Quantum Error Correction based on Quantum Latin Squares

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PhD Thesis

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Submitted: April 11, 2024

This Research has been supported by the Atlantic Technological University under the Postgraduate Research Training Programme in Modelling and Computation for Health and Society (MOCHAS PRTP).

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Atlantic Technological University

Contents

Abstract

Writing a compelling abstract for a Ph.D. thesis requires conciseness, clarity, and the ability to convey the significance of your research. Here are some tips to help you craft an effective abstract. (1) Clearly State the Problem: Begin by clearly stating the problem or question your research addresses. Be concise and specific about the problem you are investigating. (2) Highlight the Objective: Clearly state the main objective of your research. What are you contributing to the field of study? Make it evident how your work fits into the broader context. (3) Provide a Brief Overview of Methods: Mention the key methods used in your research. Briefly explain the tools or frameworks you used to address the problem. However, avoid going into excessive detail. (4) Present Key Results: Summarize the main findings and results of your research. Highlight any breakthroughs, novel insights, or contributions your work has made to the field. (5) Contextualize the Significance: Communicate the significance and relevance of your research. Explain how your findings contribute to and address gaps in the field. (6) Use Concise and Accessible Language: Write in a clear, concise, and accessible language. Avoid unnecessary jargon that may be unclear to readers outside your specific subfield of study. Remember that the abstract serves as a short summary of your entire Ph.D. thesis. It provides readers with a quick overview of your research. Striking the right balance between conciseness and informativeness is key. Put your best foot forward without overstating your findings.

About the Author

When writing about yourself, highlight your academic background, research interests, accomplishments, and any relevant experiences. Here are some tips to help you craft an effective blurb. (1) Start with a Strong Opening: Begin with a concise and engaging statement that captures the reader's attention. For example, you might start with a brief description of your academic journey or research focus. (2) Highlight Your Academic Background: Provide a brief overview of your academic qualifications, including your degree(s) and any relevant academic honors or awards you have received. (3) Emphasize Your Research Interests: Clearly articulate your research interests and the topics you are passionate about. Briefly mention any specific areas of mathematics or related fields that you have focused on in your thesis work. (4) Showcase Your Accomplishments: Highlight any significant achievements or contributions you have made in your field. This could include publications, conference presentations, research projects, or collaborations. (5) Include Relevant Experiences: Mention any relevant academic or professional experiences that have shaped your research interests and expertise. This could include internships, research assistantships, teaching positions, or involvement in academic societies.

Chapter 1

Introduction

A PhD, or Doctor of Philosophy, is the highest academic degree awarded by universities to individuals who have made a significant original contribution to knowledge in their field. It is typically earned after several years of advanced study and research beyond the master's degree level. In many countries, including the United States and the United Kingdom, it is considered a prerequisite for a career in academia, research, or certain specialized fields.

Formally, a PhD involves conducting independent research under the guidance of a supervisor, culminating in the production of a written thesis or dissertation that presents the original research findings. The thesis must demonstrate the candidate's ability to critically analyze existing literature, formulate a research question or hypothesis, design and execute appropriate research methodologies, and draw meaningful conclusions from the data gathered.

PhD programs vary in length and structure, but they typically require at least three to five years of full-time study. In addition to the research work, candidates may also be required to complete coursework, pass comprehensive exams, and participate in teaching or other academic activities.

Upon successful completion of all requirements and the defense of the thesis before a panel of experts in the field (known as the thesis defense or viva voce), the candidate is awarded the PhD degree. This degree signifies not only mastery of a specific subject area but also the ability to contribute new knowledge through original research.

1.1 About the Introduction Chapter

In the introduction, you should describe what your thesis is about, how the thesis is organised, and what the reader can expect as they read down through it. The most important aspect of the introduction is to set the context for the rest of the thesis. You should sign-post for the reader the most important parts of your work and where they appear in the document. In LaTeX, you should refer to sections, tables, and figures using commands rather than specifying a page or saying "the table below". The `ref` command will keep track of changes to the layout to the document. So, for example, I can just refer to Section 2 rather than worrying where that might move to in future. Note to refer to an item in the bibliography, you use the `cite` command, which should generally be placed after a `tilde`(~) rather than a space [1].

1.2 What Should be Included Here

The introduction chapter of a Ph.D. thesis serves as the gateway to your research and sets the stage for the reader to understand the context, significance, and objectives of your study. Here's a comprehensive guide on what to include in the introduction chapter.

Background and Context

Provide an overview of the broader field of study and the specific research area your thesis addresses. Discuss the historical background, key concepts, theories, and previous research relevant to your topic.

Research Problem and Motivation

Clearly articulate the research problem or question your thesis aims to address. Explain why this problem is significant and worthy of investigation. Discuss any gaps or limitations in existing literature that your research seeks to fill.

Objectives and Research Questions

State the objectives of your research and the specific research questions you seek to answer. These objectives should be clear, specific, and aligned with the overall purpose of your study.

Scope and Limitations

Define the scope of your research by outlining what is included and excluded from your study. Discuss any limitations or constraints that may impact the interpretation or generalizability of your findings.

Conceptual Framework or Theoretical Framework

If applicable, introduce the conceptual framework or theoretical framework that underpins your research. Explain the theoretical perspectives, models, or frameworks you will use to guide your analysis and interpretation.

Methodology

Provide an overview of the research methodology and approach you have adopted. Discuss the research design, data collection methods, analytical techniques, and any other procedures used to conduct your study.

