**Indian Institute of Technology, Kanpur**
**Probability Theory, 18–19**

SCRIBE

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# Transformation of Random Variables

## Discrete Random Variables

Suppose we have a discrete random variable $X$, we need to find a random variable $Y$ such that $Y = h(X)$, where $h$ is a function $h : \mathbb{R} \to \mathbb{R}$.

**Theorem 4.1.** Consider two random variables X and Y such that $Y = h(X)$. We can say that the set $h(S_X) = \left\{ h(x) \,\middle|\, x \in S_X \right\}$ is the support for the random variable $Y$ *i.e.* $S_Y = h(S_X)$.

*Proof.* Let $y \in h(S_X)$ *i.e.* there is some $x^*\ast$ such that $h(x^*) = y$, then

$$\begin{aligned} \mathbb{P}\left[\, Y = y \,\right] &= \mathbb{P}\left[\, h(X) = y \,\right] \\ &= \mathbb{P}\left[\, X = h^{-1}(y) \,\right] \\ &> 0 \end{aligned}$$

Hence, for every $y \in h(S_X)$, the probability $\mathbb{P}\left[\, Y = y \,\right]$ is non-zero.

$$\begin{aligned} \mathbb{P}\left[\, Y \in h(S_X) \,\right] &= \mathbb{P}\left[\, h(X) \in h(S_X) \,\right] \\ &= \mathbb{P}\left[\, X \in S_X \,\right] \quad = \quad 1 \end{aligned}$$

Since for every $y \in h(S_X)$, the probability $\mathbb{P}\left[\, Y = y \,\right]$ is non-zero and the probability $\mathbb{P}\left[\, Y \in h(S_X) \,\right] = 1$, we can say that $h(S_X)$ is the support for the random variable $Y$. ☐

For $y \in S_Y = h(S_X)$,

$$\begin{aligned} \mathbb{P}\left[\, Y = y \,\right] &= \mathbb{P}\left[\, h(X) = y \,\right] \\ &= \mathbb{P}\left[\, X \in h^{-1}(y) \,\right] \\ &= \sum_{x \in h^{-1}(y)} f_X(x) \end{aligned}$$

Hence, we can finally write the pmf of the random variable $Y$ where $Y = h(X)$ with support $S_Y = h(S_X) = \left\{ h(x) \,\middle|\, x \in S_X \right\}$

$$f_Y(y) \quad = \quad \begin{cases} \sum_{x \in h^{-1}(y)} f_X(x) & \text{if } y \in S_Y \\ 0 & \text{else} \end{cases}$$

## Absolutely Continuous Variable

Consider a random variable $X$ and a random variable $Y = h(X)$. As discussed in the previous section, if $X$ is discrete, $Y$ is also discrete. However, this is not true for an absolutely continuous variable. Consider the following example.

**Example 4.1.** Let $X$ be an absolutely continuous r.v. with pdf

$$f_X(x) \quad = \quad \begin{cases} \frac{1}{2} & \text{if } -1 < x < 1 \\ 0 & \text{else} \end{cases}$$

Let $Y = \lfloor X \rfloor$. Then, we can write $Y$ as

$$Y = \begin{cases} -1 & \text{if } -1 < X < 0 \\ 0 & \text{if } 0 \le X < 1 \end{cases}$$

Therefore, we can say

$$\begin{aligned} \mathbb{P}[\,Y = -1\,] &= \mathbb{P}[\,-1 < X < 0\,] \\ &= \int_{-1}^{0} f_x(x)dx \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \mathbb{P}[\,Y = 0\,] &= \mathbb{P}[\,0 \le X < 1\,] \\ &= \int_{0}^{1} f_x(x)dx \\ &= \frac{1}{2} \end{aligned}$$

Clearly, $Y$ is discrete with support $S_Y = \{-1, 0\}$, and pmf

$$f_Y(y) = \begin{cases} \frac{1}{2} & \text{if } y \in -1, 0 \\ 0 & \text{else} \end{cases}$$

We now give a formula to transform Absolutely Continuous Random variables (say $X$ and $Y$) such that $Y = h(X)$, for which the support is a collection of disjoint intervals $S_X = \bigcup_n S_{n,X}$, such that for each set $S_{n,X}$, $h_n : S_{n,x} \longrightarrow \mathbb{R}$ is strictly monotone with inverse function $h^{-1}(y)$ with $\frac{d}{dy} h^{-1}(y)$ is continuous.

Let $h(S_{n,X} = \{h(x) \mid x \in S_{n,X}\}$, then the pdf for $Y$ $(f_Y)$ is written as

$$f_Y(y) \quad = \quad \sum_n f_Y\left(h_n{}^{-1}(y)\right) \left| \frac{d}{dy} h_n{}^{-1}(y) \right| \mathbb{I}[\,y \in h(S_{n,X})\,]$$

**Exercise 4.1.** Let X be an absolutely continuous r.v. with pdf

$$f_X(x) \quad = \quad \begin{cases} \frac{|x|}{2} & \text{if } -1 < x < 1 \\ \frac{x}{3} & \text{if } 1 < x < 2 \\ 0 & \text{else} \end{cases}$$

Let $Y = X^2$, then

(i) find the pdf of $Y$ and hence the CDF of $Y$

(ii) find the CDF of $Y$ and hence the pdf of $Y$