Computing Lifshitz Field Theory Correlation Functions Using the AdS/CFT Correspondence

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Abstract

We use Anti-de Sitter spacetime/Conformal Field Theory (AdS/CFT) correspondence to determine quantum field theory correlation function—expectation value of products of fields—in two field theories, by performing a calculation in the dual gravity theory. I.e., we compute the correlation functions holographically. The first theory is the canonical one. We solve the Klein-Gordon equation in Anti-de Sitter spacetime or asymptotically so which gives us two independent solutions with undetermined coefficients. The path integral depends on these coefficients, which correspond to a choice of boundary condition on the AdS side and a choice of sources on the field theory side. On the field theory side, the correlation functions coincide with derivatives of the path integral with respect to the sources. We perform the corresponding dual gravity calculation. After completing the calculation for the anonical AdS/CFT setting, we perform the analogous computation for a novel field theory of interest in condensed matter physics, which possesses a Lifshitz multicritical point.

Introduction

The AdS/CFT correspondence is a relation between a gravity theory with certain boundary conditions and a field theory in the presence of corresponding sources. It is the most concrete realization of holography in quantum gravity -- the idea that all information content of a gravitational theory in any region of spacetime can be equivalently represented by a non-gravitational theory living in the boundary of that region. The AdS/CFT correspondence has the property that whenever one side is strongly coupled, the other side is weakly coupled and vice versa. Thus, this correspondence provides a way to calculate physical quantities that would be inaccessible by direct computation on one side alone.
Here, we try to understand how it works in general, and then use it to study Lifshitz field theory, a theory with applications to 2D strongly coupled electron systems.

Correlator in AdS

The Klein Gordon Equation

The quantized version of the relativistic energy-momentum relation, the Klein Gordon equation is a Lorentz covariant relativistic wave equation for spinless particles.

In general, $p.p = -m^2$

In QM, $\boldsymbol{p}_\mu = -i\hbar\nabla_\mu , \mu = 0,1,2,3$

From the non-quantum mechanical equation, $\eta_{\mu\nu}p^\mu p^\nu + m^2 = 0$

We obtain the Klein Gordon equation by promoting p^μ **to an operator** $\boldsymbol{p}^\mu$ **and letting the left equation act on the wavefunction** ϕ.

$$((-i\hbar)^2 g_{\mu\nu}\nabla^\mu\nabla^\nu + m^2)\phi(x) = 0\ (General)$$
$$d'Alembertian\ operator\ \Box = g_{\mu\nu}\nabla^\mu\nabla^\nu = \nabla_\mu\nabla^\nu$$
$$= \nabla^2\ in\ 4D, i.e.\ 4D\ generalization\ of\ derivative$$

$\Box$ **in general coordinates in general spacetime:**

$$\Box\phi = \frac{1}{\sqrt{-g}}\partial_\mu\left(g^{\mu\nu}\sqrt{-g}\,\partial_\upsilon\phi\right), where\ g_{\mu\nu} is\ the\ metric, g^{\mu\nu} = inverse\ of\ that\ metric,$$
$$g = determinant\ of\ the\ metric$$

Thus, the KG equation becomes: $(-\Box + \boldsymbol{m^2})\phi(x) = 0$

d'Alembertian Operator in AdS$_{d+1}$

The metric, $ds^2 = \frac{r^2}{L^2}\left(\eta_{\mu\nu}dx^\mu dx^\upsilon\right) + \frac{L^2}{r^2}dr^2$, **where** $\eta_{\mu\nu} = diag(-1,1,1,\ldots,1)$

[For AdS$_5$, $ds^2 = \frac{r^2}{L^2}\left(-dt^2 + d\overrightarrow{x^2}\right) + \frac{L^2}{r^2}dr^2 = g_{\mu\nu}dx^\mu dx^\upsilon$, **where** $\vec{x} = (x^1, x^2, x^3)$**]**

With $z = \frac{L^2}{r}$, **so** $\frac{dz}{z} = -\frac{dr}{r}$, $\quad ds^2 = \frac{L^2}{z^2}\left(\eta_{\mu\nu}{}^{(d)}dx^\mu dx^\upsilon + dz^2\right)$

