Differential equations

$\boxed{\text{Differential equation}}$ = equation that relates one or more function(s) and their derivatives

→ Describe relationships between the magnitude of a quantity and the rate of change of that quantity

Types of differential equations

$\boxed{\text{Ordinary differential equation (ODE)}}$ = equation that relates a function of one variable $f(x)$ and its derivative $\frac{df}{dx}$

Ex. Newton's second law of motion

$\mathrm{F(x(t),v(t),t)}$ = force….function of position x, velocity v, and time t

Force = mass m · acceleration a

$$F(x, v, t) = m\frac{dv}{dt}$$

Differential equations

$\boxed{\text{Differential equation}}$ = equation that relates one or more function(s) and their derivatives

→ Describe relationships between the magnitude of a quantity and the rate of change of that quantity

<u>Types of differential equations</u>

$\boxed{\text{Ordinary differential equation (ODE)}}$ = equation that relates a function of one variable $f(x)$ and its derivative $\frac{df}{dx}$

$\boxed{\text{Partial differential equation (PDE)}}$ = equation that relates a multivariate function $f(x, y, \ldots)$ and its partial derivatives $\frac{\partial f(x,y,\ldots)}{\partial x}, \frac{\partial f(x,y,\ldots)}{\partial y}, \ldots$

<u>Ex.</u> Heat equation

$\mathrm{u}(\mathrm{x}, \mathrm{y}, \mathrm{z}, \mathrm{t})$ = temperature at a point in space & time α = thermal diffusivity [$L^2 T^{-1}$]

$\frac{\partial u}{\partial t} = \alpha\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$ Newton's notation: $\dot{u} = \alpha\nabla^2 u$ (∇^2 = Laplace operator)

Properties of differential equations

A linear, first-order ODE of the function $\mathrm{x(t)}$ is one that can be put in the **standard form**:

$$\underbrace{\frac{dx}{dt} + p(t)x}_{\text{System}} = \underbrace{q(t)}_{\text{Input signal}}$$

→ A solution $\mathrm{x(t)}$ is a **system response** or **output signal**

The equation is **homogenous** if q is the null signal, $\mathrm{q(t) = 0}$. This corresponds to letting the system evolve in isolation:

$$\frac{dx}{dt} + p(t)x = 0$$

Properties of differential equations

The linear, first-order, homogenous ODE is **separable**, so can be solved analytically:

$$\frac{dx}{dt} + p(t)x = 0$$

	General case: $\frac{dx}{dt} + p(t)x = 0$	Example: $\frac{dx}{dt} + 2tx = 0$
Separate:	$\frac{dx}{x} = -p(t)\,dt$	$\frac{dx}{x} = -2t\,dt$
Integrate:	$\ln\lvert x\rvert = -\int p(t)\,dt + c$	$\ln\lvert x\rvert = -t^2 + c$
Exponentiate:	$\lvert x\rvert = e^c\, e^{-\int p(t)\,dt}$ $x = C\, e^{-\int p(t)dt}$	$\lvert x\rvert = e^c\, e^{-t^2}$ $x = C\, e^{-t^2}$

Properties of differential equations

$\boxed{\text{Linearity}}$ = linear in the unknown function and its derivative (exponent on derivative = 1)

Ex. Steady-state equation for bedrock river profile evolution by the stream power law (more on this later)

$$0 = U - KA^m \left|\frac{dz}{dx}\right|^n$$

Linear: n = 1, Non-linear: n ≠ 1

$\boxed{\text{Order}}$ = the degree of the derivative

Ex. First-order differential equation → stream power law for bedrock river incision: $\frac{\partial z}{\partial t} = -KA^m \left|\frac{\partial z}{\partial x}\right|^n$

Second-order differential equation → hillslope diffusion equation: $\frac{\partial z}{\partial t} = K_d \frac{\partial^2 z}{\partial x^2}$