Significance and Contributions

Clearly articulate the significance of your research and the potential contributions it makes to the field. Explain how your study advances knowledge, addresses gaps in the literature, or has practical implications.

Organizational Structure

Outline the structure of your thesis by briefly describing the contents of each chapter. Provide a roadmap that helps the reader navigate through your thesis and understand the sequence of your argumentation.

Literature Review

While the main literature review may be presented in a separate chapter, briefly summarize the key literature that informs your research in the introduction. Highlight the most relevant theories, concepts, and empirical studies that provide context for your study.

Engage the Reader

Write in a clear, engaging, and concise manner to capture the reader's interest. Use compelling language and examples to draw the reader into the topic and motivate them to continue reading. Remember, the introduction chapter sets the tone for your entire thesis and should provide a comprehensive overview of your research topic, objectives, methodology, and significance. It should be well-structured, focused, and persuasive, laying the foundation for the reader to understand and appreciate the rest of your work.

Chapter 2

Literature

Quantum bits

The fundamental concept of the classical computation and classical information is *bit*. A bit is the smallest unit of information in classical computing and can take one of two values: 0 or 1. It is the basic building block of classical computers and is used to represent and store information in binary form. However, in quantum computing, the basic unit of information is called a *quantum bit* or *qubit*. Schumacher coined the term quantum bit or qubit in 1995 [2] as a two-level quantum system in which information is encoded using two orthogonal states. Just like its classical cousin, the qubit can take a value of either 0 or 1. However, unlike a classical bit, a qubit can exist in a superposition of states, meaning it can be in a state of 0, 1, or both simultaneously. Physically, qubits can be represented as any two-level quantum systems such as

- The spin of a particle in a magnetic field where up means 0 and down means 1 or

- The polarization of a single photon where horizontal polarization means 1 and vertical polarization means 0.

We can make a quantum computer out of light as well. In both cases 0 and 1 are the only possible states. Geometrically, qubits can be visualized using a shape called the Bloch sphere, an instrument named after Swiss physicist Felix Bloch as in figure 2.1.

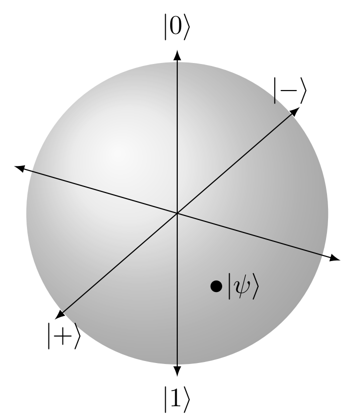


Figure 2.1: Bloch Sphere representation for a Qubit.

Mathematically, the qubit is treated as a mathematical object and the state of a qubit is treated as a vector in a two-dimensional complex vector space, as in equation 2.1

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle \tag{2.1}$$

where α is the probability amplitude of the state $|0\rangle$ and β is the probability amplitude of the state $|1\rangle$.

A probability amplitude is a complex number c whose squared magnitude $|c|^2$ gives the probability of finding the quantum system in its associated orthogonal state. The probabilities $|\alpha|^2$ and $|\beta|^2$ must therefore sum to one, so qubits are in that sense normalized. We define the inner product $\langle a|b\rangle = \sum_{i=0}^{n} a_i^* b_i$ for a and b in $\mathbb{C}^n$, leading to a Hilbert space. The states $|0\rangle$ and $|1\rangle$ form a basis for the Hilbert space. Thus, we can write

$$|0\rangle = \begin{bmatrix}1\\0\end{bmatrix} \text{ and } |1\rangle = \begin{bmatrix}0\\1\end{bmatrix}$$

Systems of Qubits

Two or more qubits can be combined in a single system. Where $\mathcal{H}_1$ and $\mathcal{H}_2$ are two Hilbert spaces, the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ is the Hilbert space of the combined system. In a combined system, qubits can become entangled in the sense that measurements of the qubits become correlated. Suppose we denote the four orthogonal computational states of a two qubit system as $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. The system is correlated when the state vector

$\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$ cannot be decomposed into a tensor product of two independent state vectors, one for each qubit.

Physical and Logical Qubits

We distinguish between physical qubits and logical qubits. A physical qubit is a physical quantum object with two levels. Various physical systems can be used to realize physical qubits such as the two different polarizations of a photo. A logical qubit is a mathematical object used to model a physical qubit. Where physical qubits are fragile, a logical qubit might be encoded by several physical qubits.

Unitary Operators and Unitary Error Bases (UEB)

Unitary operators are also discussed in 1977 in book quantum Mechanics by Cohen [3].

Operators

In quantum mechanics, operators are used to represent physical observables(i.e. energy, momentum, position, etc.) and mathematically they these physical observables are represented by Hermitian operators. For instance, the operator corresponding to energy in the Hamiltonian operator.

$$\hat{H} = \sum_{i=1}^{N}(-\frac{\hbar^2}{2m_i}\nabla_i^2 + V_i) \tag{2.2}$$

where i is an index over all particles of the system. The result $\hat{H}$ is the Hamiltonian equal to the individual sum of all particles of the system 2.2. The average value of an observable A, operator denoted by $\hat{A}$ for a quantum molecular state $\psi(r)$ is given by the 'expectation value' formula:

$$\langle A \rangle = \int \psi^*(r) A \psi(r) dr \tag{2.3}$$

In Equation 2.3, we calculate the expectation value of the observable A or operator $\hat{A}$. These operators show certain mathematical properties as summarized below:

- The sum and difference of two operators $hatA$ and $\hat{B}$ are also operators, given by equation.

$$(\hat{A} + \hat{B})f = \hat{A}f + \hat{B}f \tag{2.4}$$

- The product of two operators is defined by

$$(\hat{A}\hat{B})f = \hat{A}(\hat{B}f) \tag{2.5}$$

- Two operators are equal if

$$\hat{A}f = \hat{B}f \tag{2.6}$$

  for all functions f in equation 2.6.

