Some Connections between Partially Asymmetric Exclusion Process and Permutation Tableaux

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Abstract

This project aims to study the connection between the Partially Asymmetric Exclusion Process (PASEP) and some combinatorial structures (Permutation Tableaux-PT). The PASEP is a foundational model for interacting particles in statistical mechanics. Notably, these particles exhibit bidirectional motion, wherein discrete probabilities control transitions in both left and right directions. A Markov chain (PT chain) on the permutation tableaux was introduced by Corteel and Williams in 2012. The PT chain can project to the PASEP model and gives a novel method to compute the stationary distribution of the PASEP. Our primary focus lies in studying the proof of the stationary distribution of the PT chain and how it projects to the PASEP.

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1 Historical Background

The asymmetric simple exclusion process (ASEP) is a well-known interacting particle system introduced by Frank Spitzer in 1970 in probability theory. Since its introduction, numerous articles have been published in physics and mathematics literature, solidifying its status as a fundamental stochastic model for studying transportation phenomena.

The ASEP is described by a continuous-time Markov process on the state space $S = \{0, 1\}^{\mathbb{Z}}$, where 1s represent particles and 0s represent holes. The process has positive parameters p and q, with the constraint $p + q = 1$. Each particle waits for an exponentially distributed random time with a mean one and then attempts to jump one site to the right with probability p or one site to the left with probability q. The jump can only occur if the target site is unoccupied; otherwise, the particle waits for another exponentially distributed time. Importantly, all particles behave independently of each other.

In addition to the ASEP, the partially asymmetric simple exclusion process (PASEP) has been extensively studied by researchers to understand various transitions and phenomena. The PASEP exhibits boundary-induced phase transitions, spontaneous symmetry breaking, and phase separation. It is considered a fundamental model for various processes such as biopolymerization, traffic flow, and formation of shocks, and it also appears in sequence alignment problems in computational biology. Hence, we will study the permutation tableaux that project to the PASEP to understand its properties and mechanisms.

2 Markov Chains

During the early 20th century, Andrey Markov conducted an in-depth study of Markov processes and published his groundbreaking work in 1906. However, Markov processes in continuous time, such as the Poisson process, were already known before his contributions. Markov's interest lay in exploring an extension of independent random sequences, which was prompted by a disagreement with Pavel Nekrasov. Nekrasov argued that independence was necessary for the weak law of large numbers to hold. However, Markov's initial paper on Markov chains, published in 1906, revealed that, under certain conditions, the average outcomes of the Markov chain would converge to a fixed vector of values. This finding established a weak law of large numbers without relying on the independence assumption, which had been commonly believed to be essential for such mathematical laws to hold.

A Markov chain is a mathematical system that transitions from one state to another based on probabilistic rules. The defining characteristic of a Markov chain is that the future states are solely determined by the current state and time elapsed, regardless of how the process reached its present state. In other words, the probability of transitioning to a particular state depends only on the current state. These chains can represent various systems with diverse state spaces, such as letters, numbers, weather conditions, baseball scores, or stock performances.

Mathematically, Markov chains can be modeled using finite state machines, and they find extensive applications in various fields, including statistics, information theory, economics, game theory, queueing theory, genetics, and finance. While Markov chains can be discussed with any size of state space, initial theory and most applications focus on cases with a finite or countably infinite number of states. The versatility and usefulness of Markov chains, especially in the context of random walks, make them a prolific tool in mathematical analysis and practical problem-solving.

2.1 Markov Property

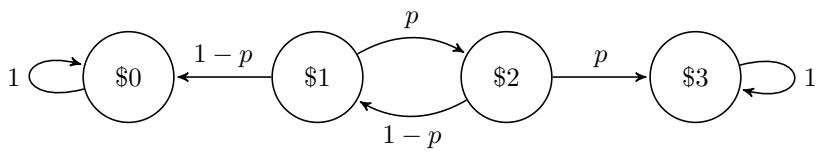
Let us examine the definition of a Markov chain, often referred to as the Markov property, and is of utmost importance. A stochastic process $\{X_n, n = 0, 1, 2, \ldots\}$, with a finite or countable state space, is said to be a Markov chain if for all states $i_0, i_1, \ldots, i_{n-1}, i, j$, and all $n \geq 0$,

$$P\{X_{n+1} = j \mid X_0 = i_0, X_1 = i_1, \ldots, X_{n-1} = i_{n-1}, X_n = i\} = P\{X_{n+1} = j \mid X_n = i\}.$$

Let us remember the famous Gambler's Ruin problem. Imagine Allan and Beth, two friends, decide to play a series of independent games, each wagering \$1. Allan starts the game with \$2 and Beth with \$1. A fair coin flip determines the outcome of each game. If the coin comes up heads, Allan wins \$1 from Beth, and if it comes up tails, Beth wins \$1 from Allan. Remember, there are no ties allowed in this game.

Let us denote the probability of Allan winning \$1 as p. Since the coin is fair, the probability of Beth winning \$1 is $1 - p$. The game continues until one of the players goes broke (has \$0). To analyze their progress, we define the random variable X_n, which represents the amount of money Allan has after playing n games. The initial state is $X_0 = \$2$ since Allan starts with \$2. We can observe Allan's holdings after each game as $X_0, X_1, X_2, \ldots$, forming a chain of states. Since our probabilities $P\{X_2 = \$2 \mid X_1 = \$1, X_0 = \$2\}$ is actually equal to $P\{X_2 = \$2 \mid X_1 = \$1\}$ where it satisfies the Markov property.

The state space for X_n consists of the possible dollar amounts Allan can have after n games: $\{0, 1, 2, 3\}$. Therefore, our Markov chain is a finite-state chain. Now, we can draw it as a Markov diagram as follows.



Consequently, finite-state chains are represented diagrammatically as shown. Additionally, we will examine the Partially Asymmetric Exclusion Process (PASEP) as a finite-state chain and illustrate its corresponding diagrams in a similar manner.

The conditional probabilities specified in the context of the Gambler's Ruin problem (p, q) are referred to as the one-step transition probabilities of the chain, or simply the transition probabilities. It is crucial to acknowledge that these probabilities are conditional, as they define the likelihood of the next state member, denoted by X_{n+1}, being in a particular state, given the current state X_n of the chain.

2.2 Transition Matrix of Markov Chains

As you may recall, the one-step transition probabilities of the chain play a vital role in our analysis. It is imperative to recognize that these probabilities are conditional, as they characterize the likelihood of the next state member, denoted by X_{n+1}, assuming a specific state, given the current state X_n of the chain.

Now, let $X_0, X_1, X_2, \ldots$ be a finite-state, time-homogeneous Markov chain, and let the states of the chain be indexed by the non-negative integers $1, 2, \ldots, s$. The (one-step) transition matrix of the Markov chain is the $s \times s$ matrix P whose (i, j)th entry is given by

$$p_{ij} = P\{i \to j\} = P\{X_{n+1} = j \mid X_n = i\}$$

for $i = 1, \ldots, s$ and $j = 1, \ldots, s$.

As a reminder, let us revisit the Gambler's Ruin problem, and its transition matrix can be represented as follows:

$$P = \begin{bmatrix} P\{0 \to 0\} & P\{0 \to 1\} & P\{0 \to 2\} & P\{0 \to 3\} \\ P\{1 \to 0\} & P\{1 \to 1\} & P\{1 \to 2\} & P\{1 \to 3\} \\ P\{2 \to 0\} & P\{2 \to 1\} & P\{2 \to 2\} & P\{2 \to 3\} \\ P\{3 \to 0\} & P\{3 \to 1\} & P\{3 \to 2\} & P\{3 \to 3\} \end{bmatrix}$$

And if we represent all the transition probabilities in matrix form:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1-p & 0 & p & 0 \\ 0 & 1-p & 0 & p \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Understanding the motivation behind representing a Markov chain in matrix form may appear ambiguous. However, this matrix representation allows us to conveniently and compactly capture the transition probabilities between states. By organizing the probabilities in a matrix, we gain a systematic way of analyzing the dynamics of the Markov chain over multiple time steps. The matrix elements correspond to the probabilities of transitioning from one state to another after a certain number of steps (given by the row and column indices).

