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Quantum computations: algorithms and error correction

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The first part of the article ($\S1-\S6$) is devoted mainly to one goal, namely, to showing the computational capacity of a quantum computer using the problem of a stabilizer in the group \mathbf{Z}^k as an example. The discrete logarithm and decomposition into prime factors can be reduced to this problem. The formal definition of a quantum computer (more precisely, a quantum scheme) appears in $\S4$. All necessary information from computing theory and quantum mechanics is contained in $\S2$ and $\S3$.

The reading of the second part can begin with the theory of one-to-one quantum codes (§8.1, §8.2 and §9). This theory is sufficiently complete, simple and self-contained. However, in the present paper it is considered as a means of solving the perturbation problem, that is, constructing a quantum computer from unreliable (inaccurate, being subject to perturbations) elements. To this topic we devote §7, §8.3, §10 and §11. Interesting results are obtained in §10.1 and §11. The rest is a rather tedious technical preparation. (So far the problem in hand has been considered only at the physical level of rigour. The formal approach revealed one subtle point where a naive argument may lead to an error, see §10.1.) The initial formulation of the problem is not rigorous and rests on an intuitive picture of the way a real-world computational device might work. This is formalized in §7, but the final formulation of the problem (the so-called polynomial error correction system, see Definition 10.4) appears only in the middle of §10.1.

§0. Introduction

The structure of atoms, the chemical bonds in molecules and the various physical properties of solids all follow, in principle, from the fundamental laws of quantum mechanics discovered 70 years ago. Many of these phenomena are well understood and lend themselves to numerical computations. Theoretical physicists have developed various approximate methods of solving quantum mechanical problems. However, in the general case the modelling of quantum processes requires impossibly high computation volumes.

Only recently has it become clear that this difficulty has a reverse positive side. If it is impossible (within an acceptable time) to model quantum mechanics using an ordinary computer, then one could try to use some quantum system to solve difficult computational problems. First of all, one must obviously formulate a simple and sufficiently general mathematical model of quantum computations. Apparently, this idea was first put forward by Manin in [1]. It has been demonstrated by a number of authors how to carry out ordinary computations using quantum devices [2]–[4]. But it was only in 1985 that Deutsch stated the general definition of the so-called quantum Turing machine [5]. In 1989 he formulated another more convenient model, namely, quantum schemes [6]. The equivalence of these models was proved by Yao [7].

It is customary to regard a computational problem as difficult if it has no polynomial algorithm [8]. An algorithm is called polynomial if the number of computational steps increases no faster than some power of the length of the input of the problem. It is clear that this definition depends on the model of computations used. It turns out that the majority of abstract computational models (the Turing machine, a machine with random access memory (RAM), and so on) are polynomially equivalent to one another. In other words, a machine can model

another one with polynomial deceleration in the worst possible case. For example, if a machine with random access memory requires t steps to complete a computation, then a Turing machine will need $O(t^2)$ steps to do it.

The above computational models can be realized physically. (In practice any modern computer can be regarded as a realization of RAM). Conversely, classical mechanics can be modelled on a computer within a time depending polynomially on the number of degrees of freedom. However, this is not the case with quantum mechanics. We consider a system with two basis states $|0\rangle$ and $|1\rangle$. Such a system, independently of its physical realization, is called a quantum bit or qbit. The general quantum state of a qbit is a superposition (linear combination) $c_0|0\rangle + c_1|1\rangle$ with complex coefficients c_0, c_1 . A system of n qbits has 2^n basis states. The general state is a vector in a 2^n -dimensional complex space. The transformation (evolution) of a state in a given time interval can be described by a unitary matrix of dimension $2^n \times 2^n$. A natural way of modelling consists in multiplying such matrices, which requires exponential computation volumes.

A quantum computation is a sequence of elementary unitary transformations (operations), each affecting a small number of qbits. The input word can be encoded as a basis vector, and the result of the computation is defined probabilistically with the aid of projection operators (see Definition 4.1). In this new computational model any quantum system can be modelled in polynomial time. In principle, a quantum computer can be realized physically, but this is unlikely to be achieved in the near future. (On some experimental progress in this direction see [9], [10].) The present article is devoted to two general problems independent of the concrete physical realization:

- 1) Can a quantum computer solve difficult 'classical' problems (in combinatorics, number theory, and so on)?
- 2) To what extent is a quantum computer stable under perturbations and imperfections in its elements? Can the computation process be set up in such a way that a moderate perturbation will not affect the result?

In 1994 Shor designed polynomial quantum algorithms for the discrete logarithm and the decomposition of integers into prime factors [11]. Both problems are regarded as difficult.¹ This conjecture constitutes the foundation of so-called public key cryptographic systems [12]. (This means that a quantum computer could 'break' the corresponding codes.) However, no effective quantum algorithm for an NP-complete problem has been found so far.

Let us recall the definition of the discrete logarithm. Let p be a prime number and let ζ be a primitive root in the field of remainders modulo p. Then the map $x \mapsto \zeta^x$ is an isomorphism from the additive group of remainders modulo p-1 to the multiplicative group of remainders modulo p. The inverse map is called the discrete logarithm.

We shall describe a polynomial quantum algorithm for a considerably more general problem, namely, finding the stabilizer of a given element relative to the action of an Abelian group G on a finite set. The case $G = (\mathbf{Z}_p)^k$ was considered by Grigoriev [14]. The problem in the general form was solved by the present

¹To date not a single natural computational problem has been shown to be difficult. However, there are many problems for which no polynomial algorithms have been found.

author in [15]. The key point is a procedure for measuring an eigenvalue of a unitary operator (see §5). Simon's method [13] is also used to generate a random element of the group of characters.

The perturbation problem is of paramount significance for the practical realization of a quantum computer. In the general case a sequence of L operations yields the correct result with probability $\geqslant 1-\varepsilon$ if each operation is performed with accuracy $\varepsilon/(2L)$ (see Proposition 4.2 or [16]). Under real-life conditions qbits will be subject to perturbations even if nothing is done to them. As a result, the quantum state will become corrupted. This situation is not very encouraging. It would be desirable to preserve the quantum state and to carry out arbitrarily long quantum computations involving operations with fixed accuracy. In other words, we need to learn how to model accurate operations by means of inaccurate ones.

For classical computations the perturbation problem was solved long ago. The simplest method of increasing reliability consists in storing several copies of each bit and comparing them periodically. Unfortunately, this method is inapplicable in the quantum case, because a quantum state cannot be copied (the operator of copying $|\xi\rangle \mapsto |\xi\rangle \otimes |\xi\rangle$ is not linear).

A more general method of combating perturbations is based on error correction codes. The classical theory of such codes is very extensive [17]. The simplest classical code REP(n) is based on repetition: 0 is represented by a word consisting of n zeros and 1 by a word consisting of n units. A unitary embedding of the space of states of m qbits into the space of states of n qbits is called a quantum code of type (n, m). The first example of an error-correcting quantum code was designed by Shor [18]. Soon after this, Calderbank and Shor [19] and also Steane [20] proposed a rather general method of constructing quantum codes. (The existence of so-called 'good' quantum codes was also proved in [19]). Various examples of quantum codes are contained in [21]–[28]. Finally, Calderbank, Rains, Shor and Sloane [29] defined a very large and natural class of quantum codes including all earlier examples. We shall call these codes symplectic because they are connected with the symplectic group over the field \mathbf{F}_2 . A more traditional name is additive codes. An example of a non-additive code was constructed in [30].

In the present article we do not consider questions such as the carrying capacity of a quantum channel [31]–[34], the optimality of quantum coding [19], [25], [35], [36], quantum code enumerators [37], [38], the use of 'entangled states' or 'quantum teleportation' [39]–[41]. We restrict ourselves just to two far from optimal but interesting examples. An infinite sequence of so-called torus codes TOR(k) [42] is constructed in §9.4. An important property of such codes is that they are codes with local verification.

Let us emphasize that we intend to use codes to prevent errors in computations rather than to transmit information (or quantum states). This will dictate very specific requirements for the codes themselves and for the computing procedures connected with them. In particular, the error correction procedure must be stable under new errors which may occur in the course of its realization. Moreover, we need to learn how to carry out operations involving coded qbits without decoding them explicitly (or they will fail to be protected against perturbations). Only some codes enable us to do this.

The principal solution of the problem of perturbations in quantum computations was obtained by Shor [43]. He proved that a computation of length L can be carried out if the accuracy of each operation is $(\log L)^{-c}$ (c = const). In the present article this result is strengthened. We shall demonstrate that fixed accuracy is sufficient. To this end we use so-called *cascade codes*. Similar (but technically distinct) constructions were proposed in [44]–[46]. Other results on this subject are contained in [47], [48].

§1. Abelian problem on the stabilizer

Let G be a group acting on a finite set M. We assume that this action and the group operations in G are easy to compute. Compute the stabilizer of a given element $a \in M$. This problem (which still needs to be made more precise) includes many interesting cases, for example, isomorphism of graphs. Unfortunately, to date we are not in a position to solve the problem in its full generality. We shall assume instead that G is an Abelian group. Since any finitely-generated Abelian group is a factor group \mathbf{Z}^k , we can assume without loss of generality that $G = \mathbf{Z}^k$. The less general case $G = (\mathbf{Z}_p)^k$ was studied by Grigoriev in [14], where he constructed an algorithm that is polynomial in k and k0 (but not in $\log k$ 2). Under the same assumptions we shall obtain a fully polynomial algorithm, that is, an algorithm which is polynomial in $k + \log k$ 2.

To state the problem as a computational one we shall assume that M is identified with a subset of the Boolean cube $\mathbf{B}^n = \{0,1\}^{n,2}$ Each element $g = (m_1, \ldots, m_k) \in \mathbf{Z}^k$ can be represented by a binary word³ of length size $(g) = \sum_{i=1}^k O(\log(|m_i|+1))$.

Let us proceed to the precise formulation. The Abelian problem on the stabilizer (more precisely, a representative of this problem) can be defined by specifying the following objects:

- two positive integers k and n. The number k + n is called the dimension of the problem;
- an element $a \in \mathbf{B}^n$;
- a function $F: \mathbf{Z}^k \times M \to M \ (a \in M \subseteq \mathbf{B}^n)$ such that F(0,x) = x and F(g+h,x) = F(g,F(h,x)) for any $g,h \in \mathbf{Z}^k, x \in M$.

The function F is to be regarded as an oracle, that is, a 'black box' which receives an input word $(g,x) \in \mathbf{Z}^k \times \mathbf{B}^n$ and immediately produces an answer $y \in \mathbf{B}^n$ such that y = F(g,x) for any $g \in \mathbf{Z}^k$ and $x \in M$. (If $x \notin M$, then the oracle may give no answer or give an arbitrary answer.) It is assumed that one can resort to F in the course of a classical or quantum computation, see §2 and §4.1. The formulation of the problem involves no explicit description of M. It suffices to know that such a set exists.

Remark. In all sensible applications (see the examples below) F can be computed in polynomial time. A quantum computer can do the work on its own, so there

 $^{^{2}}$ A situation as natural as above but more general could be considered: each element of M has several representations in \mathbf{B}^{n} and the equivalence of these representations can be verified with the aid of some effective procedure. Unfortunately, our method does not work in this case.

³Obviously, this representation is not unique. However, size(g) can differ at most by a constant multiplier in various 'natural' representations.

is no need to resort to an oracle. We shall describe this situation in more precise terms. Let poly denote any function increasing no faster than a polynomial, that is, $\operatorname{poly}(x) = x^{O(1)}$. We assume that the function $(g,x) \mapsto F(g,x)$ can be computed in time $\operatorname{poly}(\operatorname{size}(g) + n)$ using a Turing machine. For a fixed function poly of polynomial growth this condition defines the restricted Abelian problem on the stabilizer, that is, a subclass of representatives of this problem. In this case it is possible to construct a polynomial quantum algorithm whose input is the description of a Turing machine which can compute F.

The set $\operatorname{St}_F(a)=\{g\in \mathbf{Z}^k: F(g,a)=a\}$ is called the stabilizer of a with respect to F. It is a subgroup of finite index $\leqslant |M|\leqslant 2^n$ in \mathbf{Z}^k . It follows that $\operatorname{St}_F(a)$ is isomorphic to \mathbf{Z}^k and has a basis $\{g_1,\ldots,g_k\}$ of polynomial dimension, that is, $\sum_{j=1}^k\operatorname{size}(g_j)\leqslant\operatorname{poly}(n+k)$. Any such basis is admissible as a solution of the stabilizer problem. There exists an effective procedure which makes it possible to check if two sets $\{g_1,\ldots,g_k\}$ and $\{g'_1,\ldots,g'_k\}$ are bases in the same subgroup of rank k in \mathbf{Z}^k . Moreover, there exists a rather simple polynomial algorithm for reducing a basis to a unique (for a given subgroup $A\subseteq \mathbf{Z}^k$) canonical form. The canonical basis consists of the columns of a matrix $(m_{ij},i,j=1,\ldots,k)$ uniquely characterized by the following conditions:

$$\begin{split} m_{ij} &= 0 & \text{if } i > j, \\ m_{ii} &> 0, \\ 0 &\leqslant m_{ij} < m_{ii} & \text{if } i < j. \end{split} \tag{1}$$

It follows that the construction of an arbitrary basis (of polynomial dimension) is equivalent to the construction of the canonical basis.

The computational problems of 'factorization' and the 'discrete logarithm' can be reduced to an Abelian problem on the stabilizer. Let M be the ring of integers modulo q, and let G be the group of invertible elements in M. If $g_1, \ldots, g_k \in G$, then the action of \mathbf{Z}^k on M can be defined as follows: $F_{g_1,\ldots,g_k}: (m_1,\ldots,m_k,x) \mapsto g_1^{m_1},\ldots,g_k^{m_k}x \ (m_i \in \mathbf{Z},x \in M)$. We shall consider two cases.

- 1. Decomposition of a whole number into prime factors. Let k = 1. The stabilizer of $1 \in M$ with respect to F_g gives the order of an element g in G. There exists a randomized reduction of the problem of 'factorization' to the problem of the 'order of an element' [49]. (The reduction is briefly presented in Shor's paper [11].)
- 2. The discrete logarithm. Let q be a prime number. Then G is a cyclic group. We fix a primitive element $\zeta \in G \cong \mathbf{Z}_{q-1}$ and take an arbitrary element $g \in G$. The set $P = \{(m,r) \in \mathbf{Z}^2 : \zeta^m g^r = 1\}$ is the stabilizer of 1 with respect to $F_{\zeta,g}$. If a basis of the subgroup $P \subseteq \mathbf{Z}^2$ is known, then it is easy to find an element of the form $(m,-1) \in P$. Then $\zeta^m = g$, that is, m is the discrete logarithm of g with base ζ .

§2. Classical models of computations

2.1. Boolean schemes and sequences of operations. From now on we shall mainly deal with Boolean functions, that is, functions of the form $f: \mathbf{B}^n \to \mathbf{B}^m$. We choose a set \mathcal{A} of such functions, which will be used to construct other functions. This set, called the *basis*, must contain the identity function $I = I_{\mathbf{B}} : \mathbf{B} \to \mathbf{B}$

(for technical convenience). The following standard basis is usually used: $\mathcal{C} = \{I, 0, \neg, \wedge\}$ (where \neg is negation and \wedge is the 'and' function). This basis is *complete*, that is, any Boolean function can be represented as a superposition of elements of the basis.

Let $F: \mathbf{B}^n \to \mathbf{B}^m$ be an arbitrary function. A *Boolean scheme* for F is a procedure which transforms an input word $x \in \mathbf{B}^n$ into an output word $y = F(x) \in \mathbf{B}^m$ using auxiliary Boolean variables z_1, \ldots, z_l according to the following rules:

- 1) copy x into (z_1, \ldots, z_n) ;
- 2) compute the variables z_{n+1}, \ldots, z_l one after another, applying the basis functions $f_1, \ldots, f_L \in \mathcal{A}$ $(f_i : \mathbf{B}^{n_i} \to \mathbf{B}^{m_i})$ to the values computed so far (at a given moment):

$$(z_{k_{i-1}+1},\ldots,z_{k_i}) = f_i(z_{\alpha(i,1)},\ldots,z_{\alpha(i,n_i)}), \qquad \alpha(i,1),\ldots,\alpha(i,n_i) \leqslant k_{i-1}, \quad (2)$$

where $k_0 = n$ and $k_i = k_{i-1} + m_i$;

3) read off y from $(z_{\beta(1)}, \ldots, z_{\beta(m)})$.

It follows that a Boolean scheme is determined by the sequence of functions $f_1, \ldots, f_L \in \mathcal{A}$ and the numbers $\alpha(i,j), \beta(j)$. We call L the dimension of the scheme. For more generality we can assume that F is a partially defined function $\mathbf{B}^n \to \mathbf{B}^m$, that is, a function of the form $N \to \mathbf{B}^m$, where $N \subseteq \mathbf{B}^n$. In this case the output word y must be equal to F(x) for all $x \in N$.

Example 2.1. We consider the composition of two binary numbers of length 1:

$$F: (x_1, x_2) \mapsto (x_1 \land x_2, (x_1 \lor x_2) \land \neg (x_1 \land x_2)).$$

This function can be represented by the following Boolean scheme in the basis $A = \{I, \neg, \land, \lor\}$:

$$(z_1, z_2) = x$$
, $z_3 = z_1 \wedge z_2$, $z_4 = z_1 \vee z_2$, $z_5 = \neg z_3$, $z_6 = z_4 \wedge z_5$, $y = (z_3, z_6)$.

The above definition of a Boolean scheme is the generally accepted one. Now we are going to define a slightly different but equivalent computational model, which will prove more convenient to generalize later on. In the new model the values of z_i can change at each step (instead of being computed once and for all). This bears a much closer resemblance to the way a real computer works.

Let $\Delta = \{1, \ldots, l\}$ be the memory to be used in the computation. Each memory element (bit) $i \in \Delta$ corresponds to a Boolean variable z_i . Any ordered set of bits is called a register. With each register $A = (j_1, \ldots, j_n)$ we associate a variable $z_A = (z_{j_1}, \ldots, z_{j_n})$ taking values in \mathbf{B}^n . If an arbitrary Boolean function $g: \mathbf{B}^n \to \mathbf{B}^n$ is given, then the action $g[A]: z_A \mapsto g(z_A)$ on the set of states of A can be defined. We can consider g[A] as a map on the whole set $\Gamma = \mathbf{B}^{\Delta}$ of memory states. Such maps are called operations.

We shall consider a more general situation: let Δ and Δ' be two memories, let $A \subseteq \Delta$ and $A' \subseteq \Delta'$ be two registers, and let $g: \mathbf{B}^{|A|} \to \mathbf{B}^{|A'|}$ be a Boolean function. We assume that $\Delta \setminus A = \Delta' \setminus A'$. Then $g[A'; A]: \mathbf{B}^{\Delta} \to \mathbf{B}^{\Delta'}$. Thus, g[A] is an abbreviation for g[A; A].

As a basis we fix a set \mathcal{A} of Boolean functions. We introduce the notation $\nabla : \mathbf{B} \to \mathbf{B}^2 : a \mapsto (a, a)$. The new model is a procedure of the following form:

- 1) enter the input word into some register $X = \Delta_0$;
- 2) perform some operations $g[A'_1; A_1], \ldots, g[A'_L; A_L]$ $(g_i \in A \cup \{\nabla\})$ one after another; it is assumed that $A_i \subseteq \Delta_{i-1}, \Delta_i = (\Delta_{i-1} \setminus A_i) \cup A'_i$;
- 3) read the output word from some register $Y \subseteq \Delta_L$.

Example 2.1 can be represented as the following composition of operations (which should be read from right to left):

$$\wedge [6;4,5] \quad \neg [5] \quad \nabla [3,5;3] \quad \vee [4;1,2] \quad \wedge [3;3,4] \quad \nabla [2,4;2] \quad \nabla [1,3;1].$$

It is clear that any Boolean scheme of dimension L can be transformed into a sequence of cL operations (where c is the largest number of bits acted upon by a basis function). Conversely, any sequence of L operations can be modelled by a Boolean scheme of dimension L. It follows that the two models are equivalent.

Schemes and algorithms. A Boolean scheme always works with input words of fixed length. However, computational problems are, as a rule, formulated for words of variable length. For example, we shall consider the problem of addition of binary numbers $(x, x') \mapsto x + x'$. Any reasonable computational problem F can be represented as a sequence of functions $F|_s : \mathbf{B}^s \to \mathbf{B}^{\text{poly}(s)}$, each of which corresponds to a certain dimension s of the input word. Each function $F|_s$ requires its own Boolean scheme. If a polynomial algorithm exists for F, then each function $F|_s$ can be computed by means of a scheme Φ_s of dimension poly(s). Indeed, a computer (more precisely, a Turing machine) with limited resources (memory \times time $\leq t \times t$) can be modelled by a Boolean scheme of dimension $O(t^2)$.

The converse is false, in general. The existence of a polynomial scheme Φ_s for each function $F|_s$ does not necessarily mean that F as a whole is polynomially computable (or even just computable). For this to happen it should be possible to construct Φ_s effectively. More precisely, the function $s \mapsto \Phi_s$ should be computable by a Turing machine in polynomial time. A family of functions (Φ_s) satisfying this condition is called *homogeneous*. It means that a Turing machine generates a scheme and the scheme computes the function.

This two-level construction is particularly well suited for defining non-standard computational models, including quantum computations. In §4 we shall define quantum schemes, that is, some theoretical devices to compute Boolean functions. As with any combinatorial object, a quantum scheme can be described (encoded) with the aid of a binary word.⁴ By a quantum algorithm for F we shall understand the usual (classical) algorithm consisting of a quantum scheme Ψ_s for each function $F|_s$.

2.2. Reversible computations. The models defined above admit irreversible operations, as do real-world computers. Indeed, even clearing (setting to zero) a single bit is an irreversible action. In this section we shall demonstrate that such operations can be avoided. The notion of a reversible computation was introduced

⁴In combinatorial terms we can describe the *device* rather than the performance of a quantum scheme.

by Lecerf [50] and then, independently, by Bennett [51]. It serves as an important link between Boolean schemes and quantum computations.

First of all, a reversible computation should use only bijective maps $g_i: \mathbf{B}^n \to \mathbf{B}^n$, that is, permutations of Boolean cubes. A simple but important example is the bijection $\bigoplus_n: (u,v) \mapsto (u,v\oplus u)$ on \mathbf{B}^{2n} . (Here ' \oplus ' denotes bitwise addition modulo 2.) It is obvious that the operation $\bigoplus_n [U,V]$ is equivalent to applying $\bigoplus = \bigoplus_1$ to each pair of bits. The map \bigoplus makes it possible to copy one bit into another given that the initial value of the other bit is zero. Here is a more general example: we consider an arbitrary function $F: N \to \mathbf{B}^m$ $(N \subseteq \mathbf{B}^n)$. The following map is a bijection:

$$F_{\oplus} : N \times \mathbf{B}^m \to N \times \mathbf{B}^m : (u, v) \mapsto (u, v \oplus F(u)).$$
 (3)

Now it is clear how to model any Boolean scheme by a sequence of reversible operations. First of all, we need a sufficiently large supply of 'pure' (with value zero) bits. We add them to the input word and carry out the following sequence of operations:

$$(f_i)_{\oplus} [\alpha(i,1), \ldots, \alpha(i,n_i), k_{i-1}+1, \ldots, k_i]$$
 $(i=1,\ldots, L).$

However, this computation is not fully reversible. The point is that apart from the input word it produces some 'rubbish', that is, the auxiliary bits do not remain pure. Without this rubbish the computation cannot be performed in the reverse order to return from the output word to the input one.⁵ We shall see that the rubbish prevents one from using a classical sequence of operations as a program for a quantum computer. This is why we pay so much attention to reversibility. Fortunately, it is possible to avoid the appearance of rubbish.

Now we are in a position to state the precise definition of a reversible computation, that is, a computation without rubbish. In the constructions below we shall assume that the memory Δ is the union of two non-intersecting registers: an input-output register X and a working register W. Thus, each memory state can be written as (x, w), where $x \in \mathbf{B}^X$ and $w \in \mathbf{B}^W$.

Definition 2.2. Let $G: N \to M$ $(N, M \subseteq \mathbf{B}^n)$ be an arbitrary bijection. We say that a reversible scheme, that is, a sequence of bijective operations $g_i[A_i]$ (i = 1, ..., L), represents G if their composition $g_L[A_L] \cdots g_1[A_1]$ transforms (x, 0) into (G(x), 0) for each $x \in N$.

Remark. We assume that any bit permutations are 'cost-free', that is, they are not included in the dimension of the scheme. This convention is equivalent to admitting operations of the form g[A'; A] and not just g[A]. On the other hand, two bits can be interchanged as follows:

$$[1 \leftrightarrow 2] = \bigoplus [1, 2] \bigoplus [2, 1] \bigoplus [1, 2]. \tag{4}$$

⁵Real-world computers also produce a certain kind of 'rubbish', namely, heat. The irreversibility of elementary computational operations determines the theoretical lower bound for the amount of heat produced [52]. (Present-day computers exceed this bound by many orders of magnitude.)

Proposition 2.3. We assume that $F: N \to \mathbf{B}^m$ $(N \subseteq \mathbf{B}^n)$ is computable in a basis A by means of a Boolean scheme of dimension L. Then F_{\oplus} can be represented in the basis $A_{\oplus} = \{f_{\oplus}: f \in A\}$ by a reversible scheme of dimension 2L + m.

Proof. As shown above, a Boolean scheme can be replaced by a sequence of L operations in the basis A_{\oplus} . The action of this sequence of operations can be written as $G[U,W]:(u,0)\mapsto (u,w)$, where w contains F(x) in a certain subregister $Y\subseteq W$. To construct a reversible scheme for F_{\oplus} we need an auxiliary (working) register V of dimension m. Now any memory state can be written as (u,v,w), where u,v and w are the contents of U,V and W, respectively. The desired reversible computation is given by the composition of maps

$$(G[U,W])^{-1} \bigoplus_{m} [Y,V] \quad G[U,W] : (u,v,0) \mapsto (u,v \oplus F(u),0) \quad (u \in N, v \in \mathbf{B}^m).$$

Indeed, the map G[U,W] computes a word F(u), the map $\bigoplus_m [Y,V]$ adds it to v (modulo 2), and $(G[U,W])^{-1}$ removes the rubbish, that is, makes w equal to zero. (We observe that $\bigoplus = I_{\oplus} \in \mathcal{A}_{\oplus}$, since $I \in \mathcal{A}$.)

Proposition 2.4. Let $G: N \to M$ $(N, M \subseteq \mathbf{B}^n)$ be a bijection. We assume that G and G^{-1} are computable in a basis A by means of Boolean schemes of dimension L and L', respectively. Then G can be represented by a reversible scheme of dimension 2L + 2L' + 2n in the basis A_{\oplus} .

Proof. Let X be the input-output register, and let Y be an auxiliary register of the same dimension n. It is necessary to add one more working register W, which will be used in the reversible schemes for G_{\oplus} and $(G^{-1})_{\oplus}$. According to Proposition 2.3, these schemes have dimension 2L + n and 2L' + n, respectively. The reversible scheme for G has the form

$$[X \leftrightarrow Y] \quad (G^{-1})_{\oplus}[Y,X] \quad G_{\oplus}[X,Y].$$

Indeed, $(x, 0) \mapsto (x, G(x)) \mapsto (0, G(x)) \mapsto (G(x), 0)$.

Corollary. \mathcal{C}_{\oplus} is a complete basis for reversible computations.

