Reputation, Learning and Externalities in Frictional Markets

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Abstract

I introduce a dynamic model of random search where ex ante heterogeneous agents with unknown abilities match with a variety of projects. There is incomplete yet symmetric information about the agents' types. Interpreting the posterior belief about the agents' ability as their reputation, I study the outcomes of the economy (namely the endogenous matching sets and the steady-state distributions) when the success or failure of the projects create feedback effects: reputational externalities and spillovers in the population of projects. In the former case when the meeting rate of each agent is inversely impacted by the distribution of other agents' reputation, the proportion of agents who are both high ability and inactive is inefficiently high, and the projects suffer from early termination. When there are positive spillovers from the low-type to the high-type projects, increased levels of search frictions could save the market from breakdown caused by the rational neglect of spillover effect in the agents' matching decisions.

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1 Introduction

Much of the theoretical literature on experimentation is about learning the other party's (namely the project's) type. In this paper, however, I shift the attention to learning the self-types. Specifically, the agents in my paper do not know their type, and the only way to learn it is by matching with projects and observing the output of their partnerships.

There are natural instances where agents *learn* their type through the course of their partnerships with other parties. For example, firms learn about their productivity while they are matched with workers. Colleges learn about the quality of their teaching staff while students are enrolled in their programs. Venture capitalists learn about their ability and the quality of their post-investment services while investing in startups.[1] Common in all these cases is the cost of maintaining the partnership and the *tangible* created surplus (such as the high-quality output of production in the first case, students' accomplishment in the second case, and the startups' success in the last case). These tangible gains from partnerships can be isomorphically captured by the choice of the *production function* in the matching markets (e.g. Shimer and Smith (2000)).

However, when agents hold incomplete information about their types, there is also an *intangible* gain due to the learning, that cannot be nested in the former construct. Because, what is now used as an input to the production function is no longer the static type of the agent, but a dynamic state process that reflects the agent's belief of her own type. Specifically, in addition to the tangible gains, there are now information gains from agent's matching choices, as present matches convey information about agent's type, that in turn can be used in future choices of projects.

In this paper, agents are ex ante endowed with high or low immutable types $\theta \in \{L, H\}$, that are hidden to everyone in the economy. On the other hand, there are heterogenous projects with observable types denoted by q. The individuals on the two sides of this economy randomly meet each other subject to search frictions and decide to form a partnership or not. Once a (one-to-one) match is formed there will be a random success event whose arrival density depends on the types of both parties. Agents continuously update their belief about their underlying type during the course of their matches. Therefore, interpreting the posterior belief as their reputation (denoted by π), there will be a range of realized reputations at every point in time.

Whenever an agent pairs up with a project, a learning opportunity is created for the entire market as well as that particular agent about her type. Since maintaining the match is costly, the agent effectively solves a stopping time problem, by which it weighs the value of the match $v(\pi, q)$ (as a function of its current reputation π and the type of the project q) against the reservation value $w(\pi)$ – the value of holding current reputation while being

[1] Sørensen (2007) teases out the positive treatment effect of the VCs' involvement in their portfolio companies from the concerns regarding the sorting and selection. In addition, when it comes to VCs' underlying ability Gompers and Lerner (1999) argues that empirical evidence favors the learning model (symmetric but incomplete information) compared to the signalling model (asymmetric information).

unmatched, that is called the reputation value function throughout the paper. Because of the search friction framework these two functions are intertwined in the equilibrium. That is the reputation function is simply the expected discounted value of future surpluses that the agent extracts, and the matching value function is the solution to the free-boundary problem with the exit option w. The continuation region of this free-boundary problem (with endogenous exit option w) determines the equilibrium matching sets $\mathcal{M}$.

The central innovation of this paper is to study the optimality and shape of these matching sets when agents have long-run incentives and learn about their ability as they match. Section 2 of the paper explains the baseline economy and studies the equilibrium tuple $\langle w, v, \mathcal{M} \rangle$. The paper's main points of the departure from the established search theory literature (e.g. Shimer and Smith (2000)) is the Bayesian learning force in the Bellman equation for v, and from the experimentation literature (e.g. Keller et al. (2005) and Bonatti and Hörner (2017)) is the endogeneity of the outside option w.

In section 2, I show within the space of increasing and differentiable value functions in reputation, there is a unique equilibrium tuple. The equilibrium matching sets are connected subsets of the real line with endogenous end-points, at which the equilibrium value of the match coincides with the equilibrium reservation value of the partnering agent. The main techniques behind the proof include (i) representing the reputation function w as a solution to a maximization problem featuring the matching value functions v in the objective and the matching sets $\mathcal{M}$ as the choice variables; and (ii) the application of two necessary conditions (namely *majorant* and *superharmonic* properties) in the free-boundary problem to pin down the shape of the matching sets $\mathcal{M}$.

Furthermore, it is shown in section 2 that lower values of search frictions (equivalently higher contact rates) increase agents' reputation building incentives and hence enlarge the equilibrium matching sets. All the equilibrium analyses in this section enjoy the closed-form expressions – because I restrict the space of project types to a bi-valued set, i.e. $q \in \{a, b\}$. This assumption is relaxed in the online appendix B.2, in which I show that if the success intensity function is monotone and supermodular with respect to the agent's and project's types, then the results of the baseline model (with binary type space for projects) are robust under the general type setting for q.

Building on the baseline model, in section 3 I study the reputational externality. In particular, the meeting rates in the baseline model exhibit no interaction effects among agents, whereas in section 3 it accounts for higher meeting rates for more reputable agents. This is achieved via the choice of *reputation weights* in the matching technology function. As a result of this externality, the more reputable agents slow down the meeting rate of the less reputable ones, which amounts to under-learning of self-types and early termination of projects. Specifically, it turns out the equilibrium threshold to sever the match would be tighter than what is socially optimal. Through a comparative static exercise on the choice of the reputation weight function, I further show flattening the weighting curve reduces the agents' tolerance (namely tightens the equilibrium termination point), by lowering the equilibrium value of reputation building. Equivalently, this means in the markets where there is not a price for reputation and agents' reputation manifest itself through the rate of their contacts, *steeper* reputational incentive is a force toward efficiency.

The baseline model also lends itself to an economy where there is a spillover from successful low-type projects to the creation of high-type opportunities. Specifically, in section 4 I no longer treat the mass of available a- and b-projects as the exogenous primitives of the economy. In that, the high-type projects become available only as a result of successful breakthroughs in the low-types. Reflecting on the initial examples, firms invest in their low-skilled employees by training them, and contribute to the creation of high-skilled workers who might leave their organizations and thus the full benefits of their growth cannot be extracted by the pioneering recruiters. Alternatively, venture capitalists invest in early-stage ideas and turn them to promising ventures creating significant knowledge spillovers, whose gains due to the weaknesses in the intellectual property laws cannot be fully appropriated by the original investors.[2] At a high level one would naturally expect that agents rationally under-appreciate the matches with low-types, because they cannot fully extract the spillover gains. At the extreme there is a region where no matches with low-types are formed, therefore there will be no high-type opportunities and the total matching activity shuts down. I show contrary to its perceived nature, higher search frictions could save the market from breakdown and resurrect the efficiency (that is search frictions counteract the spillover externality). This happens because higher search frictions decrease the opportunity cost of matching with low-types and hence relax the incentive constraint to match with such projects.

Related literature. The Bayesian learning force in the agents' decision problem in this paper is based on the Poisson arrival of breakthroughs, and in that sense the paper is related to the strategic experimentation literature with Poisson news processes, initiated by Keller et al. (2005), and expanded in the follow-up works of Keller and Rady (2010) and Keller and Rady (2015). The main strategic tension in these works are the free-riding and the so called "encouragement effect" among players, as the background type is *common* across agents. Therefore, each agent's learning path conveys information to the others about the underlying hidden type. However, the subjects of learning in the present paper are the *independent* self-types of the agents and thus both of the above strategic forces are absent. Instead, the central strategic tension in section 3 is the limited mass of matching opportunities and the impact of reputation on the players' meeting rates. And the strategic force studied in section 4 is that each agent does not internalize the positive externality of matching with low-type projects to the creation of high-type opportunities for the other agents.

In the context of reputation building (when the information about the persistent or dynamic self-type is incomplete) and interpreting the reputation as the posterior belief, this paper is related to Holmström (1999) and Board and Meyer-ter-Vehn (2013). However the

[2]The public policy report about the New Zealand's government efforts to stimulate the venture capital industry by Lerner et al. (2005) highlights many of these issues.

kind of economic engagement that releases informative signals in both of these papers is the agent's effort, and in the current study is about her matching choices.

The analysis of this paper has also the flavor of the literature on learning in labor markets such as the works by Jovanovic (1979), Moscarini (2005) and Li and Weng (2017). Aside from differences in the context and motivation, the subject of learning in these studies is the *match specific* parameter, and not the underlying types of the agents. Therefore, the information released over the present match has no bearing on the future matches and naturally the reputational aspects are absent. In regard to agents learning their type while matching with partners this paper is also related to Anderson and Smith (2010) and Eeckhout and Weng (2010). There are substantive differences between the information structure in these works and the current study. For example, there is *two-sided* incompleteness of information in the former paper that complicates the Bayesian updating process, and hence led to imposing assumptions about the observability of the output of current matches with previous partners. The latter paper studies the competitive equilibria absent the search frictions, which is a central element in my paper. Lastly, different from all the previous research, the baseline model in my paper is tractable enough that can be applied to interesting and important variations in the matching technology function (e.g. reputation weighting in section 3) and could also lend itself to different ways of endogenizing the projects' population (e.g. spillovers in section 4).

2 Baseline Economy

In this section, I describe the elements of an economy populated by a unit mass of long-lived agents and a continuum of projects.

Agents. The agents in this side of the market are long-lived individuals, who care about their reputation, which is the market posterior belief about their type $\theta \in \{L, H\}$. Given the market filtration $\boldsymbol{I} = \{\mathcal{I}_t\}$, $\pi_t = \mathsf{P}\left(\theta = H | \mathcal{I}_t\right)$ refers to the time t reputation of a generic agent. The σ-field $\mathcal{I}_t$ aggregates all information that market participants hold at time $t \in \mathbb{R}_+$. Initially each agent has high type independently with probability p.

Projects. The entities on the other side of this economy are treated as projects that are picked by the agents. Specifically, they have no bargaining power against agents.[3] Each project is endowed with a type $q \in \{a, b\}$, which is publicly observable. The (unnormalized) mass of type-q projects is φ_q for $q \in \{a, b\}$, exogenously replenished and held constant.

[3]This assumption makes the analysis substantially simpler, yet it downplays the strategic role of "project owners" in the equilibrium outcomes. However, given the paper's focus on the agents' side and their reputational concerns, such an abstraction seems plausible. Also from the empirical standpoint, for example in the venture capital literature it is shown that firms can continue their projects without their original entrepreneurs; see Ueda (2004) and the references therein such as Gorman and Sahlman (1989) and Hellmann and Puri (2002).

Matching and partnerships. Pairwise meetings between agents on two sides of this market take place. The meetings are subject to search frictions with the meeting rate $\kappa > 0$, and the matching technology is *quadratic* à la Chade et al. (2017), that is the probability with which an agent meets a type-q project over the period $\mathrm{d}t$ is approximately equal to $\kappa\varphi_q \mathrm{d}t$. Furthermore, the matches are one-to-one, that is both parties have capacity constraint over the number of partners they can match with.

Output and reputation. Given the partnership between a type-θ agent and a type-q project, the success arrives at the rate $\lambda_q(\theta)$, where $\lambda_q(H) = \bar{\lambda}_q$ and $\lambda_q(L) = \underline{\lambda}_q$, with normalized payoff of one.[4] The agent has to cover the flow cost of project $c > 0$ that is common across all matches. In return, she receives the right to terminate the partnership at her will, so conceptually a stopping time problem is solved by each agent ex post to every partnership formation. The flow cost c captures both the running cost of maintaining the partnership and learning about the self-type θ. I assume there is a mechanism in the market that tracks the output of each partnership and records the Bayes-updated posterior of every agent during her match. This information is reflected in the market filtration $\boldsymbol{I}$. The posterior dynamics for the reputation process thus follows

$$\mathrm{d}\pi_t = -\pi_t(1-\pi_t)\Delta_q \mathrm{d}t, \tag{2.1}$$

prior to the success, where $\Delta_q := \bar{\lambda}_q - \underline{\lambda}_q$. For the purpose of simplicity, I assume the breakthroughs are *conclusive* in the sense that $\underline{\lambda}_q = 0$, that is the success never happens to a low-type agent. Therefore upon the success event π_t immediately jumps up to one.[5] Furthermore, without loss of generality it is assumed $\lambda_b := \bar{\lambda}_b > \lambda_a := \bar{\lambda}_a$. Also, I assume $p > c/\lambda_b$ throughout, because otherwise there are cases in which even the high-type projects are not worth matching.

Figure 1 summarizes the dynamic timeline for a typical agent, who starts the cycle with reputation π, and after some exponential random time meets a project randomly drawn from the population of unmatched ones. A decision to accept or reject the contacted project is made by the agent. Upon rejection, the agent returns to the initial node, and conditioned on acceptance an investment problem with the flow cost of c is solved. Finally, a success or a failure at the terminal node guides the entire market participants to rationally update their beliefs about the ongoing agent, and she returns back to the pool of unmatched agents. Failure simply means severing the match before the success arrival.

[4] The choice of Poisson processes to model the breakthroughs is more natural when news arrive in discrete and randomly separated instants, than the Wiener process treatment of experimentation (e.g. see Bolton and Harris (1999) and Pourbabaee (2020)).

[5] In the online appendix B.2, I relax this assumption and study the general case, where the success is not necessarily conclusive and there is a continuum of projects with the type space $[a, b]$ distributed according to an *arbitrary* CDF function ϕ.

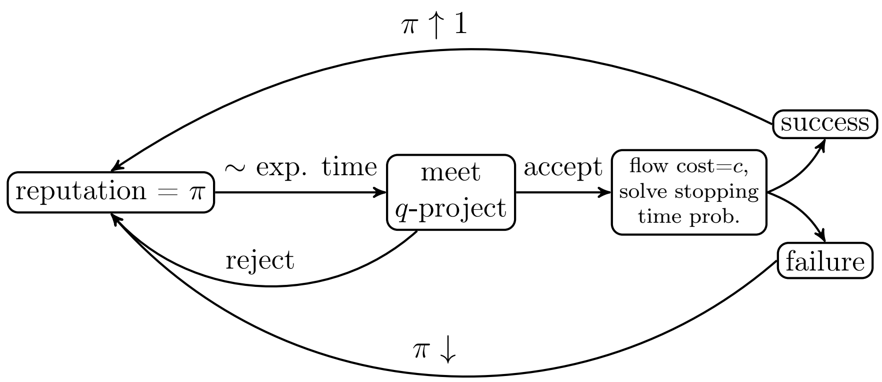


Figure 1: Decision timeline for a typical agent

2.1 Value Functions and Matching Sets

The rate of time preference for agents in this economy is $r > 0$. Let $w(\pi)$ be the value of holding reputation π, when the agent is *unmatched*. This function shall be treated as the agent's outside option and is weighed against the *matching value function* upon the meetings. The matching value function when a reputation-π individual pairs up with a type-q project is $v(\pi, q)$, that is the expected value of discounted future payoffs generated by this partnership. Therefore, a match is *profitable* if $v(\pi, q) > w(\pi)$, in that case I say $(q, \pi) \in \mathcal{M} \subseteq \{a, b\} \times [0, 1]$, where $\mathcal{M}$ is called the matching set. Also, understood from the context, $\mathcal{M}(\pi)$ (resp. $\mathcal{M}_q$) refers to the π (resp. q) *section* of this two dimensional set. In addition, often in the paper I use the indicator function $\chi_q(\pi)$ to denote whether a reputation-π agent matches with a q-project, that is whether $(q, \pi) \in \mathcal{M}$ or not. Recall that φ denotes the mass of available projects in the economy (that are so far treated exogenously as the primitives of the model). Below, I invoke a standard dynamic programming analysis for $w(\pi)$:

$$
\begin{aligned}
w(\pi) \approx \kappa \sum_{q \in \mathcal{M}(\pi)} \big(w(\pi) + [v(\pi, q) - w(\pi)] \big) \varphi_q \mathrm{d}t + \kappa \sum_{q \in \{a,b\} \setminus \mathcal{M}(\pi)} w(\pi) \varphi_q \mathrm{d}t \\
+ (1 - \kappa \varphi(\{a, b\}) \mathrm{d}t)(1 - r \mathrm{d}t) w(\pi).
\end{aligned}
$$

The first term in the *rhs* is the expected value of payoffs generated from all *acceptable* matches, taking into account that the next project with type q arrives at the rate of $\kappa \varphi_q$. The second term is the expected payoff over all *denied* partnerships, and the third term simply refers to the discounted payoff conditioned on receiving no proposal over the period $\mathrm{d}t$. Accounting for these three sources, the following Bellman equation for the reputation value function w is resulted:

$$rw(\pi) = \kappa \sum_{q \in \mathcal{M}(\pi)} \left[v(\pi, q) - w(\pi) \right] \varphi_q. \tag{2.2}$$

Next, I inspect the matching value function $v(\pi, q)$. Imagine a partnership of an agent with an initial reputation π and a type-q project. Let σ represent the random exponential time of success with the unit payoff and the arrival intensity of λ_q if $\theta = H$. Therefore, the matching value function $v(\cdot, q)$ is an endogenous outcome of a free-boundary problem with the outside option w. In that, the agent selects an optimal stopping time τ, upon which she stops backing the project, taking into account the project's success payoff and her reputation value w:

$$\begin{aligned} v(\pi, q) = \sup_{\tau} \Bigg\{ &\mathsf{E} \left[e^{-r\sigma} - c \int_0^{\sigma} e^{-rs} \mathrm{d}s + e^{-r\sigma} w(\pi_\sigma); \sigma \leq \tau \right] \\ &+ \mathsf{E} \left[-c \int_0^{\tau} e^{-rs} \mathrm{d}s + e^{-r\tau} w(\pi_\tau); \sigma > \tau \right] \Bigg\}. \end{aligned} \tag{2.3}$$

The exit option upon the stopping time τ is the agent's reservation value of holding reputation π_τ. The corresponding HJB representation for this stopping time problem is

$$rv(\pi, q) = \max \Big\{ rw(\pi), -c + \lambda_q \pi \big(1 + w(1) - v(\pi, q) \big) - \lambda_q \pi (1 - \pi) v'(\pi, q) \Big\}. \tag{2.4}$$

The above HJB is presented in the variational form, that is the first expression in the *rhs* is the value of stopping – refusing the match and holding on to the outside option w – and the second expression represents the Bellman equation over the *continuation region* $\mathcal{M}_q$, on which $v(\pi, q) > w(\pi)$. The first term in the Bellman equation is the flow cost of the project borne by the agent, the second term is the expected flow of created surplus, and the last term captures the marginal reputation loss due to the lack of success.[6] Induced by the above stopping time problem, the matching set $\mathcal{M}$ can thus be interpreted as the continuation set for the free-boundary problem (2.4), namely

$$\mathcal{M} = \left\{ (q, \pi) \in \{a, b\} \times [0, 1] : v(\pi, q) > w(\pi) \right\}, \tag{2.5}$$

and on the stopping region $\mathcal{M}^c$, the matching value function equals the agent's reputation function, i.e. $v(\pi, q) = w(\pi)$.

[6]Henceforth, all the derivatives are w.r.t π, that is for example $v'(\pi, q)$ points to the π-partial of v, unless the underlying variable is explicitly mentioned.

2.2 Equilibrium Construction

The goal of this section is to progressively suggest the necessary conditions pinning down the equilibrium outcome and finally express the properties of the endogenous matching sets.[7]

Definition 2.1 (Stationary equilibrium)**.** Given the mass φ for the unmatched projects, the tuple $\langle w, v, \mathcal{M} \rangle$ constitutes a stationary equilibrium, if (i) given v and $\mathcal{M}$, the reputation value function w satisfies (2.2); (ii) Given w, the matching value function v and the matching set $\mathcal{M}$ together solve the free-boundary system (2.4) and (2.5).

The two-way feedback between the reputation function w and the matching variables $\langle v, \mathcal{M} \rangle$ are portrayed in figure 2. The link connecting w to the $\langle v, \mathcal{M} \rangle$ block is upheld by the stopping time problem (2.3), and its recursive representation (2.4). The opposite link from the matching variables block to w is supported by the Bellman equation for the reputation function (2.2). Then, the stationary equilibrium is formally the fixed point to the endogenous loops of figure 2.

$$\begin{tikzcd} \boxed{\langle w \rangle} \arrow[r, bend left] & \boxed{\langle v, \mathcal{M} \rangle} \arrow[l, bend left] \end{tikzcd}$$

Figure 2: Equilibrium feedbacks

Next lemma uses (2.2) to express the reputation value function in terms of v and $\mathcal{M}$, and thereby provides a partial characterization of matching sets *only* in terms of the matching value functions.

Lemma 2.2. *An agent with reputation π accepts both types of projects, namely $\pi \in \mathcal{M}_a \cap \mathcal{M}_b$ iff*

$$v(\pi, a)\left(1 - \frac{1}{1 + r^{-1}\kappa\varphi_a}\right) < v(\pi, b) < v(\pi, a)\left(1 + \frac{1}{r^{-1}\kappa\varphi_b}\right). \tag{2.6}$$

In addition, $\pi \in \mathcal{M}_b \cap \mathcal{M}_a^c$ iff the upper bound is achieved, $\pi \in \mathcal{M}_a \cap \mathcal{M}_b^c$ iff the lower bound is achieved, and $\pi \in \mathcal{M}_a^c \cap \mathcal{M}_b^c$ iff the upper and lower bounds coincide, which is only the case where all value functions are zero.

Proof. An equivalent representation for (2.2) is

$$w(\pi) = \frac{r^{-1}\kappa\big[v(\pi, a)\varphi_a\chi_a(\pi) + v(\pi, b)\varphi_b\chi_b(\pi)\big]}{1 + r^{-1}\kappa\left[\varphi_a\chi_a(\pi) + \varphi_b\chi_b(\pi)\right]}. \tag{2.7}$$

[7]The term "equilibrium" is used loosely in this section, because the analysis essentially uncovers the agents' optimal response absent any strategic interactions. However, since the results in this section set out the stage for the upcoming analysis where there *is* strategic tension between agents, the use of term equilibrium is justified.

Then one can check with $\chi_a(\pi) = \chi_b(\pi) = 1$ in this representation both of the conditions $v(\pi, a) > w(\pi)$ and $v(\pi, b) > w(\pi)$ are satisfied, and hence the *if* part is established. For the *only if* direction, assume $\pi \in \mathcal{M}(a) \cap \mathcal{M}(b)$, then it must be that $\chi_a(\pi) = \chi_b(\pi) = 1$. Replacing this in (2.7) and simplifying $v(\pi, b) > w(\pi)$ results in the first inequality in (2.6). Similarly, simplifying $v(\pi, a) > w(\pi)$ leads the second inequality in (2.6). The proofs of the remaining claims follow the same logic. □

Intuitively, this lemma asserts that the ratio $v(\pi, b)/v(\pi, a)$ always lies in a bounded interval for $\pi \in \mathcal{M}_a \cup \mathcal{M}_b$. At its maximum where it reaches the upper bound, the agents do not accept a-projects and alternatively, when it hits the lower bound, the agents only choose the a-projects. This analysis renders much of the results in the next proposition on the equilibrium shape of the matching sets.

