The Application of Linear Programming to Extremal Combinatorics

Math 529: April 11, 2021

Jon Noel

Aaron Slobodin

1. Introduction

In this paper, we verify the counterexamples presented in [Wag19] which are used to disprove open conjectures in extremal combinatorics. Broadly speaking, each refuted conjecture makes claims about the maximum size of some combinatorial object. Given this, the author of [Wag19] phrase each conjecture as a linear program (LP) with constraints specific to the combinatorial object and an objective function which seeks to maximize the size of said object. Each LP is then used to find a counterexamples via brute force search.

The structure of this paper is as follows. Section 2 includes example combinatorial problems (derived from the course homework) and their corresponding phrasal as LPs. Section 3 is broken into eight subsections, each of which introduce the conjecture to be refuted, the corresponding result from [Wag19], and our own verification of said result.

All computational results were found on the author's 2012 4GB MacBook Pro with a 2.5 GHz Intel Dual-Core i5 processor. The LPs were solved using a Gurobi solver ([GO21]). We mention the specifications of the author's computer not to brag but to observe that you do not need a powerful computer to produce meaningful solutions to the problems discussed here.

For completeness, we note that Section 3.4 (discussing the forbidden trace problem) and Section 3.7 (discussing rainbow matchings) of [Wag19] and their corresponding results are not included in this paper.

2. Warm Up

Below we include two example combinatorial problem and their corresponding LP's.

2.1. **Example 1.**

Given the definition of an antichain, a natural question one might ask is: "what is the largest size of an antichain in $2^{[n]}$?" The answer to this question is given by Sperner's Theorem, stated as follows.

Theorem 2.1. *If $\mathcal{A} \subseteq 2^{[n]}$ is an antichain, then $|\mathcal{A}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$.*

However, lets suppose that we know nothing of Sperner's Theorem. By framing the problem as an LP, we can run the LP on different values of n to find a sampling of known optimal solutions. We can then look for patterns in these solutions and perhaps derive a answer to our question.

For a fixed n, we wish to find an antichain $\mathcal{A}$ of $2^{[n]}$ of maximum cardinality. For each $A \in 2^{[n]}$, let $x_A \in \{0, 1\}$ be an indicator variable giving the truth value of the statement "$A \in \mathcal{A}$." As $\mathcal{A}$ is an antichain, for each pair of subsets $A, B \in 2^{[n]}$ we add the constraint $x_A + x_B \leq 1$ if $A \subset B$ or $B \subset A$. As we wish $\mathcal{A}$ to have maximum size, our objective function is simply $\sum_{A\subseteq[n]} x_A$. Put compactly, our LP is as follows:

$$\begin{aligned} \text{Maximize:} \quad & \sum_{A\in 2^{[n]}} x_A \\ \text{Subject to:} \quad & x_A + x_B \leq 1 \quad \forall A, B \in 2^{[n]} : A \subset B \text{ or } B \subset A \end{aligned}$$

See `Example_1.py` for an implementation of this LP. Below we include the outputs of this LP for a variety of values of n.

$n = 3$	$n = 4$	$n = 5$
$\{1, 2, 3\}$	$\{12, 13, 14, 23, 24, 34\}$	$\{123, 124, 125, 134, 135, 145, 234, 235, 245, 345\}$

Note that here and throughout the rest of this paper, we have adopted the notation used in [Wag19]. That is, the element 125 in the above set actually corresponds with the set $\{1, 2, 5\}$. The same goes for the other elements.

Upon inspection, for each n, we see that the LP produced $\binom{[n]}{\lfloor n/2\rfloor}$ as the maximum antichain. Given this, and that said construction holds for all n, we might conjecture that the maximum size of an antichain in $2^{[n]}$ is at least $\binom{[n]}{\lfloor n/2\rfloor}$; a bound tight with Theorem 2.1. This LP cannot prove that this bound is tight generally. However it was useful in finding a generalized construction.

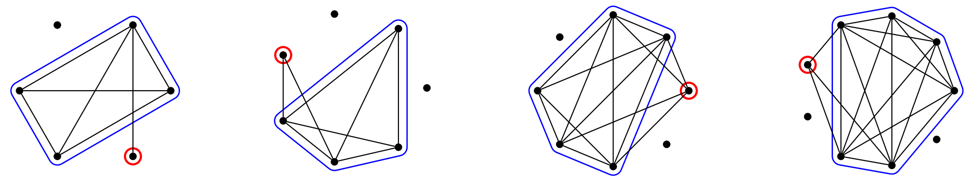
2.2. **Example 2.** Fix n and $m \leq \binom{n}{2}$. Suppose we wish to find the maximum number of triangles in an n vertex graph with m edges. The answer to this question follows from the Kruskal-Katona Theorem (see assignment 4). However, lets proceed as in Example 2.1 and attempt to gain insight to this problem by using a LP.

The graph G we hope to construct is a subgraph of K_n. For each $e \in \binom{[n]}{2}$, i.e. an edge of K_n, let x_e be an indicator variable giving the truth value of the statement "$e \in E(G)$." For each $t \in \binom{[n]}{3}$, i.e. a triangle of K_n, let y_t be an indicator variable giving the truth value of the statement "t is in G." If a triangle t is in G then the edges of t are necessarily in G. Therefore, for each $t \in \binom{[n]}{3}$ with edges t_1, t_2, t_3, we add the constraint $x_{t_1} + x_{t_2} + x_{t_3} \geq 3y_t$. As there are m edges in G, we add the constraint $\sum_{e\in\binom{[n]}{2}} x_e = m$. As we wish to maximize the number of triangles in G, our objective function is simply $\sum_{t\in\binom{[n]}{3}} y_t$. Put compactly, our LP is as follows:

$$\begin{aligned} \text{Maximize:} \quad & \sum_{t\in\binom{[n]}{3}} y_t \\ \text{Subject to:} \quad & (1) \quad \sum_{e\in\binom{[n]}{2}} x_e = m \\ & (2) \quad x_{t_1} + x_{t_2} + x_{t_3} \geq 3y_t \quad \forall t \in \binom{[n]}{3}, \text{ where } t_1, t_2, t_3 \text{ are the edges of } t \end{aligned}$$

See `Example_2.py` for an implementation of this LP. Below we include the graphs corresponding with the output of this LP for a variety of values of n and m.