Let $X_0, X_1, X_2, \ldots$ be a time-homogeneous Markov chain. For any positive integer k, the k-step transition probabilities are defined by

$$P_{ij}^{(k)} = P\{X_{n+k} = j \mid X_n = i\}$$

where i and j range across the states of the chain (typically $1, \ldots, s$). For $k = 1$, we will typically revert to the previous notation: $P_{ij}^{(1)} = P\{i \to j\}$.

Let's calculate the probability $P\{X_{n+2} = \$2 \mid X_n = \$2\}$ in the Gambler's Ruin problem, using the Markov property, we can express the right-hand side as:

$$\begin{aligned} P\{X_2 = \$2 \mid X_0 = \$2\} = P\{X_{n+1} = \$1 \mid X_n = \$2\} \cdot P\{X_{n+2} = \$2 \mid X_n = \$2, X_{n+1} = \$1\} \\ + P\{X_{n+1} = \$2 \mid X_n = \$2\} \cdot P\{X_{n+2} = \$2 \mid X_n = \$2, X_{n+1} = \$2\} \\ + P\{X_{n+1} = \$3 \mid X_n = \$2\} \cdot P\{X_{n+2} = \$2 \mid X_n = \$2, X_{n+1} = \$3\} \end{aligned}$$

By the Markov property, we can further simplify the expression as:

$$\begin{aligned} P\{X_2 = \$2 \mid X_0 = \$2\} &= P\{X_{n+1} = \$1 \mid X_n = \$2\} \cdot P\{X_{n+2} = \$2 \mid X_{n+1} = \$1\} \\ &\quad + P\{X_{n+1} = \$1 \mid X_n = \$2\} \cdot P\{X_{n+2} = \$2 \mid X_{n+1} = \$1\} \\ &\quad + P\{X_{n+1} = \$1 \mid X_n = \$2\} \cdot P\{X_{n+2} = \$2 \mid X_{n+1} = \$1\} \\ &= (1-p) \cdot p + 0 \cdot 0 + 1 \cdot 0 \\ &= (1-p) \cdot p \end{aligned}$$

Hence, $P\{X_2 = \$2 \mid X_0 = \$2\} = (1-p)p$.

As we can see, it will be challenging to compute P^n directly for larger n. Therefore, we resort to using the transition matrix to simplify the process. The Chapman-Kolmogorov equation provides an efficient method to calculate the transition probabilities for any positive integers m and n and any states i and j. The equation is given by:

$$P_{ij}^{(m+n)} = \sum_{k=1}^{s} P_{ik}^{(m)} \cdot P_{kj}^{(n)}$$

Now let us prove the Chapman-Kolmogorov equation. We start by considering the conditional probability on the left-hand side:

$$P_{ij}(n+m) = \Pr(X_{m+n} = j \mid X_0 = i) = \Pr\left(\bigcup_{k \in S} [X_n + m = j, X_n = k] \mid X_0 = i\right)$$

Next, we use the definition of countable additivity to express the above probability as a sum:

$$= \sum_{k \in S} \Pr(X_{n+m} = j, X_n = k \mid X_0 = i) \quad \text{(Definition of Countable Additivity)}$$

Then, applying the chain rule for probability, we can split the joint probability in the sum:

$$= \sum_{k \in S} \Pr(X_{n+m} = j \mid X_n = k, X_0 = i) \times \Pr(X_n = k \mid X_0 = i) \quad \text{(Chain Rule for Probability)}$$

Since the Markov property holds for the given Markov chain, we can further simplify the expression:

$$= \sum_{k \in S} \Pr(X_{n+m} = j \mid X_n = k) \times \Pr(X_n = k \mid X_0 = i) \quad \text{(Markov Property)}$$

Finally, by the definition of transition probabilities for a homogeneous Markov chain, we have:

$$= \sum_{k \in S} p_{ik}(n) \times p_{kj}(m) \quad \text{(Transition Probabilities of Homogeneous Markov Chain)}$$

Thus, we have successfully derived the Chapman-Kolmogorov equation.

Corollary:

Let P be the one-step transition matrix of a Markov chain. Then, the k-step transition probabilities are the entries of the matrix P^k. In other words,

$$P_{ij}^{(k)} = P\{X_k = j \mid X_0 = i\}$$

where $P_{ij}^{(k)}$ represents the (i, j)th entry of P^k.

After deriving the Chapman-Kolmogorov equation, we can utilize it to conveniently ascertain alterations in the transition matrix. Leveraging the transition matrix enables us to overcome the computational complexities that arise when attempting direct calculations for larger values of n. Specifically, we can observe that $P\{X_{n+2} = \$2 \mid X_n = \$2\}$ can be determined by simply examining the entry at the intersection of the third row and third column of the transition matrix which is equal to $(1 - p)p$.

$$P^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1-p & (1-p)p & 0 & p^2 \\ (1-p)^2 & 0 & (1-p)p & p \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2.3 Initial Distribution of Markov Chains

Up to this point, our analysis still needs to address the initial distribution. The initial distribution bears significant importance in the context of Markov chains, serving as a fundamental pillar for forecasting future states and elucidating the long-term behavior of the chain. By explicitly defining the probabilities associated with distinct initial states, one obtains valuable insights into the subsequent evolution of the system and its likelihood of transitioning to diverse states over time. The initial distribution sets the trajectory for the Markov chain's evolution.

It is a vital determinant of its steady-state behavior, which we will examine later, particularly when analyzing the chain's convergence to a stationary distribution. Thus, the reasonable consideration and accurate specification of the initial distribution are critical for understanding and effectively modeling various stochastic processes, making it an indispensable aspect of Markov chain analysis.

Example 1: Consider a coin-flipping game modeled by a two-state Markov chain, where state 1 represents "heads" and state 2 represents "tails." Let the initial distribution be denoted as $\mathbf{v_0} = [0.6, 0.4]$, representing the probabilities of starting the game with "heads" and "tails," respectively.

Upon flipping the coin once, we seek to compute the probabilities of obtaining "heads" and "tails" after the first flip. To accomplish this, we perform a matrix multiplication between the initial distribution vector $\mathbf{v_0}$ and the given transition matrix $\mathbf{P}$.

After the first flip, the updated probabilities, denoted as $\mathbf{v_1}$, may be expressed as

$$\mathbf{v_1} = \mathbf{v_0}\mathbf{P} = [0.6, 0.4] \begin{bmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{bmatrix} = [0.6 \times 0.7 + 0.4 \times 0.2, 0.6 \times 0.3 + 0.4 \times 0.8] = [0.58, 0.42]$$

.

Thus, following the first coin flip, there is now a 58% chance of obtaining "heads" and a 42% chance of obtaining "tails."

Theorem 1: Let $X_0, X_1, \ldots, X_n, \ldots$ be a Markov chain with state space $\{1, \ldots, s\}$ and one-step transition matrix P. Let $\mathbf{v_0} = [v_{01}, v_{02}, \ldots, v_{0s}]$ be a $1 \times s$ vector specifying the initial distribution of the chain, i.e., $v_{0k} = P(X_0 = k)$ for $k = 1, \ldots, s$. If $\mathbf{v_1}$ denotes the vector of marginal (i.e., unconditional) probabilities associated with X_1, then

$$\mathbf{v_1} = \mathbf{v_0}\mathbf{P}$$

More generally, if $\mathbf{v_n}$ denotes the $1 \times s$ vector of marginal probabilities for X_n, then

$$\mathbf{v_n} = \mathbf{v_0}\mathbf{P^n}$$

Let us prove this, The formula $\mathbf{v_1} = \mathbf{v_0}\mathbf{P}$ can be established. Now consider $\mathbf{v_2}$, the vector of unconditional probabilities for X_2. We have

$$\mathbf{v_2} = \mathbf{v_1}\mathbf{P}$$

The substitution $\mathbf{v_1} = \mathbf{v_0}\mathbf{P}$ then yields $\mathbf{v_2} = (\mathbf{v_0}\mathbf{P})\mathbf{P} = \mathbf{v_0}\mathbf{P}^2$. Continuing by induction, we have for general n that $\mathbf{v_n} = \mathbf{v_{n-1}}\mathbf{P} = (\mathbf{v_0}\mathbf{P}^{n-1})\mathbf{P} = \mathbf{v_0}\mathbf{P}^n$, as claimed. ■

2.4 Regular Chains

2.4.1 Regular Markov Chains

A finite-state Markov chain with one-step transition matrix P is said to be a regular chain if there exists a positive integer n such that all of the entries of the matrix P^n are positive. In other words, for a regular Markov chain, there is some positive integer n such that every state can be reached from every state (including itself) in exactly n steps.

