Reputation, Learning and Project Choice in Frictional Economies

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Abstract

I introduce a dynamic model of learning and random meetings between a long-lived agent with unknown ability and heterogenous projects with observable types. There is incomplete yet symmetric information about the agent's ability. She needs to accept the contacting projects and create success to learn her type. Alternatively, lack of success during a match leads to a reputational loss followed from Bayesian learning, in that the reputation is interpreted as the posterior belief about the agent's ability. Developing a self-type learning framework with endogenous outside option, I find the optimal matching strategy of the agent, that determines what types of projects the agent with a certain level of reputation will accept. Comparing with a perfect information benchmark, I show learning incentives lead to larger matching sets in the optimum.

JEL classification: D81; D83; O31

Keywords: Reputation; Learning; Search and Matching

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1 Introduction

Much of the theoretical literature on experimentation and project choice is about learning the other party's (namely the project's) type. In this paper, however, I shift the attention to learning the self-type. Specifically, the agent in my paper does not know her type, and the only way to learn it is by matching with projects and observing the output of their partnerships.

There are natural instances where agents *learn* their type through the course of their matches with other parties. For example, firms learn about their productivity while they are matched with workers. Colleges learn about the quality of their teaching staff while students are enrolled in their programs. Venture capitalists learn about their ability and the quality of their post-investment services while investing in the startups.[1] Common in all these cases is the cost of maintaining the partnership and the *tangible* created surplus (such as the high-quality output of production in the first case, students' accomplishment in the second case, and the startups' success in the last case). These tangible gains from partnerships can be isomorphically captured by the choice of the *production function* in the matching markets (e.g., Shimer and Smith (2000)).

However, when the agent holds incomplete information about her type, there is also an *intangible* gain due to the learning, that cannot be nested in the former construct. Because, what is now used as an input to the production function is no longer the static type of the agent, but a dynamic state process that reflects the agent's belief of her own type. Specifically, in addition to the tangible gains, there are now information gains from agent's matching choices, as present matches convey information about the agent's ability, that in turn can be used in future choices of projects. The basic research question that I ask and answer in this paper is: presented with heterogenous projects, that differ in their expected payoff (tangible margin) and speed of learning (intangible margin), what is the agent's optimal project selection policy as a function of her reputation?

In this economy, the agent is ex ante endowed with a high or low *immutable* type $\theta \in \{L, H\}$, that is hidden to everyone. On the other hand, there are heterogenous projects with observable types denoted by q. The agent randomly meets the projects subject to the

[1] For example, Sørensen (2007) teases out the positive treatment effect of the VCs' involvement in their portfolio companies from the concerns regarding the sorting and selection. In addition, when it comes to VCs' underlying ability Gompers and Lerner (1999) argues that the empirical evidence favors the learning model (symmetric but incomplete information) compared to the signalling model (asymmetric information).

search frictions and decide to accept them or not. Once a (one-to-one) match is formed, there will be a random success event whose arrival intensity depends on the types of both parties. Agent continuously updates her belief about the underlying type during the course of her matches. Therefore, I interpret the posterior belief as her reputation and denote it by π.

Whenever the agent pairs up with a project, a learning opportunity is created about her type. Since maintaining the match is costly, the agent effectively solves a stopping time problem, by which she weighs the *matching value function* $v(\pi, q)$ (that is a function of her current reputation π and the type of the project q) against the reservation value $w(\pi)$ – the value of holding current reputation while being unmatched, that is called the *reputation value function* throughout the paper. Because of the random meetings framework, these two functions are intertwined in the equilibrium. That is the reputation function is simply the expected discounted value of future surpluses that the agent extracts, and the matching value function is the solution to the free-boundary problem with the exit option w. The continuation region of this free-boundary problem determines the optimal matching set $\mathcal{M}$.

The central innovation of this paper is to study the optimality and shape of these matching sets when the agent has long-run incentives and learn her ability as she selects and matches with the projects. Specifically, I find and study the properties of the optimal tuple $\langle w, v, \mathcal{M}\rangle$. The main point of the departure from the experimentation literature (e.g., Keller et al. (2005) and Bonatti and Hörner (2017)) is the endogeneity of the outside option w, that determines the type of acceptable projects in the agent's optimal policy. In addition, the subject of learning in the experimentation literature is the project's type, whereas in my paper the learning is about the self-type and projects provide the context for learning and a source of creating surplus.

1.1 Summary of Results

In section 2, I introduce the dynamic learning and project selection model. Three main objects in the study of agent's optimal policy are the matching value function $v(\pi, q)$, the reputation or reservation value function $w(\pi)$, and the matching set $\mathcal{M}$. We will see how the agent's optimal policy can be translated to a fixed-point solution of a system that connects the above three elements.

To highlight the impact of learning on the optimal matching sets, I first explore the *no learning* benchmark in section 3. In particular in this section, I let the agent's true type to be equal to her reputation (i.e., a number $\pi \in [0, 1]$), as opposed to a background hidden binary variable $\theta \in \{L, H\}$. This will shut down the learning channel, that is the Bayesian learning force is absent in the associated Bellman equation. Then, I investigate the optimal matching sets, indicating the levels of reputation with which the agent is willing to match with a certain type of project. I show the sections of the matching set are connected subsets of the real line. Additionally, we will see that the optimal value functions are locally concave with kinks on the boundary of the matching sets.

Next, in section 4, I find the fixed-point tuple $\langle w, v, \mathcal{M} \rangle$ in the complete model that features learning. Similar to the no learning benchmark, we will see that the matching sets are connected. Hence, higher levels of reputation are associated with higher patience, larger distance to the endogenous separation point, and thus later termination of the match by the agent. However, in contrast with the no learning setting, the value functions now become convex in reputation. This is owed to the Bayesian learning force in the Bellman equation. As a result of this convexity, the matching sets become larger than the no learning benchmark. The intuition behind this outcome is that because of the convexity of the value function, losing reputation leads to a marginal loss that is smaller than the marginal gain of reaching a higher reputation. Hence, the Martingale nature of the belief process implies that on expectation continuing the match will be optimal on some regions, that absent of learning the agent would have turned down the match.

Also, in section 4, I show within the space of increasing and differentiable value functions in reputation, there is a unique fixed-point tuple. The main techniques behind the proof include (i) representing the reputation function w as a solution to a maximization problem featuring the matching value functions v in the objective and the matching set $\mathcal{M}$ as the choice variables; and (ii) the application of two necessary conditions (namely the *majorant* and the *superharmonic* properties) in the free-boundary problem to pin down the shape of the matching set $\mathcal{M}$.

Performing a comparative static exercise in section 4, we see that lower levels of flow cost, time discount rate, and search frictions (equivalently higher contact rates) are all associated with larger optimal matching sets.

All the analyses in this paper enjoy closed-form expressions – because I restrict the space of project types to a bi-valued set, i.e., $q \in \{a, b\}$. This assumption is relaxed in the appendix B, in which I show that if the success intensity is monotone and supermodular with respect to the agent's and project's types, then the results of the baseline model (with binary type space for projects) are robust under the general type setting for q.

1.2 Related Literature

The Bayesian learning force in the agent's decision problem in this paper is based on the Poisson arrival of breakthroughs, and in that sense the paper is related to the strategic experimentation literature with Poisson news processes, initiated by Keller et al. (2005), and expanded in the follow-up works of Keller and Rady (2010) and Keller and Rady (2015). The main strategic tension in these works are the free-riding and the so called "encouragement effect" among players, as there are multiple players and the background type is *common* across the agents. However, the subject of learning in the present paper is the type of the single agent, and thus both of the above strategic forces are absent. Also, the experimentation choices available to the agent, not only differ in their expected payoff, but also confer different learning speeds (when it comes to learning the agent's ability).

In the context of reputation building (when the information about the persistent or dynamic self-type is incomplete) and interpreting the reputation as the posterior belief, this paper is related to Holmström (1999) and Board and Meyer-ter-Vehn (2013). However the kind of economic engagement that releases informative signals in both of these papers is the agent's effort, and in the current study is about the agent's project selection.

The analysis of this paper has also the flavor of the literature on learning in labor markets such as the works by Jovanovic (1979), Moscarini (2005) and Li and Weng (2017). Aside from differences in the context and motivation, the subject of learning in these studies is the *match specific* parameter, and not the underlying types of the agents. Therefore, the information released over the present match has no bearing on the future matches and naturally the reputational aspects are absent. In regard to agents learning their type while matching with partners this paper is related to Anderson and Smith (2010) and Eeckhout and Weng (2021). There are substantive differences between the information structure in these works and the current paper. For example, there is *two-sided* incompleteness of information in the former paper that complicates the Bayesian updating process, and hence led to imposing assumptions about the observability of the output of current matches with previous partners. The latter paper studies the competitive equilibria absent the search friction, which is a central element in my paper.

There is also previous research on how agents hold *perfect* private information about themselves, and receive some form of information about the type of their partner before the match (e.g., Chade (2006), Chakraborty et al. (2010) and Liu et al. (2014)). My setting is different than these works, mainly in the sense that the agent in this paper has incomplete information about herself, and one of the motives in her matching decisions (besides receiving the tangible surplus from the projects she accepts) is learning her type.

2 Model

2.1 Agent, Projects and Dynamic Timeline

In this subsection, I describe the elements of an economy populated by a single long-lived agent and a continuum of projects.

Agent. The agent is a long-lived individual with the rate of time preference $r > 0$. She holds incomplete information about her type $\theta \in \{L, H\}$. The σ-field $\mathcal{I}_t$ aggregates all the information that is available in the economy at time $t \in \mathbb{R}_+$. The agent cares about her reputation, which is the posterior belief about her type. Given the filtration $\boldsymbol{I} = \{\mathcal{I}_t\}$, $\pi_t = \mathsf{P}\left(\theta = H | \mathcal{I}_t\right)$ refers to her time t reputation.

Projects. The entities on the other side of this economy are treated as projects that are matched with the agent. Specifically, they have no bargaining power against her.[2] Each project is endowed with a type $q \in \{a, b\}$, which is publicly observable. The (unnormalized) mass of type-q projects is φ_q for $q \in \{a, b\}$, exogenously replenished and held constant.

Meetings and project selection. The agent randomly meets the projects subject to the search frictions, with the meeting rate of $\kappa > 0$, and the matching technology is *quadratic* à la Chade et al. (2017). That is the probability with which the agent meets a type-q project over the period $\mathrm{d}t$ is approximately equal to $\kappa\varphi_q\mathrm{d}t$. Furthermore, the matches are one-to-one, that is both parties have capacity constraint over the number of partners they can accept.

Output and reputation. Given the partnership between a type-θ agent and a type-q project, the success arrives at the rate of $\lambda_q(\theta)$, where $\lambda_q(H) = \bar{\lambda}_q$ and $\lambda_q(L) = \underline{\lambda}_q$, with the normalized payoff of one.[3] The agent has to cover the flow cost of project $c > 0$ that is common across all matches. In return, she receives the right to terminate the project at

[2]This assumption makes the analysis substantially simpler, yet it downplays the strategic role of "project owners" in the optimal outcome. However, given the paper's focus on the agent's side and her reputational concerns, such an abstraction seems plausible. Also from the empirical standpoint, for example in the venture capital literature, it is shown that firms can continue their projects without their original entrepreneurs as the project owner; see Ueda (2004) and the references therein such as Gorman and Sahlman (1989) and Hellmann and Puri (2002).

[3]The choice of Poisson processes to model the breakthroughs is more natural when news arrive in discrete

her will, so conceptually a stopping time problem is solved by the agent ex post to every partnership formation. The flow cost c captures both the running cost of maintaining the match and learning about the self-type θ. I assume there is a mechanism in the economy that tracks the output of each match and records the Bayes-updated posterior of the agent. This information is reflected in the filtration $\boldsymbol{I}$. The posterior dynamics for the reputation process (resulted from the Bayes law) follows

$$\mathrm{d}\pi_t = -\pi_t(1 - \pi_t)\Delta_q \mathrm{d}t, \tag{2.1}$$

prior to the success, where $\Delta_q := \bar{\lambda}_q - \underline{\lambda}_q$. For the purpose of simplicity, I assume the breakthroughs are *conclusive* in the sense that $\underline{\lambda}_q = 0$, that is the success never happens to a low-type agent. Therefore, upon the success, π_t immediately jumps up to one.[4] Furthermore, without loss of generality it is assumed $\lambda_b := \bar{\lambda}_b > \lambda_a := \bar{\lambda}_a$. Also, to make the setting non-trivial, one has to assume $c < \lambda_a < \lambda_b$ (absent of this no matching activity takes place).

Figure 1 summarizes the dynamic timeline for a typical agent, who starts the cycle with reputation π, and after some exponentially distributed time meets a project randomly drawn from the population of available ones. A decision to accept or reject the contacting project is made by the agent. Upon rejection, the agent returns to the initial node, and conditioned on acceptance an investment problem with the flow cost of c is solved. Finally, I interpret the success as an event in which the breakthrough happens before the agent stops the project, thereby rationally updating her belief upwards. And the failure refers to the case where the project is terminated before the success arrival, thus the agent returns to the unmatched status with a lower reputation.

2.2 Value Functions and Matching Sets

Let $w(\pi)$ be the value of holding reputation π, when the agent is *unmatched*. This function shall be treated as the agent's outside option and is weighed against the matching value function upon the meetings. The matching value function when the agent with reputation π matches with a type-q project is $v(\pi, q)$, that is the expected value of discounted future

and randomly separated instants, than the Wiener process treatment of experimentation (e.g., see Bolton and Harris (1999) and Pourbabaee (2020)).

[4] In the appendix B, I relax this assumption and study the general case, where the success is not necessarily conclusive and there is a continuum of projects with the type space $[a, b]$, distributed according to an *arbitrary* CDF function ϕ.

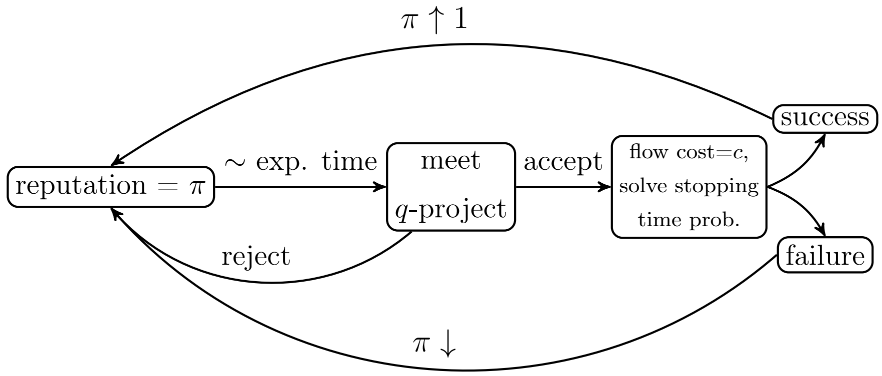


Figure 1: Decision timeline for a typical agent

payoffs generated by this match. Therefore, a match is optimal if $v(\pi, q) > w(\pi)$, in that case I say $(q, \pi) \in \mathcal{M} \subseteq \{a, b\} \times [0, 1]$, where $\mathcal{M}$ is called the matching set. Also, understood from the context, $\mathcal{M}(\pi)$ (resp. $\mathcal{M}_q$) refers to the π (resp. q) *section* of this two dimensional set. In addition, often in the paper I use the indicator function $\chi_q(\pi)$ to denote whether the agent with reputation π accepts a type-q project, that is whether $(q, \pi) \in \mathcal{M}$ or not. Recall that φ denotes the mass of available projects in the economy (that are treated exogenously as the primitives of the model). Below, I invoke a standard dynamic programming analysis for $w(\pi)$:

$$\begin{aligned} w(\pi) \approx \kappa \sum_{q \in \mathcal{M}(\pi)} \big(w(\pi) + [v(\pi, q) - w(\pi)]\big)\varphi_q \mathrm{d}t + \kappa \sum_{q \in \{a,b\} \setminus \mathcal{M}(\pi)} w(\pi)\varphi_q \mathrm{d}t \\ + (1 - \kappa\varphi(\{a, b\})\mathrm{d}t)\,(1 - r\mathrm{d}t)w(\pi). \end{aligned}$$

The first term in the *rhs* is the expected value of payoffs generated from all *acceptable* matches, taking into account that the next project with type q arrives at the rate of $\kappa\varphi_q$. The second term is the expected payoff over all *denied* projects, and the third term simply refers to the discounted payoff conditioned on receiving no proposal over the period $\mathrm{d}t$. Accounting for these three sources, the following Bellman equation for the reputation value function w is resulted:

$$rw(\pi) = \kappa \sum_{q \in \mathcal{M}(\pi)} \big(v(\pi, q) - w(\pi)\big)\varphi_q. \tag{2.2}$$

Next, I inspect the matching value function $v(\pi, q)$. Imagine a match between the agent with an initial reputation π and a type-q project. Let σ represent the random exponential time of success with the unit payoff and the arrival intensity of λ_q if $\theta = H$. Therefore, the matching value function $v(\cdot, q)$ is an endogenous outcome of a free-boundary problem with the outside option w. In that, the agent selects an optimal stopping time τ, upon which she stops backing the project, taking into account the project's success payoff and her reputation value w:

$$\begin{aligned} v(\pi, q) = \sup_{\tau} \Bigg\{ \mathsf{E}\left[e^{-r\sigma} - c \int_0^{\sigma} e^{-rs} \mathrm{d}s + e^{-r\sigma} w(\pi_\sigma); \sigma \leq \tau \right] \\ + \mathsf{E}\left[-c \int_0^{\tau} e^{-rs} \mathrm{d}s + e^{-r\tau} w(\pi_\tau); \sigma > \tau \right] \Bigg\}. \end{aligned} \tag{2.3}$$

Formally in the above stopping time problem, if the success happens before the agent stops backing the project (namely when $\sigma \leq \tau$), the agent collects the discounted unit payoff, has paid the flow cost until time σ, and successfully leaves the project with the updated reputation function $w(\pi_\sigma)$, where π_σ is the updated posterior belief reflecting the successful exit. In the current setting of the conclusive breakthroughs, $\pi_\sigma = 1$. On the other hand, if the agent stops the project before the success realization (namely when $\tau < \sigma$), then she has just paid the flow cost up until time τ, and leaves with the updated reputation function $w(\pi_\tau)$, reflecting the fact that the success has not happened until time τ. Therefore, the exit option at the stopping time τ is the agent's reservation value of holding reputation π_τ.

