The Fourier Series of Music

Robert Fustero

$$f(q,p) = \left\langle R_{x_1} \middle| R_{x_2} \right\rangle$$

The Fustero Function generates the continuous sequence,

$R_{x_1} = \frac{p^x}{q^n}, R_{x_2} = \frac{q^{n+1}}{p^x}$. Where $n = \left\lfloor x \cdot \frac{\ln(p)}{\ln(q)} \right\rfloor$

What does an exponential waveform have to do with music and the notes we recognize? I discovered a mathematical function that displays this key insight into the unit of musical harmony. The motivation for finding this function was to see if there was a linear way to describe all the ratios in the chromatic scale (Pythagorean tuning). This came from a simple hypothesis, "The simpler the ratio the better the sound". When you list all the ratios in order from simplest (smallest numerator and denominator) to most complex a clear pattern emerges.

$$P1, P8, P5, P4, M2, m7, M6, m3, M3, m6, M7, m2, A4, D5$$

$$\frac{3^0}{2^0}, \frac{2^1}{3^0}, \frac{3^1}{2^1}, \frac{2^2}{3^1}, \frac{3^2}{2^3}, \frac{2^4}{3^2}, \frac{3^3}{2^4}, \frac{2^5}{3^3}, \frac{3^4}{2^6}, \frac{2^7}{3^4}, \frac{3^5}{2^7}, \frac{2^8}{3^5}, \frac{3^6}{2^9}, \frac{2^{10}}{3^6}$$

Every ratio pair has a power of 3^x in the numerator and denominator AND every pair of ratios with 3^x in the numerator and 3^x denominator always equaled 2. In fact, this can be viewed as one particular case of the equation (eq 1) below where p=3 and q =2.

$$\frac{p^x}{q^n} \cdot \frac{q^{n+1}}{p^x} = q \quad \text{Eq 1}$$

In this python code, you can see how this pattern extends far beyond just beyond the Pythagorean ratios.

```
p=3
q=2
x=0
n=0
limit = 12
while x < limit:  #limit can go beyond six
        rat = (p**x)/(q**n)
        print('3^{}/2^{} = {}'.format(x,n,rat))
        n = n+1
        rat = ((q**n)/(p**x))
        print('2^{}/3^{} = {}'.format(n,x,rat))
        x = x+1
        if ((p**x)/(q**n)) > 2:
                n= n+1
3^0/2^0 = 1.0
2^1/3^0 = 2.0
3^1/2^1 = 1.5
2^2/3^1 = 1.3333333333333333
3^2/2^3 = 1.125
2^4/3^2 = 1.7777777777777777
3^3/2^4 = 1.6875
2^5/3^3 = 1.1851851851851851
3^4/2^6 = 1.265625
2^7/3^4 = 1.5802469135802468
3^5/2^7 = 1.8984375
2^8/3^5 = 1.0534979423868314
3^6/2^9 = 1.423828125
2^10/3^6 = 1.4046639231824416
3^7/2^11 = 1.06787109375
2^12/3^7 = 1.8728852309099222
3^8/2^12 = 1.601806640625
2^13/3^8 = 1.2485901539399482
3^9/2^14 = 1.20135498046875
2^15/3^9 = 1.6647868719199308
3^10/2^15 = 1.802032470703125
2^16/3^10 = 1.1098579146132872
3^11/2^17 = 1.3515243530273438
2^18/3^11 = 1.4798105528177163
```

As you can see – if the ratio, Rx1 (3^x/2^n), is greater than 2 – you add 1 to 'n' term and the function continues.

