-2}\big).$$

We should now substitute the decomposition (B.24) for $\mathfrak{g}^2$ while the one for $\mathfrak{g}^{-2} = \mathfrak{g}(-\mathbb{X})^2$ is found by sending $\mathbb{X} \to -\mathbb{X}$, such that $z^i \to -z^i$ and $y^i \to -y^i$. Since the coefficients $G_\pm$ are expressed in terms of $|z^2|$ and $|y^2|$ only, this has the net effect of sending $G_\mu \to -G_\mu$. Noting that $\Upsilon\Omega(M)\Upsilon = \Omega^3(M) = -\Omega(M)$ for M odd, the final form of the Wess-Zumino term is a sum of

$$\begin{aligned}\mathrm{i}\kappa\frac{T}{2}\operatorname{str}\big(F_\tau\mathfrak{g}^2\mathcal{K}F_\sigma^{st}\mathcal{K}^{-1}\mathfrak{g}^{-2}\big) &= \mathrm{i}\kappa\frac{T}{2}(G_+^2 - G_-^2)\operatorname{str}\big(F_\tau\mathcal{K}F_\sigma^{st}\mathcal{K}^{-1}\big)\\ &\quad - \mathrm{i}\kappa\frac{T}{2}G_\mu G_\nu\operatorname{str}\big(\Sigma_\nu F_\tau\Sigma_\mu\mathcal{K}F_\sigma^{st}\mathcal{K}^{-1}\big)\end{aligned}$$

and, remembering $\Sigma_\nu\Sigma_+ = -\Sigma_+\Sigma_\nu$,

$$\begin{aligned}-\kappa\dot{x}_+\frac{T}{2}\operatorname{str}\big(\Sigma_+\chi\sqrt{\mathbb{1}+\chi^2}\mathfrak{g}^2\mathcal{K}F_\sigma^{st}\mathcal{K}^{-1}\mathfrak{g}^{-2}\big) &= -\kappa\dot{x}_+\frac{T}{2}(G_+^2 - G_-^2)\operatorname{str}\big(\Sigma_+\chi\sqrt{\mathbb{1}+\chi^2}\mathcal{K}F_\sigma^{st}\mathcal{K}^{-1}\big)\\ &\quad - \kappa\dot{x}_+\frac{T}{2}G_\mu G_\nu\operatorname{str}\big(\Sigma_\nu\Sigma_+\chi\sqrt{\mathbb{1}+\chi^2}\Sigma_\mu\mathcal{K}F_\sigma^{st}\mathcal{K}^{-1}\big).\end{aligned}$$

Grouping terms, we have an explicit expression (2.49) for the Wess-Zumino term of the Lagrangian,

$$\begin{aligned}\mathscr{L}_{\mathrm{WZ}} &= \kappa\frac{T}{2}(G_+^2 - G_-^2)\operatorname{str}\big([\mathrm{i}F_\tau - \dot{x}_+\Sigma_+\chi\sqrt{\mathbb{1}+\chi^2}]\mathcal{K}F_\sigma^{st}\mathcal{K}^{-1}\big)\\ &\quad - \kappa\frac{T}{2}G_\mu G_\nu\operatorname{str}\big(\Sigma_\nu[\mathrm{i}F_\tau - \dot{x}_+\Sigma_+\chi\sqrt{\mathbb{1}+\chi^2}]\Sigma_\mu\mathcal{K}F_\sigma^{st}\mathcal{K}^{-1}\big).\end{aligned}$$

Solving $C_2 = 0$

Moving on to C_2, the following holds thanks to the Σ identites (B.22):

$$\begin{aligned}\operatorname{str}\big(\pi^2\big) &= -\frac{1}{8}\pi_+\pi_-\operatorname{str}\big(\{\Sigma_+,\Sigma_-\}\big) + \frac{1}{4}\pi_\mu\pi_\nu\operatorname{str}\big(\{\Sigma_\mu,\Sigma_\nu\}\big)\\ &= -\frac{1}{8}\pi_+\pi_-\operatorname{str}\big(2\Upsilon\big) + \frac{1}{4}\pi_\mu\pi_\nu\frac{1}{2}\operatorname{str}\Big(\begin{pmatrix}2\delta^{\mu\nu}\mathbb{1}_4 & 0\\ 0 & 0\end{pmatrix} \text{ or } \begin{pmatrix}0 & 0\\ 0 & 2\delta^{\mu\nu}\mathbb{1}_4\end{pmatrix}\Big)\\ &= 2\pi_+\pi_- + \pi_\mu^2.\end{aligned}$$

The extra 1/2 factor comes from averaging between the times when $\mu \le 4$ and $\mu \ge 4$. By substituting (B.28) we can now write C_2 as a quadratic in π_-:

$$C_2 = \operatorname{str}\big(\pi^2 + g^2 A_\sigma^{(2)}A_\sigma^{(2)}\big) \equiv \operatorname{str}\big(\pi^2\big) + g^2\mathcal{A}^2$$

$$0 = 2\pi_+\pi_- + \pi_\mu^2 + g^2\mathcal{A}^2 = 2G_+^{-1}(p_+ + \frac{1}{2}\pi_-G_-)\pi_- + \pi_\mu^2 + g^2\mathcal{A}^2$$
$$0 = \pi_-^2 G_- + \pi_- 2p_+ + G_+(\pi_\mu^2 + g^2\mathcal{A}^2).$$

The solutions can be manipulated in the form found in [8, p. 45], namely

$$\begin{aligned}\pi_- &= \frac{-2p_+ \pm \sqrt{4p_+^2 - 4G_-G_+(\pi_\mu^2 + g^2\mathcal{A}^2)}}{2G_-} \\ &= \frac{-p_+ \pm \sqrt{p_+^2 - G_-G_+(\pi_\mu^2 + g^2\mathcal{A}^2)}}{G_-} \times \frac{-p_+ \mp \sqrt{p_+^2 - G_-G_+(\pi_\mu^2 + g^2\mathcal{A}^2)}}{-p_+ \mp \sqrt{p_+^2 - G_-G_+(\pi_\mu^2 + g^2\mathcal{A}^2)}} \\ &= \frac{p_+^2 - p_+^2 + G_-G_+(\pi_\mu^2 + g^2\mathcal{A}^2)}{-G_-p_+ \mp G_-\sqrt{p_+^2 - G_-G_+(\pi_\mu^2 + g^2\mathcal{A}^2)}} = -\frac{G_+(\pi_\mu^2 + g^2\mathcal{A}^2)}{p_+ \pm \sqrt{p_+^2 - G_-G_+(\pi_\mu^2 + g^2\mathcal{A}^2)}}.\end{aligned}$$

We should disregard the minus solution in the last expression. For small tension and vanishing transversal momenta, the denominator would present a singularity leading to a non-physical value of π_-.

Simplifying $\mathcal{A}^2$

To compute $\mathcal{A}^2 = \mathrm{str}\big(A_\sigma^{(2)}A_\sigma^{(2)}\big)$, we first need to find $A_\sigma^{(2)}$. According to (2.33),

$$A_{\mathrm{e},\sigma} = \underbrace{-\frac{\mathrm{i}}{4}x'_-\mathfrak{g}^{-1}\Sigma_-\mathfrak{g}}_{1} - \underbrace{\mathfrak{g}^{-1}B_\sigma\mathfrak{g}}_{2} - \underbrace{\mathfrak{g}^{-1}\mathfrak{g}'}_{3}.$$

where $\mathfrak{g}' = \partial_\sigma\mathfrak{g}$. Now we use $A_\sigma^{(2)} = \frac{1}{2}[A_{\mathrm{e},\sigma} - \Omega(A_{\mathrm{e},\sigma})]$ and $\mathfrak{g}^{\pm1}\mathcal{K} = \mathcal{K}(\mathfrak{g}^{\pm1})^{st}$ (B.35) to compute

$$1^{(2)} = \frac{1}{2}\left[\frac{\mathrm{i}}{4}\mathfrak{g}^{-1}\Sigma_-\mathfrak{g}x'_- + \mathcal{K}\frac{\mathrm{i}}{4}\left(\mathfrak{g}^{-1}\Sigma_-\mathfrak{g}\right)^{st}x'_-\mathcal{K}^{-1}\right] = \frac{\mathrm{i}}{8}x'_-\left[\Sigma_-\mathfrak{g}^2 + \mathfrak{g}\mathcal{K}\Sigma_-\mathcal{K}^{-1}\mathfrak{g}^{-1}\right] = \frac{\mathrm{i}}{8}x'_-\left[\Sigma_-\mathfrak{g}^2 + \mathfrak{g}^2\Sigma_-\right],$$
$$2^{(2)} = \frac{1}{2}\left[\mathfrak{g}^{-1}B_\sigma\mathfrak{g} + \mathcal{K}\left(\mathfrak{g}^{-1}B_\sigma\mathfrak{g}\right)^{st}\mathcal{K}^{-1}\right] = \frac{1}{2}\left[\mathfrak{g}^{-1}B_\sigma\mathfrak{g} + \mathfrak{g}\mathcal{K}B_\sigma^{st}\mathcal{K}^{-1}\mathfrak{g}^{-1}\right],$$
$$3^{(2)} = \frac{1}{2}\left[\mathfrak{g}^{-1}\mathfrak{g}' + \mathcal{K}\left(\mathfrak{g}^{-1}\mathfrak{g}'\right)^{st}\right] = \frac{1}{2}\left[\mathfrak{g}^{-1}\mathfrak{g}' + \mathfrak{g}'\mathfrak{g}\mathfrak{g}^{-1}\right].$$

Therefore, as stated in [8],

$$A_\sigma^{(2)} = -\left(\frac{\mathrm{i}}{8}x'_-\left[\Sigma_-\mathfrak{g}^2 + \mathfrak{g}^2\Sigma_-\right] + \frac{1}{2}\left[\mathfrak{g}^{-1}B_\sigma\mathfrak{g} + \mathfrak{g}\mathcal{K}B_\sigma^{st}\mathcal{K}^{-1}\mathfrak{g}^{-1}\right] + \frac{1}{2}\left[\mathfrak{g}^{-1}\mathfrak{g}' + \mathfrak{g}'\mathfrak{g}^{-1}\right]\right).$$

For clarity, we can write $A_\sigma^{(2)} = -\frac{\mathrm{i}}{8}x'_-\mathcal{B} - \frac{1}{2}\mathcal{C}$ such that we need to evaluate three terms in

$$\mathcal{A}^2 = -\frac{1}{64}x'^2_-\,\mathrm{str}\big(\mathcal{B}^2\big) + \frac{\mathrm{i}}{8}x'_-\,\mathrm{str}\big(\mathcal{B}\mathcal{C}\big) + \frac{1}{4}\mathrm{str}\big(\mathcal{C}^2\big).$$

Term quadratic in x'_-

This calculation is straightfoward using (B.24):

$$\begin{aligned}\mathrm{str}\big(\mathcal{B}^2\big) &= \mathrm{str}\left(\Sigma_-\mathfrak{g}^2\Sigma_-\mathfrak{g}^2 + \mathfrak{g}^2\Sigma_-\Sigma_-\mathfrak{g}^2 + \Sigma_-\mathfrak{g}^2\mathfrak{g}^2\Sigma_- + \mathfrak{g}^2\Sigma_-\mathfrak{g}^2\Sigma_-\right) \\ &= \mathrm{str}\left(\mathbb{1}_8 + \mathfrak{g}^4 + \mathfrak{g}^4 + \mathbb{1}_8\right) = 2\,\mathrm{str}\big(\mathfrak{g}^4\big) = 2\,\mathrm{str}\big((\mathfrak{g}^2)^2\big) \\ &= 2\,\mathrm{str}\left((G_+\mathbb{1}_8 + G_-\Upsilon + G_\mu\Sigma_\mu)^2\right) = 4G_+G_-\,\mathrm{str}\big(\Upsilon\big) = 32G_+G_-.\end{aligned}$$