Throughout the paper, I seek to construct equilibria with increasing[8] value functions in π. Specifically, in the baseline model and its proceeding extensions the focus is given to increasing functions $v(\cdot, q)$ and $w(\cdot)$ in π.

The representation of $w(\pi)$ in (2.7) and lemma 2.2 allow us to express the equilibrium w as the output of a maximization problem over the space of all Borel measurable indicator functions $\chi_q(\pi)$ (similar idea to lemma 1 of Shimer and Smith (2000)):

$$w(\pi) = \max_{\chi} \left\{ \frac{r^{-1}\kappa \left[v(\pi, a)\varphi_a \chi_a(\pi) + v(\pi, b)\varphi_b \chi_b(\pi) \right]}{1 + r^{-1}\kappa \left[\varphi_a \chi_a(\pi) + \varphi_b \chi_b(\pi) \right]} \right\}. \tag{2.8}$$

An important consequence of the above representation is that if $v(\cdot, a)$ and $v(\cdot, b)$ are increasing in π, then $w(\cdot)$ becomes increasing in π as well. The reverse direction is the result of the following lemma.

Lemma 2.3. *The matching value functions $\{v(\cdot, q) : q \in \{a, b\}\}$ are increasing in π if and only if $w(\cdot)$ is increasing in π.*

Toward the equilibrium construction, I now analyze the Bellman equation for the matching value functions. In the sequel, I repeatedly use the general solution form for the Bellman equation (2.4) on the continuation region $\mathcal{M}_q$, in that $\gamma(q)$ is the constant dependent on the appropriate boundary conditions:

$$v(\pi, q) = -\frac{c}{r} + \frac{\lambda_q}{r + \lambda_q} \left(1 + w(1) + \frac{c}{r} \right) \pi + \gamma(q) \left(1 - \pi \right)^{1 + r/\lambda_q} \pi^{-r/\lambda_q}. \tag{2.9}$$

To further examine the essence of the stopping time problem (2.4), I highlight two *necessary* conditions that the optimal matching value function and the continuation region must satisfy.[9] The dynamics of the reputation process can be compactly represented by $\mathrm{d}\pi_t = (1 - \pi_{t^-}) \left[\mathrm{d}\iota_t - \lambda_q \pi_{t^-} \mathrm{d}t \right]$, where ι is the success indicator process, that is $\iota_t := 1_{\{t \geq \sigma\}}$.

[8] I use the word increasing to refer to a non-decreasing function.

[9] These two conditions are standard in the literature of optimal stopping and can be found in chapter 2 of Peskir and Shiryaev (2006).

The infinitesimal generator associated with this stochastic process is $\mathcal{L}_q : C^1[0,1] \to C^1[0,1]$, where for a generic $u \in C^1[0,1]$:[10]

$$[\mathcal{L}_q u](\pi) = \lambda_q \pi\big(1 + w(1) - u(\pi)\big) - \lambda_q \pi(1-\pi)u'(\pi). \tag{2.10}$$

For every candidate equilibrium tuple $\langle w, v, \mathcal{M}\rangle$, the following two conditions must hold for all $\pi \in [0,1]$ and $q \in \{a,b\}$:

(i) *Majorant property*: $v(\pi, q) \geq w(\pi)$.

(ii) *Superharmonic property*: $[\mathcal{L}_q v](\pi, q) - rv(\pi, q) - c \leq 0$.

The first condition simply means that in every partnership the agent has the option to terminate the match, thus enjoying her reputation value w. The second condition means *on expectation* a typical agent *loses* if she decides to keep the partnership on the stopping region. Exploiting these two conditions, the following proposition establishes a set of descriptive properties of equilibrium $\mathcal{M}$, when the value functions are increasing and belonging to $C^1[0,1]$. It is important to recall that because of the continuity of value functions the sections of the matching sets, $\mathcal{M}_a$ and $\mathcal{M}_b$, are open subsets of $[0,1]$. So, to characterize them, it is sufficient to identify their boundary points. For this I employ lemma 2.2 and the above two optimality conditions in conjunction with $\lambda_b > \max\{\lambda_a, c\}$ to identify these boundary points. As it turns out there appear two distinct equilibrium regimes: *low* and *high cost*, that respectively correspond to $\lambda_a - c > \frac{\kappa\varphi_b(\lambda_b - c)}{r+\lambda_b+\kappa\varphi_b}$ and $\lambda_a - c \leq \frac{\kappa\varphi_b(\lambda_b - c)}{r+\lambda_b+\kappa\varphi_b}$.

Proposition 2.4 (Endogenous matching sets)**.** *In every stationary equilibrium with increasing value functions in $C^1[0,1]$, the following properties hold:*

(i) In both regimes $1 \in \mathcal{M}_b$, and $1 \in \mathcal{M}_a$ only in the low cost regime.

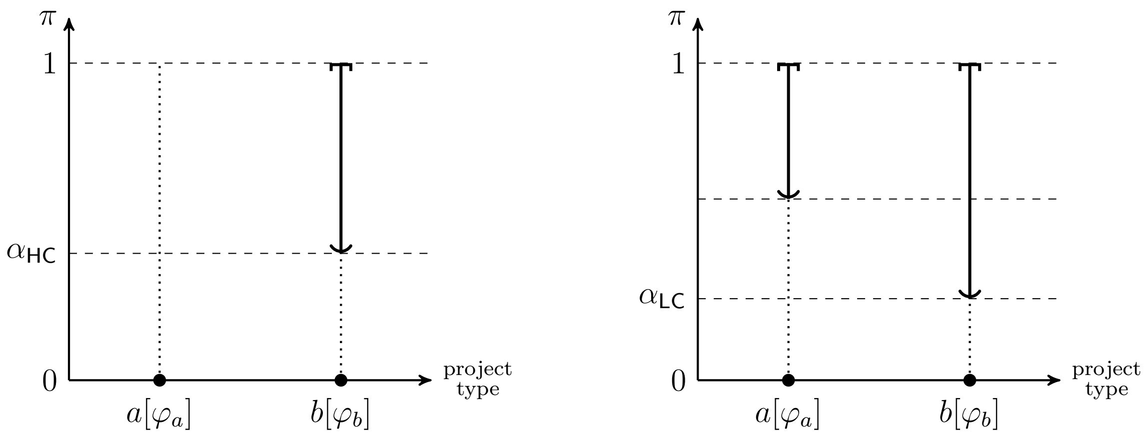
(ii) In both regimes the matching set $\mathcal{M}_b$ is a connected subset of $[0,1]$.

(iii) In the high cost regime $\mathcal{M}_a = \emptyset$ and in the low cost regime $\mathcal{M}_a$ is a connected subset of $\mathcal{M}_b$.

Figure 3 illustrates the equilibrium matching sets in both cost regimes. There are a few points related to this result that should be raised. First, it is the comparison between the expected flow payoff of matching with a-projects and the opportunity cost of forgoing the wait for the next b-project that determines the cost regime:

$$\text{low cost regime} \Leftrightarrow \lambda_a - c > \underbrace{\frac{\kappa\varphi_b(\lambda_b - c)}{r+\lambda_b+\kappa\varphi_b}}_{\substack{\text{opportunity cost of forgoing} \\ \text{the wait for a b-project}}}. \tag{2.11}$$

[10]Space of continuously differentiable functions on $(0,1)$ with continuous extension to the boundary $\{0,1\}$.



(a) high cost regime (b) low cost regime

Figure 3: Equilibrium matching sets

For instance, as the share of available b-projects (φ_b) increases, the opportunity cost of matching with a-projects goes up, and consequently, agents become more reluctant to partner with a-types. Second, one can verify that lowering the cost c increases the expected flow payoff of matching with a-projects more than it does the opportunity cost component, thereby enhancing the *variety* of accepted projects. Therefore, to the extent that these results speak to the venture capitalists' investment decisions, the equilibrium response observed in the matching sets confirms the prevalence of the investment approach "*spray and pray*" that arises due to the cost-reducing technological shocks, mentioned in Ewens et al. (2018). Third, this model suggests a method to endogenize the *tolerance for failure* (see Tian and Wang (2014) and Manso (2011)) by relating it to the agent's reputation.[11] The equilibrium observation in figure 3 on *connectedness* of the matching sets advances the idea that agents with higher reputation have higher tolerance for failure. In other words, the distance to the endogenous separation point α is larger for a more reputable agent than a less reputable one. Furthermore, when it comes to cross-project comparison, the agents show more tolerance toward b-projects – that confer faster success time on average. Fourth, in light of $\mathcal{M}_a \subset \mathcal{M}_B$ the model offers the testable prediction that the agents who exit the market and do not engage in further partnerships made their last few engagements in the high-growth projects (i.e. b-types). Formally, in both panels of figure 3 we see that the endogenous termination point α is the lower boundary point of $\mathcal{M}_b$ (not $\mathcal{M}_a$), at which the matching value function $v(\cdot, b)$ *smoothly* meets the zero function (as shown in the proof of

[11]Specifically, in Tian and Wang (2014) VCs learn about the quality of the startup over the course of the match, whereas reflecting in my model the startup's quality is observable and the learning is about the VC's ability. Consequently, the approach here suggests one way to endogenize the *tolerance parameter* in Tian and Wang (2014).

the previous proposition).[12] Also in the proof, it is established that in equilibrium

$$\alpha = \frac{c}{\lambda_b \left((1 + w(1)\right)}, \tag{2.12}$$

where $w(1)$ is the value of holding the maximum reputation, i.e. $\pi = 1$, in each cost regime. In the high cost regime $w(1)$ only depends on the b-parameters, because $\mathcal{M}_a = \emptyset$, whereas in the low cost regime it takes the a-related parameters into account as well. Some easy-to-verify comparative statics (for instance in the former case) are $\frac{\partial \alpha}{\partial c} > 0$, $\frac{\partial \alpha}{\partial \lambda_b} < 0, \frac{\partial \alpha}{\partial \varphi_b} < 0$ and $\frac{\partial \alpha}{\partial \kappa} < 0$.

Having known the form of the matching sets that are sustained in the equilibrium, I can now state the main theorem related to the decentralized matching strategy of the agents, namely the fixed point outcome of figure 2.

Theorem 2.5 (Stationary equilibrium, existence and uniqueness)**.** *There exists a unique stationary equilibrium in the space of continuously differentiable and increasing payoff functions in each cost regime. Furthermore, for large values of discount rate r, this equilibrium is unique in the larger space of $L^\infty[0, 1]$.*

The substantial result of this theorem is that there always exists an equilibrium tuple in which the value functions are increasing and continuously differentiable in reputation. Furthermore, there is not a possibility for multiple equilibria of such kind. However, the possibility of other equilibria with non-increasing value functions can not be ruled out unless the discount rate is large enough so that a contraction type theorem can be applied.

2.3 Steady-state Reputation and Social Surplus

In the identified equilibrium with matching sets expressed in proposition 2.4, the steady-state reputation of agents take value in $\{\alpha, 1\}$. Specifically, they either reach the maximum reputation $\pi = 1$, or their reputation stick at the lower endogenous point α. Consequently, there will be *four* types of agents in the steady-state: a mass of $n(\alpha)$ individuals stuck in α, in addition to three other groups with maximum reputation, $n(1)$ unmatched, $m_a(1)$ matched to the a-projects and $m_b(1)$ matched to the b-projects. Inflow outflow equations together with the Bayesian consistency at the steady-state amount to:

$$\begin{aligned} n(\alpha) + n(1) + m_a(1) + m_b(1) = 1 \quad & \kappa n(1)\varphi_a \chi_a(1) = \lambda_a m_a(1) \\ \alpha n(\alpha) + n(1) + m_a(1) + m_b(1) = p \quad & \kappa n(1)\varphi_b \chi_b(1) = \lambda_b m_b(1) \end{aligned} \tag{2.13}$$

On the first column, the first equation simply says that the total mass of agents is one, and the second equation states that in the steady-state the average ability of agents must be equal to the initial average ability p. On the second column, the first (resp. second) expression

[12] It is shown in the proof of proposition 2.4, that the smooth pasting and value matching at α is ensued in spite of the Poissonian environment and the absence of diffusion processes.

equates the inflow to the group of agents matching with a-projects (resp. b-projects) to its outflow (that is the rate at which these projects experience success thus their corresponding agent exit their position and become unmatched). The above group of equations uniquely pin down the steady-state masses, and hence render the following characterization of the social surplus function S:

$$rS = n(1)w(1) + \sum_{q\in\{a,b\}} m_q(1)v(1,q)\chi_q(1). \tag{2.14}$$

Since the matching technology function is quadratic and the matching decisions made by the agents do not alter the exogenous mass of projects (i.e. φ_a and φ_b), the economy presents no externality and hence the equilibrium outcome is constrained efficient, subject to the search frictions.

For example in the high cost equilibrium regime the social surplus function follows:

$$rS_{\mathsf{HC}} = \frac{p-\alpha_{\mathsf{HC}}}{1-\alpha_{\mathsf{HC}}}\frac{\kappa\varphi_b/\lambda_b}{1+\kappa\varphi_b/\lambda_b}\left(\lambda_b - c\right). \tag{2.15}$$

In an ideal informational setting, there is no uncertainty about the agents' types, and thus only the high-type agents with the mass of p will match and the others stay inactive, corresponding to the maximum social surplus with $\alpha = 0$. In particular, a lower distance between the steady-state reputation measure and the perfect learning benchmark is associated with lower values of equilibrium α_{HC} and corresponds to higher welfare outcomes following from (2.15). Additionally from the surplus expression in (2.15) it falls out that higher search frictions (meaning fewer meetings and smaller κ) reduces the social surplus directly and indirectly via raising the equilibrium termination point α.

3 Reputational Externality

In this section, I aim to examine the equilibrium outcome when there is a reputational externality at play. Specifically, I ask what are the indirect impacts of a reputable actor on the *remaining* body of agents? In the frictional economies where there is not a price for reputation, one would expect agents with higher reputations are exposed to more contacts. For example, in the context of two-sided market of venture capitalists and startups, there are empirical evidences about the *individual benefits* associated with higher reputation among VCs. The findings include the theory of grandstanding, and lower pay-for-performance for smaller and younger VC firms toward the goal of establishing a reputation and enjoying a *higher deal flow* (see Gompers (1996) and Gompers and Lerner (1999)). Relatedly, by dissecting investment-level data Nanda et al. (2020) finds that initial success confers preferential access to deal flow and perpetuates the early superior performances made by successful VCs. Building on the previous baseline results, in this section I investigate how this connection between higher reputation and higher meeting rates manifests itself in the equilibrium.

3.1 Equilibrium with Long-lived Agents

To capture the aforementioned interaction effect, I would propose a different matching technology. Thus far, the matching function was assumed *uniformly* quadratic. That is over every period $\mathrm{d}t$ the total mass of meetings between projects and agents was $\kappa(\varphi_a + \varphi_b)\mathrm{d}t$, and it was *uniformly* distributed among the unit mass of agents. Holding the total rate of contacts constant, now I assume this flow is not uniformly distributed among agents, rather it contacts more (resp. less) reputable individuals with higher (resp. lower) probability, according to the *reputation weight* function $\psi(\cdot)$. Specifically, let $\boldsymbol{\pi}_{\boldsymbol{\infty}}$ be the stationary distribution of agents' reputation. That is the rate with which q-projects meet an agent with reputation π is

$$\kappa\varphi_q \frac{\psi(\pi)}{\mu}, \quad \text{where} \quad \mu := \mathsf{E}\left[\psi(\boldsymbol{\pi}_{\boldsymbol{\infty}})\right].^{13}$$

I assume ψ satisfies some regularity conditions. Specifically, it belongs to the following space

$$\Psi := \Big\{\psi : [0,1] \to [0,1] \Big| \psi(0) = 0, \psi(1) = 1, \psi' \geq 0, \psi'' \leq 0\Big\}. \tag{3.1}$$

It is expected that any hope to prove a uniqueness theorem such as the one in theorem 2.5 without having much more restrictive assumptions on $\psi(\cdot)$ is doomed to fail. This is mainly because the analogue of proposition 2.4 – in which we prove the connectedness of the matching sets – for the general reputation weight function is very complicated and requires making a collection of assumptions on $\psi(\cdot)$ in conjunction with other primitives. However, to a large extent such an analysis is futile in this context, because alternatively I propose an equilibrium that exists for every $\psi(\cdot)$ satisfying the above minimal conditions. Consequently, we can perform the comparative statics on this equilibrium with respect to the choice of $\psi \in \Psi$.

Inspired by the analysis in section 2, I conjecture that there exists an equilibrium featuring $\mathcal{M}_b = (\alpha_e, 1]$, $\mathcal{M}_a \subset \mathcal{M}_b$ and $1 \in \mathcal{M}_a$ iff

$$\lambda_a - c > \frac{\kappa\varphi_b(\lambda_b - c)}{\mu(r + \lambda_b) + \kappa\varphi_b}. \tag{3.2}$$

In this equilibrium the value of holding the maximum reputation is

$$w(1) = \max_{\chi} \left\{ \frac{r^{-1}\kappa\big[\varphi_b\left(\lambda_b - c\right)\left(r + \lambda_a\right)\chi_b(1) + \varphi_a\left(\lambda_a - c\right)\left(r + \lambda_b\right)\chi_a(1)\big]}{\left(r + \lambda_a\right)\left(r + \lambda_b\right)\mu + \kappa\varphi_b\left(r + \lambda_a\right)\chi_b(1) + \kappa\varphi_a\left(r + \lambda_b\right)\chi_a(1)} \right\}. \tag{3.3}$$

The steady-state distribution of reputation across agents follows the system in equation (2.13), except that the inflow terms (on the second column) are now adjusted by a factor of

[13] Notice that μ is the *steady-state* average reputation weight, and is not the current population average of reputation weights, i.e. $\int_0^1 \psi(\pi_{it})\mathrm{d}i$. This assumption simplifies the equilibrium analysis, particularly by letting us to focus on the time-independent termination policies, i.e. constant α over time.

$\psi(1)/\mu = \mu^{-1}$. Also, the equilibrium α_e is the fixed point of the following system:

$$\mu = \frac{1-p}{1-\alpha}\psi(\alpha) + \frac{p-\alpha}{1-\alpha}, \tag{3.4a}$$

$$\alpha = \frac{c}{\lambda_b\,(1+w(1))}. \tag{3.4b}$$

Relation 3.4a is owed to the presence of $n(\alpha) = \frac{1-p}{1-\alpha}$ agents with reputation α and the remaining $n(1) + m_a(1) + m_b(1) = \frac{p-\alpha}{1-\alpha}$ agents with reputation one in the steady-state. And equation 3.4b simply expresses the endogenous termination point in line with the analysis offered for (2.12). I refer to any equilibrium with the above features as *normal* equilibrium.

Proposition 3.1. *In the described economy with reputational externality,*

(i) there always exists a normal equilibrium with $\alpha_e < p$.

(ii) The normal equilibria are Pareto ranked. Specifically, the α_e for the most (least) preferred equilibrium is increasing with respect to the pointwise order on $\psi \in \Psi$.

Part (i) ensures the existence of the normal equilibrium under the new choice of the matching technology function that exhibits the reputational externality. In light of that, we can safely claim that the sort of matching sets depicted in figure 3 are applicable in this case as well. Particularly, the normal equilibria require the matching sets to be connected and hence the outcome of learning in the economy at the steady-state can be characterized by examining the masses at the endpoints, i.e. $\pi \in \{1, \alpha_e\}$.

Emboldened by the existence of normal equilibria, the analogue of the results based on proposition 2.4 would apply in this section too, with the change of $\kappa\varphi_q$ to $\kappa\varphi_q/\mu$ in all expressions. Specifically, when it comes to cost regime determination, the characterization (2.11) changes to (3.2). In a meaningful contrast with the baseline model – where the reputational externality was absent – the agents' equilibrium response to whether match with a-projects depends on the *average reputation score* μ of the whole body of agents. Specifically, any increase in the equilibrium value of μ lowers the opportunity cost of forgoing the option to wait for b-projects, that in turn relaxes the constraint for matching with a-types. Therefore, *flattening* the extent of reputational externalities would encourage agents toward the a-projects.

To sharpen the meaning behind *flattening the reputational externality*, I investigate the effect of the choice of ψ as a *parameter* picked from the space Ψ endowed with the pointwise partial order, that is $\psi_2 \succsim \psi_1$ if $\psi_2(x) \geq \psi_1(x)\ \forall x \in [0,1]$ (see figure 4). Inspired by this figure, I say ψ_2 is *flatter* than ψ_1, because the marginal return to a higher reputation in ψ_2 is smaller than ψ_1. In part (ii) of the previous proposition, it is shown that the equilibrium termination point α_e is increasing w.r.t to $\succsim$ on Ψ. Therefore, flattening the extent of reputational externality (namely increasing ψ in a pointwise manner), reduces the agents' tolerance (i.e. increases the equilibrium α_e), by lowering the equilibrium value of reputation

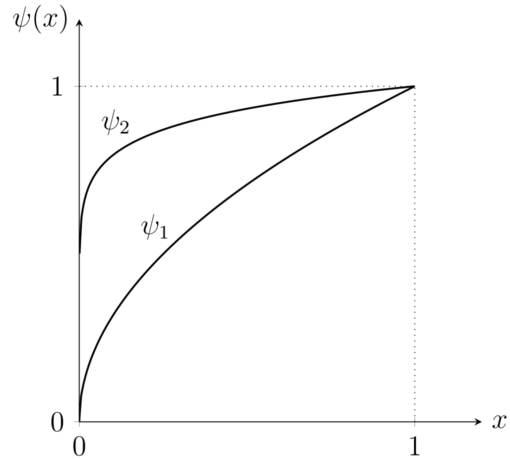


Figure 4: Weighting functions

building (i.e. $w(1)$). In addition, it relaxes the constraint for matching with a-projects in equation (3.2). The following line summarizes the result of this comparative static exercise:

$$\psi \uparrow \;\; \Rightarrow \;\; w(1) \downarrow, \; \mu_e \uparrow \;\; \text{and } \alpha_e \uparrow$$

Because of the reputational externality, one would expect *under-learning* in the equilibrium outcome relative to the social optimum. That is the reputable group of agents receive a higher than the socially optimal share of matching proposals, leaving the lesser-known agents with fewer contacts, thereby lowering their reservation value w. This also means the fraction of high-ability agents who failed their initial attempts and thus stuck in the lower barrier is larger than the socially optimal level.

The comparison of the steady-state equilibrium social surplus with the steady-state social optimum in the current environment of *long-lived* agents ignores the previous costs borne by them on the course of their matches (starting from p and ending at α_e). This is owed to the fact that in the steady-state there will be no agents with a reputation in $(\alpha_e, p]$. Therefore, in the next subsection, I will allow for exogenous birth and death of agents to obtain a *non-degenerate* stationary economy, justifying the comparison of the steady-state equilibrium outcome with the steady-state social optimum, by the means of having a continuous distribution of agents on $(\alpha_e, p]$. This tweak helps us to understand the spirit of the reputational externality and the extent to which the decentralized outcome under-appreciates the gains from more tolerance.

