Indian Institute of Technology, Kanpur
Probability Theory, 18–19

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SCRIBE

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Introduction to Probability Theory

Introduction

The term probability is related to the degree of certainty about a particular event of interest. We use the Kolmogorov Definition of Probability. In order to completely define Probability, we need to have knowledge of a few other terms.

Definition 1.1 (Sample Space)**.** The set of all possible outcomes on occurence of an event is known as the Sample Space. It is denoted by Ω

Definition 1.2 (Event Space)**.** The collection of all events, or subset of possible outcomes (Ω) is known as the event space. An event space is also known as a σ-field or a σ-algebra (denoted by $\mathcal{F}$) if it follows the following conditions

1. (Surety of an event) $\Omega \in \mathcal{F}$
2. (Closure under complementation) For any event A, $A \in \mathcal{F} \iff \overline{A} \in \mathcal{F}$
3. (Closure under union) For any events A_1, A_2, $A_1, A_2 \in \mathcal{F} \implies A_1 \cup A_2 \in \mathcal{F}$

Note. Closure under intersection is implied as $A_0 \cap A_2 = \overline{\overline{A_1} \cup \overline{A_2}}$

Probability Function

We can now formally define Probaility or Probability Measure

Definition 1.3 (Probability Measure)**.** Probability Measure defines a function on a σ-field — $\mathbb{P} : \mathcal{F} \longrightarrow [\,0, 1\,]$ which satisfies the following conditions

1. $\mathbb{P}\,[\,\Omega\,] = 1$
2. If $\{A_n\}$ is a set of pairwise disjoint events $\in \mathcal{F}$, then

$$\mathbb{P}\left[\bigcup_n A_n\right] = \sum_n \mathbb{P}\,[\,A_n\,]$$

Result 1.3.1 (Inclusion-Exclusion Principle)**.** For events $\{E_n\}_{n\in[\,N\,]}$, we can say

$$\begin{aligned}\mathbb{P}\left[\bigcup_{n\in[\,N\,]} E_n\right] &= p_1 - p_2 \cdots + (-1)^{n-1} p_n \\ \text{where } p_r &= \sum_{1\le i_1 < \cdots < i_r \le n} \mathbb{P}\,[\,E_{i_1} \cap E_{i_2} \cdots \cap E_{i_r}\,]\end{aligned}$$

Result 1.3.2. For events $\{E_n\}_{n\in[\,N\,]}$,

(i)

$$\mathbb{P}\left[\bigcup_{n\in[\,N\,]} E_n\right] \quad \leq \quad \sum_{n\in[\,N\,]} \mathbb{P}\left[\,E_n\,\right] \qquad \text{(Boole's Inequality)}$$

(ii)

$$\mathbb{P}\left[\bigcap_{n\in[\,N\,]} E_n\right] \quad \geq \quad \sum_{n\in[\,N\,]} \mathbb{P}\left[\,E_n\,\right] - (n-1) \qquad \text{(Bonferroni's Inequality)}$$

Definition 1.4 (Probability Space)**.** The triplet $(\Omega, \mathcal{F}, \mathbb{P})$ of a set of outcomes, a valid σ-field and a valid probability measure constitutes a probability space.

Definition 1.5. Let $\{E_n\}_{n\geq 1}$ be a sequence of events. Then the sequence is said to be

(i) $(E_n \uparrow)$ *increasing* if $E_n \subset E_{n+1}$, $n = 1, 2 \ldots$. In this case, $\lim_{n\to\infty} E_n = \bigcup_{n\geq 1} E_n$

(ii) $(E_n \downarrow)$ *decreasing* if $E_{n+1} \subset E_n$, $n = 1, 2 \ldots$. In this case, $\lim_{n\to\infty} E_n = \bigcap_{n\geq 1} E_n$

The sequence is said to be a *monotone* if either $E_n \uparrow$ or $E_n \downarrow$

Result 1.5.1. Let $\{E_n\}_{n\geq 1}$ be a monotone sequence of events, then

$$\mathbb{P}\left[\lim_{n\to\infty} E_n\right] \quad = \quad \lim_{n\to\infty} \mathbb{P}\left[\,E_n\,\right]$$

Exercise 1.1. Prove Result 1.5.1

Definition 1.6. For a collection of events $\{E_n\}_{n\in[\,N\,]}$,

(i) the events are said to be mututally (or pairwise) exclusive if $E_i \cap E_j = \phi \;\forall\, i \neq j$

(ii) the collection is said to be exhaustive if $\mathbb{P}\left[\bigcup_{n\in[\,N\,]} E_n\right] = 1$