The corresponding HJB representation for this stopping time problem is

$$rv(\pi, q) = \max \Big\{ rw(\pi), -c + \lambda_q \pi \big(1 + w(1) - v(\pi, q)\big) - \lambda_q \pi (1 - \pi) v'(\pi, q) \Big\}. \tag{2.4}$$

The above HJB is presented in the variational form, that is the first expression in the *rhs* is the value of stopping – denying the project and holding on to the outside option w – and the second expression represents the Bellman equation over the *continuation region* $\mathcal{M}_q$, on which $v(\pi, q) > w(\pi)$. The first term in the Bellman equation is the flow cost of the project borne by the agent, the second term is the expected flow of created surplus, and the last term captures the marginal reputation loss due to the lack of success.[5] Induced by the above stopping time problem, the matching set $\mathcal{M}$ can thus be interpreted as the continuation set for the free-boundary problem (2.4), namely

$$\mathcal{M} = \{(q, \pi) \in \{a, b\} \times [0, 1] : v(\pi, q) > w(\pi)\}, \tag{2.5}$$

and on the stopping region $\mathcal{M}^c$, the matching value function equals the agent's reputation function, i.e., $v(\pi, q) = w(\pi)$.

[5]Henceforth, all the derivatives are with respect to π, that is for example $v'(\pi, q)$ points to the π-partial of v, unless the underlying variable is explicitly mentioned.

The overarching goal of this paper is to study the optimal outcome, which is the solution to the following fixed-point problem: the tuple $\langle w, v, \mathcal{M} \rangle$ constitutes an optimal outcome, if (i) given v and $\mathcal{M}$, the reputation value function w satisfies (2.2) and (ii) given w, the matching value function v and the matching set $\mathcal{M}$ together solve the free-boundary system (2.4) and (2.5).

The two-way feedback between the reputation function w and the matching variables $\langle v, \mathcal{M} \rangle$ are plotted in figure 2. The link connecting w to the $\langle v, \mathcal{M} \rangle$ block is upheld by the stopping time problem (2.3), and its recursive representation (2.4). The opposite link from the matching variables block to w is supported by the Bellman equation for the reputation function in (2.2). The optimal outcome is formally the fixed-point to the endogenous loops of this figure.

$$
\begin{tikzcd}
\boxed{\langle w \rangle} \arrow[rr, bend left] & & \boxed{\langle v, \mathcal{M} \rangle} \arrow[ll, bend left]
\end{tikzcd}
$$

Figure 2: Equilibrium feedbacks

Toward our ultimate goal, which is finding the optimal tuple $\langle w, v, \mathcal{M} \rangle$ satisfying the system (2.2), (2.4) and (2.5), in the next section I study the simpler benchmark case of *no learning*. Specifically, I assume the agent's type is equal to her reputation π, and this variable does **not** evolve based on a learning procedure. That is types are persistent and the learning channel is shut down in the agent's matching decisions.

3 No Learning Benchmark

In contrast with our original model, where the agent's underlying type is the binary variable $\theta \in \{L, H\}$, and π reflects the posterior belief, here I assume the actual underlying type is $\pi \in [0, 1]$ and it stays *constant* over time. Specifically, when a type-π agent selects a type-q project, the success arrives with the rate of $\lambda_q \pi$.

The underlying reason behind studying this benchmark case is to understand how the learning process impacts the optimal matching sets. That is first, we explore the endogenous matching set $\mathcal{M}$, when the agent only seeks the tangible gains from success, and then we see how the results change when the intangible gains due to learning enter.

In that direction, the Bellman equation for the reputation function $w(\pi)$ remains the same as (2.2). The major changes happen in the Bellman equation for the matching value function. First, the Bayesian learning component that includes the π-derivative of $v(\pi, q)$ is no longer present. Second, the exit option at the time of success is $1 + w(\pi)$ instead of $1 + w(1)$. This is owed to the fact that the agent's type is persistent and she leaves the match with the same reputation that she entered. Formally, the no learning Bellman equation for the matching value function is:

$$rv(\pi, q) = \max\left\{rw(\pi), -c + \lambda_q \pi\big(1 + w(\pi) - v(\pi, q)\big)\right\}\ . \tag{3.1}$$

The endogenous matching set also keeps the form of equation (2.5), that is $(q, \pi) \in \mathcal{M}$ iff $v(\pi, q) > w(\pi)$.

Throughout the paper and particularly in the next proposition we seek the fixed-points in which the endogenous reputation function $w(\cdot)$ is increasing π. Next proposition identifies the *unique* tuple $\langle w, v, \mathcal{M}\rangle$, in the space of increasing continuous functions, solving the system (2.2), (3.1) and (2.5). Before that, I define a terminology.

Definition 3.1. We say that the economy is in the *low cost regime* if $\lambda_a - c > \frac{\kappa\varphi_b(\lambda_b - c)}{r + \kappa\varphi_b + \lambda_b}$, and otherwise is in the *high cost regime.*

Proposition 3.2 (Optimum with no learning)**.** *In the space of increasing continuous functions on the unit interval* $[0, 1]$*, there exists a unique fixed-point* $\langle w, v, \mathcal{M}\rangle$ *satisfying the system of conditions* (2.2), (3.1) *and* (2.5). *The fixed-point tuple further satisfies the following properties:*

(i) $\mathcal{M}_a \subseteq \mathcal{M}_b$*, and* $\mathcal{M}_b$ *is a connected subset of* $[0, 1]$*.*

(ii) In both cost regimes, $1 \in \mathcal{M}_b$*. In addition,* $1 \in \mathcal{M}_a$ *only in the low cost regime.*

(iii) In the high cost regime $\mathcal{M}_a = \emptyset$*, and in the low cost regime* $\mathcal{M}_a$ *is a connected subset.*

The graphs in figure 3 summarize the findings in this proposition. They indicate what kinds of projects an agent with a certain type (say π) is willing to accept in each cost regime. Because of monotonicity of success intensity in π, the matching sets are connected and increasing subsets of $[0, 1]$. Also, as a result of $\lambda_a < \lambda_b$, a match with a type-a project is optimal if it is also optimal for the agent to accept a b-type project, thus $\mathcal{M}_a \subseteq \mathcal{M}_b$.

Furthermore, in the high cost regime, the opportunity cost of matching with the a-types is larger than the expected gains of the match, and hence $\mathcal{M}_a$ becomes empty. Specifically, denying the inferior projects, and waiting for an opportunity to meet a superior one leads to a higher expected payoff in the high cost regime. It is indeed the comparison between the expected flow payoff of matching with the a-projects and the opportunity cost of forgoing the wait for the next b-project that underlies the cost regime (and the status of $\pi = 1$) determination:

$$\text{low cost regime} \Leftrightarrow \lambda_a - c > \underbrace{\frac{\kappa\varphi_b(\lambda_b - c)}{r + \kappa\varphi_b + \lambda_b}}_{\substack{\text{opportunity cost of forgoing} \\ \text{the wait for a b-project}}} .$$

For instance, as the share of available b-projects (φ_b) increases, the opportunity cost of matching with a-projects goes up, and consequently, agents become more reluctant to select the a-types.

Additionally, lower levels of the meeting rate κ (i.e., higher search frictions) decrease this opportunity cost, and shift the incentives toward matching with the a-projects.[6]

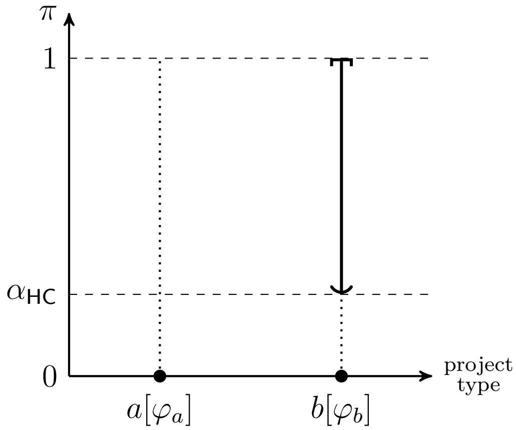
One can also verify that lowering the cost c increases the expected flow payoff of matching with a-projects more than it does the opportunity cost component, thereby enhancing the *variety* of accepted projects. Therefore, to the extent that these results speak to the venture capitalists' investment decisions, the optimal response observed in the matching sets confirms the prevalence of the investment approach "*spray and pray*" that arises due to the cost-reducing technological shocks, mentioned in Ewens et al. (2018).

In both cost regimes, we can take the lower end of $\mathcal{M}_b$ (denoted by α_{LC} in the low cost regime and α_{HC} in the high cost regime) as a proxy for the size of the optimal matching sets. At these points the matching value function becomes equal to $w(\alpha)$ which is zero, thereby identifying the endpoints. The Bellman equation for $v(\cdot, q)$ in (3.1) implies that

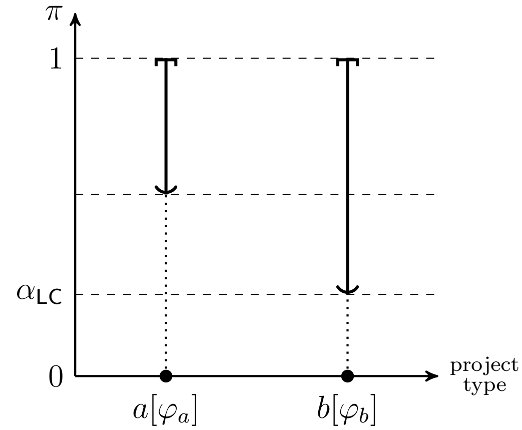
$$\alpha_{\mathsf{LC}} = \alpha_{\mathsf{HC}} = \frac{c}{\lambda_b} . \tag{3.2}$$

Figure 4 draws the value functions in the low cost regime, where the learning channel is shut and the agent's actual type is equal to her reputation π. In that α_a (resp. α_b) is the lower end of $\mathcal{M}_a$ (resp. $\mathcal{M}_b$). The two important features in these graphs are the local concavity and the kinks on the boundary points of the matching sets, both of which will disappear in the learning model, as we see in the next section.

[6] The labor market analogue of this outcome is that in the markets with high search frictions, firms turn down the low-skilled workers less often. Because on expectation, it takes longer for them to find and hire the high-skilled workers.



(a) high cost regime



(b) low cost regime

Figure 3: Optimal matching sets

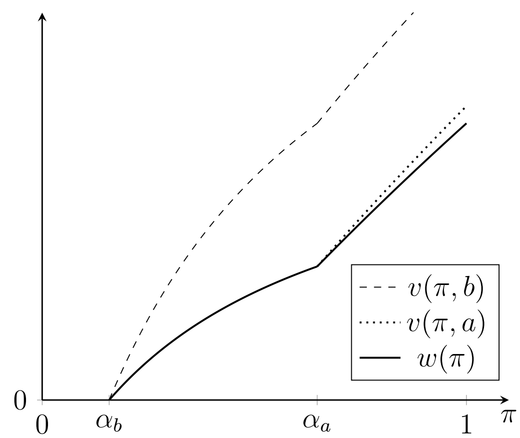


Figure 4: Value functions in the low cost regime (no learning benchmark)

Taking these result as a stepping stone for the complete model, in the next section I investigate the solution to the fixed-point problem of section 2, where the agent's underlying type is a binary variable $\theta \in \{L, H\}$, and π reflects the dynamic posterior belief that evolves during the course of the match. We will see that the overall shape of the optimal matching sets remain the same as in figure 3, with important differences that highlight the impact of learning and incomplete information on the agent's matching choices.

4 Optimality *with* Learning

In this section, I characterize the fixed-point tuple $\langle w, v, \mathcal{M} \rangle$ satisfying the set of conditions (2.2), (2.4) and (2.5). In contrast with the previous section, the agent's type θ belongs to the binary set $\{L, H\}$, and π is the dynamic state variable, representing the posterior belief of the event $\{\theta = H\}$, which refers to the agent's reputation.

The main stretch in the analysis comes from the incomplete information about the underlying type $\theta \in \{L, H\}$, that creates learning opportunities during the match between the agent and a project. It is precisely this combination of learning the self-type (i.e., the intangible gain) and creating tangible output that underlie the agent's optimal decision. On one hand, selecting the b-projects leads to a higher expected payoff (as $\lambda_b > \lambda_a$), thus favoring the tangible side of the agent's decision. On the other hand, selecting an a-project decreases the reputation at a slower rate during an unsuccessful match (that is apparent from the Bayes law in equation (2.1)), hence reinforcing the intangible side.

Comparing the Bellman equation in (2.4) with its no learning counterpart in equation (3.1), one also notices two differences. First, lack of success during an unsuccessful match pushes down the reputation π (as a result of Bayesian learning) and this inflicts a "*negative pressure*" on the matching value function $v(\cdot, q)$, via the π-derivative term. Second, once the agent succeeds in a match, she leaves with a higher reservation value $w(1)$ than her starting level $w(\pi)$. This effect introduces a "*positive pressure*" on the matching value function. Therefore, it is not clear from the outset which force would dominate in shaping the optimal matching sets. Our finding is that the positive one, originating from establishing a high reputation and harvesting its benefits in the future projects, ultimately dominates and that in turn expands the matching sets relative to the no learning optimum.

The intuition behind this is that Bayesian learning renders the matching value function convex, and therefore because of the Martingale nature of the belief process $\{\pi_t\}$, losing reputation leads to a marginal loss that is smaller than the marginal gain of reaching a higher reputation. Therefore, the possibility of learning about the self-type makes the agent to accept projects at reputation levels that she would have otherwise denied .

To further examine the essence of the stopping time problem (2.4), I highlight two *necessary* conditions that the optimal matching value function and the continuation region must satisfy.[7] The dynamics of the reputation process can be compactly represented by

[7] These two conditions are standard in the literature of optimal stopping and can be found in chapter 2 of Peskir and Shiryaev (2006).

