財務工程 - Week6

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Github 作業繳交列表

https://docs.google.com/spreadsheets/d/1ZGO3ciyAhqZerbG3wUZd9zhHIxgqh8mvHbw0sqpQGN4/edit?usp=sharing

Martingale

Martingale

- Martingale 源於法文，原意是指馬的籠套，同時也指一種逢輸就加倍賭註，直到贏為止的惡性賭博方法 (double strategy)。
- Martingale 簡單來說，是公平賭局 (fair game)，假設一個人已賭了 n 次，正準進行第 n+1 次賭博。如果賭局沒有做任何詐賭的手腳，此人的運氣應當與他以前的賭博經歷無關，用 Xn 表示他在賭完第 n 次後擁有的賭本數為

$$E[\, X_{n+1} \mid X_1, X_2, \ldots, X_n \,] = X_n$$

- 即賭博的期望收穫為 0，僅能維持原有財富不變，就可以認為這種賭博在統計上是公平的。
- Harrison 及 Kreos (1979) 提出 Martingale 求衍生性金融商品的定價。

EXAMPLE 13.2.2 Consider the stochastic process $\{Z_n - n\mu, n \geq 1\}$, where $Z_n \equiv \sum_{i=1}^{n} X_i$ and $X_1, X_2, \ldots,$ are independent random variables with mean μ. As

$$\begin{aligned} E[Z_{n+1} - (n+1)\mu \mid X_1, X_2, \ldots, X_n] &= E[Z_{n+1} \mid X_1, X_2, \ldots, X_n] - (n+1)\mu \\ &= Z_n + \mu - (n+1)\mu \\ &= Z_n - n\mu, \end{aligned}$$

$\{Z_n - n\mu, n \geq 1\}$ is a martingale with respect to $\{I_n\}$, where $I_n \equiv \{X_1, X_2, \ldots, X_n\}$.

Theorems about mean, variance

- Properties of mean, variance for one random variable X, where a and b are constant:
  - $E[aX+b] = aE[X] + b$
  - $Var(aX+b) = a^2 Var(X)$
  - $Var(X) = E[X^2] - (E[X])^2$
- Theorem. Let X and Y be independent random variables and let g and h be real valued functions of a single real variable.

$$E[g(X)h(Y)] = E[g(X)]E[h(Y)].$$

- Theorem. For random variables $X_1, X_2, \ldots, X_n$, defined on the same sample space, and for constants $a_1, a_2, \ldots, a_n$, we have

$$E\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i E(X_i).$$

Brownian motion

- Brownian motion is a stochastic process $\{X(t), t \geq 0\}$ with the following properties.

  1. $X(0) = 0$, unless stated otherwise.

  2. for any $0 \leq t_0 < t_1 < \cdots < t_n$, the random variables

$$X(t_k) - X(t_{k-1})$$

  for $1 \leq k \leq n$ are independent.

  3. for $0 \leq s < t$, $X(t) - X(s)$ is normally distributed with mean $\mu(t-s)$ and variance $\sigma^2(t-s)$, where μ and $\sigma \neq 0$ are real numbers.

- The existence and uniqueness of such a process is guaranteed by Wiener's theorem.
- This process will be called a (μ, σ) Brownian motion with drift μ and variance σ^2.
- Although Brownian motion is a continuous function of t with probability one, it is almost nowhere differentiable.
- The $(0, 1)$ Brownian motion is called the Wiener process.

Geometric Brownian Motion and Brownian Motion

- Consider geometric Brownian motion

$$Y(t) \triangleq e^{X(t)}.$$

  - $X(t)$ is a (μ, σ) Brownian motion.

$$dX = \mu\, dt + \sigma\, dW.$$

stochastic calculus

Stochastic Calculus

- 隨機分析 (stochastic calculus) 是機率論的一個分支。主要內容有伊藤積分 (Ito integral) 與隨機微分方程。
- 隨機模型是指含有隨機成分的模型。與確定性模型的不同處，用下例子解釋：

在賭場裡賭大小，如果有人認為三次連開大第四次必然開小，那麼此人所用的既是確定性模型。但常識告訴我們第四次的結果並不一定與之前的結果相關聯。

Ito Process

- The stochastic process $X = \{ X_t, t \geq 0 \}$ that solves

$$X_t = X_0 + \int_0^t a(X_s, s)\, ds + \int_0^t b(X_s, s)\, dW_s, \quad t \geq 0$$

  is called an Ito process.

  - X_0 is a scalar starting point.
  - $\{ a(X_t, t) : t \geq 0 \}$ and $\{ b(X_t, t) : t \geq 0 \}$ are stochastic processes satisfying certain regularity conditions.
  - $a(X_t, t)$: the drift.
  - $b(X_t, t)$: the diffusion.