Term linear in x'_-

We will need to use the fact that $\operatorname{str}\big(\Sigma_-\mathfrak{g}'\mathfrak{g}^n\big)=0$ for all $n\in\mathbb{Z}$. To show this, notice

$$\begin{aligned}\operatorname{str}\big(\Sigma_-\mathfrak{g}^{n+1}\big) &= \operatorname{str}\big(\mathfrak{g}^{-1}\Sigma_-\mathfrak{g}^{n}\big) = \operatorname{str}\big(\Sigma_-\mathfrak{g}^{n-1}\big) = \operatorname{str}\big(\Sigma_-\mathfrak{g}^{n-3}\big) = \ldots\\ \partial_\sigma\operatorname{str}\big(\Sigma_-\mathfrak{g}^{n+1}\big) &= \operatorname{str}\big(\Sigma_-\mathfrak{g}'\mathfrak{g}^{n}\big) + \operatorname{str}\big(\Sigma_-\mathfrak{g}\mathfrak{g}'\mathfrak{g}^{n-1}\big) + \operatorname{str}\big(\Sigma_-\mathfrak{g}^2\mathfrak{g}'\mathfrak{g}^{n-2}\big) + \ldots\\ &= \operatorname{str}\big(\Sigma_-\mathfrak{g}'\mathfrak{g}^{n}\big) + \operatorname{str}\big(\Sigma_-\mathfrak{g}'\mathfrak{g}^{n-2}\big) + \operatorname{str}\big(\Sigma_-\mathfrak{g}'\mathfrak{g}^{n-4}\big) + \ldots\end{aligned}$$

Different scenarii for n odd or even should be considered and the proof then follows by strong induction on n. To calculate this term, we will need to find the supertrace of the product of

$$\mathcal{B}=\mathfrak{g}\Sigma_-\mathfrak{g}^{-1}+\mathfrak{g}^{-1}\Sigma_-\mathfrak{g},\qquad \mathcal{C}=\mathfrak{g}^{-1}\mathfrak{g}'+\mathfrak{g}'\mathfrak{g}^{-1}+\mathfrak{g}^{-1}B_\sigma\mathfrak{g}+\mathfrak{g}\mathcal{K}B_\sigma^{st}\mathcal{K}^{-1}\mathfrak{g}^{-1}.$$

We get

$$\begin{aligned}\operatorname{str}\big(\mathcal{B}\mathcal{C}\big) &= \operatorname{str}(\mathfrak{g}\Sigma_-\mathfrak{g}^{-2}\mathfrak{g}'+\mathfrak{g}\Sigma_-\mathfrak{g}^{-1}\mathfrak{g}'\mathfrak{g}^{-1}+\mathfrak{g}\Sigma_-\mathfrak{g}^{-2}B_\sigma\mathfrak{g}+\mathfrak{g}\Sigma_-\mathcal{K}B_\sigma^{st}\mathcal{K}^{-1}\mathfrak{g}^{-1}\\ &\qquad +\mathfrak{g}^{-1}\Sigma_-\mathfrak{g}'+\mathfrak{g}^{-1}\Sigma_-\mathfrak{g}\mathfrak{g}'\mathfrak{g}^{-1}+\mathfrak{g}^{-1}\Sigma_-B_\sigma\mathfrak{g}+\mathfrak{g}^{-1}\Sigma_-\mathfrak{g}^2\mathcal{K}B_\sigma^{st}\mathcal{K}^{-1}\mathfrak{g}^{-1})\\ &= \operatorname{str}(\Sigma_-B_\sigma\mathfrak{g}^4+\cancel{\Sigma_-B_\sigma^{st}}+\cancel{\Sigma_-B_\sigma}+\mathcal{K}\Sigma_-(\mathfrak{g}^{-2}B_\sigma\mathfrak{g}^2)^{st}\mathcal{K}^{-1}) = 2\operatorname{str}\big(\Sigma_-B_\sigma\mathfrak{g}^4\big)\end{aligned}$$

The two extra terms separately vanish because we can write B_α as a sum of commutators (B.32) such that $\Sigma_-\chi=\chi\Sigma_-$ implies $\operatorname{str}\big(\Sigma_-B_\alpha\big)=\operatorname{str}\big(\Sigma_-B_\alpha^{st}\big)=0$. We are left to calculate

$$\begin{aligned}\operatorname{str}\big(\Sigma_-B_\sigma\mathfrak{g}^4\big) &= \operatorname{str}\big(\Sigma_-B_\sigma(G_+\mathbb{1}_8+G_-\Upsilon+G_\mu\Sigma_\mu)^2\big)\\ &= 2G_+G_-\operatorname{str}\big(\Upsilon\Sigma_-B_\sigma\big)+2G_+G_\mu\operatorname{str}\big(\Sigma_\mu\Sigma_-B_\sigma\big)+2G_-G_\mu\operatorname{str}\big(\Upsilon\Sigma_\mu\Sigma_-B_\sigma\big)\\ &\quad +G_\mu G_\nu\operatorname{str}\big(\{\Sigma_\mu,\Sigma_\nu\}\Sigma_-B_\sigma\big).\end{aligned}$$

The second and third terms both vanish because $\Sigma_\pm B_\sigma=B_\sigma\Sigma_\pm$ while $\Sigma_\pm\Sigma_\mu=-\Sigma_\mu\Sigma_\pm$. The last term, if non-zero, will contain a 4×4 block which satisfies the Clifford algebra relation such that

$$\{\Sigma_\mu,\Sigma_\nu\}\propto\delta_{\mu\nu}(\mathbb{1}_8\pm\Upsilon)$$

contributing two terms which vanish for the exact reasons the second and third did. In the end,

$$\operatorname{str}\big(\mathcal{B}\mathcal{C}\big)=4G_+G_-\operatorname{str}\big(\Upsilon\Sigma_-B_\sigma\big)=-4G_+G_-\operatorname{str}\big(\Sigma_+B_\sigma\big)\overset{\text{(B.33)}}{=}4G_+G_-\operatorname{str}\big(\Sigma_+\chi\chi'\big).$$

Term independent of x'_-

This term can be calculated as above (see [8], but adding the three yields the final result

$$\begin{aligned}\mathcal{A}^2 &= -x_-'^2G_+G_-+\mathrm{i}x'_-G_+G_-\operatorname{str}\big(\Sigma_+\chi\chi'\big)+\frac{z'^iz'^i}{(1-\frac{|z|^2}{4})}+\frac{y'^iy'^i}{(1+\frac{|y|^2}{4})}+\frac{1}{2}\operatorname{str}\big(B_\sigma B_\sigma\big)\\ &\quad -\frac{1}{2}G'_\mu G_\nu\operatorname{str}\big([\Sigma_\mu,\Sigma_\nu]B_\sigma\big)+\frac{1}{2}(G_+^2-G_-^2)\operatorname{str}\big(B_\sigma\mathcal{K}B_\sigma^{st}\mathcal{K}^{-1}\big)\\ &\quad +G_+G_-\operatorname{str}\big(\Upsilon B_\sigma\mathcal{K}B_\sigma^{st}\mathcal{K}^{-1}\big)+\frac{1}{2}G_\mu G_\nu\operatorname{str}\big(\Sigma_\mu B_\sigma\Sigma_\nu a\mathcal{K}B_\sigma^{st}\mathcal{K}^{-1}\big).\end{aligned}\tag{B.36}$$

Simplifying $\mathbf{p}_-$

The starting expression for $\mathbf{p}_-$ is (2.37). Substituting the decomposition of $\mathcal{P}$, we get

$$\begin{aligned}\mathbf{p}_- = &\frac{\mathrm{i}}{2}\operatorname{str}\big(\frac{\mathrm{i}}{2}\mathcal{P}_+\Sigma_+\Sigma_+\mathfrak{g}(\mathbb{1}_8+2\chi^2)\mathfrak{g}\big) && (1)\\ &+\frac{\mathrm{i}}{2}\operatorname{str}\big(\frac{\mathrm{i}}{2}\mathcal{P}_-\Sigma_-\Sigma_+\mathfrak{g}(\mathbb{1}_8+2\chi^2)\mathfrak{g}\big) && (2)\\ &+\frac{\mathrm{i}}{2}\operatorname{str}\big(\frac{1}{2}\mathcal{P}_\mu\Sigma_\mu\Sigma_+\mathfrak{g}(\mathbb{1}_8+2\chi^2)\mathfrak{g}\big) && (3)\\ &+\frac{\mathrm{i}}{2}\operatorname{str}\big(\mathcal{P}_{\mathbb{1}}\mathrm{i}\mathbb{1}_8\Sigma_+\mathfrak{g}(\mathbb{1}_8+2\chi^2)\mathfrak{g}\big). && (4)\end{aligned}$$

Since $\mathcal{P}_{\mathbb{1}}$ does not appear in the constructed Lagrangian (2.24), we can set (4) to zero without affecting the discussion. Each other term can now be evaluated remembering the decomposition of $\mathfrak{g}$ and $\mathfrak{g}^2$, and the various commutation relations of $\Sigma_\pm$, Σ_μ and χ. Starting with the first term,

$$\begin{aligned}(1) &= -\frac{1}{4}\mathcal{P}_+\operatorname{str}\big(\mathfrak{g}^2(\mathbb{1}_8+2\chi^2)\big)\\ -\frac{4}{\mathcal{P}_+}(1) &= \operatorname{str}\big(G_+\mathbb{1}_8(\cancel{\mathbb{1}_8}+2\chi^2)\big)+\operatorname{str}\big(G_-\Upsilon(\mathbb{1}_8+2\chi^2)\big)+\operatorname{str}\big(G_\mu\Sigma_\mu(\cancel{\mathbb{1}_8}+2\chi^2)\big)\\ &= 2G_+\operatorname{str}\big(\chi^2\big)+8G_-+2G_-\operatorname{tr}\big(\chi^2\big)+2G_\mu\operatorname{str}\big(\Sigma_\mu\chi^2\big)\\ (1) &= -\frac{1}{2}\mathcal{P}_+G_+\operatorname{str}\big(\chi^2\big)-2\mathcal{P}_+G_--\frac{1}{2}\mathcal{P}_+G_-\operatorname{tr}\big(\chi^2\big)-\frac{1}{2}\mathcal{P}_+G_\mu\operatorname{str}\big(\Sigma_\mu\chi^2\big).\end{aligned}$$

However, from the κ-symmetry gauge fixed form (1.123) of χ, we see that

$$\chi^2=\left(\begin{array}{cc|cc} -\eta\eta^\dagger & 0 & 0 & 0\\ 0 & \theta^\dagger\theta & 0 & 0\\ \hline 0 & 0 & \theta\theta^\dagger & 0\\ 0 & 0 & 0 & -\eta^\dagger\eta\end{array}\right) \tag{B.37}$$

which directly implies

$$\operatorname{tr}\big(\chi^2\big)=0,\qquad \operatorname{str}\big(\chi^2\big)=2\operatorname{tr}\Big(\theta^\dagger\theta-\eta\eta^\dagger\Big). \tag{B.38}$$

We also can calculate $\Sigma_\mu\chi^2$ more explicitly. Taking $\mu\le 4$ for example,

$$\Sigma_i\chi^2=\left(\begin{array}{c|c}\gamma^i & 0\\ \hline 0 & 0\end{array}\right)\left(\begin{array}{cc|cc} -\eta\eta^\dagger & 0 & 0 & 0\\ 0 & \theta^\dagger\theta & 0 & 0\\ \hline 0 & 0 & \theta\theta^\dagger & 0\\ 0 & 0 & 0 & -\eta^\dagger\eta\end{array}\right)=\left(\begin{array}{c|c}\gamma^i\begin{pmatrix}-\eta\eta^\dagger & 0\\ 0 & \theta^\dagger\theta\end{pmatrix} & 0\\ \hline 0 & 0\end{array}\right).$$

