Machine Learning for Large-Scale Data Analysis and Decision Making (MATH80629A)
Winter 2022

Week #3 - **Summary**

Announcement

- **Practical Lab session** on week #4 (January 26)
- TA virtual office hour is on **Fridays at 1-2 pm**.
  Zoom link
  Meeting ID: 899 6153 5982
  Passcode: 290256
- Team Registration, due: **January 26, 2022**
  Fill the **form.**
- Looking for team-members, check the shared **spreedsheet**.
- You can also post on **Piazza.**

Today

- **Second Quiz** on Gradescope!
- Summary of Machine learning fundamental
- Q&A
- Hands-on session

Quiz 1

Login to your Gradescope account

Models for supervised learning

- (Mostly) linear models
- Focus on **classification**

1. **Non-Probabilistic Models**
   - Nearest Neighbor, Support Vector Machines (SVMs)
2. **Probabilistic Models**
   - Naive Bayes

Supervised learning

Train Data

$$\underbrace{\begin{array}{c} \\ x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array}\begin{array}{c}\text{Nb.bed.} \quad \text{Area} \quad \text{Neigh.} \quad \cdot \quad \cdot \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 100 & 1 & .2 & .5 \\ 3 & 200 & 0 & .1 & .2 \\ 1 & 150 & 1 & .4 & .1 \\ 2 & 210 & 2 & .5 & 1.1 \end{bmatrix}\end{array}}_{X} \quad \underbrace{\begin{array}{c} \\ y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{array}\begin{array}{c}\text{Sell-ability} \\ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}\end{array}}_{Y}$$

Task

$$f : \mathbb{R}^n \to \{0, 1, 2\}$$

Test Data

$$\underbrace{\begin{array}{c} \\ x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array}\begin{array}{c}\text{Nb.bed.} \quad \text{Area} \quad \text{Neigh.} \quad \cdot \quad \cdot \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 50 & 1 & .3 & .8 \\ 1 & 100 & 1 & .5 & 1.4 \\ 4 & 170 & 0 & .7 & .4 \\ 1 & 120 & 3 & .9 & .5 \end{bmatrix}\end{array}}_{X^{new}} \quad \underbrace{\begin{array}{c} \\ y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{array}\begin{array}{c}\text{Sell-ability} \\ \begin{bmatrix} ? \\ ? \\ ? \\ ? \\ ? \end{bmatrix}\end{array}}_{Y^{new}}$$

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Supervised learning

Train Data

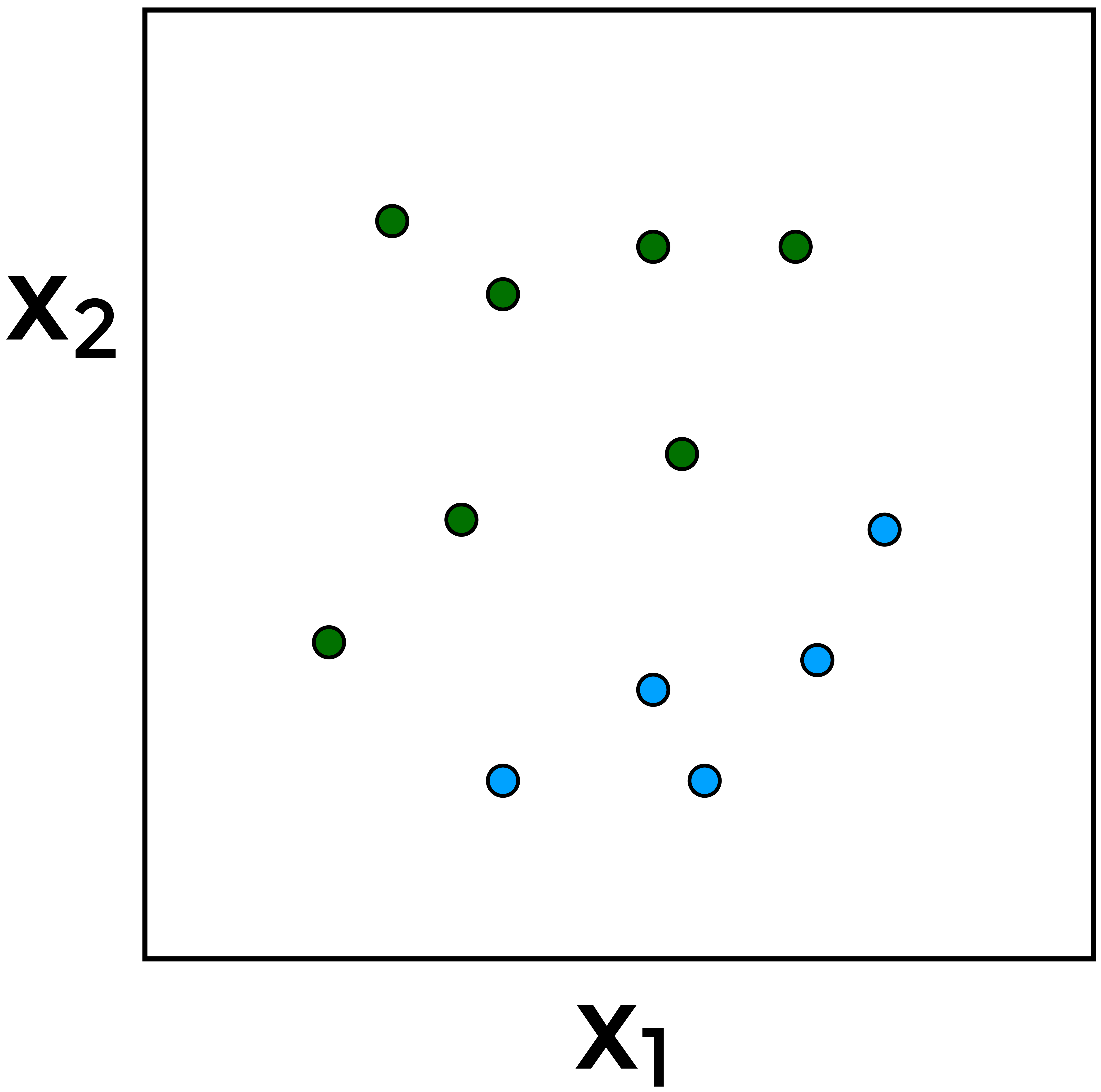
$$\underbrace{\begin{array}{c c} & \begin{array}{ccccc} \text{Nb.bed.} & \text{Area} & \text{Neigh.} & \cdot & \cdot \end{array} \\ \begin{array}{c} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 100 & 1 & .2 & .5 \\ 3 & 200 & 0 & .1 & .2 \\ 1 & 150 & 1 & .4 & .1 \\ 2 & 210 & 2 & .5 & 1.1 \end{bmatrix} \end{array}}_{X} \quad \underbrace{\begin{array}{c c} & \text{Sell-ability} \\ \begin{array}{c} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{array} & \begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \\ 1 \end{bmatrix} \end{array}}_{Y}$$

Task

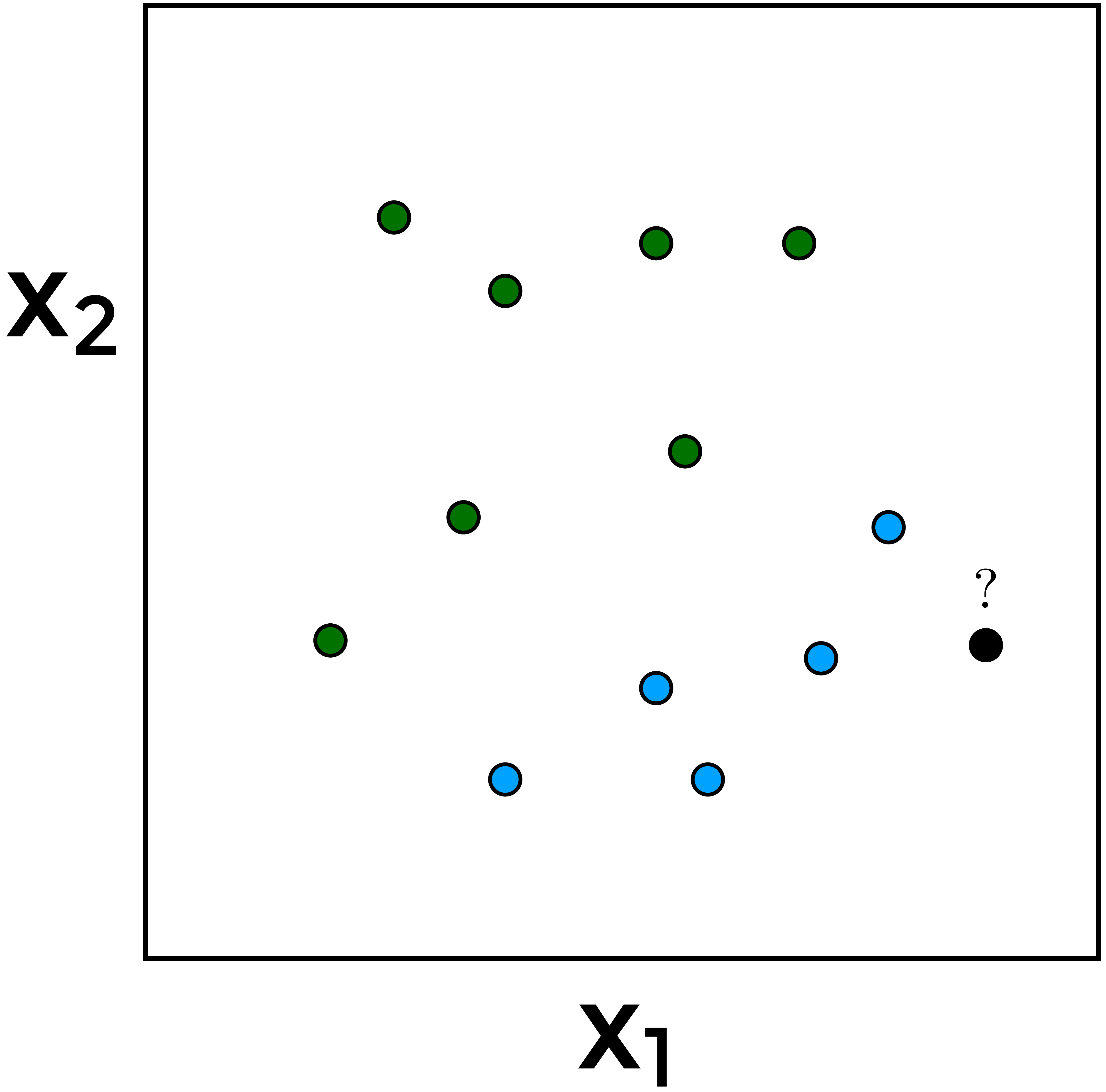
Models $\boxed{f}$ $\mathbb{R}^n \rightarrow \{0, 1, 2\}$