- A shorthand is the following stochastic differential equation (SDE) for the Ito differential dX_t,

$$dX_t = a(X_t, t)\, dt + b(X_t, t)\, dW_t.$$

  - Or simply

$$dX_t = a_t\, dt + b_t\, dW_t.$$

  - This is Brownian motion with an *instantaneous* drift a_t and an *instantaneous* variance b_t^2.

- X is a martingale if $a_t = 0$.

- From calculus, we would expect $\boxed{\int_0^t W\, dW} = W(t)^2/2$.
- But $W(t)^2/2$ is not a martingale, hence wrong!
- The correct answer is $\boxed{[\,W(t)^2 - t\,]/2.}$
- An equivalent form of $dX_t = a(X_t, t)\, dt + b(X_t, t)\, dW_t$

$$\boxed{dX_t = a_t\, dt + b_t \sqrt{dt}\; \xi,}$$

where $\xi \sim N(0, 1)$.

For example, $\int W\,dW$ can be approximated as follows:

$$\sum_{k=0}^{n-1} W(t_k)[\,W(t_{k+1}) - W(t_k)\,]$$

$$= \sum_{k=0}^{n-1} \frac{W(t_{k+1})^2 - W(t_k)^2}{2} - \sum_{k=0}^{n-1} \frac{[\,W(t_{k+1}) - W(t_k)\,]^2}{2}$$

$$= \frac{W(t)^2}{2} - \sum_{k=0}^{n-1} \frac{[\,W(t_{k+1}) - W(t_k)\,]^2}{2}.$$

Because the second term above converges to $t/2$

$$\int_0^t W\,dW = \frac{W(t)^2}{2} - \frac{t}{2}.$$

Ito integral

Ito integral

- 伊藤積分(英語:Itō integral)是將積分的概念擴展到隨機過程中，像布朗運動就可以用伊藤積分進行分析。
- 伊藤積分，可將一個隨機過程(被積分函數)對另一個隨機過程(積分變數)進行積分。積分變數一般是布朗運動。從 0 到 t 的積分結果是一個隨機變數。

Trading and the Ito Integral

- Consider an Ito process

$$dS_t = \mu_t \, dt + \sigma_t \, dW_t.$$

  - S_t is the vector of security prices at time t.

- Let ϕ_t be a trading strategy denoting the quantity of each type of security held at time t.
  - Hence the stochastic process $\phi_t S_t$ is the value of the portfolio ϕ_t at time t.

- $\phi_t \, dS_t \triangleq \phi_t(\mu_t \, dt + \sigma_t \, dW_t)$ represents the change in the value from security price changes occurring at time t.

$\boldsymbol{\phi}_t \, d\boldsymbol{S}_t \triangleq \boldsymbol{\phi}_t(\mu_t \, dt + \sigma_t \, dW_t)$ represents the change in the value from security price changes occurring at time t.

- The equivalent Ito integral,

$$G_T(\boldsymbol{\phi}) \triangleq \int_0^T \boldsymbol{\phi}_t \, d\boldsymbol{S}_t = \int_0^T \boldsymbol{\phi}_t \mu_t \, dt + \int_0^T \boldsymbol{\phi}_t \sigma_t \, dW_t,$$

measures the gains realized by the trading strategy over the period $[0, T]$.

Ito's Lemma

- 在隨機分析中，伊藤引理(Ito's lemma)是一條非常重要的性質。
- 發現者為日本數學家伊藤清，他指出了對於一個隨機過程的函數作微分的規則。

$$\boxed{dX = a_t\, dt + b_t\, dW}$$

- In differential form, Ito's lemma becomes

$$df(X) = f'(X)\, a\, dt + f'(X)\, b\, dW + \frac{1}{2} f''(X)\, b^2\, dt$$

A smooth function of an Ito process is itself an Ito process.

The stochastic process $X = \{X_t, t \geq 0\}$ that solves

$$X_t = X_0 + \int_0^t a(X_s, s)\, ds + \int_0^t b(X_s, s)\, dW_s, \quad t \geq 0$$

is called an Ito process.

$$\begin{aligned} & f(X_t) \\ = \quad & f(X_0) + \int_0^t f'(X_s)\, a_s\, ds + \int_0^t f'(X_s)\, b_s\, dW \\ & + \frac{1}{2} \int_0^t f''(X_s)\, b_s^2\, ds \end{aligned}$$

for $t \geq 0$.

Lemma：小小的定理，通常是為了證明後面的定理，如果證明的篇幅很長時，可能會把證明拆成幾個部分來論述，雖然篇幅可能變多，但派絡卻很清楚。

- Ito Lemma 是從兩變數泰勒展開式開始，一階偏微分的部分跟原本的連鎖律一樣。
- 但加入一項對布朗運動的二階偏微分，原因是布朗運動一階不可微分，但二階可微。

- In differential form, Ito's lemma becomes

$$df(X) = f'(X)\,a\,dt + f'(X)\,b\,dW + \frac{1}{2}f''(X)\,b^2\,dt$$

$$dX = a_t\,dt + b_t\,dW$$

$$df(X) = f'(X)\,dX + \frac{1}{2}f''(X)(dX)^2.$$

- We are supposed to multiply out $(dX)^2 = (a\,dt + b\,dW)^2$ symbolically according to

$\times$	dW	dt
dW	dt	0
dt	0	0

  - The $(dW)^2 = dt$ entry is justified by a known result.