As a result, we have

$$(\mathrm{s})\operatorname{tr}\big(\Sigma_i\chi^2\big)=\operatorname{tr}\left(\gamma^i\begin{pmatrix}-\eta\eta^\dagger & 0\\ 0 & \theta^\dagger\theta\end{pmatrix}\right),$$

where for each γ^i the trace vanishes:

$$(\mathrm{s})\operatorname{tr}\big(\Sigma_1\chi^2\big)=\operatorname{tr}\begin{pmatrix}0&0&0&-1\\0&0&1&0\\0&1&0&0\\-1&0&0&0\end{pmatrix}\begin{pmatrix}-\eta\eta^\dagger & 0\\ 0 & \theta^\dagger\theta\end{pmatrix}=\operatorname{tr}\begin{pmatrix}0&0&\bullet&\bullet\\0&0&\bullet&\bullet\\\bullet&\bullet&0&0\\\bullet&\bullet&0&0\end{pmatrix}=0,$$

$$(\mathrm{s})\mathrm{tr}\big(\Sigma_2\chi^2\big) = \mathrm{tr}\begin{pmatrix} 0 & 0 & 0 & \mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \\ 0 & -\mathrm{i} & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} -\eta\eta^\dagger & 0 \\ 0 & \theta^\dagger\theta \end{pmatrix} = \mathrm{tr}\begin{pmatrix} 0 & 0 & \bullet & \bullet \\ 0 & 0 & \bullet & \bullet \\ \bullet & \bullet & 0 & 0 \\ \bullet & \bullet & 0 & 0 \end{pmatrix} = 0,$$

and similarly for γ^3 and γ^4. In conclusion, for all $\mu \leq 8$,

$$(\mathrm{s})\mathrm{tr}\big(\Sigma_\mu\chi^2\big) = 0. \tag{B.39}$$

The first term reduces to

$$1 = -\frac{1}{2}\pi_+ G_+\,\mathrm{str}\big(\chi^2\big) - 2\pi_+ G_-.$$

The second term in $\mathbf{p}_-$ can be calculated in a similar fashion:

$$\begin{aligned} 2 &= -\frac{1}{4}\pi_-\,\mathrm{str}\big(\Upsilon\mathfrak{g}^2(\mathbb{1}_8 + 2\chi^2)\big) \\ -\frac{4}{\pi_-}2 &= \mathrm{str}\big(G_+\Upsilon(\mathbb{1}_8 + 2\chi^2)\big) + \mathrm{str}\big(G_-(\cancel{\mathbb{1}_8} + 2\chi^2)\big) + \mathrm{str}\big(G_\mu\Upsilon\Sigma_\mu(\cancel{\mathbb{1}_8} + 2\chi^2)\big) \\ &= 8G_+ + 2G_+\cancel{\mathrm{tr}\big(\chi^2\big)} + 2G_-\,\mathrm{str}\big(\chi^2\big) + 2G_\mu\cancel{\mathrm{tr}\big(\Sigma_\mu\chi^2\big)} \\ 2 &= -2\pi_- G_+ - \frac{1}{2}\pi_- G_-\,\mathrm{str}\big(\chi^2\big). \end{aligned}$$

The third term will take a little more work, but keep faith. This time we cannot commute $\mathfrak{g}$ past the Σ_μ matrix so we must deal with the decomposition (B.24) of both $\mathfrak{g}$ factors. Starting with the left factor,

$$\begin{aligned} 3 &= \frac{\mathrm{i}}{4}\pi_\mu\,\mathrm{str}\big(\Sigma_\mu\Sigma_+\mathfrak{g}(\mathbb{1}_8 + 2\chi^2)\mathfrak{g}\big) \\ &= \frac{\mathrm{i}}{4}\pi_\mu\,\mathrm{str}\big(\Sigma_\mu\Sigma_+\mathfrak{g}_+(\mathbb{1}_8 + 2\chi^2)\mathfrak{g}\big) + \frac{\mathrm{i}}{4}\pi_\mu\,\mathrm{str}\big(\Sigma_\mu\Sigma_+\mathfrak{g}_-\Upsilon(\mathbb{1}_8 + 2\chi^2)\mathfrak{g}\big) + \frac{\mathrm{i}}{4}\pi_\mu\,\mathrm{str}\big(\Sigma_\mu\Sigma_+\mathfrak{g}_\nu\Sigma_\nu(\mathbb{1}_8 + 2\chi^2)\mathfrak{g}\big) \\ &= \underbrace{\frac{\mathrm{i}}{4}\pi_\mu\mathfrak{g}_+\,\mathrm{str}\big(\Sigma_\mu\Sigma_+(\mathbb{1}_8 + 2\chi^2)\mathfrak{g}\big)}_{3a}\underbrace{-\frac{\mathrm{i}}{4}\pi_\mu\mathfrak{g}_-\,\mathrm{str}\big(\Sigma_\mu\Sigma_-(\mathbb{1}_8 + 2\chi^2)\mathfrak{g}\big)}_{3b}\underbrace{+\frac{\mathrm{i}}{4}\pi_\mu\mathfrak{g}_\nu\,\mathrm{str}\big(\Sigma_\mu\Sigma_+\Sigma_\nu(\mathbb{1}_8 + 2\chi^2)\mathfrak{g}\big)}_{3c}. \end{aligned}$$

We now substitute the second $\mathfrak{g}$'s decomposition to get

$$\begin{aligned} 3a &= \frac{\mathrm{i}}{4}\pi_\mu\mathfrak{g}_+\,\mathrm{str}\big(\Sigma_\mu\Sigma_+(\mathbb{1}_8 + 2\chi^2)\mathfrak{g}_+\big) + \frac{\mathrm{i}}{4}\pi_\mu\mathfrak{g}_+\,\mathrm{str}\big(\Sigma_\mu\Sigma_+(\mathbb{1}_8 + 2\chi^2)\mathfrak{g}_-\Upsilon\big) + \frac{\mathrm{i}}{4}\pi_\mu\mathfrak{g}_+\,\mathrm{str}\big(\Sigma_\mu\Sigma_+(\mathbb{1}_8 + 2\chi^2)\mathfrak{g}_\nu\Sigma_\nu\big) \\ &= 0 + 0 + \frac{\mathrm{i}}{4}\pi_\mu\mathfrak{g}_\nu\mathfrak{g}_+\,\mathrm{str}\big(\Sigma_\nu\Sigma_\mu\Sigma_+(\mathbb{1}_8 + 2\chi^2)\big). \end{aligned}$$

The first term disappears because the supertrace has cyclicity and the $\mathbb{1}_8$ term is the supertrace of $\Sigma_\mu\Sigma_\pm = -\Sigma_\pm\Sigma_\mu$, while the χ^2 term vanishes for the same reason since $\Sigma_+\chi^2 = -\chi\Sigma_+\chi = \chi^2\Sigma_+$. The second term is identical except for the supertrace being replaced by a trace due to the presence of Υ. By the exact same reasoning with $\Sigma_+ \leftrightarrow \Sigma_-$,

$$\begin{aligned} 3b &= -\frac{\mathrm{i}}{4}\pi_\mu\mathfrak{g}_-\,\mathrm{str}\big(\Sigma_\mu\Sigma_-(\mathbb{1}_8 + 2\chi^2)\mathfrak{g}_+\big) - \frac{\mathrm{i}}{4}\pi_\mu\mathfrak{g}_-\,\mathrm{str}\big(\Sigma_\mu\Sigma_-(\mathbb{1}_8 + 2\chi^2)\mathfrak{g}_-\Upsilon\big) - \frac{\mathrm{i}}{4}\pi_\mu\mathfrak{g}_-\,\mathrm{str}\big(\Sigma_\mu\Sigma_-(\mathbb{1}_8 + 2\chi^2)\mathfrak{g}_\nu\Sigma_\nu\big) \\ &= 0 + 0 - \frac{\mathrm{i}}{4}\pi_\mu\mathfrak{g}_\nu\mathfrak{g}_-\,\mathrm{str}\big(\Sigma_\nu\Sigma_\mu\Sigma_-(\mathbb{1}_8 + 2\chi^2)\big). \end{aligned}$$

Finally,

$$\frac{4}{\mathrm{i}}\,\text{③c} = \pi_\mu \mathfrak{g}_\nu \operatorname{str}\big(\Sigma_\mu\Sigma_+\Sigma_\nu(\mathbb{1}_8 + 2\chi^2)\mathfrak{g}_+\big) + \pi_\mu \mathfrak{g}_\nu \operatorname{str}\big(\Sigma_\mu\Sigma_+\Sigma_\nu(\mathbb{1}_8 + 2\chi^2)\mathfrak{g}_-\Upsilon\big) + \pi_\mu \mathfrak{g}_\nu \operatorname{str}\big(\Sigma_\mu\Sigma_+\Sigma_\nu(\mathbb{1}_8 + 2\chi^2)\mathfrak{g}_\alpha\Sigma_\alpha\big)$$
$$= -\pi_\mu \mathfrak{g}_\nu \operatorname{str}\big(\Sigma_\mu\Sigma_\nu\Sigma_+(\mathbb{1}_8 + 2\chi^2)\mathfrak{g}_+\big) + \pi_\mu \mathfrak{g}_\nu \operatorname{str}\big(\Sigma_\mu\Sigma_\nu\Sigma_-(\mathbb{1}_8 + 2\chi^2)\mathfrak{g}_-\big) + 0.$$

The third term is zero because there are three matrices of type Σ_μ along with Σ_+, which anticommutes with Σ_μ. And since $\Sigma_+\chi^2 = \chi^2\Sigma_+$, this results in an overall minus sign once Σ_+ is cycled through the supertrace. The first remaining supertrace in 3c cancels the $\mathbb{1}_8$ term of 3a and results in a commutator for the χ^2 term while the second supertrace does the same for 3b. We are thus left with

$$\text{③} = \frac{\mathrm{i}}{2}\pi_\mu \mathfrak{g}_\nu \operatorname{str}\big([\Sigma_\nu, \Sigma_\mu]\chi^2(\mathfrak{g}_+\Sigma_+ - \mathfrak{g}_-\Sigma_-)\big) \tag{B.40}$$

which agrees perfectly with [8]. Summarising the current form of $\mathbf{p}_- = ① + ② + ③$:

$$① = -\frac{1}{2}\pi_+ G_+ \operatorname{str}\big(\chi^2\big) - 2\pi_+ G_-, \qquad ② = -\frac{1}{4}\pi_- G_- \operatorname{str}\big(\chi^2\big) - \pi_- G_+,$$
$$③ = \frac{\mathrm{i}}{2}\pi_\mu \mathfrak{g}_\nu \operatorname{str}\big([\Sigma_\nu, \Sigma_\mu]\chi^2(\mathfrak{g}_+\Sigma_+ - \mathfrak{g}_-\Sigma_-)\big).$$

Because we want to ultimately express the gauge-fixed Lagrangian in terms of X^M, χ and p_M exclusively, we should eliminate $\pi_\pm$ and π_μ. But we found expressions for π_μ in (B.31) and for π_- in (2.45) and can express π_+ in terms of p_+ and π_- using (2.41). This means we should write

$$\begin{aligned} ① + ② &= -\frac{1}{2}(\pi_+ G_+ - \frac{1}{2}\pi_- G_-) \operatorname{str}\big(\chi^2\big) - 2(\pi_+ G_- - \frac{1}{2}\pi_- G_+) \\ &= -\frac{p_+}{2} \operatorname{str}\big(\chi^2\big) - 2(\pi_+ G_- - \frac{1}{2}\pi_- G_+) \end{aligned}$$

where

$$\begin{aligned} \pi_+ G_- - \frac{1}{2}\pi_- G_+ &= \frac{G_-}{G_+}(\pi_+ G_+ - \frac{1}{2}\frac{G_+^2}{G_-}\pi_-) \\ &= \frac{G_-}{G_+}(\pi_+ G_+ - \frac{1}{2}\pi_- G_- + \frac{1}{2}\pi_- G_- - \frac{1}{2}\frac{G_+^2}{G_-}\pi_-) \\ &= \frac{G_-}{G_+}p_+ + \frac{G_-^2 - G_+^2}{2G_+}\pi_- \end{aligned}$$

such that

$$① + ② = -2\frac{G_-}{G_+}p_+ + \frac{G_+^2 - G_-^2}{G_+}\pi_- - \frac{1}{2}p_+ \operatorname{str}\big(\chi^2\big).$$

Adding this to ③ yields the final form of $\mathbf{p}_-$ in (2.46) which also agrees with [8] up to definition of p_+.

