SCRIBE
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Expectation of a Random Variable

1. Expectation

Definition 5.1 (Expectation)**.** The expected value (or mean) of a random variable is a weighted average of the possible values that the variable can take, each value being weighted according to the probability of that event occurring. Expectation is denoted by the symbol $\mathbb{E}$. Consider a random variable X, then

(i) if X is dicrete, with support S_X and p.m.f. f_X

$$\mathbb{E}\left[\, X \,\right] \quad = \quad \sum_{x \in S_X} x \cdot f_X(x)$$

(ii) if X is absolutely continous with p.d.f. f_X, the expectation exists and equals

$$\mathbb{E}\left[\, X \,\right] \quad = \quad \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx$$

provided $\int_{-\infty}^{\infty} |\, x \,| \cdot f_X(x)\, dx$

Result 5.1.1. We can also write a common representation of expectation for both discrete and absolutely continuous random variables

$$\mathbb{E}\left[\, X \,\right] \quad = \quad \int_{0}^{\infty} \mathbb{P}\left[\, X > t \,\right] dt - \int_{-\infty}^{0} \mathbb{P}\left[\, X < t \,\right] dt$$

Result 5.1.2 (LOTUS – Law of the Unconcious Statician)**.** If X is a random variable with p.m.f. / p.d.f. f_X and support S_X, and Y is another r.v. such that for some $g :\longrightarrow \mathbb{R}$, $Y = g(X)$, then we can write the expectation of Y as follows

$$\mathbb{E}\left[\, Y \,\right] = \mathbb{E}\left[\, g(X) \,\right] \quad = \quad \begin{cases} \sum_{x \in S_X} g(x) f_X(x) & \text{if X is discrete} \\ \\ \int_{-\infty}^{\infty} g(x) f_X(x) dx & \text{if X is absolutely continuous} \end{cases}$$

Proof. (For discrete case) Let f_Y be the p.m.f. of Y and let $S_Y = g(S_X)$ be the support of Y. Then,

$$
\begin{aligned}
\mathbb{E}[\,Y\,] &= \sum_{y \in S_Y} f_Y(y) \\
&= \sum_{y \in S_Y} y \cdot \mathbb{P}[\,g(X) = y\,] \\
&= \sum_{y \in S_Y} \left(y \sum_{x \in\, g^{-1}(y)} f_X(x) \right) \\
&= \sum_{y \in S_Y} \left(\sum_{x \in\, g^{-1}(y)} g(x) f_X(x) \right) \\
\implies \mathbb{E}[\,Y\,] &= \sum_{x \in S_X} g(x) f_X(x)
\end{aligned}
$$

Hence, the expectation is same as the form given in Result 5.1.1 for discrete case □

Exercise 5.1. Give the proof for the case where X is absolutely continuous

1.1 Properties of Expectation

Some properties of expectation are listed below

1. If X is a random variable such that $X \equiv c$, where $c \in \mathbb{R}$ is some constant, then $\mathbb{E}[\,X\,] = c$
2. (Linearity in Addition) If X and Y are two random variables, then $\mathbb{E}[\,X + Y\,] = \mathbb{E}[\,X\,] + \mathbb{E}[\,Y\,]$
3. (Linearity in Scalar Multiplication) If X is a random variable and $c \in \mathbb{R}$ is some constant, then $\mathbb{E}[\,c \cdot X\,] = c \cdot \mathbb{E}[\,X\,]$

Definition 5.2 (Variance)**.** Variance is the expectation of the squared deviation of a random variable from its mean. Informally, it measures how far a set of (random) numbers are spread out from their average value. Suppose X is a random variable, then we write the variance of X as

$$
\mathrm{Var}\,(X) \quad = \quad \mathbb{E}\left[(X - \mathbb{E}[\,X\,])^2 \right] \quad = \quad \mathbb{E}\left[\,X^2\,\right] - \mathbb{E}[\,X\,]^2
$$

2. Moments and Moment Generating Function

Definition 5.3 (Moment)**.** The r$^{\text{th}}$ moment of a random variable (say X) is the expectation of X^r and is represented by μ_r'

$$
\mu_r' \quad = \quad \mathbb{E}[\,X^r\,]
$$

Similarly, we can define Central Moment and Absolute Moment

Definition 5.4. Consider a random variable X. Then the

(i) *(Central Moment)* r^{th} central moment of X is the r^{th} moment of the random variable $X - \mathbb{E}[\,X\,]$ and is represented by $\mu_r \longrightarrow \mathbb{E}\left[\,(X - \mathbb{E}[\,X\,])^r\,\right]$

(ii) *(Absolute Moment)* r^{th} absolute moment of X is the r^{th} moment of the random variable $|\,X\,| \longrightarrow \mathbb{E}\left[\,|\,X\,|^r\,\right]$

It is clear that the expectation of a random variable is its first moment and variance is its second centered moment.

We can not define the moment generating function which is used extensively in Probability Theory for proving tail bounds and finding moments.

Definition 5.5 (Moment Generating Function)**.** The moment generating function (m.g.f.) of a random variable (say X) is defined by

$$M_X(t) \quad = \quad \mathbb{E}\left[\, e^{tX} \,\right]$$

where $t \in \mathbb{R}$ such that $\mathbb{E}\left[\, e^{tX} \,\right] < \infty$

Result 5.5.1. If for some $h(\in \mathbb{R}) > 0$, the m.g.f. $M_X(t)$ is finite $\forall\, t \in (-h, h)$, then

(i) $M_X(t)$ is differentiable (on t) any number of times in $(-h, h)$

(ii) for each $r \in \{1, 2 \cdots\}$, μ_r' exists and

$$\mu_r' \quad = \quad M_X^{(r)}(0) \quad = \quad \left[\left(\frac{d}{dt} \right)^r M_X(t) \right]_{t=0}$$

(iii) for $t \in (-h, h)$

$$M_X(t) \quad = \quad \sum_{n \geq 0} \frac{t^n}{n!} \, \mu_r'$$

(iv) for $t \in (-h, h)$

$$\begin{aligned} \mathbb{E}\,[\, X \,] \quad &= \quad \psi_X^{(1)}(0) \quad = \quad \mu_1' \\ \mathrm{Var}\,(X) \quad &= \quad \psi_X^{(2)}(0) \quad = \quad \mu_2 \end{aligned}$$

where $\psi_X(t) = \ln\left(M_X(t)\right)$. This is the Cumulant Generating Function of X