The element $\neg_{\oplus} \in \mathcal{C}_{\oplus}$ can be replaced by \neg . Thus, we obtain another complete basis

$$\mathcal{R} = \{I, \neg, \bigoplus, \wedge_{\oplus}\}. \tag{5}$$

(The map \wedge_{\oplus} is called a Toffoli gate.) We shall always use this basis when dealing with so-called relative computations, that is, computations with an oracle.

Let $F: \mathbf{B}^n \to \mathbf{B}^m$ be an arbitrary function, possibly a partially defined one. Any reversible scheme in the basis $\mathcal{R} \cup \{F_{\oplus}\}$ will be called a *reversible scheme with oracle F*. This definition is natural in view of Proposition 2.3.

In what follows we shall consider only one special case of an oracle. Let $F: \mathbf{Z}^k \times M \to \mathbf{B}^n$ be a function as in the definition of the problem on a stabilizer (see §1). We put

$$\mathcal{Z}^k_s = \big\{g \in \mathbf{Z}^k : \mathrm{size}(g), \mathrm{size}(-g) \leqslant s \big\}.$$

The set \mathcal{Z}_s^k can be identified with a certain subset of \mathbf{B}^s . We denote by $F|_{s+n}$ the restriction of F to $\mathcal{Z}_s^k \times M$. By a computation with oracle F we shall understand a computation in the basis $\mathcal{R} \cup \{(F|_{s+n})_{\oplus}\}$, where s = poly(k+n). For our quantum algorithm (see §6) we shall need the following bijection:

$$G \colon \mathbf{Z}^k \times M \to \mathbf{Z}^k \times M : (g, x) \mapsto (g, F(g, x))$$
 (6)

rather than F itself. We observe that $G^{-1}:(g,x)\mapsto (g,F(-g,x))$. The function $G|_{s+n}$ (a restriction of G) can be regarded as a partial permutation on $\mathbf{B}^s\times\mathbf{B}^n$. Proposition 2.4 makes it possible to express $G|_{s+n}$ in terms of $F|_{s+n}$.

Lemma 2.5. The function $G|_{s+n}$ can be represented by a reversible scheme of dimension O(n) with oracle $F|_{s+n}$. The element $(F|_{s+n})_{\oplus}$ appears twice in this scheme.

§3. Quantum formalism

3.1. Basic notions and notation. A classical system can be described by a finite set of states Γ . (In a theoretical computational context $\Gamma = \mathbf{B}^{\Delta}$ is the set of states of computer memory Δ .) The corresponding quantum system can be described by a unitary space $\overline{\Gamma} = \mathbf{C}^{\Gamma}$ with a standard orthonormal basis $\{|a\rangle : a \in \Gamma\}$. A pure quantum state is a unit vector $|\psi\rangle \in \overline{\Gamma}$ defined to within a phase factor c such that |c| = 1. (In other words, a quantum state is a one-dimensional subspace in $\overline{\Gamma}$.) The evolution of a quantum state (in a given time interval) is given by $|\psi\rangle \mapsto U|\psi\rangle$, where U is a unitary operator. To each bijection $G: \Gamma \to \Gamma$ there corresponds a unitary operator \widehat{G} acting by $\widehat{G}|a\rangle = |G(a)\rangle$. Such operators are called classical.

Example 3.1. A system with two-dimensional state space $\mathcal{B} = \overline{\mathbf{B}}$ is called a *qbit* (quantum bit). The Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
(7)

are unitary. The operator $\sigma_x = \widehat{\neg}$ is classical, but σ_y and σ_z are not.

The vectors in $\mathcal{N}=\overline{\Gamma}$ are typically denoted by $|\xi\rangle$ even if the symbol in brackets (in this case ξ) is never used separately. The scalar product of two vectors $|\xi\rangle, |\eta\rangle \in \mathcal{N}$ is denoted by $\langle \xi|\eta\rangle$. Thus, $\langle \xi|$ denotes the linear functional $|\eta\rangle \mapsto \langle \xi|\eta\rangle$ on \mathcal{N} . The space of such functionals is denoted by \mathcal{N}^* . In coordinate notation $|\xi\rangle = \sum_{j\in\Gamma} \xi_{(j)} |j\rangle$ and $\langle \xi| = \sum_{j\in\Gamma} \xi_{(j)}^* \langle j|$, or

$$|\xi\rangle = \begin{pmatrix} \xi_{(1)} \\ \vdots \\ \xi_{(k)} \end{pmatrix}, \qquad \langle \xi | = (\xi_{(1)}^{\star}, \dots, \xi_{(k)}^{\star}).$$

(Here the asterisk * denotes complex conjugation.) If $|\xi\rangle, |\eta\rangle \in \mathcal{N}$, then $|\xi\rangle\langle\eta|$ is an element in $\mathcal{N} \otimes \mathcal{N}^*$, so it can be regarded as a linear operator on \mathcal{N} . The result of

applying a linear operator $A: \mathcal{N} \to \mathcal{N}$ to a vector $|\xi\rangle$ is denoted by $A|\xi\rangle = |A\xi\rangle$. It follows that

$$\langle \xi | A \eta \rangle = \langle \xi | A | \eta \rangle = \langle A^{\dagger} \xi | \eta \rangle, \qquad \langle \xi | A = \langle A^{\dagger} \xi |,$$

where A^{\dagger} is the Hermitian adjoint of A.

The set of linear operators on \mathbb{N} is denoted by $\mathbf{L}(\mathbb{N})$. It is a finite-dimensional C^* -algebra called the *algebra of observables*. The group of unitary operators is denoted by $\mathbf{U}(\mathbb{N})$. The space of linear operators $\mathbb{N} \to \mathbb{N}'$ is denoted by $\mathbf{L}(\mathbb{N}, \mathbb{N}')$.

Let $\Pi_{\mathcal{M}}|\xi\rangle$ denote the orthogonal projection of $|\xi\rangle$ onto a linear subspace $\mathcal{M}\subseteq\mathcal{N}$. The projection operator $\Pi_{\mathcal{M}}$ can be represented as $\sum_{j=1}^{k}|e_{j}\rangle\langle e_{j}|$, where $(|e_{j}\rangle:j=1,\ldots,k)$ is an arbitrary orthonormal basis in \mathcal{M} .

Two things are required to understand quantum mechanics formally: the probabilistic interpretation and the relationship between a system and its subsystems. From the mathematical point of view the probabilistic interpretation is just the definition of some function called the 'probability'. Once the definition is stated, it is possible to verify that the function indeed has the usual properties of probability. For the time being, we shall just state the definition. The analogy with the classical case will be fully exposed in §5, where conditional probability will be introduced.

The classical probability $P(\mu, M) = \mu(M) = \sum_{j \in M} \mu(j)$ is a function of two arguments: a probability measure μ on Γ and a subset $M \subseteq \Gamma$. Correspondingly, quantum probability depends on the quantum state $|\xi\rangle$ and a linear subspace $\mathcal{M} \subseteq \mathcal{N} = \overline{\Gamma}$,

$$P(\xi, \mathcal{M}) = \langle \xi | \Pi_{\mathcal{M}} | \xi \rangle. \tag{8}$$

This quantity can also be represented as $\operatorname{Tr}(\rho\Pi_{\mathcal{M}})$, where $\rho = |\xi\rangle\langle\xi|$ is the *density* operator corresponding to the state $|\xi\rangle$. Here is the general definition: the density operator or *mixed state* is a positive Hermitian operator $\rho \in \mathbf{L}(\mathbb{N})$ with trace 1. The set of such operators is denoted by $\mathbf{D}(\mathbb{N})$. In this case

$$P(\rho, \mathcal{M}) = Tr(\rho \Pi_{\mathcal{M}}). \tag{9}$$

It is natural to consider density operators as linear functionals on the algebra $\mathbf{L}(\mathcal{N})$. The space $\mathbf{L}(\mathcal{N})^*$ of such functionals can be identified with $\mathbf{L}(\mathcal{N})$ by means of the complex scalar product (A, B) = Tr(AB).

Formula (9) covers classical probability. Indeed, let \mathcal{M} be the space generated by the standard vectors $|a\rangle$: $a\in M$. Moreover, let $\rho=\sum_{a\in\Gamma}\mu(a)|a\rangle\langle a|$, where μ is a probability measure on Γ . Then $\mathsf{P}(\rho,\mathcal{M})=\mathsf{P}(\mu,M)$. Quantum probability is additive, just as classical probability is. Namely, if $\mathcal{M}_1,\mathcal{M}_2$ are orthogonal subspaces, then

$$\mathsf{P}(\rho, \mathcal{M}_1 \oplus \mathcal{M}_2) = \mathsf{P}(\rho, \mathcal{M}_1) + \mathsf{P}(\rho, \mathcal{M}_2).$$

A typical problem in quantum mechanics has the following form. An initial state $\rho_0 \in \mathbf{D}(\mathcal{N})$ is given, and a unitary operator $U \in \mathbf{U}(\mathcal{N})$ acting as $\rho_0 \mapsto \rho = U\rho_0 U^{\dagger}$ is applied to ρ_0 . (This corresponds to the formula $|\psi\rangle \mapsto U|\psi\rangle$ for pure states.) We need to find the numbers $P(\rho, \mathcal{V}_j)$, where $\{\mathcal{V}_j\}$ is a set of subspaces in \mathcal{N} orthogonal

⁶Since Γ is a finite set, the probability measure is just a positive function $\mu:\Gamma\to\mathbf{R}$ such that $\sum_{j\in\Gamma}\mu(j)=1$.

to one another $(j \text{ runs through a finite set }\Omega)$. These numbers can be interpreted as the probabilities of various outcomes of a certain physical process. In this situation we say that a quantum variable z_{Ω} taking values $j \in \Omega$ with probability $P(\rho, \mathcal{V}_j)$ is defined. It is obvious that $\sum_{j \in \Omega} P(\rho, \mathcal{V}_j) + P(\rho, \mathcal{V}_\perp) = 1$, where \mathcal{V}_\perp is the orthogonal complement of $\bigoplus_{j \in \Omega} \mathcal{V}_j$. If \mathcal{A} is a predicate on Ω (that is, a function $\Omega \to \{\text{true}, \text{false}\}$), then $\text{Prob}_{\rho}[\mathcal{A}(z_{\Omega})]$ denotes the probability that $\mathcal{A}(z_{\Omega})$ is true,

$$\operatorname{Prob}_{\rho}[\mathcal{A}(z_{\Omega})] = \sum_{j \in \Omega \colon \mathcal{A}(j)} \mathsf{P}(\rho, \mathcal{V}_{j}).$$

For example, $\operatorname{Prob}_{\rho}[z_{\Omega} = k] = \mathsf{P}(\rho, \mathcal{V}_k)$. The notation $\operatorname{Prob}_{\rho}[\dots]$ is convenient because it expresses the intuitive meaning of probability.

Starting from §7, we shall consider combined classical-quantum systems. A combined space is any finite collection of unitary spaces $\mathcal{N}=(\mathcal{N}_j:j\in\Omega)$. To such a collection there corresponds the total space $\overline{\mathcal{N}}=\bigoplus_{j\in\Omega}\mathcal{N}_j$ and the algebra of observables $\mathbf{L}(\mathcal{N})=\bigoplus_{j\in\Omega}\mathbf{L}(\mathcal{N}_j)\subseteq\mathbf{L}(\overline{\mathcal{N}})$. It is known that any finite-dimensional C^* -algebra can be uniquely decomposed into a direct sum of matrix algebras. Therefore, there is a one-to-one correspondence between combined spaces and finite-dimensional C^* -algebras. It is also useful to keep the following fact in mind. Let \mathcal{L} be a unitary space and let $\mathcal{H}\subseteq\mathbf{L}(\mathcal{L})$ be a subalgebra with involution and unity. Then

$$\mathcal{L} = \bigoplus_{j} \mathcal{M}_{j} \otimes \mathcal{F}_{j}, \qquad \mathcal{H} = \bigoplus_{j} \mathbf{L}(\mathcal{M}_{j}) \otimes 1_{\mathcal{F}_{j}}. \tag{10}$$

A pure state in a combined space $\mathcal{N} = (\mathcal{N}_j : j \in \Omega)$ is an arbitrary unit vector $|\xi\rangle \in \mathcal{N}_j$ (to within a phase factor). A mixed state is an element of $\mathbf{D}(\mathcal{N}) = \mathbf{D}(\overline{\mathcal{N}}) \cap \mathbf{L}(\mathcal{N})$. By a combined subspace $\mathcal{M} \subseteq \mathcal{N}$ we understand a collection of linear subspaces $(\mathcal{M}_i \subseteq \mathcal{N}_i : j \in \Omega)$. The corresponding probability is given by (9).

Any physical process in a combined system can be considered in terms of the unitary space $\overline{\mathbb{N}}$. In this sense the notion of a combined space is unnecessary. However, it may be useful to know the 'classical degrees of freedom' in a problem.

Now we shall consider the union of two quantum subsystems with state spaces $\mathcal{N}_1 = \overline{\Gamma}_1$ and $\mathcal{N}_2 = \overline{\Gamma}_2$. The set of classical states of the whole system is equal to $\Gamma = \Gamma_1 \times \Gamma_2$, whereas the corresponding unitary space has the form $\mathcal{N} = \mathcal{N}_1 \otimes \mathcal{N}_2$. Two vectors $|\xi_1\rangle \in \mathcal{N}_1$ and $|\xi_2\rangle \in \mathcal{N}_2$ can be combined into $|\xi_1, \xi_2\rangle = |\xi_1\rangle \otimes |\xi_2\rangle \in \mathcal{N}$. It is also possible to define the tensor product of linear subspaces, linear operators, unitary operators and mixed states. It is clear that

$$P(\rho_1 \otimes \rho_2, \mathcal{M}_1 \otimes \mathcal{M}_2) = P(\rho_1, \mathcal{M}_1) P(\rho_2, \mathcal{M}_2). \tag{11}$$

(In the classical case $\rho_1 \otimes \rho_2$ corresponds to the joint distribution of two independent random variables.)

A principal difference between classical and quantum mechanics is that a quantum state of the whole system cannot, in general, be decomposed into states of its subsystems. There is not even a natural map $\mathbb{N} \to \mathbb{N}_1$. However, there is another natural map, the partial trace

$$\operatorname{Tr}_{\mathcal{N}_2} \colon \mathbf{L}(\mathcal{N}_1 \otimes \mathcal{N}_2) \to \mathbf{L}(\mathcal{N}_1), \qquad \operatorname{Tr}_{\mathcal{N}_2} X = \sum_{a,b \in \Gamma_1} \left(\sum_{c \in \Gamma_2} \langle a,c | X | b,c \rangle \right) |a\rangle\langle b|.$$
 (12)

The passage from a mixed state $\rho \in \mathbf{D}(\mathbb{N})$ to a mixed state $\rho_1 = \operatorname{Tr}_{\mathbb{N}_2} \rho \in \mathbf{D}(\mathbb{N}_1)$ can be interpreted as the exclusion of the second subsystem from consideration. (This operation is analogous to the projection of the joint distribution onto one of the variables.) The passage from ρ to ρ_1 makes sense if we are interested in probabilities of the form $\mathsf{P}(U\rho U^{\dagger}, \mathbb{M})$, where $U = U_1 \otimes U_2$ (that is, the subsystems evolve independently) and $\mathbb{M} = \mathbb{M}_1 \otimes \mathbb{N}_2$ (that is, we observe only the first subsystem). Indeed,

$$\mathsf{P}(\rho, \mathcal{M}_1 \otimes \mathcal{N}_2) = \mathsf{P}(\mathrm{Tr}_{\mathcal{N}_2} \, \rho, \mathcal{M}_1), \qquad \mathrm{Tr}_{\mathcal{N}_2} \big((U_1 \otimes U_2) \rho (U_1 \otimes U_2)^{\dagger} \big) = U_1 (\mathrm{Tr}_{\mathcal{N}_2} \, \rho) U_1^{\dagger}.$$

It remains to recall three properties of partial trace. First, any mixed state $\rho \in \mathbf{D}(\mathbb{N})$ can be represented as

$$\rho = \operatorname{Tr}_{\mathcal{N}^*} (|\xi\rangle\langle\xi|), \qquad |\xi\rangle = \sqrt{\rho} \in \mathbf{L}(\mathcal{N}) = \mathcal{N} \otimes \mathcal{N}^*. \tag{13}$$

Second, the state $\rho_1 = \operatorname{Tr}_{\mathcal{N}_2} \rho$ $(\rho \in \mathbf{D}(\mathcal{N}_1 \otimes \mathcal{N}_2))$ is pure if and only if $\rho = (|\xi_1\rangle\langle\xi_1|) \otimes \rho_2$, where $|\xi_1\rangle \in \mathcal{N}_1$ and $\rho_2 \in \mathbf{D}(\mathcal{N}_2)$. Third, the following property holds.

Lemma 3.2. Let $|\xi\rangle \in \mathcal{N}_1 \otimes \mathcal{N}_2$, let $\rho_1 = \operatorname{Tr}_{\mathcal{N}_2}(|\xi\rangle\langle\xi|)$, and let $\rho_2 = \operatorname{Tr}_{\mathcal{N}_1}(|\xi\rangle\langle\xi|)$. Then the non-zero eigenvalues of ρ_1 and ρ_2 are the same.

Proof. We diagonalize ρ_1 and write $|\xi\rangle = \sum_j |j, \eta_j\rangle$, where $|\eta_j\rangle \in \mathcal{N}_2$. Then $\langle \eta_j | \eta_k \rangle = \lambda_j \delta_{jk}$, where λ_j is the jth eigenvalue of ρ_1 .

3.2. Transformations of mixed states. The results of this chapter will, in principle, be used in the second part of the article only. Nevertheless, examples of admissible transformations (see below) are quite instructive. It makes sense to consider them right now.

Any operators of the form $\mathbf{L}(\mathcal{N}) \to \mathbf{L}(\mathcal{M})$, where \mathcal{N} and \mathcal{M} are unitary spaces, will be called *transformations*. The space of transformations will be denoted by $\mathbf{T}(\mathcal{N}, \mathcal{M})$. We shall also use the notation $\mathbf{T}(\mathcal{N}) = \mathbf{T}(\mathcal{N}, \mathcal{N})$. The identity transformation will be denoted by $I_{\mathcal{N}}$.

For any transformation $T \in \mathbf{T}(\mathcal{N}, \mathcal{M})$ we can define the adjoint transformation $T^* \in \mathbf{T}(\mathcal{M}, \mathcal{N})$ (in the sense of the scalar product $(A, B) = \mathrm{Tr}(AB)$). To each pair of operators $A \in \mathbf{L}(\mathcal{N}, \mathcal{M})$ and $B \in \mathbf{L}(\mathcal{M}, \mathcal{N})$ there corresponds the transformation $A \cdot B \in \mathbf{T}(\mathcal{N}, \mathcal{M})$ acting by $X \mapsto AXB$.

As we have already seen, transformations of the form $U \cdot U^{\dagger}$, where $U \in \mathbf{U}(\mathcal{N})$, describe the evolution of mixed states. Such transformations, which are called unitary, form a group $\mathbf{UT}(\mathcal{N}) \cong \mathbf{U}(\mathcal{N})/\mathbf{U}(1)$, where $\mathbf{U}(1) = \{c \in \mathbf{C} : |c| = 1\}$. A transformation is unitary if and only if it is an automorphism of the algebra with involution $\mathbf{L}(\mathcal{N})$.

Apart from unitary ones, there are other transformations which can be realized physically. A typical process, which we need to consider, involves the interaction of a quantum system with the surrounding medium. First the state $\rho \in \mathbf{D}(\mathbb{N})$ of the system is combined with the state of the environment $\lambda \in \mathcal{G}$ (that is, their tensor product is taken). Then the combined state evolves, which can be described by a unitary operator $U: \mathbb{N} \otimes \mathcal{G} \to \mathbb{M} \otimes \mathcal{F}$. (Here \mathbb{M} and \mathcal{F} are the spaces of states of

the system and the environment after the interaction.) Finally, the environment is excluded from consideration, which gives the state

$$\rho' = \operatorname{Tr}_{\mathcal{F}}(U(\rho \otimes \lambda)U^{\dagger}) = T\rho \in \mathbf{D}(\mathcal{M}).$$

The transformations T that can be obtained by means of this scheme are called *admissible*. We observe that λ can always be assumed to be a pure state. (It suffices to replace \mathcal{G} by $\mathcal{G} \otimes \mathcal{G}^*$ and use (13).)

Example 3.3. Let $\mathcal{N} = \mathcal{M} = \mathcal{G} = \mathcal{F} = \overline{\mathbf{B}}$. We put $\lambda = |0\rangle\langle 0|$ and $U = \widehat{\oplus} : |a,b\rangle \mapsto |a,a\oplus b\rangle$ (here ' \oplus ' denotes addition modulo 2). Then $\langle a|\rho'|b\rangle = \delta_{ab}\langle a|\rho|a\rangle$, that is, T extracts the diagonal part of the density matrix ρ . On the other hand, $T = \frac{1}{2}I_{\mathcal{B}} + \frac{1}{2}\sigma_z \cdot \sigma_z$. It follows that T can be interpreted as a random phase shift: with probability $\frac{1}{2}$ the relative phase between the states $|0\rangle$ and $|1\rangle$ changes by π (this is how σ_z acts). This is a manifestation of a general principle: the interaction with the environment plays the role of randomness for quantum systems.

The class of admissible transformations can be characterized by a very simple set of properties. Prior to stating these properties, we observe that

$$\mathbf{T}(\mathcal{N}, \mathcal{M}) = \mathbf{L}(\mathcal{M}) \otimes \mathbf{L}(\mathcal{N})^* \cong \mathcal{M} \otimes \mathcal{N}^* \otimes \mathcal{M}^* \otimes \mathcal{N} = \mathbf{L}(\mathcal{N}, \mathcal{M}) \otimes \mathbf{L}(\mathcal{N}, \mathcal{M})^*. \tag{14}$$

By analogy with the notation $A \cdot B$ we can write $\mathbf{T}(\mathcal{N}, \mathcal{M}) = \mathbf{L}(\mathcal{N}, \mathcal{M}) \cdot \mathbf{L}(\mathcal{N}, \mathcal{M})^*$. It follows that each transformation $T \in \mathbf{T}(\mathcal{N}, \mathcal{M})$ can be regarded as an operator $\check{T} : \mathbf{L}(\mathcal{N}, \mathcal{M}) \to \mathbf{L}(\mathcal{N}, \mathcal{M})$. In tensor notation (where summation over repeated indices is performed)

$$(TX)_{(ik)} = T_{(ijkl)}X_{(jl)}, \qquad (\check{T}X)_{(ij)} = T_{(ijkl)}X_{(kl)}, \qquad (T^*X)_{(jl)} = T_{(ijkl)}X_{(ik)}.$$

Theorem 3.4. A transformation $T \in \mathbf{T}(\mathbb{N}, \mathbb{M})$ is admissible if and only if the following three conditions hold:

- 1) $\operatorname{Tr}(TX) = \operatorname{Tr} X$ for any $X \in \mathbf{L}(\mathbb{N})$ or (which is equivalent) $T^*1_{\mathbb{M}} = 1_{\mathbb{N}}$;
- 2) $(TX)^{\dagger} = TX^{\dagger}$ for any $X \in \mathbf{L}(\mathbb{N})$, that is, $T^*_{(ijkl)} = T_{(klij)}$; in other words, T is a Hermitian operator;
- 3) \tilde{T} is a positive operator.

Transformations satisfying conditions 2) and 3) are called *completely positive*.

Proof of Theorem 3.4. Admissibility \Rightarrow 1)-3). Only condition 3) is not completely obvious. Suppose that $\lambda = |\xi\rangle\langle\xi|$ in the scheme described above. Then $T_{(ijkl)} = V_{(im,j)}V_{(km,l)}^*$, where $V_{(im,j)} = U_{(im,jr)}\xi_{(r)}$. It follows that \check{T} is a positive operator.

1)-3) \Rightarrow admissibility. Let $T \in \mathbf{T}(\mathcal{N}, \mathcal{M})$ satisfy conditions 1)-3). We define the canonical representation of T to be $\mathrm{Tr}_{\mathcal{F}}(V \cdot V^{\dagger})$, where $\mathcal{F} = \mathcal{M}^* \otimes \mathcal{N}$ and $V \in \mathbf{L}(\mathcal{N}, \mathcal{M} \otimes \mathcal{F})$. Namely, we put $V_{(ims,j)} = R_{(ijms)}$, where $\check{R} = \check{T}^{1/2}$. It follows that $V_{(ims,j)}V_{(kms,l)}^* = T_{(ijkl)}$. Condition 1) implies that V is a unitary embedding.

Let $\mathcal{G} = \mathcal{M} \otimes \mathcal{M}^*$ and let $|\xi\rangle \in \mathcal{G}$ be an arbitrary unit vector. We consider the operator $W: |\psi, \xi\rangle \mapsto V|\psi\rangle$ with domain $\mathcal{N} \otimes (|\xi\rangle) \subseteq \mathcal{N} \otimes \mathcal{G}$. The spaces $\mathcal{N} \otimes \mathcal{G}$ and $\mathcal{M} \otimes \mathcal{F}$ have the same dimension, so W can be extended to a unitary operator $U: \mathcal{N} \otimes \mathcal{G} \mapsto \mathcal{M} \otimes \mathcal{F}$. It remains to put $\lambda = |\xi\rangle\langle\xi|$.

Lemma 3.5. Let $T \in \mathbf{T}(\mathbb{N}, \mathbb{M})$ be a completely positive transformation, and let $V \in \mathbf{L}(\mathbb{N}, \mathbb{M} \otimes \mathbb{F})$ and $W \in \mathbf{L}(\mathbb{N}, \mathbb{M} \otimes \mathbb{F})$ be its representations, that is, $T = \mathrm{Tr}_{\mathcal{F}}(V \cdot V^{\dagger}) = \mathrm{Tr}_{\mathcal{F}}(W \cdot W^{\dagger})$. We assume that $\dim \mathcal{F} \leqslant \dim \mathcal{G}$. Then there is a unitary embedding $U : \mathcal{F} \to \mathcal{G}$ such that $W = (1_{\mathbb{M}} \otimes U)V$.

Proof. The operators V and W can be considered as vectors in $\mathbf{L}(\mathcal{N}, \mathcal{M}) \otimes \mathcal{F}$ and $\mathbf{L}(\mathcal{N}, \mathcal{M}) \otimes \mathcal{G}$. (To emphasize this we shall use the notation $|V\rangle$ and $|W\rangle$.) It is obvious that

$$\operatorname{Tr}_{\mathcal{F}}(|V\rangle\langle V|) = \operatorname{Tr}_{\mathcal{G}}(|W\rangle\langle W|) = \check{T}.$$

By Lemma 3.2 the non-zero eigenvalues of

$$A = \operatorname{Tr}_{\mathbf{L}(\mathcal{N},\mathcal{M})}(|V\rangle\langle V|)$$
 and $B = \operatorname{Tr}_{\mathbf{L}(\mathcal{N},\mathcal{M})}(|W\rangle\langle W|)$

are the same. It follows that there is a unitary embedding $U: \mathcal{F} \to \mathcal{G}$ transforming A into B. The operator $1_{\mathbf{L}(\mathcal{N},\mathcal{M})} \otimes U$ transforms $|V\rangle$ into $|W\rangle$.

The following proposition, which is used in the theory of quantum codes, can be interpreted as the impossibility of observation without perturbation. (\mathcal{M} is to be regarded as the observed object and \mathcal{F} as the space of states of the observer.)

Proposition 3.6. Let $T \in \mathbf{T}(\mathcal{M}, \mathcal{M} \otimes \mathcal{F})$ be an admissible transformation such that $\mathrm{Tr}_{\mathcal{F}}T = I_{\mathcal{M}}$. Then there exists $\lambda \in \mathbf{D}(\mathcal{F})$ such that $TX = X \otimes \lambda$ (for any $X \in \mathbf{L}(\mathcal{M})$).