B.4 Two-index field Lagrangian

To derive (2.65), we need to start by finding a momentum matrix $\pi_\perp$ such that

$$\operatorname{str}\big(\pi_\perp \dot{\mathbb{X}}\big) = p_\mu \dot{x}^\mu = P_{\alpha\dot{\alpha}} Z^{\alpha\dot{\alpha}} + P_{a\dot{a}} Y^{a\dot{a}}.$$

This way we can properly read off what the momenta conjugate to $Z^{\alpha\dot{\alpha}}$ and $Y^{a\dot{a}}$. Comparing the components of $\mathbb{X}$ in terms of x^μ and $Z^{\alpha\dot{\alpha}}, Y^{a\dot{a}}$ given in (1.129), it is easy to read off

$$\begin{aligned} Z^{3\dot{3}} &= \frac{1}{2}(z_1 - \mathrm{i}z_2), & Z^{3\dot{4}} &= \frac{1}{2}(z_3 - \mathrm{i}z_4), \\ Z^{4\dot{3}} &= -\frac{1}{2}(z_3 + \mathrm{i}z_4), & Z^{4\dot{4}} &= \frac{1}{2}(z_1 + \mathrm{i}z_2). \end{aligned} \tag{B.41}$$

Thus, the components of the conjugate momentum should be

$$\begin{aligned} P_{3\dot{3}} &= p_1 + \mathrm{i}p_2, & P_{3\dot{4}} &= p_3 + \mathrm{i}p_4, \\ P_{4\dot{3}} &= -p_3 + \mathrm{i}p_4, & P_{4\dot{4}} &= p_1 + \mathrm{i}p_2, \end{aligned} \tag{B.42}$$

since we don't want independent directions to mix. If we start with a generic momentum matrix in terms of 2×2 blocks

$$\pi_\perp = \begin{pmatrix} 0 & \pi_z & 0 & 0 \\ \pi_z^\dagger & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{i}\pi_y \\ 0 & 0 & \mathrm{i}\pi_y^\dagger & 0 \end{pmatrix}, \tag{B.43}$$

given the matrix Z in two-index form, we are looking for $\pi_\perp$ such that

$$\mathrm{tr}\left(\pi_z^\dagger Z\right) = \mathrm{tr}\left[\begin{pmatrix} \pi_z^1 & \pi_z^2 \\ \pi_z^3 & \pi_z^4 \end{pmatrix}^\dagger \begin{pmatrix} Z^{3\dot{4}} & -Z^{3\dot{3}} \\ Z^{4\dot{4}} & -Z^{4\dot{3}} \end{pmatrix}\right] = \frac{1}{2}P_{\alpha\dot{\alpha}}Z^{\alpha\dot{\alpha}}.$$

This is satisfied if

$$\pi_z^\dagger = \frac{1}{2}\begin{pmatrix} P_{3\dot{4}} & P_{4\dot{4}} \\ -P_{3\dot{3}} & -P_{4\dot{3}} \end{pmatrix} \iff \pi_z = \frac{1}{2}\begin{pmatrix} -P_{4\dot{3}} & -P_{4\dot{4}} \\ P_{3\dot{3}} & P_{3\dot{4}} \end{pmatrix}. \tag{B.44}$$

The same holds for the S^5 coordinates. Based on the forms of $P^{\alpha\dot{\alpha}}$ and $\mathrm{P}^{a\dot{a}}$, it is easy to see that $\pi_\perp = \dfrac{1}{2}p_\mu\Sigma_\mu$. Indeed, the same holds for $\mathbb{X} = \frac{1}{2}x^\nu\Sigma_\nu$ by definition such that

$$\begin{aligned} \mathrm{str}\left(\pi_\perp\dot{\mathbb{X}}\right) &= \frac{1}{4}p_\mu\dot{x}^\nu\,\mathrm{str}\left(\Sigma_\mu\Sigma_\nu\right) = \frac{1}{8}p_\mu\dot{x}^\nu\,\mathrm{str}\left(\{\Sigma_\mu,\Sigma_\nu\}\right) = \frac{1}{8}p_i\dot{z}^j\,\mathrm{str}\left(\{\Sigma_i,\Sigma_j\}\right) + \frac{1}{8}p_{4+i}\dot{y}^j\,\mathrm{str}\left(\{\Sigma_{4+i},\Sigma_{4+j}\}\right) \\ &= \frac{1}{8}p_i\dot{z}^j 2\delta^{ij}\,\mathrm{tr}(\mathbb{1}_4) - \frac{1}{8}p_{4+i}\mathrm{i}^2\dot{y}^j 2\delta^{ij}\,\mathrm{tr}(\mathbb{1}_4) = p_i\dot{z}^i + p_{4+i}\dot{y}^i = p_\mu\dot{x}^\mu \end{aligned}$$

exactly as desired. Similar calculations yield $\mathrm{str}\left(\pi_\perp\pi_\perp\right) = p_\mu p_\mu$ and the other terms in the two-index Lagrangian (2.65).

B.5 Lagrangian mode decomposition

Having derived the quadratic Lagrangian (2.57) we will now derive the diagonal form of $\mathbb{L}_2$.

Given the ever-so-useful identity

$$\epsilon^{ac}\epsilon_{bd} = \delta^a_b\delta^c_d - \delta^a_d\delta^c_b,$$

we can see the effect of switching which factor of a generic product has upper and lower indices:

$$A^{a\dot{a}}B_{b\dot{b}} = A_{c\dot{c}}B^{d\dot{d}}\epsilon^{ac}\epsilon^{\dot{a}\dot{c}}\epsilon_{bd}\epsilon_{\dot{b}\dot{d}} = \left(A_{d\dot{c}}B^{d\dot{d}}\delta^a_b - A_{b\dot{c}}B^{a\dot{d}}\right)\left(\delta^{\dot{a}}_{\dot{b}}\delta^{\dot{c}}_{\dot{d}} - \delta^{\dot{a}}_{\dot{d}}\delta^{\dot{c}}_{\dot{b}}\right)$$

$$= \delta^a_b \delta^{\dot{a}}_{\dot{b}} A_{c\dot{c}} B^{c\dot{c}} - \delta^a_b A_{c\dot{b}} B^{c\dot{a}} - \delta^{\dot{a}}_{\dot{b}} A_{b\dot{c}} B^{a\dot{c}} + A_{b\dot{b}} B^{a\dot{a}}.$$

In particular this means that for any two-index contraction (setting $a = b$ and $\dot{a} = \dot{b}$ above),

$$A^{a\dot{a}} B_{a\dot{a}} = (2)(2) A_{c\dot{c}} B^{c\dot{c}} - (2) A_{c\dot{a}} B^{c\dot{a}} - (2) A_{a\dot{c}} B^{a\dot{c}} + A_{a\dot{a}} B^{a\dot{a}} = A_{a\dot{a}} B^{a\dot{a}}.$$

This index notation consistency will be particularly useful as we will often equate contractions of the type $a^{\dagger a\dot{a}} a_{a\dot{a}} = a^{\dagger}_{a\dot{a}} a^{a\dot{a}}$ when working through the quantisation procedure.

Let us first look at the bosonic fields. The momenta canonically conjugate to $Y^{a\dot{a}}$ and $Z^{\alpha\dot{\alpha}}$ in the Hamiltonian formalism are determined by the Poisson structure showcased by the kinetic part of $\mathscr{L}$ (as well as the matching two-field indices). We can simply read off the equal-τ commutation relations as the analogue of $[X^i, P_j] = \mathrm{i}\hbar \delta^i_j \mathbb{1}$:

$$[Y^{a\dot{a}}(\tau,\sigma), P_{b\dot{b}}(\tau,\sigma')] = \mathrm{i}\delta^a_b \delta^{\dot{a}}_{\dot{b}} \delta(\sigma-\sigma')\mathbb{1}, \qquad [Z^{\alpha\dot{\alpha}}(\tau,\sigma), P_{\beta\dot{\beta}}(\tau,\sigma')] = \mathrm{i}\delta^\alpha_\beta \delta^{\dot{\alpha}}_{\dot{\beta}} \delta(\sigma-\sigma')\mathbb{1}, \tag{B.45}$$

(Here $\mathbb{1}$ is the identity operator in the relevant Hilbert space.) The quadratic Lagrangian can manifestly be partitioned as

$$\mathbb{L}_2 = \int d\sigma\, \mathscr{L}_2 = \mathbb{L}_{\mathrm{AdS}_5} + \mathbb{L}_{S^5} + \mathbb{L}_\theta + \mathbb{L}_\eta.$$

We will start by computing

$$\mathbb{L}_{\mathrm{AdS}_5} = \int d\sigma \left(P_{a\dot{a}} \dot{Y}^{a\dot{a}} - \frac{1}{4} P_{a\dot{a}} P^{a\dot{a}} - Y_{a\dot{a}} Y^{a\dot{a}} - Y'_{a\dot{a}} Y'^{a\dot{a}} \right) = \int d\sigma\, P_{a\dot{a}} \dot{Y}^{a\dot{a}} - \mathbb{H}_{\mathrm{AdS}_5}. \tag{B.46}$$

Just as for the Klein-Gordon field quantisation, we should first use the canonical commutation relations (2.59) to find similar relations for the ladder operators $a^{a\dot{a}}$ and its conjugate $a^\dagger_{a\dot{a}}$. Using the identity

$$\epsilon^{ca}\epsilon^{\dot{c}\dot{a}}\epsilon_{ab}\epsilon_{\dot{a}\dot{b}} = (-\delta^c_a)(-\delta^{\dot{c}}_{\dot{a}}) = \delta^c_a \delta^{\dot{c}}_{\dot{a}}, \tag{B.47}$$

we can invert the Fourier mode decompositions for $Y^{a\dot{a}}$ and $P_{a\dot{a}}$ to get

$$\begin{aligned} a^{a\dot{a}}(\tau,p) &= \int \frac{d\sigma}{\sqrt{2\pi}} \frac{1}{\sqrt{\omega_p}} \left(\omega_p Y^{a\dot{a}}(\tau,\sigma) + \frac{\mathrm{i}}{2} P^{a\dot{a}}(\tau,\sigma) \right) e^{-\mathrm{i}p\sigma}, \\ a^\dagger_{a\dot{a}}(\tau,p) &= \int \frac{d\sigma}{\sqrt{2\pi}} \frac{1}{\sqrt{\omega_p}} \left(\omega_p Y_{a\dot{a}}(\tau,\sigma) - \frac{\mathrm{i}}{2} P_{a\dot{a}}(\tau,\sigma) \right) e^{\mathrm{i}p\sigma}. \end{aligned} \tag{B.48}$$

Suppressing the dependence on τ, these expressions and (2.59) together imply

$$\begin{aligned} [a^{a\dot{a}}(p), a^\dagger_{b\dot{b}}(p')] &= \frac{1}{2\pi} \frac{1}{\sqrt{\omega_p \omega_{p'}}} \int d\sigma \int d\sigma' \left[\omega_p Y^{a\dot{a}}(\sigma) + \frac{\mathrm{i}}{2} P^{a\dot{a}}(\sigma), \omega_{p'} Y_{b\dot{b}}(\sigma') - \frac{\mathrm{i}}{2} P_{b\dot{b}}(\sigma') \right] e^{-\mathrm{i}(p\sigma - p'\sigma')} \\ &= \frac{1}{2\pi} \frac{1}{\sqrt{\omega_p \omega_{p'}}} \int d\sigma \int d\sigma' \left(-\frac{\mathrm{i}}{2}\omega_p [Y^{a\dot{a}}(\sigma), P_{b\dot{b}}(\sigma')] + \frac{\mathrm{i}}{2}\omega_{p'} [P^{a\dot{a}}(\sigma), Y_{b\dot{b}}(\sigma')] \right) e^{-\mathrm{i}(p\sigma - p'\sigma')} \\ &= \frac{1}{2\pi} \frac{1}{\sqrt{\omega_p \omega_{p'}}} \int d\sigma \int d\sigma' \left(-\frac{\mathrm{i}}{2}\omega_p \mathrm{i}\delta^a_b \delta^{\dot{a}}_{\dot{b}} \delta(\sigma-\sigma')\mathbb{1} - \frac{\mathrm{i}}{2}\omega_{p'} \mathrm{i}\delta^a_b \delta^{\dot{a}}_{\dot{b}} \delta(\sigma-\sigma')\mathbb{1} \right) e^{-\mathrm{i}(p\sigma - p'\sigma')} \\ &= \delta^a_b \delta^{\dot{a}}_{\dot{b}} \frac{\omega_p + \omega_{p'}}{2\sqrt{\omega_p \omega_{p'}}} \mathbb{1} \frac{1}{2\pi} \int d\sigma e^{-\mathrm{i}\sigma(p-p')} = \delta^a_b \delta^{\dot{a}}_{\dot{b}} \frac{\omega_p + \omega_{p'}}{2\sqrt{\omega_p \omega_{p'}}} \delta(p-p')\mathbb{1} \end{aligned}$$

which effectively yields the equal-time commutator relation for the AdS_5 ladder operators:

$$[a^{a\dot{a}}(\tau,p), a^{\dagger}_{b\dot{b}}(\tau,p')] = \delta^a_b \delta^{\dot{a}}_{\dot{b}} \delta(p-p')\mathbb{1}. \tag{B.49}$$

The above calculation differs by a sign from those for same-operator commutators which instead satisfy

$$[a^{a\dot{a}}(\tau,p), a^{b\dot{b}}(\tau,p')] = [a^{\dagger}_{a\dot{a}}(\tau,p), a^{\dagger}_{b\dot{b}}(\tau,p')] = 0.$$