So, $\quad g_{\mu\nu} = diag(-\frac{L^2}{z^2}, \frac{L^2}{z^2}, \frac{L^2}{z^2}, \ldots, \frac{L^2}{z^2}) = \frac{L^2}{z^2}\eta_{\mu\nu}{}^{(d+1)}$, $\sqrt{-g} = (\frac{L}{z})^{d+1}$

$$\Box\phi = \frac{1}{\sqrt{-g}}\partial_\mu\left(g^{\mu\nu}\sqrt{-g}\,\partial_\upsilon\phi\right)$$
$$\Box_{(d+1)}\phi = (\frac{z}{L})^{d+1}\partial z[(\frac{L}{z})^{d-1}\partial_z\phi] + (\frac{z^2}{L^2})\Box_{(d)}\phi$$

Klein Gordon Equation in AdS$_{d+1}$

$$(-\Box_{(d+1)} + m^2)\phi(x) = 0$$
$$-[(\frac{z}{L})^{d+1}\partial_z[(\frac{L}{z})^{d-1}\partial_z\phi] + (\frac{z^2}{L^2})\Box_{(d)}\phi] + m^2\phi = 0$$

In momentum space: $-(\frac{z}{L})^{d+1}\partial_z[(\frac{L}{z})^{d-1}\partial z\tilde{\phi}] + (\frac{z^2}{L^2})p_{(d)}{}^2\tilde{\phi} + m^2\tilde{\phi} = 0$

$$z^2\,\partial_z{}^2\tilde{\phi} - (d-1)z\partial_z\tilde{\phi} - (z^2p^2 + m^2L^2)\tilde{\phi} = 0$$

The above equation is in the form of a modified or hyperbolic Bessel function, and has the solution: $\tilde{\phi}(p,z) = z^{d/2}Z_\upsilon(pz)$, **where** $\upsilon = \sqrt{\frac{d^2}{4} + m^2L^2}$ **and** $Z_\upsilon(pz)$ **is the linear combination of** $I_\upsilon(pz)$ **and** $k_\upsilon(pz)$.

For regularity in the deep interior ($z \to \infty$**) of AdS$_{d+1.}$ we take,** $Z_\upsilon = k_\upsilon$. **So** $\tilde{\phi}(p,z) = z^{d/2}k_\upsilon(pz)$

Calculation of Correlation Function

There is an identification between the path integrals of the gravity theory (with source for the KG field) and the field theory (with source for the dual operator).

$$Z_{grav}[\phi; \alpha(\vec{x},t) = J(\vec{x},t)] = Z_{CFT}[Source\ for\ O(\vec{x},t)\ is\ J(\vec{x},t)]$$

We can calculate correlators of the field theory operator by performing a dual KG calculation on the gravity side.

$$S_{grav} = S_{KG} = \frac{1}{4k^2}\int d^{d+1}x\sqrt{-g}\phi(-\Box + m^2)\phi - \frac{1}{4k^2}\int_{r=R} d^dx\sqrt{-h}\,\phi n^\mu\partial_\mu\phi$$

For a particular choice of phi with d= 3, $m^2L^2 = -2, \Delta_+ = 2, \Delta_- = 1$:

$$\phi(\mathrm{r} \to \infty, \vec{x}, \mathrm{t}) = \frac{\alpha(\vec{x},\mathrm{t})\mathrm{L}^2}{r} + \frac{\beta(\vec{x},\mathrm{t})\mathrm{L}^4}{r^2} + \cdots$$
$$S_{KG} + S_{bdy} = \frac{L^2}{4k^2}\int d^3\,x\alpha\beta,$$

where the boundary term $S_{bdy} = \frac{-1}{4k^2}\int d^3x\sqrt{-h}\,\phi^2$

Varying the field, we get $\delta S_{KG} + \delta S_{bdy} = \frac{L^2}{2k^2}\int d^3\,x\beta\delta\alpha$

So now, the one point function is: $\langle\vartheta(\vec{x},t)\rangle = \frac{1}{i}\frac{\delta}{\delta\alpha(\vec{x},t)}e^{S_{grav}}\Big|_{\alpha=0} = \frac{\delta S_{grav}}{\delta\alpha(\vec{x},t)}\Big|_{\alpha=0}$

$$= \frac{L^2}{2k^2}\beta(\vec{x},t)$$

For arbitrary d and Δ**, it is:** $\langle\vartheta(\vec{x},t)\rangle = (2\Delta - d)\,\frac{L^{d-1}}{2k^2}\beta(\vec{x},t)$

And the two-point function is: $i\langle\vartheta(x),\vartheta(y)\rangle = (2\Delta - d)\,\frac{L^{d-1}}{2k^2}\frac{\delta\beta(x)}{\delta\alpha(y)}\Big|_{\alpha=0}$