$n = 6, m = 7$ $\quad$ $n = 7, m = 8$ $\quad$ $n = 8, m = 14$ $\quad$ $n = 9, m = 18$



Observe that for each fixed n and m, the optimal construction of G comprises a clique (as indicated by the blue regions) and some 'pendant' vertex (as indicated by the red circles) which is adjacent to some number of vertices of the clique. This matches the construction as given by Kruskal-Katona. Though the LP cannot be used to prove this construction is always optimal, it did quickly provide known optimal solution which easily exposed a recurring underlying structure.

3. Main Results

3.1. **Antichains of fixed diameter.** Let $\mathcal{F} \subseteq 2^{[n]}$. The *diameter* of $\mathcal{F}$ is defined as

$$\operatorname{diam}(\mathcal{F}) = \max_{A,B\in\mathcal{F}} \{|(A \setminus B) \cup (B \setminus A)|\}.$$

[Wag19] considered but was unable to refute the following conjecture of Frankl regarding antichains of bounded diameter.

Conjecture 3.1. *Let $n > d$ be positive integers. Suppose that $\mathcal{F} \subseteq 2^{[n]}$ is an antichain with diam$(\mathcal{F}) \leq d$. Then $|\mathcal{F}| \leq \binom{n}{\lfloor d/2 \rfloor}$.*

This problem can be phrased with the following LP. For a fixed n and d, we wish to find an antichain $\mathcal{F}$ of $2^{[n]}$ of maximum cardinality where $\operatorname{diam}(\mathcal{F}) \leq d$. For each $A \in 2^{[n]}$, let $x_A \in \{0, 1\}$ be an indicator variable giving the truth value of the statement "$A \in \mathcal{F}$." As $\mathcal{F}$ is an antichain, for each pair of subsets $A, B \in 2^{[n]}$ we add the constraint $x_A + x_B \leq 1$ if $A \subset B$ or $B \subset A$. To ensure that $\mathcal{F}$ has diameter at most d, for each $A, B \in 2^{[n]}$ where $|(A \setminus B) \cup (B \setminus A)| > d$, we add the constraint $x_A + x_B \leq 1$. As we wish $\mathcal{F}$ to have maximum cardinality, our objective function is simply $\sum_{A\in 2^{[n]}} x_A$. Put compactly, our LP is as follows:

$$\begin{aligned}
\text{Maximize:} \quad & \sum_{A\in 2^{[n]}} x_A \\
\text{Subject to:} \quad & (1) \quad x_A + x_B \leq 1 \quad \forall A, B \in 2^{[n]} : A \subset B \text{ or } B \subset A \\
& (2) \quad x_A + x_B \leq 1 \quad \forall A, B \in 2^{[n]} : |(A \setminus B) \cup (B \setminus A)| > d
\end{aligned}$$

See `Conjecture.1.py` for an implementation of this LP. [Wag19] verified the conjecture for $(n, d) = (10, 3), (8, 5), (8, 7)$ with antichains of size 10, 28, and 56. Running our own computation we can confirm these results. To verify, run `Conjecture.1.py` as is.

3.2. **k-chain free families.** A *k-chain* is a collection of k sets $A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_k$ totally ordered under inclusion. [Wag19] refuted the following conjecture of Frankl with regards to the maximum size of a k-chain-free family $\mathcal{F} \subseteq 2^{[n]}$.

Conjecture 3.2. *Let $n > d \geq \ell$ be positive integers and define $s =$ $\min\{\ell - 1, \lfloor d/2 \rfloor\}$. If $\mathcal{F} \subseteq 2^{[n]}$ is a $(\ell + 1)$-chain-free family with diam$(\mathcal{F}) \leq d$ then*

$$|\mathcal{F}| \leq \sum_{\lfloor d/2 \rfloor \geq i \geq \lfloor d/2 \rfloor - s} \binom{n}{i}.$$

This problem can be phrased with the following LP. For a fixed n, d, and ℓ, we wish to find a $(\ell+1)$-chain-free family $\mathcal{F}$ of $2^{[n]}$ of maximum cardinality where $\operatorname{diam}(\mathcal{F}) \leq d$. For each $A \in 2^{[n]}$, let $x_A \in \{0,1\}$ be an indicator variable giving the truth value of the statement "$A \in \mathcal{F}$." To ensure that $\mathcal{F}$ has diameter at most d, for each $A, B \in 2^{[n]}$ where $|(A \setminus B) \cup (B \setminus A)| > d$, we add the constraint $x_A + x_B \leq 1$. For each $(\ell+1)$-chain $A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_{\ell+1}$ of $2^{[n]}$, we add the constraint $x_{A_1} + x_{A_2} + \cdots + x_{A_{\ell+1}} \leq \ell$ to ensure $\mathcal{F}$ does not contain an $(\ell+1)$-chain. As we wish $\mathcal{F}$ to have maximum size, our objective function is simply $\sum_{A \in 2^{[n]}} x_A$. Put compactly, our LP is as follows:

$$\text{Maximize:} \quad \sum_{A \in 2^{[n]}} x_A$$

$$\begin{aligned} \text{Subject to:} \quad &(1) \quad x_A + x_B \leq 1 \quad \forall A, B \in 2^{[n]} : |(A \setminus B) \cup (B \setminus A)| > d \\ &(2) \quad x_{A_1} + x_{A_2} + \cdots + x_{A_{\ell+1}} \leq \ell \quad \forall (\ell+1)\text{-chains } A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_{\ell+1} \text{ of } 2^{[n]} \end{aligned}$$

Note that, like many of the LPs in [Wag19], stating the LP is quite simple however implementing it is a bit more tricky. For instance, Given $2^{[n]}$, it is not entirely clear how to find all of the $(\ell+1)$-chains. In our implementation, we wrote a recursion which, given some super set of size $\geq \ell$, we found strings of subsets totally ordered under inclusion which form an $(\ell+1)$-chain. Observe that as $\emptyset \in 2^{[n]}$, we required our super sets to be of size $\geq \ell$ as opposed to $\geq \ell + 1$.

