Section 1.1

We classify Des by their type, order, and linearity.

- Ordinary: If a DE contains a single independent variable.
- Partial: If a DE contains only partial derivatives of 2 or more variables

Order: The highest order derivative appearing in the equation.

The Normal Form of a DE is the standard way of writing a DE, so the highest order term is isolated on one side of the equation.

$$y' = \frac{f(x,y)}{x}$$
$$x = 0 \text{ is "a point of singularity"}$$

Linearity:

An n^{th} order ODE is linear if we can write it in the following form:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots$$

Where these terms are all functions depending at most on x. They can be constant or zero.

$(y')^n - Non\ Linear$
$\cos(x+y) - Non\ Linear$
$(y)(y') - Non\ Linear$

First Order ODE's Normal Form:

$$y' = f(x,y)$$
$$f(x,y) = -P(x)y + Q(x)$$

First Order Non-Linear DE's Normal Form:

$$y' = f(x,y)$$
$$f(x,y) \text{ depends on } y \text{ in a NL way}$$

Second Order Linear ODE's Normal Form:

$$y'' = f(x,y,y')$$
$$f(x,y,y') = a(x)y + b(x)y' + c(x)$$

Solutions of DEs

A solution is a function that satisfies the DE. We also want to know for which values the solution is valid for. The interval I of a solution to an ODE is called the "interval of definition".

Families of Solutions

The set of solutions $y = Ce^x$ is a family of solutions with parameter C.

A general solution of a DE is a function $y = y(x)$ with arbitrary parameters that includes all possible solutions of the DE.

A particular solution is one solution of the DE without parameters.

Section 1.2

Initial-Value Problems

The problem of solving a DE with info about conditions at a point.

Explicit and Implicit Solutions

An explicit solution to a DE is a function $y = y(x)$ written in terms of the independent variable (& constants) only.

An implicit solution of a DE is a relation $G(x,y) = 0$ such that there is at least one function PHI that satisfies both the relation and the DE.

An implicit solution is a relation which can be converted to the given DE.

Uniqueness of Solutions to IVPS

- Existence: Does the DE $y' = f(x,y)$ have any solutions? If so, do any of the solution curves pass through (x_0, y_0)?
- Uniqueness: When is there exactly one curve through the point (x_0, y_0)?

Theorem 1.2.1

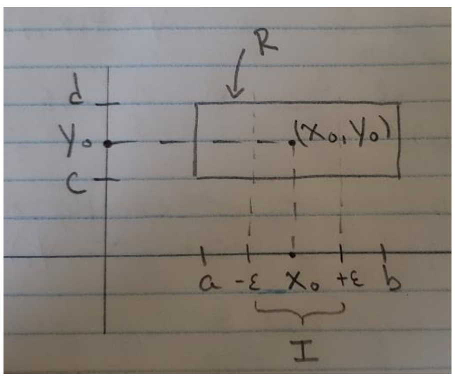
Suppose we have a first order IVP

$$y' = f(x,y) \quad y(x_0) = y_0$$

Let a, b, c, d be real numbers such that

$$a < x_0 < b \ \ \& \ \ c < y_0 < d$$

Let R be the rectangle $[a,b] \times [c,d]$



If $f(x,y)$ and $\frac{df}{dy}(x,y)$ are continuous at and near (x_0, y_0), then there exists a unique solution curve that passes through (x_0, y_0).

Section 2.1

Direction Fields – Sep 15 Note

2.1.2 Autonomous DEs

An ODE in which the independent variable does not appear explicitly. $y' = f(y)$

Sketching Solution Curves for ADEs

Regions where $f(y) > 0$ solutions $y(x)$ is increasing. Graphically, solutions rising.

Regions where $f(y) < 0$ solutions $y(x)$ is decreasing. Graphically, solutions falling.

If $f(y_0) = 0$ at a y-value then $f(y_0) = 0$ for all values x. The constant function $y(x) = y_0$ will be a solution to the DE. Graphically, a horizontal line.

Define: If y_0 is a number where $f(y_0) = 0$, y_0 is called an "equilibrium point" or "critical point".

Note: Solution curves do not cross.

Theorem 1.2.1 ; if $f(x,y)$ and $\frac{df}{dy}$ are continuous at and near (x_0, y_0), then the IVP $y' = f(x,y)$ $y(x_0) = y_0$ has a unique solution.

