Type I Improper Integrals

$\int^{\infty} = f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$

$\int_{-\infty} = f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx$

Convergent if limit exists
Divergent if limit is $\pm\infty$ or undefined

Type II Improper Integrals

$\int_a^b = f(x)\,dx = \lim_{c\to a+} \int_c^b f(x)\,dx$
If f(x) is discontinuous at a

$\int_a^b = f(x)\,dx = \lim_{c\to b-} \int_a^c f(x)\,dx$
If(x) is discontinuous at b

Mixed Improper Integrals

Simply split up the integral to do one from 0-1 and the other from 1-∞

P-Integrals

Type I Integrals: (# to infinity)

Divergent if	$P \leq 1$
Convergent if	$P > 1$

Type II Integrals:

Divergent if	$P \geq 1$
Convergent if	$P < 1$

Telescopic Sums

Repeatedly add and subtract terms, cancelling each other out, and leaving us with only a few terms at the start/end.

Geometric Sums

$\sum ar^i = a(1-R^n)/(1-R)$

Common Series Converge/Diverge

$\sum 1/n^2$	Converges
$\sum 1/n$	Diverges
$\sum \tan^{-}(n)/n$	Diverges

Sequences

"indexed, ordered, infinite set of values"

eg: 1, 3, 5, 7, 9 ...
$a_n = 2n-1$
$\{a_n\}^{\infty}_{n=1}$ "Explicit Expressions"

For a recursively defined sequence, we need both initial values and recurrence relation (pattern).

Convergence of Sequences

$\lim_{n\to\infty} a_n$ If exists: Convergent
If ∞/DNE: Divergent

Squeeze Theorem

If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} C_n = k$

And if $a_n \leq b_n \leq C_n$ then;

$\lim_{n\to\infty} b_n = k$

Special Cases

1) $\lim_{n\to\infty} r^n$ ∞ if r>1 DNE if $r \leq -1$
1 if r=1 0 if -1<r<1

Alternating Sequences

Convergent iff $b_n = 0$

$\lim_{n\to\infty} (-1)^n b_n$

Recursively Defined Sequences and Limits

If bounded above and increasing : Conv.
If bounded below and decreasing : Conv.

Proof by induction:
1. Find base case that shows the sequence starts ($a_1 > 0$)
2. Induction step proving what you want

Series

"A series is an infinite sum"

- Telescopic series go to infinity so look at the first few terms.
- Geometric series either converge or diverge based on the value of R.

The Divergence Test

If $\lim_{n\to\infty} a_n \neq 0$
$\sum a_n$ series diverges

Integral Comparison

If $\sum a_n$, $a_n \geq 0$ and $f(n) = a_n$

Where f(x) is positive, cont, decreasing

If $\int f(x)\,dx$ converges >> $\sum a_n$ converges

P-Series

Divergent if	$P \leq 1$
Convergent if	$P > 1$

Series Comparison Test

Literally compare shit that you don't know to other shit that you do know

Limit Comparison Test

lim of a_n/b_n tells you their relationship.

$0<k<\infty$	Both converge / diverge
$k = 0$	$b_n >> a_n$ for n large
$k = \infty$	$a_n >> b_n$ for n large
DNE	no info

Alternating Series Test

If $\sum (-1)n\, b_n$, $|b_n| > 0$ then if

1) b_n is decreasing
2) $\lim b_n = 0$

$\sum$ converges

Alternating Series Error Estimates

$|S_\infty - S_m| \leq b_{m+1}$

$|b_n| <$ error   solve for n, subtract 1

The Three Possibilities

1) Absolute Convergence
$\sum|a_n|$ converges and a_n converges

2) Conditional Convergence
$\sum|a_n|$ diverges but a_n converges

3) $\sum a_n$ diverges

Ratio Test

$\sum a_n$ look at $\lim |a_{n+1} / a_n|$

$k > 1$	diverges
$k < 1$	converges
$k = 1$	no info

Ratio test is blind to polynomials, rational functions, and n^p expressions.

Root Test

$\sum a_n$ consider $\lim \sqrt[n]{|a_n|}$

$k > 1$	diverges
$k < 1$	converges
$k = 1$	no info

Only use when $a_n = (\ )^n$

Power Series

Basically Ratio Test it.
If lim < 1: R = infinity
If lim > 1: R = 0

MacLawrin Series

$F(x) = \sum_0^\infty C_n x^n$
$F^n(0) = n!C_n$
Theorem: $F(x) = \sum \frac{F^n(0)}{n!} x^n$
(make a table)

Taylor Series

F(x) = $\sum_0^\infty \frac{F^n(a)}{n!}(x-a)^n$
(make a table)

Special MacLawrin Series

$$\frac{1}{1-x} = \sum_0^\infty x^n$$

$$e^x = \sum_0^\infty \frac{x^n}{n!}$$

$$\cos(x) = \sum_0^\infty (-1)^n (\frac{x^{2n}}{(2n)!})$$

$$\sin(x) = \sum_0^\infty (-1)^n \left(\frac{x^{2n+1}}{(2n+1)!}\right)$$

Taylor Polynomials

F(x) = $\sum_0^\infty \frac{F^n(a)}{n!}(x-a)^n = T_m(x)$

T_m(x) approximates F(x) better and better over a larger region. T_m is the best fit m^{th} order polynomial about x=a.
(Make a table)

Taylor Remainder Theorem

$$|F(x) - T_m(x)| = R_m(x) \leq \frac{M|x-a|^{m+1}}{(m+1)!}$$

Binomial Series

$(a+b)^m$
$= a^m + \binom{m-1}{1}(a^{m-1}b) + \binom{m-2}{2}(a^{m-2}b)$ ...