For this to be a linear function we have to get rid of the 'if statement'. So the question turns into – what does 'n' equal?

$$\frac{3^x}{2^n} \leq 2$$

Rx1 must always be less than or equal to 2

$$\frac{3^x}{2^{n+1}} \leq 1$$

Rx2 must always hold this relationship in order for eq1 to always hold true

$$2^{n+1} \geq 3^x$$

$$\ln\left(2^{n+1}\right) \geq \ln\left(3^x\right)$$

$$(n+1)\ln(2) \geq x\ln(3)$$

$$n \geq x \cdot \frac{\ln(3)}{\ln(2)} - 1$$

$$n = \left\lceil x \cdot \frac{\ln(3)}{\ln(2)} - 1 \right\rceil$$

$$n = \left\lfloor x \cdot \frac{\ln(3)}{\ln(2)} \right\rfloor$$

$$f(q,p) = \langle R_{x_1} | R_{x_2} \rangle$$

The Fustero Function generates the continuous sequence,

$R_{x_1} = \frac{p^x}{q^n}, R_{x_2} = \frac{q^{n+1}}{p^x}$. Where $n = \left\lfloor x \cdot \frac{\ln(p)}{\ln(q)} \right\rfloor$

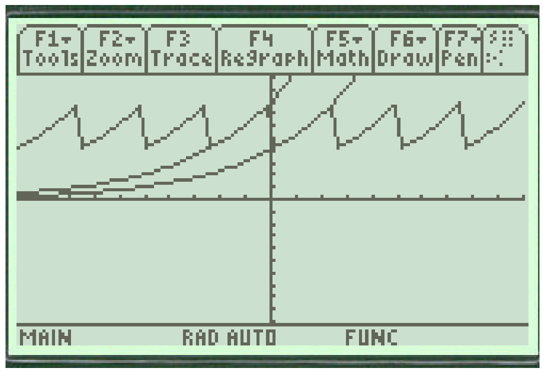
$$\prod_{x=0}^{l} R_{x_1} \cdot R_{x_2} = q^{l+1}$$

For any real number 'q' and 'p' > 0. The Fustero Function generates a linear sequence of ratios strictly less than or equal to q, whose terms $R_{x_1} \cdot R_{x_2}$ = q. The upper and lower bound are given by the first pair of ratios (when x=0)

$R_{0_1} \leq R_{l_1}$ *and* $R_{0_2} \leq R_{l_2}$

For the particular case of q=2 and p=3 where the 'x' integer values intersect the function from 0 to 6... This produces a tuple of ratios that describes all the Pythagorean Ratios from the smallest numerator/denominator to the largest.

One thing to notice about this function is that it is periodic. The q^n term consistently divides p^x in such a way that every time x increases by $\frac{\ln(q)}{\ln(p)}$, the exponential resets.



$$R_{x_1} = \begin{cases} p^x & 0 < x < \dfrac{\ln(q)}{\ln(p)} \\ q \cdot p^x & \dfrac{-\ln(q)}{\ln(p)} < x < 0 \end{cases}$$

The Fourier Series Expansion of R_{x_1}

The period of this function can be written as $L = \frac{\ln(q)}{\ln(p)}$

To calculate the Fourier Coefficients we must get rid of the <u>Floor Function</u> by finding the equivalent integral over the period – L to L. From the proof above, this is now trivial to see.

$$a_0 = \frac{1}{L}\int_{-L}^{L} \frac{p^x}{q^{\lfloor x \cdot \ln(p)/\ln(q) \rfloor}}\, dx$$

$$= \frac{1}{L} \cdot \left(\int_{-L}^{0} q \cdot p^x dx + \int_{0}^{L} p^x dx\right)$$

$$= \frac{q \cdot \ln(p)}{\ln(q)} \cdot \int_{-L}^{0} p^x dx + \frac{\ln(p)}{\ln(q)} \cdot \int_{0}^{L} p^x\, dx$$

$$= \frac{q \cdot p^x}{\ln(q)} \Big|_{-L}^{0} + \frac{p^x}{\ln(q)} \Big|_{0}^{L}$$

$$= \frac{q}{\ln(q)} - \frac{q \cdot p^{-L}}{\ln(q)} + \frac{p^L}{\ln(q)} - \frac{1}{\ln(q)}$$

$$a_0 = \frac{q-1}{\ln(q)} + \frac{q-1}{\ln(q)} = \frac{2(q-1)}{\ln(q)}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} \frac{p^x}{q^{\lfloor x\cdot \ln(p)/\ln(q)\rfloor}} \cos\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{1}{L}(\int_{-L}^{0} qp^x \cos\left(\frac{n\pi x}{L}\right) dx + \int_{0}^{L} p^x \cos\left(\frac{n\pi x}{L}\right) dx)$$

$$= \frac{q\ln(p)}{\ln(q)}\int_{-L}^{0} p^x \cos\left(\frac{n\pi x}{L}\right) dx + \frac{\ln(p)}{\ln(q)}\int_{0}^{L} p^x \cos\left(\frac{n\pi x}{L}\right) dx$$

