The QR Factorisation - Least Squares

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Outline

1. The QR Decomposition: Householder; Givens; Gram-Schmidt
2. Least squares problems with applications (image processing)
3. Some linear problems with a Kronecker structure (Sylvester, Lyapunov).

The QR decomposition

Let A be a real $m \times n$ matrix and assume that $m \geq n$. Then a QR factorization of the matrix A, consists in computing an orthogonal matrix $Q \in \mathbb{R}^{m \times m}$ and an upper $m \times n$ triangular matrix R such that

$$A = QR.$$

The well known transformations that can compute such a decomposition are the Householder transformation, the Givens and fast Givens transformations and the Gram-Schmidt process (the classical one and the modified version).

The Householder transformation

Let $v \in \mathbb{R}^m$ be a nonzero vector. The associated Householder matrix is defined by

$$H_v = I - 2\frac{vv^T}{v^T v}.$$

It can be easily shown that H_v is symmetric and orthogonal. Remark also that

$$H_v\, v = -v, \text{ and } H_v x = x \text{ if } x \in span\{v\}^{\perp}.$$

Let x be any nonzero vector in $\mathbb{R}^m$, then we would like to find a vector v such that

$$H_v x = \alpha e_1,$$

where e_1 is the first unit vector of $\mathbb{R}^m$. The vector v can be chosen as follows

$$v = x + sign(x_1)\|x\|_2 e_1.$$

This simple determination of v makes the Householder reflexions very useful. Notice that applying the Householder transformation H_v on a matrix A leads to the following expression

$$H_v A = (I - 2\frac{vv^T}{v^T v}) = A + vw^T$$

where $w = -\frac{2}{v^T v} A^T v$.

As an example, we take $x = (3, 1, 5, 1)^T$ and compute $v = (9, 1, 5, 1)^T$, then

$$H_v.x = (-6, 0, 0, 0)^T.$$

Let us see now how to use Householder transformations to get a QR decomposition. Let us do that on a matrix A of dimension 5×3 whose columns are denoted by $a_1^{(i)}$: $A = A^{(1)} = [a_1^{(1)}, a_2^{(1)}, a_3^{(1)}]$

Step 1: we look for a vector $v^{(1)} = a_1^{(1)} + sign(a_1)\|a_1^{(1)}\|_2 e_1 \in \mathbb{R}^5$ and the corresponding Householder matrix H_1 such that

$$H_1 A = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & \star & * \\ 0 & \star & * \\ 0 & \star & * \end{pmatrix}.$$

For the second step, we look for the Householder matrix $H_2 = diag(I_1, \widetilde{H}_2)$ such that

$$\widetilde{H}_2 \begin{pmatrix} \star \\ \star \\ \star \\ \star \end{pmatrix} = \begin{pmatrix} \times \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

and then

$$H_2 H_1 A = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & * \end{pmatrix},$$

The 4×4 Householder matrix $\widetilde{H}_2$ is defined by

$$\widetilde{H}_2 = I_4 - 2\frac{v^{(2)}(v^{(2)})^T}{(v^{(2)})^T v^{(2)}},$$

where

$$v^{(2)} = a_2^{(2)} + sign((a_2^{(2)}))_1 \|a_2^{(2)}\|_2 e_1^{(2)} \in \mathbb{R}^4$$

and $a_2^{(2)}$ is the first column of the matrix $A^{(2)}$ obtained by deleting the first row and the first column of the matrix $H_1 A$. In the same way, we define the Householder matrix $H_3 = diag(I_2, \widetilde{H}_3)$ such that

$$H_3 H_2 H_1 A = R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{13} \\ 0 & 0 & r_{33} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Setting $Q = H_1 H_2 H_3$, the matrix Q is orthogonal and we have

$$A = QR.$$

We notice that the upper triangular part of A could be overwritten by the upper triangular matrix R, while the Householder vectors $v^{(j)}$ could be stored in the lower triangular part of the matrix A as follows

$$A = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ v_2^{(1)} & r_{22} & r_{13} \\ v_3^{(1)} & v_3^{(2)} & r_{33} \\ v_4^{(1)} & v_4^{(2)} & v_4^{(3)} \\ v_5^{(1)} & v_5^{(2)} & v_5^{(3)} \end{pmatrix}.$$

The Householder method requires $2m^2(n - m/3)$ arithmetic operations if the matrix Q is not required explicitly. Other transformations such as givens or fast Givens could also be used to compute the QR factorization of a matrix A, see [Golub]. The Householder transformation shows that the QR factorization exists.