- The identity operator $\hat{I}$ is defined by equation.

$$\hat{I}f = f \tag{2.7}$$

  Identity element operator does not change anything when it operates on a function.

- The associative law also holds for operators as in equation.

$$\hat{A}(\hat{B}\hat{C}) = (\hat{A}\hat{B})\hat{C} \tag{2.8}$$

- The commutative law does not hold for operators. In general, $\hat{A}\hat{B} \neq \hat{B}\hat{A}$. It is convenient to define the quantity as in equation.

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} \tag{2.9}$$

  which is called commutator of $\hat{A}$ and $\hat{B}$. Note that the order matters, so that $[\hat{A}, \hat{B}] = -[\hat{B}, \hat{A}]$. If $\hat{A}$ and $\hat{B}$ commute, then $[\hat{A}, \hat{B}] = 0$.

- The n-th power of an operator $\hat{A}^n$ is defines as n successive applications of the operator, e.g.

$$\hat{A}^3 f = \hat{A}\hat{A}\hat{A}f \tag{2.10}$$

- The exponential of an operator $\exp(\hat{A})$ is defined by the power series expansion as in equation.

$$\exp(\hat{A}) = \hat{1} + \hat{A} + \frac{\hat{A}^2}{2!} + \frac{\hat{A}^3}{3!} + \ldots \tag{2.11}$$

Linearity and Antilinearity

Linear Operators

In quantum mechanics most of the operators used are linear operators, The operator is called linear if it satisfies two conditions:

1. Additivity: $\hat{A}(f+g) = \hat{A}f + \hat{A}g$
2. Homogeneity: $\hat{A}(cf) = c\hat{A}f$

Where f and g are functions and c is a constant.

The only other category of operators relevant to quantum mechanics is the set of antilinear operators, for which

$$\hat{A}(\lambda f + \mu g) = \lambda^* \hat{A}f + \mu^* \hat{A}g \tag{2.12}$$

Time-reversal operators are examples of antilinear operators.

Unitary Operators

Unitary operator can be defined as an operator U is a unitary if it is bounded linear operator ($U : H \to H$) on Hilbert space and its inverse U^{-1} is equal to its adjoint $U^\dagger$ (also denoted as U^*). Unitary operator must satisfy the following property:

$$UU^\dagger = U^\dagger U = I \tag{2.13}$$

Where I is the identity operator ($I : H \to H$). The condition $U^\dagger U = I$ defines an isometry and the other condition $UU^\dagger = 1$ defines a co-isometry. Thus a unitary operator is a bounded linear operator that is both an isometry and a co-isometry [4] Unitary operator U is *Surjective* and it preserves the inner product of the Hilbert space, H. We can write it as for all vectors A and B in H we get:

$$\langle UA, UB \rangle_H = \langle A, B \rangle_H$$

The product of two unitary operators is also unitary. For instance, If U and V are unitary operators, then UV is also unitary. From equation 2.13, we can write:

$$UU^\dagger = U^\dagger U = I$$

and

$$VV^\dagger = V^\dagger V = I$$

Then we can write:

$$(UV)(UV)^\dagger = (UV)(V^\dagger U^\dagger) = 1$$

$$(UV)^\dagger (UV) = (V^\dagger U^\dagger)(UV) = 1$$

The product of operator UV is unitary also as in show in above example.

The unitary operators are the isomorphism of the Hilbert space because they preserve the basic structure of the Hilbert space.

Unitary Matrices

If a basis for a finite dimensional Hilbert space is chosen, then the unitary operator can be represented as a unitary matrix. A unitary matrix is a square matrix U (n x n) that satisfies the following condition:

$$U^\dagger U = UU^\dagger = I_n \tag{2.14}$$

This is equivalent to saying that both the rows and the columns of U form an orthonormal basis in $\mathbb{C}^n$ with respect to the standard inner product. The U is also a normal matrix whose eigen values lie on the unit circle.

Definition (Unitary Error Basis)**.** *A set U of n^2 unitary $n \times n$ matrices is called a unitary error basis if and only if U is orthonormal with respect to the inner product $\langle A, B\rangle = Tr(A^\dagger B)/n$ where $Tr(M)$ is the trace of the matrix M and $M^\dagger$ is the conjugate transpose of M.*

Unitary error bases perform a primitive job in quantum error correcting codes and further applications of quantum information processing, such as teleportation and super-dense coding schemes [5].

Mutually Unbiased Bases (MUB)

Definition (Mutually Unbiased Bases)**.** *Two orthonormal bases U and V of $\mathbb{C}^n$ are mutually unbiased if and only if $|\langle u|v\rangle|^2 = \frac{1}{n}$ for all u in U and all v in V.*

Mutually unbiased bases embed the complementary concept of quantum formalism that relates to [6]. Two orthonormal bases of Hilbert space are said to be mutually unbiased if the measurement of one state does not disclose the result of the other state [7]. Transition probability from any state of the first basis to any state of the second basis is independent of the two chosen states [7].

Mutually Unbiased Bases and Unitary Error Bases

The idea of mutually unbiased bases was posed by Schwinger in 1960 [8] and play an important role in quantum information theory [9]. They are used in quantum state estimation [10], [11], quantum cryptography protocols, and quantum error correcting codes [12].