When manipulating ladder operators, it is easier to express

$$\begin{aligned}
Y^{a\dot{a}}(\tau,\sigma) &= \int \frac{dp}{\sqrt{2\pi}} \frac{1}{2\sqrt{\omega_p}} \left(a^{a\dot{a}}(\tau,p) + a^{\dagger a\dot{a}}(\tau,-p) \right) e^{\mathrm{i}p\sigma}, \\
P_{a\dot{a}}(\tau,\sigma) &= \int \frac{dp}{\sqrt{2\pi}} \mathrm{i}\sqrt{\omega_p} \left(a^{\dagger}_{a\dot{a}}(\tau,p) - a_{a\dot{a}}(\tau,-p) \right) e^{-\mathrm{i}p\sigma}, \\
Z^{\alpha\dot{\alpha}}(\tau,\sigma) &= \int \frac{dp}{\sqrt{2\pi}} \frac{1}{2\sqrt{\omega_p}} \left(a^{\alpha\dot{\alpha}}(\tau,p) + a^{\dagger \alpha\dot{\alpha}}(\tau,-p) \right) e^{\mathrm{i}p\sigma}, \\
P_{\alpha\dot{\alpha}}(\tau,\sigma) &= \int \frac{dp}{\sqrt{2\pi}} \mathrm{i}\sqrt{\omega_p} \left(a^{\dagger}_{\alpha\dot{\alpha}}(\tau,p) - a_{\alpha\dot{\alpha}}(\tau,-p) \right) e^{-\mathrm{i}p\sigma}.
\end{aligned} \tag{B.50}$$

The first term of the AdS_5 part of the Lagrangian (B.46) is

$$\begin{aligned}
\int d\sigma\, P_{a\dot{a}} \dot{Y}^{a\dot{a}} &= \frac{\mathrm{i}}{2} \int d\sigma \int \frac{dp}{\sqrt{2\pi}} \int \frac{dp'}{\sqrt{2\pi}} \sqrt{\frac{\omega_{p'}}{\omega_p}} \left(a^{\dagger}_{a\dot{a}}(p') - a_{a\dot{a}}(-p') \right) \left(\dot{a}^{a\dot{a}}(p) + \dot{a}^{\dagger a\dot{a}}(-p) \right) e^{-\mathrm{i}\sigma(p'-p)} \\
&= \frac{\mathrm{i}}{2} \int dp \int dp' \sqrt{\frac{\omega_{p'}}{\omega_p}} \left(a^{\dagger}_{a\dot{a}}(p') - a_{a\dot{a}}(-p') \right) \left(\dot{a}^{a\dot{a}}(p) + \dot{a}^{\dagger a\dot{a}}(-p) \right) \delta(p-p') \\
&= \frac{\mathrm{i}}{2} \int dp \left(a^{\dagger}_{a\dot{a}}(p) - a_{a\dot{a}}(-p) \right) \left(\dot{a}^{a\dot{a}}(p) + \dot{a}^{\dagger a\dot{a}}(-p) \right) \\
&= \frac{\mathrm{i}}{2} \int dp \left(a^{\dagger}_{a\dot{a}}(p)\dot{a}^{a\dot{a}}(p) - a_{a\dot{a}}(-p)\dot{a}^{a\dot{a}}(p) + a^{\dagger}_{a\dot{a}}(p)\dot{a}^{\dagger a\dot{a}}(-p) - a_{a\dot{a}}(-p)\dot{a}^{\dagger a\dot{a}}(-p) \right) \\
&= \frac{\mathrm{i}}{2} \int dp \left(a^{\dagger}_{a\dot{a}}(p)\dot{a}^{a\dot{a}}(p) - a_{a\dot{a}}(p)\dot{a}^{\dagger a\dot{a}}(p) \right) + \frac{\mathrm{i}}{2} \int dp \left(a^{\dagger}_{a\dot{a}}(p)\dot{a}^{\dagger a\dot{a}}(-p) - a_{a\dot{a}}(p)\dot{a}^{a\dot{a}}(-p) \right).
\end{aligned}$$

The second term is

$$\begin{aligned}
\frac{1}{4} \int d\sigma\, P_{a\dot{a}} P^{a\dot{a}} &= -\frac{1}{4} \int d\sigma \int \frac{dp}{\sqrt{2\pi}} \int \frac{dp'}{\sqrt{2\pi}} \sqrt{\omega_p \omega_{p'}} \left(a^{\dagger}_{a\dot{a}}(p) - a_{a\dot{a}}(-p) \right) \left(a^{\dagger a\dot{a}}(-p') - a^{a\dot{a}}(p') \right) e^{-\mathrm{i}\sigma(p-p')} \\
&= -\frac{1}{4} \int dp\, \omega_p \left(a^{\dagger}_{a\dot{a}}(p) a^{\dagger a\dot{a}}(-p) - a_{a\dot{a}}(-p) a^{\dagger a\dot{a}}(-p) - a^{\dagger}_{a\dot{a}}(p) a^{a\dot{a}}(p) + a_{a\dot{a}}(-p) a^{a\dot{a}}(p) \right) \\
&= \frac{1}{4} \int dp\, \omega_p \left(a_{a\dot{a}}(p) a^{\dagger a\dot{a}}(p) + a^{\dagger}_{a\dot{a}}(p) a^{a\dot{a}}(p) \right) - \frac{1}{4} \int dp\, \omega_p \left(a^{\dagger}_{a\dot{a}}(p) a^{\dagger a\dot{a}}(-p) + a_{a\dot{a}}(p) a^{a\dot{a}}(-p) \right).
\end{aligned}$$

The third term is

$$\begin{aligned}
\int d\sigma\, Y_{a\dot{a}} Y^{a\dot{a}} &= \frac{1}{4} \int d\sigma \int \frac{dp}{\sqrt{2\pi}} \int \frac{dp'}{\sqrt{2\pi}} \frac{1}{\sqrt{\omega_p \omega_{p'}}} \left(a_{a\dot{a}}(-p) + a^{\dagger}_{a\dot{a}}(p) \right) \left(a^{a\dot{a}}(p') + a^{\dagger a\dot{a}}(-p') \right) e^{-\mathrm{i}\sigma(p-p')} \\
&= \frac{1}{4} \int dp\, \frac{1}{\omega_p} \left(a_{a\dot{a}}(-p) a^{a\dot{a}}(p) + a_{a\dot{a}}(-p) a^{\dagger a\dot{a}}(-p) + a^{\dagger}_{a\dot{a}}(p) a^{a\dot{a}}(p) + a^{\dagger}_{a\dot{a}}(p) a^{\dagger a\dot{a}}(-p) \right) \\
&= \frac{1}{4} \int dp\, \frac{1}{\omega_p} \left(a_{a\dot{a}}(p) a^{\dagger a\dot{a}}(p) + a^{\dagger}_{a\dot{a}}(p) a^{a\dot{a}}(p) \right) + \frac{1}{4} \int dp\, \frac{1}{\omega_p} \left(a^{\dagger}_{a\dot{a}}(p) a^{\dagger a\dot{a}}(-p) + a_{a\dot{a}}(p) a^{a\dot{a}}(-p) \right).
\end{aligned}$$

The fourth and final term is

$$\begin{aligned}\int d\sigma\, Y'_{a\dot{a}}Y'^{a\dot{a}} &= \frac{1}{4}\int d\sigma \int \frac{dp}{\sqrt{2\pi}}\int \frac{dp'}{\sqrt{2\pi}}\frac{pp'}{\sqrt{\omega_p\omega_{p'}}}\left(a_{a\dot{a}}(-p)+a^{\dagger}_{a\dot{a}}(p)\right)\left(a^{a\dot{a}}(p')+a^{\dagger a\dot{a}}(-p')\right)e^{-\mathrm{i}\sigma(p-p')}\\ &= \frac{1}{4}\int dp\,\frac{p^2}{\omega_p}\left(a_{a\dot{a}}(-p)a^{a\dot{a}}(p)+a_{a\dot{a}}(-p)a^{\dagger a\dot{a}}(-p)+a^{\dagger}_{a\dot{a}}(p)a^{a\dot{a}}(p)+a^{\dagger}_{a\dot{a}}(p)a^{\dagger a\dot{a}}(-p)\right)\\ &= \frac{1}{4}\int dp\,\frac{p^2}{\omega_p}\left(a_{a\dot{a}}(p)a^{\dagger a\dot{a}}(p)+a^{\dagger}_{a\dot{a}}(p)a^{a\dot{a}}(p)\right)+\frac{1}{4}\int dp\,\frac{p^2}{\omega_p}\left(a^{\dagger}_{a\dot{a}}(p)a^{\dagger a\dot{a}}(-p)+a_{a\dot{a}}(p)a^{a\dot{a}}(-p)\right).\end{aligned}$$

Adding the four gives

$$\begin{aligned}\mathbb{L}_{\mathrm{AdS}_5} &= \frac{\mathrm{i}}{2}\int dp\left(a^{\dagger}_{a\dot{a}}(p)\dot{a}^{a\dot{a}}(p)-a_{a\dot{a}}(p)\dot{a}^{\dagger a\dot{a}}(p)\right)+\frac{\mathrm{i}}{2}\int dp\left(a^{\dagger}_{a\dot{a}}(p)\dot{a}^{\dagger a\dot{a}}(-p)-a_{a\dot{a}}(p)\dot{a}^{a\dot{a}}(-p)\right)\\ &\quad -\frac{1}{4}\int dp\left(a_{a\dot{a}}(p)a^{\dagger a\dot{a}}(p)+a^{\dagger}_{a\dot{a}}(p)a^{a\dot{a}}(p)\right)\left(\omega_p+\frac{1+p^2}{\omega_p}\right)\\ &\quad -\frac{1}{4}\int dp\left(a^{\dagger}_{a\dot{a}}(p)a^{\dagger a\dot{a}}(-p)+a_{a\dot{a}}(p)a^{a\dot{a}}(-p)\right)\left(-\omega_p+\frac{1+p^2}{\omega_p}\right).\end{aligned}$$

Remembering the dispersion is $\omega_p=\sqrt{1+p^2}$ for unit mass fields, the bottom line vanishes such that

$$\begin{aligned}\mathbb{L}_{\mathrm{AdS}_5} &= \frac{\mathrm{i}}{2}\int dp\left(a^{\dagger}_{a\dot{a}}(p)\dot{a}^{a\dot{a}}(p)-a_{a\dot{a}}(p)\dot{a}^{\dagger a\dot{a}}(p)\right)+\frac{\mathrm{i}}{2}\int dp\left(a^{\dagger}_{a\dot{a}}(p)\dot{a}^{\dagger a\dot{a}}(-p)-a_{a\dot{a}}(p)\dot{a}^{a\dot{a}}(-p)\right)\\ &\quad -\frac{1}{2}\int dp\,\omega_p\left(a_{a\dot{a}}(p)a^{\dagger a\dot{a}}(p)+a^{\dagger}_{a\dot{a}}(p)a^{a\dot{a}}(p)\right).\end{aligned}$$

To simplify the form of the Lagrangian further, we should use commutation relations between $a^{a\dot{a}}(p)$ and $a^{\dagger}_{a\dot{a}}(p)$. But the first two kinetic terms would involve commutation relations between $a^{a\dot{a}}(p)$ and $\dot{a}^{\dagger}_{a\dot{a}}(p)$. In the Heisenberg picture,

$$\dot{a}^{a\dot{a}}(\tau,p)=\mathrm{i}[\mathbb{H}_{\mathrm{AdS}_5}(\tau),a^{a\dot{a}}(\tau,p)] \tag{B.51}$$

where $\mathbb{H}_{\mathrm{AdS}_5}$ is the quadratic Hamiltonian operator corresponding to the AdS_5 degrees of freedom. We can read off $\mathbb{H}_{\mathrm{AdS}_5}$ as the third term in $\mathbb{L}_{\mathrm{AdS}_5}$, and using (B.49) which implies

$$a^{a\dot{a}}(p)a^{\dagger}_{a\dot{a}}(p)+a^{\dagger}_{a\dot{a}}(p)a^{a\dot{a}}(p)=2a^{\dagger}_{a\dot{a}}(p)a^{a\dot{a}}(p)+[a^{a\dot{a}}(p),a^{\dagger}_{a\dot{a}}(p)], \tag{B.52}$$

we find

$$\mathbb{H}_{\mathrm{AdS}_5}=\int dp\,\omega_p\left(a^{\dagger}_{a\dot{a}}(p)a^{a\dot{a}}(p)+\frac{1}{2}[a^{a\dot{a}}(p),a^{\dagger}_{a\dot{a}}(p)]\mathbb{1}\right). \tag{B.53}$$