Proof. Without loss of generality we can assume that $T = V \cdot V^{\dagger}$, where $V : \mathcal{M} \to \mathcal{M} \otimes \mathcal{F}$ is a unitary embedding. (It suffices to replace T by its canonical representation.) Then for any $|\psi\rangle \in \mathcal{M}$ the state $\mathrm{Tr}_{\mathcal{F}}(|V\psi\rangle\langle V\psi|) = \mathrm{Tr}_{\mathcal{F}}T(|\psi\rangle\langle\psi|) = |\psi\rangle\langle\psi|$ is pure, so $|V\psi\rangle = |\psi,\xi(\psi)\rangle$. By linearity $|\xi(\psi)\rangle = |\xi\rangle$ is independent of $|\psi\rangle$. It follows that $TX = X \otimes (|\xi\rangle\langle\xi|)$.

Finally, we shall consider transformations of combined states. Let $\mathbb{N} = (\mathbb{N}_j : j \in \Omega)$ and $\mathbb{M} = (\mathbb{M}_k : k \in \Theta)$ be combined spaces. Then $\mathbf{T}(\mathbb{N}, \mathbb{M}) = \bigoplus_{j,k} \mathbf{T}(\mathbb{N}_j, \mathbb{M}_k)$. The natural embedding $\mathbf{T}(\mathbb{N}, \mathbb{M}) \to \mathbf{T}(\overline{\mathbb{N}}, \overline{\mathbb{M}})$ is defined by $P \mapsto Q_{\mathbb{M}}PQ_{\mathbb{N}}^*$. Here $Q_{\mathbb{M}} : \mathbf{L}(\mathbb{M}) \to \mathbf{L}(\overline{\mathbb{M}})$ is the natural embedding of algebras with involution. (For example, the transformation $T \in \mathbf{T}(\mathcal{B})$ in Example 3.3 is the image of the classical identity transformation $I_{\mathbf{B}}$, that is, $T = Q_{\mathbf{B}}I_{\mathbf{B}}Q_{\mathbf{B}}^*$.) A combined transformation $P \in \mathbf{T}(\mathbb{N}, \mathbb{M})$ is called admissible if $Q_{\mathbb{M}}PQ_{\mathbb{N}}^*$ is an admissible transformation.

Admissible classical transformations have the form

$$T_{(ijkl)} = t_{ij}\delta_{ik}\delta_{jl}, \qquad t_{ij} \geqslant 0, \quad \sum_{i} t_{ij} = 1$$
 (15)

(no summation over repeated indices is performed).

Example 3.7. Let $\mathcal{N} = \bigoplus_{j \in \Omega} \mathcal{V}_j$ be a unitary space represented as a sum of subspaces orthogonal to one another. We put

$$MEAS_{\Omega} \colon \mathbf{L}(\mathcal{N}) \to \mathbf{L}(\mathcal{N}) \times \Omega : \quad \rho \mapsto \sum_{j \in \Omega} (\Pi_{\mathcal{V}_j} \rho \Pi_{\mathcal{V}_j}, j). \tag{16}$$

This admissible transformation is called the *measurement* of the quantum variable z_{Ω} . (The combined space $\mathbf{L}(\mathcal{N}) \times \Omega$ corresponds to the union of the classical and quantum subsystems; the state of the classical subsystem is called the measurement result.) The measurement MEAS_{\Omega} applied to a state $\rho \in \mathbf{D}(\mathcal{N})$ gives a result j with probability $\text{Tr}(\Pi_{\mathcal{V}_j}\rho\Pi_{\mathcal{V}_j}) = \mathbf{P}(\rho,\mathcal{V}_j)$. At the same time the quantum subsystem jumps to a new state $\rho' = \mathbf{P}(\rho,\mathcal{V}_j)^{-1}\Pi_{\mathcal{V}_j}\rho\Pi_{\mathcal{V}_j}$. In physics such a process is called the reduction of the quantum state.

3.3. Accuracy. In the theory of quantum computations (as in real life) the operators, quantum states, and so on, are given to within some accuracy. Therefore it becomes necessary to define metrics on the corresponding spaces. To begin with, we shall define a natural metric on the space of probability distributions.

Suppose that some device is to generate a random variable with probability distribution w(a): $a \in \Gamma$. If the performance of the device is subject to an error occurring with probability $\leq \varepsilon$, the actual distribution \widetilde{w} will satisfy the condition $\sum_{a \in \Gamma} |\widetilde{w}(a) - w(a)| \leq 2\varepsilon$. Conversely, if this condition is satisfied, then it is possible to claim that the probability of an error does not exceed 2ε . Thus, a natural metric on the space of distributions is given by the ℓ_1 -norm $||v||_1 = \sum_{a \in \Gamma} |v(a)|$.

The accuracy of $|\xi\rangle \in \overline{\Gamma}$ can be characterized by the ordinary (Euclidean) norm $||\xi\rangle|| = \sqrt{\langle \xi | \xi \rangle}$. There are two natural norms on the space of linear operators $\mathbf{L}(\Gamma)$: the ordinary operator norm

$$||A|| = \sup_{|\xi\rangle \neq 0} \frac{||A|\xi\rangle||}{||\xi\rangle||}$$

and the following trace norm, which is dual to the latter:

$$||A||_{1} = \sup_{B \neq 0} \frac{|\operatorname{Tr} AB|}{||B||} = \inf \left\{ \sum_{i} ||\xi_{j}\rangle|| \, ||\langle \eta_{j}|| : \sum_{i} |\xi_{j}\rangle \langle \eta_{j}| = A \right\} = \operatorname{Tr} \sqrt{A^{\dagger} A}. \quad (17)$$

The norm $\|\cdot\|$ characterizes the accuracy of unitary operators, while $\|\cdot\|_1$ is suitable for mixed states (because it is similar to the ordinary ℓ_1 -norm, see also Proposition 3.8 below).

There is some (apparently, unavoidable) inconsistency in the definition of the norms of pure and mixed states. Accuracy is preserved as one passes from pure states to mixed ones. If $|\xi\rangle$, $|\eta\rangle$ are unit vectors, then

$$\||\xi\rangle\langle\xi| - |\eta\rangle\langle\eta|\|_{1} = 2\sqrt{1 - |\langle\xi|\eta\rangle|^{2}} \leqslant 2\||\xi\rangle - |\eta\rangle\|. \tag{18}$$

However, the passage in the opposite direction (according to (13)) involves a loss of accuracy. If $\rho, \gamma \in \mathbf{D}(\mathbb{N})$, then

$$\|\sqrt{\rho} - \sqrt{\gamma}\|_2 \leqslant \sqrt{\|\rho - \gamma\|_1}, \quad \text{where} \quad \|A\|_2 = \sqrt{\operatorname{Tr} A^{\dagger} A}.$$

(The norm $\|\cdot\|_2$ is the ordinary Euclidean norm in $\mathcal{N} \otimes \mathcal{N}^*$.) This estimate cannot be improved.

The norms $\|\cdot\|$ and $\|\cdot\|_1$ have the following basic properties:

$$||AB|| \le ||A|| ||B||, \qquad ||AB||_1, ||BA||_1 \le ||B|| ||A||_1, \qquad |\operatorname{Tr} A| \le ||A||_1.$$
 (19)

They are also well behaved with respect to the tensor product. If $A \in \mathbf{L}(\mathbb{N})$, $B \in \mathbf{L}(\mathbb{M})$ and $C \in \mathbf{L}(\mathbb{N} \otimes \mathbb{M})$, then

$$||A \otimes B|| \le ||A|| \, ||B||, \qquad ||A \otimes B||_1 \le ||A||_1 ||B||_1, \qquad ||\operatorname{Tr}_{\mathfrak{M}} C||_1 \le ||C||_1.$$
 (20)

Proposition 3.8. Let Ω be a set of subspaces of $\mathbb N$ orthogonal to one another. Then for any pair of mixed states ρ, γ

$$\sum_{\mathcal{V} \in \Omega} |\mathsf{P}(\rho, \mathcal{V}) - \mathsf{P}(\gamma, \mathcal{V})| \leqslant \|\rho - \gamma\|_{1}.$$

Proof. The left-hand side of this inequality can be represented as $\text{Tr}((\rho - \gamma)B)$, where $B = \sum_{v \in \Omega} (\pm \Pi_v)$. It is clear that $||B|| \leq 1$. Now use the properties (19) of the norms.

A natural norm on the space of transformations is given by

$$||T||_1 = \sup_{X \neq 0} \frac{||TX||_1}{||X||_1} = \sup_{X \neq 0} \frac{||T^*X||}{||X||}.$$

Unfortunately, this norm is unstable relative to the tensor product. Example: we consider the transformation $T:|i\rangle\langle j|\mapsto |j\rangle\langle i|$ (i,j=0,1). It is clear that $||T||_1=1$, however $||T\otimes I_{\mathcal{B}}||_1=2$. (Apply the transformation $T\otimes I_{\mathcal{B}}$ to the operator $X=\sum_{i,j}|i,i\rangle\langle j,j|$.) This is why we shall use another norm on the space of transformations $T(\mathcal{N},\mathcal{M})$:

$$||T||_{\diamondsuit} = \inf\{||A|| \, ||B|| : \operatorname{Tr}_{\mathcal{F}}(A \cdot B^{\dagger}) = T\}, \qquad A, B \in \mathbf{L}(\mathcal{N}, \mathcal{M} \otimes \mathcal{F}). \tag{21}$$

Here \mathcal{F} is an arbitrary unitary space of dimension $\geqslant (\dim \mathcal{N})(\dim \mathcal{M})$. The following result implies that this is indeed a norm.

Proposition 3.9. $||T||_{\diamondsuit} = ||T \otimes I_{\mathcal{G}}||_{1} \geqslant ||T||_{1}$, where dim $\mathcal{G} \geqslant \dim \mathcal{N}$.

Proof. The inequality $||T||_{\diamondsuit} \le ||T \otimes I_{\S}||_{1}$ is the only non-obvious fact. We take $||T||_{\diamondsuit} = 1$ without loss of generality. The infimum in (21) is attained when ||A|| = ||B|| = 1. Let $\mathcal{K} = \operatorname{Ker}(A^{\dagger}A - 1_{\mathcal{N}})$, $\mathcal{L} = \operatorname{Ker}(B^{\dagger}B - 1_{\mathcal{N}})$, and

$$E = \big\{ \mathrm{Tr}_{\mathfrak{M}}(A \rho A^{\dagger}) : \rho \in \mathbf{D}(\mathfrak{K}) \big\}, \qquad F = \big\{ \mathrm{Tr}_{\mathfrak{M}}(B \gamma B^{\dagger}) : \gamma \in \mathbf{D}(\mathcal{L}) \big\}.$$

The number ||A|| ||B|| is minimal with respect to infinitesimal variations of the scalar product $\delta\langle\cdot|\cdot\rangle = \langle\cdot|Z|\cdot\rangle$ on \mathfrak{F} . It follows that no Hermitian operator $Z\in\mathbf{L}(\mathfrak{F})$ exists such that $\mathrm{Tr}\, XZ>\mathrm{Tr}\, YZ$ for any $X\in E$ and $Y\in F$. Since E and F are compact convex sets, $E\cap F\neq\varnothing$.

Let $\operatorname{Tr}_{\mathfrak{M}}(A\rho A^{\dagger}) = \operatorname{Tr}_{\mathfrak{M}}(B\gamma B^{\dagger})$, where $\rho \in \mathbf{D}(\mathfrak{K})$ and $\gamma \in \mathbf{D}(\mathfrak{L})$. We shall represent ρ and γ as $\rho = \operatorname{Tr}_{\mathfrak{G}}(|\xi\rangle\langle\xi|)$ and $\gamma = \operatorname{Tr}_{\mathfrak{G}}(|\eta\rangle\langle\eta|)$, where $|\xi\rangle, |\eta\rangle \in \mathfrak{N} \otimes \mathfrak{G}$ are unit vectors. We put $X = |\xi\rangle\langle\eta|$. Then $\|(T \otimes I_{\mathfrak{G}})X\|_1 = \|X\|_1 = 1$.

From (21) and Proposition 3.9 it follows that

$$||T \otimes R||_{\Diamond} = ||T||_{\Diamond} ||R||_{\Diamond}, \qquad ||TR||_{\Diamond} \leqslant ||T||_{\Diamond} ||R||_{\Diamond}. \tag{22}$$

The norm of an admissible transformation is always equal to one. If V and W are unitary embeddings, then $||V \cdot V^{\dagger} - W \cdot W^{\dagger}||_{\diamondsuit} \leq 2||V - W||$.

Lemma 3.10. Let $T, P \in \mathbf{T}(\mathbb{N}, \mathbb{M})$ be admissible transformations and let V and W be their canonical representations. Then $||V - W|| \leq (\dim \mathbb{M}) \sqrt{||T - P||_{\diamondsuit}}$.

Proof. To each transformation $R \in \mathbf{T}(\mathbb{N}, \mathbb{M})$ there corresponds an operator $\check{R} : \mathbf{L}(\mathbb{N}, \mathbb{M}) \to \mathbf{L}(\mathbb{N}, \mathbb{M})$. With the latter we can, in turn, associate the operator $\check{R}^{\triangleright} : \mathbb{N} \to \mathbb{M} \otimes \mathbb{M}^* \otimes \mathbb{N}$ with components $(\check{R}^{\triangleright})_{(ikl,j)} = R_{(ijkl)}$. In this notation $V = (\check{T}^{1/2})^{\triangleright}$ and $W = (\check{P}^{1/2})^{\triangleright}$. It is easy to see that for any $R \in \mathbf{T}(\mathbb{N}, \mathbb{M})$

$$\|\check{R}\| \leqslant (\dim \mathcal{M}) \|R\|_{\diamond}, \qquad \|\check{R}^{\triangleright}\| \leqslant (\dim \mathcal{M})^{1/2} \|\check{R}\|.$$

It follows that

$$||V - W|| \le (\dim \mathcal{M})^{1/2} ||\check{T}^{1/2} - \check{P}^{1/2}||$$

$$\le (\dim \mathcal{M})^{1/2} ||\check{T} - \check{P}||^{1/2} \le (\dim \mathcal{M}) \sqrt{||T - P||_{\diamondsuit}}.$$

§4. Quantum models of computations

4.1. Definitions and basic properties. We intend to define a quantum model similar to an irreversible (that is, generating rubbish) classical computation with reversible operations. A reversible quantum computation will be defined later on. First of all, we shall discuss which actions over a quantum system can be realized physically. Starting from this point, we shall consider systems with the classical set of states $\Gamma = \mathbf{B}^{\Delta} = \mathbf{B}^{l}$ (where $\Delta = \{1, \dots, l\}$ is the set of bits). The corresponding unitary space has the form $\overline{\Gamma} = \mathcal{B}^{\otimes \Delta}$.

We consider a unitary operator $U \in \mathbf{U}(\mathbb{B}^{\otimes n})$ and a register $A \subseteq \Delta$ of dimension n. In this situation U[A] denotes the action of U on $\mathbb{B}^{\otimes A}$. This action can be extended to the whole space of states $\mathbb{B}^{\otimes \Delta}$ by taking its tensor product with the identity operator $1[\Delta \setminus A] \in \mathbf{U}(\mathbb{B}^{\otimes (\Delta \setminus A)})$. As in the classical case, operators of the form $U[A] \in \mathbb{B}^{\otimes \Delta}$ are called *operations*. It is possible to realize such operators physically if n is small.

For each $y \in \mathbf{B}^n$ we consider the *standard space*

$$\mathcal{W}_{y}[A] = (|y\rangle[A]) \otimes \mathcal{B}^{\otimes(\Delta\backslash A)} \subseteq \mathcal{B}^{\otimes\Delta}. \tag{23}$$

These subspaces are orthogonal to one another. In this way a standard quantum variable z_A taking values in \mathbf{B}^n can be defined. For a given state $|\xi\rangle$ of the quantum system we can measure the value of z_A , that is, set up a physical procedure producing a result y with probability $P(|\xi\rangle, W_y[A])$. The measurement will destroy the quantum state, so it should be performed at the very end of the computation.

We shall proceed to a more formal discussion. Suppose that the memory Δ consists of an input register X and a working register W. The output register $Y \subseteq \Delta$ is arbitrary (here |X| = n and |Y| = m). The classical memory states can be written as (x, w), where $x \in \mathbf{B}^X$ and $w \in \mathbf{B}^W$.

Definition 4.1. Let \mathcal{A} be a set of unitary operators (a basis) and let $0 < \varepsilon < \frac{1}{2}$ be an arbitrary constant. We say that a quantum scheme (that is, a sequence of operations) $U_1[A_1], \ldots, U_L[A_L]$ ($U_i \in \mathcal{A}$) computes a function $F: N \to \mathbf{B}^m$ ($N \subseteq \mathbf{B}^n$) with probability of error $\leqslant \varepsilon$ if

$$\forall x \in N \quad \mathsf{P}\big(|\xi(x)\rangle, \, \mathcal{W}_{F(x)}[Y]\big) \geqslant 1 - \varepsilon, \quad \text{where} \quad |\xi(x)\rangle = U_L[A_L] \cdots U_1[A_1]|x,0\rangle.$$

The probability of error can be made as small as desired by repeating the computation several times. Indeed, we can take k different copies of the memory and for each of them we can perform the same computation with the same input word $x \in N$. Thanks to (11) the corresponding output words y_1, \ldots, y_k can be regarded as independent random variables. If more than half of the y_i coincide and are equal to y, then this y is taken as the final result. This 'voting procedure' can be represented by $y = \text{MAJ}(y_1, \ldots, y_k)$, where MAJ is a (partially defined) function called the majority function. If the computation is set up in this way, then the total probability of an error does not exceed $\sum_{j\geqslant k/2} \binom{k}{j} \varepsilon^j (1-\varepsilon)^{k-j} \leqslant \lambda^k$, where $\lambda = 2(\varepsilon(1-\varepsilon))^{1/2} < 1$. It follows that at the cost of increasing the size of the scheme polynomially the probability of an error can be reduced to $\exp(-p(n))$, where p is an arbitrary function of polynomial growth. The original choice of ε is not important, $\varepsilon = \frac{1}{3}$ being the usual choice.

Remark. The function MAJ can be computed by a Boolean scheme of dimension $O(k \log k)$. However, to remain in the framework of Definition 4.1 we must represent the operator $\widehat{\text{MAJ}}_{\oplus}$ in the basis \mathcal{A} . This is possible, for example, in the complete quantum basis Ω , see below.

Proposition 4.2. Suppose that a quantum scheme of dimension L computes a function F with probability of error $\leq \varepsilon$. If each operation of this scheme is performed with accuracy δ , then the resulting probability of error does not exceed $\varepsilon + 2L\delta$. (It follows that an accuracy $\delta < \text{const } L^{-1}$ is sufficient for the scheme to work reliably.)

Proof. Let $U_1, \ldots, U_L, \widetilde{U_1}, \ldots, \widetilde{U_L}$ be unitary operators. If $\|\widetilde{U}_j - U_j\| \leq \delta$, then $\|\widetilde{U}_L \cdots \widetilde{U}_1 - U_L \cdots U_1\| \leq L\delta$. Now we can use (18) and Proposition 3.8.

Choice of the basis. In this article we use the standard basis $Q = \{1_{\mathcal{B}}, S, K, \widehat{\bigoplus}, \widehat{\wedge}_{\oplus}\}$, where

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1\\ 1 & -1 \end{pmatrix}, \qquad K = \begin{pmatrix} 1 & 0\\ 0 & i \end{pmatrix}. \tag{24}$$

If an oracle F is given, then we supplement the basis with the operator \widehat{F}_{\oplus} .⁷ We shall see that Ω is a complete basis in the approximate sense, see Theorem 4.8 below. (Roughly speaking, it generates an everywhere dense set of unitary operators.) On the completeness of some other bases, see [53].

⁷If F is a partially defined function, then the operator \widehat{F}_{\oplus} is also a partially defined one. (A partially defined unitary operator is a unitary linear map from one subspace to another.)

Modelling of classical probability. (Compare with Example 3.3.) To model a classical probabilistic computation we must be able to generate random bits. We take a quantum bit in the state $S|0\rangle = 2^{-1/2}(|0\rangle + |1\rangle)$ and copy it into another qbit with the aid of $\widehat{\oplus}$. (Warning: the operator $\widehat{\oplus}$ copies each basis state appearing in the quantum superposition, rather than the superposition as a whole!) In this way we obtain a two-bit quantum state $|\psi\rangle = 2^{-1/2}(|0,0\rangle + |1,1\rangle)$. Now we destroy one copy (or we simply refrain from using it in what follows). This situation can be described as a passage to the density operator of the remaining qbit

$$\rho' = \mathrm{Tr}_2 \big(|\psi\rangle \langle \psi| \big) = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}.$$

This density operator corresponds to the classical probability measure $\mu(0) = \mu(1) = \frac{1}{2}$.

Reversible quantum computation. Definition 2.2 can immediately be extended to the quantum case. From the technical point of view it is useful to restate this definition in terms of partially defined operators. A partially defined unitary operator on \mathbb{N} is a unitary operator $U: \mathcal{L} \to \mathbb{M}$, where $\mathcal{L}, \mathbb{M} \subseteq \mathbb{N}$. For example, $\tau: |0\rangle \mapsto |0\rangle$ is a partially defined unitary operator on $B^{\otimes k}$ (k is arbitrary). The set of partially defined unitary operators on \mathbb{N} will be denoted by $\check{\mathbf{U}}(\mathbb{N})$.

The notion of accuracy is applicable to partially defined unitary operators. Let $U: \mathcal{L} \to \mathcal{M}$ and $\widetilde{U}: \widetilde{\mathcal{L}} \to \widetilde{\mathcal{M}}$ be partially defined unitary operators on \mathcal{N} . We shall say that \widetilde{U} represents U with accuracy δ if $\mathcal{L} \subseteq \widetilde{\mathcal{L}}$, $\mathcal{M} \subseteq \widetilde{\mathcal{M}}$ and $\|(\widetilde{U}-U)|\xi\rangle\| \leq \delta \|\xi\rangle\|$ for any $|\xi\rangle \in \mathcal{L}$. We observe that \widetilde{U}^{-1} represents U^{-1} with the same accuracy.

Let $\Delta = X \cup W$ be the memory (a set of qbits) and let |X| = n and $X \cap W = \emptyset$.

Definition 4.3. We say that a quantum scheme (that is, a sequence of operations) $U_1[A_1], \ldots, U_L[A_L]$ represents an operator $U \in \check{\mathbf{U}}(\mathcal{B}^{\otimes n})$ (with accuracy δ) if the product $U_L[A_L] \cdots U_1[A_1]$ represents the partially defined operator $U[X] \otimes \tau[W]$ (with accuracy δ).

A quantum scheme can therefore *represent* an operator, but *compute* a Boolean function.

Exercise. (Compare with Proposition 2.4.) Suppose that functions G and G^{-1} can be computed by means of known quantum schemes Ψ and Ψ' with probability of error $\leq \frac{1}{3}$. Represent the operator $U = \widehat{G}$ with arbitrary accuracy $\delta > 0$. (The methods of §2.2 work in the quantum case, one only needs to deal with the accuracy and the probability of error with sufficient precision.)

Effect of rubbish. In the classical formulation the problem of reversibility looks quite artificial. If rubbish appears for some operations, it can simply be ignored. The result will be the same as for a reversible computation. This is not so in the quantum case. Let $G: \mathbf{B}^n \to \mathbf{B}^n$ be a bijection. Suppose that a quantum scheme uses the operator \widehat{G} as a subroutine. What will happen if we replace \widehat{G} by another classical operator $V: |x,0\rangle \mapsto |G(x),g(x)\rangle$ (where g(x) is the rubbish)? The operator V transforms a quantum state $|\xi\rangle = \sum_x c_x |x\rangle$ into the state $|\psi\rangle = \sum_x c_x |G(x),g(x)\rangle$. Since the rubbish is ignored, we must compute the partial

trace with respect to the second variable. Thus, we obtain the mixed state

$$\rho' = \sum_{x,y:g(x)=g(y)} c_x^{\star} c_y |G(x)\rangle \langle G(y)|,$$

which clearly differs from the 'correct' state $\rho = G|\xi\rangle\langle\xi|G^{\dagger}$. Indeed, $\rho = \rho'$ (for all $|\xi\rangle$) only if the rubbish g(x) is independent of x.

4.2. Construction of various operators from the elements of a basis. The operators S and K (see (24)) generate a finite group $E \subseteq U(\mathcal{B})$ of order 192. This group contains the Pauli matrices (7) (for example, $\sigma_x = \widehat{\gamma} = SK^2S$) and the constant operator $\zeta = (SK)^3 = \frac{1+i}{\sqrt{2}}$. Factorizing E relative to its centre $\{\zeta^k : k = 0, \ldots, 7\}$, we obtain the corresponding group of unitary transformations $\mathrm{ESp}_2(1) \subseteq \mathbf{UT}(\mathcal{B})$ of order 24.8 The group $\mathrm{ESp}_2(1)$ is isomorphic to the group of characteristic rotations of the cube, the Pauli matrices $\sigma_x, \sigma_y, \sigma_z \in \mathbf{L}(\mathcal{B}^{\otimes n})$ transforming like the basis unit vectors. The transformations $S \cdot S^{\dagger}$ and $K \cdot K^{\dagger}$ act on the Pauli matrices as follows:

$$S \cdot S^{\dagger} : (\sigma_x, \sigma_y, \sigma_z) \mapsto (\sigma_z, -\sigma_y, \sigma_x), \qquad K \cdot K^{\dagger} : (\sigma_x, \sigma_y, \sigma_z) \mapsto (\sigma_y, -\sigma_x, \sigma_z).$$
 (25)

Any classical operator can be represented in the basis $\widehat{\mathcal{R}} = \{1_{\mathcal{B}}, \sigma_x, \widehat{\bigoplus}, \widehat{\wedge}_{\oplus}\}$. (This follows from the completeness of the classical reversible basis \mathcal{R} .) It follows that any classical operator can be represented in the basis \mathcal{Q} .

Now we shall describe one very useful construction, namely, operators with quantum control. Let U be a partially defined unitary operator on $\mathcal{B}^{\otimes n}$, that is, $U: \mathcal{N} \to \mathcal{M} \ (\mathcal{N}, \mathcal{M} \subseteq \mathcal{B}^{\otimes n})$. We shall define a new operator $\Lambda(U): \mathcal{B} \otimes \mathcal{N} \to \mathcal{B} \otimes \mathcal{M}$ with the aid of the formula

$$\Lambda(U)|a,\xi\rangle = \begin{cases} |0,\xi\rangle & \text{if } a = 0, \\ |1\rangle \otimes U|\xi\rangle & \text{if } a = 1. \end{cases}$$
 (26)

This formula means that U is applied or not depending on the value of the additional control bit (1 or 0). In this definition the control bit is a quantum one. An operator with classical control $\Lambda_c(U)$ will be defined in §7 (more precisely, it will be a transformation rather than an operator). The difference is that $\Lambda(U)$ feels the phase factor between the different states of the control bit. The significance of this property will be made clear by the 'Important example' in §5.

Examples $\Lambda(\sigma_x) = \widehat{\bigoplus}$, $\Lambda(\widehat{\bigoplus}) = \widehat{\wedge}_{\oplus}$, $\Lambda(i) = K$. (In the last case *i* is regarded as an operator on $\mathbf{C} = \mathcal{B}^{\oplus 0}$.) The following identities can be used to construct more complex operators of the form $\Lambda(U)$:

$$\Lambda(UV) = \Lambda(U)\Lambda(V), \qquad \Lambda(V^{-1}UV)[1, A] = V^{-1}[A] \Lambda(U)[1, A] V[A]. \tag{27}$$

⁸The meaning of the symbol ESp₂(1) will be clarified in §9.1.