Example 2: Consider a Markov chain that models the weather in a city with three states: (1) sunny, (2) cloudy, and (3) rainy. The one-step transition matrix P for this Markov chain is as follows:

$$P = \begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.4 & 0.4 & 0.2 \\ 0.2 & 0.3 & 0.5 \end{bmatrix}$$

Notice that every entry of P is positive, and each row sums up to 1, indicating a regular Markov chain. Now, let us consider the two-step transition matrix P^2:

$$P^2 = \begin{bmatrix} 0.5 & 0.33 & 0.17 \\ 0.46 & 0.34 & 0.20 \\ 0.38 & 0.33 & 0.29 \end{bmatrix}$$

Again, every entry of P^2 is positive, confirming that the Markov chain is regular.

Since this Markov chain is regular, it means that from any weather state, the city's weather can transition to any other weather state in precisely two steps.

2.4.2 Irreducible Markov Chains

An irreducible Markov chain is closely related to the concept of a regular Markov chain. A regular Markov chain is a type of Markov chain where all states are positive recurrent, and there exists a positive integer d such that any state can return to itself in exactly d steps with positive probability. In other words, a regular Markov chain has a well-defined periodicity where the chain eventually enters a cycle of length d and continues to revisit that cycle.

The connection between irreducible and regular Markov chains lies in that an irreducible Markov chain can be seen as a particular case of a regular Markov chain with periodicity $d = 1$. In an irreducible Markov chain, every state is reachable from every other state, which implies that there are no separate cycles or disjoint sets of states. As a result, the periodicity of an irreducible Markov chain is trivial $d = 1$, and thus, it is regular.

In summary, while an irreducible Markov chain is a specific property indicating that all states are interconnected, a regular Markov chain is a broader concept encompassing both positive recurrent states and periodic behavior. Irreducible Markov chains can be seen as a subset of regular Markov chains with periodicity $d = 1$".

2.4.3 Steady States of Regular Chains

What's so special about regular Markov chains? The transition matrices of regular Markov chains exhibit a rather interesting property. Let's consider a Markov chain that models the behavior of customers in a supermarket with three states: (1) browsing, (2) shopping, and (3) checkout. The one-step transition matrix P for this Markov chain is given by:

$$P = \begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.2 & 0.7 & 0.1 \\ 0.1 & 0.2 & 0.7 \end{bmatrix}$$

Raising P to the 100th power yields P^{100}:

$$P^{100} = \begin{bmatrix} 0.2 & 0.6 & 0.2 \\ 0.2 & 0.6 & 0.2 \\ 0.2 & 0.6 & 0.2 \end{bmatrix}$$

As we can observe, every row of P^{100} is identical: approximately $[0.2, 0.6, 0.2]$. Raising P to even higher powers will also result in the same matrix up to several decimal places. That is, P^{101}, P^{102}, and so on, will be roughly equal to P^{100}.

This property of regular Markov chains holds true for many other examples as well, and it is a fundamental result in the theory of Markov chains known as the Steady-State Theorem.

Theorem 2: Let P be the one-step transition matrix of a finite-state, regular Markov chain. Then, the matrix limit $\Pi = \lim_{n\to\infty} P^n$ exists. Furthermore, the rows of the limiting matrix Π are identical, and all its entries are positive.

2.4.4 Computations on Regular Chains

Theorem: Let P be the one-step transition matrix of a regular Markov chain on the state space $\{1, \ldots, s\}$. The steady-state distribution of the Markov chain is the unique solution $\pi = [\pi_1, \ldots, \pi_s]$ to the system of equations formed by $\pi P = \pi$ and $\pi_1 + \ldots + \pi_s = 1$.

Let w be any $1 \times s$ vector satisfying the two conditions $wP = w$ and $\sum w_i = 1$. Similar to earlier derivations, we have $wP^2 = (wP)P = wP = w$, and by induction, $wP^n = w$ for any positive integer n. Taking the limit of both sides as $n \to \infty$, the Steady-State Theorem implies that $w\pi = w$.

Now expand $w\pi$:

$$w\pi = w \begin{bmatrix} \pi_1 \\ \pi_2 \\ \vdots \\ \pi_s \end{bmatrix} = \begin{bmatrix} w_1\pi_1 + w_2\pi_1 + \ldots + w_s\pi_1 \\ w_1\pi_2 + w_2\pi_2 + \ldots + w_s\pi_2 \\ \vdots \\ w_1\pi_s + w_2\pi_s + \ldots + w_s\pi_s \end{bmatrix} = \begin{bmatrix} \sum w_i\pi_1 \\ \sum w_i\pi_2 \\ \vdots \\ \sum w_i\pi_s \end{bmatrix} = \begin{bmatrix} \sum w_i\pi_1 \\ \sum w_i\pi_2 \\ \vdots \\ \sum w_i\pi_s \end{bmatrix} = \begin{bmatrix} \sum w_i \\ \sum w_i \\ \vdots \\ \sum w_i \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = w$$

Since $\sum w_i = 1$ by assumption, we have $w\pi = \pi$. It was established above that $w\pi = w$, and so we conclude that $w = \pi$, as originally claimed.

Example 3: Consider the Markov chain model for snowy days (S) and non-snowy or "green" days (G) in New York City. The one-step transition matrix is given by:

$$P = \begin{bmatrix} 0.964 & 0.036 \\ 0.224 & 0.776 \end{bmatrix}$$

Since all the entries of P are positive, this is a regular Markov chain. We can determine the steady-state probabilities $\pi = [\pi_1, \pi_2]$ using the equations in theorem, which are written out long-hand as follows:

$$0.964\pi_1 + 0.224\pi_2 = \pi_1$$
$$0.036\pi_1 + 0.776\pi_2 = \pi_2$$
$$\pi_1 + \pi_2 = 1$$

Substituting $\pi_2 = 1 - \pi_1$ into the first equation gives $0.964\pi_1 + 0.224(1 - \pi_1) = \pi_1$. Solving for π_1 produces $\pi_1 = 0.224/0.260 \approx 0.8615$, and then $\pi_2 = 1 - 0.8615 \approx 0.1385$. For the season to which this model applies, in the long run, New York City has at least 50 mm of snow on approximately 86.15

It's essential to note that the top two equations alone, i.e., those provided by the relationship $\pi P = \pi$, do not uniquely determine the value of the vector π. The first equation is equivalent to $0.224\pi_2 = 0.036\pi_1$ (subtract $0.964\pi_1$ from both sides), but so is the second equation (subtract $0.776\pi_2$ from both sides). The final equation, requiring the entries of π to sum to 1, is necessary to obtain a unique solution.

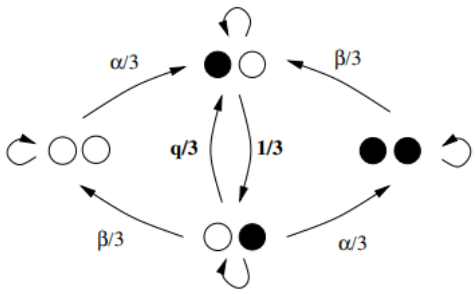
3 Partially Asymmetric Exclusion Process

3.1 Introduction

Finally, we consider a one-dimensional lattice of N sites, where each site i $(1 \leq i \leq N)$ is either occupied by a particle $(\tau_i = 1)$ or is empty $(\tau_i = 0)$. At most one particle can occupy a given site. During each infinitesimal time interval dt, each particle in the system has a probability dt of jumping to the next site on its right (for particles on sites $1 \leq i \leq N - 1$), and a probability qdt of jumping to the next site on its left (for particles on sites $2 \leq i \leq N$). Additionally, a particle is added at site $i = 1$ with probability αdt if site 1 is empty, and a particle is removed from site N with probability βdt if this site is occupied.