Mutually unbiased bases can be constructed using mutually orthogonal Latin squares and vice-versa [13], [14]. The eigen-bases of the well-known

Pauli operators $\hat{\sigma}_x$, $\hat{\sigma}_y$, and $\hat{\sigma}_z$ constitute mutually unbiased bases, where each vector in a given basis shares an identical degree of overlap with every vector in the other bases [15]. Note that the Pauli matrices are d^2 order unitary error bases with $d = 2$ but they can be generalized to higher order dimensions [5].

$$\sigma_I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},\ \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Unitary error bases, also known as unitary operator bases, are the basic primitive in quantum information. They were introduced in the mid 1990s, having great use in the area of constructing quantum error correcting codes [16]–[18]. Quantum teleportation protocols, dense coding protocols and error correction can be precisely mathematically characterized as unitary error bases [5].

Mutually unbiased bases and unitary error bases are fundamental primitives in quantum information theory. They can be constructed using Latin squares and quantum Latin squares. Considering their importance, Musto has introduced a way of constructing mutually unbiased bases using orthogonal quantum Latin squares, proving the reciprocal construction of quantum Latin squares from mutually unbiased bases [19].

Unitary error bases can also be constructed from quantum Latin squares and a family of Hadamard matrices [20] via one of three known methods, as follows. First, the *shift and multiply* method uses classical Latin squares alongside a set of Hadamard matrices [21]. Second, the *Hadamard* method uses two mutually unbiased bases. Finally, the *Algebraic* method uses a finite group possessing a projective representation satisfying certain properties [17].

2.1 Guidelines for writing

A comprehensive literature review is a critical component of any academic research, including a Ph.D. thesis. It serves several purposes, as follows.

Establishing Context

Provide background information on the topic of study to contextualize your research within existing knowledge and scholarship. Identify key concepts, theories, and methodologies relevant to your research.

Identifying Gaps and Opportunities

Identify gaps, inconsistencies, or contradictions in the existing literature that your research aims to address. Highlight areas where further research is needed or opportunities for innovation exist.

Synthesizing Previous Research

Summarize and synthesize findings from previous studies, organizing them thematically or chronologically. Evaluate the strengths and weaknesses of previous research methodologies, data sources, and analytical approaches.

Demonstrating Scholarly Engagement

Demonstrate your familiarity with the current state of research in your field and your ability to critically evaluate and synthesize existing literature. Establish your credibility as a researcher by showing that you are building upon established knowledge and contributing to ongoing scholarly conversations.

Supporting Methodological Choices

Justify your research methodology by explaining how it builds upon or diverges from previous approaches. Discuss the suitability of different research methods and theoretical frameworks for addressing your research questions.

Highlighting Significance and Contribution

Emphasize the significance of your research by showing how it fills a gap, extends existing knowledge, or offers new insights. Articulate the specific contribution your research makes to the field and how it advances scholarship.

Citing Sources Appropriately

Ensure that you accurately cite all sources consulted in your literature review according to the citation style guidelines required by your institution or discipline.

Maintaining Focus and Organization

Keep your literature review focused and well-organized, with clear sections or subsections devoted to different themes, theories, or methodologies. Provide transitions and connections between different sections to maintain coherence and flow.

Remaining Critical and Objective

Remain critical and objective in your evaluation of previous research, acknowledging both strengths and limitations. Avoid bias and strive to present a balanced and comprehensive overview of the literature.

Updating and Revising

Continuously update and revise your literature review as your research progresses and new relevant studies are published. Be open to incorporating feedback from peers, advisors, and reviewers to strengthen your literature review.

By including these elements in your literature review, you can effectively situate your research within the broader scholarly conversation, demonstrate your expertise in the field, and lay the groundwork for your own empirical investigation.

2.2 Maxwell's Equations

Maxwell's equations are a set of four fundamental equations that describe the behavior of electric and magnetic fields, as well as their interaction with matter. They are the cornerstone of classical electromagnetism and are essential for understanding a wide range of phenomena in physics and engineering.

Gauss's Law for Electricity

Gauss's Law for Electricity is one of the four fundamental equations in classical electromagnetism, formulated by Carl Friedrich Gauss. It describes the relationship between the electric flux through a closed surface and the electric charge enclosed within that surface. Mathematically, Gauss's Law for Electricity is expressed as:

$$\oint_S \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{\text{enc}}}{\varepsilon_0}$$

Here's an explanation of the key components of Gauss's Law for Electricity:

- $\oint_S$ represents a closed surface integral over a closed surface S. This means that we are summing the electric field ($\mathbf{E}$) over all infinitesimal areas ($d\mathbf{A}$) of the closed surface S.
- $\mathbf{E}$ is the electric field vector at each point on the surface. It represents the force experienced by a unit positive charge placed at that point.
- $d\mathbf{A}$ is a vector representing an infinitesimal area element of the surface. It is oriented perpendicular to the surface at each point.
- Q_{enc} is the total electric charge enclosed within the closed surface S. This includes the sum of all positive and negative charges within the enclosed region.
- ε_0 is the permittivity of free space, a fundamental constant in electromagnetism. It represents the ability of a material to permit the formation of an electric field in response to an applied electric field.

In simpler terms, Gauss's Law for Electricity states that the total electric flux through a closed surface is proportional to the total electric charge enclosed within that surface, with the constant of proportionality being the permittivity of free space. In other words, it quantifies how much electric field passes through a closed surface due to the presence of electric charges inside that surface.