Example 4.4 (Fredkin's gate).

$$\Lambda(\leftrightarrow) \colon \mathcal{B}^{\otimes 3} \to \mathcal{B}^{\otimes 3} \ : \quad |a, b, c\rangle \mapsto \left\{ \begin{array}{ll} |0, b, c\rangle & \text{if } a = 0, \\ |1, c, b\rangle & \text{if } a = 1. \end{array} \right. \tag{28}$$

This operator can be represented as follows (compare with (4)):

$$\Lambda(\leftrightarrow)[1,2,3] \ = \ \widehat{\wedge}_{\oplus}[1,2,3] \quad \widehat{\wedge}_{\oplus}[1,3,2] \quad \widehat{\wedge}_{\oplus}[1,2,3].$$

Lemma 4.5. Let U be a (partially defined) unitary operator on $\mathbb{B}^{\otimes n}$ such that $U|0\rangle = |0\rangle$. Then $\Lambda(U)$ can be represented by a quantum scheme of dimension 6n+1 in the basis $\Omega \cup \{U\}$. The element U appears once in this scheme.

Proof. Let A and B be two disjoint registers of dimension n. The partially defined operator $\Lambda(U)[1,A]\otimes \tau[B]$ can be represented as

$$\Lambda(\leftrightarrow)[1, A, B] \quad U[A] \quad \Lambda(\leftrightarrow)[1, A, B].$$

Using this lemma, we can represent all operators $\Lambda(U)$ such that $U \in \Omega$, except $\Lambda(S)$. Unfortunately, $\Lambda(S)$ can only be represented approximately, which gives rise to some technical inconvenience. The situation can be simplified if Ω is replaced by another basis

$$Q' = \{1_{\mathcal{B}}, S', K, \widehat{\bigoplus}, \widehat{\wedge}_{\oplus}\}, \quad \text{where} \quad S' = \frac{1+i}{\sqrt{2}} S.$$
 (29)

We observe that $S' = K(S')^{-1}KS'K$ and $(S')^{-1} = (S')^7$, so $\Lambda(S') = \lambda(K)(S')^7 \times \Lambda(K)S'\Lambda(K)$. The basis Ω' generates a set of operators closed under the transformation $U \mapsto \Lambda(U)$ (and also under $U \mapsto U^{-1}$). The elements of the basis Ω' are rational numbers with denominators of the form 2^r . On the other hand, Ω and Ω' generate the same set of unitary transformations $U \cdot U^{\dagger}$. The corresponding operators U differ only by phase factors. The phase factors are not important for the physically meaningful Definition 4.1. They are significant only in the technical Definition 4.3. Besides, each element of Ω' can be represented in the basis Ω .

We intend to demonstrate that the basis Ω' (and so Ω too) is complete in the approximate sense. The completeness of Ω' follows from that of the infinite basis $\Omega' \cup \mathbf{U}(1)$. To represent the constant operator $c \in \mathbf{U}(1)$ in the basis Ω' it suffices to represent $\Lambda(c) \in \mathbf{U}(\mathfrak{B})$ to within a phase factor. Indeed,

$$\sigma_x \Lambda(c) \sigma_x \Lambda(c)^{-1} : |0\rangle \mapsto c|0\rangle.$$

Lemma 4.6. The operators S[1], S[2] and $\Lambda(K)[1,2] = \Lambda(K)[2,1]$, along with all possible phase factors, generate a dense subset in the group $U(\mathbb{B}^{\otimes 2})$.

Proof. Let $X = \Lambda(SKS)$, $X_1 = X[1,2]$ and $X_2 = X[2,1]$. The operators X_1, X_2 preserve the vectors $|00\rangle$ and $|\eta\rangle = |01\rangle + |10\rangle + |11\rangle$. We denote by $\mathcal L$ the two-dimensional subspace orthogonal to these vectors. It is clear that $X_1, X_2 \in \mathbf{U}(\mathcal L)$.

Moreover, the operators $Y_1 = X_1 X_2^{-1}$ and $Y_2 = X_2^{-1} X_1$ belong to the group $\mathbf{SU}(\mathcal{L})$. These operators commute with one another and have eigenvalues $\frac{1}{4}(1 \pm \sqrt{-15})$, which are not roots of 1. It follows that Y_1 and Y_2 generate a dense subset in $\mathbf{SU}(\mathcal{L})$.

Now we shall use the following geometric fact. Let \mathcal{M} be a unitary space of dimension ≥ 3 . We consider a subgroup $H \subseteq \mathbf{SU}(\mathcal{M})$, the stabilizer of a non-zero vector $|\xi\rangle \in \mathcal{M}$. Let V be an arbitrary unitary operator that does not preserve the subspace $(|\xi\rangle)$. Then the set of operators $H \cup V^{-1}HV$ generates the whole group $\mathbf{SU}(\mathcal{M})$.

The unitary operator $V = \Lambda(K)$ preserves the vector $|00\rangle$, but does not preserve the subspace $(|\eta\rangle)$. Consequently, $Y_1, Y_2, V^{-1}Y_1V, V^{-1}Y_2V$ generate a dense subset in $\mathbf{SU}(\mathcal{L} \oplus (|\eta\rangle))$. Finally, the operator S[1] does not preserve the subspace $(|00\rangle)$, so a dense subset can be generated in $\mathbf{SU}(\mathcal{B}^{\otimes 2})$.

This is a non-constructive result. So far it has not been understood how to approximate a given unitary matrix U algorithmically by products of S[1], S[2] and $\Lambda(K)$. To this end we need the following lemma, which was proved independently by Solovay [54] and the present author.

Lemma 4.7. Suppose that matrices $X_1, \ldots, X_r \in \mathbf{SU}(n)$ generate a dense subset in $\mathbf{SU}(n)$. There is an algorithm making it possible to approximate a given matrix $U \in \mathbf{SU}(n)$ by a product of $X_1, \ldots, X_r, X_1^{-1}, \ldots, X_r^{-1}$ with given accuracy $\delta > 0$. This algorithm is polynomial in $\log(1/\delta)$.

Outline of proof. The algorithm consists of two parts, the first being independent of U. First we shall describe the second trivial part of the algorithm.

An (R,ε) -net is a finite-dimensional set of points in $\mathbf{SU}(n)$ such that the ε -neighbourhoods of these points cover the R-neighbourhood of the unit element. Let a>2 and b>0 be some constants. Suppose that we have already constructed an (∞,b) -net Γ_0 and $(ab2^{-k},b2^{-k})$ -nets Γ_k $(1 \le k \le l = [\log_2(ab/\delta)])$. For any matrix $U \in \mathbf{SU}(n)$ we can find $A_k \in \Gamma_k$ $(k=0,\ldots,l)$ such that $A_0A_1\cdots A_k$ belongs to the $cb2^{-k}$ -neighbourhood of U (for each k). Here 1 < c < a/2 is an arbitrary constant (a correction due to the inaccuracy of computations involving real numbers).

To construct the nets Γ_0 and Γ_1 we simply need to generate sufficiently many different products of the matrices $X_1, \ldots, X_r, X_1^{-1}, \ldots, X_r^{-1}$. This step requires a fixed number of operations. The remaining nets can be constructed from Γ_1 using matrix commutators.

We choose the parameters a > 2, b > 0 and s so that the following property is satisfied: for any $\varepsilon', \varepsilon'' < b$, any $(a\varepsilon', \varepsilon')$ -net Γ' and any $(a\varepsilon'', \varepsilon'')$ -net Γ'' the set

$$[\Gamma, \Gamma']^s = \{V_1 \cdots V_s : V_j = A_j B_j A_i^{-1} B_i^{-1}, A_j \in \Gamma', B_j \in \Gamma''\}$$

is an $(a\varepsilon, \varepsilon/2)$ -net, where $\varepsilon = \varepsilon' \varepsilon''/b$.

The net $[\Gamma, \Gamma']^s$ can easily be transformed into an $(a\varepsilon, \varepsilon)$ -net $[\Gamma, \Gamma']^s_*$ of bounded capacity. Indeed, any $(R, \varepsilon/2)$ -net Γ contains an (R, ε) -net Γ_* with the following additional properties:

- (a) the distance between any two points in Γ_* is greater than $\varepsilon/3$;
- (b) Γ_* is contained in the $(R+\varepsilon)$ -neighbourhood of the unit element.

It is obvious that $|\Gamma_*| \leq O(R/\varepsilon)^{n^2-1}$ (since SU(n) is a manifold of dimension n^2-1). Such a subset Γ_* can be selected from Γ by means of an effective procedure.

It follows that the nets Γ_k $(k=2,\ldots,l)$ can be constructed according to the following recursive rules: $\Gamma_{2k} = [\Gamma_k, \Gamma_k]_*^s$ and $\Gamma_{2k+1} = [\Gamma_k, \Gamma_{k+1}]_*^s$. Each element of these nets is a product of $O((\log(1/\delta))^{\log_2(4s)})$ matrices from the set $\{X_1,\ldots,X_r,X_1^{-1},\ldots,X_r^{-1}\}$. It follows that the algorithm is polynomial in $\log(1/\delta)$.

Using these results, we can represent each operator $V \in \mathbf{U}(\mathfrak{B})$ and $\Lambda(V) \in \mathbf{U}(\mathfrak{B}^{\otimes 2})$ with accuracy δ by a scheme of dimension $\operatorname{poly}(\log(1/\delta))$ in the basis $\mathfrak{Q}' \cup \mathbf{U}(1)$. It follows that the operator $\Lambda^{n-1}(V) = \Lambda(\Lambda(\ldots V \ldots))$ can be represented by a scheme of dimension $O(n) + \operatorname{poly}(\log(1/\delta))$ (by Lemma 4.5). This operator preserves all basis vectors except $|1\ldots 1,0\rangle$ and $|1\ldots 1,1\rangle$. An operator of the form $\widehat{G}^{-1}\Lambda^{n-1}(V)\widehat{G}$ (where $G: \mathbf{B}^n \to \mathbf{B}^n$) acts on an arbitrary pair of basis vectors. Any unitary operator $U \in \mathbf{U}(\mathfrak{B}^{\otimes n})$ can be represented as a product of $2^n(2^n-1)/2$ such operators. Thus we have obtained the following result.

Theorem 4.8. Any unitary matrix $U \in \mathbf{U}(\mathfrak{B}^{\otimes n})$ can be represented with arbitrary accuracy $\delta > 0$ by a quantum scheme of dimension $\exp(O(n)) \operatorname{poly}(\log(1/\delta))$ in the basis \mathfrak{Q}' . This scheme can be constructed algorithmically in time $\exp(O(n)) \operatorname{poly}(\log(1/\delta))$.

4.3. Generalized quantum control and universal schemes. The goal of this subsection is to provide the reader with some intuition of 'quantum programming'. The problem is essentially the following. Suppose that the solution of a problem A gives rise to a subproblem B for which the algorithm is already known. In the classical case we can simply invoke the procedure realizing B. But what does it mean to invoke a procedure in the quantum case? In simple terms it is just the substitution of one scheme into another. Nevertheless, such a definition does not provide enough flexibility. The results obtained below give much more freedom and justify the use of the informal language involving procedures. However, of all these results only Lemma 4.9 will be needed later on.

First of all we shall extend the notion of an operator with quantum control. Quantum control of general form consists in selecting one operator from some family. We consider a parametric family of (partially defined) unitary operators, that is, a function of the form $\mathcal{U}: \mathbf{B}^l \to \check{\mathbf{U}}(\mathbf{B}^n)$, and define the operator

$$\Lambda(\mathcal{U}) \in \check{\mathbf{U}}(\mathcal{B}^{\otimes (l+n)}), \qquad \Lambda(\mathcal{U}) \colon |a,\xi\rangle \mapsto |a\rangle \otimes \mathcal{U}(a)|\xi\rangle. \tag{30}$$

Thanks to the following lemma we can easily pass to a subfamily or another parametrization of the same family.

Lemma 4.9. Let $f: \mathbf{B}^k \to \mathbf{B}^l$ be a (partially defined) function and let $\mathfrak{U}: \mathbf{B}^l \to \check{\mathbf{U}}(\mathfrak{B}^{\otimes n})$ be a parametric family of (partially defined) unitary operators. We consider another parametric family $\mathfrak{U} \circ f: \mathbf{B}^k \to \check{\mathbf{U}}(\mathfrak{B}^{\otimes n}): a \mapsto \mathfrak{U}(f(a))$. Suppose that f can be computed by means of a Boolean scheme of dimension L in the standard basis \mathfrak{C} . Then the operator $\Lambda(\mathfrak{U} \circ f)$ can be represented by a quantum scheme of dimension 2L+1 in the basis $\widehat{\mathfrak{R}} \cup \{\lambda(\mathfrak{U})\}$, the element $\Lambda(\mathfrak{U})$ appearing only once in this scheme.

(The proof is fully analogous to that of Proposition 2.3.)

Example 4.10. We consider the family of unitary matrices of dimension 2×2

$$\mathfrak{I}_{\delta}:$$
 binary representation of complex numbers a,b,c,d \mapsto operator $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with accuracy δ

The operator $\Lambda(\mathcal{T}_{\delta})$ makes it possible to act on a qbit by any unitary operator $U \in \mathbf{U}(\mathcal{B})$ as desired (strictly speaking, 'as desired in the quantum sense'). We shall see that $\Lambda(\mathcal{T}_{\delta})$ can be represented in the basis Ω with any accuracy.

Example 4.11. Let $\Gamma_n = \{0, \dots, 2^n - 1\}$ (it is natural to identify this set with \mathbf{B}^n). Suppose that we need to generate the quantum state

$$|\psi_n(q)\rangle = q^{-1/2} \sum_{j=0}^{q-1} |j\rangle \in \overline{\Gamma_n} = \mathcal{B}^{\otimes n}, \quad \text{where} \quad q \in \{1, \dots, 2^n\}.$$

This can be done with the aid of the following recursive procedure:

- 1. Let $q_0 = \min\{2^{n-1}, q\}$ and $q_1 = q q_0$.
- 2. We set the first bit to be in the state $(q_0/q)^{1/2}|0\rangle + (q_1/q)^{1/2}|1\rangle$.
- 3. Depending on the value x of this bit we generate the state $|\psi_{n-1}(q_x)\rangle$ involving the remaining n-1 bits.

The procedure terminates when n=0. To describe this procedure more precisely we define the functions

$$a_n: q \mapsto (q_0/q)^{1/2}, \qquad b_n: q \mapsto (q_1/q)^{1/2}, \qquad f_n: (q, x) \mapsto q_x,$$

where $q_0 = \min\{2^{n-1}, q\}$ and $q_1 = q - q_0$. We also introduce the family of partially defined unitary operators $V_n(q) : |0\rangle \mapsto |\psi_n(q)\rangle$. Then

$$\Lambda(V_n)[A, 1, \dots, n] = \Lambda(V_{n-1} \circ f_n)[A, 1, \dots, n] \quad \Lambda(\mathfrak{T} \circ (a_n, b_n, b_n, -a_n))[A, 1],$$

where A is the control register containing q. (The family of operators T was defined in the previous example. The accuracy parameter δ is omitted for simplicity.)

The existence of a universal Turing machine is a remarkable result in the classical theory of algorithms. A similar (but obviously not as deep) property holds for Boolean schemes too. We intend to construct a universal quantum scheme which can model any scheme of given dimension L in the basis Ω' . It is obvious that schemes of dimension L can be encoded in a natural way by binary words of length $O(L \log L)$. Let

Theorem 4.12. The operators $\Lambda(U_L)$, L = 1, 2, ..., can be represented by a homogeneous sequence of polynomial schemes Ψ_L in the basis Q'.

Proof. Let $\Delta_0 = \{1, \ldots, 3L\}$. (3L is the largest number of bits used in a scheme of dimension L.) We apply the operators $\Lambda(U)$ such that $U \in \Omega'$ (represented in the same basis Ω') to all possible combinations of bits from Δ_0 . We repeat this procedure L times. We have thus constructed a scheme which can model any scheme of dimension L if the appropriate values are assigned to the control bits. We denote by f the function transforming the description of the scheme into the values of the control bits. Now we use Lemma 4.9.

Remark. If we replaced the basis Q' by Q, we would be able to obtain only an approximate representation.

Corollary. The operator $\Lambda(\mathcal{T}_{\delta})$ (see Example 4.10) can be represented with accuracy $O(\delta)$ by a scheme of dimension $\operatorname{poly}(\log(1/\delta))$ in the basis Q. This scheme can be constructed algorithmically in time $\operatorname{poly}(\log(1/\delta))$.

Proof. Use Theorem 4.8 and Lemma 4.9.

§5. Measurement operators

We have just seen how to carry over many classical constructions to the theory of quantum computations. Now we shall demonstrate some fundamentally new possibilities, which are absent in the classical case.

An important physical assumption lying at the foundations of Definition 4.1 is the possibility of measuring the classical memory state at the end of a computation. More general quantum measurements performed in the course of the computation could serve as a very useful computational instrument. Unfortunately, the standard definition of a measurement (see Example 3.7) is based on the notion of a combined (quantum-classical) system, which is beyond the framework of our computational model. However, one can define a special class of unitary operators acting almost like the transformations MEAS $_{\Omega}$ in Example 3.7.

Definition 5.1. Let A and D be two disjoint registers and let $C \subseteq D$ with |C| = m. Also, let $\{\mathcal{V}_a : a \in \Omega\}$ be a family of subspaces of $\mathcal{B}^{\otimes A}$ orthogonal to one another. We put $\mathcal{N} = \bigoplus_{a \in \Omega} \mathcal{V}_a$. For each $y \in \mathbf{B}^m$ we denote by $\mathcal{W}_y = \mathcal{W}_y[C]$ the corresponding standard subspace in $\mathcal{B}^{\otimes D}$ (see (23)). The measurement operator for a quantum variable z_{Ω} is a unitary operator of the form

$$U = \sum_{a \in \Omega} \Pi_{\mathcal{V}_a} \otimes U_a : \mathcal{N} \otimes \mathcal{B}^{\otimes D} \to \mathcal{N} \otimes \mathcal{B}^{\otimes D},$$

where U_a are arbitrary unitary operators on $\mathcal{B}^{\otimes D}$. The numbers

$$P_U(a, y) = P(U_a|0\rangle, W_y) \quad (a \in \Omega, y \in \mathbf{B}^m)$$

are called *conditional probabilities*. The operator U is said to *measure* the values of a function $F: \Omega \to \mathbf{B}^m$ with probability of error not exceeding ε if $\mathsf{P}_U(a, F(a)) \geqslant 1 - \varepsilon$ for all $a \in \Omega$.

⁹The corresponding control bits are different and do not belong to Δ_0 .

¹⁰The accuracy of a matrix differs from that of its elements.

For example, the operator $\bigoplus_n [A, D]$ measures the value of z_A with zero probability of error. Any quantum computation (see Definition 4.1) can be set up as a measurement with respect to its input. To this end it suffices to copy the input word with the aid of the operator \bigoplus_n and use the copy in place of the original.

Important example. Let U be a unitary operator on a subspace $\mathbb{N} \subseteq \mathbb{B}^{\otimes n}$. The eigenvalues of this operator have the form $\lambda(\varphi) = \exp(2\pi i \varphi)$, where φ is a real number modulo 1. We denote the corresponding eigenspaces by $\mathcal{L}(U,\varphi)$. Suppose that U acts on a register A. We add one more quantum bit and assign the number 1 to it. Then

$$\Xi(U)[A,1] = S[1] \Lambda(U)[1,A] S[1]$$
(31)

is a measurement operator for the quantum variable φ . If $|\xi\rangle \in \mathcal{L}(U,\varphi)$, then $\Xi(U)|\xi,0\rangle = |\xi,\eta\rangle$, where

$$|\eta\rangle = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \lambda(\varphi) \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(1 + \lambda(\varphi)) \\ \frac{1}{2}(1 - \lambda(\varphi)) \end{pmatrix}.$$

It follows that the conditional probabilities $\mathsf{P}_{\Xi(U)}(\varphi,y)$ are equal to

$$\mathsf{P}_{\Xi(U)}(\varphi,0) = \frac{1}{2} \big(1 + \cos(2\pi\varphi) \big), \qquad \mathsf{P}_{\Xi(U)}(\varphi,1) = \frac{1}{2} \big(1 - \cos(2\pi\varphi) \big). \tag{32}$$

The general properties of measurement operators are quite simple. We shall consider measurement operators for the same quantum variable z_{Ω} but with different additional registers D.

Proposition 5.2. 1. Let U be a measurement operator. The total probability formula

$$P(U|\xi,0\rangle,\mathcal{W}_y) = \sum_{a \in \Omega} P(|\xi\rangle,\mathcal{V}_a) P_U(a,y)$$
(33)

holds for any quantum state $|\xi\rangle \in \mathbb{N}$.

- 2. A product of measurement operators is a measurement operator. Measurement operators with disjoint additional registers commute.
- 3. Let U' and U'' be measurement operators with disjoint additional registers $D' \supseteq C'$ and $D'' \supseteq C''$. Then

$$\mathsf{P}_{U'U''}\big(a,(y',y'')\big) = \mathsf{P}_{U'}(a,y')\,\mathsf{P}_{U''}(a,y''). \tag{34}$$

Proof. 1. It is clear that $U|\xi,0\rangle = \sum_{a\in\Omega} \Pi_{\mathcal{V}_a} U|\xi,0\rangle$. Consequently,

$$\mathsf{P}\big(U|\xi,0\rangle,\mathcal{W}_y\big) = \sum_{a\in\Omega} \langle \xi,0|U^\dagger \Pi_{\mathcal{V}_a} \Pi_{\mathcal{W}_y} U|\xi,0\rangle = \sum_{a\in\Omega} \langle \xi|\Pi_{\mathcal{V}_a}|\xi\rangle \, \langle 0|U_a^\dagger \Pi_{\mathcal{W}_y} U_a|0\rangle.$$

This gives the right-hand side of (33).

- 2. This follows from the definition.
- 3. This is a consequence of the general property (11) of quantum probability.

Now we go back to the example considered above. Suppose that the measurement operator $\Xi(U)$ is applied s times to the same register A and additional bits $1, \ldots, s$. We shall construct a scheme which looks through the values z_1, \ldots, z_s of these bits and counts how many of them are equal to one. Thus we shall define a procedure whose result will be the number y of units (as well as a new state of the register A). This procedure can be described by a new measurement operator $\Xi_s(U)$. Since z_1, \ldots, z_s behave like independent random variables, y/s will most likely be close to the corresponding probability $P_{\Xi(U)}(\varphi, 1)$. More precisely, for any constant $\delta > 0$ given in advance the following estimate is true:¹¹

$$\operatorname{Prob}[|y/s - \mathsf{P}_{\Xi(U)}(\varphi, 1)| > \delta] \leq 2 \exp(-c(\delta)s),$$

where $c(\delta) > 0$. It follows that we measure $\mathsf{P}_{\Xi(U)}(\varphi,1) = \frac{1}{2}(1-\cos(2\pi\varphi))$ with accuracy δ and probability of error $\leq \exp(-c(\delta)s)$. If U is replaced by iU, then $\cos(2\pi\varphi)$ is replaced by $-\sin(2\pi\varphi)$. It follows that $\cos(2\pi\varphi)$ and $\sin(2\pi\varphi)$ can be measured simultaneously. This provides enough information to reconstruct φ . We have proved the following assertion.

Lemma 5.3. Let $\delta > 0$ be a constant. For any $\varepsilon > 0$ the value of the quantum variable φ can be measured with accuracy δ and probability of error not greater than ε with the aid of a quantum scheme of dimension $O(\log(1/\varepsilon))$ in the basis $\Omega \cup \{\Lambda(U)\}$.

Unfortunately, it is difficult to measure φ with arbitrary accuracy because the cost of such a measurement (that is, the dimension of the scheme) increases polynomially with δ . The situation improves sharply if we have at our disposal the operators $\Lambda(U^k)$ for all k. More precisely, we consider the operator

$$U^{[0,r]} \colon \overline{\{0,\ldots,r\}} \otimes \mathcal{N} \to \overline{\{0,\ldots,r\}} \otimes \mathcal{N}, \qquad U^{[0,r]}|k,\xi\rangle = |k\rangle \otimes U^k|\xi\rangle, \tag{35}$$

where $r=2^{l-1}$. According to Lemma 4.9, $U^{[0,r]}$ can be transformed into $\Lambda(U^{2^j})$ $(j=0,\ldots,l-1)$. We apply Lemma 5.3 to each of the latter operators. In this way we can localize $2^j\varphi\pmod{1}$ in one of eight intervals $\left[\frac{s-1}{8},\frac{s+1}{8}\right]$ $(s=0,\ldots,7)$ with probability of error $\leqslant \varepsilon/l$. In other words, we can find one of the intervals containing the number unknown to us. To this end we must perform $O(\log(l/\varepsilon))$ operations. Using the information obtained we can determine (with the aid of a polynomial algorithm) the value of φ with accuracy $\frac{1}{8}2^{-(l-1)}=2^{-l-2}$ and probability of error $\leqslant \varepsilon$. We have obtained the following result.

Lemma 5.4. Let l be a positive integer and let $r = 2^{l-1}$. For any $\varepsilon > 0$ the value of φ can be measured with accuracy 2^{-l-2} and probability of error not greater than ε with the aid of a quantum scheme of dimension $O(l \log(l/\varepsilon)) + \operatorname{poly}(l)$ in the basis $\Omega \cup \{U^{[0,r]}\}$. (The element $U^{[0,r]}$ is used no more than $O(l \log(l/\varepsilon))$ times.)

Now we shall consider an important special case: $U = \widehat{g}$, where g is a permutation of a set $N \subseteq \mathbf{B}^n$. To each cycle of this permutation there correspond eigenvalues of the form $\exp(2\pi i \frac{p}{g})$, where q is the length of the cycle. Therefore all possible values

¹¹The notation Prob[·] was introduced in §3.1.

of φ must be rational numbers with denominators $\leq 2^n$. The smallest distance between such numbers is equal to $(2^n(2^n-1))^{-1}$. It follows that to measure φ precisely it suffices to measure it with accuracy 2^{-2n-1} . Moreover, it is possible to pass from the approximate value to the exact one in polynomial time. The algorithm of continued fractions should be used to this end. We have proved the following theorem.

Theorem 5.5. Let g be a permutation of a set $N \subseteq \mathbf{B}^n$. An eigenvalue of the operator $U = \widehat{g}$ can be measured with probability of error not greater than ε with the aid of a quantum scheme of dimension $\operatorname{poly}(n) + O(n) \log(1/\varepsilon)$ in the basis $Q \cup \{U^{[0,2^{2n}]}\}$. (The element $U^{[0,2^{2n}]}$ is used no more than $O(n \log(n/\varepsilon))$ times.) This scheme can be constructed algorithmically in time $\operatorname{poly}(n + \log(1/\varepsilon))$.

§6. Polynomial quantum algorithm for the stabilizer problem

Let (k, n, a, F) be a representative of the Abelian problem on the stabilizer and let $\operatorname{St}_F(a) = \{g \in \mathbf{Z}^k : F(g, a) = a\}$ be its solution. To simplify the notation we shall write g(a) instead of F(g, a). It is obvious that g(a) depends only on the image of g in the quotient group $E = \mathbf{Z}^k / \operatorname{St}_F(a)$. This image, which will be denoted by $\theta(g)$, can be regarded as a permutation on $N = \{g(a) : g \in \mathbf{Z}^k\} \subseteq \mathbf{B}^n$ (an orbit of a). The key idea consists in measuring the eigenvalues of the operators $\widehat{\theta(g_j)}$, $j = 1, \ldots, k$, where g_1, \ldots, g_k are the generators of the group \mathbf{Z}^k . (Obviously, these measurements are only a part of a more complex procedure.)