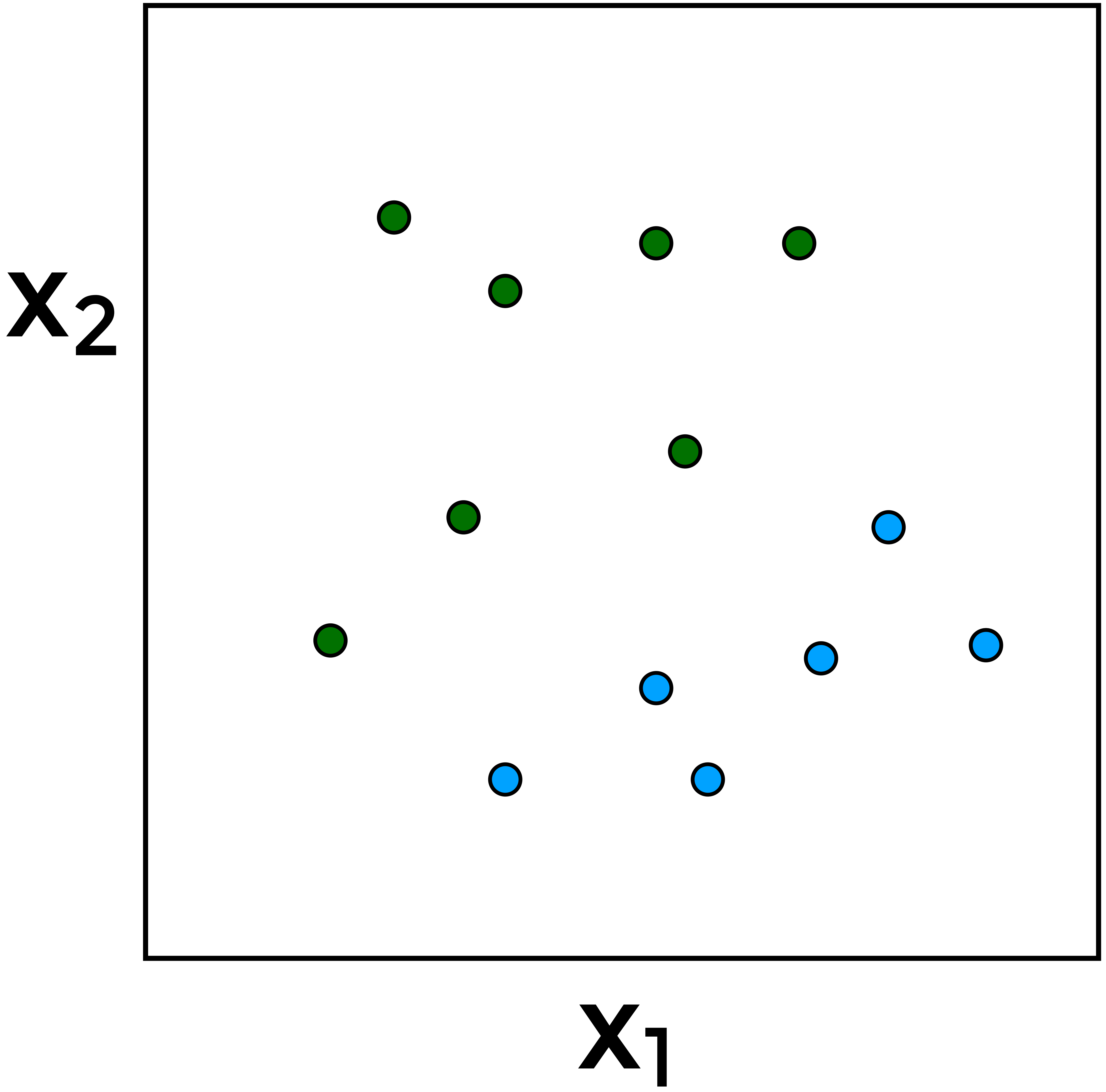
Test Data

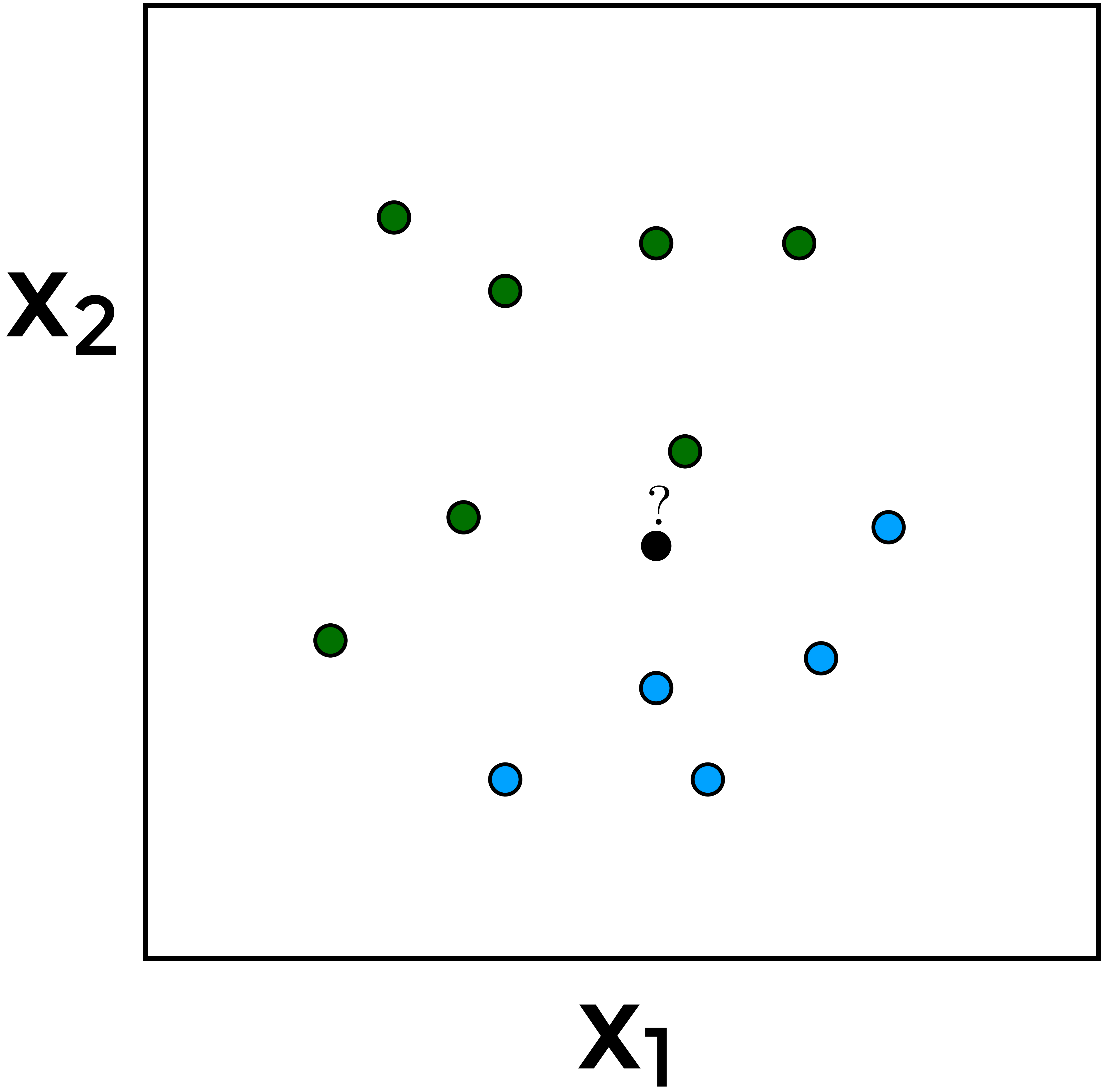
$$\underbrace{\begin{array}{c c} & \begin{array}{ccccc} \text{Nb.bed.} & \text{Area} & \text{Neigh.} & \cdot & \cdot \end{array} \\ \begin{array}{c} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 50 & 1 & .3 & .8 \\ 1 & 100 & 1 & .5 & 1.4 \\ 4 & 170 & 0 & .7 & .4 \\ 1 & 120 & 3 & .9 & .5 \end{bmatrix} \end{array}}_{X^{new}} \quad \underbrace{\begin{array}{c c} & \text{Sell-ability} \\ \begin{array}{c} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{array} & \begin{bmatrix} ? \\ ? \\ ? \\ ? \\ ? \end{bmatrix} \end{array}}_{Y^{new}}$$



