Quantum Latin Squares Overview

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1 Introduction

Latin squares are the mathematical objects that have apllications in different areas like combinatorial designs, scheduling,games such as sudoku puzzels and error correcting codes. Quantum Latin squares are objects from combinatorics that are a quantum analogue to Latin squares.
Latin squares are defined as an n-by-n array of elements of the Hilbert space Cn, such that every row and every column is an orthonormal basis. They interact with a finite-dimensional commutative C*-algebra by the linear maps being unitary. They have applications in quantum error correction, quantum key distribution, quantum state determination, and quantum pseudotelepathy. [2]

1.1 Error correcting codes

Codes can be constructed with various ways.The idea to construct codes with latin squares is not recent. Binary data is the strings of $0's$ and $1's$. Error correcting codes add some extra digits for detection and correction of errors [1] In the 1960s and 70s Latin squares found use in producing error correcting codes in classical information theory. We are going to build latin squares' analogy to quantum information theory in field of quantum error correction (QEC).

1.2 Latin Square

A latin square of order q is a q x q array whose entries are from a set of q distinct element such that every element is contained exactly once in each row and each column.

1.3 Orthogonality

Latin squares A $=(a_{ij})$ and B $=(b_{ij})$ of order q are said to be mutallay orthogonal if the q^2 ordered pairs $(a_{ij}, b_{ij}), i, j = 1, 2, 3, ..., n$ are all distinct.

$$\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} \text{ and } \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}$$

superimposing two orthogonal latin squares will give order pairs as bellow

$$\begin{array}{c} (a,a)(b,b)(c,c) \\ (b,c)(c,a)(a,b) \\ (c,b)(a,c)(b,a) \end{array}$$

2 Quantum latin squares, Orthonormal basis

definition A quantum latin square (QLS) of order n is an n -by-n array of elements of the Hilbert space $\mathbb{C}^4$, such that every row and every column is an orthonormal basis. Bellow is the QLS described in terms of computational basis elements $\{|0\rangle, |1\rangle, |2\rangle, |3\rangle\}$.

$\|0\rangle$	$\|1\rangle$	$\|2\rangle$	$\|3\rangle$
$\frac{1}{\sqrt{2}}(\|1\rangle - \|2\rangle)$	$\frac{1}{\sqrt{5}}(i\|0\rangle + 2\|3\rangle)$	$\frac{1}{\sqrt{5}}(2\|0\rangle + i\|3\rangle)$	$\frac{1}{\sqrt{2}}(\|1\rangle + \|2\rangle)$
$\frac{1}{\sqrt{2}}(\|1\rangle + \|2\rangle)$	$\frac{1}{\sqrt{5}}(2\|0\rangle + i\|3\rangle)$	$\frac{1}{\sqrt{5}}(i\|0\rangle + 2\|3\rangle)$	$\frac{1}{\sqrt{2}}(\|1\rangle - \|2\rangle)$
$\|3\rangle$	$\|2\rangle$	$\|1\rangle$	$\|0\rangle$

A classical Latin square of order n is an n-by-n array of integers in the range $\{0, ..., n-1\}$ such that every row and column contains each number exactly once. By interpreting a number $k\epsilon\{0, ..., n-1\}$ as a computational basis element $|K_i\rangle \epsilon C_n$, we can turn an array of numbers into an array of Hilbert space elements: [3]

$$\begin{array}{|c|c|c|c|} \hline 3 & 1 & 0 & 2 \\ \hline 1 & 0 & 2 & 3 \\ \hline 2 & 3 & 1 & 0 \\ \hline 0 & 2 & 3 & 1 \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|c|} \hline |3\rangle & |1\rangle & |0\rangle & |2\rangle \\ \hline |1\rangle & |0\rangle & |2\rangle & |3\rangle \\ \hline |2\rangle & |3\rangle & |1\rangle & |0\rangle \\ \hline |0\rangle & |2\rangle & |3\rangle & |1\rangle \\ \hline \end{array}$$

References

[1] *Error correcting codes and latin squares.* Apr. 23, 2021. URL: `http://ijarse.com/images/fullpdf/1517035466_J1016ijarse.pdf`.

[2] B J Musto. *Quantum Latin Squares and Quantum Functions: Applications in Quantum Information*. 2019.

[3] Benjamin Musto and Jamie Vicary. *Quantum Latin squares and unitary error bases*. 2016. arXiv: `1504.02715 [quant-ph]`.