3.2 Short-lived Agents

The nature of reputational externality is best described if we focus only on one group of projects, say the b-projects and henceforth in this section I drop the b-index from variables.

Since the focus of the forthcoming analysis is the stationary distribution of agents' reputation and its impact on the matching decisions, and not the spillovers between different types of projects, this assumption is innocuous.

The agents are short-lived. Specifically, they leave the economy exogenously at the rate of δ, and are born with the same rate with the initial reputation p. The matching function is quadratic and exhibits reputational externality normalized by the steady-state reputation score $\mu = \mathsf{E}[\psi(\boldsymbol{\pi}_\infty)]$. In addition, it is assumed $\psi \in \Psi$. I conjecture (and prove) that there exists a symmetric stationary equilibrium in which all agents terminate their matches at a common α. In light of this conjecture, denote the *cross-sectional* density function of the matched agents by $m(\pi)$ supported on $[\alpha, p]$. Let $m(1)$ and $n(1)$ be the discrete measures of the matched and unmatched agents with maximum reputation, respectively, and finally $n(\alpha)$ and $n(p)$ are the discrete measures of unmatched group at α and p. Figure 5 plots all pieces of the cross-sectional steady-state distribution of agents' reputations.

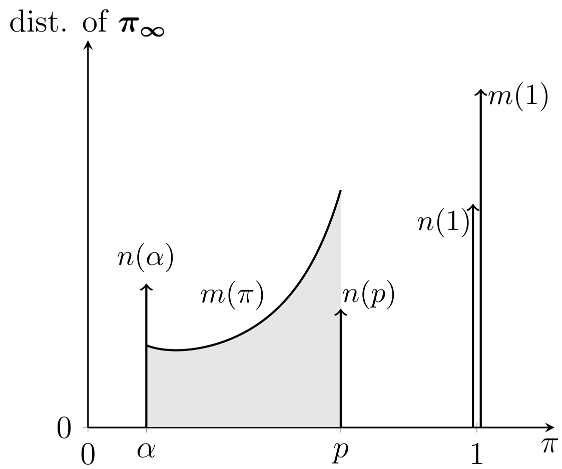


Figure 5: Steady-state cross-sectional distribution of π_∞

The inflow outflow equations at the discrete masses are:

$$\dot{m}(1) = -\lambda m(1) + \kappa\varphi\frac{n(1)}{\mu} - \delta m(1), \tag{3.5a}$$

$$\dot{n}(1) = \lambda m(1) - \kappa\varphi\frac{n(1)}{\mu} - \delta n(1) + \int_\alpha^p \lambda\pi m(\pi)\mathrm{d}\pi, \tag{3.5b}$$

$$\dot{n}(p) = -\kappa\varphi\frac{\psi(p)}{\mu}n(p) - \delta n(p) + \delta. \tag{3.5c}$$

Notice that $n(\alpha)$ is determined via the population conditions such as the conservation of zeroth and first moments. The forward equation for $m(\pi)$ is

$$\dot{m}(\pi) = -\underbrace{\lambda\pi m(\pi)}_{\substack{\text{outflow of}\\ \text{successful agents}}} + \underbrace{\lambda\partial_\pi\big(\pi(1-\pi)m(\pi)\big)}_{\text{net learning inflow}} \underbrace{-\delta m(\pi)}_{\text{exogenous exits}}. \tag{3.6}$$

The first component in the *rhs* is the outflow from $m(\pi)$ (due to the recent success events) to $n(1)$. The second term captures the net learning effect, by factoring the inflow of agents whose reputation is in $(\pi, \pi + \mathrm{d}\pi)$ and thus falling due to the lack of success and the outflow of the unsuccessful group with reputation in $(\pi - \mathrm{d}\pi, \pi)$.[14] Finally, the third term picks up the exogenous departures. In the steady-state $\dot{m}(\pi) = 0$, hence rendering a differential equation for the density function whose solution is

$$m(\pi) = m(\alpha)\left(\frac{\pi}{\alpha}\right)^{\delta/\lambda-1}\left(\frac{1-\pi}{1-\alpha}\right)^{-(\delta/\lambda+2)}, \quad \forall\pi \in [\alpha, p]. \tag{3.7}$$

The group of agents with minimum reputation at $\pi = \alpha$ are subject to two flows: the inflow from the matched ones in $(\alpha, p]$ and the outflow due to the exogenous exits at the rate of $\delta n(\alpha)$. Therefore, in the steady-state it must be that the inflow equals $\delta n(\alpha)$.

Lastly, the net inflow to the matched agents on the region $(\alpha, p]$ must equal the net outflow in the steady-state, that is:

$$\underbrace{\kappa\varphi\frac{\psi(p)}{\mu}n(p)}_{\substack{\text{new matches}\\ \text{originating from } p}} = \underbrace{\lambda\int_\alpha^p \pi m(\pi)\mathrm{d}\pi}_{\substack{\text{outflow of}\\ \text{successful agents}}} + \underbrace{\delta\int_\alpha^p m(\pi)\mathrm{d}\pi}_{\text{exogenous departure}} + \underbrace{\delta n(\alpha)}_{\substack{\text{endogenously}\\ \text{separated matches}}}. \tag{3.8}$$

Lemma A.2 in the appendix solves for the steady-state solution to the preceding distributional equations in closed form, thereby paving the way for the rest of the equilibrium analysis.

Toward finding the equilibrium, each agent stipulates the population average for ψ, say μ, and accordingly specifies the maximum attainable reputation via the mapping $\mathsf{W} : [0,1] \to \mathbb{R}_+$:

$$\mathsf{W}(\mu) := \frac{(r+\delta)^{-1}\kappa\varphi/\mu}{r+\delta+\lambda+\kappa\varphi/\mu}(\lambda - c). \tag{3.9}$$

Then, following the Bellman equation on the continuation region induced by $w(1) = \mathsf{W}(\mu)$, namely

$$rv(\pi) = \lambda - c + \lambda\big(w(1) - v(\pi)\big) - \lambda\pi(1-\pi)v'(\pi) - \delta v(\pi),$$

[14] The first two terms can also be understood in the context of Kolmogorov Forward equation (see theorem 17.4.14 of Cohen and Elliott (2015)) related to the density function of the reputation process $\mathrm{d}\pi_t = (1-\pi_{t^-})\,[\mathrm{d}\iota_t - \lambda\pi_{t^-}\mathrm{d}t]$.

each agent terminates the match at $\alpha = \mathsf{A}(w(1))$, where $\mathsf{A} : \mathbb{R}_+ \to [0,1]$ and

$$\mathsf{A}(w) := \frac{c}{\lambda(1+w)}.$$

In the symmetric stationary equilibrium the initial stipulation about μ is self-fulfilling that is $\mu = \mathsf{M}\big(\mu, \mathsf{A} \circ \mathsf{W}(\mu)\big)$, where $\mathsf{M} : [0,1]^2 \to \mathbb{R}_+$ returns the population average of reputation weights:

$$\mathsf{M}(\mu, \alpha) = \mathsf{E}\left[\psi(\boldsymbol{\pi}_{\boldsymbol{\infty}})\right] = m(1) + n(1) + \psi(p)n(p) + \int_\alpha^p \psi(\pi)m(\pi)\mathrm{d}\pi + \psi(\alpha)n(\alpha).$$

Definition 3.2 (Symmetric stationary equilibrium)**.** The symmetric stationary equilibrium in this economy with reputational externality is the set of all fixed points of the mapping $\mathsf{M}\big(\cdot, \mathsf{A} \circ \mathsf{W}(\cdot)\big)$ on the unit interval. A generic member is denoted by μ_e. Associated with the equilibrium outcome μ_e is the equilibrium termination point $\alpha_e = \mathsf{A} \circ \mathsf{W}(\mu_e)$.

In the appendix A.2.3, I show that an increase in α or μ, holding the other variable constant, *positively* shifts the steady-state distribution of $\boldsymbol{\pi}_{\boldsymbol{\infty}}$ in the sense of *second order stochastic dominance*. So, assuming a concave increasing form for $\psi(\cdot)$ one can deduce that $\mathsf{M}(\mu, \alpha)$ is an increasing function in each argument. In addition to that, the composition map $\mathsf{A} \circ \mathsf{W}$ is increasing. Therefore the mapping $\mu \mapsto \mathsf{M}\big(\mu, \mathsf{A} \circ \mathsf{W}(\mu)\big)$ is a continuous increasing function from the unit interval to itself.[15] Hence, a fixed point μ_e and $\alpha_e = \mathsf{A} \circ \mathsf{W}(\mu_e)$ exist, establishing the existence of a symmetric stationary equilibrium.

To contrast the equilibrium outcome with the socially optimal choice, I express the steady-state flow surplus of the economy in terms of the measures found in lemma A.2:

$$rS(\mu, \alpha) = (\lambda - c)m(1) + \int_\alpha^p (\lambda\pi - c)m(\pi)\mathrm{d}\pi. \tag{3.10}$$

A benevolent social planner selects an α so that jointly with its induced μ, that is the fixed point of $\mathsf{M}(\cdot, \alpha)$, maximize the social surplus $S(\mu, \alpha)$.

Definition 3.3 (Planner's problem)**.** The planner's problem is

$$\max_\alpha S(\mu, \alpha) \text{ subject to } \mu = \mathsf{M}(\mu, \alpha).$$

Remember the externality failed to be internalized in the agents' decisions is originated from the impact of their choices on μ. Therefore, it is essential to incorporate $\mu = \mathsf{M}(\mu, \alpha)$ as the constraint of the planner's problem.

Next proposition explains why the equilibrium outcome is not socially efficient, and highlights the direction along which the social surplus increases.

[15] It is clearly continuous on $(0,1]$, and it is made continuous at $\mu = 0$ by letting $\mathsf{W}(0) := \lim_{\mu\to 0} \mathsf{W}(\mu)$ and $\mathsf{M}(0,\alpha) := \lim_{\mu \to 0} \mathsf{M}(\mu, \alpha)$, where both limits exist in light of the expression (3.9) and lemma A.2.

Proposition 3.4. *Every symmetric stationary equilibrium of the economy with reputational externality is not constrained-efficient. In particular, a local reduction in the termination point α_e increases the social surplus.*

Proof. Every symmetric equilibrium is characterized by its associated pair (α_e, μ_e), in which $\alpha_e = \mathsf{A} \circ \mathsf{W}(\mu_e)$ and $\mu_e = \mathsf{M}(\mu_e, \alpha_e)$. It is further a stable equilibrium if $\partial_\mu \mathsf{M}(\mu_e, \alpha_e) < 1$. From the expression for the social surplus in (3.10) and lemma A.2 one can see that S is decreasing in μ, therefore, if $\mathsf{M}(\cdot, \alpha)$ has multiple fixed points for a given α the one with the smallest μ is the efficient one. Furthermore, this equilibrium (with the smallest μ) is stable because $\mathsf{M}(0, \alpha) > 0$, and $\mathsf{M}(\cdot, \alpha)$ *downcrosses* the 45-degree line in its first intersection.

Toward proving the constrained inefficiency, I employ a variational approach in the neighborhood of α_e. Suppose the economy is in a stable pair (α_e, μ_e), and the planner moves α_e by $\Delta\alpha$. The new smallest fixed point $\mu_e + \Delta\mu$ satisfies

$$\mu_e + \Delta\mu = \mathsf{M}(\mu_e + \Delta\mu, \alpha_e + \Delta\alpha) \approx \mathsf{M}(\mu_e, \alpha_e) + (\partial_\mu \mathsf{M})\Delta\mu + (\partial_\alpha \mathsf{M})\Delta\alpha,$$

hence $\Delta\mu \approx \frac{\partial_\alpha \mathsf{M}}{1 - \partial_\mu \mathsf{M}} \Delta\alpha$. Consequently, the change in the social surplus function would be

$$r\Delta S \approx r \left(\frac{\partial_\alpha \mathsf{M}}{1 - \partial_\mu \mathsf{M}} \partial_\mu S + \partial_\alpha S \right) \Delta\alpha.$$

Note that in every stable fixed point of $\mathsf{M}(\cdot, \alpha_e)$, $\frac{\partial_\alpha \mathsf{M}}{1 - \partial_\mu \mathsf{M}} > 0$, because M is shown to be increasing in α and due to the stability $\partial_\mu \mathsf{M} < 1$. Furthermore, $\partial_\mu S < 0$. Therefore, lowering α_e, i.e. $\Delta\alpha < 0$, leads to a strict improvement in the social surplus if $\partial_\alpha S < 0$. Relying on expression (3.10) together with lemma A.2 and applying some rearrangements lead to

$$\begin{aligned} r\partial_\alpha S(\mu_e, \alpha_e) &= (\lambda - c)\partial_\alpha m(1) - (\lambda\alpha - c)m(\alpha) \\ &= -\underbrace{\frac{\kappa\varphi\psi(p)/\mu_e}{\delta + \kappa\varphi\psi(p)/\mu_e} \frac{1-p}{(1-\alpha_e)^2} \left(\frac{p}{1-p}\right)^{-\delta/\lambda} \left(\frac{\alpha_e}{1-\alpha_e}\right)^{\delta/\lambda}}_{>0} \times \\ &\qquad \left[\frac{\delta(\lambda\alpha_e - c)}{\lambda\alpha_e} + \frac{(\lambda - c)\kappa\varphi/\mu_e}{\delta + \lambda + \kappa\varphi/\mu_e} \right]. \end{aligned}$$

Therefore, the sign of $\partial_\alpha S(\mu_e, \alpha_e)$ is the opposite of the sign of the expression in the bracket. Recalling that in the equilibrium $\alpha_e = \mathsf{A} \circ \mathsf{W}(\mu_e)$, so

$$\begin{aligned} \frac{\delta(\lambda\alpha_e - c)}{\lambda\alpha_e} + \frac{(\lambda - c)\kappa\varphi/\mu_e}{\delta + \lambda + \kappa\varphi/\mu_e} &= -\delta\mathsf{W}(\mu_e) + \frac{(\lambda - c)\kappa\varphi/\mu_e}{\delta + \lambda + \kappa\varphi/\mu_e} \\ &= -\delta\mathsf{W}(\mu_e) + \delta \lim_{r \to 0} \mathsf{W}(\mu_e) \geq 0, \end{aligned}$$

where the last inequality holds because $\mathsf{W}(\mu_e)$ is decreasing in r. This concludes that $\partial_\alpha S(\mu_e, \alpha_e) < 0$, and hence a small reduction of equilibrium α_e leads to a strict improvement of the social surplus function. $\square$

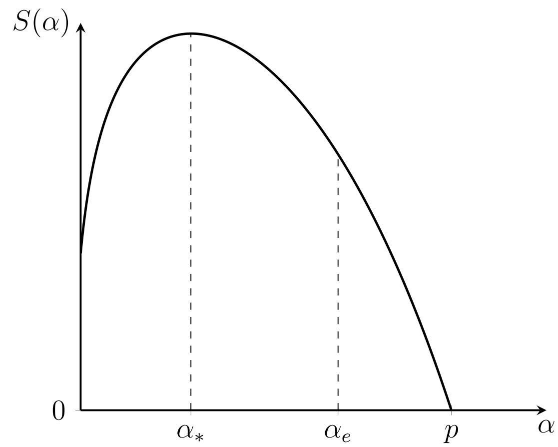


Figure 6: Social surplus with reputational externality

Figure 6 is the result of a simulation that plots the social surplus as a function of α, while implicitly satisfying $\mu = \mathsf{M}(\mu, \alpha)$ at every $\alpha \in [0, p]$. As it is expressed in this plot, the equilibrium termination point α_e is greater than the socially optimal point α_*. Hence, the equilibrium outcome is associated with early termination of projects, and predicts a lower tolerance for failure than what is socially efficient. Furthermore, in equilibrium the proportion of agents who are both high ability and inactive (that is equal to $\frac{\alpha(1-p)}{1-\alpha}$) is higher than its optimum level.

4 Project Spillovers

In the previous two sections the mass of available projects were treated exogenously. However, one could envision an economy where these masses depend on the past decisions of the agents, so they are endogenously determined in the equilibrium. Specifically, the choice of the matching sets could potentially have an impact on the supply side of this economy and particularly the mass of available projects (see figure 7).

Especially in the current section and in the context of innovation research, one can interpret the type-a projects as the ones associated with risky radical innovations with longer average time to success and the type-b projects as the safer incremental ventures with shorter average time to success. In another close reading, the a-projects can be thought as the early stage startups, and the b-projects as the late stage alternatives. At any rate, there is an spillover from *successful* a-projects to the available mass of b-projects. Formally, the stationary mass of φ_b depends on the mass of successful a-projects. Toward this construction, suppose a fraction ζ of successful a-types would spill over to the rest of economy, and gives birth to the creation of the b-projects. Therefore, in any steady-state outcome it must be

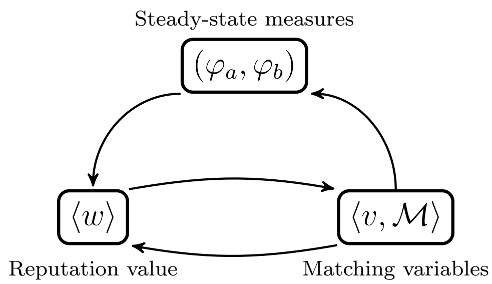


Figure 7: Equilibrium feedbacks with endogenous mass of projects

that

$$\zeta \lambda_a m_a(1) \chi_a(1) = \kappa \varphi_b n(1),$$

where the *lhs* is the rate at which the a-projects experience success multiplied by the fraction ζ entering into the pool of available b-types. And the *rhs* is the outflow rate from the pool of unmatched b-projects. So, conditioned on $\chi_a(1) = 1$, then $\varphi_b = \zeta \varphi_a$. Consequently, if

$$\lambda_a - c > \frac{\kappa \zeta \varphi_a (\lambda_b - c)}{r + \lambda_b + \kappa \zeta \varphi_a},$$

then agents match with a-projects, of which the successful ones create the b-types. Therefore, in the stationary equilibrium both types of projects coexist. I call this equilibrium the *maximum surplus equilibrium*. On the other hand, when $\lambda_a \leq c$, agents do not match with any project, so the partnerships shut down, and it is referred to as *zero surplus equilibrium*. Importantly, we observe that higher search frictions – translating to lower κ – brings down the opportunity cost of forgoing the option to wait for b-projects, and hence increases the likelihood of matching with a-types. Specifically, as it relates to the equilibrium social surplus, the search friction counters the spillover externality and hence is a force toward efficiency.

Toward a better understanding of the constrained optimum and the source of externality in this economy, I express the planner's constrained optimization problem below. The maximand is the expected social surplus of the economy and the constraints are the dynamical equations for the population of long-lived agents and projects. Let $m_q(1)$ be the mass of agents with maximum reputation connected to a q-project; $n(1)$ the mass of unmatched individuals with reputation 1; $m_q(\pi)$ the *density* of matched agents to a q-project, and finally $n(\pi)$ is the density of unmatched individuals with reputation π. All of these measures are time-dependent (even though the time index t is suppressed). Therefore, the discounted social surplus of this economy is

$$S = \int_0^\infty e^{-rt} \left(\sum_q (\lambda_q - c) m_q(1) + \int (\lambda_q \pi - c) m_q(\pi) \mathrm{d}\pi \right) \mathrm{d}t. \tag{4.1}$$

The planner chooses the time-dependent matching indicators $\chi_q(\pi)$ to maximize S subject to the following law of motions for the population measures:

$$\dot{m}_q(1) = -\lambda_q m_q(1) + \kappa\varphi_q n(1)\chi_q(1), \text{ for } q \in \{a, b\} \tag{4.2a}$$

$$\dot{n}(1) = \sum_q \lambda_q m_q(1) - \sum_q \kappa\varphi_q n(1)\chi_q(1) + \sum_q \int \lambda_q \pi m_q(\pi)\mathrm{d}\pi \tag{4.2b}$$

$$\dot{m}_q(\pi) = -\lambda_q \pi m_q(\pi) + \kappa\varphi_q n(\pi)\chi_q(\pi) + \lambda_q \partial_\pi \left(\pi(1-\pi)m_q(\pi)\right), \text{ for } q \in \{a, b\} \tag{4.2c}$$

$$\dot{n}(\pi) = -\sum_q \kappa\varphi_q n(\pi)\chi_q(\pi) \tag{4.2d}$$

The interpretation for the above distributional dynamics is similar to (3.5) and (3.6), and thus omitted. The last state constraint that should be considered in the planner's problem is

$$\dot{\varphi}_b = \underbrace{\zeta\lambda_a\left(m_a(1) + \int \pi m_a(\pi)\mathrm{d}\pi\right)}_{\substack{\text{spillover from successful } a\text{-projects} \\ \text{to } b\text{-projects}}} - \underbrace{\kappa\varphi_b\left(n(1)\chi_b(1) + \int n(\pi)\chi_b(\pi)\mathrm{d}\pi\right)}_{\text{outflow due to recent partnerships}}. \tag{4.3}$$

This equation relates the rate of change of the mass of available b-projects ($\dot{\varphi}_b$) to the inflow originated from the spillovers of the successful a-types and the outflow of the recent partnerships made with the members of the unmatched b-group.

Let $\{v_*(1,q), w_*(1), v_*(\pi,q), w_*(\pi)\}$ respectively be the co-state processes for equations (4.2a), (4.2b), (4.2c) and (4.2d); each of them shall be interpreted as the *social marginal value* of an additional member to its associated group. For example, $v_*(\pi,q)$ is the social marginal value of adding one more (π,q) match. Also, denote the co-state process for equation (4.3) by ρ. I solve for the social optimum of this economy by analyzing the current value Hamiltonian in the online appendix B.1. It is established there that from the planner's viewpoint:

$$\chi_q^*(\pi) = 1 \Leftrightarrow v_*(\pi,q) > w_*(\pi).$$

Particularly, a match between an agent with reputation π and a type-q project is socially optimal if the social marginal value of the match $v_*(\pi,q)$ exceeds the social marginal value of holding reputation $w_*(\pi)$.

Also shown in the online appendix, in the steady-state where the time derivatives are zero, the following co-state equations are resulted for social contributions:

$$\begin{aligned}
rv_*(1,q) &= \lambda_q - c + \lambda_q\left(w_*(1) - v_*(1,q)\right) + \boxed{\rho\zeta\lambda_a 1_{\{q=a\}}} \text{ if } \chi_q^*(1) = 1 \\
rw_*(1) &= \sum_q \kappa\varphi_q\big(v_*(1,q) - w_*(1)\big)\chi_q^*(1) - \boxed{\rho\kappa\varphi_b\chi_b^*(1)} \\
rv_*(\pi,q) &= \lambda_q\pi - c + \lambda_q\pi\big(w_*(1) - v_*(\pi,q)\big) - \lambda_q\pi(1-\pi)v_*'(\pi,q) + \boxed{\rho\zeta\lambda_a\pi 1_{\{q=a\}}} \text{ if } \chi_q^*(\pi) = 1 \\
rw_*(\pi) &= \sum_q \kappa\varphi_q\big(v_*(\pi,q) - w_*(\pi)\big)\chi_q^*(\pi) - \boxed{\rho\kappa\varphi_b\chi_b^*(\pi)}
\end{aligned} \tag{4.4}$$

The above differential characterization of the social value functions should be juxtaposed with the private valuations of (2.2) and (2.4). In particular, the terms in the boxes precisely characterize the sources of the departure of the social from private incentives. These terms can guide us about the profile of taxes decentralizing the social optimum. When there are spillovers from a- to b-projects the above expressions suggest the redistributive mechanism that subsidizes the cost of matching with a-projects and taxes the output of matches with b-types.