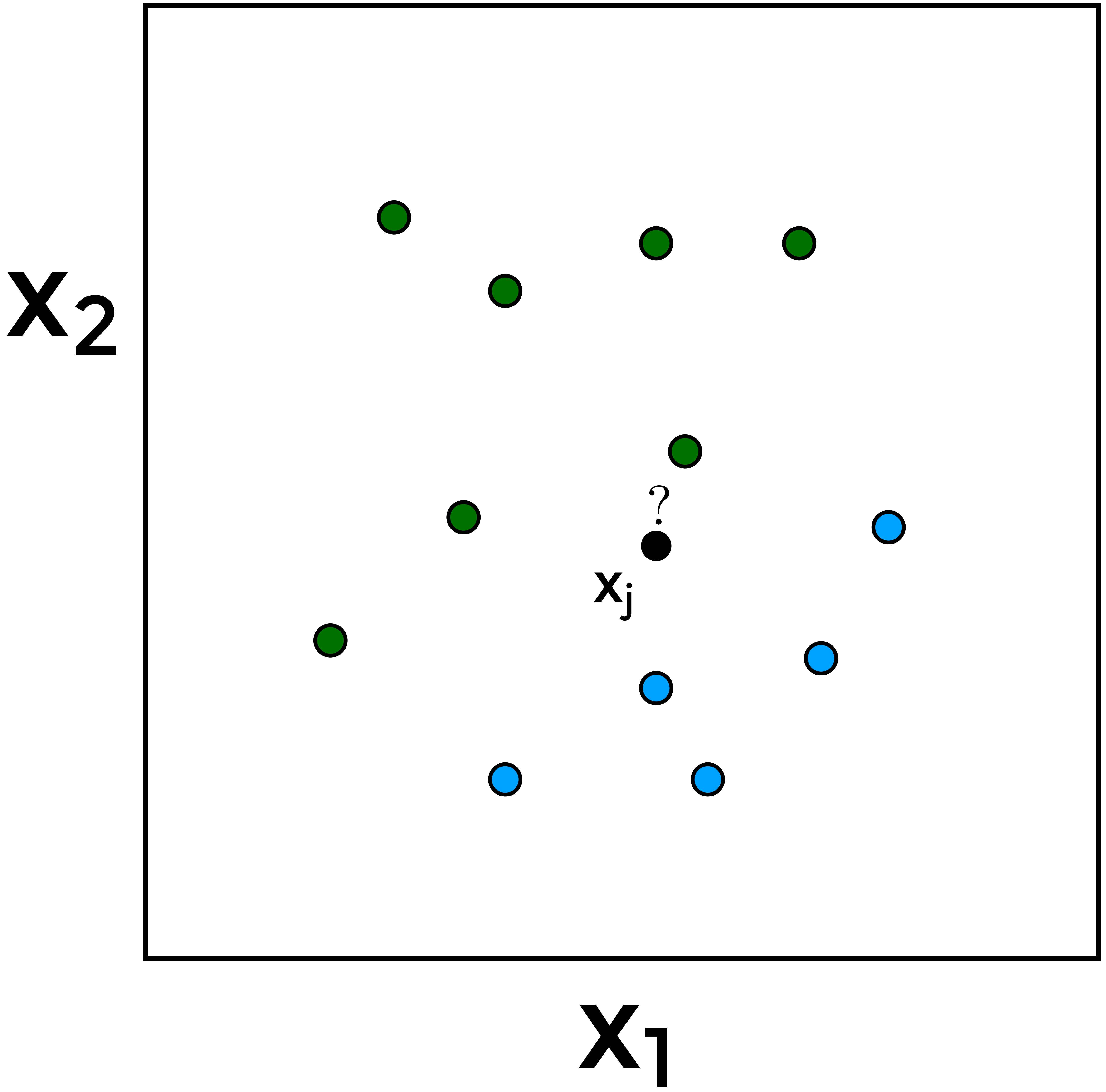
x_2
x_1


?
X_2
X_1


x_2
x_1

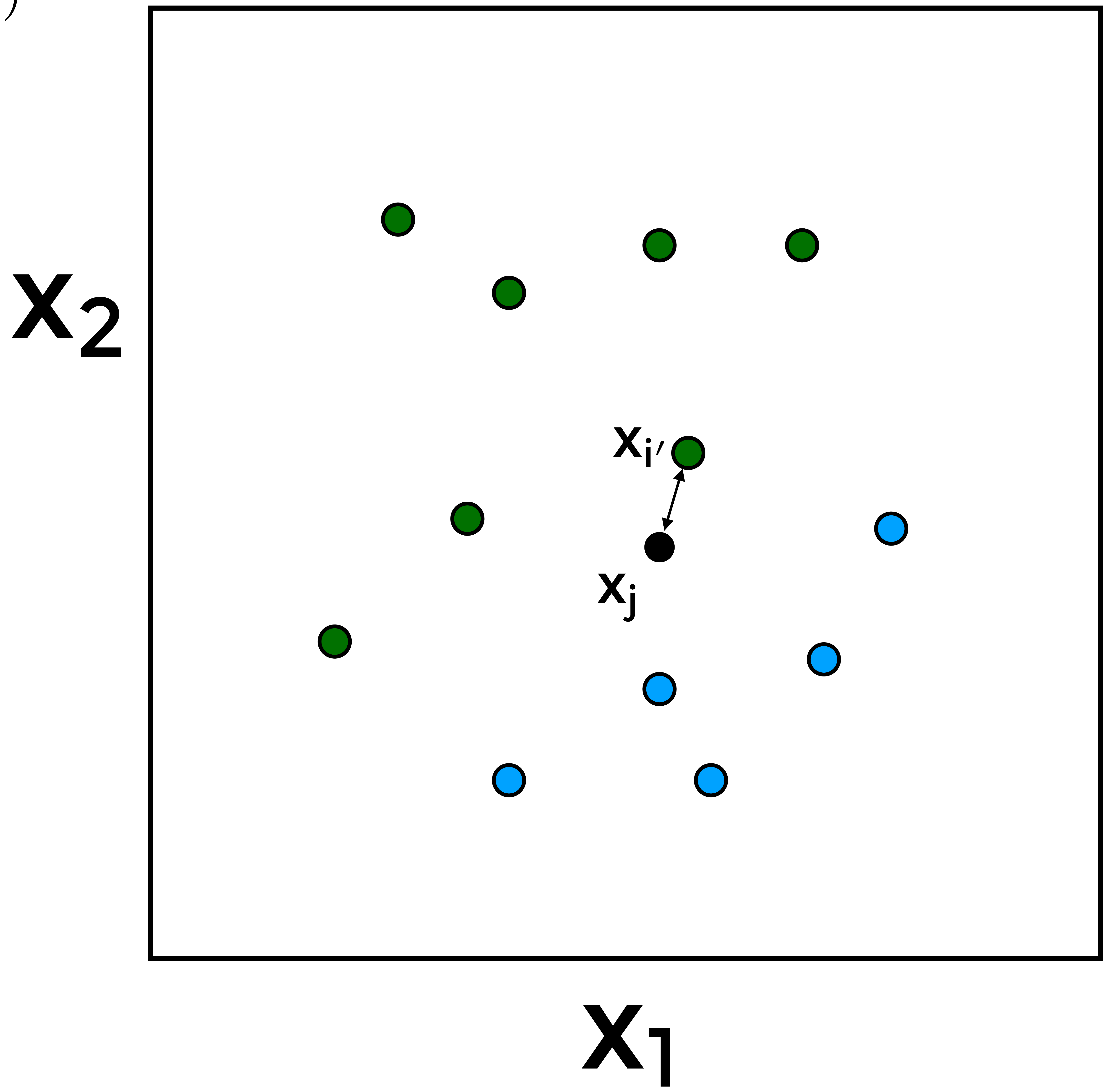


X_2
?
X_1


x_2
?
x_j
x_1

$$i' = \arg\min_i \mathsf{dist}(x_i, x_j)$$

$$y_j = y_{i'}$$



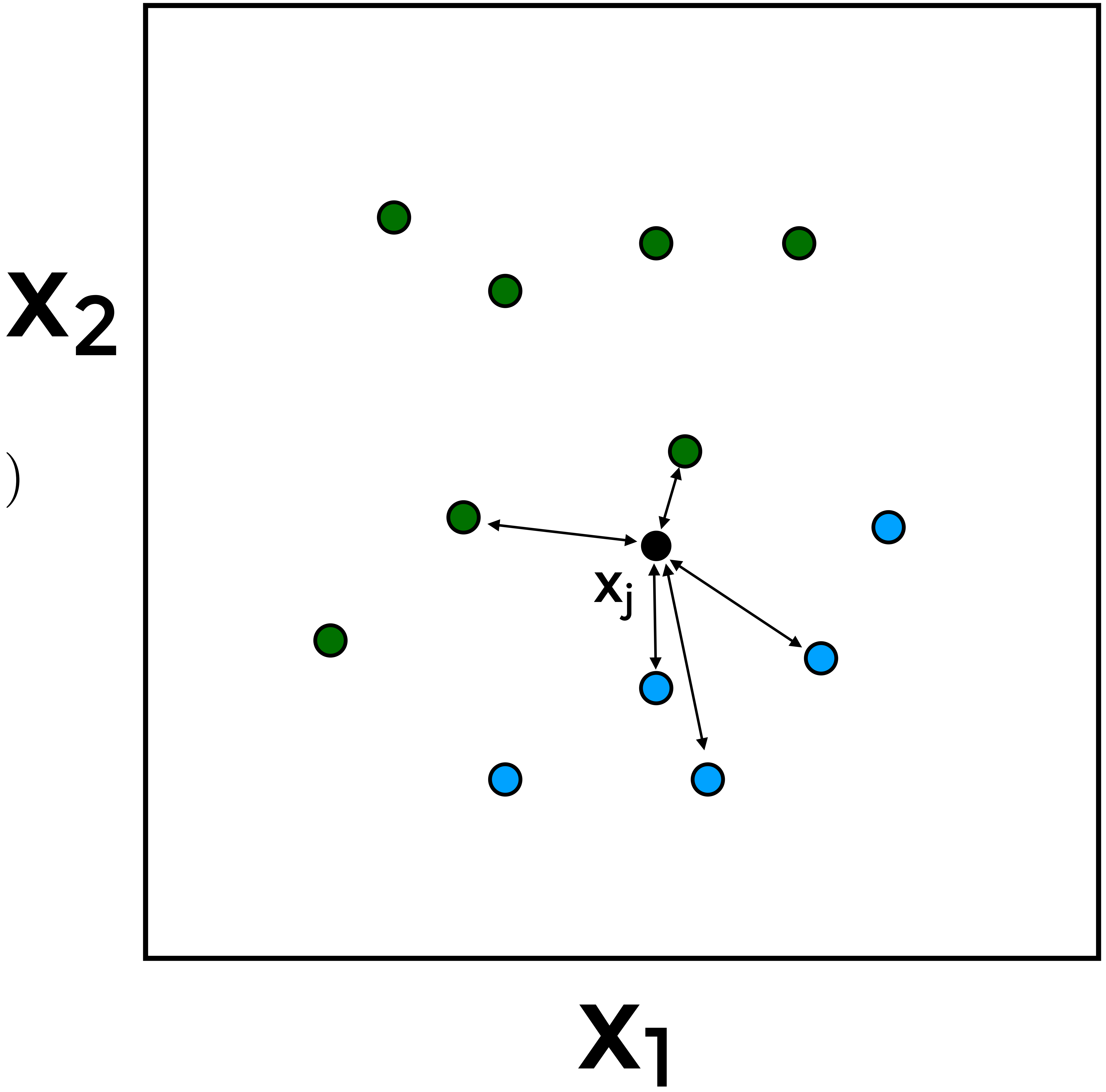
- **1-NN**

Instance classified according to its nearest neighbor

$k = 5$ (assumption)