Types of partial differential equations

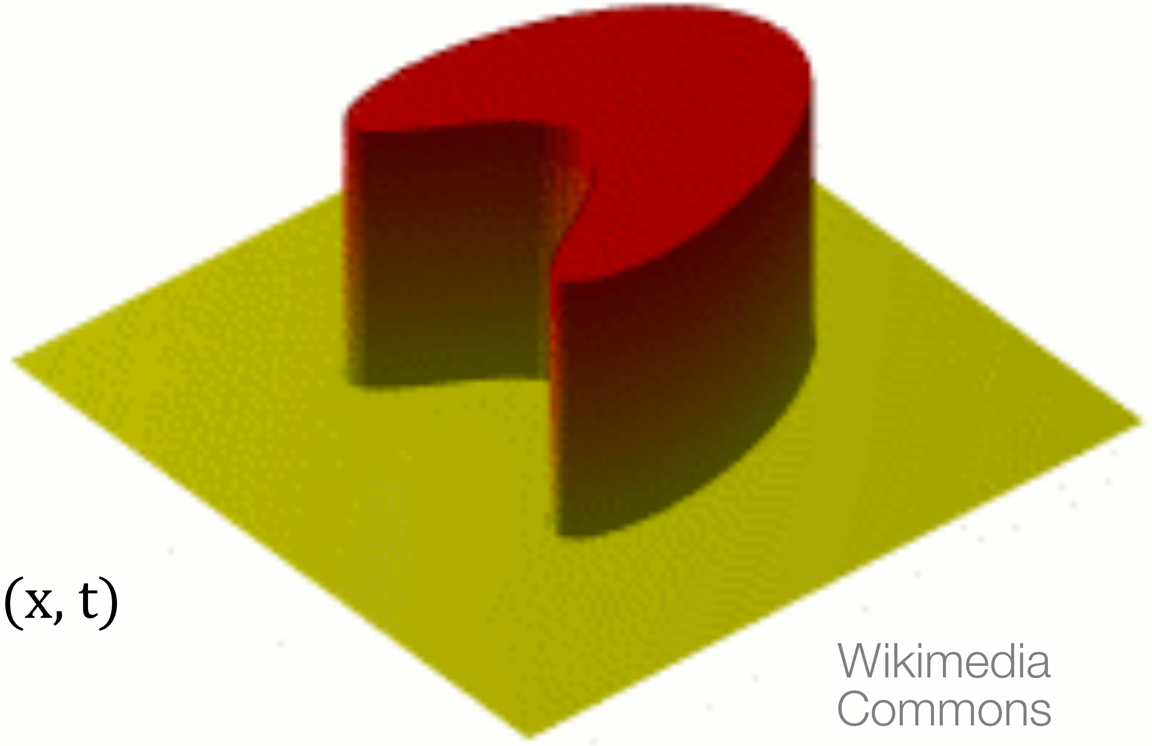
$\boxed{\text{Parabolic}}$ [e.g. heat conduction, particle diffusion] – In simplest form, a second-order, linear, constant-coefficient parabolic PDE of the function $\mathrm{u}(\mathrm{x}, \mathrm{t})$ takes the form:

$$\mathrm{A}\frac{\partial^2 \mathrm{u}}{\partial x^2} + 2B\frac{\partial^2 u}{\partial x \partial t} + C\frac{\partial^2 u}{\partial t^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial t} + F = 0$$

with coefficients satisfying $B^2 - AC = 0$

Ex. Hillslope diffusion equation: $\frac{\partial \mathrm{z}}{\partial \mathrm{t}} = \mathrm{K_d}\frac{\partial^2 \mathrm{z}}{\partial \mathrm{x}^2}$

$A = K_d$
$E = -1$
$B = C = D = F = 0$



→ Linear parabolic PDEs generally have solutions $\mathrm{u}(\mathrm{x}, \mathrm{t})$ that get progressively smoother over time

Wikimedia Commons

Types of partial differential equations

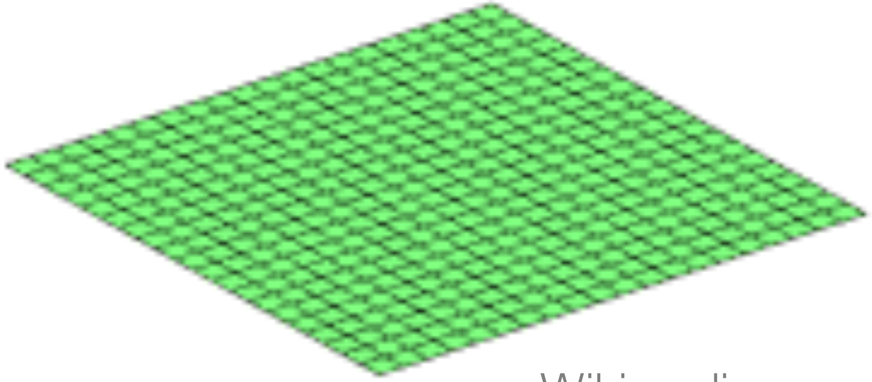
$\boxed{\text{Hyperbolic}}$ [e.g. wave equation, shallow water equations] – Solutions are "wave-like"

→ Disturbances travel at finite **propagation speed** along **characteristics** (curves/surfaces along which a PDE becomes an ODE) of the equations…not every point in space experiences a disturbance at once

A second-order, hyperbolic PDE of the function $\mathrm{u(x, t)}$ takes the form:

$$A\frac{\partial^2 u}{\partial x^2} + 2B\frac{\partial^2 u}{\partial x \partial t} + C\frac{\partial^2 u}{\partial t^2} + (\text{lower order derivative terms}) = 0$$

with coefficients satisfying $\mathrm{B^2 - AC > 0}$



Wikimedia Commons

The canonical hyperbolic equation is the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

where c is the propagation speed at which a perturbation travels (A = 1, B = 0, C = -1)

Types of partial differential equations

$\boxed{\text{Elliptic}}$ [e.g. Laplace equation, Poisson equation] – solutions do not have discontinuous derivatives anywhere, so well suited to describe **equilibrium states** and **static** (rather than dynamic) processes

Ex. Laplace's equation to describe a steady state of the heat equation

→ We will not examine elliptic PDEs in this course, but we'll see that understanding the general behavior of (in our case, parabolic and hyperbolic) PDEs is essential for selecting appropriate numerical methods to solve these equations