In the name of God

Sharif University of Technology

Department of Economics

Quantitative Economics - Spring 2024

Problem Set 2

Read and practice Chapter 17 from the Book by Sargent: Quant Econ for Julia then Answer these questions.

1 Simulating a Household

In this problem we would like to see how simulations can help us address aggregation issues in macroeconomics where heterogeneity exists or simple classic aggregation assumptions do not hold.

Consider Household i (HH_i) with preferences over consumption and leisure

$$U_i = \frac{C^{1-\sigma}}{1-\sigma} - \gamma \frac{l^{1+\phi}}{1+\phi}$$

and maximizes it subject to

$$pC = wl + T$$

where l is the labor supplied. Take $T = 1$.

1. Solve the HH_i's problem analytically.

2. Assume that $\left(\log \frac{w}{p}\right)$ is randomly distributed such that $\left(\log \frac{w}{p}\right) \tilde{} N(7, 2)$. Again simulate this economy and find the aggregate consumption. You can take $\sigma = 0.4, \phi = 1.2, \gamma = 1.3$. Plot the histogram of consumptions in this economy.

3. Now instead, assume that ϕ is randomly distributed such that $\phi \tilde{} Uniform(0.3, 3.5)$. Again simulate this economy and find the aggregate consumption. You can take $\sigma = 0.4, \gamma = 1.3, \log \frac{w}{p} = 7$. Plot the histogram of consumptions in this economy.

4. Now assume all $\frac{w}{p}, \phi$ are random with the above distributions and are independent of each other. Simulate this economy and find the aggregate consumption. You can take $\sigma = 0.4, \gamma = 1.3$. Plot the histogram of consumptions in this economy.

5. Now redo part2, with the assumption that T has an independent normal distribution $N(1, 0.5)$.

2 Apple-Orange choice of a HH

Consider a household that maximizes preference

$$U = \left(\alpha X^{1-1/\sigma} + (1-\alpha) Y^{1-1/\sigma}\right)^{\frac{\sigma}{\sigma-1}}$$

subject to the B.C.:

$$pX + qY = I$$

where X, Y are apple and oranges consumed by a household.

1. Solve the HH problem analytically.

2. Now we want to simulate the economy. Suppose α and I have beta and log-normal distributions respectively. Simulate household i's behavior and calculate the aggregate consumption of apples and oranges in the economy.

3 Regression: Behind the Scene

Recall what you have learned about ordinary least squares and what we reviewed in class. In this problem you will estimate OLS coefficients and its standard errors in a number of ways. You will experiment with how sample size, optimizer options, and other parameters change the estimates. But first we need to fabricate a data generating process (DGP).

3.1 Data Generating Process

Write a Julia/Python code to fabricate the following process:

$$y = 1 + 2x_1 - 4x_2 + \epsilon$$

where $x_1 \sim N(15, 7)$, $x_2 \sim Binomial(6, .4)$ and $\epsilon \sim N(0, 1)$ and set sample size 300. To make sure that we all work with exactly the same data, set the random generator's seed number equal to 1395. Draw the histograms.

3.2 Ordinary Least Squares (OLS)

Write down the minimization problem that OLS solves. Then write down the first order condition and solve for β. Hint: $\hat{\beta} = (X'^{-1}X'y$. What are the dimensions of y_i, X_i and β?

3.3 Maximum Likelihood (ML)

Here we want to derive a formula for ML estimator and show that it is similar to OLS estimator in a very special case. Assume $\epsilon_i \sim N(0, \sigma^2)$.

1. write the (log) likelihood contribution for a single individual observation i.
2. Write (log) likelihood for the whole sample.
3. Write the optimization problem which characterizes the maximum likelihood. Highlight why $\hat{\beta}_{ML} = \hat{\beta}_{OLS}$. Hint: $\max_{\beta,\sigma^2} LL(\beta, \sigma^2 | y, X)$.
4. How many variables are being optimized? Is this a constraint optimization? Why?

