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Conditional Probability and Independence

1. Conditional Probability

Conditional Probability simply defines the probability of an event A , given that another event B has already occurred. This is represented as $A \mid B$. We can formally define condition probability as follows.

Definition 2.1. Let A and B be two events. The conditional probability of A given B is defined by

$$\mathbb{P}[A \mid B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}$$

Note. Conditional Probability $A \mid B$ is defined only if $\mathbb{P}[B] \neq 0$

Remark. For a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $A \in \mathcal{F}$ be an event such that $\mathbb{P}[A] \neq 0$. Then, $\mathbb{P}[\cdot \mid A]$ is also a probability measure for the sample space Ω , as well as for A .

Theorem 2.1 (Total Probability Theorem). Let $\{E_n\}$ be a countable collection of mutually exclusive and exhaustive events. Then, for any other event E

$$\begin{aligned} \mathbb{P}[E] &= \sum_n \mathbb{P}[E \cap E_n] \\ &= \sum_n \mathbb{P}[E \mid E_n] \mathbb{P}[E_n] \end{aligned}$$

Proof. Since the collection $\{E_n\}$ is exhaustive, we can say that

$$\begin{aligned} \mathbb{P}[E] &= \mathbb{P}\left[E \cap \left(\bigcup_n E_n\right)\right] \\ &= \mathbb{P}\left[\bigcup_n (E \cap E_n)\right] \\ &= \sum_n \mathbb{P}[E \cap E_n] \\ &= \sum_n \mathbb{P}[E \mid E_n] \mathbb{P}[E_n] \end{aligned}$$

□

Theorem 2.2 (Bayes' Theorem). Let $S = \{E_n\}$ be a countable collection of mutually exclusive and exhaustive events such that $\mathbb{P}[E] \forall E \in S$. Then, for any j such that $E_j \in S$

$$\mathbb{P}[E_j \mid E] = \frac{\mathbb{P}[E \mid E_j] \mathbb{P}[E_j]}{\sum_n \mathbb{P}[E \mid E_n] \mathbb{P}[E_n]}$$

Exercise 2.1. Give a proof of Bayes' Theorem

2. Independence of Events

Definition 2.2. Events $E_1, E_2 \dots E_n$ are said to be

(i) pairwise independent if

$$\mathbb{P}[E_i \cap E_j] = \mathbb{P}[E_i] \mathbb{P}[E_j]$$

(ii) mutually independent if $\forall k \in \{2, 3 \dots n\}$ and distinct $d_1, d_2 \dots d_k \in \{1, 2 \dots, n\}$

$$\mathbb{P}[E_{d_1} \cap E_{d_2} \dots \cap E_{d_k}] = \mathbb{P}[E_{d_1}] \cdot \mathbb{P}[E_{d_2}] \dots \mathbb{P}[E_{d_k}]$$

Note. Pairwise Independence does not imply Mutual Independence