The subsidy $\rho\zeta\lambda_a\pi$ can be made either as a flow payment that depends on the current value of agent's reputation π, or equivalently (and more implementable) as a one-off payment that agents of a-matches receive upon the success with the face value of $\rho\zeta$. On the other hand the tax imposed on the unmatched agents is $\rho\kappa\varphi_b$, where $\varphi_b = \zeta\varphi_a$ in the steady-state level resulted from equation (4.3).

In the steady-state the redistributive schedule is budget neutral, so the planner runs no deficit or surplus. This is owed to the following accounting analysis:

$$\text{total subsidy} = \rho\zeta\lambda_a m_a(1) + \rho\zeta\lambda_a \int \pi m_a(\pi)\mathrm{d}\pi,$$

$$\text{total tax revenue} = \rho\kappa\varphi_b \left(n(1)\chi_b(1) + \int n(\pi)\chi_b(\pi)\mathrm{d}\pi \right).$$

Since in the steady-state $\dot{\varphi}_b = 0$, the above two sums are equal. Furthermore, in the steady-state of this economy, the densities should be identically equal to zero, and all the masses concentrate discretely on the boundaries. This observation hints at the condition under which intervention, namely setting $\chi_a(1) = 1$, is justified. Particularly, $\chi_a^*(1) = 1$ iff the resulting social surplus exceeds zero, which is what economy achieves when $\chi_a(1) = 0$ and $\varphi_b = \chi_b(1) = 0$. So, $\chi_a^*(1) = 1 \Leftrightarrow \lambda_a m_a(1) + \lambda_b m_b(1) > 0$. This condition translates to

$$\chi_a^*(1) = 1 \Leftrightarrow \zeta > \frac{\lambda_b(c - \lambda_a)}{\lambda_a(\lambda_b - c)}.$$

There is a very important message behind this derivation: the centralized intervention – in the form of tax and subsidy and even setting the choice of matching sets – is justified if and only if the spillovers from a- to b-projects are strong enough.[16]

Lastly, in the steady-state, the co-state process ρ, takes the following form:

$$r\rho = \kappa n(1)\big(v_*(1, b) - w_*(1) - \rho\big)\chi_b^*(1). \tag{4.5}$$

This equation confirms that that $\rho \geq 0$ and therefore the *direction* of transfers explained above is indeed correct.

[16]Relatedly in the entrepreneurship literature Lerner (2002) points to the failure of the Malaysian government to correctly predict the extent of such positive spillovers, and misguidedly making sizable upfront investments to boost the biotechnological developments in BioValley. The project earned the nickname of the "Valley of Bio-Ghosts" few years after its inception (chapter 6 in Lerner (2009)).

5 Conclusion

I study the decentralized outcome of a dynamic economy populated by agents with unknown abilities and projects with observable qualities, where individuals randomly meet each other subject to search frictions and form partnerships. Since maintaining the matches are costly, agents effectively solve stopping time problems in which they weigh the expected benefits of learning about their types and accomplishing breakthroughs against the endogenous reservation function (that is called reputation value function in the paper). The matching sets indicate what type of projects an agent with certain level of reputation is willing to match with, or alternatively what levels of reputation are profitable to match with a certain project. Within the space of increasing and differentiable value functions in reputation, I show there exists a unique equilibrium with connected matching sets, whose end-points are endogenously determined, and encode a number of messages. For example, lower levels of flow cost, search frictions and time discount rate are all associated with larger equilibrium matching sets.

Building on the baseline equilibrium framework, I study the outcomes of the economy under two important sources of externality: reputational externality and project spillovers. In the former case, the meeting rate of each agent is positively impacted by her reputation and negatively by the average reputation score of other agents. As a result of this externality, I show a local increase in agent's tolerance (equivalently a local decrease in the lower end-point of the matching sets) increases the social surplus. In addition, with a comparative static exercise on the choice of the reputation weights, I show in the markets where there is not a price for reputation and agents' reputation manifest itself in the rate of their contacts, *steeper* reputational incentive is a force toward efficiency. In the case of spillovers from successful low-type projects to the creation of available high-type opportunities, I show despite its conventional meaning, higher search friction could save the market from breakdown caused by the individually rational rejection of matching with low-type projects.

Finally, one could envision an economy that is subject to *both* sources of discussed externalities. The analysis offered in this paper suggests that steeper reputational incentives tilt the agents' decisions toward matching with low-type projects, and thus countering the spillover effect.

A Proofs

A.1 Proofs of Section 2

A.1.1 Proof of Lemma 2.3

Suppose both matching value functions, i.e. $v(\cdot, a)$ and $v(\cdot, b)$, are increasing in π. Then, the representation (2.8) implies that $w(\cdot)$ should be increasing in π. Conversely, assume $w(\cdot)$

is increasing in π, and hence almost everywhere differentiable on $[0, 1]$.[17] Recall that $v(\cdot, q)$ is the solution to the optimal stopping time problem (2.3). In that τ is the stopping time adapted to all possible future information. However, no information is released until the breakthrough time σ, and hence τ only uses the current information. This means that I can restrict the optimization space to the set of all deterministic times:

$$
\begin{aligned}
v(\pi, q) &= \sup_{\tau \in \mathbb{R}_+} V(\pi, q; \tau) \\
V(\pi, q; \tau) &:= \int_0^\tau \Big(r^{-1} c \left(e^{-rt} - 1\right) + e^{-rt}\big(1 + w(1)\big)\Big)\lambda_q \pi e^{-\lambda_q t} \mathrm{d}t \\
&\qquad + \left(1 - \pi + \pi e^{-\lambda_q \tau}\right) \left(r^{-1} c \left(e^{-r\tau} - 1\right) + e^{-r\tau} w(\pi_\tau)\right).
\end{aligned}
$$

Since w is almost everywhere differentiable, then $V(\cdot, q; \tau)$ inherits this property. Let us define $\frac{\partial V}{\partial \pi}(\pi, q; \tau) := I_1 + I_2 + I_3$, where

$$
\begin{aligned}
I_1 &:= r^{-1} c \left(\frac{\lambda_q}{r + \lambda_q}\left(1 - e^{-(r+\lambda_q)\tau}\right) - e^{-r\tau}\left(1 - e^{-\lambda_q \tau}\right)\right), \\
I_2 &:= \frac{(1 + w(1))\, \lambda_q}{r + \lambda_q}\left(1 - e^{-(r+\lambda_q)\tau}\right) - \left(1 - e^{-\lambda_q \tau}\right) e^{-r\tau} w(\pi_\tau), \\
I_3 &:= e^{-r\tau}\left(1 - \pi + \pi e^{-\lambda_q \tau}\right) w'(\pi_\tau) \frac{\partial \pi_\tau}{\partial \pi}.
\end{aligned}
$$

The expression for I_1 is zero when $\tau = 0$, and has positive derivative w.r.t τ, therefore, it is non-negative for all $\tau \geq 0$. The third term I_3 is obviously non-negative, because w is assumed increasing and due to the Bayes law $\partial \pi_\tau / \partial \pi > 0$. In regard to the second term:

$$
\begin{aligned}
I_2 &\geq \frac{(1 + w(1))\, \lambda_q}{r + \lambda_q}\left(1 - e^{-(r+\lambda_q)\tau}\right) - \left(1 - e^{-\lambda_q \tau}\right) e^{-r\tau} w(1) \\
&\geq w(1)\left(\frac{\lambda_q}{\lambda_q + r}\left(1 - e^{-(r+\lambda_q)\tau}\right) - e^{-r\tau}\left(1 - e^{-\lambda_q \tau}\right)\right).
\end{aligned}
$$

The component in the bracket above is increasing in τ and equals zero at $\tau = 0$, therefore, it is always non-negative. To sum, $\partial V / \partial \pi \geq 0$ almost everywhere and it is continuous in π, therefore V becomes increasing in π. Since $v(\pi, q) = \sup_\tau V(\pi, q; \tau)$, the matching value function $v(\cdot, q)$ must be increasing too. $\square$

[17] This is due the seminal Lebesgue theorem; see chapter 6 of Royden and Fitzpatrick (1988).

A.1.2 Proof of Proposition 2.4

Proof of part (i): At $\pi = 1$ the following fixed point system falls out of (2.8) and the rearranged version of (2.4):

$$w(1) = \max_{\chi} \left\{ \frac{r^{-1}\kappa\big(v(1,a)\varphi_a\chi_a(1) + v(1,b)\varphi_b\chi_b(1)\big)}{1 + r^{-1}\kappa\big(\varphi_a\chi_a(1) + \varphi_b\chi_b(1)\big)} \right\}, \tag{A.1a}$$

$$v(1,q) = \max\left\{ w(1), \frac{\lambda_q - c}{r+\lambda_q} + \frac{\lambda_q}{r+\lambda_q} w(1) \right\} \quad \text{for} \quad q \in \{a,b\}.. \tag{A.1b}$$

From (A.1b) it follows that

$$\chi_a(1) = 1 \Leftrightarrow rw(1) < \lambda_a - c \text{ and } \chi_b(1) = 1 \Leftrightarrow rw(1) < \lambda_b - c. \tag{A.2}$$

So there are three cases that could possibly arise from (A.2):

(a) $1 \notin \mathcal{M}_b \cup \mathcal{M}_a \Rightarrow w(1) = 0$, yet this never happens because $\lambda_b > c$ implies $v(1,b) > 0$ and hence $w(1) > 0$.

(b) $1 \in \mathcal{M}_b \cap \mathcal{M}_a^c$ so

$$w(1) = \frac{r^{-1}\kappa\varphi_b\left(\lambda_b - c\right)}{r + \lambda_b + \kappa\varphi_b}. \tag{A.3}$$

The pair $v(1,a) = w(1)$ and $v(1,b) = \left(1 + r/\kappa\varphi_b\right) w(1)$ satisfy (and is the only solution of) the fixed point system (A.1) if $\lambda_a - c \leq \frac{\kappa\varphi_b(\lambda_b - c)}{r+\lambda_b+\kappa\varphi_b}$.

(c) $1 \in \mathcal{M}_b \cap \mathcal{M}_a$ so

$$w(1) = \frac{r^{-1}\kappa\varphi_b\left(\lambda_b - c\right)\left(r+\lambda_a\right) + r^{-1}\kappa\varphi_a\left(\lambda_a - c\right)\left(r+\lambda_b\right)}{\left(r+\lambda_a\right)\left(r+\lambda_b\right) + \kappa\varphi_b\left(r+\lambda_a\right) + \kappa\varphi_a\left(r+\lambda_b\right)}. \tag{A.4}$$

If $\lambda_a - c > \frac{\kappa\varphi_b(\lambda_b - c)}{r+\lambda_b+\kappa\varphi_b}$ the above $w(1)$ satisfies (A.2). Moreover, the $v(1,a)$ and $v(1,b)$ obtained from (A.1b) once replaced as the optimization input in the *rhs* of (A.1a) confirms the $w(1)$ in (A.4), thereby closing the equilibrium loop.$\|$

Proof of part (ii): In the sequel I use the symbol $\underline{\partial} A$ to denote the lower boundary of a subset $A \subset [0,1]$. To establish the convexity of $\mathcal{M}_b$, I first derive a useful identity for any strictly positive point $x \in \mathcal{M}_a \cap \underline{\partial}\mathcal{M}_b$. Since x is a lower boundary point for $\mathcal{M}_b$, then an agent finds it optimal to terminate the match when π approaches down to x. Importantly, at this point the principles of continuous and smooth fit (Dixit (2013)) must hold. The agent's outside option just below x is equal to $w(x)$ that is supported by the option value of meeting an a-type project because $x \in \mathcal{M}_a$, so

$$v(x,b) = w(x) = \frac{\kappa\varphi_a}{r+\kappa\varphi_a} v(x,a) \quad \text{and} \quad v'(x,b) = w'(x) = \frac{\kappa\varphi_a}{r+\kappa\varphi_a} v'(x,a).$$

Now define $\Omega(x,q) := -c + \lambda_q x\,(1+w(1))$ and $\Gamma(x,q) := r + \lambda_q x$. Then, employing the HJB equations on the continuation region leads to

$$\frac{v'(x,b)}{v'(x,a)} = \frac{\lambda_a}{\lambda_b}\frac{\Omega(x,b)-\Gamma(x,b)v(x,b)}{\Omega(x,a)-\Gamma(x,a)v(x,a)}.$$

The previous two systems of equations give rise to

$$\frac{\kappa\varphi_a}{r+\kappa\varphi_a}\left(\frac{\lambda_b}{\lambda_a}\Gamma(x,a)-\Gamma(x,b)\right)v(x,a) = \frac{\kappa\varphi_a}{r+\kappa\varphi_a}\frac{\lambda_b}{\lambda_a}\Omega(x,a)-\Omega(x,b), \tag{A.5a}$$

$$\Rightarrow \frac{\kappa\varphi_a}{r+\kappa\varphi_a}\left(\frac{\lambda_b}{\lambda_a}-1\right)rv(x,a) = -c\left(\frac{\kappa\varphi_a}{r+\kappa\varphi_a}\frac{\lambda_b}{\lambda_a}-1\right)-\frac{r\lambda_b x\,(1+w(1))}{r+\kappa\varphi_a}. \tag{A.5b}$$

Now assume to the contrary that $\mathcal{M}_b$ is not connected, hence, it contains at least two separate open sets, say (x_0,x_1) and (x_2,x_3). This implies that $[x_1,x_2]\subset\mathcal{M}_a$, because otherwise w assumes zero at some point in this interval which violates the monotonicity of w. Therefore, $x_2\in\mathcal{M}_a\cap\underline{\partial}\mathcal{M}_b$, and A.5b holds at x_2. I claim that $x_0\in\mathcal{M}_a\cap\underline{\partial}\mathcal{M}_b$ too, because otherwise x_0 would be the lower boundary point at which $v(\cdot,b)$ smoothly meets the *zero* function, hence applying the continuous and smooth fit to the equation (2.9) yields

$$x_0 = \frac{c}{\lambda_b\,(1+w(1))}. \tag{A.6}$$

This expression for x_0 leads to an upper bound for $v(x_2,a)$ using (A.5b):

$$\begin{aligned}\frac{\kappa\varphi_a}{r+\kappa\varphi_a}\left(\frac{\lambda_b}{\lambda_a}-1\right)rv(x_2,a) &\leq -c\left(\frac{\kappa\varphi_a}{r+\kappa\varphi_a}\frac{\lambda_b}{\lambda_a}-1\right)-\frac{r\lambda_b x_0\,(1+w(1))}{r+\kappa\varphi_a}\\ &= \frac{c\kappa\varphi_a}{r+\kappa\varphi_a}\left(1-\frac{\lambda_b}{\lambda_a}\right)<0.\end{aligned}$$

That in turn means $v(x_2,a)<0$, hence a contradiction results. Therefore, x_0 and x_2 both belong to $\mathcal{M}_a\cap\underline{\partial}\mathcal{M}_b$. One can now apply (A.5b) at these two points and subtract their corresponding sides from each other:

$$\frac{\kappa\varphi_a}{r+\kappa\varphi_a}\left(\frac{\lambda_b}{\lambda_a}-1\right)r\big(v(x_2,a)-v(x_0,a)\big) = \frac{-r\lambda_b\,(1+w(1))}{r+\kappa\varphi_a}\,(x_2-x_0)\,. \tag{A.7}$$

The *lhs* to this equation is positive because of the monotonicity of $v(\cdot,a)$, but the *rhs* is negative, hence a contradiction is resulted, thus proving the connectedness of $\mathcal{M}_b$.$\|$

Proof of part (iii):
High cost regime: First, I show in this regime $\mathcal{M}_a$ cannot have a lower boundary point in $\mathcal{M}_b$, that is $\underline{\partial}\mathcal{M}_a\cap\mathcal{M}_b=\emptyset$. Toward the contradiction assume $\exists\ y\in\underline{\partial}\mathcal{M}_a\cap\mathcal{M}_b$. Then, a

similar analysis to the previous part yields

$$\left(\frac{\lambda_b}{\lambda_a}-1\right)rv(y,b)=-c\left(\frac{r+\kappa\varphi_b}{\kappa\varphi_b}\frac{\lambda_b}{\lambda_a}-1\right)+\frac{r\lambda_b y\,(1+w(1))}{\kappa\varphi_b}. \tag{A.8}$$

In light of lemma 2.2, such a y is a global maximum for $v(\cdot,b)/v(\cdot,a)$ on the region $w>0$. Thus, the first order derivative of $v(\cdot,b)/v(\cdot,a)$ at y will be zero, namely $v'(y,a)/v(y,a)=v'(y,b)/v(y,b)$. Since $y\in\underline{\partial}\mathcal{M}_a\cap\mathcal{M}_b$, and the second order derivatives of matching value functions exist within the interior of this set (because of the solution form (2.9)), then it must be that $\lim_{\varepsilon\downarrow 0}\left(\frac{v(y+\varepsilon,b)}{v(y+\varepsilon,a)}\right)''\le 0$. Let us denote $v''(y,q):=\lim_{\varepsilon\downarrow 0}v''(y+\varepsilon,q)$ for $q\in\{a,b\}$, then the previous second order condition implies:

$$\frac{v''(y,b)}{v(y,b)}\le\frac{v''(y,a)}{v(y,a)}\Rightarrow v''(y,b)\le\frac{r+\kappa\varphi_b}{\kappa\varphi_b}v''(y,a). \tag{A.9}$$

Next, I find an expression for the second order derivative by differentiating the HJB equation (2.4) on the continuation region:

$$\begin{aligned}rv'(y,q)=\lambda_q\big(1+w(1)-v(y,q)\big)-\lambda_q yv'(y,q)\\ -\lambda_q(1-2y)v'(y,q)-\lambda_q y(1-y)v''(y,q).\end{aligned}$$

Substituting $v'(\cdot,q)$ from the HJB in the above equation leads to

$$\begin{aligned}\lambda_q y(1-y)v''(y,q)&=\lambda_q\big(1+w(1)-v(y,q)\big)\\&\quad-\frac{\big(r+\lambda_q(1-y)\big)}{\lambda_q y(1-y)}\times\big(-c+\lambda_q y\,(1+w(1))-(r+\lambda_q y)\,v(y,q)\big)\\&=-\frac{r\,(1+w(1))}{1-y}+\frac{r(r+\lambda_q)}{\lambda_q y(1-y)}v(y,q)+\frac{c\,(r+\lambda_q(1-y))}{\lambda_q y(1-y)}.\end{aligned}$$

Plugging the second order derivatives from above into (A.9) and applying some rearrangements yield the following *equivalent* relation

$$\begin{aligned}rv(y,b)\left(\frac{\lambda_b}{\lambda_a}-1\right)\left(1+\frac{r}{\lambda_a}+\frac{r}{\lambda_b}\right)\ge\big(ry\,(1+w(1))-c(1-y)\big)\left(\frac{r+\kappa\varphi_b}{\kappa\varphi_b}\frac{\lambda_b}{\lambda_a}-1\right)\\ -\frac{cr}{\lambda_b}\left(\frac{r+\kappa\varphi_b}{\kappa\varphi_b}\frac{\lambda_b^2}{\lambda_a^2}-1\right).\end{aligned}$$

Substituting (A.8) in above and applying several regroupings amount to:

$$y\Big[\,(1+w(1))\left(\lambda_a\,(r+\lambda_b)-\kappa\varphi_b\,(\lambda_b-\lambda_a)\right)-c\left(\lambda_b+r^{-1}\kappa\varphi_b\,(\lambda_b-\lambda_a)\right)\Big]\ge cr. \tag{A.10}$$

I would then replace $w(1)$ from (A.3) in the above expression and get an equivalent condition to (A.9) that is *only* in terms of the primitives:

$$\frac{cr^2}{r+\kappa\varphi_b}\left(1+\frac{\kappa\varphi_b}{r+\lambda_b}\right)+cy\lambda_b\left(1+\frac{r}{r+\lambda_b}\frac{\kappa\varphi_b}{r+\kappa\varphi_b}\right)\\ \leq y\big(\lambda_a\left(r+\lambda_b\right)-\kappa\varphi_b\left(\lambda_b-\lambda_a\right)\big). \tag{A.11}$$

Then, I will show that the *lhs* above is always greater than the *rhs*, thus there is no $y \in \underline{\partial}\mathcal{M}_a \cap \mathcal{M}_b$. Obviously at $y=0$ the *lhs* is greater than the *rhs*. At $y=1$, the *rhs* is increasing in λ_a, so can be upper bounded when λ_a assumes its maximum level in the high cost regime, i.e. $c+\frac{\kappa\varphi_b(\lambda_b-c)}{r+\lambda_b+\kappa\varphi_b}$. Therefore the *rhs* of (A.11) at $y=1$ is upper bounded by

$$\lambda_a\left(r+\lambda_b\right)-\kappa\varphi_b\left(\lambda_b-\lambda_a\right)\leq c\left(r+\lambda_b\right).$$

However, the *lhs* of (A.11) equals $c(r+\lambda_b)$ at $y=1$. So (A.11) can never be satisfied, therefore in the high cost regime $\mathcal{M}_a$ cannot have a lower boundary point in $\mathcal{M}_b$. Given $1\notin\mathcal{M}_a$ and the monotonicity of w on $\mathcal{M}_b^c$, the only possible candidate for a non-empty $\mathcal{M}_a$ is (α_a,β_a) such that $\alpha_a<\alpha_b:=\inf\mathcal{M}_b$. Because of optimality, $v(\cdot,a)$ must smoothly meet the zero function at α_a, so similar analysis to (A.6) would imply $\alpha_a=c/\lambda_a(1+w(1))$, in that $w(1)$ follows (A.3). Furthermore, the superharmonic condition for $v(\cdot,b)$ requires that at $\pi=\alpha_a$:

$$0\geq[\mathcal{L}_bv](\alpha_a,b)-rv(\alpha_a,b)-c=\lambda_b\alpha_a\left(1+w(1)\right)-c=\left(\frac{\lambda_b}{\lambda_a}-1\right).$$

However, this never holds, because the rightmost side is positive. So the only continuation set that survives the high cost regime is $\mathcal{M}_a=\emptyset$.