Just as in regular quantum field theory, the divergent $[a^{a\dot{a}}(p),a^{\dagger}_{a\dot{a}}(p)]=4\delta(0)$ term can be considered the zero point energy from which the quanta spanned by $a^{\dagger}_{a\dot{a}}$ grow. We will see that this term in fact cancels with another term in $\mathbb{L}_{\mathrm{AdS}_5}$. Returning to the calculation at hand,

$$\begin{aligned}\dot{a}^{a\dot{a}}(p)=\mathrm{i}[\mathbb{H}_{\mathrm{AdS}_5},a^{a\dot{a}}(p)] &= \mathrm{i}\int dp'\;\omega_{p'}[a^{\dagger}_{b\dot{b}}(p')a^{b\dot{b}}(p'),a^{a\dot{a}}(p)]\\ &= \mathrm{i}\int dp'\;\omega_{p'}\left(a^{\dagger}_{b\dot{b}}(p')[a^{b\dot{b}}(p'),a^{a\dot{a}}(p)]+[a^{\dagger}_{b\dot{b}}(p'),a^{a\dot{a}}(p)]a^{b\dot{b}}(p')\right)\\ &= \mathrm{i}\int dp'\;\omega_{p'}\left(-\delta^a_b\delta^{\dot{a}}_{\dot{b}}\delta(p-p')a^{b\dot{b}}(p')\right)=-\mathrm{i}\omega_p a^{a\dot{a}}(p).\end{aligned}$$

We thus recover the usual evolution equations for the harmonic oscillator ladder operators,

$$\dot{a}^{a\dot{a}}(p) = -\mathrm{i}\omega_p a^{a\dot{a}}(p), \qquad \dot{a}^{\dagger}_{a\dot{a}}(p) = \mathrm{i}\omega_p a^{\dagger}_{a\dot{a}}(p). \tag{B.54}$$

The important takeaway is that the time derivatives of the field operators commute in the same way – up to a factor – as the operators themselves. We can now use (B.54) to evaluate the first of the kinetic terms of $\mathbb{L}_{\mathrm{AdS}_5}$. Integrating by parts,

$$\begin{aligned}\int dp\left(a^{\dagger}_{a\dot{a}}(p)\dot{a}^{a\dot{a}}(p) - a^{a\dot{a}}(p)\dot{a}^{\dagger}_{a\dot{a}}(p)\right) &= \int dp\left(a^{\dagger}_{a\dot{a}}(p)\dot{a}^{a\dot{a}}(p) + \dot{a}^{a\dot{a}}(p)a^{\dagger}_{a\dot{a}}(p)\right)\\ &= \int dp\left(2a^{\dagger}_{a\dot{a}}(p)\dot{a}^{a\dot{a}}(p) + [\dot{a}^{a\dot{a}}(p), a^{\dagger}_{a\dot{a}}(p)]\right)\end{aligned}$$

On the other hand, the second kinetic term vanishes because of commutation relations. Integrating by parts again,

$$\begin{aligned}\int dp\left(a^{\dagger}_{a\dot{a}}(p)\dot{a}^{\dagger a\dot{a}}(-p)\right) &= \frac{1}{2}\int dp\left(a^{\dagger}_{a\dot{a}}(p)\dot{a}^{\dagger a\dot{a}}(-p) + a^{\dagger}_{a\dot{a}}(p)\dot{a}^{\dagger a\dot{a}}(-p)\right)\\ &= \frac{1}{2}\int dp\ [a^{\dagger a\dot{a}}(p), \dot{a}^{\dagger}_{a\dot{a}}(-p)] = \frac{1}{2}\int dp\ \mathrm{i}\omega_p[a^{\dagger a\dot{a}}(p), a^{\dagger}_{a\dot{a}}(-p)] = 0.\end{aligned}$$

The same holds for the $a^{\dagger}_{a\dot{a}}(p)\dot{a}^{\dagger a\dot{a}}(-p)$ term giving

$$\mathbb{L}_{\mathrm{AdS}_5} = \int dp\left(\mathrm{i}a^{\dagger}_{a\dot{a}}(p)\dot{a}^{a\dot{a}}(p) + \frac{1}{2}[a^{a\dot{a}}(p), a^{\dagger}_{a\dot{a}}(p)]\right) - \mathbb{H}_{\mathrm{AdS}_5}.$$

As promised, the two $\delta(0)$ terms cancel and we are left with

$$\mathbb{L}_{\mathrm{AdS}_5} = \int dp\left(\mathrm{i}a^{\dagger}_{a\dot{a}}(p)\dot{a}^{a\dot{a}}(p) - \omega_p a^{\dagger}_{a\dot{a}}(p)a^{a\dot{a}}(p)\right). \tag{B.55}$$

Considering instead the contributions of $Z^{\alpha\dot{\alpha}}$ and $P_{\alpha\dot{\alpha}}$, an identical calculation would yield

$$\mathbb{L}_{S^5} = \int dp\left(\mathrm{i}a^{\dagger}_{\alpha\dot{\alpha}}(p)\dot{a}^{\alpha\dot{\alpha}}(p) - \omega_p a^{\dagger}_{\alpha\dot{\alpha}}(p)a^{\alpha\dot{\alpha}}(p)\right). \tag{B.56}$$

This is the end of the story for the bosonic fields. What about the fermions? The functional forms of f_p and h_p will be decided when it comes time to diagonalise the Lagrangian in the same exact form as the bosonic case. To this end, we need to evaluate

$$\begin{aligned}\mathbb{L}_{\chi} = \int d\sigma\Big(&\mathrm{i}\theta^{\dagger}_{a\dot{\alpha}}\dot{\theta}^{a\dot{\alpha}} + \theta^{\dagger}_{a\dot{\alpha}}\theta^{a\dot{\alpha}} + \frac{\kappa}{2}\theta'_{a\dot{\alpha}}\theta^{a\dot{\alpha}} - \frac{\kappa}{2}\theta'^{\dagger}_{a\dot{\alpha}}\theta^{\dagger a\dot{\alpha}}\\ &+ \mathrm{i}\eta^{\dagger}_{\alpha\dot{a}}\dot{\eta}^{\alpha\dot{a}} + \eta^{\dagger}_{\alpha\dot{a}}\eta^{\alpha\dot{a}} + \frac{\kappa}{2}\eta'_{\alpha\dot{a}}\eta^{\alpha\dot{a}} - \frac{\kappa}{2}\eta'^{\dagger}_{\alpha\dot{a}}\eta^{\dagger\alpha\dot{a}}\Big)\end{aligned} \tag{B.57}$$

and infer f_p, h_p. This time the inverse Fourier transforms of $\theta^{a\dot{\alpha}}$ and $\eta^{\alpha\dot{a}}$ are

$$\begin{aligned}a^{a\dot{\alpha}}(\tau,p) &= \frac{e^{\mathrm{i}\pi/4}\sqrt{\omega_p}}{f^*_{-p}f_p - h_{-p}h^*_p}\int\frac{d\sigma}{\sqrt{2\pi}}\left(f^*_{-p}\theta^{a\dot{\alpha}}(\tau,\sigma) - h_{-p}\theta^{\dagger a\dot{\alpha}}(\tau,\sigma)\right)e^{-\mathrm{i}p\sigma},\\ a^{\dagger}_{a\dot{\alpha}}(\tau,p) &= \frac{-e^{-\mathrm{i}\pi/4}\sqrt{\omega_p}}{f_{-p}f^*_p - h^*_{-p}h_p}\int\frac{d\sigma}{\sqrt{2\pi}}\left(h^*_{-p}\theta_{a\dot{\alpha}}(\tau,\sigma) - f_{-p}\theta^{\dagger}_{a\dot{\alpha}}(\tau,\sigma)\right)e^{\mathrm{i}p\sigma}.\end{aligned} \tag{B.58}$$

The equal-τ anti-commutator for fermionic ladder operators is

$$\begin{aligned}
\{a^{a\dot{\alpha}}(p), a^{\dagger}_{b\dot{\beta}}(p')\} &= -\sqrt{\omega_p\omega_{p'}}\frac{(f_{-p'}f^*_{p'} - h^*_{-p'}h_{p'})^{-1}}{(f^*_{-p}f_p - h_{-p}h^*_p)}\int\frac{d\sigma}{\sqrt{2\pi}}\int\frac{d\sigma'}{\sqrt{2\pi}}e^{-\mathrm{i}p\sigma+\mathrm{i}p'\sigma'} \\
&\quad\times\left((-f^*_{-p}f_{-p'})\{\theta^{a\dot{\alpha}}(\sigma),\theta^{\dagger}_{b\dot{\beta}}(\sigma')\} + (-h_{-p}h^*_{-p'})\{\theta^{\dagger a\dot{\alpha}}(\sigma),\theta_{b\dot{\beta}}(\sigma')\}\right) \\
&= \sqrt{\omega_p\omega_{p'}}\frac{(f_{-p'}f^*_{p'} - h^*_{-p'}h_{p'})^{-1}}{(f^*_{-p}f_p - h_{-p}h^*_p)}(f^*_{-p}f_{-p'} + h_{-p}h^*_{-p'})\delta^a_b\delta^{\dot{\alpha}}_{\dot{\beta}}\int\frac{d\sigma}{\sqrt{2\pi}}\int\frac{d\sigma'}{\sqrt{2\pi}}e^{-\mathrm{i}p\sigma+\mathrm{i}p'\sigma'}\delta(\sigma-\sigma')\mathbb{1} \\
&= \sqrt{\omega_p\omega_{p'}}\frac{(f_{-p'}f^*_{p'} - h^*_{-p'}h_{p'})^{-1}}{(f^*_{-p}f_p - h_{-p}h^*_p)}(f^*_{-p}f_{-p'} + h_{-p}h^*_{-p'})\delta^a_b\delta^{\dot{\alpha}}_{\dot{\beta}}\frac{1}{2\pi}\int d\sigma\ e^{-\mathrm{i}\sigma(p-p')}\mathbb{1} \\
&= \sqrt{\omega_p\omega_{p'}}\frac{(f_{-p'}f^*_{p'} - h^*_{-p'}h_{p'})^{-1}}{(f^*_{-p}f_p - h_{-p}h^*_p)}(f^*_{-p}f_{-p'} + h_{-p}h^*_{-p'})\delta^a_b\delta^{\dot{\alpha}}_{\dot{\beta}}\delta(p-p')\mathbb{1}.
\end{aligned}$$

At this point, it would make our lives much easier if we chose the mode decomposition of the fermionic fields such that f_p and h_p are real functions of p. In this case,

$$\{a^{a\dot{\alpha}}(p), a^{\dagger}_{b\dot{\beta}}(p')\} = \omega_p\frac{(f^2_{-p} + h^2_{-p})}{(f_{-p}f_p - h_{-p}h_p)^2}\delta^a_b\delta^{\dot{\alpha}}_{\dot{\beta}}\delta(p-p')\mathbb{1}.$$

Focusing on the $\theta^{a\dot{\alpha}}$ terms for example, we aim to calculate

$$\mathbb{L}_\theta = \int d\sigma\left(\mathrm{i}\theta^{\dagger}_{a\dot{\alpha}}\dot{\theta}^{a\dot{\alpha}} - \theta^{\dagger}_{a\dot{\alpha}}\theta^{a\dot{\alpha}} - \frac{\kappa}{2}\theta'_{a\dot{\alpha}}\theta^{a\dot{\alpha}} + \frac{\kappa}{2}\theta'^{\dagger}_{a\dot{\alpha}}\theta^{\dagger a\dot{\alpha}}\right) = \int d\sigma\,\mathrm{i}\theta^{\dagger}_{a\dot{\alpha}}\dot{\theta}^{a\dot{\alpha}} - \mathbb{H}_\theta.$$