5. Write the first order condition for σ^2 and derive a formula for standard error of $\hat{\beta}_{ML}$.

3.4 Estimation

Here we want to estimate β using the following 4 methods:

1. Julia/Python method for OLS regression: Call it $\hat{\beta}_{py}$
2. Use algebra to find $\hat{\beta}_{algebra} = (X'^{-1}X'y$
3. Use Julia/Python optimizer to solve for $\hat{\beta}_{optim}$ that minimizes the sum of squared residual for the sample: $\sum_i (y_i - X_i\beta)^2$. You may use `lambda` functions in Julia/Python to define SSR as a function of β. Then find $\hat{\beta}$.
4. Use Julia/Python optimizer to maximize the likelihood function in order to find $\hat{\beta}_{ML}$. Again you should define a `lambda` function in for log likelihood function, then maximize it.

Use sample size of $n = 10, 20, 50, 100, 1000, 1000$ to estimate β and plot $[\hat{\beta}_{py}, \hat{\beta}_{algebra}, \hat{\beta}_{optim}, \hat{\beta}_{ML}]$ as a function of n. You should draw 3 plots for intercept, β_1 and β_2. Interpret.

4 Monte-Carlo Simulation

We will later study Monte-Carlo simulations and their applications in this course. But here we just want to introduce this useful method. Statistical inference (significance levels and hypothesis testing) relies heavily on asymptotic properties of estimators. Most of the important results are actually implications of Law of Large Numbers (LLN) and Central Limit Theorem (CLT) that we studied in the first problem set. I hope that you are convinced that LLN and CLT actually hold in practice, but then may need really large numbers. For small samples analytical results are rarely available, aside from tests of linear restrictions in the linear regression model under normality. Small-sample results can nonetheless be obtained by performing a Monte Carlo study. In this exercise we simulate a simple Monte Carlo study to depict small sample properties of an OLS estimator. Our goal is to first see how Monte Carlo simulations work and then to assure that asymptotic properties of OLS are actually work, but in asymptotic sense!

4.1 Small-Sample Properties

Imagine the true data generating process is $y = 2 + 3x_1 - 5x_2 + \epsilon$. Each time we draw x and y variables, estimated β parameters are themselves random variables. Assume $x_1 \sim N(6, 1)$, $x_2 \sim Binomial(7, .2)$ and $\epsilon \sim N(0, 1)$.

What is the mean and standard deviation of these random variables? (Hint: short answer is that we don't know!) Why are we interested in these statistics?

Monte Carlo simulation is a solution to find properties of the estimators. Here is how to run MC simulation: For $R = 10,000$ times, draw a sample of size $N = 80$ observations from this DGP. Then use OLS to estimate parameters of the model, $(\beta_0, \beta_1, \beta_2)$, while we know that the true parameters are $(2, 3, -5)$. Then draw histogram of your estimated parameters and compare them with expected normal distribution.

Even for this very simple model, it is not easy to compute small sample properties of the OLS estimators. You need all regressors to be distributed normally to have nice distribution. What is the implied distribution of the estimators in this example? Compare the simulated distributions with Normal distribution.

In general, it would be impossible to compute small sample properties of more complicated estimators. These so called "Monte-Carlo simulations" help us to calculate not only mean and standard deviation of estimators, but the complete distribution of estimators.

4.2 Asymptotic versus Small Sample

For the model we studied so far, increase sample size from 40 to 10,000 and compare the distribution of estimated parameters. Interpret.

4.3 True Size of Test

Hypothesis tests lead to either rejection or non-rejection of the null hypothesis. Correct decisions are made if H_0 is rejected when H_0 is false or if H_0 is not rejected when H_0 is true.