Givens rotations

Givens rotations, allows us to zero many element of a vector selectively. Givens matrices are rank-two corrections of the identity matrix and are defined as follows

$$G(i,k,\theta) = \begin{pmatrix} 1 & \dots & 0 \dots & 0 \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & c \dots & s \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & -s \dots & c \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 \dots & 0 \dots & 1 \end{pmatrix},$$

where $c = \cos(\theta)$, $s = \sin(\theta)$ for some θ.

The coefficient c is located at the (i,i) and the (k,k) positions while s is the $(i,,k)$-element of the matrix located in the $G(i,k,\theta)$. The $n \times n$ matrix $G(i,k,\theta)$ is orthonormal.

If x is a vector in $\mathbb{R}^n$ and $y = G(i,k,\theta)^T x$, then we get

$$\begin{cases} y_i = cx_i - sx_k \\ y_k = sx_i + cx_k \\ y_j = x_j, \text{ for } j \neq i,k \end{cases}$$

If we want to force y_k, $k \neq i, j$, to zero, then we can set

$$c = \frac{x_i}{\sqrt{x_i^2 + x_k^2}}, \text{ and } s = -\frac{x_k}{\sqrt{x_i^2 + x_k^2}}.$$

Hence Givens rotations allows us to zero any component of any vector. Therefore, applying Givens rotation to the matrix A gives the QR decomposition where Q is the orthogonal matrix obtained as a product of Givens matrices.

Givens rotations for QR factorisations are important for particular structure such as Hessenberg matrices that appears in many problems such as in GMRES which is the most popular methods for solving large and sparse linear systems of equations.

Consider the QR decomposition of $n \times m$ matrix $A = QR$ (obtained with Householder or Givens transformations) and assume that $n \geq m$ and that A is of full rank. Let

$$Q = [Q_1, Q_2], \text{ and } R = \begin{bmatrix} R_1 \\ 0 \end{bmatrix},$$

where $Q_1 \in \mathbb{R}^{n \times m}$, $Q_2 \in \mathbb{R}^{n \times n-m}$ have orthonormal columns and $R_1 \in \mathbb{R}^{m \times m}$ the square upper triangular matrix part of R with positive entries on its diagonal. Then

$$A = [Q_1, Q_2] \begin{bmatrix} R_1 \\ 0 \end{bmatrix} = Q_1 R_1.$$

The last decomposition is unique and is called the skinny QR factorization.

The Gram-Schmidt process

Let the $\{u_1, \ldots, u_k\}$, k vectors of $\mathbb{R}^n$ assumed to be linearly independent. Then the Gram-Schmidt allows us to construct an orthonormal basis $\{q_1, \ldots, q_k\}$ of the space $F = span\{u_1, u_2, \ldots, u_k\}$. In the following, we give the modified version of the process which is more stable numerically.

- $r_{11} = \|u_1\|_2$, $q_1 = \dfrac{u_1}{r_{11}}$
- For $j = 2, \ldots, k$
  1. $\tilde{q} = u_j$
  2. for $i = 1, \ldots, j-1$
     1. $r_{ij} = \langle \tilde{q}, q_i \rangle$
     2. $\tilde{q} = \tilde{q} - r_{ij} q_i$
  3. endfor
  4. Compute $r_{jj} = \|\tilde{q}\|_2$.
  5. If $r_{jj} = 0$, stop, else
  6. $q_j = \tilde{q}/r_{jj}$.
- EndFor.

Setting $U = [u_1, \ldots, u_k]$, $Q_1 = [q_1, \ldots, q_k]$ and $R_1 = [r_{ij}]$ the $k \times k$ triangular matrix obtained from the modified Gram-Schmidt process, we get

$$u_j = \sum_{i=1}^{j} r_{ij} q_i, \; j = 1, \ldots, k,$$

and in a matrix form, we have

$$U = Q_1 R_1, \;\; \text{with} \;\; Q_1^T Q_1 = I,$$

which is called the Gram-Schmidt QR decomposition of the matrix U also called the skinny QR factorization of U. The $n \times k$ matrix Q_1 has orthonormal columns and R_1 has positive diagonal entries.