<u>Low cost regime</u>: Note that in this regime $w(1)$ follows (A.4). I first prove in equilibrium it must be that $\mathcal{M}_a\subset\mathcal{M}_b$. We have seen in the part (i) that $1\in\mathcal{M}_a\cap\mathcal{M}_b$ in this regime. To show the above set inclusion, I prove $\alpha_a:=\inf\mathcal{M}_a\in\mathcal{M}_b$, and the claim follows from the connectedness of $\mathcal{M}_b$. Toward the contradiction assume $\alpha_a<\alpha_b$, where $\alpha_b=\inf\mathcal{M}_b$. Examining the superharmonicity of $v(\cdot,b)$ on $[0,\alpha_a]$ leads to

$$\begin{aligned}\mathcal{L}_bv(\pi,b)-rv(\pi,b)-c&=\lambda_b\pi\left(1+w(1)\right)-c=\frac{\lambda_b}{\lambda_a}\lambda_a\pi\left(1+w(1)\right)-c\\&=\frac{\lambda_b}{\lambda_a}\lambda_a\left(\pi-\alpha_a\right)\left(1+w(1)\right)+\left(\frac{\lambda_b}{\lambda_a}-1\right)c.\end{aligned}$$

As π approaches α_a from below, the first term above converges to zero while the second term remains a positive constant. Therefore, $\exists\ \pi_0<\alpha_a$ such that $\mathcal{L}_bv(\pi,b)-rv(\pi,b)-c>0$ for all $\pi\in(\pi_0,\alpha_a]$. This violates the superharmonicity of $v(\cdot,b)$, so there can be no equilibrium in which the lowest boundary point $\alpha_a\notin\mathcal{M}_b$. Next, I show having $\mathcal{M}_a\subset\mathcal{M}_b$ leads to the connectedness of $\mathcal{M}_a$. Because of the optimality of $v(\cdot,b)$ the principles of continuous and smooth fit hold at $\pi = \alpha_b$ with the zero outside option. Combining this with (2.9) implies the following expression for $v(\cdot, b)$:

$$\begin{aligned} v(\pi, b) = & -\frac{c}{r} + \frac{\lambda_b}{r+\lambda_b}\left(1 + w(1) + \frac{c}{r}\right)\pi \\ & + \left(\frac{c}{r} - \frac{\lambda_b}{r+\lambda_b}\left(1 + w(1) + \frac{c}{r}\right)\alpha_b\right)\left(\frac{1-\pi}{1-\alpha_b}\right)^{1+r/\lambda_b}\left(\frac{\pi}{\alpha_b}\right)^{-r/\lambda_b}, \end{aligned} \tag{A.12}$$

with α_b following (A.6). Furthermore, the above value function is convex if and only if

$$\left(\frac{c}{r} - \frac{\lambda_b}{r+\lambda_b}\left(1 + w(1) + \frac{c}{r}\right)\alpha_b\right) \geq 0. \tag{A.13}$$

Substituting α_b yields an equivalent condition for convexity:

$$\frac{c}{r} - \frac{c}{r+\lambda} - \frac{c^2}{r(r+\lambda_b)(1+w(1))} = \frac{c}{r(r+\lambda_b)}\left(\lambda_b - \frac{c}{1+w(1)}\right) \geq 0.$$

The above condition always holds because $\lambda_b > c$ and $w(1) > 0$, therefore $v(\cdot, b)$ followed from (A.12) is a convex function. Now define $[\mathcal{D}_a v](\pi, a) := [\mathcal{L}_a v](\pi, a) - rv(\pi, a) - c$, and note that from the HJB equation

$$[\mathcal{D}_a v](\pi, a) = \frac{-\kappa\varphi_b}{r+\kappa\varphi_b}(\lambda_b - \lambda_a)\frac{rv(\pi, b) + c}{\lambda_b} + \frac{r\lambda_a\pi(1 + w(1)) - cr}{r+\kappa\varphi_b}.$$

Consequently, convexity of $v(\cdot, b)$ implies

$$\frac{\partial^2}{\partial\pi^2}[\mathcal{D}_a v](\pi, a) = \frac{-\kappa\varphi_b(\lambda_b - \lambda_a)}{(r+\kappa\varphi_b)\,\lambda_b}v''(\pi, b) < 0.$$

Therefore, $[\mathcal{D}_a v](\cdot, a)$ is a concave function in π. Were $\mathcal{M}_a$ not be connected then at least it has two disjoint components, say (x_1, x_2) and (x_3, x_4) where $x_2 < x_3$. Superharmonicity jointly with the satisfaction of Bellman equation on the continuation region require that $[\mathcal{D}_a v](\cdot, a)$ be negative just below x_1, zero on $[x_1, x_2]$, negative again on (x_2, x_3), followed by being zero on (x_3, x_4). This pattern is not consistent with the concavity of $[\mathcal{D}_a v](\cdot, a)$, therefore $\mathcal{M}_a$ must be connected. □

A.1.3 Proof of Theorem 2.5

I prove the assertion only for the high cost regime, as the proof of the other case follows the same logic, but is just lengthier. Proposition 2.4 provides necessary conditions for matching sets associated with increasing value functions in $C^1[0, 1]$. Specifically, we know from this proposition that in the high cost regime the only matching sets surviving the optimality principles are $\mathcal{M}_a = \emptyset$ and $\mathcal{M}_b = (\alpha_b, 1]$, where α_b is found via the continuous and smooth fit principles as

$$\alpha_b = \frac{c}{\lambda_b\left(1+w(1)\right)}. \tag{A.14}$$

Also, from the construction of that proposition we know that the following profile embodies the only candidate for an equilibrium with increasing $C^1[0,1]$ value functions on $(\alpha_b, 1]$:

$$\begin{aligned}
v(\pi,a) &= w(\pi) = \frac{\kappa\varphi_b}{r+\kappa\varphi_b} v(\pi,b),\\
v(\pi,b) &= -\frac{c}{r} + \frac{\lambda_b}{r+\lambda_b}\left(1+w(1)+\frac{c}{r}\right)\pi \\
&\quad + \left(\frac{c}{r} - \frac{\lambda_b}{r+\lambda_b}\left(1+w(1)+\frac{c}{r}\right)\alpha_b\right)\left(\frac{1-\pi}{1-\alpha_b}\right)^{1+r/\lambda_b}\left(\frac{\pi}{\alpha_b}\right)^{-r/\lambda_b}.
\end{aligned} \tag{A.15}$$

And all are equal to zero on $[0, \alpha_b]$. Therefore, our task here is to employ a *verification* scheme to show that the above value functions are indeed the optimal equilibrium values. I divide the proof into three steps: (a) verifying the majorizing and superharmonicity conditions; (b) using these two and applying a Martingale argument to establish the *optimality* of the above profile of the value functions; (c) for large r the Banach fixed point theorem is applied and proves the uniqueness of the identified equilibrium in the larger space of *essentially bounded measurable* functions. Lastly, only in the remaining parts of the proof, I slightly change the notation and use $v_q(\pi) := v(\pi, q)$ for $q \in \{a, b\}$.

Step (a):

Majorizing. This step is quite straightforward because in (A.15) $w = v_a$ and $v_b \geq w = \frac{\kappa\varphi_b}{r+\kappa\varphi_b} v_b$.

Superharmonicity of v_b. Obviously the superharmonic condition holds with equality on $(\alpha_b, 1]$ because of the Bellman equation. However, it needs to be checked on $[0, \alpha_b]$ as it is carried out below:

$$[\mathcal{L}_b v_b](\pi) - r v_b(\pi) - c = \lambda_b \pi\left(1+w(1)\right) - c \leq \lambda_b \alpha_b\left(1+w(1)\right) - c = 0.$$

Superharmonicity of v_a. Remember that in the high cost regime $\mathcal{M}_a = \emptyset$, thus $v_a = w$. So on $[0, \alpha_b]$:

$$\begin{aligned}
[\mathcal{L}_a v_a](\pi) - r v_a(\pi) - c &= \lambda_a \pi\left(1+w(1)\right) - c \\
&\leq \lambda_b \alpha_b\left(1+w(1)\right) - c \leq 0,
\end{aligned}$$

where in the last inequality I used the expression (A.14) for α_b. The analysis of the superharmonicity of v_a on $(\alpha_b, 1]$ however needs a little more work:

$$\begin{aligned}
[\mathcal{L}_a v_a](\pi) - r v_a(\pi) - c &= \left[\mathcal{L}_a \left(\frac{\kappa\varphi_b}{r+\kappa\varphi_b} v_b\right)\right](\pi) - \frac{r\kappa\varphi_b}{r+\kappa\varphi_b} v_b(\pi) - c \\
&= \frac{\kappa\varphi_b}{r+\kappa\varphi_b}\big([\mathcal{L}_a v_b](\pi) - r v_b(\pi) - c\big) \\
&\quad + \frac{r\lambda_a\pi}{r+\kappa\varphi_b}\left(1+w(1)\right) - \frac{rc}{r+\kappa\varphi_b} \\
&= -\frac{\kappa\varphi_b}{r+\kappa\varphi_b}\left[(\mathcal{L}_b - \mathcal{L}_a) v_b\right](\pi) + \frac{r\lambda_a\pi}{r+\kappa\varphi_b}\left(1+w(1)\right) - \frac{rc}{r+\kappa\varphi_b} \\
&= -\frac{\kappa\varphi_b}{r+\kappa\varphi_b}\left(\lambda_b - \lambda_a\right)\pi\big(1+w(1) - v_b(\pi) - (1-\pi)v_b'(\pi)\big) \\
&\quad + \frac{r\lambda_a\pi}{r+\kappa\varphi_b}\left(1+w(1)\right) - \frac{rc}{r+\kappa\varphi_b}.
\end{aligned}$$

Some straightforward manipulations similar to equation (A.13) implies the candidate v_b in (A.15) is also convex, therefore, $v_b(\pi) + (1-\pi)v_b'(\pi) \le v_b(1)$ that yields an upper bound on the above relation:

$$\begin{aligned}
[\mathcal{L}_a v_a](\pi) - r v_a(\pi) - c &\le -\frac{\kappa\varphi_b}{r+\kappa\varphi_b}\frac{r\left(\lambda_b - \lambda_a\right)\pi}{r+\lambda_b}\left(1+w(1)+\frac{c}{r}\right) + \frac{r\lambda_a\pi\left(1+w(1)\right) - cr}{r+\kappa\varphi_b} \\
&\le \left(-\frac{\kappa\varphi_b}{r+\kappa\varphi_b}\frac{r\left(\lambda_b - \lambda_a\right)}{r+\lambda_b}\left(1+w(1)+\frac{c}{r}\right) + \frac{r\lambda_a\left(1+w(1)\right) - rc}{r+\kappa\varphi_b}\right)^+.
\end{aligned}$$

In the second inequality above I used the fact that the *rhs* of the first inequality is affine in π and negative at $\pi = 0$. Now denote the argument of $(\cdot)^+$ by $\mathfrak{Z}$. It is increasing in λ_a, hence can be bounded above when λ_a is replaced with $c + rw(1)$ (its maximum value in the high cost regime):

$$\begin{aligned}
\mathfrak{Z} &\le -\frac{\kappa\varphi_b}{r+\kappa\varphi_b}\frac{r\left(\lambda_b - c - rw(1)\right)}{r+\lambda_b}\left(1+w(1)+\frac{c}{r}\right) + \frac{r\left(c+rw(1)\right)\left(1+w(1)\right) - rc}{r+\kappa\varphi_b} \\
&= -\frac{\kappa\varphi_b}{r+\kappa\varphi_b}\frac{\left(\lambda_b - c\right)\left(r+\lambda_b\right)\left(r+\kappa\varphi_b+c\right)}{r\left(\kappa\varphi_b+r+\lambda_b\right)^2} + \frac{\kappa\varphi_b}{r+\kappa\varphi_b}\frac{\left(\lambda_b - c\right)\left(r+\lambda_b\right)\left(r+\kappa\varphi_b+c\right)}{r\left(\kappa\varphi_b+r+\lambda_b\right)^2} = 0,
\end{aligned}$$

where in the second line $w(1)$ is replaced from (A.3). This concludes the superharmonicity of v_a w.r.t $\mathcal{L}_a$ on $(\alpha_b, 1]$, and hence on the entire unit interval.

Step (b): Define $\mathbf{v}_q(\iota, \pi) := v_q(\pi)1_{\{\iota=0\}} + \big(\iota + w(\pi)\big)1_{\{\iota=1\}}$, where ι is the success indicator process. Since $\mathbf{v}$ is a bounded function, for each $q \in \{a, b\}$, one can find a bounded (and hence uniformly integrable) Martingale process M^q such that:

$$e^{-rt}\mathbf{v}_q(\iota_t, \pi_t) = \mathbf{v}_q(\iota, \pi) + \int_0^t e^{-rs}\big(\mathbb{L}_q \mathbf{v}_q(\cdot,\cdot) - r\mathbf{v}_q(\cdot,\cdot)\big)(\iota_{s^-}, \pi_{s^-})\mathrm{d}s + M_t^q. \tag{A.16}$$

In that $\mathbb{L}_q \mathbf{v}_q(\iota,\pi) := [\mathcal{L}_q v_q](\pi) 1_{\{\iota=0\}}$. From the majorant condition, for every stopping time τ, we have $\mathbf{v}_q(\iota_\tau,\pi_\tau) \geq \iota_\tau + w(\pi_\tau)$, therefore

$$\begin{aligned}
e^{-r\tau}\big(\iota_\tau + w(\pi_\tau)\big) &\leq \mathbf{v}_q(\iota,\pi) + \int_0^\tau e^{-rs}\big(\mathbb{L}_q \mathbf{v}_q(\cdot,\cdot) - r\mathbf{v}_q(\cdot,\cdot)\big)(\iota_{s^-},\pi_{s^-})\mathrm{d}s + M^q_\tau \\
&\leq \mathbf{v}_q(\iota,\pi) + \int_0^\tau c e^{-rs}\mathrm{d}s + M^q_\tau,
\end{aligned}$$

wherein the second inequality I used the superharmonic property proven in step (a). Doob's optional stopping theorem implies $\mathsf{E}M^q_\tau = 0$, hence for every stopping time τ,

$$\mathbf{v}(\iota,\pi) \geq \mathsf{E}_{\pi,q,\iota}\left[e^{-r\tau}\left(\iota_\tau + w(\pi_\tau)\right) - c\int_0^\tau e^{-rs}\mathrm{d}s\right].$$

That in turn implies

$$v(\pi,q) \geq \sup_\tau \mathsf{E}_{\pi,q,\iota=0}\left[e^{-r\tau}\left(\iota_\tau + w(\pi_\tau)\right) - c\int_0^\tau e^{-rs}\mathrm{d}s\right].$$

Now for each q, let $\tau(q) := \inf\{t \geq 0 : \pi_t \notin \mathcal{M}_q \text{ or } \iota_t = 1\}$ that is the optimal stopping policy. Using this in (A.16) yields

$$\begin{aligned}
e^{-r\tau(q)}\left(\iota_{\tau(q)} + w(\pi_{\tau(q)})\right) &= e^{-r\tau(q)}\mathbf{v}_q\big(\iota_{\tau(q)},\pi_{\tau(q)}\big) \\
&= \mathbf{v}_q(\iota,\pi) + \int_0^{\tau(q)} e^{-rs}\big(\mathbb{L}_q \mathbf{v}_q(\cdot,\cdot) - r\mathbf{v}_q(\cdot,\cdot)\big)(\iota_{s^-},\pi_{s^-})\mathrm{d}s + M^q_{\tau(q)} \\
&= \mathbf{v}_q(\iota,\pi) - \int_0^{\tau(q)} c e^{-rs}\mathrm{d}s + M^q_{\tau(q)},
\end{aligned}$$

which after taking expectations of both sides amounts to

$$\mathbf{v}_q(\iota,\pi) = \mathsf{E}_{\pi,q,\iota}\left[e^{-r\tau(q)}\left(\iota_{\tau(q)} + w(\pi_{\tau(q)}\right) - c\int_0^{\tau(q)} e^{-rs}\mathrm{d}s\right],$$

thereby concluding the verification proof.

Step (c): In this part we apply the contraction mapping theorem to study equilibrium uniqueness. For every $(v_a, v_b, w) \in \big(L^\infty[0,1]\big)^3$, define

$$\begin{aligned}
\mathbb{T}_q w(\pi) &:= \sup_\tau \Big\{\mathsf{E}_q\left[e^{-r\sigma} - c\int_0^\sigma e^{-rs}\mathrm{d}s + e^{-r\sigma}w(\pi_\sigma); \sigma \leq \tau\right] \\
&\qquad + \mathsf{E}_q\left[-c\int_0^\tau e^{-rs}\mathrm{d}s + e^{-r\tau}w(\pi_\tau); \sigma > \tau\right]\Big\} \quad \text{for } q \in \{a,b\}, \\
\mathbb{T}_0[(v_a,v_b,w)](\pi) &:= r^{-1}\kappa \sum_{q\in\mathcal{M}(\pi)} \left[v_q(\pi) - w(\pi)\right]\varphi_q,
\end{aligned}$$

where E_q is the expectation w.r.t to the Poisson process with intensity λ_q and $\mathcal{M}(\pi) = \{q : v_q(\pi) > w(\pi)\}$. Define $\mathbb{T} := (\mathbb{T}_a, \mathbb{T}_b, \mathbb{T}_0)$. The goal of this part of the proof is to show the fixed point of $\mathbb{T}$ exists and is unique in $\big(L^\infty[0,1]\big)^3$. Given the definition of $\mathcal{M}(\pi)$ one can see that $\mathbb{T}_0$ and $\mathbb{T}_q$ preserve the measurability and essential boundedness. Therefore, $\mathbb{T}\big(L^\infty[0,1]\big)^3 \subset \big(L^\infty[0,1]\big)^3$. The next step is to investigate the contraction property of $\mathbb{T}$. For this, let us equip $(L^\infty[0,1])^3$ with the following norm,

$$\|(v_a, v_b, w)\|_\varsigma := \varsigma\left(\|v_a\|_\infty + \|v_b\|_\infty\right) + \|w\|_\infty,$$

where $\varsigma > 0$ is to be determined. First, I examine the contraction coefficient of $\mathbb{T}_q$. For every $w, \tilde{w} \in L^\infty[0,1]$:

$$\begin{aligned}\Big|\mathbb{T}_q[w] - \mathbb{T}_q[\tilde{w}]\Big|(\pi) \le \sup_\tau \Big\{ & \mathsf{E}_q\left[e^{-r\sigma}\left|w(\pi_\sigma) - \tilde{w}(\pi_\sigma)\right|; \sigma \le \tau\right] \\ & + \mathsf{E}_q\left[e^{-r\tau}\left|w(\pi_\tau) - \tilde{w}(\pi_\tau)\right|; \sigma > \tau\right]\Big\} \\ \le {} & \|w - \tilde{w}\|_\infty \sup_\tau \mathsf{E}_q\left[e^{-r(\tau\wedge\sigma)}\right] = \|w - \tilde{w}\|_\infty.\end{aligned}$$

Let $\phi := \varphi_a + \varphi_b$, and $v, \tilde{v} \in (L^\infty[0,1])^2$, respectively inducing the matching sets $\mathcal{M}$ and $\widetilde{\mathcal{M}}$. Then:

$$\big(\mathbb{T}_0[(v,w)] - \mathbb{T}_0[(\tilde{v},\tilde{w})]\big)(\pi) = r^{-1}\kappa\left(\sum_{q\in\mathcal{M}(\pi)}\left(v_q(\pi) - w(\pi)\right)\varphi_q - \sum_{q\in\widetilde{\mathcal{M}}(\pi)}\left(\tilde{v}_q(\pi) - \tilde{w}(\pi)\right)\varphi_q\right).$$

Partitioning the matching sets $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ yields the following equivalent expression:

$$\begin{aligned}\Big(\mathbb{T}_0[(v,w)] - \mathbb{T}_0[(\tilde{v},\tilde{w})]\Big)(\pi) = {} & r^{-1}\kappa \sum_{q\in\mathcal{M}(\pi)\setminus\widetilde{\mathcal{M}}(\pi)}\left(v_q(\pi) - \tilde{v}_q(\pi) - w(\pi) + \tilde{w}(\pi)\right)\varphi_q \\ & + r^{-1}\kappa \sum_{q\in\mathcal{M}(\pi)\cap\widetilde{\mathcal{M}}(\pi)}\left(v_q(\pi) - \tilde{v}_q(\pi) - w(\pi) + \tilde{w}(\pi)\right)\varphi_q \\ & + r^{-1}\kappa \underbrace{\sum_{q\in\mathcal{M}(\pi)\setminus\widetilde{\mathcal{M}}(\pi)}\left(\tilde{v}_q(\pi) - \tilde{w}(\pi)\right)\varphi_q}_{=0} \\ & - r^{-1}\kappa \underbrace{\sum_{q\in\widetilde{\mathcal{M}}(\pi)\setminus\mathcal{M}(\pi)}\left(\tilde{v}_q(\pi) - \tilde{w}(\pi)\right)\varphi_q}_{\ge 0} \\ \le {} & r^{-1}\kappa\phi\left(\sum_{q\in\{a,b\}}\|v_q - \tilde{v}_q\|_\infty + \|w - \tilde{w}\|_\infty\right).\end{aligned}$$

Putting together the previous bounds yields:

$$\begin{aligned}\left\|\mathbb{T}\left[(v_a,v_b,w)\right]-\mathbb{T}\left[(\tilde{v}_a,\tilde{v}_b,\tilde{w})\right]\right\|_{\varsigma} &= \varsigma\Big(\left\|\mathbb{T}_a[w]-\mathbb{T}_a[\tilde{w}]\right\|_{\infty}+\left\|\mathbb{T}_b[w]-\mathbb{T}_b[\tilde{w}]\right\|_{\infty}\Big)\\ &\quad+\left\|\mathbb{T}_0\left[(v_a,v_b,w)\right]-\mathbb{T}_0\left[(\tilde{v}_a,\tilde{v}_b,\tilde{w})\right]\right\|_{\infty}\\ &\le 2\varsigma\|w-\tilde{w}\|_{\infty}+r^{-1}\kappa\phi\left(\sum_{q\in\{a,b\}}\|v_q-\tilde{v}_q\|_{\infty}+\|w-\tilde{w}\|_{\infty}\right)\\ &= r^{-1}\kappa\phi\|v_a-\tilde{v}_a\|_{\infty}+r^{-1}\kappa\phi\|v_b-\tilde{v}_b\|_{\infty}\\ &\quad+\left(2\varsigma+r^{-1}\kappa\phi\right)\|w-\tilde{w}\|_{\infty}.\end{aligned}$$

Assume $r^{-1}\kappa\phi<1/3$, and find $\varepsilon>0$ such that $r^{-1}\kappa\phi<1/(1+\varepsilon)(3+2\varepsilon)$, and let $\varsigma=(1+\varepsilon)r^{-1}\kappa\phi$, then

$$\begin{aligned}\left\|\mathbb{T}\left[(v_a,v_b,w)\right]-\mathbb{T}\left[(\tilde{v}_a,\tilde{v}_b,\tilde{w})\right]\right\|_{\varsigma} &\le \frac{r^{-1}\kappa\phi}{\varsigma}\left(\varsigma\|v_a-\tilde{v}_a\|_{\infty}+\varsigma\|v_b-\tilde{v}_b\|_{\infty}+\overbrace{\frac{\varsigma(2\varsigma+r^{-1}\kappa\phi)}{r^{-1}\kappa\phi}}^{<1}\|w-\tilde{w}\|_{\infty}\right)\\ &\le \frac{1}{1+\varepsilon}\left\|(v_a,v_b,w)-(\tilde{v}_a,\tilde{v}_b,\tilde{w})\right\|_{\varsigma}.\end{aligned}$$

So the Banach fixed point theorem implies that there exists a unique fixed point in the space of essentially bounded measurable functions, as long as $r>3\kappa\phi$. □

A.2 Proofs of Section 3

A.2.1 Proof of Proposition 3.1

I need the following lemma to prove the proposition.