Types of Critical Points

- Asymptotically Stable (Attractor)

If all solutions of the DE that start sufficiently close to C have the property

$$\lim_{x > \infty} y(x) = 0$$

Solution curves get close to the line y = C as x > ∞

- Unstable (Repeller)

If all solutions of the DE $y(x)$ that start sufficiently Close to C move away from C as x > ∞

- Semi-stable

If solutions starting sufficiently close to C are pulled towards C on one side, and repelled on the other.

Summary of Critical Points:

Suppose C is a critical point of DE $y' = f(y)$

$y' > 0$ when close to and below C, and
$y' < 0$ when close to and above C,
then C is stable.

$y' < 0$ when close to and below C, and
$y' > 0$ when close to and above C,
then C is unstable.

$y' < 0$ when close to and below/above C,
$y' > 0$ when close to and below/above C,
then C is semi-stable.

Classifying the Critical Points of a DE
September 16 Note

Method 1: Check the sign of $f(y)$ near C.

Method 2: Graph $f(y)$

Section 2.2 - Separable DEs

$$\int_{y_0}^{y} \frac{1}{f(t)} dt = \int g(x)$$

Section 2.3 - Linear Equations

First Order:

$a_1(x)y' + a_0(x)y = g(x)$

$g(x) = 0 \quad - \quad Homogeneous$
$g(x) \neq 0 \quad - \quad Non-Homogenous$

Standard Form:

$$y' + \frac{a_0(x)}{a_1(x)}y = \frac{g(x)}{a_1(x)}$$

$\rightarrow y' + p(x)y = f(x)$

Homo: $y' + p(x)y = 0$
Constant $y = 0$ always a solution to homo

Solving First Order Linear Homo DEs

$y' + p(x)y = 0$

$y = Ce^{-\int p(x)}$

Leave out the C from the integral

Linear Superposition Principal (LSP)

For $y' + p(x)y = 0 \quad *$

If $y_1(x)$ is any particular solution to $*$ then
$y(x) = C_1 y_1(x)$ is a general solution of $*$

General Solution:

$$y' + p(x)y = q(x)$$

$$I(x) = e^{-\int p(x)}$$

$$y(x) = \frac{1}{I(x)}\left(\int I(x)q(x) \ + C\right)$$

"Transient Term" refers to term $t(x)$ in the solution $y(x)$ such that limit of $t(x) = 0$

Solving First Order Linear Non-Homo DEs

$y' + p(x)y = q(x) \neq 0$

LSP: If y_p is a particular solution of the DE and y_c is a general solution of the homo DE, then $y(x) = y_c(x) + y_p(x)$ is a general solution of the DE.

Also implied that

$y(x) = Ky_c(x) + y_p(x)$

Step 1: Put into standard form
Step 2: Compute $I(x)$
Step 3: Computer $\int I(x)q(x)$
Step 4: General solution

Section 2.7 - Linear Models

Exponential Growth/Decay

Growth:

$P' = kP$
$P = Ce^{kt}$

Decay:

$A' = kA$
$A = Ce^{kt}$

Cooling/Warming

$T' = -k(T - T_m)$

$T(t) = T_m + (T_0 - T_m)e^{-kt}$

Solution Curves:

$T<T_m$	f(t)>0
$T>T_m$	f(t)<0

Section 3.1

Linear Second Order Equations

$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$
$a_2 \neq 0$

Standard Form:

$$y'' + \frac{a_1(x)}{a_2(x)}y' + \frac{a_0(x)}{a_2(x)}y = \frac{g(x)}{a_2(x)}$$

$y'' + p(x)y' + q(x)y = f(x)$

$f(x) = 0 \quad - \quad Homogenous$
$f(x) \neq 0 \quad - \quad Non\ Homogenous$

Second Order Homo DEs:

$y'' + p(x)y' + q(x)y = 0$

Constant function $y(x) = 0$ is always a solution.

LSP: Let y_1 and y_2 be particular solutions to the homo, then $y = y_1 + y_2$ is also a soln.

Linear Independence of Functions

We say f_1, f_2 are linearly dependent if there exist constants C_1, C_2 (not both 0) such that:

$C_1f_1(x) + C_2f_2(x) = 0$ for all x in I

They are linearly independent if the only C's that work are both C's being zero.