The possible combinations of eigenvalues can be described by the group of characters on E,

$$H = \operatorname{Hom}(E, \mathbf{Q}/\mathbf{Z}) \subseteq \operatorname{Hom}(\mathbf{Z}^k, \mathbf{Q}/\mathbf{Z}) = (\mathbf{Q}/\mathbf{Z})^k.$$

Each element h of H can be represented by k rational numbers $\pmod{1}$ $\varphi_1, \ldots, \varphi_k \in \mathbf{Q}/\mathbf{Z}$ with a common denominator $q = |E| = |H| \leqslant 2^n$. Namely, $\varphi_j = (h, \theta(g_j))$, where $(\cdot, \cdot) : H \times E \to \mathbf{Q}/\mathbf{Z}$ is the natural pairing. All operators $\widehat{\theta(g)}$ such that $g \in \mathbf{Z}^k$ have the same set of eigenvectors

$$|\psi_h\rangle = \frac{1}{\sqrt{q}} \sum_{u \in E} \exp(-2\pi i (h, u)) |u(a)\rangle \qquad (h \in H).$$
 (36)

The corresponding eigenvalues are equal to $\lambda_h(g) = \exp(2\pi i(h, \theta(g)))$.

Theorem 5.5 makes it possible to measure $\varphi_j = (h, \theta(g_j))$ for each j (using the operators $\widehat{\theta(g_j)}$). These measurements do not affect each other because they apply to the same system of subspaces orthogonal to one another (see Proposition 5.2). It follows that all numbers $\varphi_j = (h, \theta(g_j))$ can be measured simultaneously. Therefore the quantum variable $h = (\varphi_1, \ldots, \varphi_k) \in (\mathbf{Q}/\mathbf{Z})^k$ can be measured with probability of error $\leq k\varepsilon$ with the aid of a scheme of dimension $k(\text{poly}(n) + O(n)\log(1/\varepsilon))$ in the basis $\Omega \cup \{\widehat{G|_{O(n)}}\}$. (The map G is defined by (6).)

Now we shall go back to the original problem. It is clear that the computation of $\operatorname{St}_F(a)$ is polynomially equivalent to that of H. (To find H means to find a polynomial subset in $(\mathbf{Q}/\mathbf{Z})^k$ generating H.) The group H is generated by sufficiently many random elements (see below). Just now we shall describe a method which makes it possible to generate one random element.

Here we shall use a new trick (cf. [13]). We prepare the base state $|a\rangle$ and measure h. It is obvious that

$$|a\rangle = \frac{1}{\sqrt{q}} \sum_{h \in H} |\psi_h\rangle.$$

According to the total probability formula (33), the probability of obtaining a result y is equal to

$$\operatorname{Prob}[h=y] = q^{-1} \sum_{h \in H} \mathsf{P}_U(h,y), \text{ where } \mathsf{P}_U(h,h) \geqslant 1 - k\varepsilon, \quad \sum_{y:y \neq h} \mathsf{P}_U(h,y) \leqslant k\varepsilon.$$

(Here U denotes a measurement operator.) It follows that

$$q^{-1}|L|(1-k\varepsilon)\leqslant \operatorname{Prob}[h\in L]\leqslant q^{-1}|L|+k\varepsilon$$
 for any $L\subseteq H$.

In this way we can generate a random element H with almost uniform distribution. Let $h_1, \ldots, h_l \in (\mathbf{Q}/\mathbf{Z})^k$ be independent random elements obtained in this way. We intend to demonstrate that they almost surely generate H if a suitable l is chosen.

All elements h_1,\ldots,h_l belong to H with probability $\geqslant 1-kl\varepsilon$. Suppose that they belong to H, but do not generate H. Then $h_1,\ldots,h_l\in L$, where L is a maximal proper subgroup of H. For each L the probability of this event does not exceed $(\operatorname{Prob}[h\in L])^l\leqslant (\frac12+k\varepsilon)^l$ (because $|L|\leqslant |H|/2$). There is a one-to-one correspondence between the maximal proper subgroups of H and the minimal non-zero subgroups of E. The number of such subgroups is less than $|E|\leqslant 2^n$. It follows that the total probability that h_1,\ldots,h_l do not generate H is less than $kl\varepsilon+2^{n-l}(1+2k\varepsilon)^l$. (The first term corresponds to the case $\{h_1,\ldots,h_l\}\not\subseteq H$ and the second one takes into account all possible subgroups L.) We put l=n+4 and $\varepsilon=(6kl)^{-1}$. Then the random elements $h_1,\ldots,h_l\in (\mathbf{Q}/\mathbf{Z})^k$ generate H with probability $\geqslant \frac23$.

Thus, the whole computation proceeds as follows. We take l=n+4 registers and in each of them we prepare the initial state $|a\rangle$. Then we perform $O(kn\log(kn))$ elementary measurements $\Xi(\widehat{g}_j^{2^s})$ $(1 \leq s \leq 2n)$ for each register. The results of these measurements are processed classically. In this way we find h_1, \ldots, h_l and, finally, the canonical basis of the stabilizer (with probability of error $\leq \frac{1}{3}$). We need to resort $O(kn^2\log(kn))$ times to the oracle F, each question presented to the oracle being of length O(n). We emphasize that the described procedure is homogeneous, that is, the corresponding quantum scheme can be constructed algorithmically in time $\operatorname{poly}(k+n)$.

§7. Computations with perturbations: the choice of a model

We proceed to a new topic, namely, combating perturbations. The problem is the following. There is an arbitrary sequence of operations representing a unitary operator or computing a Boolean function. We need to realize the same operator or the same function with the aid of another sequence of operations which is stable under perturbations. In the literal sense this is impossible. The correct formulation

of the problem (see §10.1) is based on the notion of a many-to-one code. The goal of the present section is to formulate certain assumptions (sufficiently realistic) about the character of the perturbations by choosing a suitable computational model.

In the general case the perturbation can be described by a unitary operator acting on the whole space of states of qbits and the surrounding medium. Assuming that the environment is renewed at each step, we can exclude it from consideration and use the language of transformations of density operators (see §3.2). (This assumption means that errors at different steps are statistically independent.) It proves convenient to assume from the very beginning that the quantum computer works with mixed states, and even that unperturbed elementary transformations do not necessarily have the form $U \cdot U^{\dagger}$. Such computations are called mixed.

As we mentioned in the Introduction, perturbations act on quantum bits even when no operations are performed. Because of this we are forced to consider the computational process in real time. More precisely, we must pass from sequential computations to parallel ones.

At each step of a mixed parallel computation all bits are divided into small groups (registers) A_1, \ldots, A_s of 1–3 bits in each group. An admissible transformation T_j from a given basis A is applied to each register A_j . (It is possible that almost all registers A_j consists of one bit, the corresponding transformations T_j being the identity.) The result of T_j is written into a new register A'_j whose length may differ from that of A_j . This operation is denoted by $T_j[A'_j;A_j]$. Some of the resulting bits may be ignored and others used in the next step. The sequence of such steps is called a layered scheme. Here is a formal definition.

Definition 7.1. A layered scheme of depth d is a collection of the following objects:

- a sequence of finite sets $\Delta_0, \ldots, \Delta_d$ called the *layers*; the null layer Δ_0 is called the *input* of the scheme and the last layer Δ_d is called the *output*;
- partitions of each set Δ_{i-1} $(i=1,\ldots,d)$ into ordered subsets (registers) A_{i1},\ldots,A_{is_i} as well as partitions of some sets $\Delta'_i\supseteq\Delta_i$ into registers A'_{i1},\ldots,A'_{is_i} ;
- a family of admissible transformations $T_{ij} \in \mathbf{T}(\mathcal{B}^{\otimes n_{ij}}, \mathcal{B}^{\otimes m_{ij}})$ $(n_{ij} = |A_{ij}|, m_{ij} = |A'_{ij}|)$ from some basis $A \ni I_{\mathcal{B}}$; the transformations T_{ij} (or their numbers (i,j)) are called the *elements*; the total number of elements is called the *dimension of the scheme*. (We recall that $I_{\mathcal{B}}$ denotes the identity transformation of a single qbit.)

Such a scheme Φ realizes the transformation Trans $(\Phi) = T_d \cdots T_1$, where

$$T_i\rho=\mathrm{Tr}_{\Delta_i'\setminus\Delta_i}\big(T_{i1}[A_{i1}';A_{i1}]\otimes\cdots\otimes T_{is_i}[A_{is_i}';A_{is_i}]\,\rho\big)\quad\text{for any}\quad\rho\in\mathbf{L}(\mathfrak{B}^{\otimes\Delta_{i-1}}).$$

In the presence of perturbations each of the transformations T_{ij} is replaced by an arbitrary admissible transformation \widetilde{T}_{ij} such that $\|\widetilde{T}_{ij} - T_{ij}\|_{\diamondsuit} \leq \delta$. As a result, we obtain a set Trans (Φ, δ) of transformations which can be realized by the scheme Φ with inaccurate elements. The accuracy constant δ will be assumed to be sufficiently small, but independent of the concrete layered scheme.

As the basis elements for the construction of layered schemes it is natural to take all transformations of the form $U \cdot U^{\dagger}$ such that $U \in \Omega$ and also the transformation

giving rise to the state $|0\rangle$. (We recall that Ω denotes the standard basis of quantum computations.) However, we shall use a larger basis including both quantum and classical elements. This requires a slight modification of the model described above.

All the bits in the scheme are divided into quantum and classical ones. As the elements we are allowed to use admissible combined transformations of the form

$$T \in \mathbf{T}(\mathcal{B}^{\otimes n} \times \mathbf{B}^k, \mathcal{B}^{\otimes m} \times \mathbf{B}^l) \subseteq \mathbf{T}(\mathcal{B}^{\otimes (n+k)}, \mathcal{B}^{\otimes (m+l)}).$$

(*T* can be applied to *n* quantum and *k* classical bits.) Such combined computations constitute a special case of mixed ones. Nevertheless, it is natural to assume that the perturbation leaves the classical bits in a classical state. For example, if a Boolean function f is applied to a classical register A, then the perturbed transformation can be described by the transition matrix t (see (15)). In this case the accuracy condition takes the form $\forall a \in \mathbf{B}^A$ $(1 - t_{f(a),a} \leq \delta/2)$. The number $1 - t_{f(a),a}$ can be interpreted as the probability of an error.

The basis for combined computations can be chosen as follows. We take some classical basis \mathcal{A}_c (consisting of Boolean functions) and a purely quantum basis \mathcal{A}_q (consisting of unitary operators). The corresponding combined basis \mathcal{A} will contain the following elements:

- the transformations $f \in \mathcal{A}_c \cup \{\nabla\}$ with classical bits (we recall that ∇ is the operation of copying a single bit);
- the transformations $U \cdot U^{\dagger}$ such that $U \in \mathcal{A}_q$ with quantum bits;
- the transformation generating the quantum state $|0\rangle$;
- the transformation $Q_{\mathbf{B}}^*$ mapping a quantum bit into a classical one;
- the combined transformations $\Lambda_c(U)$, where $U \in \mathcal{A}_q$, with one classical control bit and several quantum bits

$$\Lambda_{c}(U)(a,\rho) = \begin{cases}
(0,\rho) & \text{if } a = 0, \\
(1,U\rho U^{\dagger}) & \text{if } a = 1, \\
\Lambda_{c}(U) \colon \mathbf{B} \times \mathbf{L}(\mathcal{B}^{\otimes n}) \to \mathbf{B} \times \mathbf{L}(\mathcal{B}^{\otimes n}).
\end{cases}$$
(37)

In what follows we shall, in principle, work with the complete basis QC defined by the following choice of A_q and A_c :

$$\mathcal{A}_c = \mathcal{C} = \{I_{\mathbf{B}}, 0, \neg, \wedge\}, \quad \mathcal{A}_q = \mathcal{Q} = \mathcal{S} \cup \{\widehat{\wedge}_{\oplus}\}, \quad \text{where} \quad \mathcal{S} = \{1_{\mathcal{B}}, S, K, \widehat{\oplus}\}.$$
 (38)

(The operators S and K are defined by (24).) The quantum basis S is called symplectic because it generates the extended symplectic group $\mathrm{ESp}_2(n)$ (see §9.1). The corresponding combined basis SC is called the symplecto-classical basis. It is not complete, that is, does not make it possible to realize an arbitrary admissible transformation with arbitrary accuracy. However, in this basis we can perform such useful procedures as coding, decoding and error correction. This is why the problem of perturbations for a symplecto-classical basis can be considered separately. It is somewhat easier to solve than in the case of a complete basis.

§8. Quantum codes (definitions and general properties)

8.1. Basic notions and ideas. We shall consider a simplified problem on combating perturbations. Suppose that a perturbation acts at a single step only. Arbitrary transformations, namely, coding and decoding, are allowed before and after this step. Such a situation occurs when a quantum state is transmitted between two ideal devices through a noisy channel. The quantum state of an *m*-bit register is to be transmitted with a sufficiently small probability of error using an *n*-bit communication channel.

Let us recall the classical version of this problem. A classical channel can be described by a matrix of transition probabilities $(t_{ij}:x\in N,y\in N')$, where N and N' are the sets of input and output signals. There is also another rougher model: all transitions $x\mapsto y$ are divided into 'probable' and 'improbable' ones, so that the total probability (for a given x) of 'improbable' transitions does not exceed a given number ε . In this case we are dealing only with the set $E\subseteq N\times N'$ of 'probable' transitions. For example, we consider an n-bit classical channel with independent errors: $N=N'=\mathbf{B}^n$, $t_{xy}=p^{d(x,y)}(1-p)^{n-d(x,y)}$, where p is a parameter (the probability of error in a single bit) and $d(\cdot,\cdot)$ is the Hamming distance. In this case the 'probable' transitions are those with $d(x,y)\geqslant k=\mathrm{const.}$ The set of such transitions is denoted by E(n,k).

A classical code of type (N,M) is an arbitrary embedding $h:M\to N$. (The elements of M are interpreted as the messages being coded and transmitted through the channel.) It is possible to reconstruct the original message if no two different elements $x_1, x_2 \in L = \operatorname{Im} h$ are 'glued together', that is, there is no y such that $(x_1,y) \in E$ and $(x_2,y) \in E$. In the case E = E(n,k) it is said that the code corrects k errors. The task is to find codes with this property. (It is usually required that the capacity |L| of the code should be large enough; there are also other criteria for the 'quality' of codes.)

A quantum channel can be described by an admissible transformation $\widetilde{T} \in \mathbf{T}(\mathcal{N}, \mathcal{N}')$. A rougher model of a quantum channel is given by a linear *error* space $\mathcal{E} \subseteq \mathbf{L}(\mathcal{N}, \mathcal{N}')$, which has the same meaning as the set of 'probable' transitions in the classical case. We shall make these models more concrete and indicate a connection between them.

We shall consider the following class of channels: $\mathcal{N} = \mathcal{N}' = \mathcal{B}^n$, $\widetilde{T} = \widetilde{T}_1 \otimes \cdots \otimes \widetilde{T}_n$, where \widetilde{T}_j are admissible transformations with a single qbit such that $\|\widetilde{T}_j - I_{\mathcal{B}}\|_{\diamondsuit} \leq \delta$. (Here δ is a constant.) For each subset $A \subseteq \Delta = \{1, \ldots, n\}$ we define the *space* of errors with support A, that is, the space of linear operators acting only on the qbits in A

$$\mathcal{E}(A) = \{X[A] : X \in \mathbf{L}(\mathcal{B}^{\otimes |A|})\} \subseteq \mathbf{L}(\mathcal{B}^{\otimes \Delta}). \tag{39}$$

To the space of operators $\mathcal{E}(A)$ there corresponds the space of transformations $\mathfrak{R}(A) = \mathcal{E}(A) \cdot \mathcal{E}(A)^*$. We put

$$\mathcal{E}(n,k) = \sum_{A \subseteq \Delta: |A| \leqslant k} \mathcal{E}(A), \qquad \Re(n,k) = \sum_{A \subseteq \Delta: |A| \leqslant k} \Re(A). \tag{40}$$

(It is obvious that $\Re(n,k) \subseteq \mathcal{E}(n,k) \cdot \mathcal{E}(n,k)^*$.) In this situation the following result holds.

Proposition 8.1. Let $\widetilde{T} = \widetilde{T}_1 \otimes \cdots \otimes \widetilde{T}_n$, where \widetilde{T}_j is an admissible transformation with a single qbit such that $\|\widetilde{T}_j - I_{\mathcal{B}}\|_{\diamondsuit} \leq \delta$ (for each j). Then there is a transformation $T \in \mathfrak{R}(n,k)$ such that

$$||T - \widetilde{T}||_{\diamondsuit} \leqslant \sum_{l>k} {n \choose l} \delta^{l}.$$

Proof. Multiply out the brackets in the expression $\widetilde{T} = \bigotimes_{i=1}^{n} (I_{\mathcal{B}} + (\widetilde{T}_{j} - I_{\mathcal{B}})).$

Any unitary embedding $V: \mathcal{M} \to \mathcal{N}$ defined to within a phase factor is called a (one-to-one) quantum code of type $(\mathcal{N}, \mathcal{M})$. (Many-to-one codes will be defined in §8.3.) The space \mathcal{N} is called the code space and \mathcal{M} the information space. If $\mathcal{N} = \mathcal{B}^{\otimes n}$ and $\mathcal{M} = \mathcal{B}^{\otimes m}$, then V is called a code of type (n, m).

Error correction can be described by an admissible transformation $P \in \mathbf{T}(\mathcal{N}', \mathcal{M})$, see Definition 8.2 below. The error correction property of a code depends not on the embedding V itself, but on the *information subspace* $\mathcal{L} = V\mathcal{M} \subseteq \mathcal{N}$. The quantum code $V : \mathcal{M} \to \mathcal{N}$ corrects errors from a space $\mathcal{E} \subseteq \mathbf{L}(\mathcal{N}, \mathcal{N}')$ if orthogonal vectors are not 'glued together'. More precisely, for any pair of orthogonal vectors $|\xi\rangle, |\eta\rangle \in \mathcal{L}$ and any pair of operators $X, Y \subseteq \mathcal{E}$ the vectors $X|\xi\rangle$ and $Y|\eta\rangle$ are also orthogonal (see Theorem 8.3 below).

In the classical case there exists a very simple code REP(n) correcting $\lfloor \frac{n-1}{2} \rfloor$ errors (see the Introduction). Why can't we extend this example to the quantum case? Obviously, it is possible to define the embedding $\mathcal{B} \to \mathcal{B}^{\otimes n}$ by $|0\rangle \mapsto |0,\ldots,0\rangle$, $|1\rangle \mapsto |1,\ldots,1\rangle$. However, such a quantum code does not even protect from a single error or, more precisely, it cannot correct errors from $\mathcal{E}(n,1)$. Indeed, we can use the above criterion. We consider the following pair of orthogonal vectors from the information space:

$$|\xi\rangle = 2^{-1/2} (|0, \dots, 0\rangle + |1, \dots, 1\rangle), \qquad |\eta\rangle = 2^{-1/2} (|0, \dots, 0\rangle - |1, \dots, 1\rangle).$$

Let X=1 and $Y=\sigma_z[j]$ (j is arbitrary). Then $X|\xi\rangle=Y|\eta\rangle=|\xi\rangle$, that is, the criterion is not satisfied. We observe that an error of the form $\sigma_z[j]$ (a so-called phase error) has no effect on the basis states, and so it has no classical analogue. On the other hand, classical errors of the form $\sigma_x[j]$ will be corrected by the code under consideration. It follows that the difficulty in constructing quantum codes lies in the fact that classical and phase errors need to be corrected simultaneously. The simplest known example of a quantum code correcting one error (Example 9.3) is quite non-trivial.

Among classical codes the most extensively studied ones are linear codes over the field \mathbf{F}_2 . The set of code words $L \subseteq \mathbf{B}^n = \mathbf{F}_2^n$ of a linear code is characterized by the null value of some *control sums*, that is, some linear functions of the form $\mathbf{F}_2^n \to \mathbf{F}_2$. The quantum analogue of a function is an operator, and the analogue of a linear function is an operator of the form $\pm \sigma_{\gamma_1} \otimes \cdots \otimes \sigma_{\gamma_n}$ ($\gamma_j \in \{0, x, y, z\}$), where $\sigma_0 = 1_{\mathcal{B}}$. Suppose that a number of operators of this form are given. They will be called the *verification operators*. The information subspace $\mathcal{L} \subseteq \mathcal{B}^{\otimes n}$ can be defined as the set of vectors invariant under the action of these operators. This definition makes sense only if the verification operators commute with one another.

The commutation condition can be written with the aid of a classical symplectic form over \mathbf{F}_2 . This is why codes of this kind are called *symplectic*. The construction will be described in detail in §9. The next two subsections (8.2 and 8.3) are devoted to the study of the general properties of quantum codes.

8.2. One-to-one codes. Let $V: \mathcal{M} \to \mathcal{N}$ be a quantum code and let $\mathcal{L} = V\mathcal{M} \subseteq \mathcal{N}$ be the information subspace.

Definition 8.2. Let $\mathfrak{R} \subseteq \mathbf{T}(\mathcal{N}, \mathcal{N}')$. We say that V corrects errors from \mathfrak{R} if there is an admissible transformation $P \in \mathbf{T}(\mathcal{N}', \mathcal{N})$ and a function $c : \mathfrak{R} \to \mathbf{C}$ such that

$$PT\rho = c(T)\rho \quad \text{for any} \quad T \in \Re, \quad \rho \in \mathbf{L}(\mathcal{L}).$$
 (41)

P is called the correction transformation. In the case $\Re = \mathcal{E} \cdot \mathcal{E}^*$ (where $\mathcal{E} \subseteq \mathbf{L}(\mathcal{N}, \mathcal{N}')$ is a linear subspace) V is said to correct errors from \mathcal{E} , and \mathcal{E} is called the space of errors. If $\mathcal{E} = \mathcal{E}(n, k)$, then V is said to correct k errors.

We observe that if T is an admissible transformation, then c(T)=1. Indeed, $c(T)1_{\mathcal{M}}=(T(V\cdot V^{\dagger}))^*1_{\mathcal{N}'}$.

We shall see what this definition gives for a channel described by an arbitrary admissible transformation $\widetilde{T} \in \mathbf{T}(\mathcal{N}, \mathcal{N}')$. We assume that there is a $T \in \mathfrak{R}$ such that $||T - \widetilde{T}||_{\diamondsuit} \leq \varepsilon$ (cf. Proposition 8.1). Then $|c(T) - 1| \leq \varepsilon$, so

$$||P\widetilde{T}(V\cdot V^{\dagger}) - I_{\mathcal{M}}||_{\diamondsuit} \leqslant 2\varepsilon.$$

It follows that quantum codes can be used for the approximate reconstruction of a state after a perturbation.

We denote by $\mathbf{L}_0(\mathcal{M})$ the centre of the algebra $\mathbf{L}(\mathcal{M})$, that is, the set of operators of the form $c1_{\mathcal{M}}$ such that $c \in \mathbf{C}$.

Theorem 8.3. Let $\mathfrak{R} \subseteq \mathbf{T}(\mathcal{N}, \mathcal{N}')$ be a set of completely positive transformations. We define the minimal space of operators $\mathcal{E} \subseteq \mathbf{L}(\mathcal{N}, \mathcal{N}')$ such that $\mathfrak{R} \subseteq \mathcal{E} \cdot \mathcal{E}^*$. The following conditions are equivalent:

- 1) the code V corrects errors from \mathcal{E} ;
- 2) V corrects errors from \Re ;
- 3) for any pair of orthogonal vectors $|\xi\rangle, |\eta\rangle \in \mathcal{L}$ and any pair of operators $X, Y \subseteq \mathcal{E}$ the vectors $X|\xi\rangle$ and $Y|\eta\rangle$ are orthogonal;
- 4) $V^{\dagger}Y^{\dagger}XV \in \mathbf{L}_0(\mathcal{M})$ for any $X, Y \in \mathcal{E}$.

Proof. 1) \Rightarrow 2). This is obvious.

2) \Rightarrow 3). Each transformation $T \in \Re$ can be represented in the form $\sum_{j} X_{Tj} \cdot X_{Tj}^{\dagger}$. The operators X_{Tj} generate \mathcal{E} . It suffices to show that the vectors $X_{Tj}|\xi\rangle$ and $X_{Rk}|\eta\rangle$ are orthogonal for any T,j,R,k.

Let $\mathcal{G}_1, \mathcal{G}_2 \subseteq \mathcal{L}$ be complementary subspaces orthogonal to one another such that $|\xi\rangle \in \mathcal{G}_1$ and $|\eta\rangle \in \mathcal{G}_2$. We denote the corresponding projections by Π_1 and Π_2 . The operators $H_1 = P^*\Pi_1$ and $H_2 = P^*\Pi_2$ are Hermitian and positive. Moreover, $H_1 + H_2 = 1_{\mathcal{N}}$. It is obvious that

$$\sum_{j} \langle X_{Tj} \xi | H_2 | X_{Tj} \xi \rangle = \text{Tr} \big(H_2 (T | \xi) \langle \xi |) \big) = \text{Tr} \big(\Pi_2 (PT | \xi) \langle \xi |) \big) = c(T) \langle \xi | \Pi_2 | \xi \rangle = 0.$$

Consequently, each term on the left-hand side is equal to zero. Similarly, if $|\eta\rangle \in \mathcal{G}_2$, then $\langle X_{Rk}\eta|H_1|X_{Rk}\eta\rangle=0$. It follows that $|X_{Tj}\xi\rangle$ and $|X_{Rk}\eta\rangle$ are orthogonal vectors.

- 3) \Rightarrow 4). Let $Z = V^{\dagger}Y^{\dagger}XV \in \mathbf{L}(\mathcal{M})$. If $\langle \xi | \eta \rangle = 0$, then $\langle \xi | Z | \eta \rangle = 0$. It follows that Z commutes with the projections. But the linear span of all projections is equal to $\mathbf{L}(\mathcal{M})$, so Z commutes with any operator.
- $4)\Rightarrow 1$). Let $\mathcal{F}=\{XV:X\in\mathcal{E}\}$. Condition 4) makes it possible to define a Hermitian scalar product $\langle\cdot|\cdot\rangle$ on \mathcal{F} such that $B^{\dagger}A=\langle B,A\rangle 1_{\mathfrak{M}}$ (for any $A,B\in\mathcal{F}$). We can define in a natural way a linear operator $U:\mathfrak{M}\otimes\mathcal{F}\to\mathfrak{N}'$ such that $U(|\xi\rangle\otimes A)=A|\xi\rangle$ (for any $|\xi\rangle\in\mathfrak{M}$ and any $A\in\mathcal{F}$). It is obvious that U preserves the scalar product, that is, it is a unitary embedding.

Let \mathcal{K} be the orthogonal complement of the subspace $U(\mathcal{M} \otimes \mathcal{F}) \subseteq \mathcal{N}'$ and let $W: \mathcal{K} \to \mathcal{N}'$ be the natural embedding. We choose an arbitrary admissible transformation $R \in \mathbf{T}(\mathcal{K}, \mathcal{M})$ and put

$$P = (V \cdot V^{\dagger}) \left(\operatorname{Tr}_{\mathcal{F}} (U^{\dagger} \cdot U) + R(W^{\dagger} \cdot W) \right).$$

The function $c: \mathcal{E} \cdot \mathcal{E}^* \to \mathbf{C}$ can be defined by $c(X \cdot Y^{\dagger}) = \langle YV, XV \rangle$. It is easy to verify that P and c have the required properties.

