A Smooth Real-Analytic Extension of the Collatz Map and a Global Boundedness Conjecture (extended): conjectures, conditional verification, and continuous/complex-time extension

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Abstract

A real-analytic self-map of $\mathbb{R}$ is studied which coincides exactly with the classical Collatz map on the integers. The induced discrete-time dynamical system

$$x_{t+1} = f(x_t)$$

is investigated. A global boundedness conjecture for all real initial conditions is formulated, together with several weaker variants and related side problems. Basic structural properties are established: analyticity, exact integer agreement, existence and local attraction of a fixed point at the origin, and a forward-invariant contracting neighborhood. Local expansions at integers, derivative growth bounds, sign-preservation, Lyapunov-candidate analysis, and a conditional reduction of the conjecture to a finite interval-arithmetic verification are presented. A new side problem is added asking for an analytic (or meromorphic) extension of the discrete-time orbit $\{x_t\}_{t\in\mathbb{N}}$ to a function x_s defined for all real or complex s; background on fractional iteration (Abel and Schröder equations) and a precise statement of the extension problem are included. Numerical-experiment scaffolding, implementation pseudocode, and appendices provide further material for computational study.

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1 Introduction

The Collatz $(3x+1)$ map on the integers,

$$T(n) = \begin{cases} n/2, & n \text{ even}, \\ 3n+1, & n \text{ odd}, \end{cases} \qquad n \in \mathbb{Z},$$

poses an elementary yet unresolved problem: whether every positive integer reaches 1 under iteration. Embedding T in a smooth dynamical system on $\mathbb{R}$ permits analytic and numerical tools not directly available in the discrete integer setting. The present work considers the map

$$\boxed{f(x) = \frac{x}{2}\cos^2\left(\frac{\pi x}{2}\right) + (3x+1)\sin^2\left(\frac{\pi x}{2}\right)}$$

which agrees exactly with T on integers and depends smoothly on x elsewhere. The primary aim is to state and investigate a conjecture asserting boundedness of all real orbits, provide rigorous local results, and formulate concrete computational and theoretical programs towards verification. A new side problem asks for extending the discrete-time orbit $\{x_t\}$ to real or complex time via classical functional-iteration techniques; the extension problem is detailed and related to known functional equations.

2 Definition and notation

Definition 2.1. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \frac{x}{2}\cos^2\left(\frac{\pi x}{2}\right) + (3x+1)\sin^2\left(\frac{\pi x}{2}\right).$$

For an initial condition $x_0 \in \mathbb{R}$ define the forward orbit by

$$x_{t+1} = f(x_t), \qquad t = 0, 1, 2, \ldots.$$

Iterates will also be denoted by $f^{(n)}$ for the n-fold composition of f.

Throughout the manuscript x_t denotes the state at integer time t; whenever fractional or complex-time extensions are discussed the notation x_s is used for a putative extension to real or complex s.

3 Main conjecture and weaker variants

Conjecture 3.1 (Global Boundedness)**.** *For every $x_0 \in \mathbb{R}$ the forward orbit $\{x_t\}_{t\geq 0}$ is bounded:*

$$\sup_{t\geq 0} |x_t| < \infty.$$

Several weaker formulations are of independent interest:

Conjecture 3.2 (Almost-everywhere boundedness)**.** *The set $\{x_0 \in \mathbb{R} : \sup_{t\geq 0} |x_t| = \infty\}$ has Lebesgue measure zero.*

Conjecture 3.3 (Restricted-subset boundedness)**.** *For specified subsets $S \subset \mathbb{R}$ (e.g., $\mathbb{Q}$, algebraic numbers, reals with restricted fractional parts), Conjecture 3.1 holds for every $x_0 \in S$.*

Conjecture 3.4 (Non-full-measure failure)**.** *The set of initial conditions violating Conjecture 3.1 has Lebesgue measure strictly less than that of $\mathbb{R}$.*

4 Side problems and research program

The following concrete side problems form a research program surrounding Conjecture 3.1.

4.1 Periodic and preperiodic points

Problem 4.1. Classify all $x_0 \in \mathbb{R}$ and integers $n \geq 1$ for which $x_n = x_0$.

Problem 4.2. Describe all preperiodic points, i.e., all x_0 for which there exist $m < n$ with $x_m = x_n$.

4.2 Integer capture

Problem 4.3. Classify the set $\{x_0 \in \mathbb{R} : \ \exists t \geq 0, \ x_t \in \mathbb{Z}\}$.