Briefly, the Partially Asymmetric Simple Exclusion Process (PASEP) outlines the movement of particles along a one-dimensional chain with N sites, where each site can be empty or occupied by a particle. In small time steps, particles have probabilities of hopping between neighboring sites or being added/removed at the chain's ends. This dynamic behavior attaches to distinct transition probabilities tending movement in a particular direction. The process is concisely defined through these rules, encompassing particle hopping, entry, and exit. PASEP's analysis employs fundamental weights to establish a stationary distribution, providing valuable insights into particle interactions and their patterns across diverse scenarios.

Example 4: The state diagram of the Partially Asymmetric Exclusion Process (PASEP) for N = 2



Definition: Let B_N be the set of all $2N$ words in the language $\{\bullet, \circ\}^*$. The Partially Asymmetric Simple Exclusion Process (PASEP) is defined as a Markov chain on B_N with the following transition probabilities:

1. If $X = A \bullet \circ B$ and $Y = A \circ \bullet B$, then $P(X, Y) = \frac{1}{N+1}$ (particle hops right), and $P(Y, X) = \frac{q}{N+1}$ (particle hops left).
2. If $X = \circ B$ and $Y = \bullet B$, then $P(X, Y) = \frac{\alpha}{N+1}$ (particle enters from the left).
3. If $X = B\bullet$ and $Y = B\circ$, then $P(X, Y) = \frac{\beta}{N+1}$ (particle exits to the right).
4. For any other cases, $P(X, Y) = 0$ if $Y \neq X$, and $P(X, X) = 1 - \sum_{Y \neq X} P(X, Y)$.

These transition probabilities define the dynamics of the PASEP on the set of $2N$ words B_N.

In conclusion, the remarkable nature of PASEP lies in its Markovian character, efficiently capturing the intricate motion of particles within a one-dimensional chain. This characteristic has profound implications for comprehending particle behavior across diverse scenarios.

3.2 Calculations on PASEP

First, let us explore the significance of establishing the irreducibility of the PASEP model. The essence of proving irreducibility for the PASEP lies in demonstrating that every state can be reached from any other state with a positive probability. Upon close examination of the specified transition probabilities governing the PASEP, the following observations can be made:

1. Particle transitions involving movements to the left and right (cases 1 and 2 as defined) allow particles to traverse the lattice seamlessly. This characteristic ensures the accessibility of any state from any other state.
2. Transitions involving particle entry from the left and particle exit to the right (cases 3 and 4 as defined) are essential in enhancing the lattice's connectivity. By enabling particles to move in both directions, these transitions further guarantee the reachability of all states.
3. The remaining cases (case 5 as defined) assume a crucial role in upholding the overall connectivity of the Markov chain. By preventing the formation of isolated states, these transitions maintain the linked nature of the chain.

In summation, the explicitly defined transition probabilities inherent to the PASEP model empower particles to navigate both leftward and rightward, in addition to entering and exiting the system. This inherent adaptability assures that every state can be accessed from any other state. As a result, the PASEP stands distinguished by its irreducibility—a fundamental characteristic that certifies its qualification as a legitimate and valid Markov chain.

Example 5: Now, consider the Partially Asymmetric Simple Exclusion Process (PASEP) with $N = 2$. We will analyze the irreducibility of the PASEP with this parameter setting, using the following transition matrices:

Transition matrix for P:

$$\begin{bmatrix} 1-\frac{\alpha}{3} & \frac{\alpha}{3} & 0 & 0 \\ 0 & 1-\frac{1}{3} & \frac{1}{3} & 0 \\ \frac{\beta}{3} & \frac{q}{3} & 1-\frac{\alpha+\beta+q}{3} & \frac{\alpha}{3} \\ 0 & \frac{\beta}{3} & 0 & 1-\frac{\beta}{3} \end{bmatrix}$$

Transition matrix for P^2:

$$\begin{bmatrix} \left(1-\frac{\alpha}{3}\right)^2 & \frac{\alpha\left(1-\frac{\alpha}{3}\right)}{3}+\frac{2\alpha}{9} & \frac{\alpha}{9} & 0 \\ \frac{\beta}{9} & \frac{q}{9}+\frac{4}{9} & -\frac{\alpha}{9}-\frac{\beta}{9}-\frac{q}{9}+\frac{5}{9} & \frac{\alpha}{9} \\ \frac{\beta\left(1-\frac{\alpha}{3}\right)}{3}+\frac{\beta\left(-\frac{\alpha}{3}-\frac{\beta}{3}-\frac{q}{3}+1\right)}{3} & \frac{2\alpha\beta}{9}+\frac{q\left(-\frac{\alpha}{3}-\frac{\beta}{3}-\frac{q}{3}+1\right)}{3}+\frac{2q}{9} & \frac{q}{9}+\left(-\frac{\alpha}{3}-\frac{\beta}{3}-\frac{q}{3}+1\right)^2 & \frac{\alpha\left(1-\frac{\beta}{3}\right)}{3}+\frac{\alpha\left(-\frac{\alpha}{3}-\frac{\beta}{3}-\frac{q}{3}+1\right)}{3} \\ 0 & \frac{\beta\left(1-\frac{\beta}{3}\right)}{3}+\frac{2\beta}{9} & \frac{\beta}{9} & \left(1-\frac{\beta}{3}\right)^2 \end{bmatrix}$$

These transition matrices provide insight into the dynamics of the PASEP with $N = 2$, shedding light on its irreducibility properties. After performing one more calculation, we can observe that all entries become greater than 0. This implies that the PASEP is a regular Markov chain.

4 Integer Partitions, Young Diagrams and Young Tableau

Young Tableau are pervasive combinatorial entities that play significant and motivational roles in representation theory, geometry, and algebra. Their inherent emergence is observed in examining symmetric functions, the representation theory of symmetric and complex general linear groups, Schubert's calculus of Grassmannians, and integer partition problems. It is highly connected to integer partition and symmetry groups. Unveiling and comprehending enumerative expressions for Young tableaux (and their extensions) constitute a fundamental focus in algebraic combinatorics, contributing to a deeper understanding of these structures and their applications.

4.1 Integer Partition

First, let us establish the concept of an integer partition formally. An **integer partition** of a positive integer n is denoted as $P(n)$ and defined as the set of all possible ways to represent n as a sum of positive integers, where the order of the summands does not matter. Mathematically, we can express $P(n)$ as follows:

$$P(n) = \left\{ \{a_1, a_2, \ldots, a_k\} \mid a_i \in \mathbb{Z}^+, \sum_{i=1}^{k} a_i = n \right\}$$

Here, $\mathbb{Z}^+$ denotes the set of positive integers. The elements of $P(n)$ are subsets of positive integers that add up to n. For example, the integer partition of 5, denoted as $P(5)$, includes the following sets:

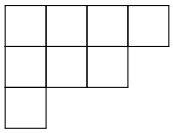
$$P(5) = \{\{5\}, \{4,1\}, \{3,2\}, \{3,1,1\}, \{2,2,1\}, \{2,1,1,1\}, \{1,1,1,1,1\}\}$$

These sets represent the various ways to partition integer 5 into sums of positive integers.

4.2 Young Diagrams

A Young diagram is a left-justified shape consisting of k rows of boxes, where k is a finite sequence of natural numbers. Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k \geq 0)$ be a partition of size $|\lambda| = \lambda_1 + \lambda_2 + \ldots + \lambda_k$, identified with the Young diagram. Here, n_i $(1 \leq i \leq k)$ represents the number of boxes in the ith row, and $n = \sum_{i=1}^{k} n_i$ is the total number of boxes in the Young diagram. The Young diagram is further defined by the number of rows, k, and the number of columns in the diagram, n_1, which is the number of boxes in the first column.

Young diagram for the partition (4, 3, 1):



Now we need to emphasize how we derive Young diagrams from integer partitions. The connection between symmetry groups and Young diagrams is established through Cayley's theorem, stating that any finite group G of order n is isomorphic to a subgroup of the symmetric group S_n. This enables us to represent any finite group as a permutation group, bridging abstract group theory with permutations. For each finite group G, there exists a subgroup of S_n where n is the order of G, representing G.

Cayley's Theorem: Every finite group G is isomorphic to a subgroup of the symmetric group S_n for some n.

The relationship between Young diagrams and symmetry groups, particularly S_n, is closely tied to Cayley's theorem, cycle structures of permutations, and the concept of conjugacy classes. A permutation's cycle structure in S_n refers to its decomposition into cycles of different lengths. For instance, a permutation like (1 2)(3 4 5) consists of two cycles, one of length 2 and the other of length 3.