Lemma A.1. *In any normal equilibrium $w(\cdot)$ is increasing iff $\{v(\cdot,a),v(\cdot,b)\}$ are increasing.*

Proof. In the normal equilibria $w(\pi)$ follows

$$w(\pi)=\max_{\chi}\left\{\frac{r^{-1}\kappa\psi(\pi)\left[v(\pi,a)\varphi_a\chi_a(\pi)+v(\pi,b)\varphi_b\chi_b(\pi)\right]}{\mu+r^{-1}\kappa\psi(\pi)\left[\varphi_a\chi_a(\pi)+\varphi_b\chi_b(\pi)\right]}\right\}.$$

Assume first, that $\{v(\cdot,a),v(\cdot,b)\}$ are increasing. It is known that the maximum of increasing functions remains increasing, therefore I have to show for any combination of χ's the *rhs* of the above expression is increasing in π. For example, let $\chi_a=\chi_b=1$, then its derivative is positively proportional to

$$r^{-1}\kappa\psi(\pi)\big(v'(\pi,a)\varphi_a+v'(\pi,b)\varphi_b\big)+r^{-1}\kappa\mu\psi'(\pi)\big(v(\pi,a)\varphi_a+v(\pi,b)\varphi_b\big)\ge 0.$$

The other permutations of χ_a and χ_b can also be checked, and one can similarly verify that for each combination, the *rhs* is increasing in π, therefore $w(\cdot)$ becomes increasing.

Conversely, now assume $w(\cdot)$ is increasing. Then the same analysis presented in lemma 2.3 implies that $\{v(\cdot, a), v(\cdot, b)\}$ are increasing.‖

Proof of part (i): For proving the existence of a normal equilibrium, I first establish the existence of a fixed point α_e to the system (3.3) and (3.4). To fix ideas, let us define the following mappings $\mathsf{M} : [0, 1] \to [0, 1]$, $\mathsf{W} : [0, 1] \to \mathbb{R}_+$ and $\mathsf{A} : \mathbb{R}_+ \to [0, 1]$:

$$\begin{aligned}
\mathsf{M}(x) &:= \frac{1-p}{1-x}\psi(x) + \frac{p-x}{1-x}, \\
\mathsf{W}(\mu) &:= \max_{\chi} \left\{ \frac{r^{-1}\kappa\left[\varphi_b\left(\lambda_b - c\right)\left(r+\lambda_a\right)\chi_b(1) + \varphi_a\left(\lambda_a - c\right)\left(r+\lambda_b\right)\chi_a(1)\right]}{\left(r+\lambda_a\right)\left(r+\lambda_b\right)\mu + \kappa\varphi_b\left(r+\lambda_a\right)\chi_b(1) + \kappa\varphi_a\left(r+\lambda_b\right)\chi_a(1)} \right\}, \\
\mathsf{A}(w) &:= \frac{c}{\lambda_b(1+w)}.
\end{aligned}$$

Then, α_e is the fixed point of $\mathbf{H} : [0, 1] \to [0, 1]$, where $\mathbf{H} := \mathsf{A} \circ \mathsf{W} \circ \mathsf{M}$. Since this map is continuous on $[0, 1]$, the existence of the fixed point is obvious. However, the normal equilibrium requires $\alpha_e < p$. For this note that $\mathbf{H}(0) > 0$ and

$$\mathbf{H}(p) = \frac{c}{\lambda_b\left(1 + \mathsf{W}(\psi(p))\right)} < \frac{c}{\lambda_b} < p.$$

The intermediate value theorem therefore implies that there always exists a normal equilibrium with $0 < \alpha_e < p$.

Next, I analyze the best-response correspondence of a typical agent. Suppose all individuals except one follow the matching strategy induced by $\mathcal{M}_b = (\alpha_e, 1]$ and $\mathcal{M}_a \subset \mathcal{M}_b$. Then, $\mu = \mathsf{M}(\alpha_e)$. Using the machinery developed in the proof of the proposition 2.4 and the previous lemma, one can easily confirm that the unique best-response of the potential deviant agent is the aforementioned matching sets $(\mathcal{M}_a, \mathcal{M}_b)$.‖

Proof of part (ii): Assuming $\psi'' \leq 0$ implies that

$$\mathsf{M}'(x) = \frac{1-p}{1-x}\left(\psi'(x) - \frac{1-\psi(x)}{1-x}\right) \geq 0.$$

Hence, the composition map becomes increasing from $[0, 1]$ to itself, because W and A are both decreasing.

Endow the space of admissible weighting functions Ψ in equation (3.1) with the pointwise order $\succsim$, i.e. $\psi_2 \succsim \psi_1$ if $\psi_2(x) \geq \psi_1(x)$, $\forall x \in [0, 1]$. So, $(\Psi, \succsim)$ becomes a partially ordered set that is used as the underlying parameter space for the fixed point map $\mathbf{H}$. With slight abuse of notation, I extend the domain of $\mathbf{H}$ as $\mathbf{H} : [0, 1] \times \Psi \to [0, 1]$. Holding x constant, $\mathbf{H}(x, \psi)$ is increasing in ψ w.r.t $\succsim$ order. Therefore, the mapping $\mathbf{H}$ is an increasing function from $[0, 1] \times \Psi$ to $[0, 1]$. Now one can apply corollaries 2.5.1 and 2.5.2 of Topkis (2011) to conclude that the set of fixed points is a complete lattice and its greatest (least) element is increasing in $\psi \in \Psi$. Finally, the lattice of fixed points, namely the space of α_e's, completely Pareto rank the equilibria. Because smaller values of α_e lead to smaller μ and hence larger $w(1)$ and $\{v(1,b), v(1,a)\}$. In addition, it is associated with larger masses of matched and unmatched agents with maximum reputation (i.e. $\pi = 1$). Therefore, the welfare ranking of equilibria coincides inversely with the ranking of the fixed points of $\mathbf{H}$. □

A.2.2 Steady-state Measures

Lemma A.2. *In the steady-state of the economy with short-lived agents and reputational externality,*

$$\int_\alpha^p m(\pi)\mathrm{d}\pi = \frac{\kappa\varphi\psi(p)/\mu}{\delta+\kappa\varphi\psi(p)/\mu}\frac{p-\alpha}{\Upsilon_2(\alpha)-\alpha\Upsilon_1(\alpha)}\left(\Upsilon_1(\alpha)-\frac{\lambda}{\delta+\lambda}\Upsilon_2(\alpha)\right), \tag{A.17a}$$

$$\int_\alpha^p \pi m(\pi)\mathrm{d}\pi = \frac{\kappa\varphi\psi(p)/\mu}{\delta+\kappa\varphi\psi(p)/\mu}\frac{\delta}{\delta+\lambda}\frac{(p-\alpha)\Upsilon_2(\alpha)}{\Upsilon_2(\alpha)-\alpha\Upsilon_1(\alpha)}, \tag{A.17b}$$

$$m(1) = \frac{\kappa\varphi\psi(p)/\mu}{\delta+\kappa\varphi\psi(p)/\mu}\frac{\kappa\varphi/\mu}{\delta+\lambda+\kappa\varphi/\mu}\frac{\lambda}{\delta+\lambda}\frac{(p-\alpha)\Upsilon_2(\alpha)}{\Upsilon_2(\alpha)-\alpha\Upsilon_1(\alpha)}, \tag{A.17c}$$

$$n(\alpha) = \frac{\kappa\varphi\psi(p)/\mu}{\delta+\kappa\varphi\psi(p)/\mu}\frac{\Upsilon_2(\alpha)-p\Upsilon_1(\alpha)}{\Upsilon_2(\alpha)-\alpha\Upsilon_1(\alpha)}, \tag{A.17d}$$

where

$$\Upsilon_i(\alpha) := \left(\frac{p}{\alpha}\right)^{\delta/\lambda-1}\left(\frac{1-p}{1-\alpha}\right)^{-(\delta/\lambda+2)} p^i(1-p) - \alpha^i(1-\alpha), \quad \textit{for } i \in \{1,2\}.$$

Proof. In the steady-state the time derivatives in (3.5) are zero, therefore (3.5a) and (3.5b) amount to:

$$n(1) = \frac{\delta+\lambda}{\kappa\varphi/\mu}m(1), \tag{A.18a}$$

$$(\delta+\lambda+\kappa\varphi/\mu)\delta m(1) = \kappa\varphi/\mu\int_\alpha^p \lambda\pi m(\pi)\mathrm{d}\pi. \tag{A.18b}$$

Also at $\pi = p$, equation (3.5c) implies that $n(p) = \delta/(\delta+\kappa\varphi\psi(p)/\mu)$. Next, the expression found in (3.7) translates to

$$\int_\alpha^p \pi m(\pi)\mathrm{d}\pi = \frac{\lambda m(\alpha)}{\delta+\lambda}\Upsilon_2(\alpha). \tag{A.19}$$

The *rhs* to (3.8) can be simplified using the steady-state ODE resulted from $\dot{m}(\pi) = 0$:

$$\begin{aligned}
\kappa\varphi\frac{\psi(p)}{\mu}n(p) = \frac{\delta\kappa\varphi\psi(p)/\mu}{\delta + \kappa\varphi\psi(p)/\mu} &= \lambda\int_\alpha^p \pi m(\pi)\mathrm{d}\pi + \delta\int_\alpha^p m(\pi)\mathrm{d}\pi + \delta n(\alpha) \\
&= \lambda\int_\alpha^p \partial_\pi\left(\pi(1-\pi)m(\pi)\right)\mathrm{d}\pi + \delta n(\alpha) \\
&= \lambda\big(p(1-p)m(p) - \alpha(1-\alpha)m(\alpha)\big) + \delta n(\alpha) \\
&= \lambda m(\alpha)\Upsilon_1(\alpha) + \delta n(\alpha).
\end{aligned} \tag{A.20}$$

Recall that because of Bayesian learning during the matches the steady-state average reputation must be equal to p:

$$m(1) + n(1) + pn(p) + \int_\alpha^p \pi m(\pi)\mathrm{d}\pi + \alpha n(\alpha) = p.$$

Simplifying this relation using (A.18b) and (A.19) implies

$$\lambda m(\alpha)\Upsilon_2(\alpha) + \alpha\delta n(\alpha) = \frac{\delta\kappa\varphi\psi(p)/\mu}{\delta + \kappa\varphi\psi(p)/\mu}p. \tag{A.21}$$

It is now straightforward to solve for $n(\alpha)$ and $m(\alpha)$ using (A.20) and (A.21), thereby obtaining (A.17d) and

$$m(\alpha) = \frac{\delta\kappa\varphi\psi(p)/\mu}{\lambda\left(\delta + \kappa\varphi\psi(p)/\mu\right)}\frac{p-\alpha}{\Upsilon_2(\alpha) - \alpha\Upsilon_1(\alpha)}. \tag{A.22}$$

Substituting $m(\alpha)$ from above into (A.19) yields the lemma's claim for $\int_\alpha^p \pi m(\pi)\mathrm{d}\pi$, i.e. equation (A.17b). Subsequently, $m(1)$ can be found from (A.18b) thus verifying (A.17c). Finally, from the second line in (A.20), one obtains the following expression

$$\int_\alpha^p m(\pi)\mathrm{d}\pi = \frac{\lambda m(\alpha)}{\delta}\left(\Upsilon_1(\alpha) - \frac{\lambda}{\delta+\lambda}\Upsilon_2(\alpha)\right),$$

that amounts to (A.17a) by replacing $m(\alpha)$ in the above expression. □

A.2.3 Stochastic Ordering Results

For a better understanding of the stochastic ordering of the steady distribution $\boldsymbol{\pi}_{\infty}$, I would first express the CDF of the density $m(\cdot)$:

$$\int_{\alpha}^{\pi} m(x)\mathrm{d}x = \frac{\kappa\varphi\psi(p)/\mu}{\delta+\kappa\varphi\psi(p)/\mu}\frac{p-\alpha}{\Upsilon_{2,1}(\alpha,p)-\alpha\Upsilon_{1,1}(\alpha,p)}\Big(\frac{\delta}{\delta+\lambda}\Upsilon_{2,1}(\alpha,\pi)+\Upsilon_{1,2}(\alpha,\pi)\Big),$$
$$\Upsilon_{i,j}(x,y) := \left(\frac{y}{x}\right)^{(\delta/\lambda-1)}\left(\frac{1-y}{1-x}\right)^{-(\delta/\lambda+2)} y^i(1-y)^j - x^i(1-x)^j. \tag{A.23}$$

In addition, using the solution found for $m(\pi)$ and the expression (A.22) for $m(\alpha)$ it is easy to verify that for $\pi \in [\alpha, p]$

$$m(\pi) = \frac{\delta\kappa\varphi\psi(p)/\mu}{\lambda\left(\delta+\kappa\varphi\psi(p)/\mu\right)}\left(\frac{\pi}{p}\right)^{(\delta/\lambda-1)}\left(\frac{1-\pi}{1-p}\right)^{-(\delta/\lambda+2)}\frac{1}{p(1-p)}, \tag{A.24}$$

therefore for a fixed μ the above density is *independent* of α.

My next goal is to show that $\mathsf{M}(\mu,\alpha)$ is increasing in each argument holding the other one constant. For this, I appeal to the theory of stochastic orders, and in particular, I employ the second order stochastic dominance. For two real-valued random variables X and Y, it is said that $X \succeq_{\mathsf{SSD}} Y$ if $\mathsf{E}u(X) \geq \mathsf{E}u(Y)$ for every increasing and concave function u. An equivalent definition is that $X \succeq_{\mathsf{SSD}} Y$ if $\mathsf{E}\left[(X-t)_-\right] \geq \mathsf{E}\left[(Y-t)_-\right]$ for every $t \in \mathbb{R}$, provided that the expectations exist.[18] The next lemma offers a sufficient condition for the second order stochastic dominance that originates from the work of Karlin and Novikoff (1963).

Lemma A.3 (Sufficient condition for SSD)**.** *Suppose the following two conditions hold:*

(i) $\mathsf{E}[X] \geq \mathsf{E}[Y]$.

(ii) There exists $t_0 \in \mathbb{R}$ such that for all $t \leq t_0$, $\mathsf{P}(X \geq t) \geq \mathsf{P}(Y \geq t)$ and for all $t > t_0$, $\mathsf{P}(X \geq t) \leq \mathsf{P}(Y \geq t)$.

Then $X \succeq_{\mathsf{SSD}} Y$.

Proof. For every $t \leq t_0$,

$$\begin{aligned}\mathsf{E}\left[(X-t)_-\right] &= -\int_0^{\infty}\mathsf{P}\left(-(X-t)_- > u\right)\mathrm{d}u = -\int_0^{\infty}\mathsf{P}\left(X < t-u\right)\mathrm{d}u \\ &= -\int_{-\infty}^{t}\mathsf{P}\left(X < z\right)\mathrm{d}z \geq -\int_{-\infty}^{t}\mathsf{P}\left(Y < z\right)\mathrm{d}z = \mathsf{E}\left[(Y-t)_-\right].\end{aligned}$$

[18]For every $r \in \mathbb{R}$, $(r)_- := \min\{r, 0\}$. The reader can refer to chapter 4 of Shaked and Shanthikumar (2007) for the proof of the equivalence.

Also, an equivalent representation for $\mathsf{E}\left[(X-t)_-\right]$ is

$$\begin{aligned}\mathsf{E}\left[(X-t)_-\right] &= \mathsf{E}\left[(X-t); X<t\right] \\ &= \mathsf{E}[X] - t - \mathsf{E}\left[X-t; X \geq t\right] = \mathsf{E}[X] - t - \int_t^{\infty} \mathsf{P}\left(X \geq z\right) \mathrm{d}z.\end{aligned}$$

Therefore,

$$\mathsf{E}\left[(X-t)_-\right] - \mathsf{E}\left[(Y-t)_-\right] = \mathsf{E}[X] - \mathsf{E}[Y] + \int_t^{\infty} \big(\mathsf{P}\left(Y \geq z\right) - \mathsf{P}\left(X \geq z\right)\big) \mathrm{d}z.$$

The first term is positive and the integral term is also positive for all $t > t_0$, so $\mathsf{E}\left[(X-t)_-\right] \geq \mathsf{E}\left[(Y-t)_-\right]$ for $t > t_0$ as well. □

I will use the technique offered in this lemma to prove that an increase in α or μ *positively* shifts the steady-state distribution of $\boldsymbol{\pi_\infty}$. This distribution is completely described by the measures found in lemma A.2. For every Borel subset $B \subset [0,1]$:

$$\mathsf{P}\left(\boldsymbol{\pi_\infty} \in B\right) = \big(m(1) + n(1)\big)\delta_1(B) + n(p)\delta_p(B) + \int_B m(\pi)\mathrm{d}\pi + n(\alpha)\delta_\alpha(B).$$

Lemma A.4. *Let $\alpha_1 \leq \alpha_2 < p$ and $\mu_1 \leq \mu_2$, then*

(i) Holding α constant, $\boldsymbol{\pi_\infty}(\mu_2) \succeq_{\mathsf{SSD}} \boldsymbol{\pi_\infty}(\mu_1)$.

(ii) Holding μ constant, $\boldsymbol{\pi_\infty}(\alpha_2) \succeq_{\mathsf{SSD}} \boldsymbol{\pi_\infty}(\alpha_1)$.

Proof. **Part (i):** I show that

$$\mathsf{P}\left(\boldsymbol{\pi_\infty}(\mu_2) \geq t\right) \begin{cases} \geq \mathsf{P}\left(\boldsymbol{\pi_\infty}(\mu_1) \geq t\right) & \forall t \leq p \\ \leq \mathsf{P}\left(\boldsymbol{\pi_\infty}(\mu_1) \geq t\right) & \forall t > p. \end{cases} \tag{A.25}$$

Note that for every $t > p$

$$\mathsf{P}\left(\boldsymbol{\pi_\infty} \geq t\right) = m(1) + n(1) = \frac{\kappa\varphi\psi(p)/\mu}{\delta + \kappa\varphi\psi(p)/\mu} \frac{\lambda}{\delta+\lambda} \frac{(p-\alpha)\Upsilon_{2,1}(\alpha,p)}{\Upsilon_{2,1}(\alpha,p) - \alpha\Upsilon_{1,1}(\alpha,p)},$$

that is obviously decreasing in μ, hence proving the second assertion in (A.25). For every $t \leq p$,

$$\mathsf{P}\left(\boldsymbol{\pi_\infty} \geq t\right) = 1 - \mathsf{P}\left(\boldsymbol{\pi_\infty} < t\right) = 1 - \left(n(\alpha) + \int_\alpha^t m(\pi)\mathrm{d}\pi\right).$$

According to (A.17d), the mass $n(\alpha)$ is decreasing in μ, so is $\int_\alpha^t m(\pi)\mathrm{d}\pi$ owing to (A.23). Hence, $\mathsf{P}\left(\boldsymbol{\pi_\infty} \geq t\right)$ must be increasing in μ for every $t \leq p$, thus establishing the first line of (A.25). Given that $\mathsf{E}\left[\boldsymbol{\pi_\infty}(\mu_2)\right] = \mathsf{E}\left[\boldsymbol{\pi_\infty}(\mu_1)\right] = p$, then both parts of the lemma A.3 are satisfied to conclude part (i).

Part (ii): Holding μ constant, for every $t \leq \alpha_2$ we have $\mathsf{P}\left(\boldsymbol{\pi}_\infty(\alpha_2) \geq t\right) = 1 \geq \mathsf{P}\left(\boldsymbol{\pi}_\infty(\alpha_1) \geq t\right)$. Alternatively, for every $t > \alpha_2$

$$\mathsf{P}\left(\boldsymbol{\pi}_\infty(\alpha) \geq t\right) = 1_{\{t \leq p\}} \left(\int_t^p m(\pi) \mathrm{d}\pi + n(p) \right) + n(1) + m(1).$$

Because of (A.24) the integral term is independent of α (for a fixed μ). This is the case for $n(p)$ as well. Therefore, it is sufficient to show that holding μ constant, $n(1) + m(1)$ is decreasing in α. This is equivalent to verifying the following expression is decreasing in α:

$$\begin{aligned}
\frac{(p-\alpha)\Upsilon_{2,1}(\alpha,p)}{\Upsilon_{2,1}(\alpha,p) - \alpha\Upsilon_{1,1}(\alpha,p)} &= \frac{(p-\alpha)\Upsilon_{2,1}(\alpha,p)}{(p-\alpha)p(1-p)\left(\frac{p}{\alpha}\right)^{\delta/\lambda-1}\left(\frac{1-p}{1-\alpha}\right)^{-(\delta/\lambda+2)}} \\
&= p\left[1 - \frac{\alpha^2(1-\alpha)}{p^2(1-p)}\left(\frac{\alpha}{p}\right)^{\delta/\lambda-1}\left(\frac{1-\alpha}{1-p}\right)^{-(\delta/\lambda+2)}\right] \\
&= p\left[1 - \left(\frac{\alpha}{1-\alpha}\right)^{\delta/\lambda+1}\left(\frac{p}{1-p}\right)^{-(\delta/\lambda+1)}\right]
\end{aligned}$$

Since $\alpha/(1-\alpha)$ is increasing in α, then the above expression is decreasing in α, so as a result of this, for every $\alpha_1 < \alpha_2 < p$ and $t > \alpha_2$, it holds that $\mathsf{P}\left(\boldsymbol{\pi}_\infty(\alpha_2) \geq t\right) \leq \mathsf{P}\left(\boldsymbol{\pi}_\infty(\alpha_1) \geq t\right)$. Hence, lemma A.3 can be applied to establish part (ii). $\square$

References

[1] Charalambos D. Aliprantis and Kim C. Border (2006). *Infinite Dimensional Analysis, A Hitch-hiker's Guide*, Springer.

[2] Axel Anderson and Lones Smith (2010). "Dynamic Matching and Evolving Reputations," *The Review of Economic Studies*, 77(1): 3–29.

[3] Simon Board and Moritz Meyer-ter-Vehn (2013). "Reputation for Quality," *Econometrica*, 81(6): 2381–2462.

[4] Patrick Bolton and Christopher Harris (1999). "Strategic Experimentation," *Econometrica*, 67(2): 349–374.

[5] Alessandro Bonatti and Johannes Hörner (2017). "Career Concerns with Exponential Learning," *Theoretical Economics*, 12(1): 425–475.

[6] Hector Chade, Jan Eeckhout, and Lones Smith (2017). "Sorting through Search and Matching Models in Economics," *Journal of Economic Literature*, 55(2): 493–544.

[7] Samuel N. Cohen and Robert J. Elliott (2015). *Stochastic Calculus and Applications*, volume 2, Springer.

[8] Avinash Dixit (2013). *The Art of Smooth Pasting*, Routledge.

[9] Jan Eeckhout and Xi Weng (2010). "Assortative Learning," *Working Paper*.

[10] Michael Ewens, Ramana Nanda, and Matthew Rhodes-Kropf (2018). "Cost of Experimentation and the Evolution of Venture Capital," *Journal of Financial Economics*, 128(3): 422–442.

[11] Paul A. Gompers (1996). "Grandstanding in the Venture Capital Industry," *Journal of Financial Economics*, 42(1): 133–156.

[12] Paul Gompers and Josh Lerner (1999). "An Analysis of Compensation in the US Venture Capital Partnership," *Journal of Financial Economics*, 51(1): 3–44.