Application to least-squares problems

We consider here the following Least Squares (LS) problem. Find a vector x such that:

$$\min_{x \in \mathbb{R}^m} \|Ax - b\|_2, \tag{1}$$

where $A \in \mathbb{R}^{n \times m}$, $x \in \mathbb{R}^m$ and $b \in \mathbb{R}^n$ and assume that $n \geq m$.

If we set $\phi(x) = \|Ax - b\|_2^2$, then ϕ is a differentiable function and the minimizers satisfy $\nabla \phi(x) = 0$ where $\nabla \phi$ denotes the gradient of ϕ.

Assume that A has a full rank m. Then there exists a unique solution x_{LS} of the LS problem (1) which is the unique solution of the symmetric positive definite linear system

$$A^T A x_{LS} = A^T b.$$

Let us see now how to solve the LS problem (1) by using the QR-decomposition.

Assume that $A = QR$ where Q is an $n \times n$ orthonormal and R is an $n \times m$ upper triangular matrix. Then

$$\|Ax - b\|_2 = \|Q(Rx - \tilde{b})\|_2$$

where $\tilde{b} = Q^T b$. And since Q is orthogonal, we get

$$\|Ax - b\|_2 = \|Rx - \tilde{b}\|_2.$$

Setting $R = \begin{pmatrix} R_1 \\ 0 \end{pmatrix}$, where R_1 is the square upper triangular part of R and $\tilde{b} = \begin{pmatrix} \tilde{b}_1 \\ \tilde{b}_2 \end{pmatrix}$, Let us see now how to solve the LS problem (1) by using the QR-decomposition.

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we get

$$\|Ax - b\|_2^2 = \|R_1 x - \tilde{b}_1\|_2^2 + \|\tilde{b}_2\|_2^2$$

and then the minimum solves $R_1 x_{LS} = \tilde{b}_1$ and the corresponding residual $r_{LS} = b - Ax_{LS}$ has the following norm

$$\rho_{LS} = \|r_{LS}\|_2 = \|\tilde{b}_2\|_2.$$

Another way of computing the LS solution is the use of the SVD decomposition of the matrix A. For that, we have the following result.

Theorem

Assume that the matrix A has the SVD decomposition $A = U\Sigma V^T$. Then, the solution x_{LS} of the problem (1) is given as follows

$$x_{LS} = \sum_{i=1}^{r} \frac{u_i^T b}{\sigma_i} v_i$$

and

$$\rho_{LS}^2 = \|b - Ax_{LS}\|_2^2 = \sum_{i=r+1}^{n} (u_i^T b)^2,$$

we get

$$\|Ax - b\|_2^2 = \|R_1 x - \tilde{b}_1\|_2^2 + \|\tilde{b}_2\|_2^2$$

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proof

We have $\|Ax - b\|_2^2 = \|U(\Sigma V^T x - U^T b\|_2^2$ and since U is orthonormal, we get $\|Ax - b\|_2^2 = \|\Sigma V^T x - U^T b\|_2^2$ which can be expanded as

$$\|\Sigma V^T x - U^T b\|_2^2 = \sum_{i=1}^{r} (\sigma_i z_i - u_i^T b)^2 + \sum_{i=r+1}^{n} (u_i^T b)^2.$$

Therefore, the minimum is obtained for $z_i = (u_i^t b / \sigma_i)$.
Notice that the solution x_{LS} can also be written in term of the pseudo-inverse as

$$x_{LS} = A^+ b = V \Sigma^+ U^T b = \sum_{i=1}^{r} \frac{u_i^T b}{\sigma_i} v_i$$

Tikhonov Regularisation for Ill-posed Problems

In many situations, the $N \times N$ matrix of the least squares problem

$$\min_x \|Ax - b\|_2,$$

is ill conditioned which means that many singular values are close to zero; this occurs in image processing. As we have seen in the previous chapter, small perturbations in the data may cause large perturbations in the least squares solution. One possibility to remedy to this situation is to use the Tikhonov regularisation procedure. We replace the original problem by the following one

$$\min_x \|Ax - b\|_2^2 + \lambda^2 \|Lx\|_2, \tag{2}$$

for some chosen regularisation parameter λ and a regularized matrix L (we will take here $L = I$).