Wronskian

Suppose f_1, f_2 are differentiable. The Wronskian of them denoted by $W(f_1, f_2)$ is defined by:

$$W(f_1, f_2) = det\begin{bmatrix} f_1 & f_2 \\ f_1' & f_2' \end{bmatrix}$$

$W(f_1, f_2) = f_1f_2' - f_2f_1'$

Theorem 3.1.3

Suppose f_1, f_2 are solutions of a homo linear second order DE on interval I.

$W(f_1, f_2) \neq 0 \quad f_1, f_2$ linearly independent

If $W(f_1, f_2) = 0$ for some X then
$W(f_1, f_2) = 0$ for all X

Theorems 3.1.1 and 3.1.5 on September 9th Note #3

Non-Homo Linear Equations

$y'' + p(x)y' + q(x)y = f(x) \qquad f(x) \neq 0$

General Solution:

$y_g = y_c + y_p$

LSP (3.1.7): $y_p = y_{p_1} + y_{p_2}$

Section 3.3 – Homo Linear Equations with Constant Coefficients

Case 1: $b^2 - 4ac > 0$

Distinct roots m_1 and m_2

$y(x) = C_1 e^{m_1 x} + C_2 e^{m_2 x}$

Case 2: $b^2 - 4ac = 0$

Repeated roots $m_1 = m_2$

$y(x) = C_1 e^{m_1 x} + C_2 x e^{m_2 x}$

Case 3: $b^2 - 4ac < 0$

Complex conjugate roots m_1 and m_2

$y(x) = e^{\alpha x}(C_1 \cos(\beta x) + C_2 \sin(\beta x))$

$\alpha = -\frac{b}{2a} \qquad \beta i = \frac{\sqrt{b^2 - 4ac}}{2a}$

2 Complex m_1 and m_2 with opposite signs but equal values:

$y(x) = C_1 \cosh(kx) + C_2 \sinh(kx)$

Eg: $4y'' - y = 0$

Section 3.4 – Method of Undetermined Coefficients

$ay'' + by' + cy = g(x)$

Comp Sol: $ay'' + by' + cy = 0$

Applies when g(x) is:
- A constant
- A polynomial
- Exponential
- Sin/Cos
- Finite sum/product of these

Polynomials:

If g(x) is a degree k polynomial,

Step 1: Find Y_c using aux eqt
Step 2: $f(x) = Ax^k + Bx^{k-1} + Cx^{k-2}$ ...

If $m_1 m_2 \neq 0$	$f(x) = Y_p$
If m_1 or $m_2 = 0$	$xf(x) = Y_p$
If $m_1 m_2 = 0$	$x^2 f(x) = Y_p$

Exponential Functions:

If $g(x) = p(x)e^{\lambda x}$

Step 1: Find Y_c using aux eqt
Step 2: $f(x) = a_k x^k + b_{k-1} x^{k-1} \ldots a_0$

$Y_I = f(x)e^{\lambda x}$

If $m_1 m_2 \neq \lambda$	$Y_p = Y_I$
If m_1 or $m_2 = \lambda$	$Y_p = x * Y_I$
If $m_1 m_2 = \lambda$	$Y_p = x^2 * Y_I$

Trig Functions: Sin

$Y_p = A\sin(x) + B\cos(x)$

If $g(x) = p(x)e^{\lambda x}\sin(x)$

Step 1: Find Y_c using aux eqt

$If\ m_1 m_2 \neq \lambda \pm i$

$Y_p = f(x)e^{\lambda x}\sin(x) + g(x)e^{\lambda x}\cos(x)$

$If\ m_1 m_2 = \lambda \pm i$

$Y_c = e^{\lambda x}(\cos(x) + \sin(x))$

$Y_p = xP_p$

Trig Functions: Cos

If $g(x) = p(x)e^{\lambda x}\cos(wx)$ > p degree k

Step 1: Find Y_c using aux eqt

$If\ m_1 m_2 \neq \lambda \pm i$ else $Y_p = xY_p$

$Y_p = e^{\lambda x}(f(x)\sin(wx) + g(x)\cos(wx))$

Section 3.5 – Variation of Parameters

$a_2(x)y'' + a_1(x) + a_0(x) = g(x)$

> Standard Form:

$(x)y'' + \frac{a_1}{a_2}(x) + \frac{a_0}{a_2}(x) = \frac{g(x)}{a_2}$

> Complete Wronskian (y1, y2)

> $u_1 = \int -\frac{y_2 f(x)}{w(y_1,y_2)} dx$

> $u_2 = \int \frac{y_1 f(x)}{w(y_1,y_2)} dx$

$y_p = u_1 y_1 + u_2 y_2$

$y = c_1 y_1 + c_2 y_2 + u_1 y_1 + u_2 y_2$

Section 3.6 – Cauchy-Euler Equations

$ax^2y'' + bxy' + cy = 0$

$Case\ 1:\ (b-a)^2 - 4ac > 0$
$Y_c = C_1 x^{m_1} + C_2 x^{m_2}$

$Case\ 2:\ (b-a)^2 - 4ac = 0$
$Y_c = x^{m_1} + x^{m_2} \ln(x)$

$Case\ 3:\ (b-a)^2 - 4ac < 0$
$Y_c = C_1 x^{\alpha + iB} + C_2 x^{\alpha - iB}$

$\alpha = \frac{a-b}{2a} \qquad B = \frac{\sqrt{4ac - (a-b)^2}}{2a}$

$Y_1 = x^\alpha \cos(B\ln(x))$

$Y_2 = x^\alpha \sin(B\ln(x))$

Section 3.8 – IVPS for 2nd-order LE

$\mathrm{m}x'' + Bx' + kx = F(t)$

$x'' + 2\lambda x' + w^2 x = f(t)$

$2\lambda = \frac{B}{m} \quad w^2 = \frac{k}{m} \quad \frac{F}{m} = f$

Free Motion: No driving force (F = 0)
Driven Motion: Driving force (F ≠ 0)

Free Motion:

$m = -\lambda \pm \sqrt{(\lambda^2 - w^2)}$

$x(t) = e^{mt}$

Free Undamped Motion

$B = F = \lambda = 0$

$m = \pm wi$

$x(t) = C_1 \cos(wt) + C_2 \sin(wt)$

$x(t) = -C_1 w \sin(wt) + 2C_2 \cos(wt)$

$Period: \frac{2\pi}{w} \quad x_0 = C_1 \quad x_0' = wC_2$

Free Damped Motion

$x'' + 2\lambda x' + w^2 x = f(t)$

$m = -\lambda \pm \sqrt{(\lambda^2 - w^2)}$

Case 1: $\lambda^2 - w^2 > 0 \mid \lambda > w$ 'Overdamped'

$x(t) = C_1 e^{m_1 t} + C_2 e^{m_2 t}$

Case 2: $\lambda^2 - w^2 = 0$ 'Critically Damped'

$m_1 = m_2 = -\lambda$

$x(t) = C_1 e^{-\lambda t} + C_2 t e^{-\lambda t}$

Case 3: $\lambda^2 - w^2 < 0 \mid \lambda < w$ 'Underdamped'

$m = -\lambda \pm i\sqrt{(\lambda^2 - w^2)} \quad m_1, m_2 = a \pm bi$

$a = -\lambda \quad b = \sqrt{(w^2 - \lambda^2)}$

$x(t) = e^{-\lambda t}(C_1 \cos(bt) + C_2 \sin(bt))$

Driven Motion

$x'' + 2\lambda x' + w^2 x = f(t) = a_0 \cos(w_0 t)$

Driven Undamped

Case 1: $w_0 \neq w$

$m_1, m_2 = \pm wi$

$$X_c = C_1 \cos(wt) + C_2 \sin(wt)$$

$$X_p = \frac{a_0}{w^2 - w_0^2} \cos(w_0 t)$$

$$C_1 = x_0 - \frac{a_0}{w^2 - w_0^2} \quad C_2 = \frac{x_1}{w}$$

Case 2: $w_0 = w$

$$x(t) = x_0 \cos(wt) + \frac{x_1}{w} \sin(wt) + \frac{a_0}{2w} t\sin(wt)$$

Driven Damped

$$F \neq \lambda \neq B \neq 0 \quad 0 < \lambda < w$$

$$X_c = e^{-\lambda t}(C_1 \cos\left(\sqrt{w^2 - \lambda^2} * t\right) + C_2 \sin\left(\sqrt{w^2 - \lambda^2} * t\right))$$

$$X_p = \frac{a_0}{(w^2 - w_0^2)^2 + 4\lambda^2 w_0^2} * \dots$$

$$((w^2 - w_0^2) \cos(w_0 t) + 2w_0 \lambda \sin(w_0 t))$$

In the long run, the system oscillates.
Periodic at $2\pi / w_0$

3.9 Boundary-value Problems Linear DEs

- No solution
- A unique Solution
- Infinitely-many solutions
- Note Oct 31 for examples