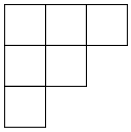
In S_n, conjugacy classes consist of permutations that are conjugate to each other. Two permutations π and ρ are conjugate if there exists a permutation g such that $\rho = g\pi g^{-1}$, indicating they share the same cycle structure.

The cycle structure of permutations in S_n corresponds to decreasing sequences of non-negative integers $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ with $\sum_{k=1}^{n} \lambda_k = n$, known as partitions of n. For example, the partition $(3, 2, 1)$ corresponds to the cycle structure (1 2 3)(4 5)(6).

These partitions are visually represented by Young diagrams, collections of boxes arranged in left-justified rows, where each row has the same or shorter length than the row above it. The rows of the Young diagram correspond to the integers in the partition.

The number of Young diagrams with n boxes is $n!$, matching the order of the symmetric group S_n. Each Young diagram uniquely represents a partition, and each partition corresponds to a distinct cycle structure of permutations in S_n.

Young diagram for the cycle (1 2 3)(4 5)(6):



5 Permutation Tableaux

5.1 Introduction

Finally, we introduce a new type of Young tableau on Young diagrams known as a permutation tableau. Let $T_{k,n}$ be a permutation tableau defined by a partition λ such that Y_λ is contained in a $k \times (n-k)$ rectangle. In other words, we arrange the boxes of the Young diagram Y_λ within this rectangular grid. Each box within Y_λ is filled with either 0 or 1, and we adhere to the following conditions to ensure a meaningful structure and interpretation:

1. Each column of the rectangular grid contains at least one 1. This ensures that there is at least one non-zero entry in every column, contributing to the tableau's significance.

2. Additionally, we enforce the constraint that no box containing 0 should have a 1 positioned directly above it within the same column, while simultaneously having a 1 to its immediate left within the same row. This rule prevents the formation of certain unwanted patterns that could undermine the tableau's distinctiveness and purpose.

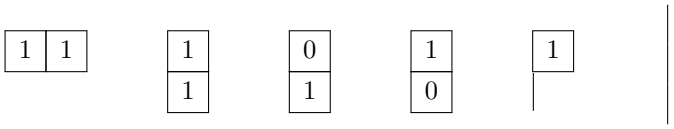
These non-negative partition conditions guarantee that the arrangement of 0's and 1's within the tableau is both well-defined and rich in combinatorial significance, allowing us to explore and analyze its properties more effectively.

Example 6: Consider the partition $\lambda = (4, 2, 1)$. We can construct a permutation tableau denoted as $T_{k,n}$ for this partition within a $k \times (n-k)$ rectangle. Let's fill the boxes of the Young tableau Y_λ with 0's and 1's while ensuring the fulfillment of the following conditions:

0	1	1	1
1	0		
0			

We shall now proceed to establish the notion of the half-perimeter of a partition denoted by λ or equivalently, that of its corresponding Young diagram Y_λ. The half-perimeter is defined as the sum of the number of rows and the number of columns present in the diagram. Additionally, within the context of the Young diagram Y_λ, we define the length of a row or column as the count of boxes contained within that specific row or column. It is pertinent to acknowledge that we permit a row to possess a length of 0 if it happens to be devoid of any boxes.

Example 7: Permutation tableaux with a half perimeter of 3:



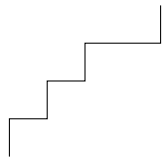
5.2 Lattice Paths in Permutation Tableaux

In the present context, Young diagrams have been established as a fundamental concept. Specifically, we commonly establish a correspondence between a Young diagram Y_λ, characterized by a half-perimeter t, and a lattice path $p(\lambda)$ of length t. This lattice path $p(\lambda)$ traverses in a sequence of unit steps south and west, originating from the north-eastern corner of Y_λ and terminating at its south-western corner. Notably, this lattice path invariably commences with a southward step.

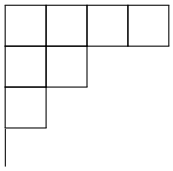
Subsequently, consider an arbitrary binary sequence $\tau \in \{0,1\}^N$. In this context, we define the Young diagram $\lambda(\tau)$ associated with a half-perimeter $N+1$. This construction is elucidated as follows: We initiate the creation of a path $p = (p_1, ..., p_{N+1}) \in \{S, W\}^{N+1}$, where $p_1 = S$, and $p_{i+1} = S$ if and only if $\tau_i = 1$. Consequently, $\lambda(\tau)$ is identified as the partition that corresponds to the path p. Remarkably, this mapping establishes a one-to-one correspondence between the set of Young diagrams with a half-perimeter of $N+1$ and the collection of N-tuples in $\{0,1\}^N$. Analogously, we denote the inverse mapping as follows: For a given Young diagram Y_λ characterized by a half-perimeter of $N+1$, we designate $\tau(\lambda)$ as the corresponding N-tuple.

To enhance clarity, we present illustrative representations for a specific partition $\lambda = (4, 2, 1, 0)$:

Lattice Path $p(\lambda)$:



Young Diagram Y_λ:



5.3 Some Statistics on Permutation Tableux

Let us establish several key statistics regarding permutation tableaux. A 1 in a tableau T is designated as "topmost" if it has exclusively 0's situated above it. If a 1 within T is not the uppermost entry, it is referred to as "superfluous." Furthermore, a 1 is termed "necessary" if it is the solitary 1 within its column. The rank $\mathrm{rk}(T)$ of a permutation tableau T is defined as the count of superfluous 1's. (Consequently, $\mathrm{rk}(T)$ is the total number of 1's in the filling minus the number of columns.)

We introduce the function $f(T)$ to denote the count of 1's found in the initial row of T. A zero within a permutation tableau is deemed "restricted" if there exists a 1 positioned above it within the same column. A row is classified as "unrestricted" if it lacks any restricted entries. The function $u(T)$ is defined as the count of unrestricted rows within T, subtracting 1 to account for the always unrestricted top row.

Now, let us formalize the definitions for the weight of a tableau T and the polynomial $F_\lambda(q)$.

The weight of a tableau T is symbolized by the monomial $\mathrm{wt}(T) := q^{\mathrm{rk}(T)}\alpha^{-f(T)}\beta^{-u(T)}$, where α and β denote constants.

Additionally, we introduce $F_\lambda(q)$ as the (Laurent) polynomial $\sum_T \mathrm{wt}(T)$, with the summation encompassing all permutation tableaux T of shape λ.

Example 8: Consider the following permutation tableau T that corresponds to the partition $\lambda = (8, 6, 5, 3, 3, 0)$:

1	1	0	1	0	0	1	1
0	0	1	1	0	1		
1	0	0	1	1			
0	0	0					
1	0	1					

We want to calculate the weight of this tableau using the given definitions:

$$\mathrm{rk}(T) = 13 - 8 = 5$$

$$f(T) = 5$$

$$u(T) = 3 - 1 = 2$$

With α and β taken from the PASEP model, the weight of tableau T is calculated as:

$$\mathrm{wt}(T) = q^{\mathrm{rk}(T)}\alpha^{-f(T)}\beta^{-u(T)} = q^5\alpha^{-5}\beta^{-2}$$

This demonstrates the step-by-step calculation of the weight for the given permutation tableau using the defined parameters and conditions.

Theorem 3: Fix $\tau = (\tau_1, \ldots, \tau_N) \in \{0,1\}^N$, and let $\lambda := \lambda(\tau)$. (Note that half-perim$(\lambda) = N + 1$.) The probability of finding the PASEP in configuration $(\tau_1, \ldots, \tau_N)$ in the steady state is

$$\frac{F_\lambda(q)}{Z_N}.$$

Here, $F_\lambda(q)$ is the weight-generating function for permutation tableaux of shape λ. Moreover, the partition function Z_N for the PASEP is equal to the weight-generating function for all permutation tableaux of half-perimeter $N + 1$.