When we expand the solution for phi near the boundary, $\tilde{\phi}(p,z) = z^{d/2}k_\upsilon(pz)$**, we get to the two point function after finding the ratio of alpha and beta.**

$$\langle\vartheta(p),\vartheta(-p)\rangle = \frac{(-1)^\upsilon L^{d-1}p^{2\upsilon}}{(2\pi)^d 2k^2[2^{\upsilon-1}\Gamma(\upsilon)]^2}$$

In position space: $\langle\vartheta(x),\vartheta(x')\rangle = \frac{CA_{d-2}}{(2\pi)^d|x-x\prime|^{2\Delta_+}}\frac{\Gamma(2\Delta_+-1,ip|x|)+\Gamma(2\Delta_--1,-ip|x|)}{(-1)^{2\Delta_+}}\Big|_0^\infty = \frac{Constant\ terms}{|x-x\prime|^{2\Delta_+}}$

Correlator in Lifshitz Field Theory

The phase diagram of certain 2D materials has a multicritical point known as a Lifshitz point. Near Lifshitz points, a material is well described by a quantum field theory where the action is

$$S = \int dt\,dx^2L\ \ where\ \ L = \left(\frac{\partial\phi}{\mathrm{d}t}\right)^2 - k(\nabla^2\phi)^2$$

For some constant k. The field has a scaling symmetry:

$$t \to t' = \lambda^2 t, \qquad x \to x' = \lambda x$$

For d=3, the AdS$_4$ metric is so that the metric is invariant under Lifshitz scale transformation : $\quad ds^2 = -\frac{r^4}{L^4}dt^2 + \frac{r^2}{L^2}d\overrightarrow{x^2} + \frac{L^2}{r^2}dr^2$

$$= -\frac{1}{v^4}dt^2 + \frac{1}{v^2}(d\overrightarrow{x^2} + L^2dv^2),$$

where $v = \frac{L}{r}$. **We set** $z = |\omega|Lv^2$ **through an intermediate co-ordinate** $y = v\sqrt{|\omega|L}$.

Klein Gordon Equation for Lifshitz metic

$$\partial_y^2\tilde{\phi}'' - \frac{3}{y}\partial_y\tilde{\phi}' - \left(\frac{k^2L}{w} + y^2\right)\tilde{\Phi} = 0$$

Solution to this Klein Gordon equation in the spacetime dual to Lifshitz theory can be defined by the corresponding boundary value $\phi_0(t,x)$ **by**

$$\phi\left(t, \boldsymbol{x}, z'{\sim}z^{d-\Delta}\phi_0(t,\boldsymbol{x})\right)\ as\ z \to \infty$$

The bulk to boundary propagator is defined as a particular solution to the KG equation when the boundary value is a delta function. That is

$$\phi(x^0, \boldsymbol{x}, z) = K_\Delta\,(x^0, \boldsymbol{x}, z, y^0, y)\text{ where }\phi_0 = (x^0, \boldsymbol{x})\ \delta^d(\boldsymbol{x} - \boldsymbol{y})$$

Then, a general solution to the KG equation can be written in terms of the bulk-to-boundary propagator and boundary value as

$$\phi(x^0, \boldsymbol{x}, z) = \int d^d\mathrm{y}\ K_\Delta\,(x^0, \boldsymbol{x}, z, y^0, \boldsymbol{y})\ \phi_0\,(y^0, \boldsymbol{y})$$

Considering translation invariance and taking it's fourier gives us the e Fourier transformed bulk-to-boundary propagator. After normalization it becomes,

$\widetilde{K_\Delta}(\omega, \boldsymbol{k}, z) = \Gamma\left(\frac{\boldsymbol{k}^2L}{4|\omega|} + \frac{3}{2}\right)e^{-\frac{z}{2}}\,\mathrm{U}\left(\frac{\boldsymbol{k}^2L}{4|\omega|} - \frac{1}{2}, -1, z\right)$ where, $\widetilde{K_\Delta}\,(\omega, \boldsymbol{k}, z)$ is the fourier transform of $K_\Delta\,(x^0 - y^0, \boldsymbol{x} - \boldsymbol{y}, z; 0,0)$ and $\mathrm{U}\left(\frac{\boldsymbol{k}^2L}{4|\omega|} - \frac{1}{2}, -1, z\right)$ is a confluent hypergeometric function.