$\mathrm{d}\pi_t = (1-\pi_{t^-})\,[\mathrm{d}\iota_t - \lambda_q \pi_{t^-}\mathrm{d}t]$, where ι is the success indicator process, that is $\iota_t := 1_{\{t\geq\sigma\}}$. The infinitesimal generator associated with this stochastic process is $\mathcal{L}_q : C^1[0,1] \to C^1[0,1]$, where for a generic $u \in C^1[0,1]$:[8]

$$[\mathcal{L}_q u]\,(\pi) = \lambda_q \pi\big(1 + w(1) - u(\pi)\big) - \lambda_q \pi(1-\pi)u'(\pi).$$

For every candidate fixed-point tuple $\langle w, v, \mathcal{M}\rangle$, the following two conditions must hold for all $\pi \in [0,1]$ and $q \in \{a,b\}$:

(i) *Majorant property*: $v(\pi,q) \geq w(\pi)$.

(ii) *Superharmonic property*: $[\mathcal{L}_q v](\pi,q) - rv(\pi,q) - c \leq 0$.

The first condition simply means that in every match the agent has the option to terminate the project, thus enjoying her reputation value w. The second condition means *on expectation* a typical agent *loses* if she decides to keep the match on the stopping region. Exploiting these two conditions, the following proposition establishes a descriptive set of properties of the optimal $\mathcal{M}$, in every fixed-point outcome where the value functions are increasing and belonging to $C^1[0,1]$. This proposition shows that all the properties of the matching sets in the no learning benchmark carries over to this setting, and the content of that is similar to proposition 3.2, even though the proof method is completely different. It is largely based on the application of the above two necessary conditions in the two relevant cost regimes.

Proposition 4.1 (Optimum with learning)**.** *In every fixed-point outcome with increasing value functions in $C^1[0,1]$, the following properties hold:*

(i) In both cost regimes, $1 \in \mathcal{M}_b$. In addition, $1 \in \mathcal{M}_a$ only in the low cost regime.

(ii) In both regimes the matching set $\mathcal{M}_b$ is a connected subset of $[0,1]$.

(iii) In the high cost regime $\mathcal{M}_a = \emptyset$, and in the low cost regime $\mathcal{M}_a$ is a connected subset of $\mathcal{M}_b$.

The shape of the optimal matching set in the current setting is also depicted in figure 3. Its only difference with the no learning benchmark is that as we will see α_{LC} is no longer equal to α_{HC}. The optimal matching sets are still connected (in spite of learning distortions). The observation in figure 3 on *connectedness* of the matching sets advances the idea that agents

[8]Space of continuously differentiable functions on $(0,1)$ with continuous extension to the boundary $\{0,1\}$.

with higher reputation have higher tolerance for failure. In other words, the distance to the endogenous separation point α is larger for a more reputable agent than a less reputable one.[9]

Furthermore, when it comes to cross-project comparison, the agent shows more tolerance toward b-projects – that confer faster success time on average. Also, in light of $\mathcal{M}_a \subseteq \mathcal{M}_B$ the model offers the testable prediction that the agents who exit the market and do not engage in further activities made their last few engagements in the high-growth projects (i.e., b-types). Formally, in both panels of figure 3, we see that the endogenous termination point α is the lower boundary point of $\mathcal{M}_b$ (not $\mathcal{M}_a$), at which the matching value function $v(\cdot, b)$ *smoothly* meets the zero function (as shown in the proof of the previous proposition).[10]

It is established in the proof that the endogenous termination point follows:

$$\alpha = \frac{c}{\lambda_b\big((1 + w(1)\big)}\,, \tag{4.1}$$

where $w(1)$ is the value of holding the maximum reputation, namely at $\pi = 1$, in each cost regime. In the high cost regime $w(1)$ only depends on the b-parameters, because $\mathcal{M}_a = \emptyset$, whereas in the low cost regime it takes the a-related parameters into account as well. Specifically, since α depends on $w(1)$, and the latter depends on the prevailing cost regime, the separation point is no longer equal across the two cost regimes.

Taking α as a proxy for the size of the matching set, we can see how the reputational incentives impact the optimal matching sets. In particular, in both cost regimes, the endogenous separation point α is *smaller* than its no learning counterpart, i.e., c/λ_b in equation (3.2). Therefore, the prospects of learning about the self-type and possibly reaching a higher reputation expand the matching sets and add more patience to the agent's continuation region. This is the distinguishing impact of learning on the matching sets. That it incorporates the chances of achieving a higher reputation in the decision on when to stop the *current* match.

In addition, one can perform a number of comparative statics on the size of the optimal matching sets. For example, it is easily verified that $\frac{\partial \alpha}{\partial c} > 0$, $\frac{\partial \alpha}{\partial r} > 0$, and $\frac{\partial \alpha}{\partial \kappa} < 0$. Namely,

[9]As it relates to the economics of venture capital, this would suggest a method to endogenize the *tolerance for failure* (see Tian and Wang (2014) and Manso (2011)) by relating it to the agent's reputation. Specifically, in Tian and Wang (2014) VCs learn about the quality of the startup over the course of the match, whereas reflecting in my model the startup's quality is observable and the learning is about the VC's ability. Consequently, the approach here suggests one way to endogenize the *tolerance parameter* in Tian and Wang (2014).

[10]It is shown in the proof of proposition 4.1, that the smooth pasting and value matching at α is ensued in spite of the Poissonian environment and the absence of diffusion processes.

lower levels of flow cost, time discount rate, and search frictions (equivalently higher contact rate) are all associated with larger optimal matching sets.

Related to the last comparative static exercise, one would expect that decreasing the meeting rates (or equivalently *increasing* the search frictions) makes the matching sets larger (thus lowering α), because the agent knows that it will take longer on average to meet an acceptable replacement, and hence spends longer time with the current project. However, this intuition is not quite correct, because a replacement (at best) would be of same quality as the current project (say a b-project), and thus the agent only suffers from the opportunity cost of time by stopping the current match and waiting for something of similar quality. In fact, the main reason that *decreasing* the search frictions makes the matching sets larger (and hence leads to a smaller α) is that the agent knows if she makes a breakthrough with the current project, she gains the maximum reputation of $w(1)$, which is an increasing function of the meeting rate κ. Therefore, she spends more time with the current match in the hope of getting a successful outcome and earning the maximum reputation. Finally, the reason that $w(1)$ is increasing in κ is that, having reached the maximum reputation, the agent now has no more learning incentives, and thus only cares about the tangible output, which comes at a faster rate, when she can meet the projects more frequently.

Having known the necessary conditions that the optimal matching sets own, I can now state the main theorem related to the existence and uniqueness of the optimal matching strategy of the agent, namely the fixed-point outcome of figure 2.

Theorem 4.2 (Existence and uniqueness of the optimum). *There exists a unique fixed-point $\langle w, v, \mathcal{M} \rangle$ in the space of continuously differentiable and increasing payoff functions in each cost regime. Furthermore, for large values of discount rate r, this fixed-point is unique in the larger space of $L^\infty[0,1]$.*

The substantial result of this theorem is that there always exists a fixed-point tuple in which the value functions are increasing and continuously differentiable in reputation. Furthermore, there is not a possibility for multiple fixed-points of such kind. However, the possibility of other fixed-points with non-increasing value functions cannot be ruled out unless the discount rate is large enough so that a contraction type theorem can be applied.

Figure 5 plots the value functions in the low cost regime. As explained in proposition 4.1, $\mathcal{M}_a \subseteq \mathcal{M}_b$, and thus the endogenous separation point in an a-match is higher than that of a b-match, namely $\alpha_a > \alpha_b$. In addition, in contrast with the no learning setting (and particularly the graphs in figure 4), the value functions exhibit no kinks, and are continuously

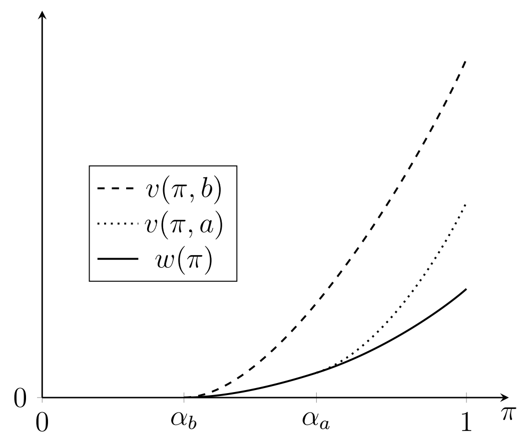


Figure 5: Value functions in the low cost regime

differentiable. This is owed to the satisfaction of the *smooth fit* principle at the boundary points of the matching sets, that in turn follows from the continuity of the rate of reputation decay (i.e., $\mathrm{d}\pi_t/\mathrm{d}t$) in the current reputation (i.e., π_t); see equation (2.1).

In addition, figure 5 confirms the convexity of the matching value functions, which is in contrast with the value functions in the no learning setting, that are locally concave according to figure 4. The convexity is the underlying reason that the matching sets in the incomplete information setting (with learning) become larger than the no learning benchmark. Essentially, since the posterior belief process $\{\pi_t\}$ is a Martingale, then a drop in π_t leads to a marginal loss that is smaller than the marginal gain followed by a rise in π_t. Hence, this creates more patience on the side of the agent and thus larger matching sets.

5 Concluding Remarks

I study the optimal project selection policy of an agent with unknown ability. The agent randomly meets the projects drawn from a heterogenous pool, that differ across the two margins of the expected payoff and the learning speed. Since maintaining the matches are costly, the agent effectively solves a stopping time problem, in which she weighs the expected benefit of learning about her type as well as accomplishing breakthroughs against the endogenous reservation function (that is called the reputation value function in the paper). The matching sets indicate what type of projects an agent with certain level of reputation is willing to accept or continue the match with. Alternatively, they describe what levels of reputation are profitable to match with a certain project. Within the space of increasing and differentiable value functions in reputation, I show there exists a unique optimum with connected matching sets, whose end-points are endogenously determined, and encode a number of messages. For example, lower levels of flow cost, search frictions and time discount rate are all associated with larger optimal matching sets.

Compared to the no learning benchmark (where there is no incomplete information about the agent's type), the optimal matching sets are larger, therefore the agent shows more patience before stopping the projects. This is owed to the convexity of the value functions in reputation, that itself is resulted from the Bayesian learning force in the corresponding Bellman equation. Essentially, the learning incentives encourage the agent to stay longer on the match.

Lastly, the framework developed in this paper to study the dynamic learning while making matching decisions can be extended to a number of important settings:

(i) One can apply our approach to a setting where the agent can increase her rate of meeting by exerting an effort. Naturally, an agent with higher reputation can afford to exert more effort, thus receiving more project proposals, and more learning opportunities. This should further convexify the matching value function, and perpetuate the initial differences in reputation across agents.

(ii) Extending the one-agent setting of this paper to an economy populated by many agents, all of whom are learning their type and making matching decisions with projects. When the meetings between the individuals on the two sides are random, the externalities associated with search frictions (e.g., congestion effect) can alter the shape of the matching sets relative to the one-agent setting of this paper.

(iii) At the expense of introducing more value functions and more elaborate analysis, one can study the shape of *equilibrium* matching sets when the project owners are also *strategic* individuals. Specifically, when an agent meets a project owner, they can bargain over the fraction of the unit surplus each receives at the time of a breakthrough. One would naturally expect more reputable agents will be able to take higher shares, and that will further reinforce the increasing relationship between reputation and patience on the match.

A Proofs

A.1 Proof of Proposition 3.2

Proof of part (i): Observe that the variational Bellman equation for $v(\pi, q)$ in (3.1) can be equivalently expressed as:

$$v(\pi, q) = \max\left\{w(\pi), \frac{\lambda_q \pi - c}{r + \lambda_q \pi} + \frac{\lambda_q \pi w(\pi)}{r + \lambda_q \pi}\right\}.$$

This representation implies that $v(\pi, q) > w(\pi)$ iff the second maximand is larger than the first, that happens when $rw(\pi) < \lambda_q \pi - c$. Hence,

$$\pi \in \mathcal{M}_q \Leftrightarrow v(\pi, q) > w(\pi) \Leftrightarrow \lambda_q \pi - c > rw(\pi). \tag{A.1}$$

This implies that $\mathcal{M}_a \subseteq \mathcal{M}_b$. Next, in any fixed-point where $w(\cdot)$ is increasing, the subset $\{\pi \in [0, 1] : w(\pi) > 0\}$ is connected. Because of the Bellman equation for $w(\cdot)$ in (2.2) and the previous implication on $\mathcal{M}_a \subseteq \mathcal{M}_b$, one has

$$\{\pi \in [0, 1] : w(\pi) > 0\} = \{\pi : v(\pi, b) > w(\pi)\} = \mathcal{M}_b,$$

and hence $\mathcal{M}_b$ must be a connected subset.

Proof of part (ii): Toward the contradiction assume $1 \notin \mathcal{M}_b$, then from the previous part $1 \notin \mathcal{M}_a$ as well. Hence, equation (2.2) implies that $w(1) = 0$. Therefore, the derivation in (A.1) implies that $\lambda_b - c \leq 0$, that is a contradiction. Therefore, $1 \in \mathcal{M}_b$ always.

Let $w_b(1)$ be the reputation function at $\pi = 1$ in an outcome where $1 \in \mathcal{M}_b \cap \mathcal{M}_a^c$. Then, the two equations in (2.2) and (3.1) yield

$$rw_b(1) = \frac{\kappa\varphi_b\left(\lambda_b - c\right)}{r + \kappa\varphi_b + \lambda_b}.$$

Subsequently, condition (A.1) implies that

$$1 \in \mathcal{M}_a \Leftrightarrow \lambda_a - c > rw_b(1).$$

Thus, only in the low cost regime $1 \in \mathcal{M}_a$.

Proof of part (iii): To show $\mathcal{M}_a$ is a connected subset, we prove that if $\pi \in \mathcal{M}_a$, then $\pi' \in \mathcal{M}_a$ for all $\pi' > \pi$. Assume $\pi \in \mathcal{M}_a$, then $\pi \in \mathcal{M}_b$ from part (i). Thus, equations (2.2) and (3.1) amount to

$$w(\pi) = \frac{\kappa\varphi_a v(\pi, a) + \kappa\varphi_b v(\pi, b)}{r + \kappa\varphi_a + \kappa\varphi_b},$$

$$v(\pi, q) = \frac{\lambda_q \pi - c}{r + \lambda_q \pi} + \frac{\lambda_q \pi w(\pi)}{r + \lambda_q \pi}, \text{ for } q \in \{a, b\},$$

hence giving the following expression for $w(\pi)$ (when $\pi \in \mathcal{M}_a$):

$$rw(\pi) = \frac{\kappa\varphi_b\left(\lambda_b\pi - c\right)\left(r + \lambda_a\pi\right) + \kappa\varphi_a\left(\lambda_a\pi - c\right)\left(r + \lambda_b\pi\right)}{\left(r + \lambda_a\pi\right)\left(r + \lambda_b\pi\right) + \kappa\varphi_b\left(r + \lambda_a\pi\right) + \kappa\varphi_a\left(r + \lambda_b\pi\right)} .$$

After some straightforward algebraic manipulations, one can confirm that $\lambda_a\pi - c > rw(\pi)$ if $\pi > c/\lambda_a$. This implies that $\mathcal{M}_a$ is a connected and *increasing*[11] subset of $[0, 1]$. Also, in the high cost regime, $1 \notin \mathcal{M}_a$, therefore it must be that $\mathcal{M}_a = \emptyset$. $\square$

A.2 Proofs of Section 4

A.2.1 Preliminaries

In this section of appendix, I present the required machinery to prove proposition 4.1 and theorem 4.2.

First, in the next lemma, I borrow from equation (2.2) to express the reputation value function w in terms of v and $\mathcal{M}$, and thereby provide a partial characterization of matching sets *only* in terms of the matching value functions.

Lemma A.1. *An agent with reputation π accepts both types of projects, namely $\pi \in \mathcal{M}_a \cap \mathcal{M}_b$ iff*

$$v(\pi, a)\left(1 - \frac{1}{1 + r^{-1}\kappa\varphi_a}\right) < v(\pi, b) < v(\pi, a)\left(1 + \frac{1}{r^{-1}\kappa\varphi_b}\right) . \tag{A.2}$$

In addition, $\pi \in \mathcal{M}_b \cap \mathcal{M}_a^c$ iff the upper bound is achieved, $\pi \in \mathcal{M}_a \cap \mathcal{M}_b^c$ iff the lower bound is achieved, and $\pi \in \mathcal{M}_a^c \cap \mathcal{M}_b^c$ iff the upper and lower bounds coincide, which is only the case when all value functions are zero.

