Mutually Unbiased Bases

Abdul Fatah

Doctor of Philosophy

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Faculty of Engineering and Computing

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Ollscoil Teicneolaíochta an Atlantaigh

Atlantic Technological University

Table of Contents

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Definition (Mutually Unbiased Bases): Two bases $\mathcal{U} = \{|u_i\rangle\}_{i=0}^{d-1}$ and $\mathcal{V} = \{|v_j\rangle\}_{j=0}^{d-1}$ of the Hilbert space $\mathbb{C}^d$ are called mutually unbiased when:

$$|\langle u_i|v_j\rangle|^2 = \frac{1}{d}$$

Definition (Set of Mutually Unbiased Bases): A set of n bases $S = \{U_i\}_{i=0}^{n-1}$ is called a set of mutually unbiased bases when for each pair of bases (U_i, U_j) in S, U_i and U_j are mutually unbiased bases.

Theorem (Horodecki [1]): There are no more than $d + 1$ mutually unbiased bases in the Hilbert space $\mathbb{C}^d$.

Proof: …

□

Theorem (Horodecki [1]): There is a set of at least three mutually unbiased bases in the Hilbert space $\mathbb{C}^d$ for $d \geq 2$.

Proof: Suppose we take $d = 2$, according to the theorem, no more than three mutually unbiased bases exist in $\mathbb{C}^2$ Hilbert space. Let's first identify the set of the three mutually unbiased bases, then show that any other basis can not be unbiased with all three of them.

$$U = \{U_0, U_1, U_2\}$$

Step 1: Defining the first basis (The computational basis)

Lets begin with a familiar orthonormal basis, the computational basis. We label this U_0:

$$U_0 = \{|0\rangle, |1\rangle\}$$

This basis corresponds to the eigenbasis of the Puali-Z operator.

Step 2: Defining a second mutually unbiased basis

A second basis, U_1, is mutually unbiased with U_0 if the squared inner product between any state from U_1 and any state from U_0 is equal to $\frac{1}{d} = \frac{1}{2}$. A good candidate is the Hadamard basis, which corresponds to the eigenbasis of the Pauli-X operator:

$$U_1 = \{|+\rangle, |-\rangle\}$$

where $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.

Let's verify the unbiased condition:

$$|\langle 0|+\rangle|^2 = \left|\left\langle 0 \middle| \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right\rangle\right|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

$$|\langle 0|-\rangle|^2 = \left|\left\langle 0 \middle| \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right\rangle\right|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

$$|\langle 1|+\rangle|^2 = \left|\left\langle 1 \middle| \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right\rangle\right|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

$$|\langle 1|-\rangle|^2 = \left|\left\langle 1 \middle| \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right\rangle\right|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

Since the condition holds for all pairs U_0 and U_1 are mutually unbiased.

Step 3: Defining a third mutually unbiased basis

Now, we seek third basis U_2 that is unbiased with both U_0 and U_1. The eigenbasis of the Pauli-Y operator works perfectly.

$$U_2 = \{|+i\rangle, |-i\rangle\}$$

where $|+i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$ and $|-i\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$. We can verify U_2 basis, the pairwise inner product with U_0 and U_1 that must equal to $\frac{1}{2}$.

$$|\langle 0|+i\rangle|^2 = \left|\left\langle \middle| \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)\right\rangle\right|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

$$|\langle 0|-i\rangle|^2 = \left|\left\langle 0 \middle| \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)\right\rangle\right|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

$$|\langle 1|+i\rangle|^2 = \left|\left\langle 1 \middle| \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)\right\rangle\right|^2 = \left|\frac{i}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

$$|\langle 1|-i\rangle|^2 = \left|\left\langle 1 \middle| \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)\right\rangle\right|^2 = \left|-\frac{i}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