See `Conjecture.2.py` for an implementation of this LP. For $n = 6, d = 5, \ell = 2$ Conjecture 3.2 gives an upper bound of 21. [Wag19] found the following counterexample for said values of size 26.

$$\mathcal{F} = \binom{[6]}{2} \cup \left(A \in \binom{[6]}{3} : 1 \in A\right) \cup \{2,3,4,5,6\}.$$

Running `Conjecture.2.py` as is obtains the following counterexample for said values of size 26.

$$\mathcal{F} = \binom{[6]}{4} \cup \left\{A \in \binom{[6]}{3} : 4 \in A\right\} \cup \{1,2,3,5,6\}$$

By inspection, this family has diameter ≤ 5. It contains no 3-chain since said chain must contain $\{1,2,3,5,6\}$ and some element of $B \in \left\{A \in \binom{[6]}{3} : 4 \in A\right\}$. As $4 \notin \{1,2,3,5,6\}$, no such chain exists. This counterexample does not appear to generalize in the same way as that of [Wag19]. The generalization of their counterexample corresponds with the following family

$$\mathcal{F} = \binom{[n]}{2} \cup \left\{A \in \binom{[n]}{3} : 1 \in A\right\}$$

which has size $(n-1)^2$. We verified their result that this lower bound holds for $(n, d, \ell) = (7,5,2), (8,5,2)$.

We verified their results of a family of size 98 (the conjecture gives 84 as best possible) for $(n, d, \ell) = (8,7,2)$. Their solution was a star centered at $\{1\}$ in layers 2 and 4 with all sets avoiding $\{1\}$ on layers 3 and 5. Our counterexample is the family

$$\mathcal{F} = \binom{[8]}{5} \cup \left\{A \in \binom{[8]}{4} : 1 \in A\right\} \cup \left\{A \in \binom{[8]}{7} : 1 \notin A\right\}$$

of the same size. The author of [Wag19] proposed a family of size 141 for $(n, d, \ell) = (9,7,2)$ but claimed it was beyond their computational limits to test if it was best possible. We have verified that it is by Cythonizing our code and running it over several days.

3.3. **Diversity of set systems.** Let n and k be positive integers with $n > 2k$. Let $\mathcal{F} \subseteq 2^{[n]}$ be an intersecting family. Define the *maximum degree* of $\mathcal{F}$ as

$$\Delta(\mathcal{F}) = \max_i |\mathcal{F}(i)| \quad \text{where} \quad \mathcal{F}(i) = \{F \in \mathcal{F} : i \in F\}.$$

Define the *diversity* of $\mathcal{F}$ as $\rho(\mathcal{F}) = |\mathcal{F}| - \Delta(\mathcal{F})$. [Wag19] refuted the following three conjectures regarding the maximum diversity of intersecting families.

Conjecture 3.3. *Let $n > 3(k-1)$. If $\mathcal{F} \subseteq \binom{[n]}{k}$ is intersecting then $\rho(\mathcal{F}) \leq \binom{n-3}{k-2}$.*

Conjecture 3.4. *Let $n = 2k+1$. If $\mathcal{F} \subseteq 2^{[n]}$ is intersecting then*

$$\rho(\mathcal{F}) \leq \sum_{i=k+1}^{2k} \binom{2k}{i}.$$

Conjecture 3.5. *Let $n = 2k$. If $\mathcal{F} \subseteq 2^{[n]}$ is intersecting and k is not a power of 2 then*

$$\rho(\mathcal{F}) \leq \frac{1}{2}\binom{2k-1}{k-1} + \sum_{i=k+1}^{2k-1} \binom{2k-1}{i}$$

and if k is a power of 2 then

$$\rho(\mathcal{F}) \leq \frac{1}{2}\left(\binom{2k-1}{k-1} - 1\right) + \sum_{i=k+1}^{2k-1} \binom{2k-1}{i}.$$

These problems can be phrased with the following LP. Note that we describe this LP for $\mathcal{F} \subseteq \binom{[n]}{k}$ but to find counter examples for Conjectures 3.4 and 3.5 it suffices to use this same LP and change every occurrence of $\binom{[n]}{k}$ by $2^{[n]}$.

For a fixed n and k, we wish to find a intersecting family $\mathcal{F}$ of $\binom{[n]}{k}$ where $\rho(\mathcal{F})$ is maximized. For each $A \in \binom{[n]}{k}$, let $x_A \in \{0,1\}$ be an indicator variable giving the truth value of the statement "$A \in \mathcal{F}$." To ensure that $\mathcal{F}$ is intersecting, for each $A, B \in \binom{[n]}{k}$ where $A \cap B = \emptyset$, we add the constraint $x_A + x_B \leq 1$. As

$$\Delta(\mathcal{F}) = \max_i |\mathcal{F}(i)| \quad \text{where} \quad \mathcal{F}(i) = \{F \in \mathcal{F} : i \in F\},$$

there exists some $i \in [n]$ such that $|\mathcal{F}(i)| \geq |\mathcal{F}(j)|$ for all $j \in [n]$. Without loss of generality, we assume that $|\mathcal{F}(1)| \geq |\mathcal{F}(i)|$ for all $i \in [n]$. Therefore, for each $i \in [n] \setminus \{1\}$, we add the constraint

$$\sum_{A \in \binom{[n]}{k} : i \in A} x_A \leq \sum_{B \in \binom{[n]}{k} : 1 \in B} x_B.$$

As we wish to maximize $\rho(\mathcal{F})$, our objective function is simply $\sum_{A \in \binom{[n]}{k} : 1 \notin A} x_A$. Put compactly, our LP is as follows:

$$\begin{aligned}
\text{Maximize:} \quad & \sum_{A \in \binom{[n]}{k} : 1 \notin A} x_A \\
\text{Subject to:} \quad & (1) \quad x_A + x_B \leq 1 \quad \forall A, B \in \binom{[n]}{k} : A \cap B = \emptyset \\
& (2) \quad \sum_{A \in \binom{[n]}{k} : i \in A} x_A \leq \sum_{B \in \binom{[n]}{k} : 1 \in B} x_B \quad \forall i \in [n] \setminus \{1\}
\end{aligned}$$

See `Conjecture_3.3.py`, `Conjecture_3.4`, and `Conjecture_3.5` for implementations of this LP for each conjecture.