Proof. An equivalent representation for equation (2.2) is

$$w(\pi) = \frac{\kappa\big(v(\pi, a)\varphi_a\chi_a(\pi) + v(\pi, b)\varphi_b\chi_b(\pi)\big)}{r + \kappa\big(\varphi_a\chi_a(\pi) + \varphi_b\chi_b(\pi)\big)} . \tag{A.3}$$

One can check that if the inequality chain (A.2) holds, then with $\chi_a(\pi) = \chi_b(\pi) = 1$ in the above representation both of the conditions $v(\pi, a) > w(\pi)$ and $v(\pi, b) > w(\pi)$ are satisfied, and hence the *if* part is established. For the *only if* direction, assume $\pi \in \mathcal{M}(a) \cap \mathcal{M}(b)$, then it must be that $\chi_a(\pi) = \chi_b(\pi) = 1$. Replacing this in (A.3) and simplifying $v(\pi, b) > w(\pi)$ results in the first inequality in (A.2). Similarly, simplifying $v(\pi, a) > w(\pi)$ leads to the second inequality in (A.2). The proofs of the remaining claims follow the same logic. $\square$

[11] A subset E of $[0, 1]$ is referred to as increasing if $\pi \in E$ implies that $\pi' \in E$ for all $\pi' > \pi$ in $[0, 1]$.

Intuitively, this lemma asserts that the ratio $v(\pi, b)/v(\pi, a)$ always lies in a bounded interval for $\pi \in \mathcal{M}_a \cup \mathcal{M}_b$. At its maximum where it reaches the upper bound, the agent does not accept a-projects and alternatively, when it hits the lower bound, the agent only accepts the a-projects. This analysis renders much of the results in the proof of proposition 4.1 about the shape of optimal matching sets.

The representation of $w(\pi)$ in equation (A.3) and lemma A.1 allow us to express the optimal w as the output of a maximization problem over the space of all Borel measurable indicator functions $\chi_q(\pi)$ (similar idea to lemma 1 of Shimer and Smith (2000)):

$$w(\pi) = \max_{\chi} \left\{ \frac{\kappa\big(v(\pi, a)\varphi_a\chi_a(\pi) + v(\pi, b)\varphi_b\chi_b(\pi)\big)}{r + \kappa\big(\varphi_a\chi_a(\pi) + \varphi_b\chi_b(\pi)\big)} \right\}. \tag{A.4}$$

An important consequence of the above representation is that if $v(\cdot, a)$ and $v(\cdot, b)$ are increasing in π, then $w(\cdot)$ becomes increasing in π as well. The reverse direction is the result of the following lemma.

Lemma A.2. *The matching value functions $\{v(\cdot, q) : q \in \{a, b\}\}$ are increasing in π if and only if $w(\cdot)$ is increasing in π.*

Proof. Suppose both matching value functions, i.e., $v(\cdot, a)$ and $v(\cdot, b)$, are increasing in π. Then, the representation (A.4) implies that $w(\cdot)$ should be increasing in π. Conversely, assume $w(\cdot)$ is increasing in π, and hence almost everywhere differentiable on $[0, 1]$.[12] Recall that $v(\cdot, q)$ is the solution to the optimal stopping time problem (2.3). In that τ is the stopping time adapted to all possible future information. However, no information is released until the breakthrough time σ, and hence τ only uses the current information. This means that I can restrict the optimization space to the set of all deterministic times:

$$\begin{aligned} v(\pi, q) &= \sup_{\tau \in \mathbb{R}_+} V(\pi, q; \tau), \\ V(\pi, q; \tau) &:= \int_0^{\tau} \Big(r^{-1}c\left(e^{-rt} - 1\right) + e^{-rt}\big(1 + w(1)\big)\Big)\lambda_q \pi e^{-\lambda_q t} \mathrm{d}t \\ &\qquad + \left(1 - \pi + \pi e^{-\lambda_q \tau}\right)\left(r^{-1}c\left(e^{-r\tau} - 1\right) + e^{-r\tau} w(\pi_\tau)\right). \end{aligned}$$

Since w is almost everywhere differentiable, then $V(\cdot, q; \tau)$ inherits this property. Let us

[12]This is due the seminal Lebesgue theorem; see chapter 6 of Royden and Fitzpatrick (1988).

define $\frac{\partial V}{\partial \pi}(\pi, q; \tau) := I_1 + I_2 + I_3$, where

$$I_1 := r^{-1}c\left(\frac{\lambda_q}{r+\lambda_q}\left(1-e^{-(r+\lambda_q)\tau}\right) - e^{-r\tau}\left(1-e^{-\lambda_q \tau}\right)\right),$$
$$I_2 := \frac{(1+w(1))\,\lambda_q}{r+\lambda_q}\left(1-e^{-(r+\lambda_q)\tau}\right) - \left(1-e^{-\lambda_q\tau}\right)e^{-r\tau}w(\pi_\tau),$$
$$I_3 := e^{-r\tau}\left(1-\pi+\pi e^{-\lambda_q\tau}\right)w'(\pi_\tau)\frac{\partial \pi_\tau}{\partial \pi}.$$

The expression for I_1 is zero when $\tau = 0$, and has positive derivative w.r.t. τ, therefore, it is non-negative for all $\tau \geq 0$. The third term I_3 is obviously non-negative, because w is assumed increasing and due to the Bayes law $\partial\pi_\tau/\partial\pi > 0$. In regard to the second term, observe that:

$$\begin{aligned} I_2 &\geq \frac{(1+w(1))\,\lambda_q}{r+\lambda_q}\left(1-e^{-(r+\lambda_q)\tau}\right) - \left(1-e^{-\lambda_q\tau}\right)e^{-r\tau}w(1) \\ &\geq w(1)\left(\frac{\lambda_q}{\lambda_q+r}\left(1-e^{-(r+\lambda_q)\tau}\right) - e^{-r\tau}\left(1-e^{-\lambda_q\tau}\right)\right). \end{aligned}$$

The last expression above is increasing in τ and equals zero at $\tau = 0$, therefore, it is always non-negative. To sum, $\partial V/\partial\pi \geq 0$ almost everywhere and it is continuous in π, therefore V becomes increasing in π. Since $v(\pi, q) = \sup_\tau V(\pi, q; \tau)$, the matching value function $v(\cdot, q)$ must be increasing too. $\square$

Toward introducing the fixed-point, I now analyze the Bellman equation for the matching value functions. In the sequel, I repeatedly use the general solution form for the Bellman equation (2.4) on the continuation region $\mathcal{M}_q$, in that $\gamma(q)$ is a constant depending on the appropriate boundary conditions:

$$v(\pi, q) = -\frac{c}{r} + \frac{\lambda_q}{r+\lambda_q}\left(1 + w(1) + \frac{c}{r}\right)\pi + \gamma(q)\,(1-\pi)^{1+r/\lambda_q}\,\pi^{-r/\lambda_q}. \tag{A.5}$$

A.2.2 Proof of Proposition 4.1

Proof of part (i): At $\pi = 1$ the following fixed-point system falls out of (A.4) and the rearranged version of (2.4):

$$w(1) = \max_\chi \left\{\frac{\kappa\big(v(1,a)\varphi_a\chi_a(1) + v(1,b)\varphi_b\chi_b(1)\big)}{r + \kappa\big(\varphi_a\chi_a(1) + \varphi_b\chi_b(1)\big)}\right\}, \tag{A.6a}$$

$$v(1, q) = \max\left\{w(1), \frac{\lambda_q - c}{r+\lambda_q} + \frac{\lambda_q}{r+\lambda_q}w(1)\right\}, \quad \text{for} \quad q \in \{a, b\}. \tag{A.6b}$$

From equation (A.6b) it follows that

$$\chi_a(1) = 1 \Leftrightarrow rw(1) < \lambda_a - c \text{ and } \chi_b(1) = 1 \Leftrightarrow rw(1) < \lambda_b - c. \tag{A.7}$$

Observe the analogy between equation (A.7) and equation (A.1) at $\pi = 1$. Therefore, the remaining justification is similar to part (ii) of proposition 3.2.

Also, for future references, in the high cost regime,

$$w(1) = \frac{r^{-1}\kappa\varphi_b\left(\lambda_b - c\right)}{r + \kappa\varphi_b + \lambda_b}\,, \tag{A.8}$$

and in the low cost regime,

$$w(1) = \frac{r^{-1}\kappa\varphi_b\left(\lambda_b - c\right)\left(r + \lambda_a\right) + r^{-1}\kappa\varphi_a\left(\lambda_a - c\right)\left(r + \lambda_b\right)}{\left(r + \lambda_a\right)\left(r + \lambda_b\right) + \kappa\varphi_b\left(r + \lambda_a\right) + \kappa\varphi_a\left(r + \lambda_b\right)}\,. \tag{A.9}$$

Proof of part (ii): In the sequel I use the symbol $\underline{\partial} A$ to denote the lower boundary of a subset $A \subseteq [0, 1]$. To establish the convexity of $\mathcal{M}_b$, I first derive a useful identity for any strictly positive point $x \in \mathcal{M}_a \cap \underline{\partial}\mathcal{M}_b$. Since x is a lower boundary point for $\mathcal{M}_b$, then the agent finds it optimal to terminate the match when π approaches down to x. Importantly, at this point the principles of continuous and smooth fit (Dixit (2013)) must hold. The agent's outside option just below x is equal to $w(x)$ that is supported by the option value of meeting an a-type project because $x \in \mathcal{M}_a$, so

$$v(x, b) = w(x) = \frac{\kappa\varphi_a}{r + \kappa\varphi_a} v(x, a) \quad \text{and} \quad v'(x, b) = w'(x) = \frac{\kappa\varphi_a}{r + \kappa\varphi_a} v'(x, a)\,. \tag{A.10}$$

Now define $\Omega(x, q) := -c + \lambda_q x\big(1 + w(1)\big)$ and $\Gamma(x, q) := r + \lambda_q x$. Then, employing the HJB equations on the continuation region leads to

$$\frac{v'(x, b)}{v'(x, a)} = \frac{\lambda_a}{\lambda_b} \frac{\Omega(x, b) - \Gamma(x, b) v(x, b)}{\Omega(x, a) - \Gamma(x, a) v(x, a)}\,.$$

The previous two systems of equations give rise to

$$\frac{\kappa\varphi_a}{r + \kappa\varphi_a}\left(\frac{\lambda_b}{\lambda_a}\Gamma(x, a) - \Gamma(x, b)\right) v(x, a) = \frac{\kappa\varphi_a}{r + \kappa\varphi_a}\frac{\lambda_b}{\lambda_a}\Omega(x, a) - \Omega(x, b)\,, \tag{A.11a}$$

$$\Rightarrow \frac{\kappa\varphi_a}{r + \kappa\varphi_a}\left(\frac{\lambda_b}{\lambda_a} - 1\right) r v(x, a) = -c\left(\frac{\kappa\varphi_a}{r + \kappa\varphi_a}\frac{\lambda_b}{\lambda_a} - 1\right) - \frac{r\lambda_b x\left(1 + w(1)\right)}{r + \kappa\varphi_a}\,. \tag{A.11b}$$

Now assume to the contrary that $\mathcal{M}_b$ is not connected, hence, it contains at least two separated *maximal* open sets (relative to $[0, 1]$), say (x_0, x_1) and (x_2, x_3).[13] This implies that $[x_1, x_2] \subseteq \mathcal{M}_a$, because otherwise w assumes zero at some point in this interval, which violates the monotonicity of w. Therefore, $x_2 \in \mathcal{M}_a \cap \underline{\partial}\mathcal{M}_b$, and A.11b holds at x_2. I claim that $x_0 \in \mathcal{M}_a \cap \underline{\partial}\mathcal{M}_b$ too, because otherwise x_0 would be the lower boundary point at which

[13] If $x_3 = 1$, then the relevant subset would be $(x_2, 1]$. This does not change any of the upcoming analysis.

$v(\cdot, b)$ smoothly meets the *zero* function, hence applying the continuous and smooth fit to the equation (A.5) yields

$$x_0 = \frac{c}{\lambda_b\left(1 + w(1)\right)}\,.$$

This expression for x_0 leads to an upper bound for $v(x_2, a)$ using (A.11b):

$$\begin{aligned}\frac{\kappa\varphi_a}{r + \kappa\varphi_a}\left(\frac{\lambda_b}{\lambda_a} - 1\right) rv(x_2, a) &\leq -c\left(\frac{\kappa\varphi_a}{r + \kappa\varphi_a}\frac{\lambda_b}{\lambda_a} - 1\right) - \frac{r\lambda_b x_0\left(1 + w(1)\right)}{r + \kappa\varphi_a} \\ &= \frac{c\kappa\varphi_a}{r + \kappa\varphi_a}\left(1 - \frac{\lambda_b}{\lambda_a}\right) < 0.\end{aligned}$$

That in turn means $v(x_2, a) < 0$, hence a contradiction results. Therefore, x_0 and x_2 both belong to $\mathcal{M}_a \cap \underline{\partial}\mathcal{M}_b$. One can now apply (A.11b) at these two points and subtract their corresponding sides from each other to obtain:

$$\frac{\kappa\varphi_a}{r + \kappa\varphi_a}\left(\frac{\lambda_b}{\lambda_a} - 1\right) r\big(v(x_2, a) - v(x_0, a)\big) = \frac{-r\lambda_b\left(1 + w(1)\right)}{r + \kappa\varphi_a}\left(x_2 - x_0\right)\,.$$

The *lhs* to this equation is positive because of the monotonicity of $v(\cdot, a)$, but the *rhs* is negative, hence a contradiction is resulted, thus proving the connectedness of $\mathcal{M}_b$.

Proof of part (iii):

High cost regime: First, I show in this regime $\mathcal{M}_a$ cannot have a lower boundary point in $\mathcal{M}_b$, that is $\underline{\partial}\mathcal{M}_a \cap \mathcal{M}_b = \emptyset$. Toward the contradiction assume $\exists\ y \in \underline{\partial}\mathcal{M}_a \cap \mathcal{M}_b$. Then, a similar analysis to the previous part and especially the one following equation (A.10) yields:

$$\left(\frac{\lambda_b}{\lambda_a} - 1\right) rv(y, b) = -c\left(\frac{r + \kappa\varphi_b}{\kappa\varphi_b}\frac{\lambda_b}{\lambda_a} - 1\right) + \frac{r\lambda_b y\left(1 + w(1)\right)}{\kappa\varphi_b}\,. \tag{A.12}$$

In light of lemma A.1, such a y is a global maximum for $v(\cdot, b)/v(\cdot, a)$ on the region where $w > 0$. Thus, the first order derivative of $v(\cdot, b)/v(\cdot, a)$ at y will be zero, namely $v'(y, a)/v(y, a) = v'(y, b)/v(y, b)$. Since $y \in \underline{\partial}\mathcal{M}_a \cap \mathcal{M}_b$, and the second order derivatives of the matching value functions exist within the interior of this set (because of the solution form (A.5)), then it must be that $\lim_{\varepsilon\downarrow 0}\left(\frac{v(y+\varepsilon,b)}{v(y+\varepsilon,a)}\right)'' \leq 0$. Let us denote $v''(y, q) := \lim_{\varepsilon\downarrow 0} v''(y + \varepsilon, q)$ for $q \in \{a, b\}$. Then, the previous second order condition implies:

$$\frac{v''(y, b)}{v(y, b)} \leq \frac{v''(y, a)}{v(y, a)} \Rightarrow v''(y, b) \leq \frac{r + \kappa\varphi_b}{\kappa\varphi_b} v''(y, a)\,. \tag{A.13}$$

Next, I find an expression for the second order derivative by differentiating the HJB equation (2.4) on the continuation region:

$$\begin{aligned}rv'(y, q) = \lambda_q\big(1 + w(1) - v(y, q)\big) &- \lambda_q y v'(y, q) \\ &- \lambda_q(1 - 2y)v'(y, q) - \lambda_q y(1 - y)v''(y, q)\,.\end{aligned}$$

Substituting $v'(\cdot, q)$ from the HJB in the above equation leads to

$$\begin{aligned}\lambda_q y(1-y)v''(y,q) &= \lambda_q\big(1+w(1)-v(y,q)\big)\\ &\qquad - \frac{\big(r+\lambda_q(1-y)\big)}{\lambda_q y(1-y)} \times \big(-c+\lambda_q y\,(1+w(1)) - (r+\lambda_q y)\,v(y,q)\big)\\ &= -\frac{r\,(1+w(1))}{1-y} + \frac{r(r+\lambda_q)}{\lambda_q y(1-y)}v(y,q) + \frac{c\,(r+\lambda_q(1-y))}{\lambda_q y(1-y)}\,.\end{aligned}$$