8.3. Many-to-one codes. Many-to-one codes are used to model schemes with accurate elements by means of schemes with inaccurate elements. The point is that a correction transformation for an ordinary code cannot be realized by a scheme with inaccurate elements. (The inaccuracy at the last step always leads to ambiguity.) The code REP(n,l) of type (n,1) is the simplest example of a many-to-one classical code. Zero is encoded by any symbol of length n containing no more than l units. One is encoded by any symbol of length n containing no more than l zeros. (This definition is correct if 2l < n.) The code REP(n,l) corrects $\lfloor \frac{n-1}{2} \rfloor - l$ errors. In the problem concerning the transmission of information (or a quantum state) many-to-one codes provide no advantage as compared to one-to-one codes.

Let N and M be finite sets. A pair (L, f), where $L \subseteq N$ and $f: L \to M$ is a surjection, is called a *many-to-one classical code* of type (N, M). A state $a \in M$ can be encoded by any state $b \in f^{-1}(a)$.

Let \mathbb{N} and \mathbb{M} be unitary spaces. A unitary map $U: \mathbb{M} \otimes \mathfrak{F} \to \mathcal{L}$, where $\mathcal{L} \subseteq \mathbb{N}$ and \mathfrak{F} is an arbitrary non-zero unitary space, is called a *many-to-one* quantum code of type (\mathbb{N},\mathbb{M}) . A quantum state $|\xi\rangle \in \mathbb{M}$ can be encoded by any mixed state $\rho \in \mathbf{D}(\mathcal{L}) \subseteq \mathbf{D}(\mathbb{N})$ such that $\mathrm{Tr}_{\mathcal{F}}(U^{\dagger}\rho U) = |\xi\rangle\langle\xi|$. The general form of the encoding transformation is determined by Proposition 3.6. The proof of Theorem 8.3 involves the construction of a many-to-one quantum code. (Later on we shall call this construction the *derived code*.)

Now we shall consider the most general situation. Let \mathbb{N} and \mathbb{M} be combined spaces. A pair (\mathcal{L}, χ) , where $\mathcal{L} \subseteq \mathbb{N}$ is a combined subspace and $\chi : \mathbf{L}(\mathbb{M}) \to \mathbf{L}(\mathcal{L})$ is an injective homomorphism of algebras with involution, is called a *many-to-one combined code* of type (\mathbb{N}, \mathbb{M}) . (χ can also be regarded as a homomorphism $\mathbf{L}(\mathbb{M}) \to \mathbf{L}(\mathbb{N})$ which does not preserve unity.) A state $\gamma \in \mathbf{D}(\mathbb{M})$ can be encoded by an arbitrary state $\rho \in \mathbf{D}(\mathcal{L}) \subseteq \mathbf{D}(\mathbb{N})$ such that $\chi^* \rho = \gamma$. In the special case of a many-to-one quantum code $\chi : X \mapsto U(X \otimes 1_{\mathcal{T}})U^{\dagger}$.

We say that a code (\mathcal{L}', χ') is *coarser* than a code (\mathcal{L}, χ) if $\mathcal{L}' \supseteq \mathcal{L}$ and the restriction of $(\chi')^*$ to the subspace $\mathbf{L}(\mathcal{L})$ is equal to χ^* . This happens if $\chi(A) = \Pi_{\mathcal{L}} \chi'(A) = \chi'(A) \Pi_{\mathcal{L}}$ for each $A \in \mathbf{L}(\mathcal{M})$.

Theorem 8.4. Let $\chi: \mathbf{L}(\mathfrak{M}) \to \mathbf{L}(\mathfrak{L})$ be an injective homomorphism of algebras with involution and let $T \in \mathbf{T}(\mathfrak{M}, \mathfrak{L})$ be an admissible transformation. T is an encoding transformation (that is, $\chi^*T = I_{\mathfrak{M}}$) if and only if

$$T(AB) = \chi(A)T(B) = T(A)\chi(B)$$
 for any $A, B \in \mathbf{L}(\mathcal{M})$. (42)

Proof. Let $\mathcal{M} = (\mathcal{M}_j : j \in \Omega)$. The space \mathcal{L} can be regarded as unitary (it only needs to be replaced by $\overline{\mathcal{L}}$). The homomorphism χ identifies \mathcal{L} with $\bigoplus_j \mathcal{M}_j \otimes \mathcal{F}_j$, where the \mathcal{F}_j are non-zero unitary spaces (see (10)). The transformation T is an encoding one if and only if it is a direct sum of encoding transformations $T_j \in \mathbf{T}(\mathcal{M}_j, \mathcal{M}_j \otimes \mathcal{F}_j)$. By Proposition 3.6 $T_j : X \mapsto X \otimes \lambda_j$, where $\lambda_j \in \mathbf{D}(\mathcal{F}_j)$.

On the other hand, (42) is equivalent to the condition $T = \sum_j T_j$, $T_j : X \mapsto X \otimes \lambda_j$, where $\lambda_j \in \mathbf{L}(\mathfrak{F}_j)$. Indeed, $\lambda_j \in \mathbf{D}(\mathfrak{F}_j)$ because T is an admissible transformation.

Theorem 8.3 can easily be generalized to the case of many-to-one combined codes. Let $\mathbb{N}=(\mathbb{N}_j:j\in\Omega)$ and $\mathbb{N}'=(\mathbb{N}'_k:k\in\Omega')$ be combined spaces, let (\mathcal{L},χ) be a code of type (\mathbb{N},\mathbb{M}) , and let $V_j:\mathcal{L}_j\to\mathbb{N}_j$ be the natural embeddings. We consider the error space $\mathcal{E}=\bigoplus_{j,k}\mathcal{E}_{kj}$ $(\mathcal{E}_{kj}\subseteq\mathbf{L}(\mathbb{N}_j,\mathbb{N}'_k))$ and introduce the following notation: $\overline{\mathcal{L}'}=\mathcal{E}\overline{\mathcal{L}}$ $(\mathcal{L}'\subseteq\mathbb{N}'$ being a combined subspace), $\mathcal{H}_{kj}=\mathcal{E}_{kj}V_j\subseteq\mathbf{L}(\mathcal{L}_j,\mathcal{L}'_k)$, and

$$\mathbf{L}_0(\mathcal{L}, \chi) = \{ A \in \mathbf{L}(\mathcal{L}) : \forall B \in \mathbf{L}(\mathcal{M}) \ (A\chi(B) = \chi(B)A) \}. \tag{43}$$

Then the conditions of the following theorem are satisfied.

Theorem 8.5. Let $\mathcal{L} = (\mathcal{L}_j : j \in \Omega)$ and $\mathcal{L}' = (\mathcal{L}'_k : k \in \Omega')$ be combined spaces and let $\overline{\mathcal{H}} = \bigoplus_{j,k} \mathcal{H}_{kj}$ $(\mathcal{H}_{kj} \subseteq \mathbf{L}(\mathcal{L}_j, \mathcal{L}'_k))$, where $\overline{\mathcal{L}'} = \overline{\mathcal{HL}}$. We consider a set of completely positive transformations $\mathfrak{R} \subseteq \mathbf{T}(\mathcal{L}, \mathcal{L}')$ such that $\mathfrak{R} \subseteq \bigoplus_{j,k} \mathcal{H}_{kj} \cdot \mathcal{H}^*_{kj}$, where the spaces \mathcal{H}_{kj} cannot be reduced. Moreover, let $\chi : \mathbf{L}(\mathcal{M}) \to \mathbf{L}(\mathcal{L})$ be a homomorphism of algebras with involution.

The following conditions are equivalent:

- 1) there is an admissible transformation $P \in \mathbf{T}(\mathcal{L}', \mathcal{M})$ and a function $c : \mathfrak{R} \to \mathbf{L}_0(\mathcal{L}, \chi)$ such that $PT\rho = \chi^*(c(T)\rho)$ for any $T \in \mathfrak{R}$ and $\rho \in \mathbf{L}(\mathcal{L})$;
- 2) there is a homomorphism of algebras with involution $\chi': \mathbf{L}(\mathfrak{M}) \to \mathbf{L}(\mathfrak{L}')$ such that $X\chi(A) = \chi'(A)X$ and $\chi(A)X^{\dagger} = X^{\dagger}\chi'(A)$ for any $X \in \overline{\mathfrak{H}}$ and $A \in \mathbf{L}(\mathfrak{M})$;
- 3) $Y^{\dagger}X \in \mathbf{L}_0(\overline{\mathcal{L}}, \chi)$ for any $X, Y \in \overline{\mathcal{H}}$.

Moreover, P and χ' are defined uniquely and are independent of \Re ; $P = (\chi')^*$.

Proof. The function c is defined uniquely: $c(T) = T^*1_{\mathcal{L}'} \in \mathbf{L}(\mathcal{L})$ for each $T \in \mathbf{T}(\mathcal{L}, \mathcal{L}')$. Without loss of generality we can take \mathfrak{R} to be a cone in $\mathbf{T}(\mathcal{L}, \mathcal{L}')$. Let $R \in \mathfrak{R}$ be an arbitrary point in the relative interior of this cone.

Then $R = \sum_{j,k,s} X_{kjs} \cdot X_{kjs}^{\dagger}$, where the operators X_{kjs} form a basis in \mathcal{H}_{kj} . In addition, $E = R1_{\mathcal{L}}$ is a non-degenerate operator, since $\overline{\mathcal{L}'} = \overline{\mathcal{HL}}$.

1) \Rightarrow 3). For each combined subspace $\mathcal{K} \subseteq \mathcal{M}$ the operator $\chi(\Pi_{\mathcal{K}})$ is a projection onto some combined subspace denoted by $\chi(\mathcal{K})$. It suffices to show that operators of the form $Y^{\dagger}X$ $(X,Y \in \overline{\mathcal{H}})$ commute with such projections.

Let $\mathcal{M}=\mathcal{K}_1\oplus\mathcal{K}_2$, where $\mathcal{K}_1,\mathcal{K}_2$ are combined subspaces orthogonal to one another, and let $\mathcal{G}_i=\chi(\mathcal{K}_i),\ E_i=R\Pi_{\mathcal{G}_i}$, and $H_i=P^*\Pi_{\mathcal{K}_i}\ (i=1,2)$. Then $\mathrm{Tr}(E_1H_2)=\mathrm{Tr}(E_2H_1)=0$. We observe that E_i and H_i are positive Hermitian operators such that $E_1+E_2=E$ and $H_1+H_2=1_{\mathcal{L}'}$. It follows that H_1 and H_2 are projections onto some subspaces $\mathcal{G}_1',\mathcal{G}_2'\subseteq\mathcal{L}'$ orthogonal to one another, and $E_i=H_iE=EH_i$. This implies first that P is unique, and second that $\overline{\mathcal{H}}\mathcal{G}_i\subseteq\mathcal{G}_i'$. Consequently, operators of the form $Y^{\dagger}X$ $(X,Y\in\overline{\mathcal{H}})$ commute with the projections $\Pi_{\mathcal{G}_i}$.

- 3) \Rightarrow 2). We put $\chi'(A) = E^{-1}(R\chi(A))$ for each $A \in \mathbf{L}(\mathcal{M})$. The desired conditions and the uniqueness of χ' are verified trivially.
 - $(2) \Rightarrow 1$). For any $B \in \mathbf{L}(\mathcal{M})$, $\rho \in \mathbf{L}(\mathcal{L})$ and $T \in \overline{\mathcal{H}} \cdot \overline{\mathcal{H}}^*$

$$T(\chi(B)\rho) = \chi'(B)(T\rho), \qquad T(\rho\chi(B)) = (T\rho)\chi'(B).$$

It follows that

$$\operatorname{Tr}((T^*1_{\mathcal{L}'})\chi(B)\rho) = \operatorname{Tr}(\chi'(B)(T\rho)) = \operatorname{Tr}((T^*1_{\mathcal{L}'})\rho\chi(B)),$$

so $T^*1_{\mathcal{L}'} = c(T)$ commutes with $\chi(B)$. If $P = (\chi')^*$, then

$$(PT\rho, B) = \operatorname{Tr}((T\rho)\chi'(B)) = \operatorname{Tr}T(\rho\chi(B)) = \operatorname{Tr}(c(T)\rho\chi(B)) = (\chi^*(c(T)\rho), B).$$

Let us go back to the situation described before Theorem 8.5. As a rule, we shall assume that $\mathcal{N}' = \mathcal{N}$ and $\mathcal{E} \ni 1_{\mathcal{N}}$. In this case the homomorphism χ' is injective. We call (\mathcal{L}', χ') a *derived code* and denote it by $\mathrm{Der}((\mathcal{L}, \chi), \mathcal{E})$. For example, $\mathrm{REP}(n,l) = \mathrm{Der}(\mathrm{REP}(n), \mathcal{E}(n,l))$. Derived codes have the following obvious properties.

- 1. If $\mathcal{E}_2 \supseteq \mathcal{E}_1$, then $Der(C, \mathcal{E}_2)$ is a coarser code than $Der(C, \mathcal{E}_1)$.
- 2. $\operatorname{Der}(C, \mathcal{E}_2 \mathcal{E}_1) = \operatorname{Der}(\operatorname{Der}(C, \mathcal{E}_1), \mathcal{E}_2)$.
- 3. Let $C = (\mathcal{L}, \chi)$ be a code of type $(\mathcal{M}, \mathcal{K} \otimes \mathcal{F})$ correcting the errors from $\mathbf{L}(\mathcal{F})$. Then $\mathrm{Der}(C, \mathbf{L}(\mathcal{F})) = (\mathcal{G} \otimes \mathcal{F}, \mu \otimes 1_{\mathcal{F}})$, where (\mathcal{G}, μ) is a code of type $(\mathcal{M}, \mathcal{K})$.

§9. Symplectic (additive) codes

9.1. Algebraic preparation. We consider the space of states $\mathcal{B} = \overline{\mathbf{B}}$ of one qbit. In the space of operators $\mathbf{L}(\mathcal{B})$ we have the standard basis consisting of the identity operator $1_{\mathcal{B}} = \sigma_0 = \sigma_{00}$ and the three Pauli matrices $\sigma_x = \sigma_{10}$, $\sigma_y = \sigma_{11}$ and $\sigma_z = \sigma_{01}$ (see (7)). The remarkable properties of this basis manifest themselves if the index set $G = \{0, x, y, z\} = \{00, 10, 11, 01\}$ is identified with the group $\mathbf{Z}_2 \times \mathbf{Z}_2$. Then

$$\sigma_{\alpha\beta}\sigma_{\alpha'\beta'}=(-i)^{\alpha\beta'-\beta\alpha'}\sigma_{\alpha+\alpha',\beta+\beta'}=(-1)^{\alpha\beta'-\beta\alpha'}\sigma_{\alpha',\beta'}\sigma_{\alpha,\beta}.$$

It follows that the algebra of operators $L(\mathcal{B})$ is equipped with G-gradation $(G = \mathbb{Z}_2 \times \mathbb{Z}_2)$. (Generalization: Let E be any finite Abelian group and let E^*

be the corresponding group of characters. Then the algebra $\mathbf{L}(\overline{E})$ is equipped with $(E \times E^*)$ -gradation.)

The operators $\sigma(\alpha_1\beta_1,\ldots,\alpha_n\beta_n)=\sigma_{\alpha_1\beta_1}\otimes\cdots\otimes\sigma_{\alpha_n\beta_n}$ form the standard basis in $\mathbf{L}(\mathbb{B}^{\otimes n})$. Each of them is both unitary and Hermitian. Their eigenvalues are equal to ± 1 . The one-dimensional subspaces $(\sigma(f)):\sigma\in G^n$ generated by these operators define the G^n -gradation of the algebra $\mathbf{L}(\mathbb{B}^{\otimes n})$. This gradation is consistent with involution, that is, if $X\in(\sigma(f))$, then $X^{\dagger}\in(\sigma(-f))$.

The skew-symmetric form

$$\omega\Big((\alpha_1\beta_1,\ldots,\alpha_n\beta_n),(\alpha_1'\beta_1',\ldots,\alpha_n'\beta_n')\Big) = \sum_{j=1}^n \alpha_j\beta_j' - \beta_j\alpha_j' \pmod{2} \tag{44}$$

is defined on G^n (which can be regarded as a 2n-dimensional space over the field \mathbf{F}_2 .) The form determines the commutation relations between the operators of the basis:

$$\sigma(f)\sigma(g) = (-1)^{\omega(f,g)}\sigma(g)\sigma(f). \tag{45}$$

The automorphisms of G^n preserving ω form the group $\operatorname{Sp}_2(n)$ (the symplectic group over \mathbf{F}_2). In what follows an important role will be played by the group $\operatorname{ESp}_2(n) \subseteq \operatorname{UT}(\mathcal{B}^{\otimes n})$ of unitary transformations preserving the G^n -gradation of the algebra $\mathbf{L}(\mathcal{B}^{\otimes n})$. The image of the natural homomorphism $\theta: \operatorname{ESp}_2(n) \to \operatorname{Aut}(G^n)$ is equal to $\operatorname{Sp}_2(n)$, and the kernel consists of transformations of the form $\sigma(f) \cdot \sigma(f)^\dagger$, where $f \in G^n$. It follows that $\operatorname{ESp}_2(n)/G^n \cong \operatorname{Sp}_2(n)$. Because of this we call $\operatorname{ESp}_2(n)$ the extended symplectic group. It is not just a semidirect product of $\operatorname{Sp}_2(n)$ and G^n . The group $\operatorname{ESp}_2(1)$ is isomorphic to the group of characteristic rotations of a cube (see §4.2). We observe that the elements of the subgroup G correspond to the rotations by π about the coordinate axes and also the identity transformation. It is not hard to show that $\operatorname{ESp}_2(n)$ is generated by the unitary operators S, K and $\widehat{\oplus}$, which can be applied to different qbits. The operator $\widehat{\oplus}$ acts on the canonical generators of the algebra $\mathbf{L}(\mathcal{B}^{\otimes 2})$ as follows:

$$(\widehat{\oplus} \cdot \widehat{\oplus})[1,2] : (\sigma_z[1], \sigma_z[2], \sigma_x[1], \sigma_x[2]) \mapsto (\sigma_z[1], \sigma_z[1]\sigma_z[2], \sigma_x[1]\sigma_x[2], \sigma_x[2]). \tag{46}$$

The images of the operators $\sigma_x[k], \sigma_z[k]$ (k = 1, ..., n) under an arbitrary isomorphism $\Upsilon: \mathbf{L}(\mathcal{B}^{\otimes n}) \to \mathcal{G}$ are called the *canonical generators* of the G^n -graded algebra with involution $\mathcal{G} \cong \mathbf{L}(\mathcal{B}^{\otimes n})$. The isomorphism Υ is uniquely characterized by the system of canonical generators $Y_{\gamma k} = \Upsilon \sigma_{\gamma}[k]$ $(\gamma = x, z; k = 1, ..., n)$. The canonical generators are unitary, they belong to the gradation spaces, and have the following properties:

$$Y_{\gamma k}^2 = 1, \quad Y_{xk}Y_{xl} = Y_{xl}Y_{xk}, \quad Y_{zk}Y_{zl} = Y_{zl}Y_{zk}, \quad Y_{xk}Y_{zl} = (-1)^{\delta_{kl}}Y_{zl}Y_{xk}.$$
 (47)

Invertible linear maps $u: \mathbf{F}_2^n \to \mathbf{F}_2^n$ form the linear group $\mathrm{GL}_2(n)$. To each such map there corresponds an operator $\widehat{u} \in \mathrm{U}(\mathcal{B}^{\otimes n})$ and a transformation $\widehat{u} \cdot \widehat{u}^{\dagger} \in \mathrm{ESp}_2(n)$. It follows that $\mathrm{GL}_2(n) \in \mathrm{ESp}_2(n)$. The group $\mathrm{GL}_2(n)$ is generated by the operator $\widehat{\oplus}$, which can be applied to various qbits.

Graded subalgebras with involution in $L(\mathfrak{B}^{\otimes n})$ correspond to linear subspaces $F \subseteq G^n$ and are denoted by $\sigma(F)$. The subalgebra $\sigma(F)$ is commutative if and only if F is a homogeneous subspace, that is, $\omega(f,g)=0$ for any $f,g\in G^n$. Acting by a suitable symplectic transformation, we can reduce any homogeneous subspace $F \subseteq G^n$ to the standard form

$$F' = \{ (0\beta_1, \dots, 0\beta_s, 0, \dots, 0) : \beta_1, \dots, \beta_s \in \mathbf{F}_2 \}, \tag{48}$$

where $s = \dim F$. The operators $\sigma_z[1], \ldots, \sigma_z[s]$ are the generators of $\sigma(F')$.

9.2. The basic construction. Let n = s + m. We define the standard embedding $V': \mathcal{B}^{\otimes m} \to \mathcal{B}^{\otimes n}$ by $|\xi\rangle \mapsto |\underbrace{0, \dots, 0}_{\xi}, \xi\rangle$.

Definition 9.1. A unitary embedding of the form $V = UV' : \mathbb{B}^{\otimes m} \to \mathbb{B}^{\otimes n}$, where $U \cdot U^{\dagger} \in \mathrm{ESp}_2(n)$, is called a *symplectic* (or *additive*) code of type (n, m). If $U = \widehat{u}S[1] \cdots S[l]$, where $u \in \mathrm{GL}_2(n)$ and $0 \leqslant l \leqslant s$, then the code is called *quasilinear*.

Symplectic codes were first defined in [29]. Quasilinear codes were considered (in slightly different terms) by Steane [20], and Shor and Calderbank [19].

Definition 9.1 is constructive but not very convenient. We shall describe symplectic codes in terms of the algebra $L(\mathcal{B}^{\otimes n})$.

The homogeneous subspace $F = uF' \subseteq G^n$, where $u = \theta(U \cdot U^{\dagger}) \in \operatorname{Sp}_2(n)$ and F' is the standard homogeneous subspace (48), is called the *characteristic subspace* of a code V. It is easily seen that the information subspace $\mathcal{L} = V\mathcal{B}^{\otimes m} \subseteq \mathcal{B}^{\otimes n}$ is the common characteristic subspace for all operators $\sigma(f)$ such that $f \in F$. Consequently, there is a homomorphism of algebras with involution $\Phi: \sigma(F) \to \mathbf{C}$ such that

$$\mathcal{L} = \{ |\xi\rangle \in \mathcal{B}^{\otimes n} : \forall X \in \sigma(F) \ (X|\xi\rangle = \Phi(X)|\xi\rangle) \}.$$

It is obvious that $\Phi(\sigma(f)) = (-1)^{\varphi(f)}$, where $\varphi(f) \in \mathbf{F}_2$. The homomorphism Φ is determined by a sequence of numbers $\varphi_j = \varphi(f_j) \in \mathbf{F}_2$ (j = 1, ..., s), where $\{f_1, ..., f_s\}$ is an arbitrary basis in F. The f_j and $X_j = (-1)^{\varphi_j} \sigma(f_j)$ are called the *verification* vectors and operators because the measurement of the eigenvalues of X_j makes it possible to verify whether or not the quantum state belongs to \mathcal{L} :

$$\mathcal{L} = \{ |\xi\rangle \in \mathcal{B}^{\otimes n} : X_j |\xi\rangle = |\xi\rangle \ (j = 1, \dots, s) \}.$$
(49)

In some cases it may be convenient to define a redundant set of verification operators (see §9.4).

A symplectic code can be completely characterized by specifying the verification operators X_1, \ldots, X_s and information operators $Y_{\gamma k} = (U \cdot U^{\dagger})\sigma_{\gamma}[s+k]$, where $\gamma = x, z$ and $k = 1, \ldots, m$. It is obvious that $Y_{\gamma k} = \pm \sigma(g_{\gamma k})$, where $g_{\gamma k}$ belongs to the subspace

$$F_{+} = \left\{ g \in G^{n} : \forall f \in F \left(\omega(f, g) = 0 \right) \right\} \supseteq F. \tag{50}$$

The information operators are defined to within multiplication by a verification operator. (The non-uniqueness is due to the fact that different operators U may

¹²We recall that $\theta : \mathrm{ESp}_2(n) \to \mathrm{Aut}(G^n)$ is the natural homomorphism.

correspond to the same code V.) In the language of invariants the information operators are the canonical generators of the F_+/F_- graded algebra with involution

$$\sigma(F_+)/(\sigma(F_+)\operatorname{Ker}\Phi) \cong \mathbf{L}(\mathcal{L}) \cong \mathbf{L}(\mathcal{B}^{\otimes m}). \tag{51}$$

Quasilinear codes can be characterized by two conditions:

- 1) each operator X_j has the form $\sigma(\alpha_1 0, \ldots, \alpha_n 0)$ or $\sigma(0\beta_1, \ldots, 0\beta_n)$;
- 2) the operators Y_{xk} have the form $\sigma(\alpha_1 0, \ldots, \alpha_n 0)$ and Y_{zk} have the form $\sigma(0\beta_1, \ldots, 0\beta_n)$.

We shall proceed to study the error correction properties of symplectic codes. Suppose that the space of errors has the form $\sigma(E)$, where $E \subseteq G^n$ is an arbitrary set. Spaces of the form $\mathcal{E}(n,k) = \sigma(E(n,k))$, where

$$E(n,k) = \{g \in G^n : |\operatorname{Supp}(g)| \leq k\},$$

$$\operatorname{Supp}(\alpha_1 \beta_1, \dots, \alpha_n \beta_n) = \{j : \alpha_i \neq 0 \text{ or } \beta_i \neq 0\}$$
(52)

are of interest in applications. For brevity, the elements of E and the corresponding operators $\sigma(g)$, where $g \in E$, will both be called *errors*.

We denote by F^* the space of linear functionals on F with values in \mathbf{F}_2 . Then

$$\mathcal{B}^{\otimes n} = \bigoplus_{h \in F^*} \mathcal{L}_h, \text{ where } \mathcal{L}_h = \{ |\xi\rangle \in \mathcal{B}^{\otimes n} : X_j |\xi\rangle = (-1)^{(h,f_j)} |\xi\rangle \ (j = 1, \dots, s) \},$$

$$\sigma(g)\mathcal{L} = \mathcal{L}_{\mu(g)}, \quad \mu(g) = \omega(\cdot, g) \in F^* \quad \text{(for any } g \in G^n). \tag{53}$$

It is obvious that $\mathcal{L}_0 = \mathcal{L}$ and $\operatorname{Ker} \mu = F_+$. It proves convenient to represent $\mu(g)$ in the coordinate form: $\mu(g) = (\mu_1(g), \dots, \mu_n(g))$, where $\mu_j(g) = \omega(f_j, g)$. By analogy with the classical theory of linear codes [17], $\mu(g)$ is called the *syndrome* of an error g.

Errors $g', g'' \in E$ are called *indistinguishable* if $\mu(g') = \mu(g'')$, that is, $g' - g'' \in F_+$. The errors are indistinguishable if and only if $V^{\dagger}\sigma(g')^{\dagger}\sigma(g'')V \neq 0$. Errors $g', g'' \in E$ are said to be *equivalent* if $g' - g'' \in F$, which occurs if and only if they are indistinguishable and $V^{\dagger}\sigma(g')^{\dagger}\sigma(g'')V \in \mathbf{L}_0(\mathcal{B}^{\otimes m})$. As applied to symplectic codes, condition 3) of Theorem 8.3 can be stated as follows: 'any two indistinguishable errors are equivalent'. In this way we have proved the following result.