- Hence $(dX)^2 = (a\,dt + b\,dW)^2 = b^2\,dt$
- This form is easy to remember because of its similarity to the Taylor expansion.

- Ito's formula $\boxed{df(X) = f'(X)\, a\, dt + f'(X)\, b\, dW + \frac{1}{2} f''(X)\, b^2\, dt}$

$$
\begin{aligned}
dY &= Y\, dX + (1/2)\, Y\, (dX)^2 \\
&= Y\, (\mu\, dt + \sigma\, dW) + (1/2)\, Y\, (\mu\, dt + \sigma\, dW)^2 \\
&= Y\, (\mu\, dt + \sigma\, dW) + (1/2)\, Y \sigma^2\, dt.
\end{aligned}
$$

- Hence

$$
\frac{dY}{Y} = (\mu + \sigma^2/2)\, dt + \sigma\, dW.
$$

Geometric Brownian Motion and Brownian Motion

- Consider geometric Brownian motion

$$Y(t) \triangleq e^{X(t)}.$$

  - $X(t)$ is a (μ, σ) Brownian motion.

$$dX = \mu\, dt + \sigma\, dW.$$

- Similarly, suppose

$$\frac{dY}{Y} = \mu\, dt + \sigma\, dW.$$

- Then $X(t) \triangleq \ln Y(t)$ follows

$$dX = \left(\mu - \sigma^2/2\right) dt + \sigma\, dW.$$

考慮股價模型為 Geometric Brownian Motion，故可寫做下列隨機微分方程 SDE

$$\frac{dY}{Y} = \mu\, dt + \sigma\, dW.$$

$$dS_t := \mu S_t dt + \sigma S_t dB_t,\ 0 \le t \le T$$

其中 S_t 為時刻 t 的股價， $\mu :=$ 股價每年的收益率期望值 (或稱 drift rate)，$\sigma :=$ 股價每年的波動度 (volatility)，B_t 為標準布朗運動。注意到這邊我們假設 μ, σ 為固定常數。

https://ch-hsieh.blogspot.com/2012/04/how-to-solve-sde-practically-4.html

$$S_t = S_0 \exp\left\{ \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma B_t \right\}$$

改寫上式

$X(t) \overset{\Delta}{=} \ln Y(t)$ follows

$$dX = \left(\mu - \sigma^2/2\right) dt + \sigma\, dW.$$

$$\Rightarrow \ln \frac{S_t}{S_0} = \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma B_t$$

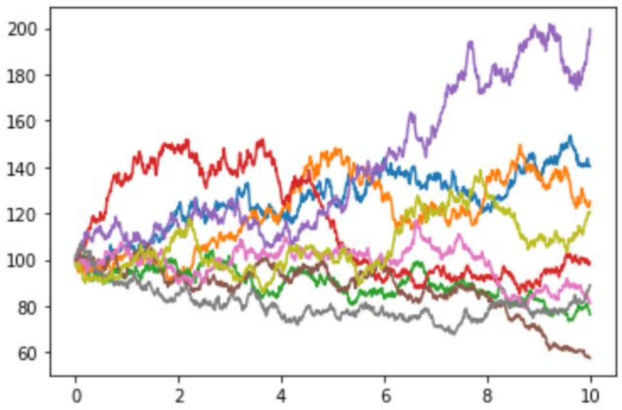
由於 B_t 為標準布朗運動，由定義可知標準布朗運動為 Gaussian process with mean 0, variance t 與 covariance $\min(s,t)$，故我們可推論

$$
\begin{aligned}
&\ln \frac{S_T}{S_0} \sim \mathcal{N}((\mu - \frac{\sigma^2}{2})T, \sigma^2 T) \\
&\Rightarrow \ln S_T \sim \mathcal{N}(\ln S_0 + (\mu - \frac{\sigma^2}{2})T, \sigma^2 T)
\end{aligned}
$$

其中 S_T 是未來時間 T 時的股價，S_0 是時間 0 時的股價。上式表明 $\ln S_T$ 服從 normal distribution，故 S_T 為Log-normal (亦即 取 Log之後為normal)，

```
[6]     return S
```

```
[7] paths = []
    for i in range(0,N-1):
        paths.append(genBrownPath(T, mu, sigma, S0, dt))
    plt.show()
```



https://colab.research.google.com/drive/1LL_m1UO_U2oHDMQhBDPjhUBANDpVhev7

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Q and A