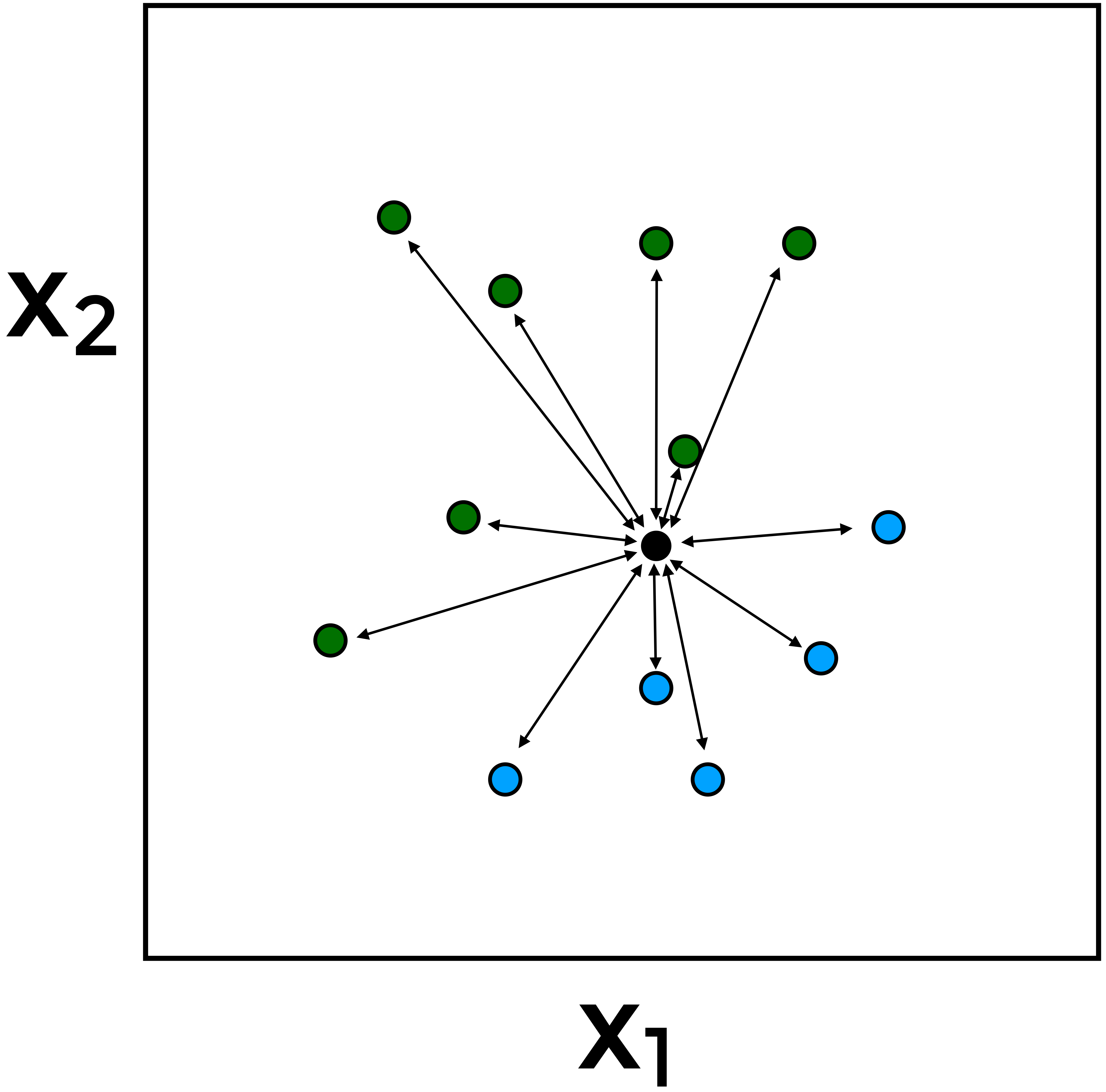
$i = \arg\operatorname{sort}_i \operatorname{dist}(x_i, x_j)$

$y_j = \operatorname{majority}(i_{:5})$



- **K-NN**

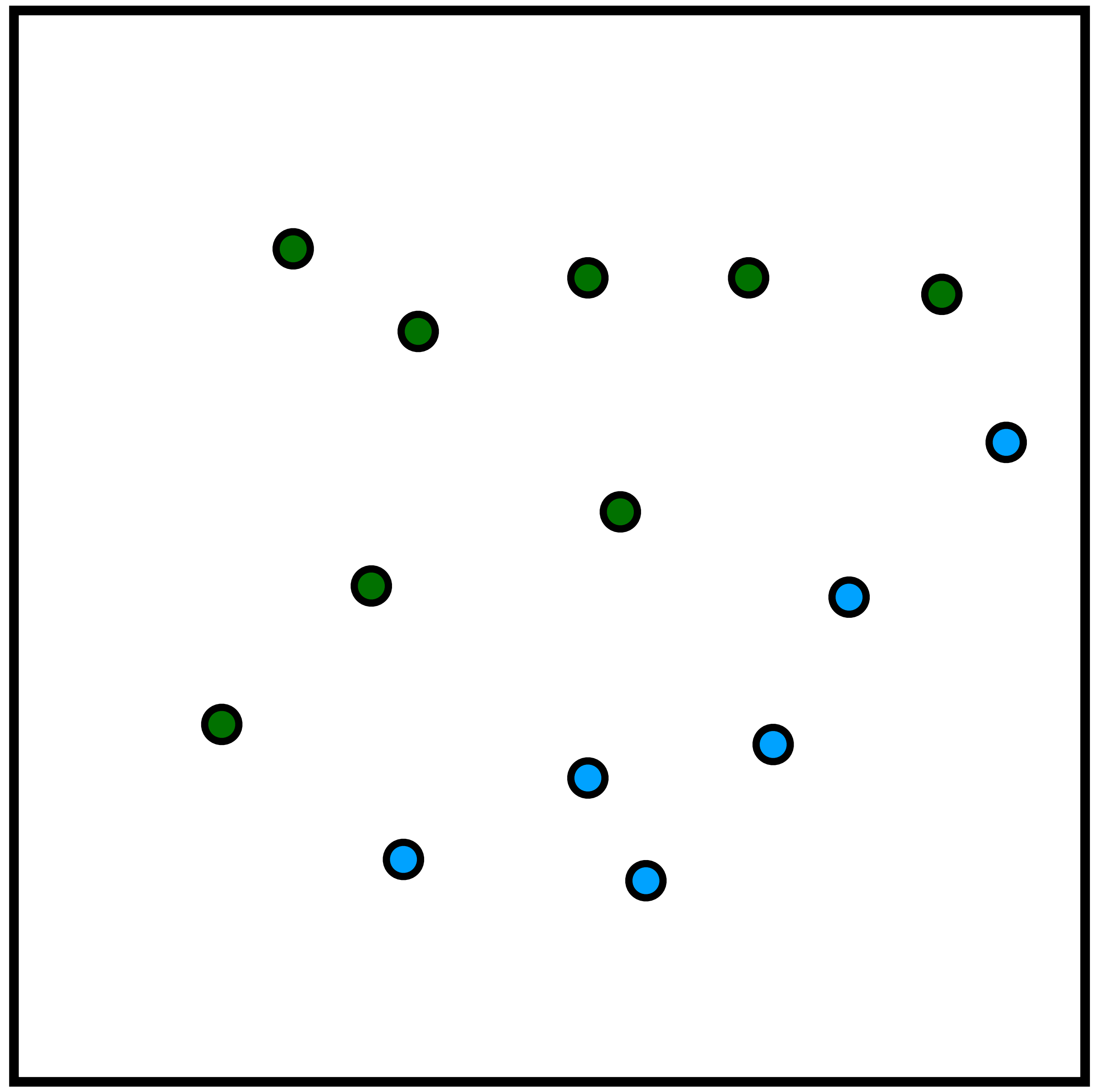
Instance classified according to the majority of its K nearest neighbors