Theorem 9.2. A symplectic code V corrects errors from $\sigma(E)$ $(E \subseteq G^n)$ if and only if

$$(E-E) \cap F_{+} \subseteq F$$
, where $E-E = \{g'-g'': g', g'' \in E\}$. (54)

(Here $F \subseteq G^n$ is the characteristic subspace of V.)

Corollary. A symplectic code corrects k errors if and only if $E(n,2k) \cap F_+ \subseteq F$.

Example 9.3. A symplectic code of type (5,1) correcting one error. (See [21], [25], where the same code is defined in a different way.)

Let f_1, \ldots, f_4 be verification vectors given by the rows of the following table:

It is obvious that $\omega(f_i, f_j) = 0$ (i, j = 1, 2, 3, 4). Any two pairs of columns are linearly independent, so $E(4, 2) \cup F_+ = \{0\}$.

9.3. Error correction procedure. Theorem 9.2 does not say anything about realizing the error correction transformation as a scheme. Errors can be corrected, for example, as follows. Suppose that a state $|\xi\rangle \in \mathcal{L}$ is subject to an error $g \in E$ and the state $|\psi\rangle = \sigma(g)|\xi\rangle$ is obtained. With the aid of the measurement operators $\Xi(X_j)$ (see (31)) we can determine the error syndrome $\mu(g)$. Thanks to condition (54) the error g itself can be reconstructed uniquely from the syndrome to within equivalence. Acting on $|\psi\rangle$ by $\sigma(g')$ ($g'-g\in F$), we obtain the vector $\pm |\xi\rangle$. The error has been corrected.

We shall present this argument in more precise terms. Let Δ be the basic register (of length n) and let Δ' and Δ'' be auxiliary registers (of length s and 2n, respectively). We shall try to construct a correction transformation $R \in \mathbf{T}(\mathfrak{B}^{\otimes n})$ as a composition of three admissible transformations:

$$R[\Delta] = CORR[\Delta; \Delta, \Delta''] \nu[\Delta''; \Delta'] MEAS[\Delta, \Delta'; \Delta].$$
 (56)

Here MEAS is the procedure of measuring the syndrome, which also consists of three stages. First the state $|0\rangle$ is entered into the auxiliary register Δ' , then the unitary operators $\Xi(X_j)[\Delta,j]$ (for $j \in \{1,\ldots,s\} = \Delta'$) are applied, and finally the auxiliary bits are made classical (with the aid of Q_B^*). The transformation MEAS $\in \mathbf{T}(\mathcal{B}^{\otimes n}, \mathcal{B}^{\otimes n} \times F^*)$ acts as follows (compare with (16)):

$$MEAS \rho = \sum_{h \in F^*} (\Pi_{\mathcal{L}_h} \rho \Pi_{\mathcal{L}_h}, h).$$
 (57)

The function $\nu: F^* \to E \subseteq G^n$ computes the error from the syndrome. To be more precise, $\nu(\mu(g)) - g \in F$ for any $g \in E$. The latter transformation $\mathrm{CORR} \in \mathbf{T}(\mathbb{B}^{\otimes n} \times G^n, \mathbb{B}^{\otimes n})$ corrects the error, that is, $\mathrm{CORR}(\rho, g) = \sigma(g)\rho\sigma(g)$.

The constructed transformation R can be represented as

$$R = \sum_{h \in F^*} W_h \cdot W_h^{\dagger}, \quad \text{where} \quad W_h = \sigma(\nu(h)) \Pi_{\mathcal{L}_h}. \tag{58}$$

We shall show that it is indeed an error correcting transformation. Let

$$T = \sum_{f,g \in E} b_{fg} \sigma(f) \cdot \sigma(g),$$

and let $\rho \in \mathbf{L}(\mathcal{L})$. Then

$$RT\rho = \sum_{\mu(f) = \mu(g) = h} b_{fg} \, \sigma(\nu(h)) \, \rho \, \sigma(\nu(h)) = \left(\sum_{f,g \in E} c_{fg} b_{fg}\right) \rho$$

(where $c_{fq} \in \mathbf{C}$ are constants), as required.

The transformation CORR can be realized by a layered scheme of depth 2 from the elements $\Lambda_c(\sigma_x)$ and $\Lambda_c(\sigma_z)$. The function ν can be realized by some scheme from classical elements, however, the dimension and depth of this scheme may turn out to be large. The transformation MEAS can be realized by a scheme of depth O(n) and dimension $O(n^2)$ from the elements $|0\rangle$, S, $\Lambda(\sigma_x)$, $\Lambda(\sigma_y)$, $\Lambda(\sigma_z)$ and Q_B^* .

In this connection, *codes with local verification* are of special interest. This is what we call infinite families of symplectic codes such that

- a) the verification vectors have supports of bounded dimension;
- b) each qbit appears in a bounded number of such supports;
- c) the number of corrected errors is unbounded.

For codes with local verification the transformation MEAS can be realized by a scheme of depth O(1) and dimension O(n). It follows that if one error occurs as the MEAS scheme is run, then only O(1) qbits will be corrupted.¹³ Thanks to this property, partial correction of errors can be realized by a scheme involving inaccurate elements, see §11.1. (Moreover, Shor's method [43] makes this possible for any symplectic code.)

9.4. Torus codes. The infinite sequence of quasilinear codes TOR(k) (k = 1, 2, ...), constructed as follows [42], is a beautiful example of a code with local verification.

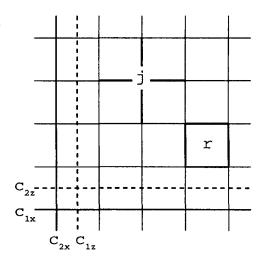


Figure 1. Torus code TOR(5)

¹³On the meaning of an 'error as a scheme is run' see §10.

We consider a square lattice of dimension $k \times k$ on a torus (see Fig. 1). Let A_0 be the set of vertices of this lattice, let A_1 be the set of edges, and A_2 the set of faces (two-dimensional cells). We associate a qbit with each edge. (There are $n=2k^2$ edges altogether.) With each face $r \in A_2$ and each vertex $j \in A_0$ we associate the verification operators

$$X_r = \prod_{l \in \text{Boundary}(r)} \sigma_x[l], \qquad X_j = \prod_{l \in \text{Star}(j)} \sigma_z[l].$$
 (59)

(There are two relations between these operators: $\Pi_{r\in A_2}X_r=1$ and $\Pi_{j\in A_0}X_j=1$.) The information operators $Y_{x1}, Y_{x2}, Y_{z1}, Y_{z2}$ are associated with the cycles c_{x1}, c_{x2} and cocycles (cuts) c_{z1}, c_{z2} (see Fig. 1): we need to take the product of the operators $\sigma_x[l]$ over all the edges appearing in the cycle or cut. The torus code TOR(k) constructed in this way is a quasilinear code of type $(2k^2, 2)$.

To verify that this definition is well posed we consider complexes of chains $C_0 \stackrel{\partial_1}{\longleftarrow} C_1 \stackrel{\partial_2}{\longleftarrow} C_2$ and cochains $C_0^* \stackrel{\partial_1^*}{\longrightarrow} C_1^* \stackrel{\partial_2^*}{\longrightarrow} C_2^*$ with coefficients in \mathbf{Z}_2 . We identify G^n with the group $C_1 \oplus C_1^*$ by the obvious formula

$$G^n \ni (\alpha_1 \beta_1, \dots, \alpha_n \beta_n) = ((\alpha_1, \dots, \alpha_n), (\beta_1, \dots, \beta_n)) \in C_1 \oplus C_1^*$$

The verification operators (59) correspond to the subspace $F = \operatorname{Im} \partial_2 \oplus \operatorname{Im} \partial_1^* \subseteq G^n$. This is a homogeneous subspace, since $(\partial_1^* c_0', \partial_2 c_2) = 0$ for any $c_0' \in C_0^*$ and $c_2 \in C_2$. Thus

$$F = \operatorname{Im} \partial_2 \oplus \operatorname{Im} \partial_1^*, \qquad F_+ = \operatorname{Ker} \partial_1 \oplus \operatorname{Ker} \partial_2^*, \qquad F_+/F = H_1 \oplus H_1^*, \tag{60}$$

where $H_1 \cong H_1^* \cong (\mathbf{Z}^2)^2$ are the homology and cohomology groups of the torus with coefficients in \mathbf{Z}_2 . It is obvious that $Y_{x1}, Y_{x2}, Y_{z1}, Y_{z2}$ are the canonical generators of the F_+/F -graded algebra (51).

The support of any cycle or cocycle non-homologous to zero consists of no fewer than k edges. It follows that the code TOR(k) corrects $\lfloor \frac{k-1}{2} \rfloor$ errors.

Error correction, that is, the computation of ν , can be reduced to finding the smallest 1-chain given its boundary. In turn, this problem can be reduced to the problem of least weight matching, for which a polynomial algorithm is known [55].

The above construction has a generalization based on Hopf algebras [48].

§10. Error correction in the computation process: general principles

10.1. Definitions and results. We shall render concrete the problem posed in §7: "How can we model a scheme with accurate elements by one with inaccurate elements?" Difficulties in the formulation of this problem arise even in the classical case.

The first thing which comes to mind is to require that the inaccurate scheme should realize the same transformation as the accurate one. However, this is impossible, because any error at the first or the last steps will inevitably disturb the input or output information. (Errors at the intermediate steps are not so serious, because one can have the input information protected with the aid of some code at this stage.) The only way around this difficulty is to supply the input information in an

encoded form. The output information should also be encoded using a many-to-one code. (The ambiguity includes errors arising at the last step.) Finally, it is natural to encode bitwise all the intermediate results obtained as the accurate scheme is run.

In this way we have arrived at the concept of a computation involving encoded bits (classical or quantum ones). Here is the formal definition.

Definition 10.1. Let $C = (\mathcal{L}, \chi)$ and $C' = (\mathcal{L}', \chi')$ be many-to-one combined codes of type $(\mathcal{N}, \mathcal{M})$ and $(\mathcal{N}', \mathcal{M}')$, respectively. We consider admissible transformations $T \in \mathbf{T}(\mathcal{N}, \mathcal{N}')$ and $P \in \mathbf{T}(\mathcal{M}, \mathcal{M}')$. We say that T represents P in the codes C, C' if the commutative diagram

$$\mathbf{L}(\mathcal{M}') \xleftarrow{(\chi')^*} \mathbf{L}(\mathcal{L}') \xleftarrow{J'} \mathbf{L}(\mathcal{N}')$$

$$\uparrow_{P} \qquad \uparrow_{R} \qquad \uparrow_{T}$$

$$\mathbf{L}(\mathcal{M}) \xleftarrow{\chi^*} \mathbf{L}(\mathcal{L}) \xleftarrow{J} \mathbf{L}(\mathcal{N})$$
(61)

holds for some admissible transformation $R \in \mathbf{T}(\mathcal{L}, \mathcal{L}')$. (Here J and J' are the natural embeddings.) If, in addition, $\operatorname{Im} T \subseteq \mathbf{L}(\mathcal{L}')$, that is, the arrow $\mathbf{L}(\mathcal{N}) \to \mathbf{L}(\mathcal{L}')$ can also be added, then T is said to represent P in the strong sense. If commutativity holds to within ε (in the sense of the norm $\|\cdot\|_{\diamondsuit}$), then T is said to represent P with accuracy ε .

If T_1, T_2 represent P_1, P_2 with accuracy $\varepsilon_1, \varepsilon_2$, then the product T_2T_1 represents P_2P_1 with accuracy $\varepsilon_1+\varepsilon_2$. Prior to proceeding any further it is useful to understand some subtle points of the notion of 'representing with accuracy ε '. In the classical case the definition adopted by us is essentially the only possible one.

Proposition 10.2. Let $P \in \mathbf{T}(M,M')$ and $\widetilde{T} \in \mathbf{T}(N,N')$ be classical admissible transformations. We assume that \widetilde{T} represents P with accuracy ε (in some classical codes). Then there is an admissible transformation $T \in \mathbf{T}(N,N')$ representing P such that $||T-\widetilde{T}||_{\diamondsuit} \leq 2\varepsilon$.

(We leave the proof to the reader as an exercise.) The quantum analogue of this assertion is false. However, the following result holds.

Theorem 10.3. Let $P \in \mathbf{T}(\mathcal{M}, \mathcal{M}')$ and $\widetilde{T} \in \mathbf{T}(\mathcal{N}, \mathcal{N}')$ be admissible transformations and let $C = (\mathcal{L}, \chi)$ and $C' = (\mathcal{L}', \chi')$ be codes of type $(\mathcal{N}, \mathcal{M})$ and $(\mathcal{N}', \mathcal{M}')$, respectively.

- a) \widetilde{T} represents P with accuracy $O(\varepsilon)$ if and only if there are transformations T and R (not necessarily admissible) such that the diagram (61) is commutative and $||T \widetilde{T}||_{\diamondsuit} \leq O(\varepsilon)$.
- b) If \tilde{T} represents P with accuracy ε , then there is an admissible transformation $T \in \mathbf{T}(\mathcal{N}, \mathcal{N}')$ representing P such that $||T \tilde{T}||_{\diamondsuit} \leq (\dim \mathcal{M}')O(\varepsilon^{1/2})$.

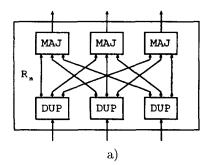
The word 'represents' can be replaced everywhere by 'represents in the strong sense'.

(The proof will be given in §10.2.)

In what follows we shall use only codes of type (n, 1). To encode several bits we must take the product of such codes, so that each bit is encoded independently.

Correction of classical errors. Suppose that we have at our disposal the classical elements from the basis \mathcal{C} realized with accuracy δ . We shall show how to represent them with accuracy $O(\delta^2)$ in the code REP(5, 1). The following (not fully rigorous) discussion is implicitly based on Proposition 10.2.

In Fig. 2a we present the scheme realizing the correction transformation $R_n \in \mathbf{T}(\mathbf{B}^n)$ for REP(n). To begin with, n copies of each bit are encoded (with the aid of a scheme DUP). Then we take one copy of each bit and compute the MAJ function. We shall show that for $n \geq 5$ the perturbed transformation \widetilde{R}_n is the identity transformation $I_{\mathbf{B}}$ of a single bit in the code REP(n, 1) with accuracy $O(\delta^2)$. Here we deal with classical transformations, so 'accuracy' means 'probability of error'.



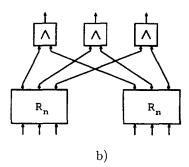


Figure 2. Classical schemes stable under errors: a) scheme R_n realizing the correction transformation for the code REP(n); b) the scheme realizing the operation \wedge in the code REP(n). (Here n=3.)

Suppose that at least n-1 input bits have value a. With probability $1-O(\delta^2)$ no more than one error will occur as the scheme is run. If the error occurs when some input bit is copied, then each block MAJ will receive no more than 2 incorrect inputs. It follows that all outputs will have value a. But if the error occurs while the function MAJ is computed, then no more than one incorrect bit will appear in the output.

To represent the transformation \wedge in the code REP(n,1) $(n \geq 5)$ it suffices to perform the operation \wedge for each pair of code bits, after correcting any possible errors in the input (see Fig. 2b). If a new error occurs (in one of the subschemes R_n or one of the elements \wedge), then no more than one output bit will be corrupted.

The elements \neg can be represented in a similar way and 0 can be represented trivially.

The underlying idea. This example suggests the idea that the accuracy of computations can be increased step-by-step by substituting the code REP(5,1) successively into itself. The accuracy $O(\delta^{2^r})$ is ensured by a code of type $(5^r,1)$. (Such codes are called *cascade codes*.) The dimension of the resulting schemes increases as a^r (a = const), that is, as a power of the logarithm of accuracy. Naturally, it is desirable to extend this discussion to the quantum case. (It is assumed that the first step, namely, the construction of schemes similar to R_n , has already been made.) Let us set about realizing this plan.

Now (at last!) it is time to state in what sense it is possible to model schemes with accurate elements by ones with inaccurate elements.

Definition 10.4. Let \mathcal{A} be a basis whose elements are realized with fixed accuracy δ . The following collection of objects is called a *polynomial error correction scheme*:

- a sequence of codes $(D_k: k=1,2,...)$ of type $(n_k,1)$ such that $n_k = \text{poly}(k)$;
- a homogeneous polynomial family of schemes $(\Psi_{tk}: t \in \mathcal{A}, k = 1, 2, ...)$ in the basis \mathcal{A} ; any transformation $T \in \text{Trans}(\Psi_{tk}, \delta)$ (that is a transformation that can be realized by the scheme Ψ_{tk} with inaccurate elements) must represent a transformation t in the code D_k with accuracy $\exp(-k)$;
- a homogeneous polynomial family of schemes Υ_{kl} in the basis \mathcal{A} ; the scheme Υ_{kl} (with inaccurate elements) must represent the identity transformation of a single bit in the codes D_k, D_l with accuracy $\exp(-\min\{k, l\})$.

In the case of combined computations we must define two sequences of codes (for encoding the classical and quantum bits, respectively). Until we arrive at concrete examples we shall assume that all bits are quantum ones.

It turns out that to construct a polynomial error correction scheme it suffices to realize the basis elements $t \in \mathcal{A}$ by means of schemes Σ_t stable under two errors¹⁴ (see Theorem 10.7 below). Roughly speaking, stability means that the scheme Σ_t will continue to work (that is, represent t in the codes in hand) when any two of its elements are altered in an arbitrary way. However, this naive definition does not make it possible to realize a cascade construction, because it does not allow the elements of Σ_t to be replaced by their representations in some code. The 'correct' definition of stability is much more involved.

We shall introduce some notation. Let Σ be a scheme with the set of elements Θ . The set of outputs Shade(Σ , q) depending on q is defined for each $q \in \Theta$. If $A \subseteq \Theta$, then Shade(Σ , A) = $\bigcup_{q \in A}$ Shade(Σ , q).

Let
$$\Delta = \{1, \ldots, n\}$$
 and $\Gamma \subseteq 2^{\Delta}$. By definition $2\Gamma = \{A \cup B : A, B \in \Gamma\}$.

A collection of schemes $\Sigma_1, \ldots, \Sigma_l$, admissible transformations P_1, \ldots, P_l and codes C_0, C_1, \ldots, C_l is called a *step-by-step realization* of an admissible transformation P in the codes C_0, C_l if

- (a) $T_i = \text{Trans}(\Sigma_i)$ represents P_i in the codes C_{i-1}, C_i ;
- (b) $P_{l} \cdots P_{1} = P$.

The sets of inputs, outputs and elements of Σ_i will be denoted by Δ_{i-1} , Δ_i and Θ_i , respectively.

Definition 10.5. Let $k \ge 0$ be an integer, let $\Gamma \subseteq 2^{\Delta_0}$ and let $\Gamma' \subseteq 2^{\Delta_l}$. A step-by-step realization is called (k, Γ, Γ') -stable if for any $B_0 \in \Gamma$ and $A = \bigcup_i A_i$ $(A_i \subseteq \Theta_i)$ such that $|A| \le k$ there are B_1, \ldots, B_l $(B_i \subseteq \Delta_i)$ such that

- 1) the code C_i corrects errors from $\mathcal{E}(B_i)$ (i = 0, ..., l);
- 2) the transformation T_i represents P_i in the codes $Der(C_{i-1}, \mathcal{E}(B_{i-1}))$, $Der(C_i, \mathcal{E}(B_i))$;
- 3) Shade(Σ_i, A_i) $\subseteq B_i$;
- 4) $B_l \in \Gamma'$.

¹⁴In the classical case stability under one error would be sufficient. This reflects the difference between Proposition 10.2 and Theorem 10.3 (b). This is the very subtle point where a discussion at the 'physical level of rigour' is not enough.

For brevity, the scheme $\Sigma = \Sigma_l \circ \cdots \circ \Sigma_1$ (the composition of the Σ_i) is sometimes called a step-by-step realization. If $P \in \mathbf{T}(\mathbb{B}^{\otimes s}, \mathbb{B}^{\otimes m})$ and bitwise coding is used, then by a (k, Γ, Γ') -stable realization in the code C we shall understand a $(k, \Gamma^s, (\Gamma')^m)$ -stable realization in the codes C^s, C^m .

Example 10.6. The scheme R_n (see Fig. 2a) can be considered as a step-by-step realization of the transformation $I_{\mathbf{B}}$. Here $P_1 = P_2 = I_{\mathbf{B}}$, $C_0 = C_2 = \text{REP}(n)$, and $C_1 = \text{REP}(n^2)$. The first step (Σ_1) consists of the subschemes DUP, and the second one (Σ_2) of the subschemes MAJ. When $n \geq 5$ this realization is $(1, \Gamma, \Gamma)$ -stable, where $\Gamma = \{A \subseteq \{1, \ldots, n\} : |A| \leq 1\}$. More interesting examples are contained in §11.

Theorem 10.7. Let C be a code of type (n,1) admitting encoding and decoding by schemes in the basis A, and let $\Gamma \subseteq 2^{\{1,\ldots,n\}}$. We assume that each transformation $t \in A$ has a $(2,2\Gamma,\Gamma)$ -stable step-by-step realization Φ_t in the code C and in the basis A. Moreover, suppose that $Trans(\Phi_I)$ represents the identity transformation $I = I_B$ in the strong sense. Then there exists a polynomial error correction system in the basis A for any sufficiently small $\delta > 0$.

Remark. If t is a unitary transformation, then the condition of $(2, 2\Gamma, \Gamma)$ -stability can be replaced by $(2, \Gamma, \Gamma)$ -stability. For classical schemes it suffices to require $(1, \Gamma, \Gamma)$ -stability.

In this way the perturbation problem has been reduced to finding a finite object with certain properties.

10.2. Proofs.

Proof of Theorem 10.3. a) Let \widetilde{T} represent P with accuracy $O(\varepsilon)$, that is, let the commutative diagram

$$\mathbf{L}(\mathcal{M}') \xleftarrow{(\chi')^{\bullet}} \mathbf{L}(\mathcal{L}') \xrightarrow{J'} \mathbf{L}(\mathcal{N}')$$

$$\uparrow_{P} \qquad \uparrow_{\tilde{R}} \qquad \uparrow_{\tilde{T}}$$

$$\mathbf{L}(\mathcal{M}) \xleftarrow{\chi^{\bullet}} \mathbf{L}(\mathcal{L}) \xrightarrow{J} \mathbf{L}(\mathcal{N})$$
(62)

be valid with accuracy $O(\varepsilon)$. We put $R = \widetilde{R} + H'(P\chi^* - (\chi')^*\widetilde{R})$ and $T = \widetilde{T} + (J'R - \widetilde{T}J)J^*$, where $H' \in \mathbf{T}(\mathcal{M}', \mathcal{L}')$ is an arbitrary encoding transformation for the code C', that is, $(\chi')^*H' = I_{\mathcal{M}'}$.

Conversely, suppose that the commutative diagram (61) holds and $||T - \tilde{T}||_{\diamondsuit} \leq O(\varepsilon)$. We put

$$\widetilde{R} = S(E^{-1/2} \cdot E^{-1/2}), \text{ where } S = (J')^* \widetilde{T} J, E = S^* 1_{\mathcal{L}'}.$$

It is obvious that \widetilde{R} is an admissible transformation and $||R - \widetilde{R}||_{\diamondsuit} \leq O(\varepsilon)$. The diagram (62) is commutative with accuracy $O(\varepsilon)$.

b) For simplicity we shall assume that all the spaces considered are unitary. In this case $\mathcal{L}' = \mathcal{M}' \otimes \mathcal{G}$ and $(\chi')^* = \operatorname{Tr}_{\mathcal{G}}$. Moreover $J = A \cdot A^{\dagger}$ and $J' = B \cdot B^{\dagger}$, where $A : \mathcal{L} \to \mathcal{N}$ and $B : \mathcal{L}' \to \mathcal{N}'$ are the natural embeddings.

Since the diagram (62) commutes approximately, it follows that

$$\|(\chi')^*(J')^*\widetilde{T}J - P\chi^*\|_{\diamondsuit} \leqslant O(\varepsilon), \qquad \|(BB^{\dagger} \otimes 1_{\mathcal{K}})XA - XA\| \leqslant O(\varepsilon),$$

where $X: \mathcal{N} \to \mathcal{N}' \otimes \mathcal{K}$ is the canonical representation of \widetilde{T} . We shall find a unitary embedding $Y: \mathcal{L} \to \mathcal{N}' \otimes \mathcal{K}$ such that

$$(\chi')^*(J')^*\operatorname{Tr}_{\mathcal{K}}(Y\cdot Y^{\dagger}) = P\chi^*, \qquad \|XA - Y\| \leqslant O(\varepsilon^{1/2}).$$

Let $V,W:\mathcal{L}\to\mathcal{M}'\otimes\mathcal{F}$ be the canonical representations of the transformations $(\chi')^*(J')^*\widetilde{T}J$ and $P\chi^*$. We observe that $(B^\dagger\otimes 1_{\mathcal{K}})XA$ is also a representation of $(\chi')^*(J')^*\widetilde{T}J$. Lemma 3.10 implies that $\|W-V\|\leqslant O(\varepsilon^{1/2})$. By Lemma 3.5 there is a unitary embedding $U:\mathcal{F}\to\mathcal{G}\otimes\mathcal{K}$ such that $(B^\dagger\otimes 1_{\mathcal{K}})XA=(1_{\mathcal{M}'}\otimes U)V$. Now we put $Y=(B\otimes 1_{\mathcal{K}})(1_{\mathcal{M}'}\otimes U)W$.

To complete the proof it remains to extend Y from $\mathcal{L} \subseteq \mathbb{N}$ to the whole space \mathbb{N} . To be more precise, we need to find a unitary embedding $Z: \mathbb{N} \to \mathbb{N}' \otimes \mathcal{K}$ such that Y = ZA and $||X - Z|| \leq O(\varepsilon^{1/2})$. Then we will be able to put $T = \text{Tr}_{\mathcal{K}}(Z \cdot Z^{\dagger})$.

Let $\mathcal{L}_{\perp} \subseteq \mathcal{N}$ be the orthogonal complement of \mathcal{L} and let $A_{\perp} : \mathcal{L}_{\perp} \to \mathcal{N}$ be the corresponding embedding. We put $Z = YA^{\dagger} + Y_{\perp}A_{\perp}^{\dagger}$, where

$$Y_{\perp} = E(E^{\dagger}E)^{-1/2}, \qquad E = (1 - YY^{\dagger})XA_{\perp} \colon \mathcal{L}_{\perp} \to \mathcal{N}' \otimes \mathcal{K}.$$

It is obvious that Z satisfies the required properties.

The version of the theorem involving a 'representation in the strong sense' can be proved in a similar way.

The proof of Theorem 10.7 will be preceded by several lemmas. We introduce the notation $\mathcal{E}(\Gamma) = \sum_{A \in \Gamma} \mathcal{E}(A)$, where $\Gamma \subseteq 2^{\{1,\dots,n\}}$.

Lemma 10.8. Suppose that T represents P in the codes $Der(C, \mathcal{E}(B)), C'$ for each $B \in 2\Gamma$. Then T represents P in the codes $Der(C, \mathcal{E}(\Gamma)), C'$. (2Γ can be replaced by Γ in the classical case and also when P is a unitary transformation.)

Proof. Let $C = (\mathcal{L}, \chi)$, $C' = (\mathcal{L}', \chi')$, $\mathcal{L}_B = \mathcal{E}(B)\mathcal{L}$, and $\mathcal{L}_{\Gamma} = \mathcal{E}(\Gamma)\mathcal{L}$.

In the general case $\mathcal{L}_{\Gamma} = \sum_{B \in \Gamma} \mathcal{L}_{B}$, so $\mathbf{L}(\mathcal{L}_{\Gamma}) \subseteq \sum_{B \in 2\Gamma} \mathbf{L}(\mathcal{L}_{B})$. The rest is obvious.