4.3 Invariant bounded sets

Problem 4.4. Describe the maximal forward-invariant bounded set

$$\Lambda_{\max} = \{x \in \mathbb{R} : \ \{f^{(n)}(x)\}_{n \geq 0} \text{ is bounded}\}.$$

Is $\Lambda_{\max}$ a finite union of intervals together with Cantor-like subsets, or does it possess a more complex topology?

4.4 Lyapunov functions

Problem 4.5. Construct a continuous function $V : \mathbb{R} \to [0, \infty)$ and constants $R, c > 0$ such that

$$V(f(x)) \leq V(x) - c \quad \text{for } |x| > R,$$

or show no such smooth V exists.

4.5 Extension to continuous / complex time (new side problem)

Problem 4.6 (Continuous and complex-time extension of orbits)**.** Does there exist a function x_s defined for all real $s \in \mathbb{R}$ (or for all complex $s \in \mathbb{C}$, meromorphic or analytic in s for each fixed initial condition), such that:

1. For integer $t \geq 0$, x_t coincides with the discrete orbit: $x_t = f^{(t)}(x_0)$.
2. The map $s \mapsto x_s$ satisfies a semigroup property: $x_{s+t} = F_s(x_t)$ for an appropriate family $\{F_s\}$ with $F_1 = f$ (equivalently x_s is an iterative flow of f).
3. The extension is analytic (or meromorphic) in s and depends appropriately on x_0.

Investigate existence, uniqueness, regularity, functional equations that characterize such extensions (Abel equation, Schröder equation), and the global behaviour of x_s as $\Re(s) \to \infty$ or $\Im(s) \to \infty$.

Remark 4.7. Problem 4.6 is analogous to extending the factorial $n!$ to the Gamma function $\Gamma(s)$: here the discrete orbit $\{x_t\}$ would be extended to a continuous/complex-time function. Classical tools include the Abel functional equation and Schröder conjugacy near fixed points; substantial obstacles include analytic continuation across different dynamical regimes and nonlinearity of f.

5 Rigorous local results

5.1 Analyticity

Proposition 5.1. *The function f is real-analytic on $\mathbb{R}$.*

Proof. f is a finite combination of polynomials and trigonometric functions (sin, cos), all entire. □

5.2 Exact integer agreement

Lemma 5.2. *For every integer n,*

$$f(n) = \begin{cases} n/2, & n \text{ even}, \\ 3n+1, & n \text{ odd}. \end{cases}$$

5.3 Fixed point at the origin and local attraction

Proposition 5.3. *$x = 0$ is a fixed point of f and is locally attracting.*

Proof. Direct substitution yields $f(0) = 0$. Differentiation gives

$$f'(x) = \frac{1}{2} + \frac{5}{2}\sin^2\left(\frac{\pi x}{2}\right) + \left(\frac{5x}{2} + 1\right)\frac{\pi}{2}\sin(\pi x),$$

thus $f'(0) = 1/2$. □

Theorem 5.4 (Local invariant interval). *There exists $r > 0$ such that $f([-r, r]) \subset (-r, r)$ and f is a contraction on $[-r, r]$.*

6 Derivative bounds and sign-preservation

Proposition 6.1. *There exists $C > 0$ with*

$$|f'(x)| \leq C(1 + |x|)$$

for all $x \in \mathbb{R}$.

Lemma 6.2 (Sign-preservation). *If $x_t \geq 0$ then $x_{t+1} \geq 0$. If $x_t \leq -\frac{1}{3}$ then $x_{t+1} \leq 0$.*

7 Local expansions near integers

Let $n \in \mathbb{Z}$ and set $x = n + u$.

7.1 Even integers ($n = 2k$)

$$f(n+u) = \frac{n}{2} + \frac{u}{2} + \left(\frac{5n}{2} + 1\right)\frac{\pi^2}{4}u^2 + O(u^3).$$

7.2 Odd integers ($n = 2k + 1$)

$$f(n+u) = 3n + 1 + 3u - \left(\frac{5n}{2} + 1\right)\frac{\pi^2}{4}u^2 + O(u^3).$$

8 Lyapunov-candidate analysis

Quadratic Lyapunov candidates fail globally; phase-weighted quadratic forms $V(x) = \alpha(x)x^2$ with 2-periodic α are plausible. The functional inequality

$$\alpha(f(x)) \left(\frac{f(x)}{x}\right)^2 \leq \alpha(x)$$

for large $|x|$ is a target for numerical and interval-based search.