Plugging the second order derivatives from above into (A.13) and applying some rearrangements yield the following *equivalent* relation

$$\begin{aligned}rv(y,b)\left(\frac{\lambda_b}{\lambda_a}-1\right)\left(1+\frac{r}{\lambda_a}+\frac{r}{\lambda_b}\right) &\geq \big(ry\,(1+w(1)) - c(1-y)\big)\left(\frac{r+\kappa\varphi_b}{\kappa\varphi_b}\frac{\lambda_b}{\lambda_a}-1\right)\\ &\qquad - \frac{cr}{\lambda_b}\left(\frac{r+\kappa\varphi_b}{\kappa\varphi_b}\frac{\lambda_b^2}{\lambda_a^2}-1\right)\,.\end{aligned}$$

Substituting (A.12) in above and applying several regroupings amount to:

$$y\Big[\,(1+w(1))\,\big(\lambda_a\,(r+\lambda_b) - \kappa\varphi_b\,(\lambda_b-\lambda_a)\,\big) - c\,\big(\lambda_b + r^{-1}\kappa\varphi_b\,(\lambda_b-\lambda_a)\big)\,\Big] \geq cr\,.$$

I would then replace $w(1)$ from (A.8) in the above expression and get an equivalent condition to (A.13) that is *only* in terms of the primitives:

$$\begin{aligned}\frac{cr^2}{r+\kappa\varphi_b}\left(1+\frac{\kappa\varphi_b}{r+\lambda_b}\right) + cy\lambda_b\left(1+\frac{r}{r+\lambda_b}\frac{\kappa\varphi_b}{r+\kappa\varphi_b}\right)&\\ \leq y\big(\lambda_a\,(r+\lambda_b) - \kappa\varphi_b\,(\lambda_b-\lambda_a)\,\big)\,.&\end{aligned} \tag{A.14}$$

Then, I will show that the *lhs* above is always greater than the *rhs*, thus the contradiction is resulted and there is no $y \in \underline{\partial}\mathcal{M}_a \cap \mathcal{M}_b$. Obviously at $y = 0$ the *lhs* is greater than the *rhs*. At $y = 1$, the *rhs* is increasing in λ_a, so can be upper bounded when λ_a assumes its maximum level in the high cost regime, i.e., $c + \frac{\kappa\varphi_b(\lambda_b - c)}{r+\kappa\varphi_b+\lambda_b}$. Therefore the *rhs* of (A.14) at $y = 1$ is upper bounded by

$$\lambda_a\,(r+\lambda_b) - \kappa\varphi_b\,(\lambda_b-\lambda_a) \leq c\,(r+\lambda_b)\,.$$

However, the *lhs* of (A.14) equals $c(r+\lambda_b)$ at $y = 1$. So (A.14) can never be satisfied, therefore in the high cost regime $\mathcal{M}_a$ cannot have a lower boundary point in $\mathcal{M}_b$. Given that $1 \notin \mathcal{M}_a$ and the monotonicity of w on $\mathcal{M}_b^c$, the only possible candidate for a non-empty $\mathcal{M}_a$ is (α_a, β_a) such that $\alpha_a < \alpha_b := \inf \mathcal{M}_b$. Because of optimality, $v(\cdot, a)$ must smoothly meet the zero function at α_a, so similar analysis to (A.16) implies that $\alpha_a = c/\lambda_a\big(1+w(1)\big)$, where $w(1)$ follows (A.8). Furthermore, the superharmonic condition for $v(\cdot, b)$ requires that at $\pi = \alpha_a$:

$$0 \geq [\mathcal{L}_b v](\alpha_a, b) - rv(\alpha_a, b) - c = \lambda_b \alpha_a \left(1 + w(1)\right) - c = \left(\frac{\lambda_b}{\lambda_a} - 1\right).$$

However, this never holds, because the rightmost side is positive. So the only continuation set that survives the high cost regime is $\mathcal{M}_a = \emptyset$.

<u>Low cost regime</u>: Note that in this regime $w(1)$ follows equation (A.9). I first prove the optimality implies that $\mathcal{M}_a \subseteq \mathcal{M}_b$. We saw in the part (i) that $1 \in \mathcal{M}_a \cap \mathcal{M}_b$ in this regime. To show the above set inclusion, I prove $\alpha_a := \inf \mathcal{M}_a \in \mathcal{M}_b$, and the claim follows from the connectedness of $\mathcal{M}_b$. Toward the contradiction assume $\alpha_a < \alpha_b$, where $\alpha_b = \inf \mathcal{M}_b$. Examining the superharmonicity of $v(\cdot, b)$ on $[0, \alpha_a]$ leads to

$$\begin{aligned}
\mathcal{L}_b v(\pi, b) - rv(\pi, b) - c = \lambda_b \pi \left(1 + w(1)\right) - c &= \frac{\lambda_b}{\lambda_a} \lambda_a \pi \left(1 + w(1)\right) - c \\
&= \frac{\lambda_b}{\lambda_a} \lambda_a \left(\pi - \alpha_a\right) \left(1 + w(1)\right) + \left(\frac{\lambda_b}{\lambda_a} - 1\right) c\,.
\end{aligned}$$

As π approaches α_a from below, the first term above converges to zero while the second term remains a positive constant. Therefore, $\exists\ \pi_0 < \alpha_a$ such that $\mathcal{L}_b v(\pi, b) - rv(\pi, b) - c > 0$ for all $\pi \in (\pi_0, \alpha_a]$. This violates the superharmonicity of $v(\cdot, b)$, and thus proving $\alpha_a \in \mathcal{M}_b$.

Next, I show having $\mathcal{M}_a \subseteq \mathcal{M}_b$ leads to the connectedness of $\mathcal{M}_a$. Because of the optimality of $v(\cdot, b)$ the principles of continuous and smooth fit hold at $\pi = \alpha_b$ with the zero outside option. Combining this with equation (A.5) implies the next expression for $v(\cdot, b)$:

$$\begin{aligned}
v(\pi, b) = &-\frac{c}{r} + \frac{\lambda_b}{r + \lambda_b} \left(1 + w(1) + \frac{c}{r}\right) \pi \\
&+ \left(\frac{c}{r} - \frac{\lambda_b}{r + \lambda_b} \left(1 + w(1) + \frac{c}{r}\right) \alpha_b\right) \left(\frac{1 - \pi}{1 - \alpha_b}\right)^{1 + r/\lambda_b} \left(\frac{\pi}{\alpha_b}\right)^{-r/\lambda_b},
\end{aligned} \tag{A.15}$$

with α_b following

$$\alpha_b = \frac{c}{\lambda_b \big(1 + w(1)\big)}\,, \tag{A.16}$$

in that $w(1)$ comes from equation (A.9). Furthermore, the above value function is convex if and only if

$$\left(\frac{c}{r} - \frac{\lambda_b}{r + \lambda_b} \left(1 + w(1) + \frac{c}{r}\right) \alpha_b\right) \geq 0\,. \tag{A.17}$$

Substituting α_b yields an equivalent condition for convexity:

$$\frac{c}{r} - \frac{c}{r + \lambda} - \frac{c^2}{r(r + \lambda_b)(1 + w(1))} = \frac{c}{r(r + \lambda_b)} \left(\lambda_b - \frac{c}{1 + w(1)}\right) \geq 0\,.$$

The above condition always holds because $\lambda_b > c$ and $w(1) > 0$, therefore $v(\cdot, b)$ followed from (A.15) is a convex function. Now define $[\mathcal{D}_a v](\pi, a) := [\mathcal{L}_a v](\pi, a) - rv(\pi, a) - c$. Observe that on the continuation region $\mathcal{M}_a$, the HJB equation implies $[\mathcal{D}_a v](\cdot, a) = 0$. Also, on $\mathcal{M}_a^c \cap \mathcal{M}_b$, one has $v(\cdot, a) = w(\cdot) = \frac{\kappa\varphi_b}{r+\kappa\varphi_b} v(\cdot, b)$, thus the HJB equation for $v(\cdot, b)$ implies that

$$[\mathcal{D}_a v](\pi, a) = \frac{-\kappa\varphi_b}{r + \kappa\varphi_b}(\lambda_b - \lambda_a)\frac{rv(\pi, b) + c}{\lambda_b} + \frac{r\lambda_a\pi(1 + w(1)) - cr}{r + \kappa\varphi_b}\,.$$

Consequently, convexity of $v(\cdot, b)$ amounts to

$$\frac{\partial^2}{\partial\pi^2}[\mathcal{D}_a v](\pi, a) = \frac{-\kappa\varphi_b(\lambda_b - \lambda_a)}{(r + \kappa\varphi_b)\,\lambda_b} v''(\pi, b) < 0\,.$$

Therefore, $[\mathcal{D}_a v](\cdot, a)$ is zero on $\mathcal{M}_a$ and a strictly concave function in π on $\mathcal{M}_a^c \cap \mathcal{M}_b$. Were $\mathcal{M}_a$ not be connected, then at least it has two disjoint components, say (x_1, x_2) and (x_3, x_4) where $x_2 < x_3$ and $[x_2, x_3] \subset \mathcal{M}_b$. Strict concavity on $[x_2, x_3]$ requires $[\mathcal{D}_a v](\cdot, a)$ to be positive on this region, which violates the superhamonicity of $v(\cdot, a)$, thus resulting the contradiction and proving the connectedness of $\mathcal{M}_a$. □

A.2.3 Proof of Theorem 4.2

I prove the assertion only for the high cost regime, as the proof of the other case follows the same logic, but is just lengthier. Proposition 4.1 provides necessary conditions for the optimal matching sets associated with increasing value functions in $C^1[0, 1]$. Specifically, we know from this proposition, that in the high cost regime, the only matching sets surviving the optimality principles are $\mathcal{M}_a = \emptyset$ and $\mathcal{M}_b = (\alpha_b, 1]$, where α_b is found via the continuous and smooth fit principles as

$$\alpha_b = \frac{c}{\lambda_b\,(1 + w(1))}\,. \tag{A.18}$$

Also, from the construction of that proposition, we know that the following profile embodies the only candidate for a fixed-point with increasing $C^1[0, 1]$ value functions on $(\alpha_b, 1]$:

$$\begin{aligned} v(\pi, a) &= w(\pi) = \frac{\kappa\varphi_b}{r + \kappa\varphi_b} v(\pi, b) \\ v(\pi, b) &= -\frac{c}{r} + \frac{\lambda_b}{r + \lambda_b}\left(1 + w(1) + \frac{c}{r}\right)\pi \\ &\quad + \left(\frac{c}{r} - \frac{\lambda_b}{r + \lambda_b}\left(1 + w(1) + \frac{c}{r}\right)\alpha_b\right)\left(\frac{1 - \pi}{1 - \alpha_b}\right)^{1+r/\lambda_b}\left(\frac{\pi}{\alpha_b}\right)^{-r/\lambda_b}\,. \end{aligned} \tag{A.19}$$

And all are equal to zero on $[0, \alpha_b]$. Therefore, our task here is to employ a *verification* scheme to show that the above value functions are indeed the optimal values that appear in the fixed-point. I divide the proof into three steps: (a) verifying the majorizing and superharmonicity conditions; (b) using these two and applying a Martingale argument to establish the *optimality* of the above profile of the value functions; (c) for large r the Banach fixed-point theorem is applied and proves the uniqueness of the identified fixed-point in the larger space of *essentially bounded measurable* functions, namely $L^\infty[0,1]$. Lastly, only in the remaining parts of the proof, I slightly change the notation and use $v_q(\pi) := v(\pi, q)$ for $q \in \{a, b\}$.

Step (a):

Majorizing. This step is quite straightforward because in equation (A.19) $w = v_a$ and $v_b \geq w = \frac{\kappa\varphi_b}{r+\kappa\varphi_b} v_b$.

Superharmonicity of v_b. Obviously the superharmonic condition holds with equality on $(\alpha_b, 1]$, because of the Bellman equation. However, it needs to be checked on $[0, \alpha_b]$ as it is carried out below:

$$[\mathcal{L}_b v_b](\pi) - r v_b(\pi) - c = \lambda_b \pi \left(1 + w(1)\right) - c \leq \lambda_b \alpha_b \left(1 + w(1)\right) - c = 0\,.$$

Superharmonicity of v_a. Recall that in the high cost regime $\mathcal{M}_a = \emptyset$, thus $v_a = w$. So on $[0, \alpha_b]$:

$$\begin{aligned}
[\mathcal{L}_a v_a](\pi) - r v_a(\pi) - c &= \lambda_a \pi \left(1 + w(1)\right) - c \\
&\leq \lambda_b \alpha_b \left(1 + w(1)\right) - c \leq 0\,,
\end{aligned}$$

where in the last inequality I used the expression (A.18) for α_b. The analysis of the superharmonicity of v_a on $(\alpha_b, 1]$ however needs a little more work:

$$\begin{aligned}
[\mathcal{L}_a v_a](\pi) - r v_a(\pi) - c &= \left[\mathcal{L}_a \left(\frac{\kappa\varphi_b}{r+\kappa\varphi_b} v_b\right)\right](\pi) - \frac{r\kappa\varphi_b}{r+\kappa\varphi_b} v_b(\pi) - c \\
&= \frac{\kappa\varphi_b}{r+\kappa\varphi_b} \big([\mathcal{L}_a v_b](\pi) - r v_b(\pi) - c\big) \\
&\qquad + \frac{r\lambda_a\pi}{r+\kappa\varphi_b} \left(1 + w(1)\right) - \frac{rc}{r+\kappa\varphi_b} \\
&= -\frac{\kappa\varphi_b}{r+\kappa\varphi_b} \left[(\mathcal{L}_b - \mathcal{L}_a) v_b\right](\pi) + \frac{r\lambda_a\pi}{r+\kappa\varphi_b} \left(1 + w(1)\right) - \frac{rc}{r+\kappa\varphi_b} \\
&= -\frac{\kappa\varphi_b}{r+\kappa\varphi_b} \left(\lambda_b - \lambda_a\right) \pi \big(1 + w(1) - v_b(\pi) - (1-\pi) v_b'(\pi)\big) \\
&\qquad + \frac{r\lambda_a\pi}{r+\kappa\varphi_b} \left(1 + w(1)\right) - \frac{rc}{r+\kappa\varphi_b}\,.
\end{aligned}$$

Some straightforward manipulations similar to the equation (A.17) imply that the candidate v_b in (A.19) is also convex. Therefore, $v_b(\pi) + (1-\pi) v_b'(\pi) \leq v_b(1)$ that yields an upper bound for the above relation:

$$
\begin{aligned}
[\mathcal{L}_a v_a](\pi) - r v_a(\pi) - c &\leq -\frac{\kappa\varphi_b}{r+\kappa\varphi_b}\frac{r\left(\lambda_b-\lambda_a\right)\pi}{r+\lambda_b}\left(1+w(1)+\frac{c}{r}\right) + \frac{r\lambda_a\pi\left(1+w(1)\right)-cr}{r+\kappa\varphi_b} \\
&\leq \left(-\frac{\kappa\varphi_b}{r+\kappa\varphi_b}\frac{r\left(\lambda_b-\lambda_a\right)}{r+\lambda_b}\left(1+w(1)+\frac{c}{r}\right) + \frac{r\lambda_a\left(1+w(1)\right)-rc}{r+\kappa\varphi_b}\right)^+ .
\end{aligned}
$$

In the second inequality above I used the fact that the *rhs* of the first inequality is affine in π and negative at $\pi = 0$. Now denote the argument of $(\cdot)^+$ by $\mathfrak{z}$. It is increasing in λ_a, hence can be bounded above when λ_a is replaced with $c + rw(1)$ (its maximum value in the high cost regime):

$$
\begin{aligned}
\mathfrak{z} &\leq -\frac{\kappa\varphi_b}{r+\kappa\varphi_b}\frac{r\left(\lambda_b-c-rw(1)\right)}{r+\lambda_b}\left(1+w(1)+\frac{c}{r}\right) + \frac{r\left(c+rw(1)\right)\left(1+w(1)\right)-rc}{r+\kappa\varphi_b} \\
&= -\frac{\kappa\varphi_b}{r+\kappa\varphi_b}\frac{\left(\lambda_b-c\right)\left(r+\lambda_b\right)\left(r+\kappa\varphi_b+c\right)}{r\left(\kappa\varphi_b+r+\lambda_b\right)^2} + \frac{\kappa\varphi_b}{r+\kappa\varphi_b}\frac{\left(\lambda_b-c\right)\left(r+\lambda_b\right)\left(r+\kappa\varphi_b+c\right)}{r\left(\kappa\varphi_b+r+\lambda_b\right)^2} = 0\,,
\end{aligned}
$$

where in the second line $w(1)$ is replaced from equation (A.8). This concludes the superharmonicity of v_a w.r.t. $\mathcal{L}_a$ on $(\alpha_b, 1]$, and hence on the entire unit interval.