For $n = 7, k = 3$ Conjecture 3.3 gives an upper bound of 4. [Wag19] found the following counterexample for said values where $\rho(\mathcal{F}) = 5$.

$$\mathcal{F} = \{235, 236, 246, 345, 456, 124, 125, 135, 136, 156\}.$$

Running `Conjecture_3.3.py` as is obtains the following counterexample for said values where $\rho(\mathcal{F}) = 5$.

$$\mathcal{F} = \{125, 126, 146, 147, 157, 245, 247, 267, 456, 567\}.$$

For $n = 7, k = 3$ Conjecture 3.4 gives an upper bound of 22. [Wag19] found a counterexample with diversity 23. For $n = 9, k = 4$ Conjecture 3.4 gives an upper bound of 93. [Wag19] found a counterexample with diversity 98. Running `Conjecture_3.4.py` as is reproduces these results. We have omitted including the families here for the sake of brevity.

For $n = 10, k = 5$ Conjecture 3.5 gives an upper bound of 193. [Wag19] found a counterexample with diversity 197. Running `Conjecture_3.5.py` as is reproduces this result. We have omitted including the counterexample for the sake of brevity.

3.4. **Multipartite intersecting families.** Let X_1 and X_2 be disjoint sets of size n_1 and n_2, respectively. Denote $\binom{X_1,X_2}{k,\ell}$ as the family of sets $S \subseteq X_1 \cup X_2$ where $|X_1 \cap S| = k$ and $|X_2 \cap S| = \ell$. A family $\mathcal{F} \subseteq \binom{X_1,X_2}{k,\ell}$ is *trivially* intersecting if it is intersecting and $\exists x$ such that $\forall A, B \in \mathcal{F}$, $x \in A \cap B$. Given this definition, a natural question is "what is the maximum size of a non-trivial intersecting family of $\binom{X_1,X_2}{k,\ell}$?" The following conjecture of Katona seeks to put an upper bound on this question

Conjecture 3.6. *If $\mathcal{F} \subseteq \binom{X_1,X_2}{k,\ell}$ is a non-trivial intersecting family then*

$$|\mathcal{F}| \leq \max\left\{\left(1 + \binom{n_1-1}{k-1} - \binom{n_1-k-1}{k-1}\right)\binom{n_2}{\ell},\right.$$
$$\left.\binom{n_1}{k}\left(1 + \binom{n_2-1}{\ell-1} - \binom{n_2-\ell-1}{\ell-1}\right)\right\}.$$

This problem can be phrased with the following LP. For fixed X_1, X_2, k, ℓ we wish to find a non-trivially intersecting family $\mathcal{F}$ of $\binom{X_1,X_2}{k,\ell}$ with maximum cardinality. For each $F \in \binom{X_1,X_2}{k,\ell}$, let x_F be an indicator variable giving the truth value of the statement "$F \in \mathcal{F}$." To ensure that $\mathcal{F}$ is intersecting, for each $F, G \in \binom{X_1,X_2}{k,\ell}$ where $F \cap G = \emptyset$, we add the constraint $x_F + x_G \leq 1$. To ensure that $\mathcal{F}$ is not trivially intersecting, for each $x \in X_1 \cup X_2$ we add the constraint $\sum_{x \notin F} x_F \geq 1$. These constraints ensure that $\mathcal{F}$ is non-trivially intersecting as, for every element $x \in X_1 \cup X_2$ (i.e. any possible fixed point), there $\exists F \in \mathcal{F}$ such that $x \notin \mathcal{F}$. As we wish to maximize $|\mathcal{F}|$, our objective function is simply $\sum_{F \in \binom{X_1,X_2}{k,\ell}} x_F$. Put compactly, our LP is as follows:

$$\text{Maximize:} \quad \sum_{F \in \binom{X_1,X_2}{k,\ell}} x_F$$

$$\text{Subject to:} \quad (1) \quad x_F + x_G \leq 1 \quad \forall F, G \in \binom{X_1, X_2}{k, \ell} : F \cap G = \emptyset$$

$$(2) \quad \sum_{x \notin F} x_F \geq 1 \quad \forall x \in X_1 \cup X_2$$

See `Conjecture_3.6.py` for an implementation of this LP. Let $X_1 = \{x_1, x_2, x_3, x_4, x_5\}$ and $X_2 = \{y_1, y_2, y_3, y_4, y_5\}$. For $n_1 = n_2 = 5, k = \ell = 2$ and X_1, X_2 as defined previously, Conjecture 3.6 gives 30 as an upper bound. Running `Conjecture_3.6.py` as is generates a family $\mathcal{F} \subseteq \binom{X_1, X_2}{k, \ell}$ of size 35 which satisfies the criteria of the conjecture. Said family is given below.

$$\begin{aligned}\mathcal{F} = &\left\{\{x_1, x_5\} \cup F : F \in \binom{X_2}{2}\right\} \cup \left\{\{x_2, x_5\} \cup F : F \in \binom{X_2}{2}\right\}\\ &\cup \left\{\{x_4, x_5\} \cup F : F \in \binom{X_2}{2}, y_5 \in F\right\} \cup \left\{\{x_3, x_5\} \cup F : F \in \binom{X_2}{2}, y_5 \in F\right\}\\ &\cup \{x_4, x_5, y_2, y_4\} \cup \{x_4, x_5, y_2, y_3\} \cup \{x_4, x_5, y_1, y_2\} \cup \{x_3, x_5, y_2, y_4\}\\ &\cup \{x_3, x_5, y_2, y_3\} \cup \{x_3, x_5, y_1, y_2\} \cup \{x_1, x_2, y_2, y_5\}\end{aligned}$$

This counterexample is different than that of [Wag19], however they are both of the same size. Upon inspection we can see that our family is intersecting. At first glance our family might appear to be trivially intersecting since almost all sets contain x_5, however the set $\{x_1, x_2, y_2, y_5\} \in \mathcal{F}$ is the one exception.