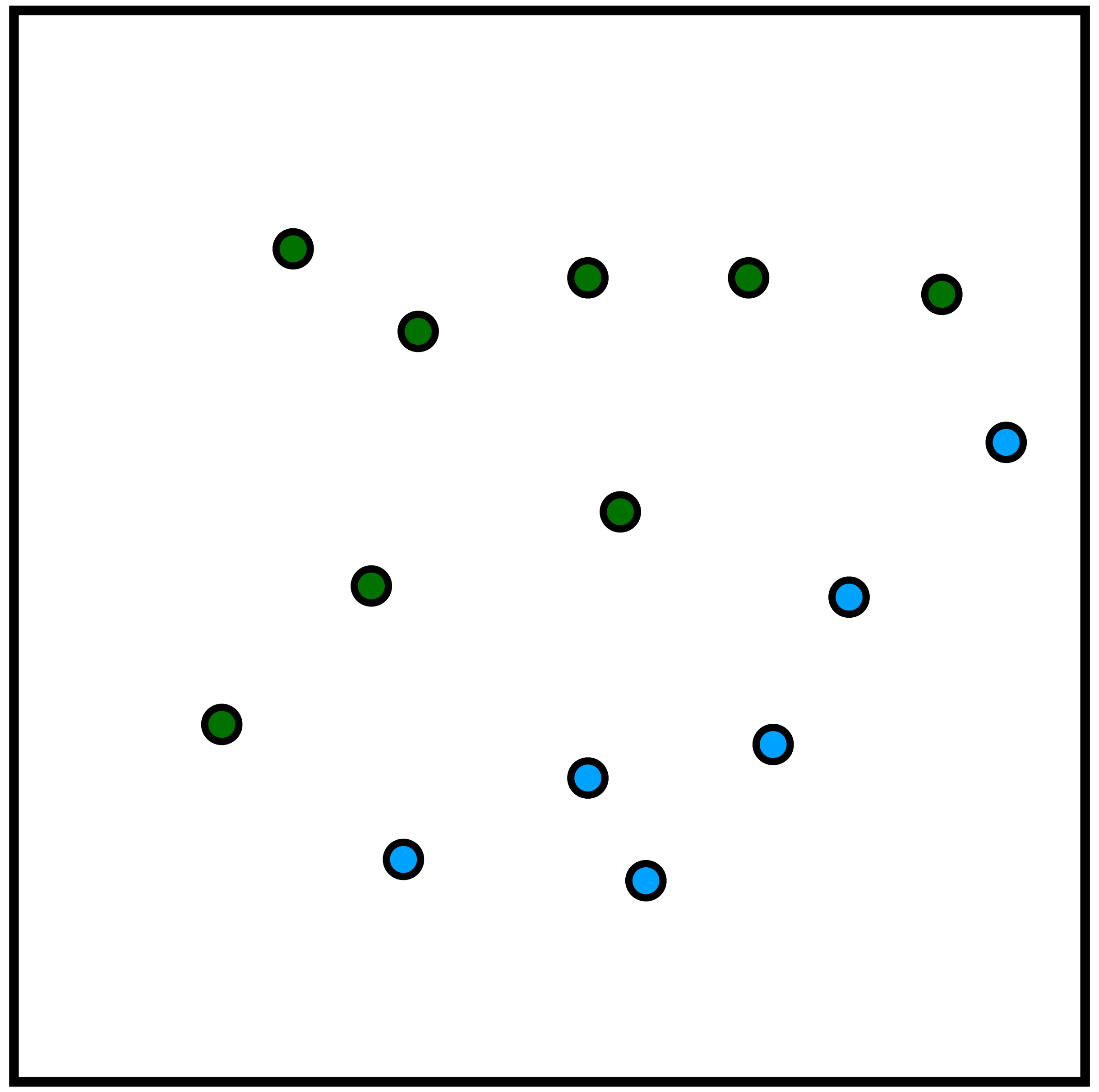
- **weighted-NN**

Instance classified according to all neighbors. The contribution of each neighbor is weighted by its distance.

Linear Classification

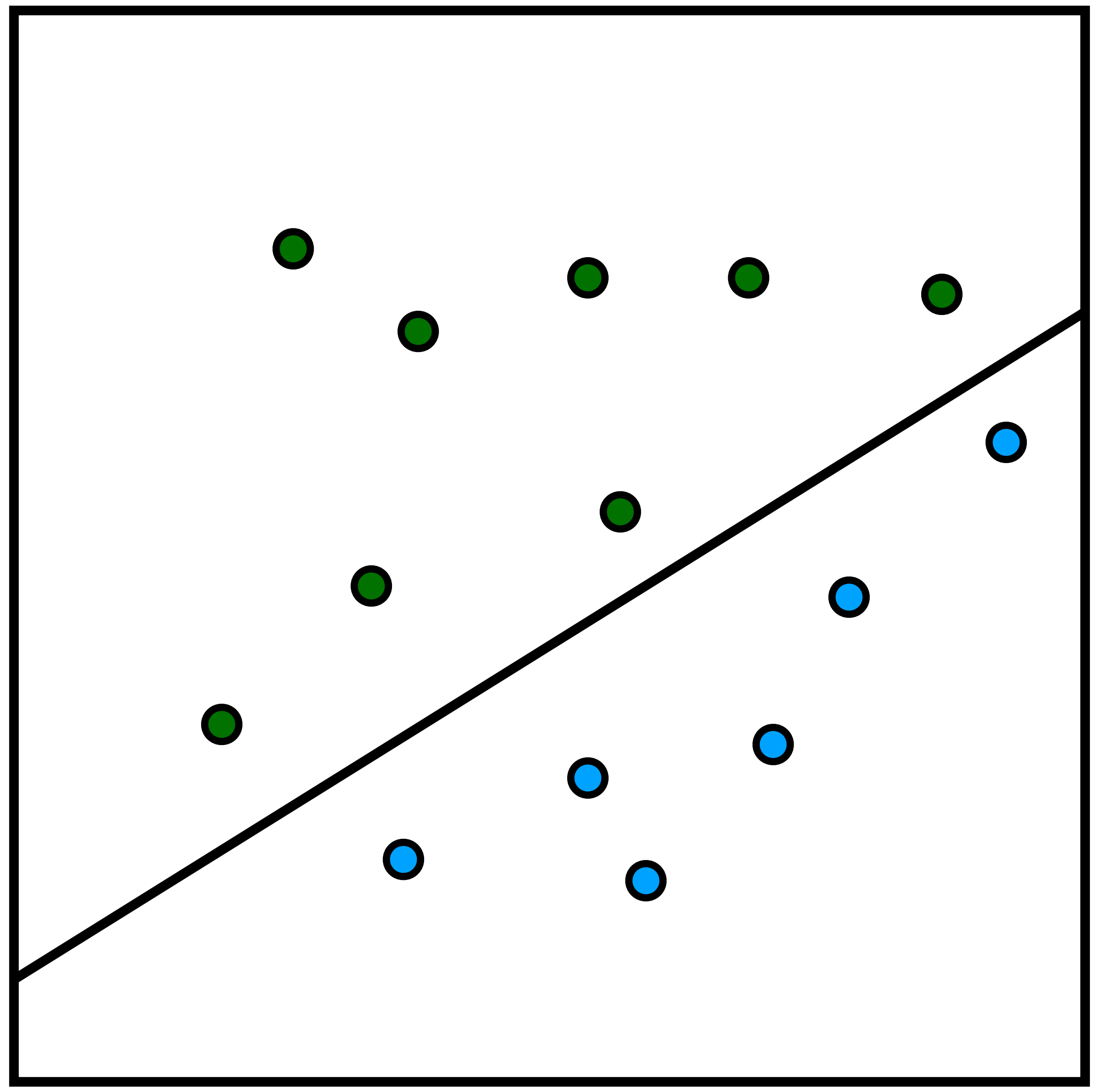


Linear Classification



$$\mathbf{y}(\mathbf{x}) = \mathbf{w}^\top \mathbf{x} + \mathbf{w}_0$$

Linear Classification



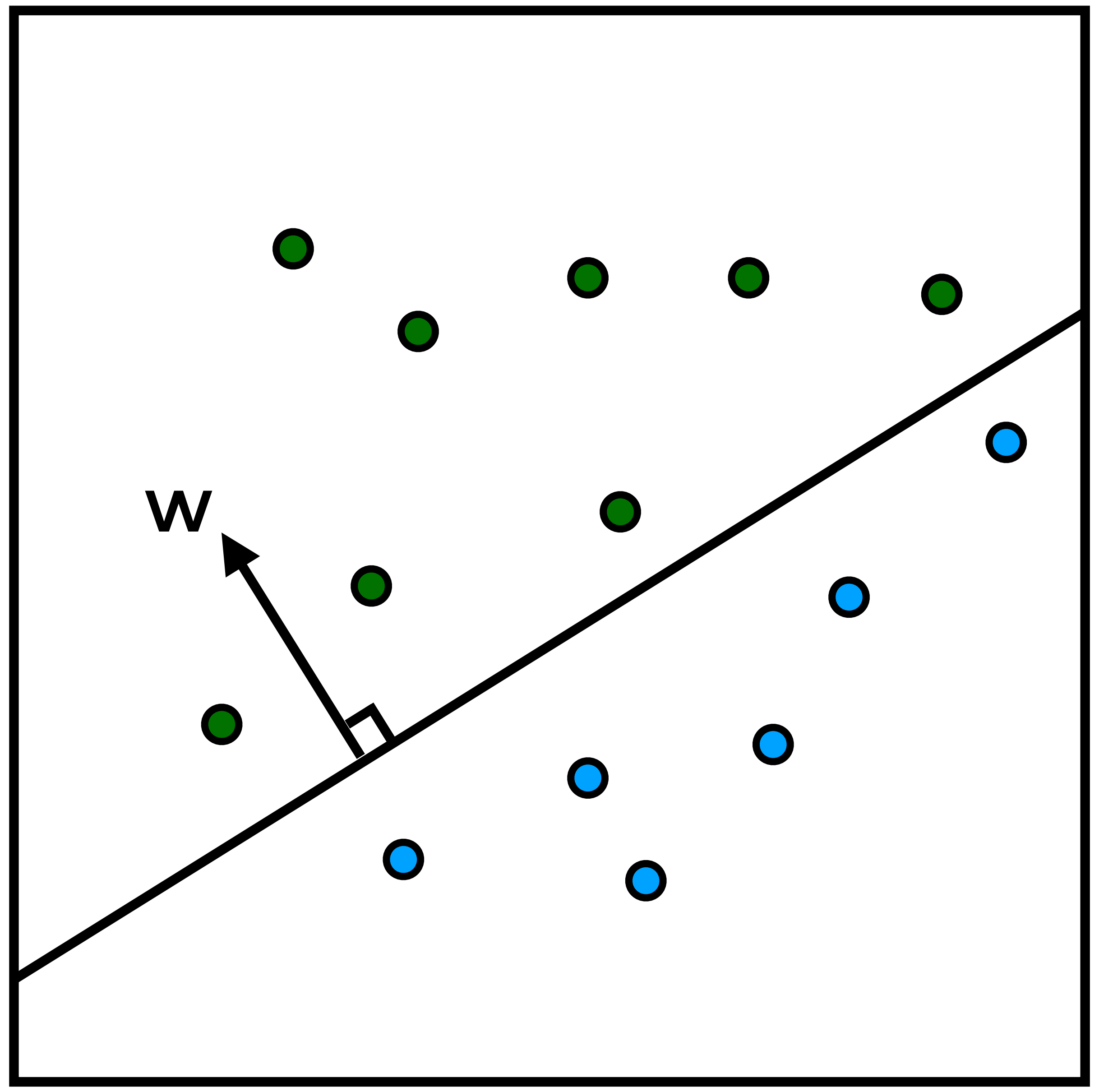
$$y(\mathbf{x}) = \mathbf{w}^\top \mathbf{x} + w_0$$