Our equation is set. Now we must calculate $\int p^x \cos\left(\frac{n\pi x}{L}\right) dx$

$$\int fg' = fg - \int f'g$$

$$f = \cos\left(\frac{\pi n\, ln(p)x}{ln(q)}\right),\ \ g' = p^x, g = \frac{p^x}{\ln(p)}$$

$$f' = \frac{d}{dx}[\cos(\frac{\pi n\, ln(p)x}{ln(q)})]$$

$$= -\sin\left(\frac{\pi n\, ln(p)x}{ln(q)}\right)\cdot \frac{d}{dx}\left[\frac{\pi n\, ln(p)x}{ln(q)}\right]$$

$$= -\sin\left(\frac{\pi n\, ln(p)x}{ln(q)}\right)\cdot \frac{\pi n\, ln(p)x}{ln(q)}\cdot \frac{d}{dx}[\mathrm{x}]$$

$$f' = \frac{-\pi n \ln(p)\sin\left(\frac{\pi n\, ln(p)x}{ln(q)}\right)}{\ln(q)}$$

$$\int p^x \cos\left(\frac{n\pi x}{L}\right) dx = \frac{p^x\cos\left(\frac{\pi n\, ln(p)x}{ln(q)}\right)}{\ln(p)} - \int \frac{-\pi n p^x \ln(p)\sin\left(\frac{\pi n\, ln(p)x}{ln(q)}\right)}{\ln(q)\ln(p)} dx$$

We now must do Integration by parts a second time for $\int \frac{-\pi n p^x \ln(p)\sin\left(\frac{\pi n\, ln(p)x}{ln(q)}\right)}{\ln(q)\ln(p)}dx$

$$\int fg' = fg - \int f'g$$

$$f = \frac{-\pi n \ln(p)\sin\left(\frac{\pi n\, ln(p)x}{ln(q)}\right)}{\ln(q)},\quad g' = \frac{p^x}{\ln(p)}, g = \frac{p^x}{\ln^2(p)}$$

$$f' = \frac{d}{dx}\left[\frac{-\pi n \ln(p)\sin\left(\frac{\pi n\, ln(p)x}{ln(q)}\right)}{\ln(q)}\right]$$

$$= \frac{-\pi n \ln(p)}{\ln(q)}\cdot\frac{d}{dx}\left[\sin\left(\frac{\pi n\, ln(p)x}{ln(q)}\right)\right]$$

$$= -\frac{\pi\cos\left(\frac{\pi n\, ln(p)x}{ln(q)}\right)\mathrm{n}\cdot\ln(\mathrm{p})\frac{d}{dx}\left[\frac{\pi n\, ln(p)x}{ln(q)}\right]}{\ln(\mathrm{q})}$$

$$= -\frac{\pi\cos\left(\frac{\pi n\, ln(p)x}{ln(q)}\right)\mathrm{n}\cdot\ln(\mathrm{p})\frac{\pi n\, ln(p)}{ln(q)}\cdot\frac{d}{dx}[x]}{\ln(\mathrm{q})}$$

$$f' = -\frac{\pi^2\mathrm{n}^2\ln^2(\mathrm{p})\cos(\frac{\pi n\, ln(p)x}{ln(q)})}{\ln^2(\mathrm{q})}$$

$$\int p^x\cos\left(\frac{n\pi x}{L}\right)dx = \frac{p^x\cos\left(\frac{\pi n\, ln(p)x}{ln(q)}\right)}{\ln(p)} - \left(\frac{-\pi n p^x\sin\left(\frac{\pi n\, ln(p)x}{ln(q)}\right)}{\ln(p)\ln(q)} - \int -\frac{\pi^2\mathrm{n}^2 p^x\cos(\frac{\pi n\, ln(p)x}{ln(q)})}{\ln^2(\mathrm{q})}dx\right)$$

$$\int p^x\cos(\frac{\pi n\, ln(p)x}{ln(q)})dx = \frac{p^x\cos\left(\frac{\pi n\, ln(p)x}{ln(q)}\right)}{\ln(p)} - \left(\frac{-\pi n p^x\sin\left(\frac{\pi n\, ln(p)x}{ln(q)}\right)}{\ln(p)\ln(q)} + \frac{\pi^2\mathrm{n}^2}{\ln^2(q)}\int p^x\cos(\frac{\pi n\, ln(p)x}{ln(q)})dx\right)$$