An example of ill-conditioned Matrices

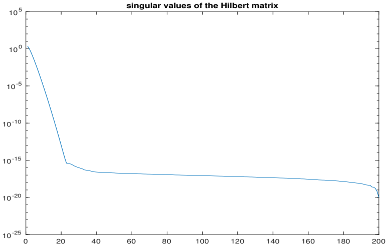

FIGURE – The singular values of the Hilbert matrix H

The minimizer of (2) is computed as the solution of the following linear system

$$A_\lambda x = A^T b, \text{ with } A_\lambda = (A^T A + \lambda^2 I). \tag{3}$$

The proof is simple since the minimizer of (2) satisfies the normal equation

$$\begin{bmatrix} A^T & \lambda I \end{bmatrix} \begin{bmatrix} A \\ \lambda I \end{bmatrix} x_\lambda = \begin{bmatrix} A^T & \lambda I \end{bmatrix} \begin{bmatrix} b \\ 0 \end{bmatrix}. \tag{4}$$

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The parameter λ should be computed such that $\|Ax - b\|_2$ is small and $\|x\|_2$ is not very large. One of the well known method for computing an "optimal" parameter is called the Generalized Cross Validation (GCV) method due to Golub & all.

$$G(\lambda) = \frac{||Ax_\lambda - \mathrm{b}||_2^2}{[tr(I - AA_\lambda^{-1}A^T)]^2} = \frac{||(I - AA_\lambda^{-1}A^T)\mathrm{b}||_2^2}{[tr(I - AA_\lambda^{-1}A^T)]^2}. \tag{5}$$

Let $A = U\Sigma V$ be the SVD of the matrix A then the GCV function G reduces to

$$G(\lambda) = \frac{\displaystyle\sum_{i=1}^{N} \left(\frac{u_i^T b}{\sigma_i^2 + \lambda^2} \right)^2}{\displaystyle\sum_{i=1}^{N} \left(\frac{1}{\sigma_i^2 + \lambda^2} \right)^2}, \tag{6}$$

where u_i is the i-th left singular vector and σ_i is the i-th singular value of the matrix A. It was shown by Golub & all that the best parameter for the problem (2) is obtained as the minimum of the minimum of the CGV function G.

After computing the optimal parameter λ, the problem (2), is equivalent to the following one

$$\min_{x} \left\| \begin{bmatrix} b \\ 0 \end{bmatrix} - \begin{bmatrix} A \\ \lambda I \end{bmatrix} x_{\lambda} \right\|_2^2 . \tag{7}$$

Notice that in image restoration, usually the blurring matrix A is ill conditioned and the right hand side is not known exactly but perturbed with an additive noise:

$$b = b_{exact} + e,$$

where e is some additive noise; usually it is a Gaussian noise with a zero mean. In that case, the computed solution tries to minimize the effect of the noise on the computed LS solution.

Theorem

Let $A = U\Sigma V^T$ be the singular value decomposition of the matrix A. Then the solution of the problem (4) can be expressed as follows

$$x_\lambda = x_{filt} = \sum_{i=1}^{N} \Phi_i \frac{u_i^T b}{\sigma_i} v_i \tag{8}$$

where Φ_i acts as filter and is given by

$$\Phi_i = \frac{\sigma_i^2}{\sigma_i^2 + \lambda^2}. \tag{9}$$

proof

Let $A = U\Sigma V^T$ and then $A^T A = V\Sigma^T \Sigma V^T$. Hence the normal equation (3) can be written as

$$V(D + \lambda^2 I)V^T x = V\Sigma U^T b, \tag{10}$$

and then

$$x_\lambda = V(D + \lambda^2 I)^{-1}\Sigma U^T b, \tag{11}$$

where $D = \sigma^T \Sigma$. Therefore $D + \lambda^2 I = diag(\sigma_i + \lambda^2)$ and we get

$$x_\lambda = x_{filt} = \sum_{i=1}^{N} \Phi_i \frac{u_i^T b}{\sigma_i} v_i.$$

The solution x_{filt} can also be expressed as

$$x_{filt} = V\Phi\Sigma^{-1}U^T b = V\Sigma_{filt}^{-1}U^T b, \tag{12}$$

where $\Sigma_{filt}^{-1} = \Phi\Sigma^{-1}$ and Φ is the diagonal matrix whose elements are $\phi_1, \ldots, \phi_N$. **remark** Notice that if we choose the filter parameters as follows

$$\phi_i = \begin{cases} 1, & i = I, \ldots, k \\ 0, & i = k+1, \ldots, N, \end{cases}$$

then we get the truncated SVD and the parameter k is the optimal truncation parameter that can also be obtained by minimizing the discrete GCV function

$$G(k) = \frac{1}{(N-k)^2} \sum_{i=k+1}^{N} (u_i^T b)^2.$$

Here, the optimal truncation parameter k_{opt} is obtained by evaluating $G(k)$ for $k = 1, \ldots, N-1$ and find the index for which $G(k)$ attains its minimum.