Step (b): Define $\mathbf{v}_q(\iota,\pi) := v_q(\pi)1_{\{\iota=0\}} + \big(\iota + w(\pi)\big)1_{\{\iota=1\}}$, where ι is the success indicator process. Since $\mathbf{v}$ is a bounded function, for each $q \in \{a, b\}$, one can find a bounded (and hence uniformly integrable) Martingale process M^q such that:

$$
e^{-rt}\mathbf{v}_q(\iota_t,\pi_t) = \mathbf{v}_q(\iota,\pi) + \int_0^t e^{-rs}\big(\mathbb{L}_q\mathbf{v}_q(\cdot,\cdot) - r\mathbf{v}_q(\cdot,\cdot)\big)(\iota_{s^-},\pi_{s^-})\mathrm{d}s + M^q_t\,. \tag{A.20}
$$

In that $\mathbb{L}_q\mathbf{v}_q(\iota,\pi) := [\mathcal{L}_q v_q](\pi)1_{\{\iota=0\}}$. From the majorant condition, for every stopping time τ, we have $\mathbf{v}_q(\iota_\tau,\pi_\tau) \geq \iota_\tau + w(\pi_\tau)$, therefore

$$
\begin{aligned}
e^{-r\tau}\big(\iota_\tau + w(\pi_\tau)\big) &\leq \mathbf{v}_q(\iota,\pi) + \int_0^\tau e^{-rs}\big(\mathbb{L}_q\mathbf{v}_q(\cdot,\cdot) - r\mathbf{v}_q(\cdot,\cdot)\big)(\iota_{s^-},\pi_{s^-})\mathrm{d}s + M^q_\tau \\
&\leq \mathbf{v}_q(\iota,\pi) + \int_0^\tau c e^{-rs}\mathrm{d}s + M^q_\tau\,,
\end{aligned}
$$

wherein the second inequality I used the superharmonic property proven in step (a). Doob's optional stopping theorem implies that $\mathsf{E}M^q_\tau = 0$, hence for every stopping time τ, one has

$$
\mathbf{v}(\iota,\pi) \geq \mathsf{E}_{\pi,q,\iota}\left[e^{-r\tau}\left(\iota_\tau + w(\pi_\tau)\right) - c\int_0^\tau e^{-rs}\mathrm{d}s\right].
$$

That in turn implies

$$
v(\pi,q) \geq \sup_\tau \mathsf{E}_{\pi,q,\iota=0}\left[e^{-r\tau}\left(\iota_\tau + w(\pi_\tau)\right) - c\int_0^\tau e^{-rs}\mathrm{d}s\right].
$$

Now for each q, let $\tau(q) := \inf\{t \geq 0 : \pi_t \notin \mathcal{M}_q \text{ or } \iota_t = 1\}$ that is the optimal stopping policy. Using this in equation (A.20) yields

$$\begin{aligned}
e^{-r\tau(q)}\left(\iota_{\tau(q)} + w(\pi_{\tau(q)})\right) &= e^{-r\tau(q)}\mathbf{v}_q\big(\iota_{\tau(q)}, \pi_{\tau(q)}\big) \\
&= \mathbf{v}_q(\iota, \pi) + \int_0^{\tau(q)} e^{-rs}\big(\mathbb{L}_q\mathbf{v}_q(\cdot,\cdot) - r\mathbf{v}_q(\cdot,\cdot)\big)(\iota_{s^-}, \pi_{s^-})\mathrm{d}s + M^q_{\tau(q)} \\
&= \mathbf{v}_q(\iota, \pi) - \int_0^{\tau(q)} ce^{-rs}\mathrm{d}s + M^q_{\tau(q)}\,,
\end{aligned}$$

which after taking expectations of both sides amounts to

$$\mathbf{v}_q(\iota, \pi) = \mathsf{E}_{\pi,q,\iota}\left[e^{-r\tau(q)}\left(\iota_{\tau(q)} + w(\pi_{\tau(q)}\right) - c\int_0^{\tau(q)} e^{-rs}\mathrm{d}s\right],$$

thereby concluding the verification proof.

Step (c): In this part we apply the contraction mapping theorem to study the uniqueness. For every $(v_a, v_b, w) \in \big(L^\infty[0,1]\big)^3$, define

$$\begin{aligned}
\mathcal{T}_q w(\pi) &:= \sup_\tau\left\{\mathsf{E}_q\left[e^{-r\sigma} - c\int_0^\sigma e^{-rs}\mathrm{d}s + e^{-r\sigma}w(\pi_\sigma); \sigma \leq \tau\right]\right. \\
&\qquad \left. + \,\mathsf{E}_q\left[-c\int_0^\tau e^{-rs}\mathrm{d}s + e^{-r\tau}w(\pi_\tau); \sigma > \tau\right]\right\}, \quad \text{for } q \in \{a,b\}\,, \\
\mathcal{T}_0[(v_a, v_b, w)](\pi) &:= r^{-1}\kappa \sum_{q\in\mathcal{M}(\pi)} \big(v_q(\pi) - w(\pi)\big)\varphi_q\,,
\end{aligned}$$

where E_q is the expectation operator w.r.t. the Poisson process with intensity λ_q and $\mathcal{M}(\pi) = \{q : v_q(\pi) > w(\pi)\}$. Define $\mathcal{T} := (\mathcal{T}_a, \mathcal{T}_b, \mathcal{T}_0)$. The goal of this part of the proof is to show that the fixed-point of $\mathcal{T}$ exists and is unique in $\big(L^\infty[0,1]\big)^3$. Given the definition of $\mathcal{M}(\pi)$, one can see that $\mathcal{T}_0$ and $\mathcal{T}_q$ preserve the measurability and essential boundedness. Therefore, $\mathcal{T}\big(L^\infty[0,1]\big)^3 \subseteq \big(L^\infty[0,1]\big)^3$. The next step is to investigate the contraction property of $\mathcal{T}$. For this, let us equip $(L^\infty[0,1])^3$ with the following norm,

$$\|(v_a, v_b, w)\|_\varsigma := \varsigma\left(\|v_a\|_\infty + \|v_b\|_\infty\right) + \|w\|_\infty\,,$$

where $\varsigma > 0$ is to be determined. First, I examine the contraction coefficient of $\mathcal{T}_q$. Observe that for every $w, \tilde{w} \in L^\infty[0,1]$:

$$\begin{aligned}
\Big|\mathcal{T}_q[w] - \mathcal{T}_q[\tilde{w}]\Big|(\pi) &\leq \sup_\tau\Big\{\mathsf{E}_q\left[e^{-r\sigma}\left|w(\pi_\sigma) - \tilde{w}(\pi_\sigma)\right|; \sigma \leq \tau\right] \\
&\qquad + \mathsf{E}_q\left[e^{-r\tau}\left|w(\pi_\tau) - \tilde{w}(\pi_\tau)\right|; \sigma > \tau\right]\Big\} \\
&\leq \|w - \tilde{w}\|_\infty \sup_\tau \mathsf{E}_q\left[e^{-r(\tau\wedge\sigma)}\right] = \|w - \tilde{w}\|_\infty\,.
\end{aligned}$$

Let $\phi := \varphi_a + \varphi_b$, and $v, \tilde{v} \in (L^\infty[0,1])^2$, respectively inducing the matching sets $\mathcal{M}$ and $\widetilde{\mathcal{M}}$. Then, one has

$$\big(\mathcal{T}_0[(v,w)] - \mathcal{T}_0[(\tilde{v},\tilde{w})]\big)(\pi) = r^{-1}\kappa \left(\sum_{q\in\mathcal{M}(\pi)} \big(v_q(\pi) - w(\pi)\big)\varphi_q - \sum_{q\in\widetilde{\mathcal{M}}(\pi)} \big(\tilde{v}_q(\pi) - \tilde{w}(\pi)\big)\varphi_q \right).$$

Partitioning the matching sets $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ yields the following equivalent expression:

$$\begin{aligned}
\Big(\mathcal{T}_0[(v,w)] - \mathcal{T}_0[(\tilde{v},\tilde{w})]\Big)(\pi) &= r^{-1}\kappa \sum_{q\in\mathcal{M}(\pi)\setminus\widetilde{\mathcal{M}}(\pi)} \big(v_q(\pi) - \tilde{v}_q(\pi) - w(\pi) + \tilde{w}(\pi)\big)\varphi_q \\
&+ r^{-1}\kappa \sum_{q\in\mathcal{M}(\pi)\cap\widetilde{\mathcal{M}}(\pi)} \big(v_q(\pi) - \tilde{v}_q(\pi) - w(\pi) + \tilde{w}(\pi)\big)\varphi_q \\
&+ \underbrace{r^{-1}\kappa \sum_{q\in\mathcal{M}(\pi)\setminus\widetilde{\mathcal{M}}(\pi)} \big(\tilde{v}_q(\pi) - \tilde{w}(\pi)\big)\varphi_q}_{=0} \underbrace{- r^{-1}\kappa \sum_{q\in\widetilde{\mathcal{M}}(\pi)\setminus\mathcal{M}(\pi)} \big(\tilde{v}_q(\pi) - \tilde{w}(\pi)\big)\varphi_q}_{\geq 0} \\
&\leq r^{-1}\kappa\phi \left(\sum_{q\in\{a,b\}} \|v_q - \tilde{v}_q\|_\infty + \|w - \tilde{w}\|_\infty \right).
\end{aligned}$$

Putting together the previous bounds amounts to:

$$\begin{aligned}
\Big\|\mathcal{T}\left[(v_a,v_b,w)\right] - \mathcal{T}\left[(\tilde{v}_a,\tilde{v}_b,\tilde{w})\right]\Big\|_\varsigma &= \varsigma\Big(\big\|\mathcal{T}_a[w] - \mathcal{T}_a[\tilde{w}]\big\|_\infty + \big\|\mathcal{T}_b[w] - \mathcal{T}_b[\tilde{w}]\big\|_\infty\Big) \\
&+ \big\|\mathcal{T}_0\left[(v_a,v_b,w)\right] - \mathcal{T}_0\left[(\tilde{v}_a,\tilde{v}_b,\tilde{w})\right]\big\|_\infty \\
&\leq 2\varsigma\|w - \tilde{w}\|_\infty + r^{-1}\kappa\phi \left(\sum_{q\in\{a,b\}} \|v_q - \tilde{v}_q\|_\infty + \|w - \tilde{w}\|_\infty \right) \\
&= r^{-1}\kappa\phi\|v_a - \tilde{v}_a\|_\infty + r^{-1}\kappa\phi\|v_b - \tilde{v}_b\|_\infty + \big(2\varsigma + r^{-1}\kappa\phi\big)\|w - \tilde{w}\|_\infty .
\end{aligned}$$

Assume $r^{-1}\kappa\phi < 1/3$. Find $\varepsilon > 0$ such that $r^{-1}\kappa\phi < 1/(1+\varepsilon)(3+2\varepsilon)$, and let $\varsigma = (1+\varepsilon)r^{-1}\kappa\phi$, then

$$\begin{aligned}
&\Big\|\mathcal{T}\left[(v_a,v_b,w)\right] - \mathcal{T}\left[(\tilde{v}_a,\tilde{v}_b,\tilde{w})\right]\Big\|_\varsigma \\
&\leq \frac{r^{-1}\kappa\phi}{\varsigma} \left(\varsigma\|v_a - \tilde{v}_a\|_\infty + \varsigma\|v_b - \tilde{v}_b\|_\infty + \overbrace{\frac{\varsigma(2\varsigma + r^{-1}\kappa\phi)}{r^{-1}\kappa\phi}}^{<1}\|w - \tilde{w}\|_\infty \right) \\
&\leq \frac{1}{1+\varepsilon}\big\|(v_a,v_b,w) - (\tilde{v}_a,\tilde{v}_b,\tilde{w})\big\|_\varsigma .
\end{aligned}$$

So the Banach fixed-point theorem implies that there exists a unique fixed-point in the space of essentially bounded measurable functions, as long as $r > 3\kappa\phi$. □

B General Type Space

The goal of this appendix is to extend the results of section 4 to the general type space for projects. Specifically, I show there always exists an *increasing* reputation function w that satisfies the agent's fixed-point problem. Suppose the projects' types are drawn from an arbitrary distribution with CDF $\phi(\cdot)$ on a bounded support $[a, b]$. The success arrival intensity takes the general form of $\lambda_q(\theta)$, for which I denote $\lambda_q(H) = \bar{\lambda}_q$ and $\lambda_q(L) = \underline{\lambda}_q$, and assume $\underline{\lambda}_q \leq \bar{\lambda}_q \leq \boldsymbol{\lambda}$ for all $q \in \mathsf{Supp}(\phi)$.

The reputation value function satisfies

$$w(\pi) = \frac{\kappa}{r} \int \big(v(\pi, q) - w(\pi)\big)^+ \phi(\mathrm{d}q)\,,$$

therefore for every measurable subset $B \subseteq [a, b]$, one can see the optimal value functions (w, v) satisfy

$$w(\pi) \geq \frac{\kappa}{r} \int_B \big(v(\pi, q) - w(\pi)\big)\phi(\mathrm{d}q) \Rightarrow w(\pi) \geq \frac{\int_B v(\pi, q)\phi(\mathrm{d}q)}{1 + \frac{\kappa}{r}\phi(B)}\,.$$

Setting $B^* = \{q : v(\pi, q) > w(\pi)\}$ to bind the above inequality, one can propose the following equivalent representation for the reputation function:

$$w(\pi) = \sup\left\{\frac{\int_B v(\pi, q)\phi(\mathrm{d}q)}{1 + \frac{\kappa}{r}\phi(B)} : \text{measurable } B \subseteq [a, b]\right\}. \tag{B.1}$$

On the other hand, given the reputation function w, the agent solves the following stopping time problem when matched with a project of type q:

$$v(\pi, q) = \sup_{\tau} \mathsf{E}\left[1_{\{\sigma\leq\tau\}}e^{-r\sigma} - c\int_0^{\sigma\wedge\tau} e^{-rs}\mathrm{d}s + e^{-r(\sigma\wedge\tau)}w(\pi_{\sigma\wedge\tau})\right]. \tag{B.2}$$

For a given q, let $\mathcal{T}_q w$ be the matching value function resulted from the above problem. Hence from (B.1) it follows that w is the fixed-point to the following operator:

$$\mathcal{A}w := \sup\left\{\frac{\int_B \mathcal{T}_q w\ \phi(\mathrm{d}q)}{1 + \frac{\kappa}{r}\phi(B)} : \text{measurable } B \subseteq [a, b]\right\}. \tag{B.3}$$

In what follows I propose the appropriate function space on which $\mathcal{A}$ will be defined, and study its fixed-points. Let $L^1[0, 1]$ be the Banach space of Lebesgue integrable functions on the unit interval, and $L^1_+[0, 1]$ be the subset of non-negative functions which is readily seen to be a cone.[14] Let $\succsim$ be the partial order induced by the cone $L^1_+[0, 1]$ on the Banach space

[14] A cone is a subset $\mathcal{K}$ of a Banach space which is (i) closed, (ii) for every $x, y \in \mathcal{K}$ and $\alpha, \beta \geq 0$: $\alpha x + \beta y \in \mathcal{K}$, and (iii) $\mathcal{K} \cap (-\mathcal{K}) = \mathbf{0}$.