$$\binom{m}{r} = \frac{m!}{r!\,(m-r)!}$$

Integration and Interaction

Surface Area of Revolution

SA = $\int_a^b 2\pi F(x)\sqrt{1+F'(x)^2}\,dx$
SA = $\int 2\pi r ds$

Rotating around y-axis but calculating in x: exchange r for x

Hydrostatics and Pressure

P = $p$gh F = PA
p = fluid density

Force on sides:
F = $\int_{top}^{bottom} p$gwh Δh

Differential Equations

Ordinary – Only have one independent
Differential – Includes derivatives
Equation – An expression

Order of ODE depends on highest order

Basic Case

Either directly integrate or use y = ce^{rx}

Y'' + w^2y = 0 y = Acos(wt)+Bsin(wt)

Seperable ODE

$$y' = f(x)g(y)$$

$$\frac{1}{g(y)}\frac{dy}{dx} = f(x)$$

$$\int \frac{1}{g(y)}dy = \int f(x)dx$$

Result: If y' = ky y = ce^{kx}

Orthogonal Trajectory

$$y'_{new} = -\frac{1}{y'_{old}}$$

If y = kx^2, integrate and solve for k = y/x^2 then plug in the new k in the y' equation. This gets you the old y'.

Exponential Growth and Decay

y = Ae^{kt} y' = rate of change

Half Life: y(⅄) = ½ y(0) = y(0)$e^{k⅄}$

Newton's Law of Cooling

$T = T_{env} + Ae^{kt}$

First Order Linear ODE

Eg: $y' + P(x)y = Q(x)$

$$y = \frac{1}{I(x)} \int I(x)Q(x)\,dx$$

$$I(x) = Ce^{\int P(x)}$$

Parametric Curves

$(x,y) = (f(t), g(t))$

$x = t \quad y = t^2 \quad : \quad y = x^2$

Methods:

1. Go back to x,y representation

Eg: $x = \ln(t) \gg t = e^x$
$y = t^2+t \gg y = e^{2x} + e^x$

2. Graph [x vs t] and [y vs t] and brute force it approximately.

Calculus on Parametric Curves

$$m = \frac{y'}{x'} \textit{ with respect to } t$$

If 0, there's a horizontal tangent. To find vertical tangent, solve separately.

Second Derivatives and Concavity

$$\frac{d^2y}{d^2x} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\left(\frac{dx}{dt}\right)}$$

Area and Integration

Parametric Area = $\int_{t_0}^{t_1} y(t)x' \; dt$

Parametric Arc Length = $\int_{t_0}^{t_1} \sqrt{(x')^2 + (y')^2} \; dt$

Parametric Surface Area: Use old formula with new ds (arc length).

Test #2 Ends Here

Polar Coordinates and Polar Graphs

$r = g(\emptyset)$

$r^2 = x^2 + y^2$
$\tan(\emptyset) = y/x \quad \emptyset$ = angle to the x-axis

eg. $(1,1) \gg (r, \tan(1/1)) \gg (\sqrt{2}, \pi/4)$

Remember, if the r is in the negatives, add a π.

Method

1. Write $r = g(\emptyset)$

2. Sketch [r vs Ø] > r vertical, Ø horizontal

3. Use above graph to sketch between "interesting points" and make polar graph

Derivatives of Polar Curves

$$\frac{dy}{dx} = \frac{r' \sin\emptyset + r\cos\emptyset}{r'\cos\emptyset - r\sin\emptyset}$$

The Third Dimension

→ Z is dependant, x/y are independent

Domain: Region in x,y plane at which f(x,y) can be evaluated
Range: Subset of Z interval

Level Sets and Multivariable Functions

The set of all points in the domain such that Z=K, K constant real number.

1) Level sets are *always* in our domain
2) If Z = f(x,y) is a functions (passes VLT) then no level sets can cross/intersect!

Limits and Continuity

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

Usual Suspects:

Cos, Sin, All Trig, Polynomials, Rational, Ln/Log, a^x, e^x {All continuous on domain}

1) Check for 0/0
2) Check common failure directions and powers
3) Squeeze it if you think limit exists

Derivatives in R^n

Partial Derivatives

$$\frac{\partial z}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

"Partial Derivative of Z with respect to Y"

Mixed Partials (2nd derivative mixed type)

Clairout's Theorem: If the 2nd partial derivatives are continuous, $f_{xy} = f_{yx}$

Tangents in Multivariable Problems

Planes: $Z = m_1x + m_2y + C$
Plug in a point to get C

$m_1 = f_x$ $\qquad$ $m_2 = f_y$

Linearization

$L(X,Y) = m_1(X-X_0) + m_2(Y-Y_0) + Z_0$

Differential Linearization

$$\partial Z = f_x(x,y)\partial x + f_y(x,y)\partial y$$

Tree Chain Rule: March 24 Note

Implicit Differentiation

$$\frac{\partial y}{\partial x} = \frac{-F_x(x,y)}{F_y(x,y)}$$

$$\frac{\partial z}{\partial x} = \frac{-F_x(x,y,z)}{F_z(x,y,z)}$$

$$\frac{\partial z}{\partial y} = \frac{-F_y(x,y,z)}{F_z(x,y,z)}$$

Gradient and Directional Derivative

Directional Derivative:

$$D_{\vec{u}}f(x,y) = \lim_{h \to 0} \frac{f(\vec{x}, h\vec{u}) - f(\vec{x})}{h}$$

U = unit vector $\qquad$ x = (x,y)
h = ||delta x||

$$D_{\vec{u}}f(x,y) = (f_x, f_y) \cdot \vec{u}$$

$$D_{\vec{u}}f(x,y) = \left|\left|f_x, f_y\right|\right| cos\theta$$

∇f *is always orthogonal to level sets*

∇f *is pointing in "direction of steepest ascent"*