$\int p^x\cos(\frac{\pi n\, ln(p)x}{ln(q)})dx$ Appears on both sides of the equation and we can now solve for it

$$\int p^x \cos(\frac{\pi n\ ln(p)x}{ln(q)})dx\ = \frac{\frac{\mathrm{\pi n ln}(p)p^x \sin(\frac{\pi n\ ln(p)x}{ln(q)})}{\ln(q)} + \ln(p)p^x \cos(\frac{\pi n\ ln(p)x}{ln(q)})}{\frac{\mathrm{\pi^2 n^2 ln^2}(p)}{\ln^2(q)} + \ln^2(p)}$$

Now plug-in the solved integral

$$\frac{q\ \ln(p)}{\ln(q)}\int p^x \cos(\frac{\pi n\ ln(p)x}{ln(q)})dx\ = \frac{q\ \ln(p)}{\ln(q)} \cdot \frac{\frac{\mathrm{\pi n ln}(p)p^x \sin(\frac{\pi n\ ln(p)x}{ln(q)})}{\ln(q)} + \ln(p)p^x \cos(\frac{\pi n\ ln(p)x}{ln(q)})}{\frac{\mathrm{\pi^2 n^2 ln^2}(p)}{\ln^2(q)} + \ln^2(p)}$$

$$= \frac{q\ \ln(p)(\frac{\mathrm{\pi n ln}(p)p^x \sin\left(\frac{\pi n\ ln(p)x}{ln(q)}\right)}{\ln(q)} + \ln(p)p^x \cos(\frac{\pi n\ ln(p)x}{ln(q)}))}{(\frac{\mathrm{\pi^2 n^2 ln^2}(p)}{\ln^2(q)} + \ln^2(p))\ln(q)}$$

Simplify

$$\frac{q\ \ln(p)}{\ln(q)}\int_{-L}^{0} p^x \cos\left(\frac{n\pi x}{L}\right)dx = \frac{qp^x(\mathrm{\pi n sin}\left(\frac{\pi n\ ln(p)x}{ln(q)}\right) + \mathrm{ln(q) cos}\left(\frac{\pi n\ ln(p)x}{ln(q)}\right))}{\ln^2(q) + \mathrm{\pi^2 n^2}}\Big|_{-L}^{0}$$

$$= \frac{(q - \cos(\pi n))\ln(q) + \pi n sin(\pi n)}{\ln^2(q) + \mathrm{\pi^2 n^2}}, \sin(\pi n) = 0$$

$$= \boldsymbol{\frac{(q - \cos(\pi n))\ln(q)}{\ln^2(q) + \pi^2 n^2}}$$

$$\frac{\ln(p)}{\ln(q)}\int_{0}^{L} p^x \cos\left(\frac{n\pi x}{L}\right)dx = \frac{\ln(p)}{\ln(q)} \cdot \frac{\frac{\mathrm{\pi n ln}(p)p^x \sin(\frac{\pi n\ ln(p)x}{ln(q)})}{\ln(q)} + \ln(p)p^x \cos(\frac{\pi n\ ln(p)x}{ln(q)})}{\frac{\mathrm{\pi^2 n^2 ln^2}(p)}{\ln^2(q)} + \ln^2(p)}$$

$$= \frac{\ln(p)(\frac{\mathrm{\pi n ln}(p)p^x \sin\left(\frac{\pi n\ ln(p)x}{ln(q)}\right)}{\ln(q)} + \ln(p)p^x \cos(\frac{\pi n\ ln(p)x}{ln(q)}))}{(\frac{\mathrm{\pi^2 n^2 ln^2}(p)}{\ln^2(q)} + \ln^2(p))\ln(q)}$$