Theorem

For the filtered solution x_{filt}, we have the following properties

1. $$\|x_{filt}\|_2^2 = \sum_{i=1}^{N} \left(\phi_i \frac{u_i^T b}{\sigma_i} \right)^2.$$
2. $$\|r_{filt}\|_2^2 = \|b - Ax_{filt}\|_2^2 = \sum_{i=1} \left((1 - \phi_i) u_i^T b \right)^2.$$

proof

The relation 1 comes directly from the expression (8) of the solution x_{filt} and by using the fact that the v_i's are orthonormal. For the second relation, we have

$$\begin{aligned}
A\,x_{filt} &= U\Sigma V^T x_{filt} && (13)\\
&= U\Sigma V^T \left(\sum_{i=1}^{N} \Phi_i \frac{u_i^T b}{\sigma_i}\, v_i \right) && (14)\\
&= U\Sigma \left(\sum_{i=1}^{N} \Phi_i \frac{u_i^T b}{\sigma_i}\, V^T v_i \right) && (15)\\
&= U \left(\sum_{i=1}^{N} \Phi_i (u_i^T b)\, V^T v_i \right). && (16)
\end{aligned}$$

Therefore

$$b - A\,x_{filt} = \sum_{i=1}^{N} (u_i^T b) u_i - \sum_{i=1}^{N} \phi_i (u_i^T b) u_i \qquad (17)$$

Some properties of the Kronecker product

Let $A = (a_{ij})$ and $B = (b_{ij})$ be $n \times p$ and $s \times q$ matrices respectively. The Kronecker product of the matrices A and B is defined as the $(ns) \times (pq)$ matrix

$$A \otimes B = (a_{ij} B).$$

The vec operator transforms the $n \times p$ matrix A to a vector $a = \mathtt{vec(A)}$ of size $np \times 1$ by stacking the columns of A, as follows

$$\mathtt{vec(A)} = [a_{11}\ a_{21} \ldots\ a_{np}]^T.$$

Some properties of the Kronecker product are given below

Theorem

1. $A \otimes (B + C) = A \otimes B + A \otimes C.$
2. $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$
3. $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}, \quad \textit{if } A, B \textit{ are nosingular.}$
4. $vec(AXB) = (B^T \otimes A)vec(X).$
5. $(A \otimes B)^T = A^T \otimes B^T.$
6. $A \otimes B = 0 \textit{ iff } A = 0 \textit{ or } B = 0.$

proof Let us give a short proof of the relation 3. We have

$$\begin{aligned}
(AXB)_k &= A(XB)_k &(19)\\
&= A\left[\sum_{i=1}^{p} b_{ik}X_i\right] &(20)\\
&= [b_{1k}A\ b_{2k}A\ \ldots b_{pk}A]\mathtt{vec}(X) &(21)\\
&= (B_b^T \otimes A)\,\mathtt{vec}(X) &(22)
\end{aligned}$$

where $X = [X_1, \ldots, X_p]$, $B = b_{i,j}]$, $i, j = 1, \ldots, p$ **and** $B_k = [b_{1k}\ \ \ldots b_{pk}]$**. Therefore,**

$$\mathtt{vec}(AXB) = \begin{bmatrix} B_1^T \otimes A \\ \vdots \\ B_p^T \otimes A \end{bmatrix} \mathtt{vec}(X).$$

But since the transpose of the columns of B **are the rows of** B^T **it follows that**

$$\mathtt{vec}(AXB) = (B^T \otimes A)\mathtt{vec}(X).$$

Theorem

Consider the SVD decompositions of the matrices $A = U_A \Sigma_A V_A^T$ *and* $B = U_B \Sigma_B V_B^T$. *Then the SVD decomposition of the Kronecker product is given as*

$$A \otimes B = U \Sigma V^T$$

with $U = U_A \otimes U_B$, $V = V_A \otimes V_B$ *and* $\Sigma = \Sigma_A \otimes \Sigma_B$.

