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Probability Theory, 18–19

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2

Conditional Probability and Independence

1. Conditional Probability

Conditional Probability simply defines the probability of an event A, given that another event B has already occurred. This is represented as $A \mid B$. We can formally define condition probability as follows.

Definition 2.1. Let A and B be two events. The conditional probability of A given B is defined by

$$\mathbb{P}\left[\, B \mid A \,\right] \quad = \quad \frac{\mathbb{P}\left[\, A \cap B \,\right]}{\mathbb{P}\left[\, B \,\right]}$$

Note. Conditional Probability $A \mid B$ is defined only if $\mathbb{P}\left[\, B \,\right] \neq 0$

Remark. For a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $A \in \mathcal{F}$ be an event such that $\mathbb{P}\left[\, A \,\right] \neq 0$. Then, $\mathbb{P}\left[\, \cdot \mid A \,\right]$ is also a probability measure for the sample space Ω, as well as for A.

Theorem 2.1 (Total Probability Theorem)**.** Let $\{E_n\}$ be a countable collection of mutually exclusive and exhaustive events. Then, for any other event E

$$\begin{aligned}
\mathbb{P}\left[\, E \,\right] &= \sum_n \mathbb{P}\left[\, E \cap E_n \,\right] \\
&= \sum_n \mathbb{P}\left[\, E \mid E_n \,\right] \mathbb{P}\left[\, E_n \,\right]
\end{aligned}$$

Proof. Since the collection $\{E_n\}$ is exhaustive, we can say that

$$\begin{aligned}
\mathbb{P}\left[\, E \,\right] &= \mathbb{P}\left[E \cap \left(\bigcup_n E_n \right) \right] \\
&= \mathbb{P}\left[\bigcup_n (E \cap E_n) \right] \\
&= \sum_n \mathbb{P}\left[\, E \cap E_n \,\right] \\
&= \sum_n \mathbb{P}\left[\, E \mid E_n \,\right] \mathbb{P}\left[\, E_n \,\right]
\end{aligned}$$

□

Theorem 2.2 (Bayes' Theorem)**.** Let $S = \{E_n\}$ be a countable collection of mutually exclusive and exhaustive events such that $\mathbb{P}\left[\, E \,\right] \; \forall \, E \in S$. Then, for any j such that $E_j \in S$

$$\mathbb{P}\left[\, E_j \mid E \,\right] \quad = \quad \frac{\mathbb{P}\left[\, E \mid E_j \,\right] \mathbb{P}\left[\, E_j \,\right]}{\sum_n \mathbb{P}\left[\, E \mid E_i \,\right] \mathbb{P}\left[\, E_i \,\right]}$$

Exercise 2.1. Give a proof of Bayes' Theorem

2. Independence of Events

Definition 2.2. Events $E_1, E_2 \dots E_n$ are said to be

(i) pairwise independent if

$$\mathbb{P}[\,E_i \cap E_j\,] \quad = \quad \mathbb{P}[\,E_i\,]\,\mathbb{P}[\,E_j\,]$$

(ii) mututally indpendent if $\forall k \in \{2, 3 \dots n\}$ and distinct $d_1, d_2 \dots d_k \in \{1, 2 \dots, n\}$

$$\mathbb{P}[\,E_{d_1} \cap E_{d_2} \cdots \cap E_{d_k}\,] \quad = \quad \mathbb{P}[\,E_{d_1}\,] \cdot \mathbb{P}[\,E_{d_2}\,] \dots \mathbb{P}[\,E_{d_k}\,]$$

Note. Pairwise Independence does not imply Mutual Independence