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$$= \frac{(\cos(\pi n)\ q - 1)\ln(q) + \pi n \sin(\pi n)\ q}{\ln^2(q) + \mathrm{\pi^2 n^2}}, \sin(\pi n) = 0$$

$$= \boldsymbol{\frac{(\cos(\pi n)\ q - 1)\ln(q)}{\ln^2(q) + \pi^2 n^2}}$$

$$a_n = \frac{q\ \ln(p)}{\ln(q)} \int_{-L}^{0} p^x \cos\left(\frac{n\pi x}{L}\right) dx + \frac{\ln(p)}{\ln(q)} \int_{0}^{L} p^x \cos\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{(q - \cos(\pi n))\ln(q)}{\ln^2(q) + \pi^2 n^2} + \frac{(\cos(\pi n)\, q - 1)\ln(q)}{\ln^2(q) + \pi^2 n^2}$$

$$a_n = \frac{(\cos(\pi n) + 1)(q - 1)\ln(q)}{\ln^2(q) + \pi^2 n^2}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} \frac{p^x}{q^{\lfloor x\cdot \ln(p)/\ln(q)\rfloor}} \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{1}{L}(\int_{-L}^{0} qp^x \sin\left(\frac{n\pi x}{L}\right) dx + \int_{0}^{L} p^x \sin\left(\frac{n\pi x}{L}\right) dx)$$

$$= \frac{q\ln{(p)}}{\ln{(q)}}\int_{-L}^{0} p^x \sin\left(\frac{n\pi x}{L}\right) dx + \frac{\ln{(p)}}{\ln{(q)}}\int_{0}^{L} p^x \sin\left(\frac{n\pi x}{L}\right) dx$$

Our equation is set. Now we must calculate $\int p^x \sin\left(\frac{n\pi x}{L}\right) dx$

$$\int fg' = fg - \int f'g$$

$$f = \sin\left(\frac{\pi n\, ln(p) x}{ln(q)}\right),\;\; g' = p^x, g = \frac{p^x}{\ln(p)}$$

$$f' = \frac{d}{dx}[\sin(\frac{\pi n\, ln(p)x}{ln(q)})]$$

$$= \cos\left(\frac{\pi n\, ln(p)x}{ln(q)}\right)\cdot \frac{d}{dx}\left[\frac{\pi n\, ln(p)x}{ln(q)}\right]$$

$$= \cos\left(\frac{\pi n\, ln(p)x}{ln(q)}\right)\frac{\pi n\, ln(p)}{ln(q)}\frac{d}{dx}[\mathrm{x}]$$

$$f' = \frac{\pi n\, ln(p)\cos\left(\frac{\pi n\, ln(p)x}{ln(q)}\right)}{ln(q)}$$

$$\int p^x \sin\left(\frac{n\pi x}{L}\right) dx = \frac{p^x \sin\left(\frac{\pi n\, ln(p)x}{ln(q)}\right)}{ln(p)} - \int \frac{\pi n p^x\, ln(p)\cos\left(\frac{\pi n\, ln(p)x}{ln(q)}\right)}{ln(q)ln(p)} dx$$

We now must do Integration by parts a second time for $\int \frac{\pi n p^x \, ln(p) \cos\left(\frac{\pi n \, ln(p)x}{ln(q)}\right)}{ln(q)ln(p)} dx$

$$\int f g' = fg - \int f' g$$

$$f = \frac{\pi n \;\; ln(p) \cos\left(\frac{\pi n \; ln(p)x}{ln(q)}\right)}{ln(q)}, g' = \frac{p^x}{ln(p)}, g = \frac{p^x}{\ln^2(p)}$$

$$f' = \frac{d}{dx}\left[\frac{\pi n \;\; ln(p) \cos\left(\frac{\pi n \; ln(p)x}{ln(q)}\right)}{ln(q)}\right]$$

$$= \frac{\pi n \; ln(p)}{ln(q)} \frac{d}{dx}\left[\cos\left(\frac{\pi n \; ln(p)x}{ln(q)}\right)\right]$$