Theorem

1. *Let* $A \in \mathbb{R}^{n \times n}$ *and* $B \in \mathbb{R}^{p \times p}$, *then if* λ *in an eigenvalue of* A *corresponding to* x *and* μ *is an eigenvalue of* B *corresponding to* y, *then* $\lambda\mu$ *is an eigenvalue of* $A \otimes B$ *corresponding to* $x \otimes y$.
2. *If* A *and* B *are positive (semi)-definite, then* $A \otimes B$ *is also positive (semi).*
3. *definite.*
4. $rank(A \otimes B) = rank(B \otimes A) = rank(A)\, rank(B)$.
5. $e^{A \otimes I + I \otimes B} = e^A \otimes e^B$.

Assume that one wants to solve a linear systems of equations $Ax = b$ where the matrix A is given as

$$A = A_2^T \otimes A_1$$

then, since $\mathtt{vec}(A_1 X A_2) = A_2^T \otimes A_1$, it follows that the equivalent matrix equation is given as

$$A_1 X A_2 = B, \tag{23}$$

where $x = \mathtt{vec}(\mathtt{X})$ and $b = \mathtt{vec}(\mathtt{B})$. For some applications, such as in image processing, equation (23) is more appropriate and more simple to solve than the corresponding linear system with the matrix A.

Consider now the more general Sylvester matrix equation

$$AX - XB = C, \tag{24}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{p\times p}$ and $C \in \mathbb{R}^{n\times p}$. Such matrix equations play a fundamental role in optimal control theory and other topics. Equation (26) van be expressed in the following form

$$(I_p \otimes A - B^T \otimes I_n)x = c, \tag{25}$$

where $x = \mathtt{vec(X)}$ and $c = \mathtt{vec(C)}$.

Consider now the more general Sylvester matrix equation

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where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{p \times p}$ and $C \in \mathbb{R}^{n \times p}$. Such matrix equations play a fundamental role in optimal control theory and other topics. Equation (26) can be expressed in the following form

$$(I_p \otimes A - B^T \otimes I_n)x = c, \tag{27}$$

where $x = \mathtt{vec}(\mathtt{X})$ and $c = \mathtt{vec}(\mathtt{C})$.

Theorem

1. *The eigenvalues of the operator $\mathcal{A}$ defined by $\mathcal{A}(X) = AX - XB$ are $\lambda_i(A) - \mu_j(B)$ where $\lambda_i(A)$ is an eigenvalue of A and $\mu_j(B)$ is an eigenvalue of B.*
2. *The matrix equation (26) has a unique solution iff $\lambda_i(A) - \mu_j(B) \neq 0$.*

The special case of the Sylvetser equation is the well known Lyapunov matrix equation given as follows

$$AX + XA^T = C.$$

This matrix equation plays an important role in model reduction techniques and in optimal control methods. Direct methods for solving small to medium Sylvester and Lyapunov equations exist and are based on the Schur decomposition of the matrices A and B. For large problems we can use the well known Krylov subspace methods. **proof** Let $Ax_i = \lambda_i x_i$

and $By_j = \mu_j y_j$. Then

$$\begin{aligned} [(I_p \otimes A) - (B^T \otimes I_n)]\,(y_j \otimes x_i) &= (y_j \otimes (Ax_i)) - (By_j \otimes x_i) && (28)\\ &= (\lambda_i - \mu_j)(y_j \otimes x_i). && (29)\end{aligned}$$

proof

Let $Ax_i = \lambda_i x_i$ and $By_j = \mu_j y_j$. Then

$$[(I_p \otimes A) - (B^T \otimes I_n)]\,(y_j \otimes x_i) = (y_j \otimes (Ax_i)) - (By_j \otimes x_i) \quad (30)$$
$$= (\lambda_i - \mu_j)(y_j \otimes x_i). \quad (31)$$

The special case of the Sylvetser equation is the well known Lyapunov matrix equation given as follows

$$AX + XA^T = C.$$

This matrix equation plays an important role in model reduction techniques and in optimal control methods. Direct methods for solving small to medium Sylvester and Lyapunov equations exist and are based on the Schur decomposition of the matrices A and B. For large problems we can use the well known Krylov subspace methods.

Notice that in the case of singular or nearly singular Sylvester or Lyapunov matrix equations, we can solve a least squares problem of the form

$$\min_X \|AX - XB - C\|_F \text{ or } \min_X \|AX + XA^T - C\|_F$$

where $\|.\|_F$ is the Frobenius norm associated to the matrix scalar product For X and Y two matrices in $\mathbb{R}^{n\times p}$, we define the following inner product

$$\langle X, Y\rangle_F = trace(X^T Y).$$