Gauss's Law for Magnetism

Gauss's Law for Magnetism states that the magnetic flux through any closed surface is always zero. Mathematically, it is expressed as:

$$\oint_S \mathbf{B} \cdot d\mathbf{A} = 0$$

Here's an explanation of the key components of Gauss's Law for Magnetism:

- $\oint_S$ represents a closed surface integral over a closed surface S. This means that we are summing the magnetic field ($\mathbf{B}$) over all infinitesimal areas ($d\mathbf{A}$) of the closed surface S.
- $\mathbf{B}$ is the magnetic field vector at each point on the surface. Unlike electric fields, magnetic fields do not have sources or sinks (monopoles), so the magnetic flux through any closed surface is always zero.

- $d\mathbf{A}$ is a vector representing an infinitesimal area element of the surface. It is oriented perpendicular to the surface at each point.

In summary, Gauss's Law for Magnetism implies that there are no magnetic monopoles (isolated north or south poles), and the magnetic flux through any closed surface is always zero, indicating that magnetic field lines neither start nor end but always form closed loops.

Faraday's Law of Induction

Faraday's Law of Induction describes how a changing magnetic field induces an electromotive force (EMF) and hence an electric current in a conducting loop. Mathematically, it is expressed as:

$$\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = -\frac{d\Phi_B}{dt}$$

Here's an explanation of the key components of Faraday's Law of Induction:

- $\oint_C$ represents a closed path integral around a closed loop C. This means that we are summing the electric field ($\mathbf{E}$) around the closed loop C.

- $\mathbf{E}$ is the induced electric field within the conducting loop. It is created by a changing magnetic flux through the loop according to Faraday's law.

- $d\boldsymbol{\ell}$ is a vector representing an infinitesimal displacement along the closed loop C.

- $d\Phi_B/dt$ represents the rate of change of magnetic flux (Φ_B) through the surface enclosed by the loop with respect to time. The negative sign indicates that the induced EMF and hence the induced electric field opposes the change in magnetic flux.

In summary, Faraday's Law of Induction states that a changing magnetic field induces an electric field and hence an electromotive force (EMF) in any closed conducting loop, producing an electric current in the loop. This phenomenon forms the basis of many practical devices, such as electric generators and transformers.

Ampere's Circuital Law (with Maxwell's addition)

Ampère's Circuital Law relates the magnetic field around a closed loop to the electric current passing through the loop. With Maxwell's addition, it accounts for the displacement current, which arises from changing electric fields. Mathematically, it is expressed as:

$$\oint_C \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 \left(I_{\text{enc}} + \varepsilon_0 \frac{d\Phi_E}{dt} \right)$$

Here's an explanation of the key components of Ampère's Circuital Law (with Maxwell's addition):

- $\oint_C$ represents a closed path integral around a closed loop C. This means that we are summing the magnetic field ($\mathbf{B}$) around the closed loop C.
- $\mathbf{B}$ is the magnetic field vector at each point along the closed loop C.
- $d\boldsymbol{\ell}$ is a vector representing an infinitesimal displacement along the closed loop C.
- μ_0 is the permeability of free space, a fundamental constant in electromagnetism.
- I_{enc} is the total current passing through the loop C. This includes both conduction current and displacement current.
- ε_0 is the permittivity of free space, another fundamental constant in electromagnetism.
- $\frac{d\Phi_E}{dt}$ represents the rate of change of electric flux (Φ_E) through the surface enclosed by the loop with respect to time. This gives rise to the displacement current, which is included in Ampère's law with Maxwell's addition.

In summary, Ampère's Circuital Law with Maxwell's addition states that the magnetic field around a closed loop is proportional to the total current passing through the loop, including both conduction current and displacement current arising from changing electric fields. This law plays a crucial role in understanding the behavior of electromagnetic fields in various physical systems.

Chapter 3

Methodology

Methodology in research refers to the systematic approach or framework used by researchers to conduct their study, gather data, analyze findings, and draw conclusions. It encompasses the strategies, techniques, procedures, and tools employed to address research questions, test hypotheses, or achieve research objectives. Methodology serves as a roadmap that guides the entire research process, ensuring rigor, reliability, and validity in the investigation.

Key components of methodology in research include the following.

Research Design

The overall plan or structure of the study, including its purpose, scope, and objectives. This may involve choosing between experimental, observational, qualitative, quantitative, or mixed methods research designs.

Data Collection Methods

The specific techniques and procedures used to gather relevant data or information for analysis. This could include surveys, interviews, experiments, observations, archival research, or data mining.

Sampling Strategy

The process of selecting a representative subset of individuals, cases, or data points from a larger population for study. This involves decisions about sample size, sampling techniques, and sampling frame.

Data Analysis Techniques

The methods used to analyze and interpret the collected data. This may include statistical analysis, qualitative coding, content analysis, thematic analysis, or other analytical approaches.

Validity and Reliability Considerations

Measures taken to ensure the validity and reliability of the research findings. This includes strategies to minimize bias, control for confounding variables, establish internal and external validity, and assess the consistency and repeatability of results.

Ethical Considerations

Adherence to ethical principles and guidelines in conducting research, particularly concerning the protection of human subjects, confidentiality, informed consent, and avoiding conflicts of interest.

Timeline and Resources

Planning and scheduling the research activities, allocating resources (such as funding, personnel, and equipment), and establishing milestones and deadlines for completion.

Theoretical Framework

The underlying theoretical perspectives, concepts, or models that inform the research design and guide data collection, analysis, and interpretation.