The kinetic term of $\mathbb{L}_\theta$ is

$$\begin{aligned}
\int d\sigma\,\theta^{\dagger}_{a\dot{\alpha}}\dot{\theta}^{a\dot{\alpha}} &= \int d\sigma\int\frac{dp}{\sqrt{2\pi}}\int\frac{dp'}{\sqrt{2\pi}}\frac{1}{\sqrt{\omega_p\omega_{p'}}}\left(f_p a^{\dagger}_{a\dot{\alpha}}(p) + h_{-p}a_{a\dot{\alpha}}(-p)\right)\left(f_{p'}\dot{a}^{a\dot{\alpha}}(p') + h_{-p'}\dot{a}^{\dagger a\dot{\alpha}}(-p')\right)e^{\mathrm{i}\sigma(p'-p)} \\
&= \int dp\int dp'\frac{1}{\sqrt{\omega_p\omega_{p'}}}\left(f_p a^{\dagger}_{a\dot{\alpha}}(p) + h_{-p}a_{a\dot{\alpha}}(-p)\right)\left(f_{p'}\dot{a}^{a\dot{\alpha}}(p') + h_{-p'}\dot{a}^{\dagger a\dot{\alpha}}(-p')\right)\delta(p-p') \\
&= \int dp\,\frac{1}{\omega_p}\left(f_p a^{\dagger}_{a\dot{\alpha}}(p) + h_{-p}a_{a\dot{\alpha}}(-p)\right)\left(f_p\dot{a}^{a\dot{\alpha}}(p) + h_{-p}\dot{a}^{\dagger a\dot{\alpha}}(-p)\right) \\
&= \int dp\,\frac{1}{\omega_p}\left(f_p^2 a^{\dagger}_{a\dot{\alpha}}(p)\dot{a}^{a\dot{\alpha}}(p) + h_p^2 a_{a\dot{\alpha}}(p)\dot{a}^{\dagger a\dot{\alpha}}(p)\right) + \int dp\,\frac{f_p h_{-p}}{\omega_p}\left(a^{\dagger}_{a\dot{\alpha}}(p)\dot{a}^{\dagger a\dot{\alpha}}(-p) + a_{a\dot{\alpha}}(-p)\dot{a}^{a\dot{\alpha}}(p)\right) \\
&= \int dp\,\frac{1}{\omega_p}\left(\left(f_p^2 + h_p^2\right)a^{\dagger}_{a\dot{\alpha}}(p)\dot{a}^{a\dot{\alpha}}(p) - h_p^2\{\dot{a}_{a\dot{\alpha}}(p), a^{\dagger a\dot{\alpha}}(p)\}\right) \\
&\quad + \int dp\,\frac{f_p h_{-p}}{\omega_p}\left(a^{\dagger}_{a\dot{\alpha}}(p)\dot{a}^{\dagger a\dot{\alpha}}(-p) + a_{a\dot{\alpha}}(-p)\dot{a}^{a\dot{\alpha}}(p)\right)
\end{aligned}$$

and so, in anticipation of wanting the fermionic part of the Lagrangian to resemble (B.55) and (B.56), we restrict our consideration to functions satisfying

$$f_p = f_{-p}, \qquad h_p = -h_{-p}, \qquad f_p^2 + h_p^2 = \omega_p. \tag{B.59}$$

In turn this simplifies the anti-commutation relation to the usual

$$\{a^{a\dot{\alpha}}(p), a^{\dagger}_{b\dot{\beta}}(p')\} = \delta^a_b\delta^{\dot{\alpha}}_{\dot{\beta}}\delta(p-p')\mathbb{1}. \tag{B.60}$$

This way, the kinetic term $\int d\sigma\, \theta^{\dagger}_{a\dot{\alpha}}\dot{\theta}^{a\dot{\alpha}}$ can be integrated by parts as we did with the bosonic degrees of freedom. We first need to find the commutation relations involving time derivatives of the field operators just as we did in (B.51). The relevant fermionic Hamiltonian $\mathbb{H}_\theta$ has terms

$$\begin{aligned}
\int d\sigma\, \theta^{\dagger}_{a\dot{\alpha}}\theta^{a\dot{\alpha}} &= \int d\sigma \int \frac{dp}{\sqrt{2\pi}} \int \frac{dp'}{\sqrt{2\pi}} \frac{1}{\sqrt{\omega_p\omega_{p'}}} \left(f_p a^{\dagger}_{a\dot{\alpha}}(p) + h_{-p} a_{a\dot{\alpha}}(-p)\right) \left(f_{p'} a^{a\dot{\alpha}}(p') + h_{-p'} a^{\dagger a\dot{\alpha}}(-p')\right) e^{\mathrm{i}\sigma(p'-p)} \\
&= \int dp\, \frac{1}{\omega_p} \left(f_p^2 a^{\dagger}_{a\dot{\alpha}}(p) a^{a\dot{\alpha}}(p) + h_p^2 a_{a\dot{\alpha}}(p) a^{\dagger a\dot{\alpha}}(p)\right) + \int dp\, \frac{f_p h_{-p}}{\omega_p} \left(a^{\dagger}_{a\dot{\alpha}}(p) a^{\dagger a\dot{\alpha}}(-p) - a_{a\dot{\alpha}}(p) a^{a\dot{\alpha}}(-p)\right),
\end{aligned}$$

$$\begin{aligned}
\int d\sigma\, \theta^{a\dot{\alpha}}\theta'_{a\dot{\alpha}} &= \int d\sigma \int \frac{dp}{\sqrt{2\pi}} \int \frac{dp'}{\sqrt{2\pi}} \frac{e^{-\mathrm{i}\pi/2}\mathrm{i}p}{\sqrt{\omega_p\omega_{p'}}} \left(f_{p'} a^{a\dot{\alpha}}(p') + h_{-p'} a^{\dagger a\dot{\alpha}}(-p')\right) \left(f_p a_{a\dot{\alpha}}(p) + h_{-p} a^{\dagger}_{a\dot{\alpha}}(-p)\right) e^{\mathrm{i}\sigma(p'+p)} \\
&= \int dp\, \frac{p}{\omega_p} \left(f_{-p} a^{a\dot{\alpha}}(-p) + h_p a^{\dagger a\dot{\alpha}}(p)\right) \left(f_p a_{a\dot{\alpha}}(p) + h_{-p} a^{\dagger}_{a\dot{\alpha}}(-p)\right) \\
&= \int dp\, \frac{p}{\omega_p} \left(f_p^2 a_{a\dot{\alpha}}(-p) a^{a\dot{\alpha}}(p) + f_p h_{-p} a^{a\dot{\alpha}}(-p) a^{\dagger}_{a\dot{\alpha}}(-p) + f_p h_p a^{\dagger a\dot{\alpha}}(p) a_{a\dot{\alpha}}(p) - h_p^2 a^{\dagger}_{a\dot{\alpha}}(p) a^{\dagger a\dot{\alpha}}(-p)\right) \\
&= \int dp\, \frac{p f_p h_p}{\omega_p} \left(a^{\dagger}_{a\dot{\alpha}}(p) a^{a\dot{\alpha}}(p) - a_{a\dot{\alpha}}(p) a^{\dagger a\dot{\alpha}}(p)\right) - \int dp\, \frac{p}{\omega_p} \left(h_p^2 a^{\dagger}_{a\dot{\alpha}}(p) a^{\dagger a\dot{\alpha}}(-p) + f_p^2 a_{a\dot{\alpha}}(p) a^{a\dot{\alpha}}(-p)\right) \\
&= \int dp\, \frac{p f_p h_p}{\omega_p} \left(2 a^{\dagger}_{a\dot{\alpha}}(p) a^{a\dot{\alpha}}(p) - \{a_{a\dot{\alpha}}(p), a^{\dagger a\dot{\alpha}}(p)\}\right) - \int dp\, \frac{p}{\omega_p} \left(h_p^2 a^{\dagger}_{a\dot{\alpha}}(p) a^{\dagger a\dot{\alpha}}(-p) + f_p^2 a_{a\dot{\alpha}}(p) a^{a\dot{\alpha}}(-p)\right),
\end{aligned}$$

$$\begin{aligned}
\int d\sigma\, \theta^{\dagger a\dot{\alpha}}\theta'^{\dagger}_{a\dot{\alpha}} &= e^{\mathrm{i}\pi/2} \int d\sigma \int \frac{dp}{\sqrt{2\pi}} \int \frac{dp'}{\sqrt{2\pi}} \frac{(-\mathrm{i}p)}{\sqrt{\omega_p\omega_{p'}}} \left(f_{p'} a^{\dagger a\dot{\alpha}}(p') + h_{-p'} a^{a\dot{\alpha}}(-p')\right) \left(f_p a^{\dagger}_{a\dot{\alpha}}(p) + h_{-p} a_{a\dot{\alpha}}(-p)\right) e^{-\mathrm{i}\sigma(p'+p)} \\
&= \int dp\, \frac{p}{\omega_p} \left(f_{-p} a^{\dagger a\dot{\alpha}}(-p) + h_p a^{a\dot{\alpha}}(p)\right) \left(f_p a^{\dagger}_{a\dot{\alpha}}(p) + h_{-p} a_{a\dot{\alpha}}(-p)\right) \\
&= \int dp\, \frac{p}{\omega_p} \left(f_p^2 a^{\dagger}_{a\dot{\alpha}}(-p) a^{\dagger a\dot{\alpha}}(p) + f_p h_{-p} a^{\dagger a\dot{\alpha}}(-p) a_{a\dot{\alpha}}(-p) + f_p h_p a^{a\dot{\alpha}}(p) a^{\dagger}_{a\dot{\alpha}}(p) - h_p^2 a^{a\dot{\alpha}}(p) a_{a\dot{\alpha}}(-p)\right) \\
&= \int dp\, \frac{p f_p h_{-p}}{\omega_p} \left(a^{\dagger}_{a\dot{\alpha}}(p) a^{a\dot{\alpha}}(p) - a^{a\dot{\alpha}}(p) a^{\dagger}_{a\dot{\alpha}}(p)\right) - \int dp\, \frac{p}{\omega_p} \left(f_p^2 a^{\dagger}_{a\dot{\alpha}}(p) a^{\dagger a\dot{\alpha}}(-p) + h_p^2 a_{a\dot{\alpha}}(p) a^{a\dot{\alpha}}(-p))\right) \\
&= \int dp\, \frac{p f_p h_{-p}}{\omega_p} \left(2 a^{\dagger}_{a\dot{\alpha}}(p) a^{a\dot{\alpha}}(p) - \{a^{a\dot{\alpha}}(p), a^{\dagger}_{a\dot{\alpha}}(p)\}\right) - \int dp\, \frac{p}{\omega_p} \left(f_p^2 a^{\dagger}_{a\dot{\alpha}}(p) a^{\dagger a\dot{\alpha}}(-p) + h_p^2 a_{a\dot{\alpha}}(p) a^{a\dot{\alpha}}(-p)\right).
\end{aligned}$$

Adding the terms,

$$\begin{aligned}
\mathbb{H}_\theta &= \int d\sigma\, \theta^{\dagger}_{a\dot{\alpha}}\theta^{a\dot{\alpha}} + \frac{\kappa}{2}\left(\int d\sigma\, \theta^{a\dot{\alpha}}\theta'_{a\dot{\alpha}} - \int d\sigma\, \theta^{\dagger a\dot{\alpha}}\theta'^{\dagger}_{a\dot{\alpha}}\right) \\
&= \int dp\, \frac{1}{\omega_p} \left(f_p^2 a^{\dagger}_{a\dot{\alpha}}(p) a^{a\dot{\alpha}}(p) + h_p^2 a_{a\dot{\alpha}}(p) a^{\dagger a\dot{\alpha}}(p)\right) + \int dp\, \frac{f_p h_{-p}}{\omega_p} \left(a^{\dagger}_{a\dot{\alpha}}(p) a^{\dagger a\dot{\alpha}}(-p) - a_{a\dot{\alpha}}(p) a^{a\dot{\alpha}}(-p)\right) \\
&\quad + \int dp\, \frac{\kappa}{2} \left(\frac{p f_p h_p}{\omega_p} - \frac{p f_p h_{-p}}{\omega_p}\right) \left(2 a^{\dagger}_{a\dot{\alpha}}(p) a^{a\dot{\alpha}}(p) - \{a^{a\dot{\alpha}}(p), a^{\dagger}_{a\dot{\alpha}}(p)\}\right) \\
&\quad + \int dp\, \frac{p}{\omega_p}\frac{\kappa}{2} \left(h_p^2 - f_p^2\right) a_{a\dot{\alpha}}(p) a^{a\dot{\alpha}}(-p) + \int dp\, \frac{p}{\omega_p}\frac{\kappa}{2} \left(f_p^2 - h_p^2\right) a^{\dagger}_{a\dot{\alpha}}(p) a^{\dagger a\dot{\alpha}}(-p) \\
&= \int dp\, \frac{1}{\omega_p} \left((f_p^2 - h_p^2) a^{\dagger}_{a\dot{\alpha}}(p) a^{a\dot{\alpha}}(p) + h_p^2 \{a^{a\dot{\alpha}}(p), a^{\dagger}_{a\dot{\alpha}}(p)\}\right) + \int dp\, \kappa \frac{p f_p h_p}{\omega_p} \left(2 a^{\dagger}_{a\dot{\alpha}}(p) a^{a\dot{\alpha}}(p) - \{a^{a\dot{\alpha}}(p), a^{\dagger}_{a\dot{\alpha}}(p)\}\right) \\
&\quad + \int dp\, \frac{1}{\omega_p} \left(f_p h_p - \frac{\kappa}{2} p \left(f_p^2 - h_p^2\right)\right) \left(a_{a\dot{\alpha}}(p) a^{a\dot{\alpha}}(-p) - a^{\dagger}_{a\dot{\alpha}}(p) a^{\dagger a\dot{\alpha}}(-p)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \int dp\, \frac{1}{\omega_p} \left(f_p^2 - h_p^2 + 2\kappa p f_p h_p\right) a^\dagger_{a\dot{\alpha}}(p) a^{a\dot{\alpha}}(p) + \int dp\, \frac{1}{\omega_p} \left(h_p^2 - \kappa p f_p h_p\right) \{a^{a\dot{\alpha}}(p), a^\dagger_{a\dot{\alpha}}(p)\} \\
&\quad + \int dp\, \frac{1}{\omega_p} \left(f_p h_p - \frac{\kappa}{2} p \left(f_p^2 - h_p^2\right)\right) \left(a_{a\dot{\alpha}}(p) a^{a\dot{\alpha}}(-p) - a^\dagger_{a\dot{\alpha}}(p) a^{\dagger a\dot{\alpha}}(-p)\right)
\end{aligned}
$$