Unbiased condition satisfied U_2 and U_0, now lets repeat for U_2 and U_1

$$|\langle+|+i\rangle|^2 = \left|\left\langle \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\right|\frac{1}{\sqrt{2}}(|0\rangle+i|1\rangle)\right\rangle\Bigg|^2 = \left|\frac{1+i}{2}\right|^2 = \frac{|1|^2+|1|^2}{2^2} = \frac{1}{2}$$

$$|\langle+|-i\rangle|^2 = \left|\left\langle \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\right|\frac{1}{\sqrt{2}}(|0\rangle-i|1\rangle)\right\rangle\Bigg|^2 = \left|\frac{1-i}{2}\right|^2 = \frac{|1|^2+|1|^2}{2^2} = \frac{1}{2}$$

$$|\langle-|+i\rangle|^2 = \left|\left\langle \frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)\right|\frac{1}{\sqrt{2}}(|0\rangle+i|1\rangle)\right\rangle\Bigg|^2 = \left|\frac{1-i}{2}\right|^2 = \frac{|1|^2+|1|^2}{2^2} = \frac{1}{2}$$

$$|\langle-|-i\rangle|^2 = \left|\left\langle \frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)\right|\frac{1}{\sqrt{2}}(|0\rangle-i|1\rangle)\right\rangle\Bigg|^2 = \left|\frac{1+i}{2}\right|^2 = \frac{|1|^2+|1|^2}{2^2} = \frac{1}{2}$$

Thus, we have found three mutually unbiased bases for $d = 2$

Step 4: Proving a fourth mutually unbiased basis is impossible

Now we show that it's impossible to construct a fourth basis, U_3, that is unbiased with U_0, U_1, and U_2.

For any new basis to be unbiased with U_0, its states must be of the form:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\varphi}|1\rangle)$$

Where φ is some real phase.

For this state to also be unbiased with U_1, we must have

$$|\langle+|\varphi\rangle|^2 = \frac{1}{2}$$

$$\left|\frac{1}{2}(\langle 0| + \langle 1|)(|0\rangle + e^{i\varphi}|1\rangle)\right|^2 = \frac{1}{2}$$

$$\left|\frac{1+e^{i\varphi}}{2}\right|^2 = \frac{1}{2} \quad \forall \quad e^{i\varphi} = \cos(\varphi) + i\sin(\varphi)$$

$$|(1+\cos(\varphi) + i\sin(\varphi))|^2 = 2$$

$$(1+\cos(\varphi))^2 + (\sin(\varphi))^2 = 2$$

$$1 + 2\cos(\varphi) + \cos^2(\varphi) + \sin^2(\varphi) = 2$$

$$1 + 1 + 2\cos(\varphi) = 2$$

$$\cos(\varphi) = 0$$

This means the phase angle φ must be either $\frac{\pi}{2}, \frac{3\pi}{2}$ (or its equivalent, $-\frac{\pi}{2}$).

Finally, for the state to also be unbiased with U_2, we check the condition for $\varphi = \frac{\pi}{2} \Rightarrow |\psi\rangle = |0\rangle + i|1\rangle$, So,

$$|\langle +i|\psi\rangle|^2 = \left|\frac{1}{2}(\langle 0| - i\langle 1|)(|0\rangle + i|1\rangle)\right|^2 = \left|\frac{1-i^2}{2}\right|^2 = \left|\frac{2}{2}\right|^2 = 1$$

Since its not equal to $\frac{1}{2}$, the state ψ for phase $\varphi = \frac{\pi}{2}$ is not unbiased with U_2, the same logic applies to $\varphi = \frac{3\pi}{2}$

The only possible candidates for for new mutually unbiased bases were the eigenbases of Pauli-X, and Pauli -Y operators, which we already proved. Therefore, no 4th basis for d=2 can exist.

$\square$

Bibliography

[1] P. Horodecki, Ł. Rudnicki, and K. \ifmmode \dot{Z}\else Ż\fi{}yczkowski, “Five Open Problems in Quantum Information Theory,” *PRX Quantum*, vol. 3, no. 1, p. 10101, 2022, doi: 10.1103/PRXQuantum.3.010101.