In the classical case $\mathbf{L}(\mathcal{L}_{\Gamma}) \subseteq \sum_{B \in \Gamma} \mathbf{L}(\mathcal{L}_B)$.

If P is a unitary transformation, then we can assume that $P = I_{\mathcal{M}}$. Let $T = \sum_{s} X_{s} \cdot X_{s}^{\dagger}$ and let \mathcal{E}_{T} be the space generated by the operators X_{s} . For each $B \in \Gamma$ the code $C_{B} = \operatorname{Der}(C, \mathcal{E}(B))$ corrects errors from \mathcal{E}_{T} , and C' is a coarser code than $\operatorname{Der}(C_{B}, \mathcal{E}_{T}) = \operatorname{Der}(C, \mathcal{E}_{T}\mathcal{E}(B))$. It follows that the homomorphism χ' satisfies condition 2) of Theorem 8.5 if we put $\overline{\mathcal{H}} = \mathcal{E}_{T}\mathcal{E}(B)$. Now we can take the sum over $B \in \Gamma$ and convert the previous argument.

Lemma 10.9. Let Σ and $\widetilde{\Sigma}$ be schemes with the same combinatorial structure but with different sets of elements A. We assume that $T = \operatorname{Trans}(\Sigma)$ represents P in the codes C, C', and C' corrects the errors from $\mathcal{E} = \mathcal{E}(\operatorname{Shade}(\Sigma, A))$. Then $\widetilde{T} = \operatorname{Trans}(\widetilde{\Sigma})$ represents P in the codes $C, \operatorname{Der}(C', \mathcal{E})$.

Proof. Let $P \in \mathbf{T}(\mathcal{M}, \mathcal{M}')$, $T, \tilde{T} \in \mathbf{T}(\mathcal{N}, \mathcal{N}')$, $C = (\mathcal{L}, \chi)$, $C' = (\mathcal{L}', \chi')$, let \mathcal{F} be the space of states of qbits from Shade (Σ, A) , and let \mathcal{K} be the space of states of the remaining outputs of the scheme Σ . Then $\mathrm{Der}(C', \mathcal{E}) = (\mathcal{G} \otimes \mathcal{F}, \mu \otimes 1_{\mathcal{F}})$, where (\mathcal{G}, μ) is a code of type $(\mathcal{M}, \mathcal{K})$. It is obvious that $\mathrm{Tr}_{\mathcal{F}} \tilde{T} = \mathrm{Tr}_{\mathcal{F}} T$. If $\rho \in \mathbf{D}(\mathcal{L})$, then $T\rho \in \mathbf{D}(\mathcal{L}') \subseteq \mathbf{D}(\mathcal{G} \otimes \mathcal{F})$, so $\mathrm{Tr}_{\mathcal{F}} \tilde{T} \rho = \mathrm{Tr}_{\mathcal{F}} T \rho \in \mathbf{D}(\mathcal{G})$. It follows that $\tilde{T} \rho \in \mathbf{D}(\mathcal{G} \otimes \mathcal{F})$. (Here we use the fact that \tilde{T} is a positive transformation.) It is obvious that $\mu^* \mathrm{Tr}_{\mathcal{F}} \tilde{T} = P\chi^*$.

The following lemma (like the previous ones) is valid for schemes with generalized bits: the space of states of each bit is an arbitrary combined space. In the case in hand it is important that the resulting estimate is independent of the dimensions of these spaces.

Lemma 10.10. Let $\Sigma = \Sigma_l \circ \cdots \circ \Sigma_1$ be a $(k, 2\Gamma, \Gamma)$ -stable step-by-step realization of a transformation P in codes C, C'. Then any transformation $\widetilde{T} \in \operatorname{Trans}(\Sigma, \delta)$ represents P in the codes $\operatorname{Der}(C, \mathcal{E}(\Gamma)), \operatorname{Der}(C', \mathcal{E}(\Gamma'))$ with accuracy $b\delta^{k+1}$. The number b depends only on the dimension of Σ . $(2\Gamma$ can be replaced by Γ in the classical case and also when P is a unitary transformation.)

Proof. With each element $q \in \Theta$ of Σ we associate an arbitrary admissible transformation \widetilde{T}_q such that $\|\widetilde{T}_q - T_q\|_{\diamondsuit} \leq \delta$. We introduce the notation $T(A) = \operatorname{Trans}(\Sigma(A))$, where $A \subseteq \Theta$ and the scheme $\Sigma(A)$ is obtained from Σ by the replacement $T_q \mapsto \widetilde{T}_q$ for all $q \in A$. Then $\widetilde{T} = T(\Theta)$.

Let $|A| \leq k$. From the definition of a stable realization and Lemma 10.9 it follows that T(A) represents P in the codes $\mathrm{Der}(C,\mathcal{E}(B_0)),\mathrm{Der}(C',\mathcal{E}(\Gamma'))$ for each $B_0 \in 2\Gamma$. Therefore, T(A) represents P in the codes $\mathrm{Der}(C,\mathcal{E}(\Gamma)),\mathrm{Der}(C',\mathcal{E}(\Gamma'))$ (see Lemma 10.8). Now we use the identity

$$\widetilde{T} = T(\Theta) = \sum_{E \subseteq \Theta} W(E), \quad \text{where} \quad W(E) = \sum_{A \subseteq E} (-1)^{|E| - |A|} T(A).$$

It is obvious that $||W(E)||_{\diamondsuit} \le \delta^{|E|}$. We put $T = \sum_{|E| \le k} W(E)$ and apply T. Sorem 10.3 a).

Now we shall introduce an important construction, namely, the substitution of one code into another. Let $C=(\mathcal{L},\chi)$ be a code of type (n,1) and $D=(\mathcal{G},\mu)$ an arbitrary code of type $(\mathcal{N},\mathcal{B})$. We define a code $C \triangleleft D$ of type $(\mathcal{N}^{\otimes n},\mathcal{B})$ as follows: one (quantum) bit is represented by n bits with the aid of the code C and then each of these bits is encoded by a generalized bit (with state space \mathcal{N}) with the aid of the code D. In formal language, $C \triangleleft D = (\mathcal{K}, \nu)$, where $\mathbf{L}(\mathcal{K}) = \mu^{\otimes n} \mathbf{L}(\mathcal{L})$ and $\nu = \mu^{\otimes n} \chi$.

Here is one more symbol: $C \triangleleft_{\Gamma} D = \operatorname{Der}(C \triangleleft D, \mathcal{E}(\Gamma))$, where $\Gamma \subseteq 2^{\{1,\dots,n\}}$. We emphasize that the code $C \triangleleft_{\Gamma} D$ is coarser than $\operatorname{Der}(C, \mathcal{E}(\Gamma)) \triangleleft D$. If $T \in \mathbf{T}(\mathbb{N})$ represents $I_{\mathcal{B}}$ in the code D in the strong sense, then $T^{\otimes n}$ represents $I_{\mathcal{B}}$ in $C \triangleleft_{\Gamma} D$, $\operatorname{Der}(C, \mathcal{E}(\Gamma)) \triangleleft D$ (also in the strong sense).

Proof of Theorem 10.7. Let D_0 be the identity code and let $D_{r+1} = C \triangleleft_r D_r$ (r = 0, 1, ...). (The codes D_r are called *cascade* codes.) In the basis A we shall

construct schemes Ψ_{tr} $(t \in A)$ with the following property: any transformation $\widetilde{T} \in \text{Trans}(\Psi_{tr}, \delta)$ represents t in the code D_r with accuracy $\varepsilon_r = c\delta^{(3/2)^r}$, where c = const. Moreover, we require that any $\widetilde{T} \in \text{Trans}(\Psi_{Ir}, \delta)$ should represent I in D in the strong sense (with the same accuracy).

We choose c to be large enough. The scheme Ψ_{t0} will consist of one element t. Induction on r: suppose that all schemes Ψ_{ur} such that $u \in \mathcal{A}$ have already been constructed. The scheme $\Psi_{t,r+1}$ can be constructed on the basis of the step-by-step realization $\Phi_t = \Sigma = \Sigma_l \circ \cdots \circ \Sigma_1$ of t. Namely, $\Psi_{t,r+1} = \Psi$ (if $t \neq I$) or $\Psi_{t,r+1} = \Psi \circ \Psi \circ \Psi$ (if t = I), where

$$\Psi = \Sigma' \triangleleft \Psi_{*r}, \qquad \Sigma' = \Sigma_l \circ I^{s_{l-1}} \circ \cdots \circ \Sigma_1 \circ I^{s_0}.$$

Here $s_{i-1} = |\Delta_{i-1}|$ is the number of inputs of Σ_i . The symbol $\forall \Psi_{*r}$ denotes the operation of replacing all the elements of a scheme (in this case Σ') according to the rule $u \mapsto \Psi_{ur}$. We shall demonstrate that any transformation $\widetilde{T} \in \text{Trans}(\Psi, \delta)$ represents T in the code D_{r+1} with accuracy ε_{r+1} .

The transformation T can be described as follows. We replace each element $q \in \Theta'$ of Σ' (more precisely, the corresponding transformation $u_q \in A$) by an arbitrary transformation $T_q \in \operatorname{Trans}(\Psi_{u_q,r},\delta)$. If the resulting scheme is denoted by $\widetilde{\Omega}$, then $\widetilde{T} = \operatorname{Trans}(\widetilde{\Omega})$. (The scheme $\widetilde{\Omega}$ works with generalized bits, each consisting of n^r ordinary bits.) For each $q \in \Theta'$ there is an admissible transformation T_q representing u_q in the code D_r such that $||T_q - \widetilde{T}_q||_{\varphi} \leqslant O(\varepsilon_r^{1/2})$. (Applying Theorem 10.3 b), we have taken into account that $\dim \mathcal{M}' = 2^m \leqslant \operatorname{const}$, where m is the number of outputs of q.) If $u_q = I$, then T_q represents I in the strong sense. Replacing \widetilde{T}_q by T_q , we obtain a new scheme Ω . It is a $(2, 2\Gamma, \Gamma)$ -stable step-by-step realization of t in the code $C \triangleleft D_r$. Now it remains to use Lemma 10.10.

If t=I, then $\operatorname{Trans}(\Omega)$ represents I in the code D_{r+1} in the strong sense. It follows that $\widetilde{T} \in \operatorname{Trans}(\widetilde{\Omega})$ represents I in D_{r+1} in the strong sense with accuracy $O(\varepsilon_r^{1/2})$. The accuracy can be increased to $\varepsilon_{r+1} = O(\varepsilon_r^{3/2})$ by replacing Ψ by $\Psi \circ \Psi \circ \Psi$.

Thus, the first two conditions in Definition 10.4 are satisfied (to within replacing k by $r \sim \log k$). As regards the last condition, we put $\Upsilon_{r,r+1} = \Sigma_{\rightarrow} \triangleleft \Psi_{*r}$ and $\Upsilon_{r+1,r} = \Sigma_{\leftarrow} \triangleleft \Psi_{*r}$, where Σ_{\rightarrow} and Σ_{\leftarrow} are the encoding and decoding schemes for the code C. An arbitrary scheme Υ_{kl} can be constructed as a composition of several schemes $\Upsilon_{r,r+1}$ or $\Upsilon_{r+1,r}$.

§11. Error correction: concrete procedures

We are now in a position to prove the main result.

Theorem 11.1. For any sufficiently small $\delta > 0$ there exist polynomial error correction systems in the symplecto-classical basis SC and the complete basis QC.

A symplecto-classical version of this theorem was proved independently by the author, whereas an idea from Shor's paper [43] is used in the realization of the Toffoli element.

The proof is fully constructive; however, the resulting degree of the polynomial in the polynomial system is rather high. We shall not even undertake to write

it down. Theorem 11.1 demonstrates that it is in principle possible to construct reliable quantum schemes from unreliable (inaccurate) elements. Error correction procedures suitable for practical purposes will need further research.

11.1. The symplecto-classical case. In the previous section it was shown how to correct errors in a classical computation. Now we shall assume that all the classical elements in the basis SC are accurate. (Otherwise it would be necessary to correct classical errors in each procedure described below.) The proof of Theorem 11.1 is based on Theorem 10.7. So as not to be overwhelmed by details, we shall appeal to the intuitive meaning of the stability of a scheme, rather than the formal Definition 10.5. The intermediate codes C_1, \ldots, C_{l-1} and the partition of the scheme Σ into steps $\Sigma_1, \ldots, \Sigma_l$ will be described only in one case, as required in the definition of a step-by-step realization.

Let $1 \ll s' \ll s \ll k$, that is, let s', s/s' and k/s be large enough. (By analysing the whole proof we can assign concrete values to s, s' and k.) We choose TOR(k) as the basic code C (see the formulation of Theorem 10.7). Just like any symplectic code, it admits encoding and decoding by schemes of polynomial dimension in the basis S. Out of the two information qbits of the code TOR(k) we shall use only one, for example, that corresponding to the information operators Y_{x1} and Y_{z1} .

We put $\Gamma = \{B \subseteq \Delta : |B| \leqslant s\}$ and $\Gamma' = \{B \subseteq \Delta : |B| \leqslant s'\}$, where $\Delta = \{1, \ldots, 2k^2\}$ is the set of code qbits. The identity transformation $I_{\mathcal{B}}$ will be realized by a $(2, 2\Gamma, \Gamma')$ -stable scheme Σ . All the remaining elements of the basis will be realized in a $(2, \Gamma', \Gamma)$ -stable way. To make them $(2, 2\Gamma, \Gamma)$ -stable we need first to reduce the number of errors in the input with the aid of the scheme Σ (see Fig. 2b).

 $(2, 2\Gamma, \Gamma')$ -stable step-by-step realization of $I_{\mathcal{B}}$. (See Fig. 3.) We consider the scheme $\Sigma = \Sigma_{11} \circ \cdots \circ \Sigma_1$. The first 9 steps $(\Sigma_1, \ldots, \Sigma_9)$ are the MEAS schemes (see §9.3) measuring the syndromes. Each of the measured syndromes h_i $(i=1,\ldots,9)$ is written in a separate register. The classical scheme Σ_{10} finds an i such that $h_i = h_{i+1} = h_{i+2}$ (the existence of such an i will be proved below) and computes the corresponding error $g = \nu(h_i)$. The scheme $\Sigma_{11} = \text{CORR}$ corrects this error.

 $(2, 2\Gamma, \Gamma')$ -stability follows from the following argument. Suppose that two errors occur as the scheme Σ is run. We can always find an i such that the subschemes $\Sigma_i, \Sigma_{i+1}, \Sigma_{i+2}$ work correctly. It follows that $h_i = h_{i+1} = h_{i+2}$. Conversely, let $h_i = h_{i+1} = h_{i+2}$ (such an i may not be unique). It is obvious that at least one of the subsystems $\Sigma_i, \Sigma_{i+1}, \Sigma_{i+2}$ must work without an error (since there are only two errors). It follows that h_i is a correct syndrome corresponding to the jth measurement ($i \leq j \leq i+2$). The subsequent measurements may not only give incorrect results, but also corrupt some of the code qbits. Nevertheless (this is a key point in the whole construction!), the depth of the scheme $\Sigma_9 \circ \cdots \circ \Sigma_1$ is bounded because TOR(k) are codes with local verification. Consequently, each of the two possible errors can corrupt only O(1) qbits. It follows that 2s old errors (which existed prior to applying the procedure Σ) in the code qbits will be corrected and only $O(1) \leq s'$ new ones will appear. (Remark: the scheme $\Sigma_{11} = CORR$ has depth O(1) too, and the classical scheme Σ_{10} is assumed to be absolutely reliable.)

The above argument can be made rigorous by specifying the intermediate codes C_1, \ldots, C_{10} from the definition of a step-by-step realization. Let $n = 2k^2$ be the

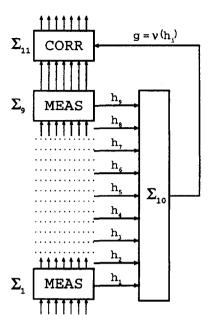


Figure 3. $(2, 2\Gamma, \Gamma')$ -stable step-by-step realization of the transformation $I_{\mathfrak{B}}$ in the code TOR(k)

number of code qbits of the code C. We define the combined space $\mathcal{N} = \mathcal{B}^{\otimes n} \times G^n$ and the space of errors

$$\mathcal{E} = \left(\sigma(g) \otimes |g\rangle : g \in E(n,2s)\right) \subseteq \mathbf{L}(\mathcal{B}^{\otimes n}) \otimes \overline{G^n} = \mathbf{L}(\mathcal{B}^{\otimes n}, \overline{\mathbb{N}}).$$

(The set E(n,2s) is defined by (52).) We put $C_{10}=D=\mathrm{Der}(C,\mathcal{E})$. In other words, the code D can be obtained from C by introducing an arbitrary error $\sigma(g)\in\mathbf{L}(\mathcal{B}^{\otimes n})$ with support $\mathrm{Supp}(g)\in 2\Gamma$, where g is entered simultaneously into an additional classical register. The code C_i $(i=1,\ldots,9)$ can be obtained from D by replacing the contents g of the classical register by i copies of the syndrome $\mu(g)$. Using these codes, one can give a precise meaning to the words 'the correct syndrome corresponding the the jth measurement' (see the previous paragraph). The construction of the sets B_1,\ldots,B_{11} from Definition 10.5 is left to the reader as an exercise. We also observe that $\mathrm{Trans}(\Sigma)$ represents $I_{\mathcal{B}}$ in C in the strong sense.

Conventions. In what follows by stability we shall understand $(2, \Gamma', \Gamma)$ -stability. By a realization of a unitary operator U we shall understand a realization of the transformation $U \cdot U^{\dagger}$.

Stable realization of the operator $\widehat{\oplus}$. Let A', A'' be two sets of code qbits, each set representing its own information qbit. We apply the operator $\widehat{\oplus}$ to each pair of code qbits, one of which belongs to A' and the other one to A''. Then the verification operators are transformed by one another, and the information

operators $Y'_{\alpha 1} = Y_{\alpha 1}[A']$ and $Y''_{\alpha 1} = Y_{\alpha 1}[A'']$ are transformed according to

$$Y'_{x1} \mapsto Y'_{x1} Y''_{x1}, \quad Y''_{x1} \mapsto Y''_{x1}, \quad Y'_{z1} \mapsto Y'_{z1}, \quad Y''_{z1} \mapsto Y'_{z1} Y''_{z1},$$

as required. (This is a general property of quasilinear codes.) The stability of this scheme is obvious.

Stable realization of the operator S. We apply S to each code qbit. Then we permute the qbits by reflecting the grid (see Fig. 1) relative to the diagonal with a shift by $\frac{1}{2}$ a period. (Then the vertices will turn into faces, faces into vertices, and edges into edges.) The information operators Y_{x1} and Y_{z1} will be interchanged, as required.

Stable realization of the transformation $Q_{\mathbf{B}}$. Let A_1 be the input register and A_2,\ldots,A_5 the auxiliary registers. Into these registers we enter the state $|0\rangle$ represented in the code C. We copy the basic register into the auxiliary ones, applying the operators $\widehat{\bigoplus}[A_1,A_i]$ (i=2,3,4,5). (In fact, in place of $\widehat{\bigoplus}$ itself we should use the stable realization of this operator in the code C as described above.) To each register A_1,\ldots,A_5 we apply the decoding transformation and then the transformation $Q_{\mathbf{B}}$, obtaining five classical bits a_1,\ldots,a_5 . We take $\mathrm{MAJ}(a_1,\ldots,a_5)$ to be the result. It is easily seen that the whole procedure is stable under two errors.

Stable realization of the state $|0\rangle$. We shall represent $|0\rangle$ in the code C. We make a copy (with the aid of the stable realization of $\widehat{\bigoplus}$). To this copy we apply a stable realization of $Q_{\mathbf{B}}$. If 0 is obtained, then everything is all right, and if 1 is the result, then everything must be repeated from the beginning. If 1 is obtained again, then we must repeat it the third time over. (This is always enough, since we assume that no more than two errors can occur.)

Stable realization of the operator K. To begin with, we shall solve an auxiliary problem: we shall realize (in a stable way) the states $|n_{\pm}\rangle = 2^{-1/2}(|0\rangle \pm i|1\rangle)$. These are the eigenstates of the operator $\sigma_y = -i\sigma_z\sigma_x$. The problem will be solved if we can measure the eigenvalues of $\sigma_z\sigma_x$, which are equal to $\pm i$. In other words, we need to find a stable realization of the transformation

$$P: \mathbf{L}(\mathcal{B}) \to \mathbf{L}(\mathcal{B}) \times \mathbf{B} : \rho \mapsto (\Pi_{+}\rho\Pi_{+}, 0) + (\Pi_{-}\rho\Pi_{-}, 1),$$

where Π_{\pm} are the projections onto the subspaces $(|\eta_{\pm}\rangle)$.

Using the procedures described above, we can construct a stable realization of $W = \Lambda(\sigma_z \sigma_x)$,

$$W[1,2] = S[2] \widehat{\bigoplus} [1,2] S[2] \widehat{\bigoplus} [1,2].$$

We represent P as (see (31))

$$P[1, 2; 1] = R[2] (W \cdot W^{\dagger})[2, 1] (S \cdot S)[2] (|0\rangle\langle 0|)[2],$$

where $R = Q_{\mathbf{B}}(S \cdot S)(K^{\dagger} \cdot K)$. At this moment we have no stable realization of R at our disposal. However, if R is realized in an unstable way, the error involved in it can only corrupt the classical bit 2, but not the quantum bit 1 (represented in the code C). Thus we must repeat the measurement five times and take the MAJ function of the five results.

Now we observe that $W|\xi,\eta_{+}\rangle = K|\xi\rangle \otimes |\eta_{+}\rangle$ (for any $|\xi\rangle \in \mathcal{B}$). It follows that if the state $|\eta_{+}\rangle$ is prepared in the second qbit, then W acts on the first qbit in the same way as K.

Elements with classical control. We have already constructed stable realizations Φ_U for all basis symplectic operators $U \in S$. To realize the transformation $\Lambda_c(U)$ we need to use a control bit in all the elements of Φ_U . In doing so we shall go beyond the framework of the basis, because some of the elements will have two control bits (one of which has already been present in the scheme Φ_U). This situation can be rectified using the identity

$$\Lambda_c(\Lambda_c(U))[1,2,A] = \Lambda_c(U)[3,A] \wedge [3;1,2].$$

11.2. The case of a complete basis. Essentially, it remains to construct a stable realization of the Toffoli element $\widehat{\wedge}_{\oplus}$. As in the case of the operator K, the problem can be solved in two stages. First we construct the state

$$|\psi\rangle = \frac{1}{2} (|000\rangle + |010\rangle + |100\rangle + |111\rangle). \tag{63}$$

The construction described is due to Shor [43].

Stable realization of the state $|\psi\rangle$ in the code C. The states $|\psi_+\rangle = |\psi\rangle$ and $|\psi_-\rangle = \sigma_x[3]|\psi\rangle$ are eigenstates of the operator $U:|a,b,c\rangle \mapsto (-1)^{ab+c}|a,b,c\rangle$. To solve the problem it suffices to measure the eigenvalue of this operator in a stable way. Indeed, let $P \in \mathbf{T}(\mathcal{B}^{\otimes 3}, \mathcal{B}^{\otimes 3} \times \mathbf{B})$ be the measurement transformation. Then

$$P \big(|\xi, \xi, \xi\rangle \langle \xi, \xi, \xi| \big) = \frac{1}{2} \big(|\psi_+\rangle \langle \psi_+|, 0 \big) + \frac{1}{2} \big(|\psi_-\rangle \langle \psi_-|, 1 \big),$$

where $|\xi\rangle = S|0\rangle$. It follows that if we apply P to the state $|\xi, \xi, \xi\rangle$, then with probability $\frac{1}{2}$ we obtain $|\psi_{+}\rangle$ (and we shall certainly know about it).

It is easily seen that $U \cdot U^{\dagger}$ is a symplectic transformation. We denote by Φ its stable realization in the code C. We replace each of the elements T_q of Φ by $\Lambda(T_q)$ using a separate control qbit. The resulting scheme will be denoted by Φ' . The transformation $T' = \operatorname{Trans}(\Phi')$ represents $\Lambda(U) \cdot \Lambda(U)^{\dagger}$ in the code $D \times C^3$, where D is a code of the form $|a\rangle = |a, \ldots, a\rangle$. (As many copies of one control qbit of $\Lambda(U)$ are made as there are elements in the scheme Φ .) However, we observe that D does not protect against errors of the form $\sigma_z[j]$.

An eigenvalue of U (represented in the code C) can be measured with the aid of the transformation

$$Q_{\mathbf{B}}[0] \quad (S \cdot S)[0] \quad R_{\leftarrow}[0; A_0] \quad T'[A_0, A_1, A_2, A_3] \quad (|\xi\rangle\langle\xi|)[A_0].$$
 (64)

Here R_{\leftarrow} is the decoding transformation for D, and $(|\xi\rangle\langle\xi|)[A_0]$ is the state $|\xi\rangle = S|0\rangle$ represented in the code D and written in the register A_0 . It can be prepared, for example, as follows: take the state $|\xi\rangle$ in C and reproduce it with the aid of the operator $\widehat{\bigoplus}$ (obtaining $|\xi\rangle$ in the code $D \triangleleft C$), and then decode each resulting 'copy'. The scheme (64) is almost stable: two errors can corrupt only the classical bit 0. Therefore, the measurement is to be repeated five times, followed by taking the MAJ function.

In what follows, instead of $|\psi\rangle$ we shall use the state

$$|\zeta\rangle = S[3]|\psi\rangle = 2^{-3/2} \sum_{a,b,c} (-1)^{abc} |a,b,c\rangle.$$
 (65)

We also observe that $\widehat{\wedge}_{\oplus} = S[3]WS[3]$, where $W: |a, b, c\rangle \mapsto (-1)^{abc}|a, b, c\rangle$.

Stable realization of the transformation $W \cdot W^{\dagger}$. We consider the unitary embedding $V : \mathcal{B}^{\otimes 3} \to \mathcal{B}^{\otimes 6}$

$$V[1,2,3,4,5,6;1,2,3] = \widehat{\bigoplus}[1,4] \widehat{\bigoplus}[2,5] \widehat{\bigoplus}[3,6] (|\zeta\rangle[4,5,6]),$$

apply the transformation $V \cdot V^{\dagger}$ to an arbitrary state $\rho \in \mathbf{D}(\mathcal{B}^{\otimes 3})$, and measure the values of the last three bits (with numbers 4, 5, 6). This procedure can be described by a transformation $T \in \mathbf{T}(\mathcal{B}^{\otimes 3}, \mathcal{B}^{\otimes 3} \times \mathbf{B}^{3})$ such that

$$T \colon \rho \mapsto \frac{1}{8} \sum_{\alpha,\beta,\gamma} (W_{\alpha\beta\gamma} \rho W_{\alpha\beta\gamma}^{\dagger}, \alpha, \beta, \gamma),$$

where

$$W_{\alpha\beta\gamma}: |a,b,c\rangle \mapsto (-1)^{(a+\alpha)(b+\beta)(c+\gamma)}|a,b,c\rangle.$$

Now, to obtain the required state $W\rho W^{\dagger}$ it remains to apply one of the operators $Y_{\alpha\beta\gamma} = WW_{\alpha\beta\gamma}^{-1}$ (depending on the result of measuring (α,β,γ)). It is easy to verify that the transformations $Y_{\alpha\beta\gamma} \cdot Y_{\alpha\beta\gamma}^{\dagger}$ are symplectic, so they can be realized in a stable way.

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