Theorem: Assume that P, Q, F are continuous on [x_0 , x_1]. The BVP

$$y'' + Py' + Qy = f$$

$$y(x_0) = y_0 \quad y(x_1) = y_1$$

Has a unique solution iff the homo BVP

$$y(x_0) = 0 \quad y(x_1) = 0$$

Has $y_c(x) = 0$ as the only solution.

Euler's Law: Note Nov 1

Section 4.1 – The Laplace Transform

$$f(t) = 1 \quad > \quad \frac{1}{s} \qquad f(t) = t^n \quad > \quad \frac{n!}{s^{n+1}}$$

$$f(t) = \sin(kt) \quad > \quad \frac{k}{k^2 + s^2}$$

$$f(t) = \cos(kt) \quad > \quad \frac{s}{s^2 + k^2}$$

$$f(t) = \sinh(kt) \quad > \quad \frac{k}{s^2 - k^2}$$

$$f(t) = \cosh(kt) \quad > \quad \frac{s}{s^2 - k^2}$$

$$f(t) = \mathrm{e}^{-at} \quad > \quad \frac{1}{s + a}$$

Solve a DE by:
1) Convert DE on a function x(t) to an algebraic equation on a function X(s)
2) Solve the algebraic equation for X(s)
3) Use Inverse Laplace to get a solution

$$F(s) = \int_0^\infty e^{-st} f(t) dt$$

If this converges, F(s) is the Laplace transform of f(t).

$$L\{f, g\} = L\{f\} + L\{g\}$$

$$f(t) = L^{-1}\{F(s)\}$$

$$L\{f'\} = sF(s) - f(0)$$

$$L\{f''\} = s^2 F(s) - sf(0) - f'(0)$$

$$ax'' + bx' + cx = g(t)$$

$$X(s) = \frac{a(x_0' + sx_0) + bx_0}{as^2 + bs + c} + \frac{L\{g\}}{as^2 + bs + c}$$

General Solution for X(t) of an IVP:

1) Apply Laplace to DE and apply initial conditions
2) Solve for X(s)
3) Inverse Laplace on X(s) to get x(t)

Translations of the Laplace Transform

$L\{e^{at} * f(t)\} = F(s-a)$

Laplace Transform of an Integral

$$L\left\{\int_0^t f(T)dT\right\} = \frac{F(s)}{s}$$

Dirac Delta Function

$L\,\{d_e(t-t_0)\} = \lim_{e \to 0} L\{d_e(t-t_0)\}$

$L\,\{d_e(t-t_0)\} = e^{-st_0}$

$L\,\{d_e(t)\} = 1$

Separable Partial Diff. Equations

$Case\ 1)\ \lambda < 0: No\ Solution, Not\ Possible$
$C_1 \& \ C_2 have\ to\ be\ 0,\ \ therefore\ F(x) = 0$

$Case\ 2)\ \lambda = 0:\ \ F(x) = 0$

$Case\ 3)\ \lambda > 0:\ \ F(0\ \&\ L) = 0, C_1 = 0$

$if\ \lambda = \left(\frac{n\pi}{L}\right)^2\ \&\ C_2 \neq 0, then:$

$F(x) = C_2 \sin\left(\frac{n\pi}{L}x\right)$

$G(t) = Ce^{-\left(\frac{n\pi}{L}\right)^2 t} \quad \lambda = \left(\frac{n\pi}{L}\right)^2$

Summary:

$$\frac{du}{dt} = \frac{d^2u}{dx^2}$$

$$u(x,t) = Ce^{-\left(\frac{n\pi}{L}\right)^2 t} * \sin\left(\frac{n\pi}{L}x\right)$$

13.2 – 13.4 Not typed up. Too many symbols.