$L^1[0,1]$, that is $w_2 \gtrsim w_1$ if $w_2(\pi) \geq w_1(\pi), \forall \pi \in [0,1]$. Then, it follows from (B.2) that $\mathcal{T}_q$ is a *positive* and *monotone* operator, that is letting $\mathbf{0}$ to be the zero element of $L^1[0,1]$, then $\mathcal{T}_q \mathbf{0} \gtrsim \mathbf{0}$, and $\mathcal{T}_q w_2 \gtrsim \mathcal{T}_q w_1$ for $w_2 \gtrsim w_1$ in $L^1_+[0,1]$. Further, it can easily be verified that $\mathcal{A}$ inherits *positivity* and *monotonicity* from the collection $\{\mathcal{T}_q : q \in [a,b]\}$. Next, I show without loss of generality, we can restrict the search for the fixed-point to the bounded region of all $w \in L^1_+[0,1]$ where $\|w\|_\infty \leq \boldsymbol{\lambda}/r$.[15]

Lemma B.1. *For every $w \in L^1_+[0,1]$,*

$$\|\mathcal{T}_q w\| \leq \max\left\{\|w\|, \frac{\boldsymbol{\lambda}}{r+\boldsymbol{\lambda}}(1+\|w\|)\right\}, \quad \phi - \textit{almost surely.}$$

Proof. For every $q \in \mathsf{Supp}(\phi)$,

$$\begin{aligned}
\mathcal{T}_q w(\pi) &= \sup_\tau \mathsf{E}\left[1_{\{\sigma \leq \tau\}} e^{-r\sigma} - c\int_0^{\sigma\wedge\tau} e^{-rs}\mathrm{d}s + e^{-r(\sigma\wedge\tau)} w(\pi_{\sigma\wedge\tau})\right] \\
&\leq \sup_\tau \mathsf{E}\left[1_{\{\sigma \leq \tau\}} e^{-r\sigma} + e^{-r(\sigma\wedge\tau)} w(\pi_{\sigma\wedge\tau})\right] \\
&\leq \sup_\tau \mathsf{E}\Big[1_{\{\sigma \leq \tau\}} e^{-r\sigma} + e^{-r(\sigma\wedge\tau)} \|w\|\Big].
\end{aligned}$$

Let $\boldsymbol{\sigma} \overset{d}{=} \mathsf{exp}(\boldsymbol{\lambda})$, then $\sigma \succeq \boldsymbol{\sigma}$ in the sense of the first order stochastic dominance.[16] Since the random variable inside the last expectation above is decreasing in σ, then

$$\mathcal{T}_q w(\pi) \leq \sup_\tau \mathsf{E}\Big[1_{\{\boldsymbol{\sigma} \leq \tau\}} e^{-r\boldsymbol{\sigma}} + e^{-r(\boldsymbol{\sigma}\wedge\tau)} \|w\|\Big].$$

The above stopping time problem has no state variable (such as π), and it maintains its memory-less property inherited from the exponential distribution of $\boldsymbol{\sigma}$. Therefore, the optimal stopping rule is either $\tau = 0$ or $\tau = \infty$, that implies:

$$\mathcal{T}_q w(\pi) \leq \max\Big\{\|w\|, \mathsf{E}\left[e^{-r\boldsymbol{\sigma}}\right]\big(1+\|w\|\big)\Big\}.$$

Since $\mathsf{E}\left[e^{-r\boldsymbol{\sigma}}\right] = \boldsymbol{\lambda}/\big(r+\boldsymbol{\lambda}\big)$ the claim follows. □

I use the previous lemma to limit the search for the space of fixed-points. The next result summarizes this claim.

Lemma B.2. *Any fixed-point of $\mathcal{A}$ (if exists) is order bounded above by the constant function $\boldsymbol{\lambda}/r$.*

[15] Henceforth, if not stated explicitly all norms are the sup-norm.

[16] The term $\mathsf{exp}(\lambda)$ denotes an exponential random variable with the rate λ.

Proof. First, observe that the supremum in (B.3) is achieved by $B_w = \{q : [\mathcal{T}_q w](\pi, q) > w(\pi)\}$ for any candidate fixed-point w. Then, for any such candidate one has

$$\left(1 + \frac{\kappa}{r}\phi(B_w)\right) w(\pi) = \int_{B_w} \mathcal{T}_q w(\pi)\phi(\mathrm{d}q)\,.$$

Therefore, using the result of the previous lemma one obtains

$$\left(1 + \frac{\kappa}{r}\phi(B_w)\right) \|w\| \leq \max\left\{\|w\|, \frac{\boldsymbol{\lambda}}{r+\boldsymbol{\lambda}}(1 + \|w\|)\right\} \phi(B_w)\,. \tag{B.4}$$

Assume to the contrary that $\|w\| > \boldsymbol{\lambda}/r$, then $\max\left\{\|w\|, \frac{\boldsymbol{\lambda}}{r+\boldsymbol{\lambda}}(1 + \|w\|)\right\} = \|w\|$, and (B.4) implies that

$$\left(1 + \frac{\kappa}{r}\phi(B_w)\right) \|w\| \leq \|w\|\phi(B_w)\,. \tag{B.5}$$

Canceling $\|w\|$ from both sides amounts to $1 + \frac{\kappa}{r}\phi(B_w) \leq \phi(B_w)$. Since, it was assumed $\|w\| > \boldsymbol{\lambda}/r$, then $\phi(B_w) > 0$. On the other hand $\phi(B_w) \leq 1$. These two together with (B.5) yield the contradiction and hence prove the lemma. □

Definition B.3 (Regular and strongly-minihedral cones: Krasnoselskij (1964) sections 1.5 and 1.7)**.** A Banach space partially ordered by a cone is called *regularly partially ordered*, if any monotone-increasing sequence, *order-bounded* from above, converges in norm to a limit point. A cone which generates a regular partial ordering is called a regular cone. A cone is said to be *strongly minihedral* if every order bounded subset has a least upper bound (i.e., order supremum).

Now consider the Banach space of integrable functions $L^1[0,1]$, and the positive cone of $L^1_+[0,1] = \{f \in L^1[0,1] : f(x) \geq 0\ \forall x \in [0,1]\}$. This cone is regular, and for any monotone increasing sequence $\{f_n\} \subset L^1[0,1]$ such that $f_1 \precsim f_2 \precsim \ldots$ and order bounded from above, one has $\|f_n - f\|_{L^1} \to 0$ where $f(x) = \sup_n f_n(x)$ *for every* $x \in [0,1]$ (Dominated convergence theorem). In addition $L^1_+[0,1]$ is strongly minihedral (page 52 Krasnoselskij (1964)).

Let $\langle \mathbf{0}, \boldsymbol{\lambda}/r\rangle := \left\{f \in L^1_+[0,1] : \mathbf{0} \precsim f \precsim \boldsymbol{\lambda}/r\right\}$ be the order interval of non-negative L^1 functions, order bounded above by the constant function $\boldsymbol{\lambda}/r$. In light of the lemma B.1, we have $\mathcal{T}_q : \langle \mathbf{0}, \boldsymbol{\lambda}/r\rangle \to \langle \mathbf{0}, \boldsymbol{\lambda}/r\rangle$ for every $q \in [a,b]$ and hence $\mathcal{A} : \langle \mathbf{0}, \boldsymbol{\lambda}/r\rangle \to \langle \mathbf{0}, \boldsymbol{\lambda}/r\rangle$. At this stage, I can apply part (a) of theorem 4.1 in Krasnoselskij (1964) to conclude the existence of a fixed-point of $\mathcal{A}$ in $\langle \mathbf{0}, \boldsymbol{\lambda}/r\rangle$, because the mapping $\mathcal{A}$ is monotonic in a strongly minihedral cone space. However, the mere existence of the fixed-point is far from enough. In particular, we want to know whether there exists an increasing fixed-point for $\mathcal{A}$. To answer such question, I will need to dig deeper into the mapping $\mathcal{A}$, beyond its monotonicity. In doing so, I shall construct a monotone sequence of functions, and show it converges in the L^1 sense to a fixed-point of $\mathcal{A}$.

Fix $w_0 := \mathbf{0}$ and recursively define $w_n = \mathcal{A}w_{n-1}$. Therefore, $\{w_n\} \subset \langle \mathbf{0}, \boldsymbol{\lambda}/r \rangle$ is an increasing sequence, order bounded from above, hence converges in L^1 to $w_\infty \in \langle \mathbf{0}, \boldsymbol{\lambda}/r \rangle$, where $w_\infty(\pi) = \sup_n w_n(\pi)$ for each $\pi \in [0,1]$ (because of the regularity of the $L^1_+[0,1]$ cone). The conceptual merit of this recursive construction is summarized in the following two points:

(i) Say a property $\star$ is owned by w_0 and is preserved by the mapping $\mathcal{A}$. Then, it holds along the sequence $\{w_n\}$.

(ii) If $\star$ is stable under the L^1 limit, then w_∞ holds this property.

Therefore, if $\mathcal{A}$ is L^1 continuous along the sequence $\{w_n\}$, then w_∞ becomes the fixed-point and the presumptive property $\star$ will be inherited to w_∞.

Proposition B.4. *For the sequence $\{w_n\}$ defined above, it holds that $\|\mathcal{A}w_n - \mathcal{A}w_\infty\|_{L^1} \to 0$, and as a result $w_\infty = \mathcal{A}w_\infty$.*

Proof. First note that for every $\pi \in [0,1]$, we have

$$\begin{aligned}
\mathcal{A}w_\infty(\pi) - \mathcal{A}w_n(\pi) &= \sup\left\{\frac{\int_B \mathcal{T}_q w_\infty(\pi)\phi(\mathrm{d}q)}{1+\frac{\kappa}{r}\phi(B)} : B \subseteq [a,b]\right\} \\
&\qquad - \sup\left\{\frac{\int_B \mathcal{T}_q w_n(\pi)\phi(\mathrm{d}q)}{1+\frac{\kappa}{r}\phi(B)} : B \subseteq [a,b]\right\} \\
&\le \sup\left\{\frac{\int_B \left(\mathcal{T}_q w_\infty - \mathcal{T}_q w_n\right)(\pi)\phi(\mathrm{d}q)}{1+\frac{\kappa}{r}\phi(B)} : B \subseteq [a,b]\right\} \\
&\le \int_0^1 \left(\mathcal{T}_q w_\infty - \mathcal{T}_q w_n\right)(\pi)\phi(\mathrm{d}q)\,,
\end{aligned}$$

where in the last line, I used the fact that $w_\infty \succsim w_n$ and the monotonicity of the operator $\mathcal{T}_q$. Therefore, the L^1-norm can be bounded above as:

$$\begin{aligned}
\|\mathcal{A}w_\infty - \mathcal{A}w_n\|_{L^1} &= \int_0^1 \big(\mathcal{A}w_\infty(\pi) - \mathcal{A}w_n(\pi)\big)\mathrm{d}\pi \\
&\le \int_0^1\int_0^1 \left(\mathcal{T}_q w_\infty - \mathcal{T}_q w_n\right)(\pi)\phi(\mathrm{d}q)\mathrm{d}\pi = \int_0^1 \|\mathcal{T}_q w_\infty - \mathcal{T}_q w_n\|_{L^1}\phi(\mathrm{d}q)\,.
\end{aligned}$$

For the last equality relation, I used the fact that the integrand is positive and uniformly bounded above by $\boldsymbol{\lambda}/r$ to apply Fubini's theorem and exchange the order of integrations.

Since the integrand of the last integral is uniformly bounded (over all $q \in [a,b]$), then one can apply the Lebesgue dominated convergence theorem to get:

$$\begin{aligned}\lim_{n\to\infty} \|\mathcal{A}w_\infty - \mathcal{A}w_n\|_{L^1} &\le \lim_{n\to\infty} \int_0^1 \|\mathcal{T}_q w_\infty - \mathcal{T}_q w_n\|_{L^1} \phi(\mathrm{d}q) \\ &= \int_0^1 \lim_{n\to\infty} \|\mathcal{T}_q w_\infty - \mathcal{T}_q w_n\|_{L^1} \phi(\mathrm{d}q)\,.\end{aligned}$$

Next, I propose a method to upper bound $(\mathcal{T}_q w_\infty - \mathcal{T}_q w_n)(\pi)$, and hence its L^1-norm. For this let G represent the random variable inside the expectation operator in the definition of $\mathcal{T}_q w$, then

$$\begin{aligned}\big(\mathcal{T}_q w_\infty - \mathcal{T}_q w_n\big)(\pi) &= \sup_\tau \mathsf{E}_\pi\left[G(\sigma, w_\infty; \tau)\right] - \sup_\tau \mathsf{E}\left[G(\sigma, w_n; \tau)\right] \\ &\le \sup_\tau \mathsf{E}_\pi\left[e^{-r(\sigma\wedge\tau)}\,(w_\infty - w_n)(\pi_{\sigma\wedge\tau})\right] \\ &\le \mathsf{E}_\pi\left[e^{-r\sigma}\,(w_\infty - w_n)\,(\pi_\sigma)\right] + \sup_\tau \mathsf{E}_\pi\left[e^{-r\tau}\,(w_\infty - w_n)\,(\pi_\tau); \tau < \sigma\right]\,.\end{aligned}$$

Therefore, the L^1-norm is bounded by

$$\begin{aligned}\|\mathcal{T}_q w_\infty - \mathcal{T}_q w_n\|_{L^1} \le{}& \underbrace{\int_0^1 \mathsf{E}_\pi\left[e^{-r\sigma}\,(w_\infty - w_n)\,(\pi_\sigma)\right] \mathrm{d}\pi}_{\mathcal{I}_1 :=} \\ &+ \underbrace{\int_0^1 \sup_\tau \mathsf{E}_\pi\left[e^{-r\tau}\,(w_\infty - w_n)\,(\pi_\tau); \tau < \sigma\right] \mathrm{d}\pi}_{\mathcal{I}_2 :=}\,.\end{aligned}$$

The integrands of both integrals are bounded by $\boldsymbol{\lambda}/r$, hence applying the Lebesgue dominated convergence theorem twice for the first integral implies

$$\lim_{n\to\infty} \mathcal{I}_1 = \int_0^1 \lim_{n\to\infty} \mathsf{E}_\pi\left[e^{-r\sigma}\,(w_\infty - w_n)\,(\pi_\sigma)\right] \mathrm{d}\pi = \int_0^1 \mathsf{E}_\pi\left[\lim_{n\to\infty} e^{-r\sigma}\,(w_\infty - w_n)\,(\pi_\sigma)\right] \mathrm{d}\pi = 0\,,$$

because w_∞ is the pointwise supremum of the sequence $\{w_n\}$. To show the convergence for the second integral, first note that for every given $\varepsilon > 0$ one can find $T > 0$ such that

$$\sup_\tau \mathsf{E}_\pi\left[e^{-r\tau}\,(w_\infty - w_n)\,(\pi_\tau); \tau < \sigma\right] \le \sup_{\tau \le T} \mathsf{E}_\pi\left[e^{-r\tau}\,(w_\infty - w_n)\,(\pi_\tau); \tau < \sigma\right] + \varepsilon\,,$$

uniformly over all π. This is owed to the uniform boundedness of $(w_\infty - w_n)$ by $\boldsymbol{\lambda}/r$. Next, because of the property of the supremum for every $\varepsilon > 0$, there exist $\tau_{n,\pi}$ (possibly depending on n and π) such that

$$\sup_{\tau \le T} \mathsf{E}_\pi\left[e^{-r\tau}\,(w_\infty - w_n)\,(\pi_\tau); \tau < \sigma\right] \le e^{-r\tau_{n,\pi}}\,(w_\infty - w_n)\,(\pi_{\tau_{n,\pi}})\mathsf{P}_\pi\,(\tau_{n,\pi} < \sigma) + \varepsilon\,.$$

Therefore,

$$\begin{aligned}\mathcal{I}_2 &\le \int_0^1 e^{-r\tau_{n,\pi}}\left(w_\infty - w_n\right)(\pi_{\tau_{n,\pi}})\mathsf{P}_\pi\left(\tau_{n,\pi} < \sigma\right)\mathrm{d}\pi + 2\varepsilon \\ &= \int_0^1 e^{-r\tau_{n,\pi}}\left(w_\infty - w_n\right)(\pi_{\tau_{n,\pi}})\left(\pi e^{-\bar{\lambda}_q\tau_{n,\pi}} + (1-\pi)e^{-\underline{\lambda}_q\tau_{n,\pi}}\right)\mathrm{d}\pi + 2\varepsilon\,.\end{aligned}$$

Because of the Bayes-law, $\pi_{\tau_{n,\pi}} = \frac{\pi e^{-\Delta_q\tau_{n,\pi}}}{1-\pi+\pi e^{-\Delta_q\tau_{n,\pi}}}$, where $\Delta_q := \bar{\lambda}_q - \underline{\lambda}_q$. Leveraging this relation and applying the change of variable to the above integral lead to

$$\begin{aligned}\mathcal{I}_2 - 2\varepsilon &\le \int_0^1 (w_\infty - w_n)(x)\frac{e^{(\bar{\lambda}_q - 2\underline{\lambda}_q - r)\tau_{n,x}}}{\left(1-x+xe^{\Delta_q\tau_{n,x}}\right)^3}\mathrm{d}x \\ &\le \int_0^1 (w_\infty - w_n)(x)e^{(\bar{\lambda}_q - 2\underline{\lambda}_q - r)\tau_{n,x}}\mathrm{d}x\,,\end{aligned} \tag{B.6}$$

where in the last inequality I used the fact that $1-x+xe^{\Delta_q\tau_{n,x}} \ge 1$. Since, $\tau_{n,x} \le T$ the last integrand in (B.6) is uniformly bounded for all x and n. Hence, one can apply the Lebesgue dominated convergence theorem again and obtain

$$\lim_{n\to\infty}\mathcal{I}_2 \le \int_0^1 \lim_{n\to\infty}(w_\infty - w_n)(x)e^{(\bar{\lambda}_q - 2\underline{\lambda}_q - r)\tau_{n,x}}\mathrm{d}x + 2\varepsilon = 2\varepsilon\,.$$

Since, this relation holds for every $\varepsilon > 0$, then $\lim_{n\to\infty}\mathcal{I}_2 = 0$. This establishes the L^1 convergence of $\mathcal{A}w_n$ to $\mathcal{A}w_\infty$ and thus proves $w_\infty = \mathcal{A}w_\infty$. □

A very important property owned by w_0 and preserved under $\mathcal{A}$ is being increasing in π. In the next lemma, using the techniques from the coupling of probability measures and stochastic dominance, I show that $\mathcal{A}w$ is increasing in π when w is.