$$= \frac{\pi n \; ln(p)(-\sin\left(\frac{\pi n \; ln(p)x}{ln(q)}\right)) \frac{d}{dx}\left[\frac{\pi n \; ln(p)x}{ln(q)}\right]}{ln(q)}$$

$$= -\frac{\pi n \; ln(p) \frac{\pi n \; ln(p)}{ln(q)} \sin\left(\frac{\pi n \; ln(p)x}{ln(q)}\right) \frac{d}{dx}[x]}{ln(q)}$$

$$f' = -\frac{\pi^2 n^2 \ln^2(p) \sin(\frac{\pi n \; ln(p)x}{ln(q)})}{\ln^2(q)}$$

$$\int p^x \sin\left(\frac{n\pi x}{L}\right) dx = \frac{p^x \sin\left(\frac{\pi n \; ln(p)x}{ln(q)}\right)}{\ln(p)} - \left(\frac{\pi n p^x \cos\left(\frac{\pi n \; ln(p)x}{ln(q)}\right)}{\ln(p)\ln(q)} - \int -\frac{\pi^2 n^2 p^x \sin\left(\frac{\pi n \; ln(p)x}{ln(q)}\right)}{\ln^2(q)} dx\right)$$

$$\int p^x \; \sin\left(\frac{\pi n \; ln(p)x}{ln(q)}\right) dx = \frac{p^x \sin\left(\frac{\pi n \; ln(p)x}{ln(q)}\right)}{\ln(p)} - \left(\frac{\pi n p^x \cos\left(\frac{\pi n \; ln(p)x}{ln(q)}\right)}{\ln(p)\ln(q)} + \frac{\pi^2 n^2}{\ln^2(q)} \int p^x \sin\left(\frac{\pi n \; ln(p)x}{ln(q)}\right) dx\right)$$

$\int p^x \sin\left(\frac{\pi n \; ln(p)x}{ln(q)}\right) dx$ Appears on both sides of the equation and we can now solve for it

$$\int p^x \sin\left(\frac{\pi n\, ln(p)x}{ln(q)}\right) dx = \frac{\ln(p)p^x \sin\left(\frac{\pi n\, ln(p)x}{ln(q)}\right) - \frac{\pi n \ln(p) p^x \cos\left(\frac{\pi n\, ln(p)x}{ln(q)}\right)}{\ln(q)}}{\frac{\pi^2 n^2 \ln^2(p)}{\ln^2(q)} + \ln^2(p)}$$

Now plug-in the solved integral

$$\frac{q \ln(p)}{\ln(q)} \int_{-L}^{0} p^x \sin\left(\frac{n\pi x}{L}\right) dx = \frac{q \ln(p)(\ln(p)p^x \sin\left(\frac{\pi n\, ln(p)x}{ln(q)}\right) - \frac{\pi n \ln(p) p^x \cos\left(\frac{\pi n\, ln(p)x}{ln(q)}\right)}{\ln(q)})}{(\frac{\pi^2 n^2 \ln^2(p)}{\ln^2(q)} + \ln^2(p))\ln(q)}$$

Simplify

$$= \frac{q p^x (\sin\left(\frac{\pi n\, ln(p)x}{ln(q)}\right) - \pi n \cos\left(\frac{\pi n\, ln(p)x}{ln(q)}\right))}{\ln^2(q) + \pi^2 n^2} \Big|_{-L}^{0}$$

$$= \frac{\sin(\pi n)\, ln(q) - \pi n q + \pi n \cos(\pi n)}{\ln^2(q) + \pi^2 n^2}, \sin(\pi n) = 0$$

$$= \boldsymbol{\frac{-\pi n q + \pi n \cos(\pi n)}{\ln^2(q) + \pi^2 n^2}}$$

$$\frac{\ln(p)}{\ln(q)} \int_{0}^{L} p^x \sin\left(\frac{n\pi x}{L}\right) dx = \frac{\ln(p)(\ln(p)p^x \sin\left(\frac{\pi n\, ln(p)x}{ln(q)}\right) - \frac{\pi n \ln(p) p^x \cos\left(\frac{\pi n\, ln(p)x}{ln(q)}\right)}{\ln(q)})}{(\frac{\pi^2 n^2 \ln^2(p)}{\ln^2(q)} + \ln^2(p))\ln(q)}$$