Decision

$$(\mathbf{w}^\top \mathbf{x} + w_0) > 0 \implies \text{green}$$

$$(\mathbf{w}^\top \mathbf{x} + w_0) < 0 \implies \text{blue}$$

Linear Classification



decision boundary: $y(\mathbf{x}) = 0$

take two points on the boundary: $\mathbf{x}_a, \mathbf{x}_b$

then: $\mathbf{w}^\top \mathbf{x}_a + w_0 = \mathbf{w}^\top \mathbf{x}_b + w_0$

$\implies \mathbf{w}^\top (\mathbf{x}_a - \mathbf{x}_b) = 0$

$\implies$ $\mathbf{w}$ is perpendicular to the decision boundary

$\mathbf{w}$ represents the orientation of the decision boundary

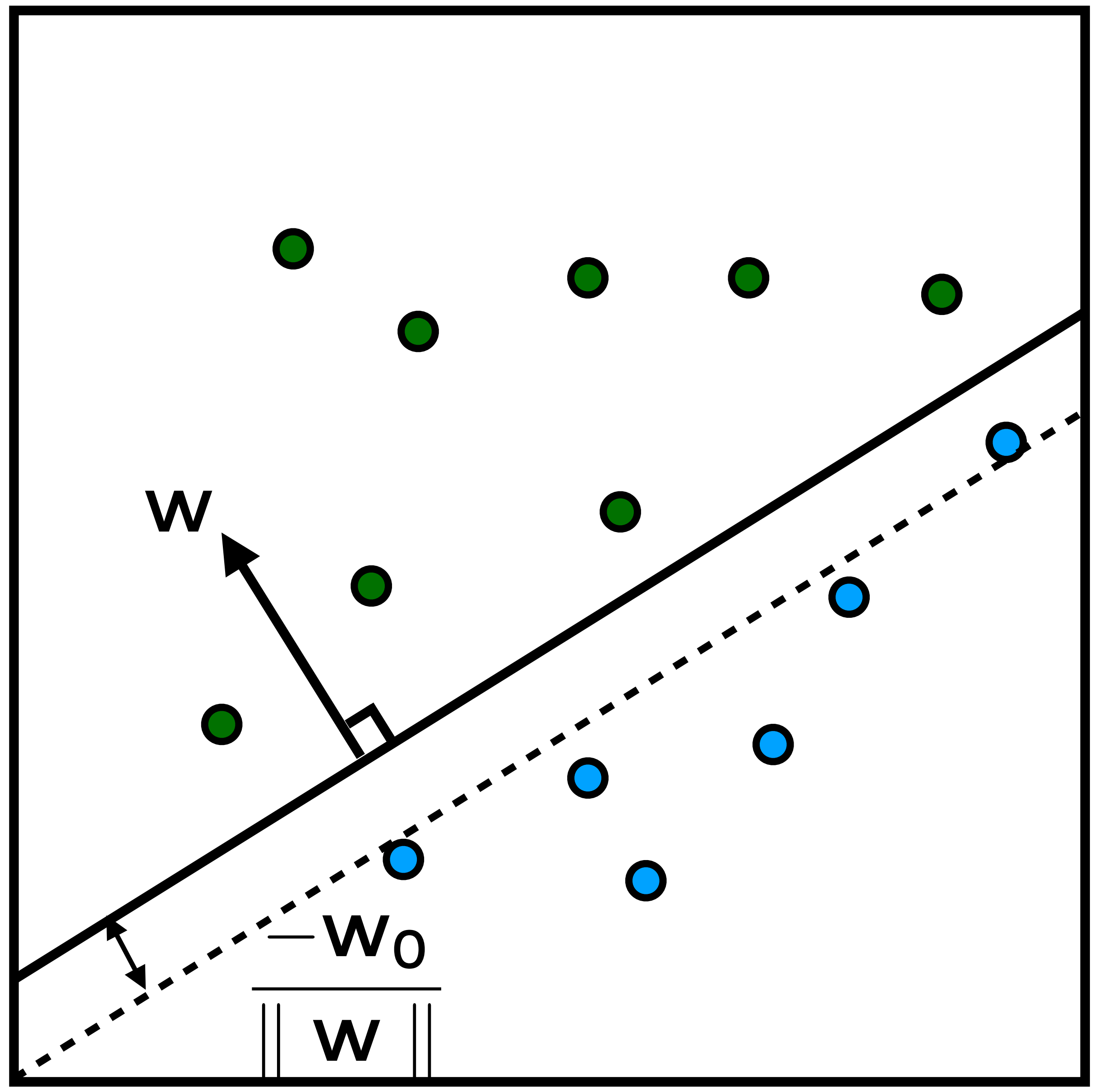
$$y(\mathbf{x}) = \mathbf{w}^\top \mathbf{x} + w_0$$

Decision

$$(\mathbf{w}^\top \mathbf{x} + w_0) > 0 \implies \text{green}$$

$$(\mathbf{w}^\top \mathbf{x} + w_0) < 0 \implies \text{blue}$$

Linear Classification



$$y(\mathbf{x}) = \mathbf{w}^\top \mathbf{x} + w_0$$

Decision

$$(\mathbf{w}^\top \mathbf{x} + w_0) > 0 \implies$$ (green)

$$(\mathbf{w}^\top \mathbf{x} + w_0) < 0 \implies$$ (blue)

w_0 is a scalar

you can think of it like an intercept

take $\mathbf{x}'$ as the closest point on the decision boundary to the origin

$\mathbf{x}' = \beta \mathbf{w}$

$$\implies y(\mathbf{x}') = \mathbf{w}^\top \mathbf{x}' + w_0$$

$$\implies y(\mathbf{x}') = \mathbf{w}^\top (\beta \mathbf{w}) + w_0$$

$$\implies 0 = \beta \| \mathbf{w} \|^2 + w_0$$

$$\implies \beta = \frac{-w_0}{\| \mathbf{w} \|^2}$$

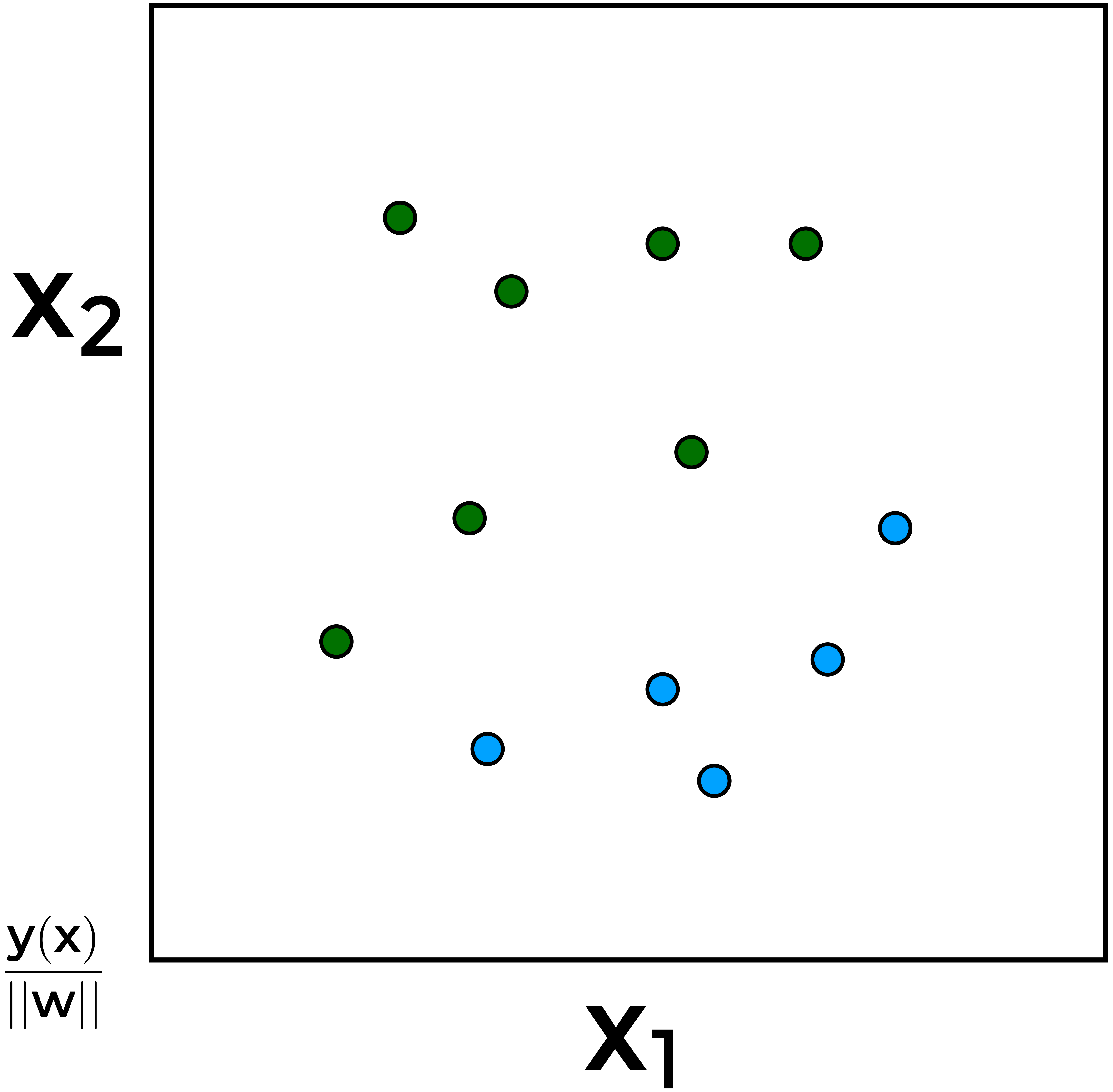
Then you know that the distance from the origin to $\mathbf{x}'$ is:

$$\| \mathbf{x}' \| = \| \beta \mathbf{w} \|$$

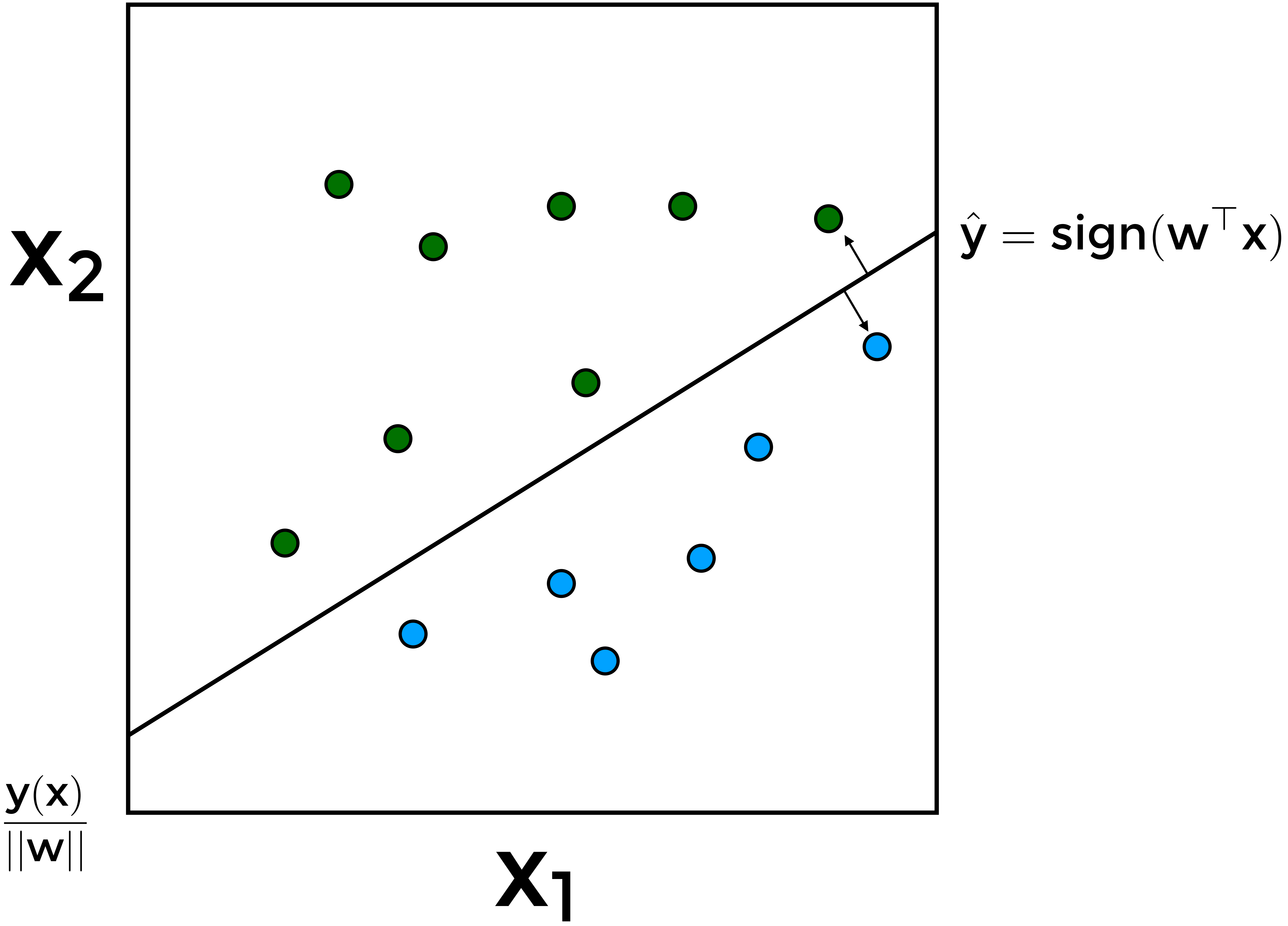
$$\implies \| \mathbf{x}' \| = \beta \| \mathbf{w} \|$$

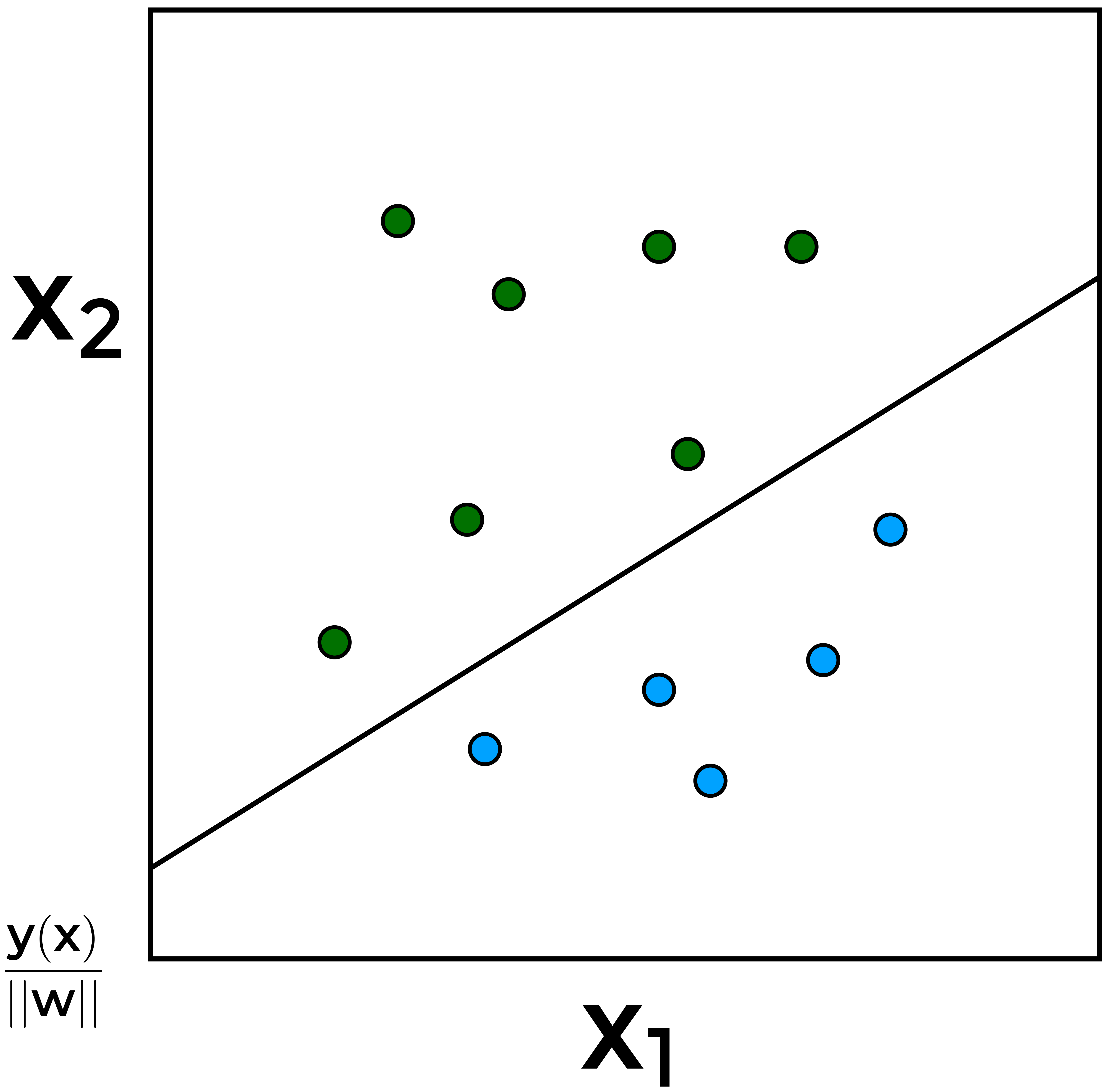
$$\implies \| \mathbf{x}' \| = \frac{-w_0}{\| \mathbf{w} \|^2} \| \mathbf{w} \|$$