To write the Hamiltonian in diagonal form, the last term should vanish. If we impose $f_p^2 - h_p^2 = 1$ along with (B.59), this leads to

$$
\frac{\kappa}{2} p = f_p h_p \implies h_p = \frac{p\kappa}{2 f_p} \tag{B.61}
$$

and

$$
(f_p^2 - h_p^2) + (f_p^2 + h_p^2) = 1 + \omega_p \implies f_p = \pm\sqrt{\frac{1+\omega_p}{2}}. \tag{B.62}
$$

The first integral gets a prefactor of

$$
\frac{1}{\omega_p} \left(f_p^2 - h_p^2 + 2\kappa p f_p h_p\right) = \frac{1}{\omega_p} \left(1 + p^2\right) = \omega_p
$$

as desired. For definiteness, we can choose $\kappa = 1$ and the positive root of f_p. Thus, the fermionic mode decomposition is (2.62) with the functions defined as

$$
f_p = \sqrt{\frac{1+\omega_p}{2}}, \qquad h_p = \frac{p}{2 f_p} \implies f_p^2 = 1 + h_p^2 = \omega_p - h_p^2, \tag{B.63}
$$

The Hamiltonian is now

$$
\mathbb{H}_\theta = \int dp\, \omega_p\, a^\dagger_{a\dot{\alpha}}(p) a^{a\dot{\alpha}}(p) + \int dp\, \frac{1}{\omega_p} \left(h_p^2 - \kappa p f_p h_p\right) \{a^{a\dot{\alpha}}(p), a^\dagger_{a\dot{\alpha}}(p)\}.
$$

The Heisenberg picture evolution is still given by

$$
\dot{a}^{a\dot{a}}(\tau, p) = \mathrm{i}[\mathbb{H}_\theta(\tau), a^{a\dot{a}}(\tau, p)]. \tag{B.64}
$$

Commutators are not useful with fermions so we will need to make use of the identity

$$
[AB, C] = A\{B, C\} - \{A, C\}B
$$

and the anti-commutation relations (B.60) to find

$$
[a^\dagger_{b\dot{\beta}}(p') a^{b\dot{\beta}}(p'), a^{a\dot{a}}(p)] = a^\dagger_{b\dot{\beta}} \cdot 0 - \delta^a_b \delta^{\dot{\alpha}}_{\dot{\beta}} \delta(p - p') a^{b\dot{\beta}}(p') = -a^{a\dot{\alpha}}(p') \delta(p - p').
$$

Putting this to use, we get

$$
\begin{aligned}
\dot{a}^{a\dot{a}}(p) &= \mathrm{i} \int dp'\, \omega_{p'}\, [a^\dagger_{b\dot{\beta}}(p') a^{b\dot{\beta}}(p'), a^{a\dot{a}}(p)] \\
&\qquad + \mathrm{i} \int dp\, \frac{1}{\omega_{p'}} \left(h_{p'}^2 - \kappa p' f_{p'} h_{p'}\right) \left([a^\dagger_{b\dot{\beta}}(p') a^{b\dot{\beta}}(p'), a^{a\dot{a}}(p)] + [a^{b\dot{\beta}}(p') a^\dagger_{b\dot{\beta}}(p'), a^{a\dot{a}}(p)]\right) \\
&= -\mathrm{i} \int dp'\, \omega_{p'}\, a^{a\dot{\alpha}}(p') \delta(p - p') + \mathrm{i} \int dp\, \frac{1}{\omega_{p'}} \left(h_{p'}^2 - \kappa p' f_{p'} h_{p'}\right) \left(-a^{a\dot{\alpha}}(p') \delta(p - p') + a^{a\dot{\alpha}}(p') \delta(p - p')\right) \\
&= -\mathrm{i} \omega_p a^{a\dot{\alpha}}(p).
\end{aligned}
$$

Thus bosons and fermions satisfy the same evolution equation. We can now simplify the kinetic term:

$$\begin{aligned}\int d\sigma\, \mathrm{i}\theta^\dagger_{a\dot\alpha}\dot\theta^{a\dot\alpha} &= \int dp\, \mathrm{i}a^\dagger_{a\dot\alpha}(p)\dot a^{a\dot\alpha}(p) - \int dp\ \frac{\mathrm{i}}{\omega_p}h_p^2\{\dot a_{a\dot\alpha}(p), a^{\dagger a\dot\alpha}(p)\} + \int dp\ \frac{\mathrm{i}f_p h_{-p}}{\omega_p}\left(a^\dagger_{a\dot\alpha}(p)\dot a^{\dagger a\dot\alpha}(-p) + a_{a\dot\alpha}(-p)\dot a^{a\dot\alpha}(p)\right)\\ &= \int dp\, \mathrm{i}a^\dagger_{a\dot\alpha}(p)\dot a^{a\dot\alpha}(p) - \int dp\ h_p^2\{a_{a\dot\alpha}(p), a^{\dagger a\dot\alpha}(p)\} + \int dp\, f_p h_p\left(a^\dagger_{a\dot\alpha}(p) a^{\dagger a\dot\alpha}(-p) - a_{a\dot\alpha}(-p)\dot a^{a\dot\alpha}(p)\right).\end{aligned}$$

Notice the last integral vanishes since the operators commute while $f_p h_p$ is odd in p. Subtracting from the kinetic term the Hamiltonian,

$$\mathbb{L}_\theta = \int dp\ \left(\mathrm{i}a^\dagger_{a\dot\alpha}(p)\dot a^{a\dot\alpha}(p) - \omega_p\, a^\dagger_{a\dot\alpha}(p)a^{a\dot\alpha}(p)\right) - \int dp\ \left[h^2 + \frac{1}{\omega_p}\left(h_p^2 - \kappa p f_p h_p\right)\right]\{a^{a\dot\alpha}(p), a^\dagger_{a\dot\alpha}(p)\}.$$

But the functional forms (B.63) of h_p and f_p imply

$$h_p^2 - \kappa p f_p h_p = \frac{p^2}{4f_p^2} - \frac{p^2}{2} = \frac{p^2}{4f_p^2}\left(1 - 2f_p^2\right) = -\frac{p^2}{4f_p^2}\omega_p = -h_p^2\omega_p$$

so that the second term cancels. Finally, we get the desired form for one fermionic part of the Lagrangian:

$$\mathbb{L}_\theta = \int dp\left(\mathrm{i}a^\dagger_{a\dot\alpha}(p)\dot a^{a\dot\alpha}(p) - \omega_p a^\dagger_{a\dot\alpha}(p)a^{a\dot\alpha}(p)\right). \tag{B.65}$$

Similarly,

$$\mathbb{L}_e ta = \int dp\left(\mathrm{i}a^\dagger_{\alpha\dot a}(p)\dot a^{\alpha\dot a}(p) - \omega_p a^\dagger_{\alpha\dot a}(p)a^{\alpha\dot a}(p)\right). \tag{B.66}$$

Compiling the bosonic and fermionic parts we find the full $\mathbb{L}_2$ (2.65).

To derive the total momentum (2.71), we compute similar terms to before. For bosons,

$$\begin{aligned}\int d\sigma\, P_{a\dot a}Y'^{a\dot a} &= \frac{\mathrm{i}}{2}\int d\sigma\int\frac{dp}{\sqrt{2\pi}}\int\frac{dp'}{\sqrt{2\pi}}\sqrt{\frac{\omega_{p'}}{\omega_p}}(\mathrm{i}p)\left(a^\dagger_{a\dot a}(p') - a_{a\dot a}(-p')\right)\left(a^{a\dot a}(p) + a^{\dagger a\dot a}(-p)\right)e^{-\mathrm{i}\sigma(p'-p)}\\ &= -\frac{1}{2}\int dp\ p\left(a^\dagger_{a\dot a}(p) - a_{a\dot a}(-p)\right)\left(a^{a\dot a}(p) + a^{\dagger a\dot a}(-p)\right)\\ &= -\frac{1}{2}\int dp\ p\left(a^\dagger_{a\dot a}(p)a^{a\dot a}(p) - a_{a\dot a}(-p)a^{a\dot a}(p) + a^\dagger_{a\dot a}(p)a^{\dagger a\dot a}(-p) - a_{a\dot a}(-p)\dot a^{\dagger a\dot a}(-p)\right)\\ &= -\frac{1}{2}\int dp\ p\left(a^\dagger_{a\dot a}(p)a^{a\dot a}(p) + a_{a\dot a}(p)a^{\dagger a\dot a}(p)\right) - \frac{1}{2}\int dp\ p\left(a^\dagger_{a\dot a}(p)a^{\dagger a\dot a}(-p) + a_{a\dot a}(p)a^{a\dot a}(-p)\right).\end{aligned}$$

The second integral vanishes because the operators with opposite arguments commute while p is obviously odd. So we are simply left with

$$\int d\sigma\, P_{a\dot a}Y'^{a\dot a} = -\int dp\ p\ a^\dagger_{a\dot a}(p)a^{a\dot a}(p) - \frac{1}{2}\int dp\ p\,[a_{a\dot a}(p), a^{\dagger a\dot a}(p)],$$

which features a divergence reflecting ordering ambiguity in quantisation. In the case of fermions,

$$\begin{aligned}\int d\sigma\, \mathrm{i}\theta^\dagger_{a\dot\alpha}\dot\theta^{a\dot\alpha} &= \int d\sigma\int\frac{dp}{\sqrt{2\pi}}\int\frac{dp'}{\sqrt{2\pi}}\frac{\mathrm{i}(\mathrm{i}p')}{\sqrt{\omega_p\omega_{p'}}}\left(f_p a^\dagger_{a\dot\alpha}(p) + h_{-p}a_{a\dot\alpha}(-p)\right)\left(f_{p'}a^{a\dot\alpha}(p') + h_{-p'}a^{\dagger a\dot\alpha}(-p')\right)e^{\mathrm{i}\sigma(p'-p)}\\ &= -\int dp\ \frac{p}{\omega_p}\left(f_p a^\dagger_{a\dot\alpha}(p) + h_{-p}a_{a\dot\alpha}(-p)\right)\left(f_p a^{a\dot\alpha}(p) + h_{-p}a^{\dagger a\dot\alpha}(-p)\right)\end{aligned}$$

$$= -\int dp\,\frac{p}{\omega_p}\left(f_p^2 a^\dagger_{a\dot\alpha}(p)a^{a\dot\alpha}(p) - h_p^2 a_{a\dot\alpha}(p)a^{\dagger a\dot\alpha}(p)\right) + \int dp\,\frac{pf_p h_{-p}}{\omega_p}\left(a^\dagger_{a\dot\alpha}(p)a^{\dagger a\dot\alpha}(-p) + a_{a\dot\alpha}(-p)a^{a\dot\alpha}(p)\right)$$

and once again the second integral vanishes (this time because the operators *anti*-commute and the prefactor is even). Finally, using $f_p^2 + h_p^2 = \omega_p$,

$$\int d\sigma\,\mathrm{i}\theta^\dagger_{a\dot\alpha}\dot\theta^{a\dot\alpha} = -\int dp\,p\;a^\dagger_{a\dot\alpha}(p)a^{a\dot\alpha}(p) + \int dp\;p\,h_p^2\{a_{a\dot\alpha}(p), a^{\dagger a\dot\alpha}(p)\}$$

where we can again ignore the divergent term. Similar expression holds for $Z^{\alpha\dot\alpha}$ and $\eta^{\alpha\dot a}$, yielding (2.71).

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