A variation of intersecting families are *two-sided* intersecting families. $\mathcal{F}$ is a two-sided intersecting family if $\mathcal{F}$ is intersecting and $\exists F_{11}, F_{12}, F_{21}, F_{22} \in \mathcal{F}$ such that $F_{11} \cap F_{12} \cap X_1 = \emptyset$ and $F_{21} \cap F_{22} \cap X_2 = \emptyset$. Given this variation, Katona also made the following conjecture.

Conjecture 3.7. *If $\mathcal{F} \subseteq \binom{X_1, X_2}{k, \ell}$ is a two-sided intersecting family then*

$$\begin{aligned}|\mathcal{F}| = \max\Bigg\{&\left(\binom{n_2 - 1}{\ell - 1} - \binom{n_2 - \ell - 1}{\ell - 1}\right)\binom{n_1}{k} + 1 + \binom{n_1}{k} - \binom{n_1 - k}{k},\\ &\left(\binom{n_1 - 1}{k - 1} - \binom{n_1 - k - 1}{k - 1}\right)\binom{n_2}{\ell} + 1 + \binom{n_2}{\ell} - \binom{n_2 - \ell}{\ell}\Bigg\}.\end{aligned}$$

This problem can be phrased with the following LP. For fixed X_1, X_2, k, ℓ we wish to find a two-sided intersecting family $\mathcal{F}$ of $\binom{X_1, X_2}{k, \ell}$ with maximum cardinality. For each $F \in \binom{X_1, X_2}{k, \ell}$, let x_F be an indicator variable giving the truth value of the statement "$F \in \mathcal{F}$." To ensure that $\mathcal{F}$ is intersecting, for each $F, G \in \binom{X_1, X_2}{k, \ell}$ where $F \cap G = \emptyset$, we add the constraint $x_F + x_G \leq 1$. To ensure that $\mathcal{F}$ is a two-sided family, we arbitrarily pick two disjoint sets L_1, L_2 of $\binom{X_1}{k}$ and two disjoint sets R_1, R_2 of $\binom{X_2}{\ell}$. Then, for each $S \in \{L_1, L_2, R_1, R_2\}$ we add the constraint $\sum_{S \subseteq F} x_F \geq 1$. These constraints ensure that $\mathcal{F}$ is two-sided since they force the existence of $F_{11}, F_{12}, F_{21}, F_{22} \in \mathcal{F}$ such that

(1) $F_{11} \cap X_1 = L_1$ and $F_{12} \cap X_1 = L_2$ which means $F_{11} \cap F_{12} \cap X_1 = \emptyset$, and
(2) $F_{21} \cap X_2 = R_2$ and $F_{22} \cap X_2 = R_2$ which means $F_{21} \cap F_{22} \cap X_2 = \emptyset$.

As we wish to maximize $|\mathcal{F}|$, our objective function is simply $\sum_{F \in \binom{X_1, X_2}{k, \ell}} x_F$. Put compactly, our LP is as follows:

$$\begin{aligned}&\text{Maximize:} && \sum_{F \in \binom{X_1, X_2}{k, \ell}} x_F\\ &\text{Subject to:} && (1) \quad x_F + x_G \leq 1 \quad \forall F, G \in \binom{X_1, X_2}{k, \ell} : F \cap G = \emptyset\\ & && (2) \quad \sum_{S \subseteq F} x_F \geq 1 \quad \forall S \in \{L_1, L_2, R_1, R_2\}\end{aligned}$$

See `Conjecture_3.7.py` for an implementation of this LP. Let $X_1 = \{x_1, x_2, x_3, x_4, x_5\}$ and $X_2 = \{y_1, y_2, y_3, y_4, y_5\}$. Additionally, let

$$L_1 = \{x_1, x_2\}, \quad L_2 = \{x_3, x_4\}, \quad R_1 = \{y_2, y_3\}, \quad \text{and} \quad R_2 = \{y_4, y_5\}.$$

For $n_1 = n_2 = 5, k = \ell = 2$ and $X_1, X_2, L_1, L_2, R_1, R_2$ as defined above, Conjecture 3.7 gives 28 as an upper bound. Running `Conjecture_3.7.py` as is generates a family $\mathcal{F} \subseteq \binom{X_1, X_2}{k, \ell}$ of size 35 which satisfies the criteria of the conjecture. Said family is given below.

$$\begin{aligned}\mathcal{F} = &\left\{\{x_1, x_2\} \cup F : F \in \binom{X_2}{2}, F \cap \{y_1, y_2\} \neq \emptyset\right\} \cup \left\{\{x_1, x_4\} \cup F : F \in \binom{X_2}{2}\right\} \\ &\cup \left\{\{x_1, x_5\} \cup F : F \in \binom{X_2}{2}, F \cap \{y_1, y_2\} \neq \emptyset\right\} \cup \left\{\{x_1, x_3\} \cup F : F \in \binom{X_2}{2}\right\} \\ &\cup \{x_3, x_4, y_1, y_2\}.\end{aligned}$$