Limitations and Delimitations

Acknowledgment of the constraints, limitations, and boundaries of the study, including factors that may impact the generalizability or applicability of the findings.

Iterative Process

Recognizing that research methodology is often iterative, with researchers refining their approach based on preliminary findings, feedback, and emerging insights.

Overall, methodology in research provides a systematic and structured framework for conducting rigorous and valid investigations, ensuring that research findings are credible, trustworthy, and meaningful.

3.1 Common Methodologies in Mathematics Research

In mathematics research, several methodologies are commonly used to investigate problems, develop theories, and prove conjectures. Some of the most common methodologies include the following.

Analytical Methods

Analytical methods involve rigorous reasoning and logical deduction to study mathematical objects and phenomena. This often includes developing mathematical models, formulating conjectures, and proving theorems using deductive reasoning and formal logic.

Computational Methods

Computational methods involve using computers to perform numerical simulations, calculations, and experiments. This approach is particularly useful for solving complex mathematical problems, generating data, and exploring mathematical structures that are difficult to analyze analytically.

Experimental Methods

Experimental methods involve conducting experiments or observations to gather empirical data and test hypotheses in mathematics. While less common in pure mathematics, experimental mathematics is often used in applied and computational mathematics to validate conjectures, explore patterns, and discover new phenomena.

Algorithmic Methods

Algorithmic methods involve designing and analyzing algorithms to solve mathematical problems efficiently. This includes developing algorithms for numerical computation, optimization, graph theory, cryptography, and other areas of mathematics.

Statistical Methods

Statistical methods involve analyzing data and making inferences using statistical techniques such as regression analysis, hypothesis testing, and Bayesian inference. This approach is commonly used in mathematical statistics, probability theory, and data analysis.

Geometric Methods

Geometric methods involve studying mathematical objects and structures using geometric techniques and visualizations. This includes methods from differential geometry, algebraic geometry, topology, and geometric group theory.

Topological Methods

Topological methods involve studying properties of spaces and mappings that are preserved under continuous deformations, such as homeomorphisms and homotopies. This approach is commonly used in algebraic topology, differential topology, and geometric topology.

Combinatorial Methods

Combinatorial methods involve studying discrete structures and counting techniques to solve problems in mathematics. This includes combinatorial enumeration, graph theory, combinatorial optimization, and discrete mathematics.

Modeling and Simulation

Modeling and simulation involve constructing mathematical models to describe real-world phenomena and simulating their behavior using mathematical tools and techniques. This approach is commonly used in mathematical modeling, applied mathematics, and mathematical physics.

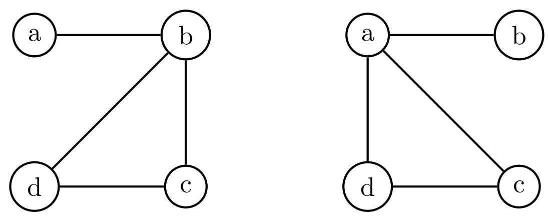
Interdisciplinary Approaches

Many mathematical research projects involve interdisciplinary collaboration with researchers from other fields, such as physics, biology, engineering, economics, and computer science. These projects often require combining mathematical methods with domain-specific knowledge and techniques to address complex problems.

These methodologies are not mutually exclusive, and researchers often combine multiple approaches to tackle mathematical problems from different angles. The choice of methodology depends on the nature of the problem, the available resources, and the researcher's expertise and preferences.

3.2 Graphical Models

LaTeXhas a lot of nice functionality for creating various types of plots and diagrams. Check out the nice graphs in Figure 3.1.



G_1 G_2

Figure 3.1: Nice pictures

3.3 Vector Spaces

A **vector space** V over a field $\mathbb{F}$ is a set equipped with two operations:

- Vector addition: $+ : V \times V \to V$, denoted as $(\mathbf{v}, \mathbf{w}) \mapsto \mathbf{v} + \mathbf{w}$.
- Scalar multiplication: $\cdot : \mathbb{F} \times V \to V$, denoted as $(\lambda, \mathbf{v}) \mapsto \lambda\mathbf{v}$.

These operations must satisfy the following properties for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $\lambda, \mu \in \mathbb{F}$:

1. **Addition is commutative**: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
2. **Addition is associative**: $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
3. **Additive identity**: There exists a vector $\mathbf{0} \in V$ such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v} \in V$.
4. **Additive inverse**: For every vector $\mathbf{v} \in V$, there exists a vector $-\mathbf{v} \in V$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.

5. **Scalar multiplication is distributive over vector addition**: $\lambda(\mathbf{v} + \mathbf{w}) = \lambda\mathbf{v} + \lambda\mathbf{w}$.

6. **Scalar multiplication is distributive over scalar addition**: $(\lambda + \mu)\mathbf{v} = \lambda\mathbf{v} + \mu\mathbf{v}$.

7. **Scalar multiplication is associative**: $(\lambda\mu)\mathbf{v} = \lambda(\mu\mathbf{v})$.

8. **Scalar multiplication identity**: $1 \cdot \mathbf{v} = \mathbf{v}$, where 1 is the multiplicative identity in $\mathbb{F}$.

3.4 Dirac Notation

Dirac notation, also known as bra-ket notation, is a powerful and concise mathematical notation used extensively in quantum mechanics. It was introduced by the physicist Paul Dirac and provides a convenient way to represent vectors, linear operators, and inner products in quantum mechanics. Here's a breakdown of the components of Dirac notation:

Ket notation $|\psi\rangle$: A ket is represented by a column vector enclosed within vertical bars. It represents a state vector in a complex vector space. For example, $|\psi\rangle$ could represent the state of a quantum system.