Lemma B.5. *Let w be an increasing function in π, then $\mathcal{A}w$ becomes increasing in π as well.*

Proof. Fix q and suppose $\pi_2 \ge \pi_1$. Define the random variables

$$\sigma_i \overset{d}{=} \pi_i\mathsf{exp}(\bar{\lambda}_q) + (1-\pi_i)\mathsf{exp}(\underline{\lambda}_q), \quad i \in \{1,2\}\,, \tag{B.7}$$

as the exponential time of success arrivals under π_1 and π_2. One can easily check $\sigma_1 \succeq \sigma_2$ in the sense of the first order stochastic dominance (see the supplementary appendix). Therefore, for every *decreasing* function f we will have $\mathsf{E}[f(\sigma_2)] \ge \mathsf{E}[f(\sigma_1)]$. Recall the definition of $\mathcal{T}_q$:

$$\begin{aligned}\mathcal{T}_q w(\pi) &= \sup_\tau \mathsf{E}_\pi\left[G(\sigma;\tau)\right] \\ G(\sigma;\tau) &:= 1_{\{\sigma\le\tau\}}e^{-r\sigma} - c\int_0^{\sigma\wedge\tau} e^{-rs}\mathrm{d}s + e^{-r(\sigma\wedge\tau)}w(\pi_{\sigma\wedge\tau})\,.\end{aligned}$$

The first two terms in G are clearly decreasing in σ, so for every $q \in [a,b]$ and τ, one has

$$\mathsf{E}\left[1_{\{\sigma_2 \le \tau\}} e^{-r\sigma_2} - c\int_0^{\sigma_2 \wedge \tau} e^{-rs}\mathrm{d}s\right] \ge \mathsf{E}\left[1_{\{\sigma_1 \le \tau\}} e^{-r\sigma_1} - c\int_0^{\sigma_1 \wedge \tau} e^{-rs}\mathrm{d}s\right] . \tag{B.8}$$

The proof for monotonicity of the last term in G is a bit more tricky, because $\pi_{\sigma\wedge\tau}$ is not just a function of σ, but it also depends on the initial belief π. So, let us define $\boldsymbol{w}(\pi,\sigma;\tau) := e^{-r(\sigma\wedge\tau)}w(\pi_{\sigma\wedge\tau})$ where π is the initial belief. To proceed, I need to define σ_1 and σ_2 on the *same* probability space, because the analysis to be presented needs more than the application of the first order stochastic dominance. For this, I use the Strassen theorem (Lindvall (2002) chapter 4) to find the coupling $(\hat{\sigma}_1, \hat{\sigma}_2)$ such that $\hat{\sigma}_i \overset{d}{=} \sigma_i$ for $i = 1, 2$, and crucially $\hat{\sigma}_1 \ge \hat{\sigma}_2$ almost surely. It is proven in the supplementary section that for every τ, $\boldsymbol{w}$ is increasing in π and decreasing in σ (while holding π constant), therefore

$$\begin{aligned}
\mathsf{E}_{\pi_2}\left[e^{-r(\sigma_2\wedge\tau)}w(\pi_{\sigma_2\wedge\tau})\right] &= \mathsf{E}\left[\boldsymbol{w}(\pi_2, \hat{\sigma}_2; \tau)\right] && \text{(B.9a)}\\
&\ge \mathsf{E}\left[\boldsymbol{w}(\pi_1, \hat{\sigma}_2; \tau)\right] && \text{(B.9b)}\\
&\ge \mathsf{E}\left[\boldsymbol{w}(\pi_1, \hat{\sigma}_1; \tau)\right] && \text{(B.9c)}\\
&= \mathsf{E}_{\pi_1}\left[e^{-r(\sigma_1\wedge\tau)}w(\pi_{\sigma_1\wedge\tau})\right] . && \text{(B.9d)}
\end{aligned}$$

In (B.9a) and (B.9d), I used the fact that coupling preserves the marginal distributions. In (B.9b), I apply the increasing property of $\boldsymbol{w}$ in π, and in (B.9c) its decreasing property in σ.

Combining (B.8) and (B.9) implies that for every τ and $q \in [a,b]$, one has $\mathsf{E}_{\pi_2}[G(\sigma_2;\tau)] \ge \mathsf{E}_{\pi_1}[G(\sigma_1;\tau)]$. Therefore, applying the supremum on both sides (w.r.t. τ) yields $\mathcal{T}_q w(\pi_2) \ge \mathcal{T}_q w(\pi_1)$. From this and expression (B.3), it is now straightforward to conclude that $\mathcal{A}w(\pi_2) \ge \mathcal{A}w(\pi_1)$. □

Now we are in a position to establish the existence of a fixed-point that is increasing in π, the proof of which follows from the previous lemma and the fact that increasing property is closed under the L^1 limit.

Theorem B.6. *The operator $\mathcal{A}$ has an increasing fixed-point function.*

For a candidate increasing fixed-point w, we can now assure that if $w(\pi') > 0$ for some π', then $w(\pi'') > 0$ for all $\pi'' > \pi'$. This means once w exceeds zero it will never fall back to zero again, therefore the union of all matching sets over $q \in [a,b]$ must be an *increasing set* in $[0,1]$. That is

$$\bigcup_{q\in[a,b]} \mathcal{M}_q = \bigcup_{q\in[a,b]} \{\pi : \mathcal{T}_q w(\pi) > w(\pi)\} ,$$

is a connected subset of $[0,1]$, that includes $\pi = 1$. Hence, there exists an endogenous point α such that

$$\bigcup_{q\in[a,b]} \mathcal{M}_q = (\alpha, 1] \ .$$

Next, I show how α is determined. Its location is important because it represents the point of endogenous exit from the economy. In particular, an agent with a lower reputation than α would no longer accept projects. In the next proposition, I show under some natural assumptions, α is the boundary point of the stopping time problem that a typical agent solves when is matched to the *best* type of the projects, i.e., $q = b$. For this I present two notions. The profile of the intensity of success arrival, namely $\lambda = \{(\underline{\lambda}_q, \bar{\lambda}_q) : q \in [a,b]\}$, is called monotone if $\underline{\lambda}_q$ and $\bar{\lambda}_q$ are increasing in q. It satisfies the *increasing-differences* if $\bar{\lambda}_{q''} - \underline{\lambda}_{q''} \geq \bar{\lambda}_{q'} - \underline{\lambda}_{q'}$ for every $q'' > q'$ in $[a,b]$.

Proposition B.7. *Assume the profile λ is monotone and satisfies the increasing-differences. Then, α is the lowest boundary point of $\mathcal{M}_b$, and is the unique fixed-point of*

$$\alpha = \frac{c}{\Delta_b\left(1 + w\left(\frac{\bar{\lambda}_b\alpha}{\Delta_b\alpha+\underline{\lambda}_b}\right)\right)} - \frac{\underline{\lambda}_b}{\Delta_b} \, . \tag{B.10}$$

Proof. Assume by contradiction that α does not belong to the closure of $\mathcal{M}_b$ (i.e., $\alpha \notin \mathsf{cl}(\mathcal{M}_b)$), and there exists $q < b$ such that $\alpha = \inf \mathcal{M}_q$. That is an agent matched with a project of type q, terminates the match as her reputation nears α. The principles of optimality requires smooth and continuous fit at α, namely $v'(\alpha, q) = v(\alpha, q) = 0$. From the Bellman equation, for every $\pi \in \mathcal{M}_q$ it must be that

$$rv(\pi, q) = -c + \left(\bar{\lambda}_q\pi + \underline{\lambda}_q(1-\pi)\right)\left(1 + w \circ \jmath(\pi) - v(\pi, q)\right) - \pi(1-\pi)\Delta_q v'(\pi, q) \, .$$

In that $\jmath$ returns the posterior *after* the success has taken place at time t, namely:

$$\jmath(\pi_{t^-}) := \frac{\bar{\lambda}_q\pi_{t^-}}{\bar{\lambda}_q\pi_{t^-} + \underline{\lambda}_q(1-\pi_{t^-})} \, .$$

In the baseline model, the success event was *conclusive* thus $\jmath(\pi) = 1$ for every $\pi \in (0,1]$. The optimality principles at $\pi = \alpha$ imply

$$c = \left(\alpha\Delta_q + \underline{\lambda}_q\right)\left(1 + w \circ \jmath(\alpha)\right) . \tag{B.11}$$

Furthermore, since $\alpha \notin \mathsf{cl}(\mathcal{M}_b)$ then $v(\alpha, b) = w(\alpha) = 0$ and superharmonicity implies that

$$0 > \mathcal{L}_b v(\alpha, b) - rv(\alpha, b) - c = (\alpha\Delta_b + \underline{\lambda}_b)\big(1 + w \circ \jmath(\alpha)\big) - c \, .$$

Replacing (B.11) in the above inequality and canceling c from both sides amount to

$$0 > \frac{\alpha\Delta_b + \underline{\lambda}_b}{\alpha\Delta_q + \underline{\lambda}_q} - 1\,.$$

However the *rhs* of the above inequality is positive because of the monotonicity and increasing-differences, hence the contradiction is resulted. Therefore, it must be that $\alpha = \inf \mathcal{M}_b$.

On the uniqueness of α, note that the *lhs* of (B.10) is increasing in α, while the *rhs* is decreasing – because w is an increasing function. Therefore, α is *uniquely* determined by this equation. □

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B.1 Supplementary Proofs for Appendix B

Proof for $\sigma_1 \succeq \sigma_2$. For the two random variables defined in equation (B.7) we have

$$\mathsf{P}(\sigma_i > t) = \pi_i e^{-\bar{\lambda}_q t} + (1-\pi_i) e^{-\underline{\lambda}_q t}\,,$$

therefore,

$$\mathsf{P}(\sigma_1 > t) - \mathsf{P}(\sigma_2 > t) = (\pi_2 - \pi_1)\left(e^{-\underline{\lambda}_q t} - e^{-\bar{\lambda}_q t}\right) \geq 0\,,$$

because $\bar{\lambda}_q \geq \underline{\lambda}_q$ for every $q \in [a,b]$. Therefore, $\sigma_1 \succeq \sigma_2$, in the sense of FOSD.$\|$

Properties of the transformed function $\boldsymbol{w}$. Here I prove the properties claimed about the function $\boldsymbol{w}$, namely the fact that it is increasing in π (initial belief) and decreasing in σ (success arrival time).

Decreasing in σ. Remember that $\boldsymbol{w}(\pi,\sigma;\tau) := e^{-r(\sigma\wedge\tau)} w(\pi_{\sigma\wedge\tau})$. Fix the initial belief π (as well as τ and q), then $\boldsymbol{w}$ is constant on $[\tau,\infty)$. Further, it is decreasing on $[0,\tau]$, because $\bar{\lambda}_q \geq \underline{\lambda}_q$ so the posterior belief about $\{\theta = H\}$ falls more as the elapsed time to success gets longer. Formally, because of Bayesian learning $\mathrm{d}\pi_\sigma/\mathrm{d}\sigma \leq 0$. To verify this, let us decompose π_σ as

$$\pi_\sigma = \pi_{\sigma^-} + \Delta\pi_\sigma = \pi_{\sigma^-} + \frac{\bar{\lambda}_q - \underline{\lambda}_q}{\pi_{\sigma^-}\left(\bar{\lambda}_q - \underline{\lambda}_q\right) + \underline{\lambda}_q}\pi_{\sigma^-}(1-\pi_{\sigma^-})\,,$$

where the first term π_{σ^-} is the posterior belief just before the success arrival and the second term $\Delta\pi_\sigma$ is the amount that the posterior jumps up at the time of success. Define $\Delta_q := \bar{\lambda}_q - \underline{\lambda}_q \geq 0$, then again because of the Bayes law, one has:

$$\pi_{\sigma^-} = \frac{\pi e^{-\Delta_q \sigma}}{1-\pi+\pi e^{-\Delta_q\sigma}} \Rightarrow \frac{\mathrm{d}\pi_{\sigma^-}}{\mathrm{d}\sigma} = -\Delta_q \pi_{\sigma^-}(1-\pi_{\sigma^-}) < 0\,.$$

Differentiating $\Delta\pi_\sigma$ w.r.t. π_{σ^-} yields:

$$\frac{\mathrm{d}\Delta\pi_\sigma}{\mathrm{d}\pi_{\sigma^-}} = \frac{\Delta_q\Big((1-2\pi_{\sigma^-})(\pi_{\sigma^-}\Delta_q + \underline{\lambda}_q) - \pi_{\sigma^-}(1-\pi_{\sigma^-})\Delta_q\Big)}{(\pi_{\sigma^-}\Delta_q + \underline{\lambda}_q)^2}\,.$$

I can now use the previous two relations to take the total derivative of π_σ w.r.t. σ:

$$\begin{aligned}\frac{\mathrm{d}\pi_\sigma}{\mathrm{d}\sigma} &= \left(1 + \frac{\partial\Delta\pi_\sigma}{\partial\pi_{\sigma^-}}\right)\frac{\mathrm{d}\pi_{\sigma^-}}{\mathrm{d}\sigma} \\ &= \frac{\underline{\lambda}_q(\underline{\lambda}_q + \Delta_q)}{(\pi_{\sigma^-}\Delta_q + \underline{\lambda}_q)^2}\frac{\mathrm{d}\pi_{\sigma^-}}{\mathrm{d}\sigma} \leq 0\,.\end{aligned} \tag{B.12}$$

Lastly, for $\sigma \in [0, \tau]$,

$$\frac{\mathrm{d}\boldsymbol{w}}{\mathrm{d}\sigma} = -re^{-r\sigma} w(\pi_\sigma) + e^{-r\sigma} w'(\pi_\sigma) \frac{\mathrm{d}\pi_\sigma}{\mathrm{d}\sigma} \leq 0\,,$$

because of (B.12) and the fact that w is assumed increasing on $[0, 1]$ and hence is a.e. differentiable with positive derivative. Therefore, $\boldsymbol{w}$ becomes decreasing in σ.$\|$

Increasing in π. To show that $\boldsymbol{w}$ is increasing in π, I must hold σ fixed, thus it remains to show $w(\pi_{\sigma\wedge\tau})$ is increasing in the initial belief π. It is pretty straightforward to show that the posterior belief at any time, for the Poissonian environment that we have, is increasing in the initial belief, hence the proof readily follows from the increasing property of w.$\|$