There are also two possible incorrect decisions: (1) rejecting H_0 when H_0 is true, called a type I error, and (2) non-rejection of H_0 when H_0 is false, called a type II error. Ideally the probabilities of both errors will be low, but in practice decreasing the probability of one type of error comes at the expense of increasing the probability of the other. The classical hypothesis testing solution is to fix the probability of a type I error at a particular level, usually 0.05, while leaving the probability of a type II error unspecified. Define the size of a test or significance level

$$\alpha = \Pr[\text{type I error}] = \Pr[\text{reject } H_0 | H_0 \text{ true}]$$

The power of a test is defined to be

$$\text{Power} = 1 - \Pr[\text{type II error}] = \Pr[\text{reject } H_0 | H_a \text{ true}]$$

Now think about the 10,000 draw from the DGP and imagine that each time you want to test the null hypothesis that $\beta_1 = 5$. Each time you simulate the data, you may use t-test for this hypothesis. What is the true size and power of this test? If you know the distribution of the error terms you may find analytical solution for the power and size of the test. Or if you have large enough sample, you may use the asymptotic theories to determine the size of the test. The problem arises when neither are available. For instance, in this case you have only $N = 80$ observations, not large enough to use asymptotic properties, and you do not know the distrbution of the error term. Therefore, you need Monte Carlo Simulations to find the true size and power of the test.

If you simulate the DGP enough times, then the **true size** or **actual size** of the test statistic is simply the fraction of replications for which the test statistic falls in the rejection region. Ideally, this is close to the nominal size, which is the chosen significance level of the test. For example, if testing at 5% the nominal test size is 0.05 and the true size is hopefully close to 0.05. The **power** of a test is calculated as the fraction of replications for that the null hypothesis is rejected.

4.4 Number of Replications

Numerous simulations are needed to determine actual test size, because this depends on behavior in the tails of the distribution rather than the center. If R simulations are run for a test of true size α, then the proportion of times the null hypothesis is correctly rejected is an outcome from R binomial trials with mean α. What is the variance of this binomial trial?

Find the 95% interval for the test size α? Hint: You may use CLT.

Then show that a mere 100 simulations is not enough since, for example, this interval is $(0.007, 0.093)$ when $\alpha = 0.05$. For $10,000$ simulations the 95% interval is much more precise, equalling $(0.008, 0.012)$, $(0.046, 0.054)$, $(0.094, 0.106)$, and $(0.192, 0.208)$ for α equal to, respectively, 0.01, 0.05, 0.10, and 0.20. This is why in this example we used $R = 10,000$ simulations.

4.5 Endogeneity

So far, we assumed that x_1 and x_2 are exogenous. Here we will study the problem that arises from endogeneity. Imagine that there is an exogenous variable that affect both x_2 and the y. Think about wage regression, in which the left hand side variable is wage of workers and x_2 is years of schooling. We know that many unobservable factors[1] such as ability, family properties,

[1] Some econometricians, prefer to call this term *unobserved* as some of them are intrinsically *observable* but in practice are not observed for the econometricians. For instance some family properties are generally observable but may not be accessible for the researcher.

etc, affect both. Call these variables z. Hence, imagine that the true data generating process is

$$
\begin{aligned}
x_1 \sim & N(6,1) \\
z \sim & \text{ Binomial}(7, .2) \\
x_2 \sim & \text{ Binomial}(3z, .6) \\
\epsilon = & 13z + \mu_2 \\
y = & 2 + 3x_1 - 5x_2 + \epsilon
\end{aligned}
$$

where μ_1, μ_2 and ϵ are assumed to independently drawn from standard normal distribution. However, you as the researcher, only observe (y, x_1, x_2) for each worker. Repeat paets 1, 2 and 3 for this dgp. Compare and discuss your results.

What is the problem of OLS estimator? Does this problem get solved asymptotically? (i.e. by increasing sample size, N.)

5 Simulator Class

1. Write a Simulator class in Julia/python that can run Monte-Carlo simulation for a given function using a determined pdf.

5.1 Frequency Simulator Class

1. Now inherit a Frequency Simulator class from your Simulator class that can run a frequency simulator.

5.2 Important Sampling Simulator Class

1. Now inherit a Frequency Simulator Simulator class from your Simulator class that can run a frequency simulator.