This counterexample is different than that of [Wag19], however they are both of the same size. Too see that the family is two-sided, observe that

$$\{x_1, x_2, y_1, y_2\}, \{x_3, x_4, y_1, y_2\} \in \mathcal{F} \quad \text{and} \quad \{x_1, x_2, y_1, y_2\} \cap \{x_3, x_4, y_1, y_2\} \cap X_1 = \emptyset$$

and

$$\{x_1, x_4, y_4, y_5\}, \{x_3, x_4, y_1, y_2\} \in \mathcal{F} \quad \text{and} \quad \{x_1, x_4, y_4, y_5\} \cap \{x_3, x_4, y_1, y_2\} \cap X_2 = \emptyset.$$

The author of [Wag19] used their counterexample for Conjecture 3.6 and 3.7 to derive a generalized two-sided intersecting family $\mathcal{F}$ of $\binom{X_1, X_2}{2,2}$ where $X_1 = \{x_1, x_2, \ldots, x_m\}$ and $X_2 = \{y_1, y_2, \ldots y_m\}$. Note that this family, being two-sided, is also non-trivially intersecting. This family has size $\geq 3m^2 - 10m + 10$. They also established an even more general two-sided intersecting family $\mathcal{F} \subseteq \binom{X_1, X_2}{k,k}$. We omit this result but encourage the reader to see Proposition 3.11 in [Wag19] for the complete result; it is a fantastic use of LPs to derive generalized solutions.

3.5. **EKR property.** A family *has the EKR property* if its largest intersecting subfamily is trivially intersecting. Frankl-Han-Huang-Zhoa made the following conjecture regarding the EKR property.

Conjecture 3.8. *Suppose $n = n_1 + \cdots + n_d$ and $k \geq k_1 + \cdots + k_d$ where $n_i > k_i \geq 0$ are integers for $1 \leq i \leq d$. Let $X_1 \cup \cdots \cup X_d$ be a partition of $[n]$ with $|X_i| = n_i$ for all i and let*

$$\mathcal{H} = \left\{F \subseteq \binom{[n]}{k} : |F \cap X_i| \geq k_i \quad \forall 1 \leq i \leq d\right\}.$$

If $n_i \geq 2k_i$ for all i and $n_i > k - \sum_{j=1}^{d} k_j + k_i$ for all but at most one $i \in [d]$ such that $k_i > 0$, then $\mathcal{H}$ has the EKR property.

This problem can be phrased with the following LP. For a fixed $n = n_1 + \cdots + n_d$ and $k \geq k_1 + \cdots + k_d$ where $n_i > k_i \geq 0$ with some partition $X_1 \cup \cdots \cup X_d$ of $[n]$ where $|X_i| = n_i$ for all i, we wish to find the maximum size of an intersecting family $\mathcal{F}$ of $\mathcal{H}$ (where $\mathcal{H}$ is defined as above). This is a mouthful to say but the LP is actually quite straightforward. For each $F \in \mathcal{H}$ let x_F be an indicator variable giving the truth value of the statement "$F \in \mathcal{F}$." To ensure that $\mathcal{F}$ is intersecting, for each $F, G \in \mathcal{H}$ where $F \cap G = \emptyset$, we add the constraint

$x_F + x_G \leq 1$. As we wish to maximize $|\mathcal{F}|$, our objective function is simply $\sum_{F\in\mathcal{H}} x_F$. Put compactly, our LP is as follows:

$$\begin{aligned} \text{Maximize:} \quad & \sum_{F\in\mathcal{H}} x_F \\ \text{Subject to:} \quad & x_F + x_G \leq 1 \quad \forall F, G \in \mathcal{H} : F \cap G = \emptyset \end{aligned}$$

See `Conjecture_3.8.py` for an implementation of this LP. Observe that

$$d = 2, \quad n_1 = n_2 = 4, \quad , k_1 = k_2 = 1, \quad \text{and} \quad k = 4$$

and

$$\mathcal{H} = \left\{ F \subseteq \binom{[8]}{4} : |F \cap \{1,2,3,4\}| \geq 2, |F \cap \{5,6,7,8\}| \geq 1 \right\}$$

satisfy the requirements of the conjecture. The largest trivially intersecting family in $\mathcal{H}$ has size 30 [Wag19]. Running `Conjecture_3.8` as is produces the following intersecting family of order 34

$$\begin{aligned} \mathcal{F} = \{&1235, 1236, 1237, 1238, 1245, 1246, 1247, 1248, 1256, 1267, 1268, \\ &1278, 1345, 1346, 1347, 1348, 1358, 1368, 1378, 1467, 1468, 2345, 2346, \\ &2347, 2348, 2356, 2367, 2368, 2378, 2458, 2468, 2478, 3467, 3468\}. \end{aligned}$$

Amusingly, this is actually the first counterexample that we found which matches exactly with [Wag19]. I would also like to note that this is, by far, the most notationally specific conjecture in this paper and yet it has the simplest LP and one of the smallest counterexamples.

3.6. **s-subset-regular set systems.** A family $\mathcal{F} \subseteq 2^{[n]}$ is *s-subset-regular* if every set of size s lies in the same number of elements in $\mathcal{F}$. The following theorem by Ihringer and Kupavskii gives an upper bound on the cardinality of s-subset-regular k-uniform intersecting families.

Theorem 3.9. *Fix $s \geq 1$. A s-subset-regular k-uniform intersecting family $\{ \subseteq 2^{[n]}$ satisfies*

$$|\mathcal{F}| \leq \frac{\binom{n}{k}}{1 + \frac{\binom{n-k}{k}}{\binom{n-k-s-2}{k-s-2}}}.$$

The theorem is known to be tight for $(n, k, s) = (7, 3, 1)$ and $(9, 4, 1)$ [Wag19]. Ihringer and Kupavskii asked whether there exists other parameters for which the theorem is tight. [Wag19] showed the existance of such a family for the parameters $(11, 5, 3)$.