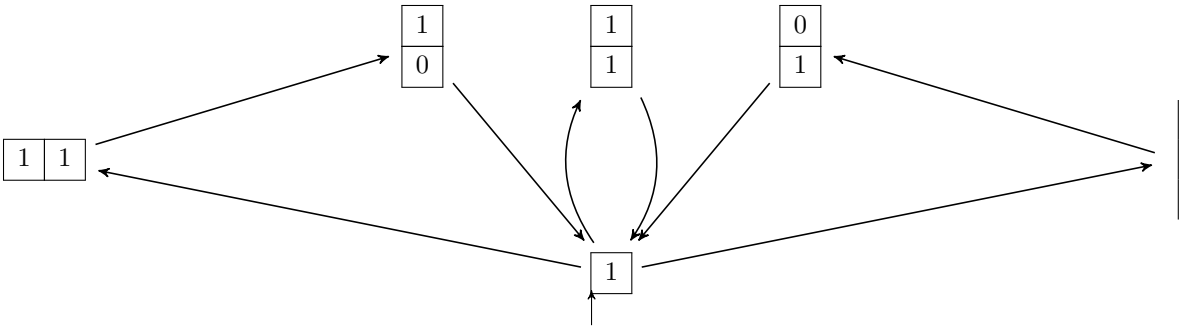
6 PT(Permutation Tableaux) Chain

6.1 Introduction

In conclusion, we unveil our central creation: a Markov chain operating on permutation tableaux that provides a mapping onto the PASEP (Partially Asymmetric Simple Exclusion Process). This newly introduced chain shall be referred to as the PT chain.

Definition: Let pr be a projection operator that maps a state of the Permutation Tableau (PT), denoted by T, to a state of the Partially Asymmetric Simple Exclusion Process (PASEP). Given a permutation tableau T of shape λ and half-perimeter $N+1$, we define the projection $\mathrm{pr}(T) := \tau(\lambda)$. Consequently, $\mathrm{pr}(T)$ represents a state within the PASEP characterized by N sites.

Example 9: The state diagram of the PT chain for $N = 2$.



The transition probabilities are defined straightforwardly: if a transition $S \to T$ exists in the Permutation Tableau (PT) chain, then the probability probPT$(S \to T)$ is equal to probPASEP$(\mathrm{pr}(S) \to \mathrm{pr}(T))$ in the Partially Asymmetric Simple Exclusion Process (PASEP).

We establish the complete set of feasible transitions within the PT chain. This set comprises four distinct types of transitions, each mirroring one of the four transition categories present in the PASEP. The determination of transition probabilities within the PT chain adheres to direct correspondence with those in the PASEP: given a transition $S \to T$ within the PT chain, the probability probPT$(S \to T)$ is defined as probPASEP$(\mathrm{pr}(S) \to \mathrm{pr}(T))$.

In the forthcoming discussion, we shall assume that S represents a permutation tableau possessing a half-perimeter of $N + 1$, characterized by the shape $\lambda = (\lambda_1, \ldots, \lambda_m, \ldots, \lambda_t)$ where $\lambda_1 \geq \ldots \geq \lambda_m > 0$, and $\lambda_r = 0$ for $r > m$.

6.2 Transition Probabilities of PT Chains

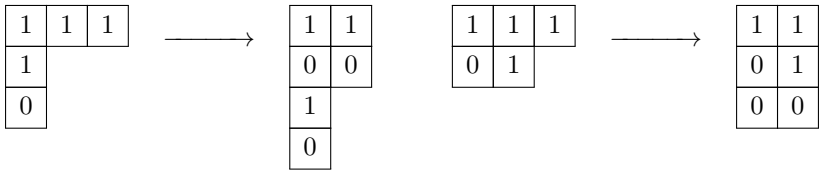
6.2.1 Particle enters from the left

Let us examine the scenario of a particle entering from the left, which transpires when the rightmost column of Tableau S has a length of 1. This occurrence prompts a corresponding transition within the PT chain, signifying a particle entering from the left in the PASEP.

We now define a new permutation tableau T as follows: delete the rightmost column of S and add a new all-zero row of length $\lambda_1 - 1$ to S, inserting it as far south as possible (subject to the constraint that the lengths of the rows of a permutation tableau must weakly decrease). Clearly adding an all-zero row in this way results in a new permutation tableau, since there is no way to introduce the forbidden pattern (condition (2) in the definition of a permutation tableau), and each column will still contain at least one 1.

We define $\text{prob}(S \to T) = \frac{\alpha}{N+1}$. Since we have removed a 1 in the top row but have not affected the unrestricted rows or superfluous 1's, we have that $\text{wt}(T) = \alpha \cdot \text{wt}(S)$, and therefore $\text{wt}(S) \cdot \text{prob}(S \to T) = \text{wt}(T) \cdot \frac{1}{N+1}$.

Example 10: To illustrate this concept, consider the following permutation tableau transformation:



This sequence exemplifies the particle entry from the right, showcasing the transformation of Tableau S into distinct tableau configurations through the prescribed operations.

6.2.2 Particle Movement to the Right

If some row $\lambda_j > \lambda_{j+1}$ in S (resp. if $\lambda_t > 0$), then there is a transition in PT from S that corresponds to the $(j-1)$-st black particle (resp. $(t-1)$-st black particle) in $\text{pr}(S)$ hopping to the right in the PASEP.

We now define a new permutation tableau T as follows, based on the rightmost entry of the j-th row of S.

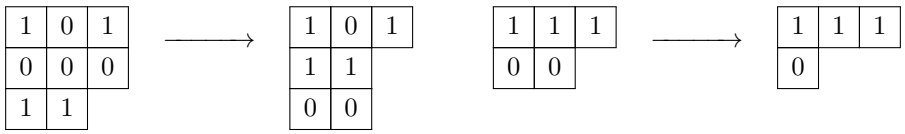
Case 1: Elimination of Rightmost 0 in the j-th Row

Suppose that the rightmost entry of the j-th row is a 0. This forces the j-th row to contain only 0's. Then we define a new tableau T by deleting the j-th row of S and adding a new row of $\lambda_j - 1$ 0's, inserting it as far south as possible. Clearly adding and deleting all-zero rows in this fashion results in a permutation tableau, so this operation is well defined.

We define $\text{prob}(S \to T) = \frac{1}{N+1}$. If $\lambda_j > 1$ then we have not affected the number of superfluous 1's, unrestricted rows, or 1's in the top row, and so $\text{wt}(T) = \text{wt}(S)$. It follows that $\text{wt}(S) \cdot \text{prob}(S \to T) = \frac{\text{wt}(T)}{N+1}$.

In the special case that $\lambda_j = 1$, then $\text{wt}(T) = \beta^{-1} \cdot \text{wt}(S)$, and therefore $\text{wt}(S) \cdot \text{prob}(S \to T) = \frac{\beta \cdot \text{wt}(T)}{N+1}$. Note that if this special case occurs, then $\text{pr}(T)$ ends with a black particle. And given such a T, there is only one such S with such a transition $S \to T$.

Example 11: To illustrate this concept, consider the following permutation tableau transformation:



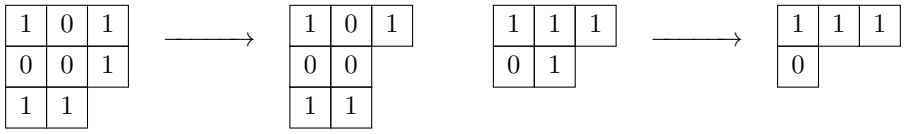
Case 2: Exclusion of Superfluous 1 from the Rightmost Entry

Continuing, suppose that the rightmost entry of the j-th row of S is a superfluous 1.

Then we define a new tableau T by deleting that 1 from S; because the column containing that 1 had at least two 1's to begin with, our new tableau will be a permutation tableau.

We define $\text{prob}(S \to T) = \frac{1}{N+1}$. Our operation has affected only the number of superfluous 1's, so $\text{wt}(T) = q^{-1} \cdot \text{wt}(S)$, and therefore $\text{wt}(S) \cdot \text{prob}(S \to T) = \frac{q \cdot \text{wt}(T)}{N+1}$.

Example 12: To illustrate this concept, consider the following permutation tableau transformation:



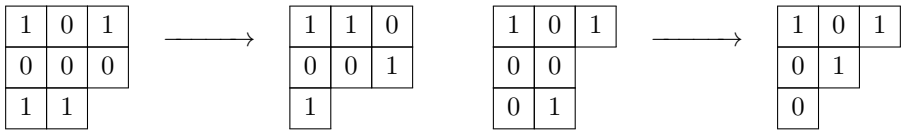
Case 3: Introduction of a Necessary 1 in the Rightmost Entry

Let us focus on the scenario where the rightmost entry of the j-th row of S is a necessary 1. Then we define a new tableau T by deleting the column containing the necessary 1 and adding a new column whose length is 1 less. That new column consists entirely of 0's except for a necessary 1 at the bottom, and it is inserted as far east as possible. Such a column cannot introduce a forbidden pattern (condition (2) in the definition of permutation tableau), so T is, in fact, a permutation tableau.