Simplify

$$= \frac{p^x (\sin\left(\frac{\pi n\, ln(p)x}{ln(q)}\right) - \pi n \cos\left(\frac{\pi n\, ln(p)x}{ln(q)}\right))}{\ln^2(q) + \pi^2 n^2} \Big|_{0}^{L}$$

$$= \frac{\sin(\pi n)\, q ln(q) - \pi n \cos(\pi n) q + \pi n}{\ln^2(q) + \pi^2 n^2}, \sin(\pi n) = 0$$

$$= \boldsymbol{\frac{-\pi n \cos(\pi n) q + \pi n}{\ln^2(q) + \pi^2 n^2}}$$

$$\frac{q\ln(p)}{\ln(q)}\int_{-L}^{0} p^x \sin\left(\frac{n\pi x}{L}\right) dx + \frac{\ln(p)}{\ln(q)}\int_{0}^{L} p^x \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{-\pi n q + \pi n \cos(\pi n)}{\ln^2(q) + \pi^2 n^2} + \frac{-\pi n \cos(\pi n) q + \pi n}{\ln^2(q) + \pi^2 n^2}$$

$$b_n = \frac{-(\pi n(q-1)(\cos(\pi n) + 1)}{\ln^2(q) + \pi^2 n^2}$$

$$a_0 = \frac{2(q-1)}{\ln(q)}$$

$$a_n = \frac{(\cos(\pi n) + 1)(q-1)\ln(q)}{\ln^2(q) + \pi^2 n^2}$$

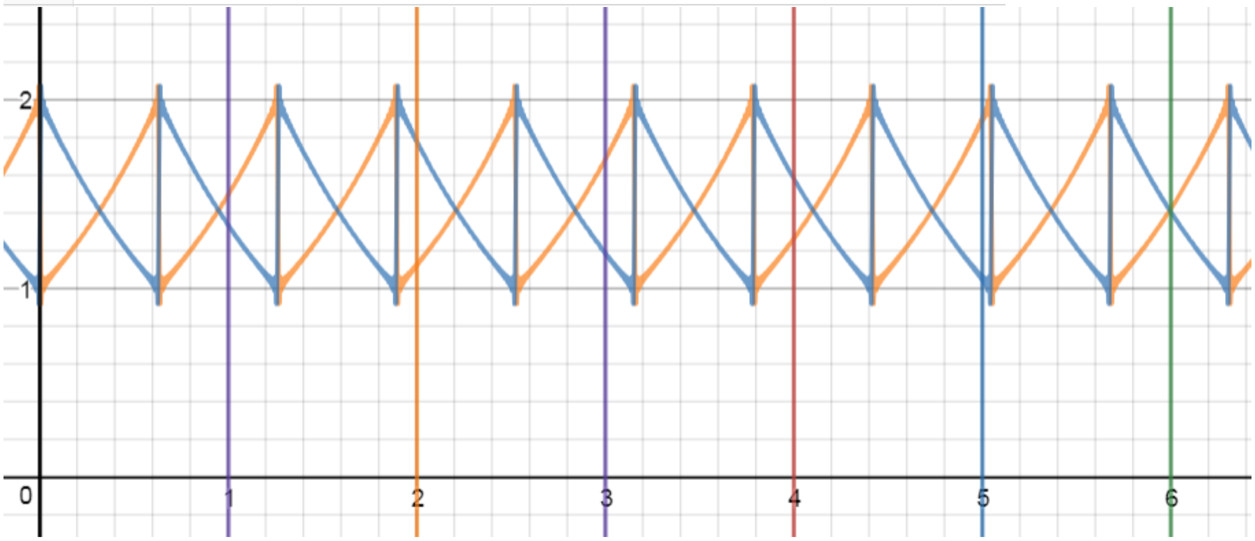
$$b_n = \frac{-(\pi n(q-1)(\cos(\pi n) + 1)}{\ln^2(q) + \pi^2 n^2}$$