Bra notation $\langle\psi|$: A bra is represented by a row vector enclosed within angular brackets. It represents the complex conjugate transpose of a ket vector. If $|\psi\rangle$ represents a state vector, then $|\psi\rangle$ represents the corresponding bra vector.

Inner product ($\langle\psi \mid \psi\rangle$): The inner product of two vectors is represented by placing a bra vector on the left and a ket vector on the right, enclosed within angular brackets. It yields a complex number and is a measure of the "overlap" between the two vectors. In quantum mechanics, the inner product is used to calculate probabilities and to determine the expectation values of observables.

Outer product ($|\psi\rangle\langle\phi|$): The outer product of two vectors is represented by placing a ket vector on the left and a bra vector on the right. It results in a linear operator known as a ket-bra or dyad, which maps one vector to another. In quantum mechanics, outer products are used to represent quantum operators.

Operators ($A|\psi\rangle$): If A is a linear operator, its action on a ket vector $|\psi\rangle$ is represented by placing the operator to the left of the ket vector. This notation shows the result of applying the operator to the state vector.

Dirac notation offers several advantages:

Clarity and Conciseness: It provides a concise representation of complex mathematical objects, making expressions and calculations easier to write and understand.

Flexibility: It can be easily extended to represent complex operations and concepts in quantum mechanics, such as composite systems, entanglement, and measurement.

Computational Efficiency: Dirac notation simplifies many calculations in quantum mechanics, such as computing inner products, expectation values, and transition probabilities.

Overall, Dirac notation is a fundamental tool in quantum mechanics, enabling physicists to express and manipulate quantum states and operations in a clear and efficient manner.

Chapter 4

Quantum Latin Squares

In this chapter we will explain quantum Latin squares with examples at later stage, First we discuss Latin squares, their historical context, and their relationship with Quasi-group and Cayley table(*comment: Yet I need to study more about quasi-group and cayley table*).

In this thesis the notations are as follows, the set of natural numbers is denoted $\mathbb{N}$ and the set of natural numbers including zero is denoted $\mathbb{N}_0$. The set of complex numbers is denoted $\mathbb{C}$. The set $\{k \in \mathbb{N}_0 \mid k < n\}$ of natural numbers (including zero) less than n is denoted $[n]$. We index the rows of an array in order with the elements of $[n]$ in their natural order, likewise for the columns.

Latin Squares

Latin squares were discussed by Euler in 1782 [22] and Fisher in 1935 [23]. The latter used Latin squares to design modern hypothesis testing experiments. Since then Latin squares have been used in number theory, statistical analysis, game theory, computer science, coding theory, cryptography, and quantum information theory [24].

Definition (Latin Square)**.** *Let S be a set of order $n \in \mathbb{N}$. An $n \times n$ array is called a Latin square of order n if each element of S appears exactly once in each row and exactly once in each column of the array.*

Suppose we construct a Latin square of order n with elements from $[n]$. Then each row and each column contains the elements of $[n]$ in some order. Here $n = 4$ and $[n] = \{0, 1, 2, 3\}$.

0	1	2	3
1	0	3	2
2	3	0	1
3	2	1	0

Figure 4.1: Latin Square

The Figure 4.1 is a Latin square of order 4. Each row and each column contain unique elements, there is not any repetition of elements in same row and same column.

Quantum Latin squares

Quantum Latin squares are combinatorial structures analogous to Latin squares. A quantum Latin square is an array of n rows and n columns containing elements of C^n such that each row and each column forms an orthonormal basis for C^n. As above, we define the inner product by $\langle a|b\rangle = \sum_{i=0}^{n} a_i^* b_i$ for a and b in $\mathbb{C}^n$ [25].

Definition (Quantum Latin Square)**.** *Let Q be an $n \times n$ array of elements of $\mathbb{C}^n$ for some $n \in \mathbb{N}$. Then Q is a quantum Latin square of order n if each row and each column of Q forms an orthonormal basis of $\mathbb{C}^n$ according to the inner product $\langle a|b\rangle = \sum_{i=0}^{n-1} a_i^* b_i$ for a and b in $\mathbb{C}^n$. That is, distinct elements of a row or column are orthogonal to each other, and each element of the array is normalized.*

Quantum Latin squares were introduced by Musto and Vicary in 2016 [20]. Let $B_n = \{|i\rangle \mid i \in [n]\}$ be a set of orthonormal quantum state vectors in a quantum state space of dimension n. Using this notation, we often call B_n the computational basis. We can represent any state vector in the space as a linear combination of the elements of B_n over $\mathbb{C}$. Then we can easily convert any classical Latin square over $[n]$ to a quantum Latin square with the map $[n] \to B_n : i \to |i\rangle$. Note that if the Latin square is over some other set of order n, we can easily map the elements of the set to the elements of $[n]$ and then use the above map to convert the Latin square to a quantum Latin square. Below is an example of order 3.

$$\begin{array}{|c|c|c|c|}\hline 0 & 1 & 2 & 3 \\ \hline 1 & 0 & 3 & 2 \\ \hline 2 & 3 & 0 & 1 \\ \hline 3 & 2 & 1 & 0 \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|c|}\hline |0\rangle & |1\rangle & |2\rangle & |3\rangle \\ \hline |1\rangle & |0\rangle & |3\rangle & |2\rangle \\ \hline |2\rangle & |3\rangle & |0\rangle & |1\rangle \\ \hline |3\rangle & |2\rangle & |1\rangle & |0\rangle \\ \hline \end{array}$$

Figure 4.2: Latin Square evolve to quantum Latin Square

However, some quantum Latin squares are not the result of such a simple map. Below we give an example from Musto's paper [20].

