Diversified Risk Parity

Farshad Noravesh
email: bmmturbo@icloud.com
Python: github.com/farshad-finance
Java: github.com/farshad-nsh
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Dimension reduction can be very useful in portfolio optimization methods such as Markowitz which is based on minimum variance. As there are countless assets, the combination of assets to invest is very big and so the decision for the rate of asset allocation is quite challenging. Although contemporary dimension reduction methods are very strong and useful, but there are very few applications like [3] in portfolio optimization methods such as Markowitz method. They could use NMF method since prices are not negative, but their reduction is only on time since daily price series is huge and can be reduced to lower time dimension to capture long time dynamics.

Another perspective that can be useful for portfolio management is to hedge or immunize a portfolio against moves in the principal components. For example, suppose the goal is to hedge the value of a portfolio against movements in the first k principal components.

[4] used conditional value at risk as a risk measure and took advantage of risk parity paradigm and principal component analysis to form a factor model to get the entire return distribution.

It is very known that factor models play an important role in financial models. An example could be capital asset pricing model (CAPM):

$$r_i = r_f + \beta_i R_M + \varepsilon_i , E(\varepsilon_i) = 0$$

Or the general arbitrage pricing theory model (APT):

$$r_i = \alpha_i + \beta_{1i} F_1 + \ldots + \beta_{\Box} F_n + \varepsilon_i$$

which has the return of i-th risky asset as a linear combination of n risk factors including a constant risk term and an error term. The constant term is constant because it is riskless and not influenced by variable risk factors. The betas are called factor sensitivities and is a measure of how sensitive the risky asset is to the known factor. The alpha and betas can be estimated using ordinary least square method.
Sometimes these factors are heuristics such as interest rates or factors in Fama-French Model which are very tangible in economics. On the other hand factors can be obtained using PCA without the goal of having tangible explanation or description. The essence of these principal components will be well understood by introducing the paradigm of risk parity. Risk Parity (RP) is an asset allocation strategy that allocates the weights according to risk characteristics of asset classes. The idea is is to diminish the concentrated risk from one market regime by obtaining assets based on their respective amount of risk.
The most important approach to RP is Equal Risk Contribution Strategy (ERC). The methodology of [7] is reviewed here for the reader. The failure of mean-variance strategy due to estimation errors in the estimated mean lead to researchers and investors to use risk-based strategies that do not have to estimate the expected mean, but only the covariance structure. Therefore, strategies like ERC are generally accepted as robust in the literature due to their good performance over the 2008 financial

crisis period and many Hedge Fund rely on that for its maturity and rigorous theory. It is highly related to ideas of machine learning in terms of both dimension reduction and classification.

Variance is the most popular risk measure due to its computational simplicity and easy interpretation.

Given $\sigma^2(w)$ is the portfolio variance, then

$$\sigma^2(w) = w' \Sigma w$$

Definition Let w be vector of asset weights and σ(w) be the portfolio risk measure, then the marginal risk contribution of the i'th asset is the first derivative of the risk measure with respect to its weight ω_i such that:

$$MRC_i(w) = \frac{\partial \sigma(w)}{\partial \omega_i}$$

MRC gives an infinitesimal change in the whole portfolio risk caused by the ith component.

Definition Let σ(**w**) be the portfolio's risk measure. Then the risk contribution of the ith component, $RC_i(w)$ is:

$$RC_i(w) = \omega_i \, MRC_i(w)$$

So, the marginal and total risk contributions of the asset i become

$$MRC_i = \frac{(\Sigma w)_i}{\sqrt{w^T \Sigma w}}$$
$$TRC_i = \omega_i \frac{(\Sigma w)_i}{\sqrt{w^T \Sigma w}}$$

Suppose, there are n asset classes and set the risk budgets $(b_1, b_2, \ldots, b_n)$ and targeted risk contributions are $(TRC_1, TRC_2, \ldots, TRC_n)$ to general risk measure R.

Then the risk budgeting portfolio is given as;

$$\begin{aligned} TRC_1(\omega_1, \omega_2, \ldots, \omega_n) &= b_1 \\ TRC_2(\omega_1, \omega_2, \ldots, \omega_n) &= b_2 \\ &\ldots \\ TRC_n(\omega_1, \omega_2, \ldots, \omega_n) &= b_n \end{aligned}$$

Thus, the optimization problem is

$$w_{RB} = argmin \sum_{i=1}^{n} \left(\omega_i \frac{\partial R(w)}{\partial \omega_i} - b_i R(w) \right)^2 = argmin \sum_{i=1}^{n} \left(\omega_i (\Sigma w)_i - b_i w^T \Sigma w \right)^2$$

subject to the following constraints:

$$\sum_{i=1}^{n} \omega_i = 1, \sum_{i=1}^{n} b_i = 1, \omega_i, b_i \geqslant 0$$