The Klein-Gordon Action is :

$$S_{cl}(\phi) = \int\frac{\mathrm{d}\omega\ \mathrm{d}^{\mathrm{d}-1}k}{(2\pi)^d}\upsilon\widetilde{\phi}_0(\omega, \boldsymbol{k})\tilde{G}_\varepsilon(\omega, \boldsymbol{k})\ \widetilde{\phi}_0(-\omega, -\boldsymbol{k})$$
$$\text{Where }\tilde{G}_\varepsilon(\omega, \boldsymbol{k}) = -\frac{\eta}{2}\sqrt{h}\widetilde{K}(-\omega, -\boldsymbol{k}, z)n^z\partial_z\widetilde{K}(\omega, \boldsymbol{k}, z)|_{z=\varepsilon}$$

Calculation of Correlation functions

We define the Lifshitz field theory operator $\langle\mathcal{O}_\Delta(\omega, \boldsymbol{k})\rangle = -\tilde{G}_{\mathrm{ren}}\,(\omega, \boldsymbol{k})\ \widetilde{\phi_0}(-\omega, -\boldsymbol{k})$ **and** $\langle\mathcal{O}_\Delta(\omega, \boldsymbol{k})\mathcal{O}_\Delta(\omega, \boldsymbol{k})\rangle = \tilde{G}_{\mathrm{ren}}\,(\omega, \boldsymbol{k})$ where $\tilde{G}_{\mathrm{ren}}\,(\omega, \boldsymbol{k})$ is the renormalized $\tilde{G}_\varepsilon(\omega, \boldsymbol{k})$

After adding counterterms and taking ε→ 0. Near z = 0, the bulk-to-boundary propagator $\tilde{k}$(ω,k,z) **takes the form:**

$$\widetilde{K_\Delta}(\omega, \boldsymbol{k}, z) = 1 - \frac{1}{4}\left(\frac{\boldsymbol{k}^2L}{|\omega|}\right)\omega + \frac{1}{64}\left[-2\boldsymbol{0} + 8\left(\frac{\boldsymbol{k}^2L}{|\omega|}\right) - 3\left(\frac{\boldsymbol{k}^2L}{|\omega|}\right)^2 + \left(4 - \left(\frac{\boldsymbol{k}^2L}{|\omega|}\right)\right)^2\left(2\gamma + \psi\left(\frac{3}{2} + \frac{\boldsymbol{k}^2L}{4|\omega|}\right)\right)\right]z^2 + \frac{\boldsymbol{1}}{64}\left(4 - \left(\frac{\boldsymbol{k}^2L}{|\omega|}\right)^{\boldsymbol{2}}\right)z^{\boldsymbol{2}}\log(z^2) + \mathcal{O}(z^3)$$

where ψ(x) **is the digamma function and** $\gamma \simeq 0.577$ **is the Euler-Mascheroni constant. Substituting for** $\tilde{G}$ **gives us the two point function:**

$$\langle\,\mathcal{O}_{\Delta_+}(t_1, \boldsymbol{x_1})\mathcal{O}_{\Delta_+}(t_2, \boldsymbol{x_2})\rangle \sim \frac{c}{|\boldsymbol{x_1} - \boldsymbol{x_2}|^8}\ \textbf{for large}\ x = |\boldsymbol{x_1} - \boldsymbol{x_2}|.$$

Discussion and Conclusion

The overall scaling $1\,/\,\lambda^{2\Delta_+} = 1/\lambda^8$under scale transformation $(t,x) \to (\lambda^2 t, \lambda x)$ is exactly as expected for an operator of the computed scaling dimension $\Delta_+ = 4$. The purely spatial power law result C= constant at large x is surprising. We had expected to find $C = C(x^2\,/\,t)\ with\ C \to 0\ as\ t = t_1 - t_2 \to 0$, so that equal time correlators vanish known as ultralocality—a property exhibited by certain other field theories with critical points in the same "university class" as that of Lifshitz field theory. Had we found that, then the same would have been the case here: $\langle\,\mathcal{O}_{\Delta_+}(t_1, \boldsymbol{x_1})\mathcal{O}_{\Delta_+}(t_2, \boldsymbol{x_2})\rangle$ = 0 for $t_1 = t_2$. But, that's not what we found. Instead, large distance correlators are independent of $t_1$and t_2. Therefore, our Lifshitz field theory is not ultralocal. This is surprising and merits further investigation.

References

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