We define $\text{prob}(S \to T) = \frac{1}{N+1}$. If the column in S containing the necessary 1 has length at least 2, then $\text{wt}(T) = \text{wt}(S)$, because we have not changed the number of superfluous 1's, the 1's in the top row, or the number of unrestricted rows. Thus $\text{wt}(S) \cdot \text{prob}(S \to T) = \frac{\text{wt}(T)}{N+1}$.

In the special case that the column in S containing the necessary 1 has length exactly 2, $\text{wt}(T) = \alpha^{-1} \cdot \text{wt}(S)$, because the new tableau will have a new 1 in the top row. Thus $\text{wt}(S) \cdot \text{prob}(S \to T) = \frac{\alpha \cdot \text{wt}(T)}{N+1}$.

Example 13: To illustrate this concept, consider the following permutation tableau transformation:



6.2.3 Particle Exit to the Right

If some row $\lambda_j > \lambda_{j+1}$ in S, then there is a transition in PT from S that corresponds to the j-th black particle in $\text{pr}(S)$ hopping to the left in the PASEP. We define a new tableau T by increasing the length of the $(j+1)$-st row by 1 and filling the extra square with a 1. The result is clearly a permutation tableau.

We define $\text{prob}(S \to T) = \frac{q}{N+1}$. Since we have added one superfluous 1, we have that $\text{wt}(T) = q \cdot \text{wt}(S)$. Thus $\text{wt}(S) \cdot \text{prob}(S \to T) = \frac{\text{wt}(T)}{N+1}$.

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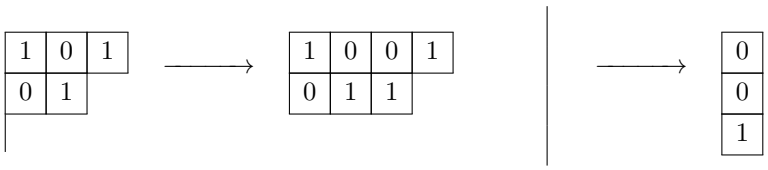
Example 14: To illustrate this concept, consider the following permutation tableau transformation: (the line has half perimeter 4)

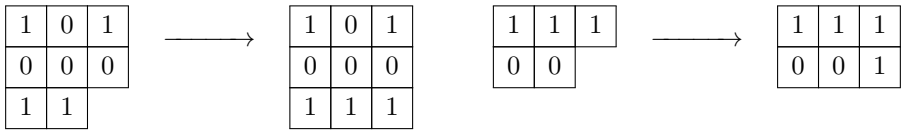


6.2.4 Particle Hop to the Left

If within tableau S, there exists a row λ_j exceeding the subsequent row λ_{j+1}, a corresponding transition within the PT chain emanates from S. This transition signifies the movement of the j-th black particle within $\text{pr}(S)$—executing a leftward hop within the PASEP. We introduce a new tableau T by augmenting the length of the $(j+1)$-st row by 1 and populating the additional cell with a 1. The resulting configuration adheres to the criteria of a permutation tableau.

The transition probability $\text{prob}(S \to T)$ is defined as $\frac{q}{N+1}$. The introduction of one superfluous 1 leads to $\text{wt}(T) = q \cdot \text{wt}(S)$. Consequently, the relationship $\text{wt}(S) \cdot \text{prob}(S \to T) = \frac{\text{wt}(T)}{N+1}$ prevails.

Example 15: To illustrate this concept, consider the following permutation tableau transformation:



6.3 Properties of PT Chains

Definition: Let M and N be Markov chains on finite sets X and Y, respectively, and let F be a surjective map from X to Y. We say that M *projects* to N if the following properties hold:

1. If x_1 and x_2 are elements in X such that $\text{prob}_M(x_1 \to x_2) > 0$, then $\text{prob}_M(x_1 \to x_2) = \text{prob}_N(F(x_1) \to F(x_2))$.

2. If y_1 and y_2 are elements in Y such that $\text{prob}_N(y_1 \to y_2) > 0$, then for each $x_1 \in X$ such that $F(x_1) = y_1$, there exists a unique $x_2 \in X$ such that $F(x_2) = y_2$ and $\text{prob}_M(x_1 \to x_2) > 0$; moreover, $\text{prob}_M(x_1 \to x_2) = \text{prob}_N(y_1 \to y_2)$.

Let $\text{prob}_M(x_0 \to x; t)$ denote the probability that, if we start at state $x_0 \in M$ at time 0, we are in state x at time t. If M projects to N, then a walk on the state diagram of M is indistinguishable from a walk on the state diagram of N in the following sense.

Proposition: Suppose that M projects to N. Let $x_0 \in X$ and $y_0, \tilde{y} \in Y$ such that $F(x_0) = y_0$. Then

$$\text{prob}_N(y_0 \to \tilde{y}; t) = \sum_{\substack{\tilde{x} \\ F(\tilde{x})=\tilde{y}}} \text{prob}_M(x_0 \to \tilde{x}; t).$$

Proof. We use induction. The base case $t = 0$ is trivially true. Suppose that the statement is true for $t-1$. Let $Y' \subset Y$ be the set of all states y' such that there is a transition $y' \to \tilde{y}$ with nonzero probability. Let $X' = \{x' \in X \mid F(x') \in Y'\}$. By the definition of projection, these are the only states in X which have a transition to a state $\tilde{x}$ such that $F(\tilde{x}) = \tilde{y}$. Furthermore, for each $x_1 \in X'$ there exists an $x_2 \in X$ such that $F(x_2) = \tilde{y}$ and $\text{prob}_M(x_1 \to x_2) > 0$, and $\text{prob}_M(x_1 \to x_2) = \text{prob}_N(y_1 \to y_2)$.

Clearly $\text{prob}_N(y_0 \to \tilde{y}; t) = \sum_{y' \in Y'} \text{prob}_N(y_0 \to y'; t-1) \cdot \text{prob}_N(y' \to \tilde{y})$, which by the induction hypothesis and the definition of projection is equal to

$$\sum_{y' \in Y'} \sum_{x' \mid F(x')=y'} \text{prob}_M(x_0 \to x'; t-1) \cdot \text{prob}_M(x' \to \tilde{x}).$$

This is equal to

$$\sum_{x' \mid F(x') \in Y'} \text{prob}_M(x_0 \to x'; t-1) \cdot \text{prob}_M(x' \to \tilde{x}).$$

But now using again the definition of projection, we see that this is equal to $\text{prob}_M(x_0 \to x; t)$, as desired. ■

Corollary 1: Suppose that M projects to N via the map F. Let $y \in Y$ and let $X' = \{x \in X \mid F(x) = y\}$. Then the steady state probability that N is in state y is equal to the steady state probabilities that M is in any of the states $x \in X'$.

Clearly, the operator pr is a surjective map from the set of permutation tableaux of half-perimeter $N+1$ to the states of the PASEP with N sites. It is clear from our definition of the PT chain that the PT chain projects to the PASEP.

Corollary 2: The projection map pr gives a projection from the PT chain on permutation tableaux of half-perimeter $N+1$ to the PASEP with N sites.

6.4 Steady States Probabilities of PT Chains

Theorem 4: Consider the PT chain on permutation tableaux of half-perimeter $N+1$ and fix a permutation tableau T (of half-perimeter $N+1$). Then the steady state probability of finding the PT chain in state T is

$$\frac{w_T(T)}{\sum_S w_T(S)}$$

where the sum is over all permutation tableaux of half-perimeter $N+1$.