[13] Michael Gorman and William A. Sahlman (1989). "What do Venture Capitalists do?" *Journal of Business Venturing*, 4(4): 231–248.

[14] Thomas Hellmann and Manju Puri (2002). "Venture Capital and the Professionalization of Start-up Firms: Empirical Evidence," *The Journal of Finance*, 57(1): 169–197.

[15] Bengt Holmström (1999). "Managerial Incentive Problems: A Dynamic Perspective," *The Review of Economic Studies*, 66(1): 169–182.

[16] Boyan Jovanovic (1979). "Job Matching and the Theory of Turnover," *Journal of Political Economy*, 87(5, Part 1): 972–990.

[17] Samuel Karlin and Albert Novikoff (1963). "Generalized Convex Inequalities." *Pacific Journal of Mathematics*, 13(4): 1251–1279.

[18] Godfrey Keller and Sven Rady (2010). "Strategic Experimentation with Poisson Bandits," *Theoretical Economics*, 5(2): 275–311.

[19] ——— (2015). "Breakdowns," *Theoretical Economics*, 10(1): 175–202.

[20] Godfrey Keller, Sven Rady, and Martin Cripps (2005). "Strategic Experimentation with Exponential Bandits," *Econometrica*, 73(1): 39–68.

[21] Mark A. Krasnoselskij (1964). *Positive Solutions of Operator Equations*, Noordhoff.

[22] Josh Lerner (2002). "When Bureaucrats Meet Entrepreneurs: the Design of Effective Public Venture Capital Programmes," *The Economic Journal*, 112(477): F73–F84.

[23] ——— (2009). *Boulevard of Broken Dreams*, Princeton University Press.

[24] Josh Lerner, David Moore, and Stuart Shepherd (2005). "A Study of New Zealand's Venture Capital Market and Implications for Public Policy," *Report to the Ministry of Research Science & Technology, LECG Limited, New Zealand.*

[25] Fei Li and Xi Weng (2017). "Efficient Learning and Job Turnover in the Labor Market," *International Economic Review*, 58(3): 727–750.

[26] Torgny Lindvall (2002). *Lectures on the Coupling Method*, Courier Corporation.

[27] Gustavo Manso (2011). "Motivating Innovation," *The Journal of Finance*, 66(5): 1823–1860.

[28] Giuseppe Moscarini (2005). "Job Matching and the Wage Distribution," *Econometrica*, 73(2): 481–516.

[29] Ramana Nanda, Sampsa Samila, and Olav Sorenson (2020). "The Persistent Effect of Initial Success: Evidence from Venture Capital," *Journal of Financial Economics*, 137(1): 231–248.

[30] Goran Peskir and Albert Shiryaev (2006). *Optimal Stopping and Free-boundary Problems*, Springer.

[31] Farzad Pourbabaee (2020). "Robust Experimentation in the Continuous Time Bandit Problem," *Economic Theory*: 1–31.

[32] Halsey L. Royden and Patrick Fitzpatrick (1988). *Real Analysis*, volume 32, Macmillan New York.

[33] Moshe Shaked and J. George Shanthikumar (2007). *Stochastic Orders*, Springer Science & Business Media.

[34] Robert Shimer and Lones Smith (2000). "Assortative Matching and Search," *Econometrica*, 68(2): 343–369.

[35] Morten Sørensen (2007). "How Smart is Smart Money? A Two-Sided Matching Model of Venture Capital," *The Journal of Finance*, 62(6): 2725–2762.

[36] Xuan Tian and Tracy Yue Wang (2014). "Tolerance for Failure and Corporate Innovation," *The Review of Financial Studies*, 27(1): 211–255.

[37] Donald M. Topkis (2011). *Supermodularity and Complementarity*, Princeton University Press.

[38] Masako Ueda (2004). "Banks versus Venture Capital: Project Evaluation, Screening, and Expropriation," *The Journal of Finance*, 59(2): 601–621.

[39] Evert Van Imhoff and Jozeff M.M. Ritzen (1988). "Optimal Economic Growth and Nonstable Population," *De Economist*, 136(3): 339–357.

B Online Appendix

B.1 Social Optimum in Section 4

The planner maximizes the present value of social surplus S that is expressed in equation (4.1), subject to the population dynamics in (4.2) and (4.3). The instruments that the planner has at her disposal is the choice of the matching sets, i.e $\big\{\chi_q(\pi) : q \in \{a,b\}$ and $\pi \in [0,1]\big\}$. The current value Hamiltonian for this problem is

$$\begin{aligned}
\mathcal{H} = & \sum_q \left[(\lambda_q - c)m_q(1) + \int (\lambda_q \pi - c)m_q(\pi)\mathrm{d}\pi\right] \\
& + \sum_q v_*(1,q)\Big[-\lambda_q m_q(1) + \kappa\varphi_q n(1)\chi_q(1)\Big] \\
& + w_*(1)\left[\sum_q \lambda_q m_q(1) - \sum_q \kappa\varphi_q n(1)\chi_q(1) + \sum_q \int \lambda_q \pi m_q(\pi)\mathrm{d}\pi\right] \\
& + \sum_q \int v_*(\pi,q)\Big[-\lambda_q \pi m_q(\pi) + \kappa\varphi_q n(\pi)\chi_q(\pi) + \lambda_q \partial_\pi\big(\pi(1-\pi)m_q(\pi)\big)\Big]\mathrm{d}\pi \\
& + \int w_*(\pi)\Big[-\sum_q \kappa\varphi_q n(\pi)\chi_q(\pi)\Big]\mathrm{d}\pi \\
& + \rho\left[\zeta\lambda_a\left(m_a(1) + \int \pi m_a(\pi)\mathrm{d}\pi\right) - \kappa\varphi_b\left(n(1)\chi_a(1) + \int n(\pi)\chi_q(\pi)\mathrm{d}\pi\right)\right].
\end{aligned}$$

Applying the integration by part implies that

$$\int v_*(\pi,q)\lambda_q\partial_\pi\big(\pi(1-\pi)m_q(\pi)\big)\mathrm{d}\pi = -\int \lambda_q \pi(1-\pi)v_*'(\pi,q)m_q(\pi)\mathrm{d}\pi.$$

Substituting this in the Hamiltonian and regrouping with respect to the population measures amount to

$$\begin{aligned}
\mathcal{H} = & \sum_q m_q(1)\Big[\lambda_q - c + \lambda_q\big(w_*(1) - v_*(1,q)\big) + \rho\zeta\lambda_a 1_{\{q=a\}}\Big] \\
& + n(1)\left[\sum_q \kappa\varphi_q\big(v_*(1,q) - w_*(1)\big)\chi_q(1) - \rho\kappa\varphi_b\chi_b(1)\right] \\
& + \sum_q \int m_q(\pi)\Big[\lambda_q\pi - c + \lambda_q\pi\big(w_*(1) - v_*(\pi,q)\big) - \lambda_q\pi(1-\pi)v_*'(\pi,q) + \rho\zeta\lambda_a\pi 1_{\{q=a\}}\Big]\mathrm{d}\pi \\
& + \int n(\pi)\Big[\sum_q \kappa\varphi_q\big(v_*(\pi,q) - w_*(\pi)\big)\chi_q(\pi) - \rho\kappa\varphi_b\chi_b(\pi)\Big]\mathrm{d}\pi.
\end{aligned}$$

The planner's optimization problem, as expressed above, features a *continuum* of control and state processes. Therefore, I appeal to the heuristic method of Van Imhoff and Ritzen (1988) (chapter 6) to interpret the integrals as the summation of discrete variables over intervals of length $\mathrm{d}\pi$. The first implication of the above representation is that from the planner's viewpoint the optimal matching indicator χ^* satisfies:

$$\chi_q^*(\pi) = 1 \Leftrightarrow v_*(\pi, q) > w_*(\pi),$$

that is a match is socially optimal if the social marginal value of the partnership (i.e. v_*) dominates the social marginal value of reputation while being unmatched (i.e. w_*).

Next, I express the co-state equations for each of the social marginal values. In that, I will use the Gâteaux derivative (see chapter 7 in Aliprantis and Border (2006)) of the Hamiltonian w.r.t the associated probability measure. For instance, to find out the derivative of $\mathcal{H}$ w.r.t $n(x)$ (for $x < 1$), define δ_x as the Dirac mass concentrated at x, then:

$$\begin{aligned}
\mathbb{D}_{n(x)}\mathcal{H} &:= \lim_{\varepsilon\to 0} \frac{\mathcal{H}[n(x) + \varepsilon\delta_x] - \mathcal{H}[n(x)]}{\varepsilon} \\
&= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int \varepsilon\delta_x(\pi) \left[\sum_q \kappa\varphi_q \big(v_*(\pi, q) - w_*(\pi)\big)\chi_q^*(\pi) - \rho\kappa\varphi_b\chi_b^*(\pi) \right] \mathrm{d}\pi \\
&= \sum_q \kappa\varphi_q \big(v_*(x, q) - w_*(x)\big)\chi_q^*(x) - \rho\kappa\varphi_b\chi_b^*(x).
\end{aligned}$$

Hence the co-state equations are ordered as follows:

$$\begin{aligned}
rv_*(1, q) - \dot{v}_*(\pi, q) &= \mathbb{D}_{m_q(1)}\mathcal{H} = \lambda_q - c + \lambda_q\big(w_*(1) - v_*(1, q)\big) + \rho\zeta\lambda_a 1_{\{q=a\}}, \\
rw_*(1) - \dot{w}_*(1) &= \mathbb{D}_{n(1)}\mathcal{H} = \sum_q \kappa\varphi_q \big(v_*(1, q) - w_*(1)\big)\chi_q^*(1) - \rho\kappa\varphi_b\chi_b(1), \\
rv_*(\pi, q) - \dot{v}_*(\pi, q) &= \mathbb{D}_{m_q(\pi)}\mathcal{H} \\
&= \lambda_q\pi - c + \lambda_q\pi\big(w_*(1) - v_*(\pi, q)\big) - \lambda_q\pi(1-\pi)v_*'(\pi, q) + \rho\zeta\lambda_a\pi 1_{\{q=a\}}, \\
rw_*(\pi) - \dot{w}_*(\pi) &= \mathbb{D}_{n(\pi)}\mathcal{H} = \sum_q \kappa\varphi_q \big(v_*(\pi, q) - w_*(\pi)\big)\chi_q^*(\pi) - \rho\kappa\varphi_b\chi_b^*(\pi).
\end{aligned}$$

The social shadow value of the mass of high-type projects, i.e. ρ, satisfies the following first order condition:

$$\begin{aligned}
r\rho - \dot{\rho} &= \frac{\partial\mathcal{H}}{\partial\varphi_b} \\
&= \kappa n(1)\big(v_*(1, b) - w_*(1) - \rho\big)\chi_b^*(1) + \int \kappa n(\pi)\big(v_*(\pi, b) - w_*(\pi) - \rho\big)\chi_b^*(\pi)\mathrm{d}\pi.
\end{aligned}$$

In the steady-state the above representation leads to (4.5).

B.2 General Type Space

The goal of this appendix is to extend the results of section 2 to the general type space for projects. Specifically, I show there always exists an *increasing* reputation function w that satisfies the agents' fixed point problem. Suppose the projects' types are drawn from an arbitrary distribution with CDF $\phi(\cdot)$ on a bounded support $[a, b]$. The success arrival intensity takes the general form of $\lambda_q(\theta)$, for which I denote $\lambda_q(H) = \bar{\lambda}_q$ and $\lambda_q(L) = \underline{\lambda}_q$, and assume $\underline{\lambda}_q \leq \bar{\lambda}_q \leq \boldsymbol{\lambda}$ for all $q \in \mathsf{Supp}(\phi)$.

The reputation value function satisfies

$$w(\pi) = \frac{\kappa}{r} \int [v(\pi, q) - w(\pi)]^+ \, \phi(\mathrm{d}q),$$

therefore for every measurable subset $B \subset [a, b]$, one can see the equilibrium value functions (w, v) satisfy

$$w(\pi) \geq \frac{\kappa}{r} \int_B [v(\pi, q) - w(\pi)] \, \phi(\mathrm{d}q) \Rightarrow w(\pi) \geq \frac{\int_B v(\pi, q)\phi(\mathrm{d}q)}{1 + \frac{\kappa}{r}\phi(B)}.$$

Setting $B^* = \{q : v(\pi, q) > w(\pi)\}$ to bind the above inequality, one can propose the following equivalent representation for the reputation value function:

$$w(\pi) = \sup \left\{ \frac{\int_B v(\pi, q)\phi(\mathrm{d}q)}{1 + \frac{\kappa}{r}\phi(B)} : \text{measurable } B \subset [a, b] \right\}. \tag{B.1}$$

On the other hand, given the reputation function w, each agent solves the stopping time problem when matched with a project of type q:

$$v(\pi, q) = \sup_\tau \mathsf{E} \left[1_{\{\sigma \leq \tau\}} e^{-r\sigma} - c \int_0^{\sigma \wedge \tau} e^{-rs} \mathrm{d}s + e^{-r(\sigma \wedge \tau)} w(\pi_{\sigma \wedge \tau}) \right]. \tag{B.2}$$

For a given q, let $\mathbb{T}_q w$ be the matching value function resulted from the above problem. Hence from (B.1) it follows that w is the fixed point to the following operator:

$$\mathcal{A}w := \sup \left\{ \frac{\int_B \mathbb{T}_q w \ \phi(\mathrm{d}q)}{1 + \frac{\kappa}{r}\phi(B)} : \text{measurable } B \subset [a, b] \right\}. \tag{B.3}$$

In what follows I propose the appropriate function space on which $\mathcal{A}$ will be defined, and study its fixed points. Let $L^1[0, 1]$ be the Banach space of Lebesgue integrable functions on the unit interval, and $L^1_+[0, 1]$ be the subset of non-negative functions which is readily seen to be a cone.[19] Let $\succsim$ be the partial order induced by the cone $L^1_+[0, 1]$ on the Banach space $L^1[0, 1]$, that is $w_2 \succsim w_1$ if $w_2(\pi) \geq w_1(\pi), \forall \pi \in [0, 1]$. Then, it follows from (B.2) that $\mathbb{T}_q$ is

[19] A cone is a subset $\mathcal{K}$ of a Banach space which is (i) closed, (ii) for every $x, y \in \mathcal{K}$ and $\alpha, \beta \geq 0$: $\alpha x + \beta y \in \mathcal{K}$, and (iii) $\mathcal{K} \cap (-\mathcal{K}) = \mathbf{0}$.

a *positive* and *monotone* operator, that is letting $\mathbf{0}$ to be the zero element of $L^1[0,1]$, then $\mathbb{T}_q\mathbf{0} \gtrsim \mathbf{0}$, and $\mathbb{T}_q w_2 \gtrsim \mathbb{T}_q w_1$ for $w_2 \gtrsim w_1$ in $L^1_+[0,1]$. Further, it can easily be verified that $\mathcal{A}$ inherits *positivity* and *monotonicity* from the collection $\{\mathbb{T}_q : q \in [a,b]\}$. Next, I show without loss of generality, we can restrict the search for the fixed point to the bounded region of all $w \in L^1_+[0,1]$ where $\|w\|_\infty \leq \boldsymbol{\lambda}/r$.[20]

Lemma B.1. *For every $w \in L^1_+[0,1]$,*

$$\|\mathbb{T}_q w\| \leq \max\left\{\|w\|, \frac{\boldsymbol{\lambda}}{r+\boldsymbol{\lambda}}(1+\|w\|)\right\}, \quad \phi - \textit{almost surely.}$$

Proof. For every $q \in \mathsf{Supp}(\phi)$,

$$\begin{aligned}
\mathbb{T}_q w(\pi) &= \sup_\tau \mathsf{E}\left[1_{\{\sigma\leq\tau\}}e^{-r\sigma} - c\int_0^{\sigma\wedge\tau} e^{-rs}\mathrm{d}s + e^{-r(\sigma\wedge\tau)}w(\pi_{\sigma\wedge\tau})\right] \\
&\leq \sup_\tau \mathsf{E}\left[1_{\{\sigma\leq\tau\}}e^{-r\sigma} + e^{-r(\sigma\wedge\tau)}w(\pi_{\sigma\wedge\tau})\right] \\
&\leq \sup_\tau \mathsf{E}\Big[1_{\{\sigma\leq\tau\}}e^{-r\sigma} + e^{-r(\sigma\wedge\tau)}\|w\|\Big].
\end{aligned}$$

Let $\boldsymbol{\sigma} \stackrel{d}{=} \mathsf{exp}(\boldsymbol{\lambda})$, then $\sigma \succeq \boldsymbol{\sigma}$ in the sense of the first order stochastic dominance.[21] Since the random variable inside the last expectation above is decreasing in σ, then

$$\mathbb{T}_q w(\pi) \leq \sup_\tau \mathsf{E}\Big[1_{\{\boldsymbol{\sigma}\leq\tau\}}e^{-r\boldsymbol{\sigma}} + e^{-r(\boldsymbol{\sigma}\wedge\tau)}\|w\|\Big].$$

The above stopping time problem has no state variable (such as π), and it maintains its memory-less property inherited from the exponential distribution of $\boldsymbol{\sigma}$, therefore, the optimal stopping rule is either $\tau = 0$ or $\tau = \infty$, that implies:

$$\mathbb{T}_q w(\pi) \leq \max\left\{\|w\|, \mathsf{E}\left[e^{-r\boldsymbol{\sigma}}\right]\big(1+\|w\|\big)\right\}.$$

Since $\mathsf{E}\left[e^{-r\boldsymbol{\sigma}}\right] = \boldsymbol{\lambda}/\big(r+\boldsymbol{\lambda}\big)$ the claim follows. □

I use the previous lemma to limit the search for the space of fixed points.

Lemma B.2. *Any fixed point of $\mathcal{A}$ (if exists) is order bounded above by the constant function $\boldsymbol{\lambda}/r$.*

Proof. First, note that the supremum in (B.3) is achieved by $B_w = \{q : \mathbb{T}_q w(\pi, q) > w(\pi)\}$ for any candidate fixed point w. Then, for any such candidate

$$\Big(1 + \frac{\kappa}{r}\phi(B_w)\Big)\, w(\pi) = \int_{B_w} \mathbb{T}_q w(\pi)\phi(\mathrm{d}q),$$

[20]Henceforth, if not stated explicitly all norms are the sup-norm.

[21]The term $\mathsf{exp}(\lambda)$ denotes an exponential random variable with the rate λ.

therefore, using the result of the previous lemma

$$\left(1+\frac{\kappa}{r}\phi(B_w)\right)\|w\| \leq \max\left\{\|w\|, \frac{\boldsymbol{\lambda}}{r+\boldsymbol{\lambda}}(1+\|w\|)\right\}\phi(B_w). \tag{B.4}$$

Assume to the contrary that $\|w\| > \boldsymbol{\lambda}/r$, then $\max\left\{\|w\|, \frac{\boldsymbol{\lambda}}{r+\boldsymbol{\lambda}}(1+\|w\|)\right\} = \|w\|$, and (B.4) implies

$$\left(1+\frac{\kappa}{r}\phi(B_w)\right)\|w\| \leq \|w\|\phi(B_w). \tag{B.5}$$

Canceling $\|w\|$ from both sides amounts to $1+\frac{\kappa}{r}\phi(B_w) \leq \phi(B_w)$. Since it was assumed $\|w\| > \boldsymbol{\lambda}/r$, then $\phi(B_w) > 0$. On the other hand $\phi(B_w) \leq 1$. These two together with (B.5) yield the contradiction and hence prove the lemma. □

Definition B.3 (Regular and strongly-minihedral cones: Krasnoselskij (1964) sections 1.5 and 1.7)**.** A Banach space partially ordered by a cone is called *regularly partially ordered*, if any monotone-increasing sequence, *order-bounded* from above, converges in norm to a limit point. A cone which generates a regular partial ordering is called a regular cone. A cone is said to be *strongly minihedral* if every order bounded subset has a least upper bound (order supremum).

Now consider the Banach space of integrable functions $L^1[0,1]$, and the positive cone of $L^1_+[0,1] = \{f \in L^1[0,1] : f(x) \geq 0 \ \forall x \in [0,1]\}$. This cone is regular, and for any monotone increasing sequence $\{f_n\} \subset L^1[0,1]$ such that $f_1 \precsim f_2 \precsim \ldots$ and order bounded from above, $\|f_n - f\|_{L^1} \to 0$ where $f(x) = \sup_n f_n(x)$ *for every* $x \in [0,1]$ (Dominated convergence theorem). In addition $L^1_+[0,1]$ is strongly minihedral (page 52 Krasnoselskij (1964)).

Let $\langle \mathbf{0}, \boldsymbol{\lambda}/r\rangle := \left\{f \in L^1_+[0,1] : \mathbf{0} \precsim f \precsim \boldsymbol{\lambda}/r\right\}$ be the order interval of non-negative L^1 functions, order bounded above by the constant function $\boldsymbol{\lambda}/r$. In light of the lemma B.1, we have $\mathbb{T}_q : \langle \mathbf{0}, \boldsymbol{\lambda}/r\rangle \to \langle \mathbf{0}, \boldsymbol{\lambda}/r\rangle$ for every $q \in [a,b]$ and hence $\mathcal{A} : \langle \mathbf{0}, \boldsymbol{\lambda}/r\rangle \to \langle \mathbf{0}, \boldsymbol{\lambda}/r\rangle$. At this stage, I can apply part (a) of theorem 4.1 in Krasnoselskij (1964) to conclude the existence of a fixed point of $\mathcal{A}$ in $\langle \mathbf{0}, \boldsymbol{\lambda}/r\rangle$, because the mapping $\mathcal{A}$ is monotonic in a strongly minihedral cone space. However, the mere existence of the fixed point is far from enough. In particular, we want to know whether there exists a continuous and/or increasing fixed point for $\mathcal{A}$. To answer such questions, I will need to dig deeper into the mapping $\mathcal{A}$, beyond its monotonicity. In doing so, I shall construct a monotone sequence of functions, and show it converges in the L^1 sense to a fixed point of $\mathcal{A}$.

Fix $w_0 := \mathbf{0}$ and recursively define $w_n = \mathcal{A}w_{n-1}$, therefore $\{w_n\} \subset \langle \mathbf{0}, \boldsymbol{\lambda}/r\rangle$ is an increasing sequence, order bounded from above, hence converges in L^1 to $w_\infty \in \langle \mathbf{0}, \boldsymbol{\lambda}/r\rangle$ where $w_\infty(\pi) = \sup_n w_n(\pi)$ for each $\pi \in [0,1]$ (because of the regularity of the $L^1_+[0,1]$ cone). The conceptual merit of this recursive construction is summarized in the following two points:

(i) Say a property $\star$ is owned by w_0 and is preserved by the mapping $\mathcal{A}$. Then, it holds along the sequence $\{w_n\}$.

(ii) If $\star$ is stable under the L^1 limit, then w_∞ holds this property.

Therefore, if $\mathcal{A}$ is L^1 continuous along the sequence $\{w_n\}$, then w_∞ becomes the fixed point and the presumptive property $\star$ will be inherited to w_∞.