This problem can be phrased with the following LP. Fix n, k, s. We wish to find a s-subset-regular k-uniform intersecting family $\mathcal{F} \subseteq 2^{[n]}$ with maximum cardinality. For each $F \in \binom{[n]}{k}$ let x_F be an indicator variable giving the truth value of the statement "$F \in \mathcal{F}$." This ensure that $\mathcal{F}$ is k-regular. To ensure that $\mathcal{F}$ is intersecting, for each $F, G \in \binom{[n]}{k}$ where $F \cap G = \emptyset$, we add the constraint $x_F + x_G \leq 1$. To ensure that $\mathcal{F}$ is s-subset-regular, for each $S \subseteq 2^{[n]}$ where $|S| = s$ we add the following constraint

$$\sum_{[s]\subseteq F\in\binom{[n]}{k}} x_F = \sum_{S\subseteq G\in\binom{[n]}{k}} x_G.$$

Note that here we use the fact that if $\mathcal{F}$ is s-subset-regular then, for each $S \subseteq 2^{[n]}$ where $|S| = s$, the number of sets $F \in \mathcal{F}$ which contain S is, without loss of generality, equal to the number of sets $G \in \mathcal{F}$ which contain $[s]$. As we wish to maximize $|\mathcal{F}|$, our objective function is simply $\sum_{F\in\mathcal{H}} x_F$. Put compactly, our LP is as follows:

$$\begin{aligned} \text{Maximize:} \quad & \sum_{F\in\mathcal{H}} x_F \\ \text{Subject to:} \quad & (1) \quad x_F + x_G \le 1 \quad \forall F, G \in \mathcal{H} : F \cap G = \emptyset \\ & (2) \quad \sum_{[s]\subseteq F \in \binom{[n]}{k}} x_F = \sum_{S\subseteq G\in\binom{[n]}{k}} x_G \quad \forall S \subseteq 2^{[n]} : |S| = s \end{aligned}$$

See `Theorem_3.9.py` for an implementation of this LP. Running `Theorem_3.9.py` as is finds a family $\mathcal{F}$ of size 66 which satisfies the criteria of the conjecture for $(11, 5, 3)$. Below is said family.

$$\begin{aligned} \mathcal{F} = \ & \{1,2,3,4,9\}, \{1,2,3,5,10\}, \{1,2,3,6,11\}, \{1,2,3,7,8\}, \{1,2,4,5,8\}, \{1,2,4,6,10\}, \\ & \{1,2,4,7,11\}, \{1,2,5,6,7\}, \{1,2,5,9,11\}, \{1,2,6,8,9\}, \{1,2,7,9,10\}, \{1,2,8,10,11\}, \\ & \{1,3,4,5,6\}, \{1,3,4,7,10\}, \{1,3,4,8,11\}, \{1,3,5,7,11\}, \{1,3,5,8,9\}, \{1,3,6,7,9\}, \\ & \{1,3,6,8,10\}, \{1,3,9,10,11\}, \{1,4,5,7,9\}, \{1,4,5,10,11\}, \{1,4,6,7,8\}, \{1,4,6,9,11\}, \\ & \{1,4,8,9,10\}, \{1,5,6,8,11\}, \{1,5,6,9,10\}, \{1,5,7,8,10\}, \{1,6,7,10,11\}, \{1,7,8,9,11\}, \\ & \{2,3,4,5,7\}, \{2,3,4,6,8\}, \{2,3,4,10,11\}, \{2,3,5,6,9\}, \{2,3,5,8,11\}, \{2,3,6,7,10\}, \\ & \{2,3,7,9,11\}, \{2,3,8,9,10\}, \{2,4,5,6,11\}, \{2,4,5,9,10\}, \{2,4,6,7,9\}, \{2,4,7,8,10\}, \\ & \{2,4,8,9,11\}, \{2,5,6,8,10\}, \{2,5,7,8,9\}, \{2,5,7,10,11\}, \{2,6,7,8,11\}, \{2,6,9,10,11\}, \\ & \{3,4,5,8,10\}, \{3,4,5,9,11\}, \{3,4,6,7,11\}, \{3,4,6,9,10\}, \{3,4,7,8,9\}, \{3,5,6,7,8\}, \\ & \{3,5,6,10,11\}, \{3,5,7,9,10\}, \{3,6,8,9,11\}, \{3,7,8,10,11\}, \{4,5,6,7,10\}, \{4,5,6,8,9\}, \\ & \{4,5,7,8,11\}, \{4,6,8,10,11\}, \{4,7,9,10,11\}, \{5,6,7,9,11\}, \{5,8,9,10,11\}, \{6,7,8,9,10\} \end{aligned}$$

Note that this example is different than that of [Wag19] but both are of equal size and both provide an affirmative answer to the question posed by Ihringer and Kupavskii.

3.7. **Matching problems.** Let $s \ge 3$ be an integer. Define $k(n, s)$ as the maximum size of a family $\mathcal{F} \subseteq 2^{[n]}$ without s pairwise disjoint members. Frankl and Tokushige made the following conjecture regarding $k(n, s)$.

Conjecture 3.10. *Let $s \ge 4$. If $n \equiv 1 \pmod{s}$ then*

$$k(n, s) = 4k(n - 2, s).$$

It is known that $k(7, 4) = 120$ [Wag19]. To disprove Conjecture 3.10, it therefore suffices to show that

$$k(9, 4) \ge 481 > 480 = 4k(7, 4).$$

This problem can be phrased with the following LP. We wish to find a family $\mathcal{F} \subseteq 2^{[9]}$ with maximum cardinality without s pairwise disjoint members. For each $F \in 2^{[n]}$ let x_F be an indicator variable giving the truth value of the statement "$F \in \mathcal{F}$." To ensure that $\mathcal{F}$ does not have 4 pairwise disjoint sets, for each quadruple of pairwise disjoint sets $A, B, C, D \in 2^{[n]}$ we add the constraint $x_A + x_B + x_C + x_D \le 3$. As we wish to maximize the cardinality of $\mathcal{F}$, our objective function is simply $\sum_{F\in 2^{[n]}} x_F$. Put compactly, our LP is as follows:

$$\text{Maximize:} \quad \sum_{F\in 2^{[n]}} x_F$$

$$\text{Subject to:} \quad x_A + x_B + x_C + x_D \leq 3 \quad \forall \text{ pairwise disjoint sets } A, B, C, D \in 2^{[n]}$$