$$\mathcal{F}\left(R_{x_1}\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

$$\mathcal{F}\left(R_{x_2}\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{-n\pi x}{L} + b_n \sin\frac{-n\pi x}{L}\right)$$

$$s_1 = \frac{a_0}{2} + \sum_{n=1}^{k} \left(a_n \cos\left(\frac{n\pi x}{\frac{\ln(q)}{\ln(p)}} \right) + b_n \sin\left(\frac{n\pi x}{\frac{\ln(q)}{\ln(p)}} \right) \right)$$

$$s_2 = \frac{a_0}{2} + \sum_{n=1}^{k} \left(a_n \cos\left(\frac{-n\pi x}{\frac{\ln(q)}{\ln(p)}} \right) + b_n \sin\left(\frac{-n\pi x}{\frac{\ln(q)}{\ln(p)}} \right) \right)$$



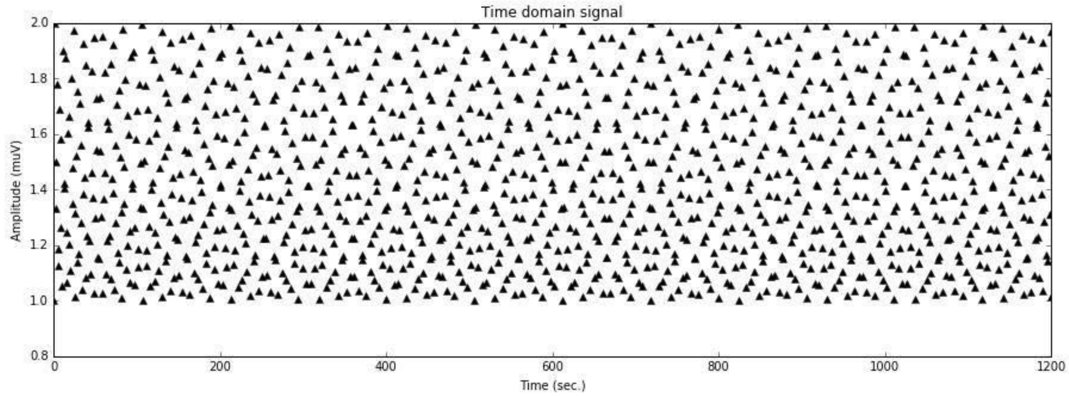
F1 Tools	F2 Setup	F3 Cell	F4 Header	F5 Del Row	F6 Ins Row

x	y1	y2	
1.	1.3333	1.5	
2.	1.7778	1.125	
3.	1.1852	1.6875	
4.	1.5802	1.2656	
5.	1.0535	1.8984	

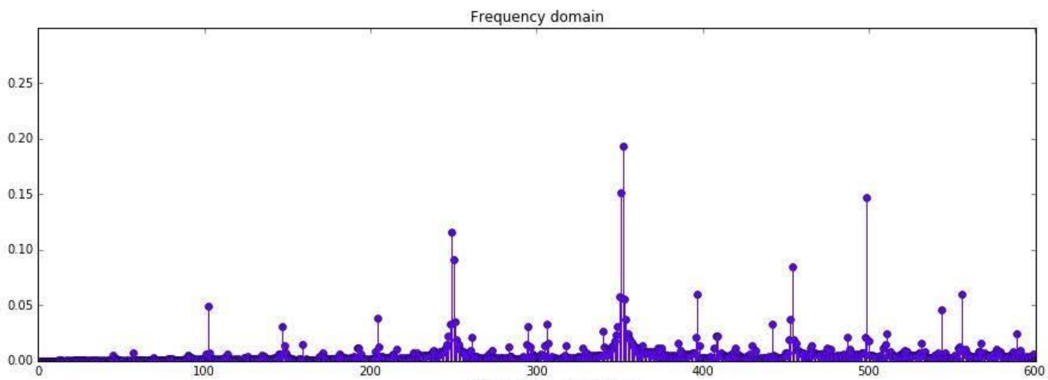
x=5.

MAIN RAD AUTO FUNC

Graph provided by https://www.desmos.com/calculator/870y0w6be6



This is a scatter plot of all the ratios when p=3 q=2 from 0 to 1200



Here is the DFT of the above plot