9 Conditional finite verification via interval arithmetic

A practical computational path is outlined:

1. Choose $R > 0$ and find a trapping neighborhood $[-R, R]$ with $f([-R, R]) \subset [-R, R]$.
2. Partition $[-M, M]$ (large M) into intervals I_k.
3. Using rigorous interval arithmetic compute $f(I_k) \subset J_k$ and verify that J_k is covered by unions of I_j's and/or leads to intervals mapping into the trapping set. Build directed graph of interval images.
4. If every interval reaches the trapping set in finitely many steps, all orbits starting in $[-M, M]$ enter the trap. Use linear-growth bounds to show tails enter $[-M, M]$.

A precise conditional theorem reduces Conjecture 3.1 to such a finite verification.

10 Continuous and complex-time extension: background and strategy

This section gives concise background on methods used to define fractional/continuous iterates of self-maps.

10.1 Schröder and Abel functional equations

Let f be analytic near a fixed point x^* with multiplier $\lambda = f'(x^*)$. The Schröder equation seeks Φ such that

$$\Phi(f(x)) = \lambda\Phi(x),$$

while the Abel equation seeks A such that

$$A(f(x)) = A(x) + 1.$$

If Φ or A exist with suitable analytic properties, fractional iterates can be defined by

$$f^{(s)}(x) = \Phi^{-1}(\lambda^s\Phi(x)) \quad \text{or} \quad f^{(s)}(x) = A^{-1}(A(x) + s).$$

These constructions are local near fixed points, and analytic continuation is required to produce global extensions.

10.2 Obstacles and global issues

Obstacles include:

- Nonlinear dependence of f on x, creating multiple dynamical regimes (near even integers, odd integers, and transition regions).
- Presence of non-hyperbolic or neutral fixed points.
- Need to analytically continue conjugacies across singularities or domains of distinct behavior.

10.3 Statement of the extension program

Problem 10.1 (Practical extension program)**.** For given initial x_0, attempt the following:

1. Identify an attracting fixed point or cycle p near which x_0 eventually lands, attempt to construct a local Schröder conjugacy and define $f^{(s)}(x)$ for $s \in \mathbb{R}$ near p.
2. Use analytic continuation and patching (overlap domains) to extend $f^{(s)}$ along orbits, controlling monodromy and branch choices.
3. Investigate whether for typical x_0 the constructed x_s extends to an entire/meromorphic function of s.

This program presents a rich interplay between complex dynamics, functional equations, and number-theoretic behavior on integers.

11 Numerical experiments and reproducible pseudocode

Implementation notes and pseudocode are given to support numerical exploration and the interval-arithmetic program; see Appendix for detailed pseudocode and sample data.

12 Generalizations and parameter families

Consider $f_\alpha(x) = \frac{x}{2}\cos^2(\frac{\pi x}{2}) + (\alpha x + 1)\sin^2(\frac{\pi x}{2})$. Dynamics as α varies are of interest; numerical bifurcation exploration is suggested.

13 Discussion and conclusion

The extended draft presents a structured research program: a clear main conjecture, a hierarchy of weaker statements, a set of concrete side problems (including the new continuous/complex-time extension), rigorous local results, an explicit conditional computational reduction, and notes on practical implementation. The program is amenable to both analytic and computer-assisted attack.

A Appendix A: Pseudocode for interval arithmetic verification

High-level pseudocode suitable for implementation in Arb, MPFI, or IntervalArithmetic.jl:

```
INPUT: M large, partition initial N, trapping radius R
Partition [-M,M] into intervals I_k
S = indices covering [-R,R]
while True:
  for k in 1..N:
    Jk = interval_eval_f(I_k)   # rigorous directed rounding
    Ak = { j : I_j intersects Jk }
  if every index reaches S in directed graph:
    report SUCCESS
    break
  else:
    refine intervals failing to reach S and continue
```

B Appendix B: Numerical experiment sketches

Python/NumPy sketches for sample orbits and max-excursion maps (non-validated):

```
import numpy as np, math
def f(x):
    return (x/2.0)*(math.cos(math.pi*x/2.0)**2) + (3.0*x+1.0)*(math.sin(math.pi*x/2.0)**2)
def orbit(x0, T):
    x = x0
    xs = [x]
    for t in range(T):
        x = f(x)
        xs.append(x)
    return xs
```

C Appendix C: Additional background references

References on fractional iteration, Abel/Schröder equations, and interval arithmetic implementations are suggested: Kuczma (functional equations), Szekeres (iterative methods), and Arb/MPFI documentation.

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