Proposition B.4. *For the sequence $\{w_n\}$ defined above, it holds that $\|\mathcal{A}w_n - \mathcal{A}w_\infty\|_{L^1} \to 0$, and as a result $w_\infty = \mathcal{A}w_\infty$.*

Proof. First note that for every $\pi \in [0,1]$,

$$\begin{aligned}
\mathcal{A}w_\infty(\pi) - \mathcal{A}w_n(\pi) &= \sup\left\{\frac{\int_B \mathbb{T}_q w_\infty(\pi)\phi(\mathrm{d}q)}{1+\frac{\kappa}{r}\phi(B)} : B \subset [a,b]\right\} \\
&\qquad - \sup\left\{\frac{\int_B \mathbb{T}_q w_n(\pi)\phi(\mathrm{d}q)}{1+\frac{\kappa}{r}\phi(B)} : B \subset [a,b]\right\} \\
&\le \sup\left\{\frac{\int_B \left(\mathbb{T}_q w_\infty - \mathbb{T}_q w_n\right)(\pi)\phi(\mathrm{d}q)}{1+\frac{\kappa}{r}\phi(B)} : B \subset [a,b]\right\} \\
&\le \int_0^1 \left(\mathbb{T}_q w_\infty - \mathbb{T}_q w_n\right)(\pi)\phi(\mathrm{d}q),
\end{aligned}$$

where in the last line, I used the fact that $w_\infty \gtrsim w_n$ and the monotonicity of the operator $\mathbb{T}_q$. Therefore, the L^1-norm can be bounded above as:

$$\begin{aligned}
\|\mathcal{A}w_\infty - \mathcal{A}w_n\|_{L^1} &= \int_0^1 \big(\mathcal{A}w_\infty(\pi) - \mathcal{A}w_n(\pi)\big)\mathrm{d}\pi \\
&\le \int_0^1\int_0^1 \left(\mathbb{T}_q w_\infty - \mathbb{T}_q w_n\right)(\pi)\phi(\mathrm{d}q)\mathrm{d}\pi = \int_0^1 \|\mathbb{T}_q w_\infty - \mathbb{T}_q w_n\|_{L^1}\phi(\mathrm{d}q).
\end{aligned}$$

For the last equality relation, I used the fact that the integrand is positive and uniformly bounded above by $\boldsymbol{\lambda}/r$ to apply Fubini's theorem and exchange the order of integrations. Since the integrand of the last integral is uniformly bounded (over all $q \in [a,b]$), then one can apply the Lebesgue dominated convergence theorem to get:

$$\begin{aligned}
\lim_{n\to\infty}\|\mathcal{A}w_\infty - \mathcal{A}w_n\|_{L^1} &\le \lim_{n\to\infty}\int_0^1 \|\mathbb{T}_q w_\infty - \mathbb{T}_q w_n\|_{L^1}\phi(\mathrm{d}q) \\
&= \int_0^1 \lim_{n\to\infty}\|\mathbb{T}_q w_\infty - \mathbb{T}_q w_n\|_{L^1}\phi(\mathrm{d}q).
\end{aligned} \tag{B.6}$$

Next, I propose a method to upper bound $\left(\mathbb{T}_q w_\infty - \mathbb{T}_q w_n\right)(\pi)$, and hence its L^1-norm. For this let G represent the random variable inside the expectation operator in the definition of $\mathbb{T}_q w$.

$$\begin{aligned}
\big(\mathbb{T}_q w_\infty - \mathbb{T}_q w_n\big)(\pi) &= \sup_\tau \mathsf{E}_\pi\left[G(\sigma, w_\infty; \tau)\right] - \sup_\tau \mathsf{E}\left[G(\sigma, w_n; \tau)\right] \\
&\le \sup_\tau \mathsf{E}_\pi\left[e^{-r(\sigma\wedge\tau)}\left(w_\infty - w_n\right)(\pi_{\sigma\wedge\tau})\right] \\
&\le \mathsf{E}_\pi\left[e^{-r\sigma}\left(w_\infty - w_n\right)(\pi_\sigma)\right] + \sup_\tau \mathsf{E}_\pi\left[e^{-r\tau}\left(w_\infty - w_n\right)(\pi_\tau); \tau < \sigma\right]
\end{aligned}$$

Therefore, the L^1-norm is bounded by

$$\|\mathbb{T}_q w_\infty - \mathbb{T}_q w_n\|_{L^1} \leq \underbrace{\int_0^1 \mathsf{E}_\pi \left[e^{-r\sigma}\left(w_\infty - w_n\right)(\pi_\sigma)\right] \mathrm{d}\pi}_{\mathcal{I}_1 :=} \\ + \underbrace{\int_0^1 \sup_\tau \mathsf{E}_\pi \left[e^{-r\tau}\left(w_\infty - w_n\right)(\pi_\tau); \tau < \sigma\right] \mathrm{d}\pi}_{\mathcal{I}_2 :=}.$$

The integrands of both integrals are bounded by $\boldsymbol{\lambda}/r$, hence applying the Lebesgue dominated convergence theorem twice for the first integral implies

$$\lim_{n\to\infty} \mathcal{I}_1 = \int_0^1 \lim_{n\to\infty} \mathsf{E}_\pi \left[e^{-r\sigma}\left(w_\infty - w_n\right)(\pi_\sigma)\right] \mathrm{d}\pi = \int_0^1 \mathsf{E}_\pi \left[\lim_{n\to\infty} e^{-r\sigma}\left(w_\infty - w_n\right)(\pi_\sigma)\right] \mathrm{d}\pi = 0,$$

because w_∞ is the pointwise supremum of the sequence $\{w_n\}$. To show the convergence for the second integral, first note that for every given $\varepsilon > 0$ one can find $T > 0$ such that

$$\sup_\tau \mathsf{E}_\pi \left[e^{-r\tau}\left(w_\infty - w_n\right)(\pi_\tau); \tau < \sigma\right] \leq \sup_{\tau\leq T} \mathsf{E}_\pi \left[e^{-r\tau}\left(w_\infty - w_n\right)(\pi_\tau); \tau < \sigma\right] + \varepsilon,$$

uniformly over all π. This is owed to the uniform boundedness of $(w_\infty - w_n)$ by $\boldsymbol{\lambda}/r$. Next, because of the property of the supremum for every $\varepsilon > 0$, there exist $\tau_{n,\pi}$ (possibly depending on n and π) such that

$$\sup_{\tau\leq T} \mathsf{E}_\pi \left[e^{-r\tau}\left(w_\infty - w_n\right)(\pi_\tau); \tau < \sigma\right] \leq e^{-r\tau_{n,\pi}}\left(w_\infty - w_n\right)(\pi_{\tau_{n,\pi}}) \mathsf{P}_\pi\left(\tau_{n,\pi} < \sigma\right) + \varepsilon.$$

Therefore,

$$\begin{aligned}\mathcal{I}_2 &\leq \int_0^1 e^{-r\tau_{n,\pi}}\left(w_\infty - w_n\right)(\pi_{\tau_{n,\pi}}) \mathsf{P}_\pi\left(\tau_{n,\pi} < \sigma\right) \mathrm{d}\pi + 2\varepsilon \\ &= \int_0^1 e^{-r\tau_{n,\pi}}\left(w_\infty - w_n\right)(\pi_{\tau_{n,\pi}}) \left(\pi e^{-\bar{\lambda}_q \tau_{n,\pi}} + (1-\pi) e^{-\underline{\lambda}_q \tau_{n,\pi}}\right) \mathrm{d}\pi + 2\varepsilon.\end{aligned}$$

Because of the Bayes-law, $\pi_{\tau_{n,\pi}} = \frac{\pi e^{-\Delta_q \tau_{n,\pi}}}{1-\pi+\pi e^{-\Delta_q \tau_{n,\pi}}}$, where $\Delta_q := \bar{\lambda}_q - \underline{\lambda}_q$. Leveraging this relation and applying the change of variable to the above integral lead to

$$\begin{aligned}\mathcal{I}_2 - 2\varepsilon &\leq \int_0^1 (w_\infty - w_n)(x) \frac{e^{(\bar{\lambda}_q - 2\underline{\lambda}_q - r)\tau_{n,x}}}{\left(1 - x + x e^{\Delta_q \tau_{n,x}}\right)^3} \mathrm{d}x \\ &\leq \int_0^1 (w_\infty - w_n)(x) e^{(\bar{\lambda}_q - 2\underline{\lambda}_q - r)\tau_{n,x}} \mathrm{d}x,\end{aligned} \tag{B.7}$$

where in the last inequality I used the fact that $\left(1 - x + x e^{\Delta_q \tau_{n,x}}\right)$ is increasing in x. Since,

$\tau_{n,x} \leq T$ the last integrand in (B.7) is uniformly bounded for all x and n. Hence, one can apply the Lebesgue dominated convergence theorem again and obtain

$$\lim_{n\to\infty} \mathcal{I}_2 \leq \int_0^1 \lim_{n\to\infty} (w_\infty - w_n)(x) e^{(\bar{\lambda}_q - 2\underline{\lambda}_q - r)\tau_{n,x}} \mathrm{d}x + 2\varepsilon = 2\varepsilon.$$

Since this relation holds for every $\varepsilon > 0$, then $\lim_{n\to\infty} \mathcal{I}_2 = 0$. This establishes the L^1 convergence of $\mathcal{A}w_n$ to $\mathcal{A}w_\infty$ and thus proves $w_\infty = \mathcal{A}w_\infty$. $\square$

A very important property owned by w_0 and preserved under $\mathcal{A}$ is being increasing in π. In the next lemma, using the techniques from the coupling of probability measures and stochastic dominance, I show $\mathcal{A}w$ is increasing in π when w is.

Lemma B.5. *Let w be an increasing function in π, then $\mathcal{A}w$ becomes increasing in π as well.*

Proof. Fix q and suppose $\pi_2 \geq \pi_1$. Define the random variables

$$\sigma_i \stackrel{d}{=} \pi_i \mathsf{exp}(\bar{\lambda}_q) + (1 - \pi_i)\mathsf{exp}(\underline{\lambda}_q), \quad i \in \{1, 2\} \tag{B.8}$$

as the exponential time of success arrivals under π_1 and π_2. One can easily check $\sigma_1 \succeq \sigma_2$ in the sense of the first order stochastic dominance (see the following appendix). Therefore, for every *decreasing* function f we will have $\mathsf{E}[f(\sigma_2)] \geq \mathsf{E}[f(\sigma_1)]$. Recall the definition of $\mathbb{T}_q$:

$$\begin{aligned} \mathbb{T}_q w(\pi) &= \sup_\tau \mathsf{E}_\pi \left[G(\sigma;\tau) \right] \\ G(\sigma;\tau) &:= 1_{\{\sigma \leq \tau\}} e^{-r\sigma} - c \int_0^{\sigma\wedge\tau} e^{-rs} \mathrm{d}s + e^{-r(\sigma\wedge\tau)} w(\pi_{\sigma\wedge\tau}). \end{aligned}$$

The first two terms in G are clearly decreasing in σ, so for every $q \in [a, b]$ and τ:

$$\mathsf{E}\left[1_{\{\sigma_2 \leq \tau\}} e^{-r\sigma_2} - c \int_0^{\sigma_2 \wedge \tau} e^{-rs} \mathrm{d}s \right] \geq \mathsf{E}\left[1_{\{\sigma_1 \leq \tau\}} e^{-r\sigma_1} - c \int_0^{\sigma_1 \wedge \tau} e^{-rs} \mathrm{d}s \right]. \tag{B.9}$$

The proof for monotonicity of the last term in G is a bit more tricky, because $\pi_{\sigma\wedge\tau}$ is not just a function of σ, but it also depends on the initial belief π. So, let us define $\boldsymbol{w}(\pi, \sigma; \tau) := e^{-r(\sigma\wedge\tau)} w(\pi_{\sigma\wedge\tau})$ where π is the initial belief value. To proceed, I need to define σ_1 and σ_2 on the *same* probability space, because the analysis to be presented needs more than the application of the first order stochastic dominance. For this, I use the Strassen theorem (Lindvall (2002) chapter 4) to find the coupling $(\hat{\sigma}_1, \hat{\sigma}_2)$ such that $\hat{\sigma}_i \stackrel{d}{=} \sigma_i$ for $i = 1, 2$, and crucially $\hat{\sigma}_1 \geq \hat{\sigma}_2$ almost surely. It is proven in the following appendix that for every τ, $\boldsymbol{w}$ is increasing in π and decreasing in σ (while holding π constant), therefore

$$\begin{aligned}
\mathsf{E}_{\pi_2}\left[e^{-r(\sigma_2\wedge\tau)}w(\pi_{\sigma_2\wedge\tau})\right] &= \mathsf{E}\left[\boldsymbol{w}(\pi_2,\hat{\sigma}_2;\tau)\right] && \text{(B.10a)}\\
&\geq \mathsf{E}\left[\boldsymbol{w}(\pi_1,\hat{\sigma}_2;\tau)\right] && \text{(B.10b)}\\
&\geq \mathsf{E}\left[\boldsymbol{w}(\pi_1,\hat{\sigma}_1;\tau)\right] && \text{(B.10c)}\\
&= \mathsf{E}_{\pi_1}\left[e^{-r(\sigma_1\wedge\tau)}w(\pi_{\sigma_1\wedge\tau})\right]. && \text{(B.10d)}
\end{aligned}$$

In (B.10a) and (B.10d), I used the fact that coupling preserves the marginal distributions. In (B.10b), I apply the increasing property of $\boldsymbol{w}$ in π, and in (B.10c) its decreasing property in σ.

Combining (B.9) and (B.10) implies that for every τ and $q \in [a,b]$: $\mathsf{E}_{\pi_2}[G(\sigma_2;\tau)] \geq \mathsf{E}_{\pi_1}[G(\sigma_1;\tau)]$, therefore, applying the supremum on both sides (w.r.t to τ) yields $\mathbb{T}_q w(\pi_2) \geq \mathbb{T}_q w(\pi_1)$. From this and expression (B.3), it is now straightforward to conclude that $\mathcal{A}w(\pi_2) \geq \mathcal{A}w(\pi_1)$. □

Now we are in a position to establish the existence of a fixed point that is increasing in π, the proof of which follows from the previous lemma and the fact that increasing property is closed under the L^1 limit.

Theorem B.6. *The operator $\mathcal{A}$ has an increasing fixed point function.*

For a candidate increasing fixed point w, we can now assure that if $w(\pi') > 0$ for some π', then $w(\pi'') > 0$ for all $\pi'' > \pi'$. This means once w exceeds zero it will never fall back to zero again, therefore the union of all matching sets over $q \in [a,b]$ must be an *increasing set* in $[0,1]$. Hence, there exists an equilibrium point α such that

$$\bigcup_{q\in[a,b]} \{\pi : \mathbb{T}_q w(\pi) > w(\pi)\} = (\alpha, 1]\,.$$

Next, I show how α is determined. Its location is important because it represents the point of endogenous exit from the market. In particular, the agents with a lower reputation than α would no longer match. In the next proposition, I show under some natural assumptions, α is the boundary point of the stopping time problem that a typical agent solves when is matched to the *best* type of the projects, i.e. $q = b$. For this I present two notions. The profile of arrival intensity $\lambda = \{(\underline{\lambda}_q, \bar{\lambda}_q) : q \in [a,b]\}$ is called monotone if $\underline{\lambda}_q$ and $\bar{\lambda}_q$ are increasing in q. It satisfies the *increasing-differences* if $\bar{\lambda}_{q''} - \underline{\lambda}_{q''} \geq \bar{\lambda}_{q'} - \underline{\lambda}_{q'}$ for every $q'' > q'$ in $[a,b]$.

Proposition B.7. *Assume the profile λ is monotone and satisfies the increasing-differences. Then, α is the lowest boundary point of $\mathcal{M}_b$, and is the unique fixed point of*

$$\alpha = \frac{c}{\Delta_b\left(1 + w\left(\frac{\bar{\lambda}_b\alpha}{\Delta_b\alpha + \underline{\lambda}_b}\right)\right)} - \frac{\underline{\lambda}_b}{\Delta_b}. \tag{B.11}$$

Proof. Assume by contradiction that $\alpha \notin \mathsf{cl}(\mathcal{M}_b)$, and there exists $q < b$ such that $\alpha = \inf \mathcal{M}_q$, that is an agent matched with a project of type q, terminates the match as her reputation nears α. The principles of optimality requires smooth and continuous fit at α, namely $v'(\alpha, q) = v(\alpha, q) = 0$. From the Bellman equation, for every $\pi \in \mathcal{M}_q$ it must be that

$$rv(\pi, q) = -c + \left(\bar{\lambda}_q \pi + \underline{\lambda}_q(1-\pi)\right)\left(1 + w \circ \jmath(\pi) - v(\pi, q)\right) - \pi(1-\pi)\Delta_q v'(\pi, q).$$

In that $\jmath$ returns the posterior *after* the success has taken place at time t, namely:

$$\jmath(\pi_{t^-}) := \frac{\bar{\lambda}_q \pi_{t^-}}{\bar{\lambda}_q \pi_{t^-} + \underline{\lambda}_q(1 - \pi_{t^-})}.$$

In the baseline model, the success event was *conclusive* thus $\jmath(\pi) = 1$ for every $\pi \in (0, 1]$. The optimality principles at $\pi = \alpha$ imply

$$c = \left(\alpha \Delta_q + \underline{\lambda}_q\right)\left(1 + w \circ \jmath(\alpha)\right). \tag{B.12}$$

Furthermore, since $\alpha \notin \mathsf{cl}(\mathcal{M}_b)$ then $v(\alpha, b) = w(\alpha) = 0$ and superharmonicity implies that

$$0 > \mathcal{L}_b v(\alpha, b) - rv(\alpha, b) - c = (\alpha \Delta_b + \underline{\lambda}_b)\left(1 + w \circ \jmath(\alpha)\right) - c.$$

Replacing (B.12) in the above inequality and canceling c from both sides amount to

$$0 > \frac{\alpha \Delta_b + \underline{\lambda}_b}{\alpha \Delta_q + \underline{\lambda}_q} - 1.$$

However the *rhs* of the above inequality is positive because of the monotonicity and increasing-differences, hence the contradiction is resulted. Therefore, it must be that $\alpha = \inf \mathcal{M}_b$.

On the uniqueness of α, note that the *lhs* of (B.11) is increasing in α, while the *rhs* is decreasing – because w is an increasing function. Therefore, upon the existence, α is *uniquely* determined by this equation. $\square$

B.3 Supplementary Proofs for B.2

Proof for $\sigma_1 \succeq \sigma_2$. For the two random variables defined in (B.8) we have

$$\mathsf{P}(\sigma_i > t) = \pi_i e^{-\bar{\lambda}_q t} + (1 - \pi_i) e^{-\underline{\lambda}_q t}$$

therefore,

$$\mathsf{P}(\sigma_1 > t) - \mathsf{P}(\sigma_2 > t) = (\pi_2 - \pi_1)\left(e^{-\underline{\lambda}_q t} - e^{-\bar{\lambda}_q t}\right) \geq 0,$$

because $\bar{\lambda}_q \geq \underline{\lambda}_q$ for every $q \in [a, b]$. Therefore, $\sigma_1 \succeq \sigma_2$, in the sense of FOSD.$\|$

Properties of the transformed function $\boldsymbol{w}$. Here I prove the properties claimed about the function $\boldsymbol{w}$, namely the fact that it is increasing in π (initial belief) and decreasing in σ (success arrival time).

Decreasing in σ. Remember that $\boldsymbol{w}(\pi, \sigma; \tau) := e^{-r(\sigma \wedge \tau)} w(\pi_{\sigma \wedge \tau})$. Fix the initial belief π (as well as τ and q), then $\boldsymbol{w}$ is clearly continuous in σ and is constant on $[\tau, \infty)$. Further, it is decreasing on $[0, \tau]$, because $\bar{\lambda}_q \geq \underline{\lambda}_q$ so the posterior belief about $\{\theta = H\}$ falls more as the elapsed time to success gets longer. Formally, because of Bayesian learning $\mathrm{d}\pi_\sigma / \mathrm{d}\sigma \leq 0$. To verify this, let us decompose π_σ as

$$\pi_\sigma = \pi_{\sigma^-} + \Delta\pi_\sigma = \pi_{\sigma^-} + \frac{\bar{\lambda}_q - \underline{\lambda}_q}{\pi_{\sigma^-}\left(\bar{\lambda}_q - \underline{\lambda}_q\right) + \underline{\lambda}_q} \pi_{\sigma^-}(1 - \pi_{\sigma^-}),$$

where the first term π_{σ^-} is the posterior belief just before the success arrival and the second term $\Delta\pi_\sigma$ is the amount that the posterior jumps up at the time of success. Define $\Delta_q := \bar{\lambda}_q - \underline{\lambda}_q \geq 0$, then again because of the Bayes law:

$$\pi_{\sigma^-} = \frac{\pi e^{-\Delta_q \sigma}}{1 - \pi + \pi e^{-\Delta_q \sigma}} \Rightarrow \frac{\mathrm{d}\pi_{\sigma^-}}{\mathrm{d}\sigma} = -\Delta_q \pi_{\sigma^-}(1 - \pi_{\sigma^-}) < 0.$$

Differentiating $\Delta\pi_\sigma$ w.r.t π_{σ^-} yields:

$$\frac{\mathrm{d}\Delta\pi_\sigma}{\mathrm{d}\pi_{\sigma^-}} = \frac{\Delta_q \Big((1 - 2\pi_{\sigma^-})(\pi_{\sigma^-}\Delta_q + \underline{\lambda}_q) - \pi_{\sigma^-}(1 - \pi_{\sigma^-})\Delta_q \Big)}{(\pi_{\sigma^-}\Delta_q + \underline{\lambda}_q)^2}.$$

I can now use the previous two relations to take the total derivative of π_σ w.r.t σ:

$$\begin{aligned} \frac{\mathrm{d}\pi_\sigma}{\mathrm{d}\sigma} &= \left(1 + \frac{\partial \Delta\pi_\sigma}{\partial \pi_{\sigma^-}}\right) \frac{\mathrm{d}\pi_{\sigma^-}}{\mathrm{d}\sigma} \\ &= \frac{\underline{\lambda}_q(\underline{\lambda}_q + \Delta_q)}{(\pi_{\sigma^-}\Delta_q + \underline{\lambda}_q)^2} \frac{\mathrm{d}\pi_{\sigma^-}}{\mathrm{d}\sigma} \leq 0. \end{aligned} \tag{B.13}$$

Lastly, for $\sigma \in [0, \tau]$,

$$\frac{\mathrm{d}\boldsymbol{w}}{\mathrm{d}\sigma} = -re^{-r\sigma} w(\pi_\sigma) + e^{-r\sigma} w'(\pi_\sigma) \frac{\mathrm{d}\pi_\sigma}{\mathrm{d}\sigma} \leq 0,$$

because of (B.13) and the fact that w is assumed increasing on $[0, 1]$ and hence is a.e differentiable with positive derivative. Therefore, $\boldsymbol{w}$ becomes decreasing in σ.‖

Increasing in π. To show that $\boldsymbol{w}$ is increasing in π, I must hold σ fixed, thus it remains to show $w(\pi_{\sigma \wedge \tau})$ is increasing in the initial belief π. It is pretty straightforward to show that the posterior belief at any time, for the Poissonian environment that we have, is increasing in the initial belief, hence the proof readily follows from the increasing property of w.‖