We will check the defining recurrences of the steady state. More precisely, it suffices to check the following. Fix a state T, let Q be the collection of all states with transitions to T, and let S be the collection

of all states with transitions from T. Then we need to prove that.

$$wt(T) = \sum_{Q \in Q} wt(Q) \cdot \text{prob}(Q \to T) \quad + wt(T) \cdot \left(1 - \sum_{S \in S} \text{prob}(T \to S)\right)$$

The above equation expresses the steady state probability of being in state T in two ways, involving two consecutive times t and $t+1$; equivalently, it encodes the condition that the transition matrix has a left eigenvector with eigenvalue 1. Combining the terms involving $wt(T)$, we get

$$\sum_{Q \in Q} wt(Q) \cdot \text{prob}(Q \to T) = wt(T) \cdot \sum_{S \in S} \text{prob}(T \to S)$$

In order to verify this, we categorize the set of states in the PT into four distinct classes. Throughout the analysis, let B represent a nonempty string of black particles, and let W represent a nonempty string of white particles. We divide the set of permutation tableaux of half-perimeter N into the following four classes:

1. States T such that $pr(T)$ has the form $BWBW \ldots BW$. (Total: $2n$ strings)
2. States T such that $pr(T)$ has the form $BWBW \ldots BWB$. (Total: $2n+1$ strings)
3. States T such that $pr(T)$ has the form $WBWB \ldots WBW$. (Total: $2n+1$ strings)
4. States T such that $pr(T)$ has the form $WBWB \ldots WB$. (Total: $2n$ strings)

For a state of type (1), there are n possible "hop right" transitions and $n-1$ possible "hop left" transitions.

Example 16: A state T of type (1), where $n = 3$, $a = 2$, $b = 0$, and $c = 1$.

1	0	1	1	0	1
0	0	0	0	1	1
1	1	0			
0	0				

For a state of type (2), there are n possible "hop right" transitions, n possible "hop left" transitions, and one "hop out to the right" transition.

Example 17: A state T of type (2), where $n = 3, a = 2, b = 0, c = 1$.

1	0	1	1	0	1
0	0	0	0	1	1
1	1	0			
0	0				

For a state of type (3), there are n "hop right" transitions, n "hop left" transitions, and one "hop in from the left" transition.

Example 18: A state T of type (3), where $n = 3, a = 2, b = 1, c = 0$.

1	0	1	1	0	1
0	0	0	0	1	
1	1	0			
0	0				

For a state of type (4), there are $n-1$ "hop right" transitions, n "hop left" transitions, one "hop in from the left," and one "hop out from the right."

Example 19: A state T of type (4), where $n = 4, a = 2, b = 2, c = 0$.

1	0	1	1	0	1
0	0	0	0	1	
1	1	0			
0	0				

As shown in the simultaneous figures, we proceed to calculate the sum $\sum_{S \in S} \text{prob}(T \to S)$ across all four cases. The computed quantities are delineated as follows:

$$(1) \quad \frac{n + q(n-1)}{N+1}$$

$$(2) \quad \frac{n + qn + \beta}{N+1}$$

$$(3) \quad \frac{n + qn + \alpha}{N+1}$$

$$(4) \quad \frac{(n-1) + qn + \alpha + \beta}{N+1}$$

6.4.1 Analysis of Transitions

Type (1) Fix a state T of type (1). The Young diagram of T will have n outer corners (south step followed by west step), and $n-1$ inner corners (west step followed by south step). If we say only "corner," this will mean an outer corner.

Of the entries in the n outer corners, a of them are 0's, b of them are necessary 1's, and $n-a-b$ are superfluous 1's.

For each 0 corner, there is a state Q such that $\text{prob}(Q \to T) > 0$ and such that $\text{wt}(Q) \cdot \text{prob}(Q \to T) = \frac{\text{wt}(T)}{N+1}$. Therefore the 0 corners contribute $\frac{\alpha \cdot \text{wt}(T)}{N+1}$ to the left-hand side of theorem.

For each corner which is a superfluous 1, there is a state Q such that $\text{prob}(Q \to T) > 0$ and such that $\text{wt}(Q) \cdot \text{prob}(Q \to T) = \frac{\text{wt}(T)}{N+1}$. Therefore these superfluous 1's contribute $\frac{(n-a-b)\cdot \text{wt}(T)}{N+1}$ to the left-hand side of theorem.

For each corner which is a necessary 1, there is a state Q such that $\text{prob}(Q \to T) > 0$ and such that $\text{wt}(Q) \cdot \text{prob}(Q \to T) = \frac{\text{wt}(T)}{N+1}$. Therefore these necessary 1's contribute $\frac{b\cdot \text{wt}(T)}{N+1}$ to the left-hand side of theorem.

For each inner corner, there is a state Q such that $\text{prob}(Q \to T) > 0$ and such that $\text{wt}(Q) \cdot \text{prob}(Q \to T) = \frac{q\cdot \text{wt}(T)}{N+1}$. Therefore these inner corners contribute $\frac{q(n-1)\cdot \text{wt}(T)}{N+1}$ to the left-hand side of theorem.

The sum of all of these contributions is $\frac{\text{wt}(T)}{N+1} \cdot (n + q(n-1))$. We see that theorem holds for states of Type (1).

Type (2) Fix a state T of type (2). The Young diagram of T will have n outer corners, and n inner corners.

Of the entries in the n outer corners, a of them are 0's, b of them are necessary 1's, and $n-a-b$ are superfluous 1's.

As before, the 0 corners contribute $\frac{a\cdot wt(T)}{N+1}$ to the left-hand side of theorem.

Similarly, the corners containing superfluous 1's contribute $\frac{(n-a-b)\cdot wt(T)}{N+1}$ to the left-hand side of theorem.

The corners containing necessary 1's contribute $\frac{b\cdot wt(T)}{N+1}$ to the left-hand side of theorem.

The n inner corners contribute $\frac{qn\cdot wt(T)}{N+1}$ to the left-hand side of theorem.

The final empty row of the Young diagram contributes $\frac{\beta\cdot wt(T)}{N+1}$ to theorem.

The sum of all of these contributions is $\frac{wt(T)}{N+1} \cdot (n + qn + \beta)$. We see that theorem holds for states of Type (2).

Type (3) Fix a state T of type (3). The Young diagram of T will have n outer corners (not counting the corner formed by the rightmost column of length 1), and n inner corners.

Of the entries in the n outer corners, a of them are 0's, b of them are necessary 1's, and $n-a-b$ are superfluous 1's.

As before, the 0 corners contribute $\frac{a\cdot wt(T)}{N+1}$ to the left-hand side of theorem.

The corners containing superfluous 1's contribute $\frac{(n-a-b)\cdot wt(T)}{N+1}$ to the left-hand side of theorem.

The corners containing necessary 1's contribute $\frac{b\cdot wt(T)}{N+1}$ to the left-hand side of theorem.

The n inner corners contribute $\frac{qn\cdot wt(T)}{N+1}$ to the left-hand side of theorem.

The (leftmost) column of length 1 contributes $\frac{\alpha\cdot wt(T)}{N+1}$ to theorem.

The sum of all of these contributions is $\frac{wt(T)}{N+1} \cdot (n + qn + \alpha)$. We see that theorem holds for states of Type (3).

Type (4) Fix a state T of type (4). The Young diagram of T will have $n-1$ outer corners (not counting the corner formed by the rightmost column of length 1), and n inner corners.

Of the entries in the $n-1$ outer corners, a of them are 0's, b of them are necessary 1's, and $n-1-a-b$ are superfluous 1's.

As before, the 0 corners contribute $\frac{a\cdot wt(T)}{N+1}$ to the left-hand side of theorem.

The corners containing superfluous 1's contribute $\frac{(n-1-a-b)\cdot wt(T)}{N+1}$ to the left-hand side of theorem.

The corners containing necessary 1's contribute $\frac{b\cdot wt(T)}{N+1}$ to the left-hand side of theorem.

The n inner corners contribute $\frac{qn\cdot wt(T)}{N+1}$ to the left-hand side of theorem.

The (leftmost) column of length 1 contributes $\frac{\alpha \cdot wt(T)}{N+1}$ to theorem.

The final empty row of the Young diagram contributes $\frac{\beta \cdot wt(T)}{N+1}$ to theorem.

The sum of all of these contributions is $\frac{wt(T)}{N+1} \cdot ((n-1) + qn + \alpha + \beta)$. We see that theorem holds for states of Type (4).

This completes the proof of the main theorem. ■

In conclusion, upon the successful establishment of theorem 4 and its accompanying corollary 2, an avenue is thereby created for the subsequent demonstration of theorem 3. ■

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