$\|0\rangle$	$\|1\rangle$	$\|2\rangle$	$\|3\rangle$
$\frac{1}{\sqrt{2}}(\|1\rangle - \|2\rangle)$	$\frac{1}{\sqrt{5}}(i\,\|0\rangle + 2\,\|3\rangle)$	$\frac{1}{\sqrt{5}}(2\,\|0\rangle + i\,\|3\rangle)$	$\frac{1}{\sqrt{2}}(\|1\rangle + \|2\rangle)$
$\frac{1}{\sqrt{2}}(\|1\rangle + \|2\rangle)$	$\frac{1}{\sqrt{5}}(2\,\|0\rangle + i\,\|3\rangle)$	$\frac{1}{\sqrt{5}}(i\,\|0\rangle + 2\,\|3\rangle)$	$\frac{1}{\sqrt{2}}(\|1\rangle - \|2\rangle)$
$\|3\rangle$	$\|2\rangle$	$\|1\rangle$	$\|0\rangle$

Figure 4.3: Example of Quantum Latin Square of order 4

The element in quantum Latin squares are quantum states which can be in superposition of states as shown in the figure 4.3.

$$\langle \psi_1 \mid \psi_2 \rangle = \sum_{i=0}^{n-1} \psi_1^* \psi_2$$

Normalized

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle) \tag{4.1}$$

$$|\psi\rangle \rightarrow \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \tag{4.2}$$

$$\Rightarrow \langle \psi| = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix} \tag{4.3}$$

$$\Rightarrow \langle \psi \mid \psi \rangle = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \tag{4.4}$$

$$= 1 \tag{4.5}$$

Orthogonal

$$|\psi_1\rangle \to \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \tag{4.6}$$

$$|\psi_2\rangle \to \begin{bmatrix} \frac{i}{\sqrt{5}} \\ 0 \\ 0 \\ \frac{2}{\sqrt{5}} \end{bmatrix} \tag{4.7}$$

$$\Rightarrow \langle\psi_1| = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix} \tag{4.8}$$

$$\Rightarrow \langle\psi_1 \mid \psi_2\rangle = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} \frac{i}{\sqrt{5}} \\ 0 \\ 0 \\ \frac{2}{\sqrt{5}} \end{bmatrix} \tag{4.9}$$

$$= (0)(\frac{1}{\sqrt{5}}) + \left(\frac{1}{\sqrt{2}}\right)(0) + (-\frac{1}{\sqrt{2}})(0) + (0)(\frac{2}{\sqrt{5}}) \tag{4.10}$$

$$= 0 \tag{4.11}$$

4.1 Code

You might consider including a section describing your algorithm and code. The minted package is great for displaying short sections of code, as in Listing 4.1.

```python
# Ian McLoughlin, 2018-02-01
# Is it Tuesday?

import datetime

if datetime.datetime.today().weekday() == 1:
  print("Yay! It is Tuesday.")
else:
  print("Unfortunately it is not Tuesday.")
```

Listing 4.1: Is it Tuesday?

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Chapter 5

Quantum Error Correction

Chapter 6

Comprehensive Analysis

You will sometimes, over the course of your research, find something particularly interesting. If you think it is worthy of a published paper in its own right, you can add a chapter about it.

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Table 6.1 is a nice table.

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Column 1	Column 2
Hello	world!

Figure 6.1: A table.

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Chapter 7

Conclusion

What Should the Conclusion Section of a PhD Thesis Include? The conclusion section of a Ph.D. thesis serves as the final chapter where you summarize the key findings of your research, discuss their implications, and reflect on the contributions of your work to the field. Below are some items you should include in the conclusion.

Summary of Key Findings

Recapitulate the main results and outcomes of your research. Summarize the key findings, discoveries, or insights obtained through your investigation.

Discussion of Research Questions or Hypotheses

Reflect on how your research has addressed the research questions or hypotheses posed at the beginning of your study. Discuss whether your findings support or refute the initial hypotheses and how they contribute to advancing knowledge in the field.

Implications and Contributions

Discuss the broader implications of your research findings. Explain how your work contributes to theoretical understanding, practical applications, or methodological advancements within your discipline. Highlight the significance of your research and its potential impact on academic research, policy-making, industry practices, or society as a whole.

Limitations and Future Directions

Acknowledge any limitations or constraints encountered during your research, such as methodological limitations, data constraints, or external factors beyond your control. Discuss opportunities for future research that arise from your findings. Identify unanswered questions, areas for further investigation, or new research directions that could build upon your work.

Reflection and Personal Insights

Reflect on your journey as a researcher throughout the Ph.D. process. Discuss any challenges, successes, or unexpected discoveries you encountered along the way. Share personal insights or lessons learned from conducting your research, including any changes in your perspectives, methodologies, or research practices.

Final Remarks

Provide a concluding statement that synthesizes the main points of your conclusion. Emphasize the significance of your research and its broader implications for the field. Express gratitude to those who have supported you during your Ph.D. journey, including mentors, advisors, colleagues, friends, and family members.

Overall, the conclusion section of a Ph.D. thesis should effectively summarize your research journey, highlight the contributions of your work, and articulate its significance within the broader scholarly context. It should leave the reader with a clear understanding of the key takeaways from your study and inspire further inquiry and exploration in the field.

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