$$\implies \| \mathbf{x}' \| = \frac{-w_0}{\| \mathbf{w} \|}$$

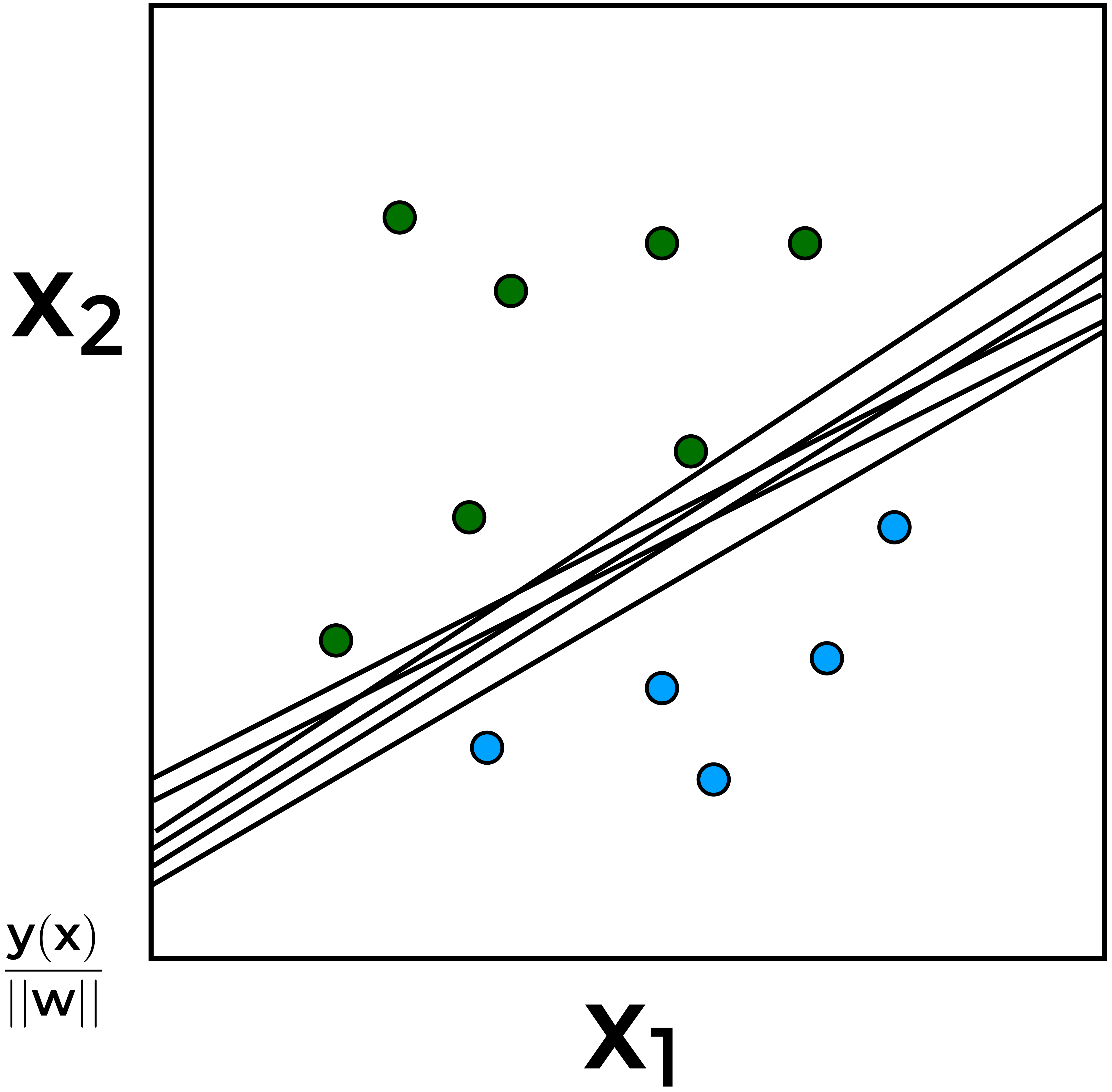


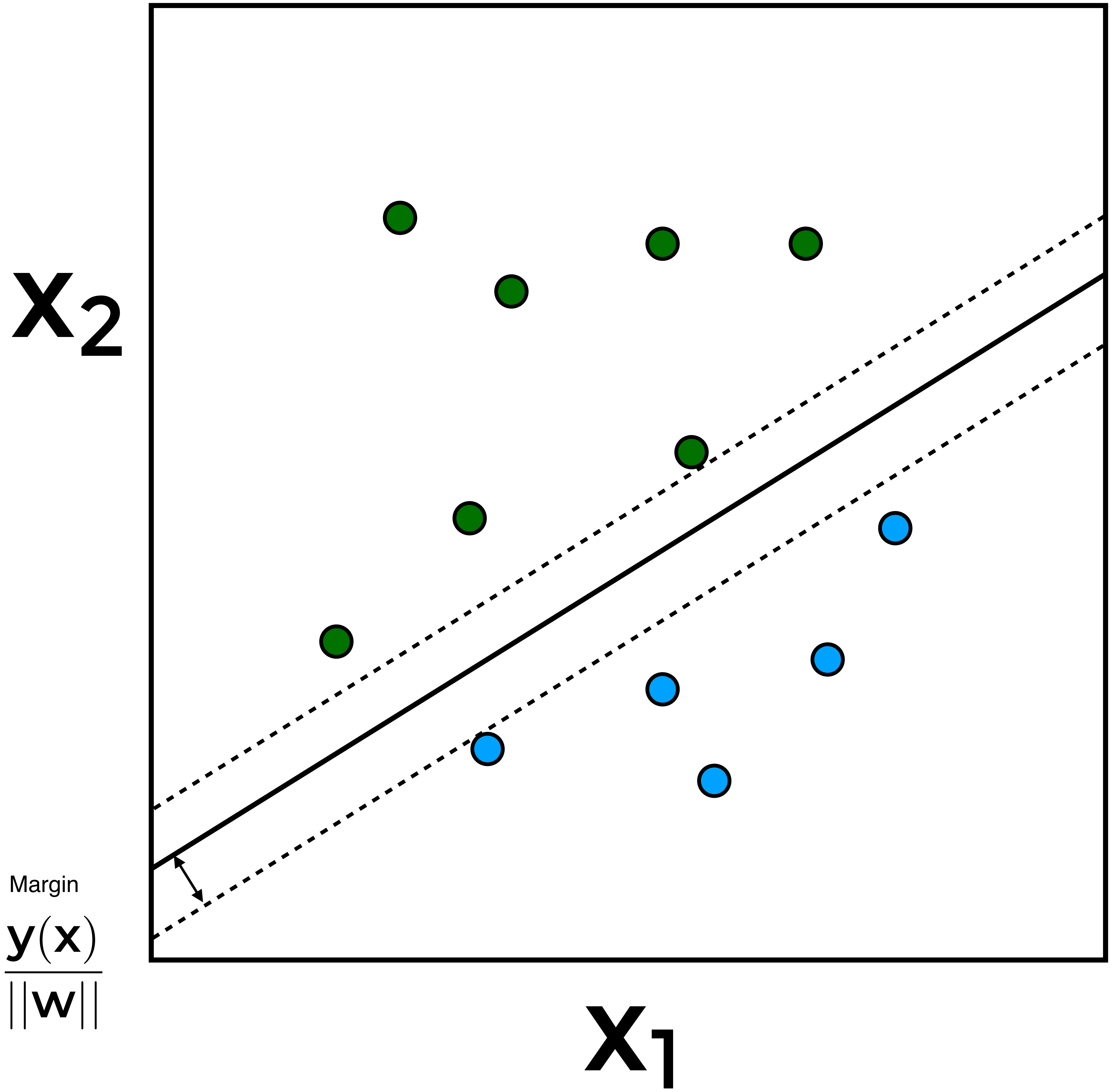
$$\frac{y(x)}{||w||}$$


$\mathbf{x}_2$
$\hat{y} = \text{sign}(\mathbf{w}^\top \mathbf{x})$
$\frac{y(\mathbf{x})}{||\mathbf{w}||}$
$\mathbf{x}_1$

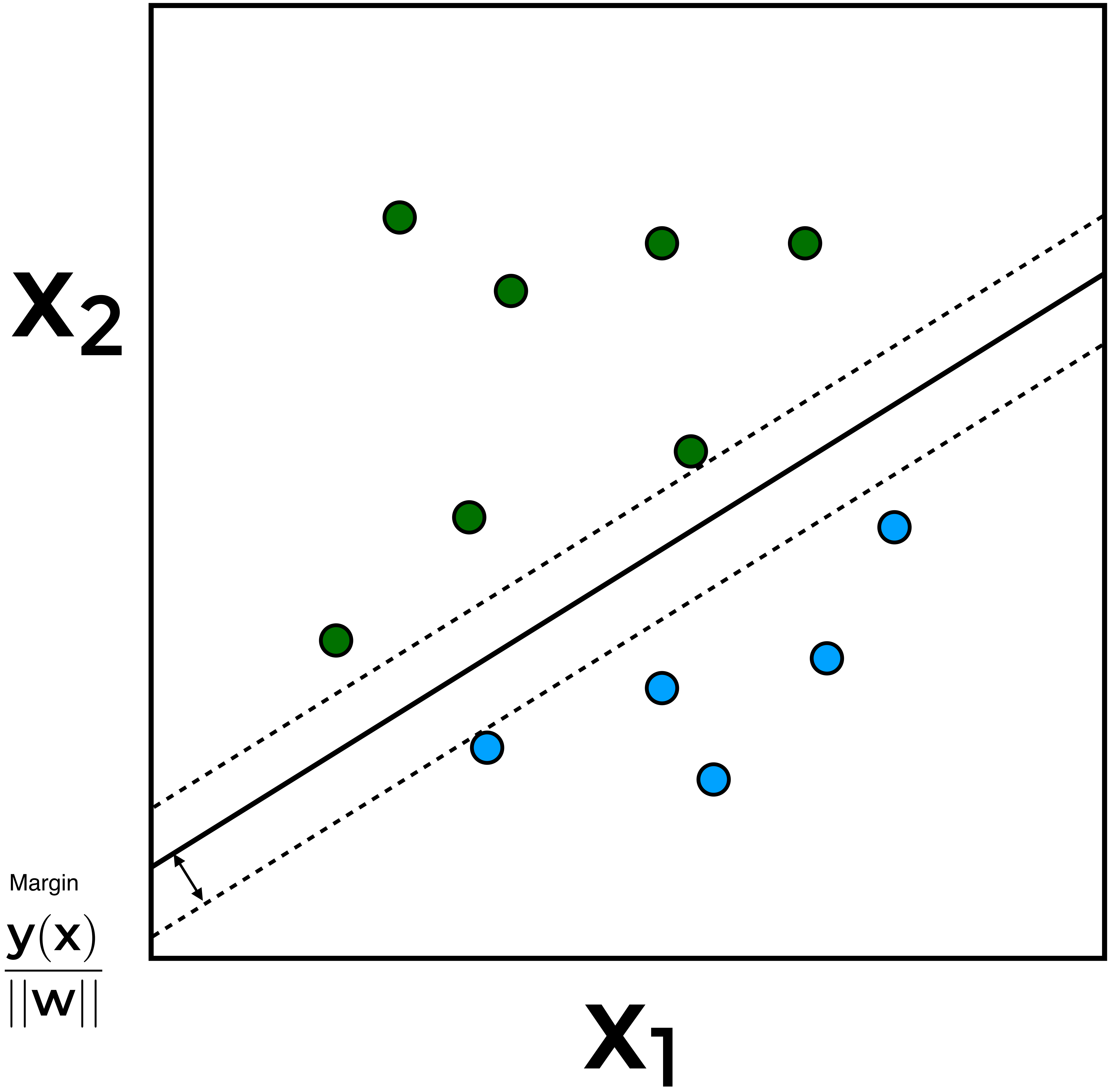


x_2
x_1
$\frac{y(x)}{||w||}$


x_2
x_1
$\frac{y(x)}{||w||}$


x_2
x_1
Margin
$\frac{y(x)}{||w||}$

The objective is to find the separating boundary that maximizes the margin



Probabilistic Models for Classification

Probabilistic Models separate Decision and Inference

Non-Probabilistic Modelling

$\mathbf{x} \longrightarrow \boxed{\text{Model}} \longrightarrow \mathbf{y}$

Probabilistic Modelling

$\mathbf{x} \longrightarrow \boxed{\text{Probabilistic Model}} \longrightarrow \mathbf{P}(\mathbf{y} = \mathbf{k}|\mathbf{x}) \longrightarrow \boxed{\text{Decision Rule}} \longrightarrow \mathbf{y}$

Probabilistic models

1. **Model the conditional directly:**

$$P(y = k|x)$$

2. **Model the joint (or the prior and the class conditionals):**

$$\underbrace{P(y = k|x)}_{\text{posterior}} \propto \underbrace{P(y = k, x)}_{\text{joint}}$$

$$= \underbrace{P(x \mid y = k)}_{\text{class conditional densities}} \underbrace{P(y = k)}_{\text{class prior}}$$

Bayes' Theorem

Probabilistic Modelling

Often
inteertwinned

1. **Posit a model: P(X, Y)**
    - How the data is generated
2. **Parametrize the distributions: P(X, Y | Parameters)**
3. **Set the objective (e.g., MLE)**
4. **Learn the parameters of the model:**
    - E.g., Naive Bayes: learn the parameters of the class conditional P(X | Y) and of the prior P(Y)
5. **Use the model (e.g., for predictions)**