Practice Questions — Week 1

Please submit your solutions to the Blackboard Assignment for your tutorial group. It is important that your scripts are as legible as possible, so you might want to consider using a scanning app — e.g. Scannable, Cam Scanner.

This week we discussed functions, their meaning and how to find their natural domain and range. (This is covered in section 0.1 Anton et al)

Reminder The roots of the quadratic polynomial (a polynomial of degree two) $P(x) = ax^2 + bx + c$, where $a \neq 0$, are computed by the formula

$$x_\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The polynomial can then be written as a product of factors

$$P(x) = a(x - x_-)(x - x_+).$$

More generally, if x_1 is a root of a polynomial $Q(x)$ of degree n, then Q can be written as $Q(x) = (x - x_1)R(x)$, where $R(x)$ is a polynomial of degree $n - 1$.

The **natural domain** of a function f defined by a formula consists of all values of x for which $f(x)$ has a well defined real value. To find it, you might use the following algorithm:

1. Whenever you see an expression of type $\sqrt{g(x)}$ (or $\sqrt[2n]{g(x)}$ with $n \in \mathbb{N}$), determine all x for which $g(x) \geq 0$.
2. Whenever you see an expression of type $\frac{h(x)}{g(x)}$, determine all x for which $g(x) \neq 0$. Beware of nested fractions and of implicit fractions, like $\tan(x) = \frac{\sin(x)}{\cos(x)}$.
3. Determine the intersection of all the parts of $\mathbb{R}$ obtained in the previous points.

The **range** of a function f consists of all values $f(x)$ it assumes when x ranges over its domain. Domains and ranges can be described as unions of non-intersecting intervals and rays, e.g., $(-\infty, -4) \cup (-4, -1] \cup [0, 3]$ or by sets of the form $\{x : x \neq 0 \text{ and } x \neq 2\}$.

Problem 1

- (a) Find $f(-1.1)$, $f(0)$, $f(\sqrt{3})$, $f(2t)$ for the function

$$f(x) = \begin{cases} \frac{1}{x} & x \leq -1 \\ 2x^2 & x > -1 \end{cases}.$$

- (b) Check that $x = 1$ is a root of the polynomial $Q(x) = x^3 - 6x^2 + 9x - 4$. Divide $Q(x)$ by $x - 1$. Use this to factorise $Q(x)$, and to determine all x for which $Q(x) \geq 0$.

Problem 2

- (a) Consider the functions $f(x) = \frac{x^2+2x}{x}$ and $g(x) = \sqrt{(x+1)^2} + 1$. Show that these functions are different. Find a ray on which they are equal.
- (b) Find the range of the function $f(x) = \frac{2x^2+1}{x^2+1}$.