In order to explain the relation between risk parity and dimension reduction methods such as PCA, a general information about diversified risk parity (DRP) based on the works by [8] is reviewed. This is a special case of risk parity (RP) with employing uncorrelated portfolios as risk sources. Applying risk parity strategy to uncorrelated risk sources and maximizing the number of risk sources in a portfolio is known as “diversified risk parity strategy”. The overlap of correlations between asset classes lead to poor diversification of RP strategy. Specifically, during the financial crisis, correlations increase significantly exceeding 90%. [9] uses the principal component analysis (PCA) to generate uncorrelated portfolios that are also called as uncorrelated risk sources. Contrary to the RP strategy based on asset classes in the previous part, focus is now on the RP approach that aims diversification based on the main risk sources(risk factors) driving the asset returns.

[9] uses PCA to construct the uncorrelated portfolios, called principal portfolios which are defined below:

Definition Principal Portfolios (PP). Let Σ be an $n \times n$ covariance matrix. By Applying principal component decomposition to Σ, the relation $E^T \Sigma E = diag(\lambda_1, \lambda_2, \ldots, \lambda_n) = \Lambda$is obtained which is equivalent to$E^{-T} \Lambda E^{-1} = diag(\lambda_1, \lambda_2, \ldots, \lambda_n) = \Sigma$. The columns of E are called principal portfolios.

Definition Principal portfolio weights (PPW). Let ω be an $n \times 1$ weight vector of original portfolio, and E is an eigenvector matrix of covariance matrix Σ of original data. Then unique vectors $\widetilde{\omega}_{PP}$satisfying: $\omega = E \widetilde{\omega}_{PP}$are called principal portfolio weights.

Using simple math, it is possible to show that the return, variance of i[th] principal portfolio, and the total variance of original portfolio is as follows:

$$\tilde{r}_{PP,i} = e_i^T R$$

$$\sigma^2(\tilde{r}_{PP,i}) = e_i^T \Sigma e_i = \lambda_i$$

$$\sigma^2(R) = \sigma^2(\widetilde{R}_{pp}) = tr(\Sigma) = \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} \sigma^2(\tilde{r}_{pp,i})$$

where$R = (r_1, r_2, \ldots, r_n)$is the return of the original portfolio.

Thus, the risk contribution of each principal portfolio to total variance can be written as

$$\frac{\sigma^2(\tilde{r}_{pp,i})}{\sigma^2(R)} = \frac{\lambda_i}{\lambda_1 + \lambda_2 + \ldots + \lambda_n}$$

Thus, the marginal risk contribution of each principal portfolio and, the risk contribution of each principal portfolio are written as follows respectively:

$$MRC_{PP} = \frac{\partial \sigma\left(\widetilde{R}_{PP}\right)}{\partial \omega_i} = \frac{1}{2\sqrt{\sum_{i=1}^{n} \widetilde{\omega}_{PP,i}^2 \lambda_i}} 2\widetilde{\omega}_{PP,i}\lambda_i = \frac{\widetilde{\omega}_{PP,i}\lambda_i}{\sigma\left(\widetilde{R}_{PP}\right)}$$

$$\widetilde{RC}_{PP,i} = \frac{1}{\sqrt{\sum_{i=1}^{n} \widetilde{\omega}_{PP,i}^2 \lambda_i}} \widetilde{\omega}_{PP,i}^2 \lambda_i = \frac{\widetilde{\omega}_{PP,i}^2 \lambda_i}{\sigma\left(\widetilde{R}_{PP}\right)}$$

Definition Let p be a discrete probability function on given set $z_1, z_2, \ldots, z_n$with $p_i = p\left(z_i\right)$the entropy of p is given as

$$H = -\sum_{i=1}^{n} p_i \log p_i$$

Definition exponential entropy of the diversification distribution is defined as

$$N_{PP,Ent} = \exp\left(-\sum_{i=1}^{n} p_{PP,i} \log p_{PP,i}\right)$$

Applying risk parity strategy to uncorrelated portfolios and distribute the portfolio risk among uncorrelated risk sources, a diversified risk parity is obtained by allocating the risk among these risk sources uniformly and have maximum risk sources. Thus, exponential of Shannon entropy should reach its maximum value.

Using the analogy explained so far and ideas in [8], the following diversification distribution p is given.

$$p_{PP,i} = \frac{1}{\sqrt{\sum_{i=1}^{n} \widetilde{\omega}_{PP,i}^2 \lambda_i}} \widetilde{\omega}_{PP,i}^2 \lambda_i \quad i = 1, 2, \ldots, n$$

Definition (Diversified risk parity)**.** If the diversification distribution is close to uniform, the strategy is called diversified risk parity.

Diversified risk parity portfolio is solution of the following optimization problem

$$argmax\, N_{PP,Ent}$$

subject to

$$\sum_{i=1}^{n} \omega_{PP,ii} = 1$$