Note that the author of [Wag19] added the following heuristics in an effort to speed up their LP. For each $F \in 2^{[n]}$ where $|F| \geq 4$ they added the constraint $x_F = 1$. For each $F \in 2^{[n]}$ where $|F| \leq 1$ they added the constraint $x_F = 0$. They justified these heuristics as it seems reasonable that a family $\mathcal{F}$ which satisfies Conjecture 3.10's criteria will be heavily biased towards large sets. We did not include these assumptions in our LP as we were curious whether or not we could find a larger counterexample than that of [Wag19]. See `Conjecture_3.10.py` for an implementation of this LP. For $(n, s) = (9, 4)$, the search in [Wag19] took under two seconds. Our search for the same values but without the heuristics speedup took several hours to run. Interestingly, both searches yielded the same counterexample of size 481. Said counterexample is given by the following family.

$$\mathcal{F} = \left\{F \in 2^{[9]} : |F| \geq 4\right\} \cup \binom{[9]}{3} \cup \left\{G \in \binom{[9]}{2} : |G \cap [2]| \geq 1\right\}.$$

3.8. **Turán-type problem.** Recall that, for graphs G and H, $\mathrm{ex}(G, H)$ denotes the maximum number of edges in a subgraph of G that contains no copy of H. De Silva et al. considered $\mathrm{ex}(G, H)$ were H is k disjoint K_r's and G is a complete multipartite graph [Wag19]. The following is known regarding such extremal number.

Theorem 3.11. *For any integers $k \leq n_1 \leq n_2 \leq \cdots \leq n_r$*

$$\mathrm{ex}(K_{n_1,\ldots,n_r}, kK_r) = \left(\sum_{1\leq i<j\leq r} n_i n_j\right) - n_1 n_2 + n_2(k-1).$$

De Silva et al. found, via construction, that

$$\mathrm{ex}\left(K_{n_1,n_2,n_3,n_4}, kK_3\right) \geq (n_1 + n_2 + n_3)n_4 + (k-1)n_3$$

and wondered whether this was the optimal construction [Wag19]. The author of [Wag19] showed that this is not the case by, as we should be used to by now, writing an LP and using it to find a better generalized construction.

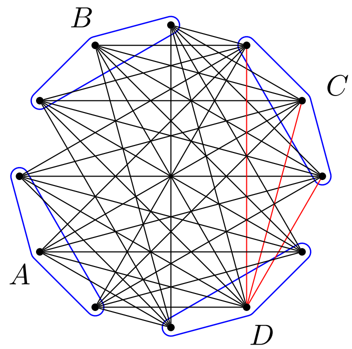
This problem can be phrased with the following LP. Fix some n_1, n_2, n_3, n_4, and k. We wish to find the subgraph G of K_{n_1,n_2,n_3,n_4} with the maximum number of edges such that G contains no kK_3. For each edge $e \in E(K_{n_1,n_2,n_3,n_4})$ let x_e be an indicator variable giving the truth value of the statement "$e \in E(G)$." To ensure that G contains no kK_3, for every collection of $3k$ edges $e_1, e_2, \ldots, e_{3k}$ forming a kK_3 we add the constraint

$$\sum_{i=1}^{3k} e_i \leq 3k - 1.$$

As we wish to maximize $|E(G)|$, our objective function is simply $\sum_{e\in E(K_{n_1,n_2,n_3,n_4})} x_e$.

See `Theorem_3.11.py` for an implementation of this LP. Note that `Theorem_3.11.py` works for arbitrary sized complete partite graph with parts of size at least 2. Below is an output of `Theorem_3.11.py` for $n_1 = n_2 = n_3 = n_4 = 3$ and $k = 2$. The four partite sets of size three are indicated by the blue. Observe that all the edges between A and B have been removed.

The same is true for C and D with the exception of a $K_{3,1}$ from C to D, as indicated by the red edges.



The construction seen above matches that of [Wag19]. This construction is generalized in [Wag19] as follows. Let A, B, C, D be the parts of the partition of size n. Remove all edges from A to B and from C to D. Add a copy of $K_{k-1,n}$ from C to D. This construction establishes that

$$\mathrm{ex}\,(k_{n,n,n,n}, kk_3) \geq 4n^2 + (k-1)n.$$

The code of `Theorem_3.11.py` is generalized to run this LP for $K_{n_1,n_2,\ldots,n_m}$ and kK_3 for arbitrary $n_1, n_2, \ldots, n_m > 2$ and k. It would be very easy to alter this code to make it work for $K_{n_1,n_2,\ldots,n_m}$ and kK_r for arbitrary $n_1, n_2, \ldots, n_m > 2$, k, and r. I had hoped to do this for this project. I think this method would have alot of promise to get a large sampling of known optimal solutions, from which a general bound for

$$\mathrm{ex}\,(k_{n_1,n_2,\ldots,n_m}, kK_r)$$

could be derived. I unfortunately could not justify the time to do so for this project with my thesis defense looming and still having more writing to do. That said, I would be happy to return to exploring this at a later date.

Conclusion

We hope the reader is now convinced (assuming they were not from the start) that linear programs have a breadth of application to research in extremal combinatorics. On a more personal note, this project has been validating for myself as a researcher. I have spent the last year studying 2-limited broadcast domination on grid like graphs. Many of my results were derived by using LPs to find a sampling of known optimal solutions from which I generalized their solutions. I know that this is “good” research but I often felt that this methodology was in some way inferior to non-computational methods. It was refreshing to read a paper which advocated for the application of LPs to combinatorics. This paper altered how I felt about my own research. Thanks for adding this paper to the pool of possible project topics Jon, I learned a lot.

References

[GO21] LLC Gurobi Optimization, *Gurobi optimizer reference manual*, 2021.

[Wag19] Adam Zsolt Wagner, *Refuting conjectures in extremal combinatorics via linear programming*, 2019.