Artificial Intelligence
ENCS 3340

Constraint Satisfaction Problems (Local Search)

Constraint Satisfaction

- Specifies structural properties of the problem
  - may depend on the representation of the problem
- The problem is defined through a set of variables and a set of domains
  - variables can have possible values specified by the problem
  - constraints describe allowable combinations of values for a subset of the variables
- state in a CSP
  - defined by **an assignment** of **values** to some or all **variables**
- solution to a CSP
  - must assign values to ALL variables
  - must satisfy ALL constraints
  - solutions may be ranked according to an **objective function**

Example1: 3-SAT

Variables:

x_1, x_2, x_3, x_4, x_5

Domains:

{True, False}

Constraints:,=and

$(x_1 \vee x_2 \vee x_4)$,

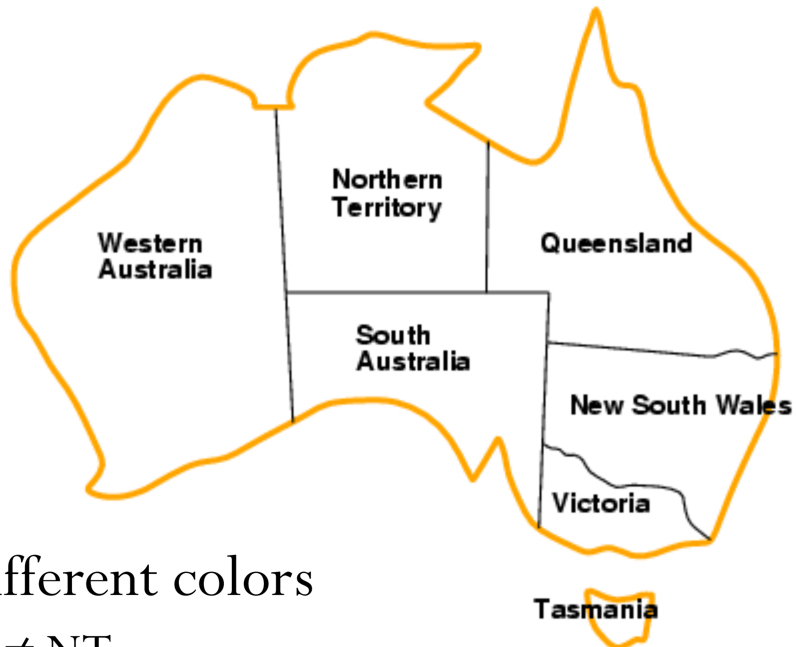
$(x_2 \vee x_4 \vee \neg x_5)$,

$(x_3 \vee \neg x_4 \vee \neg x_5)$

Suggest a solution!

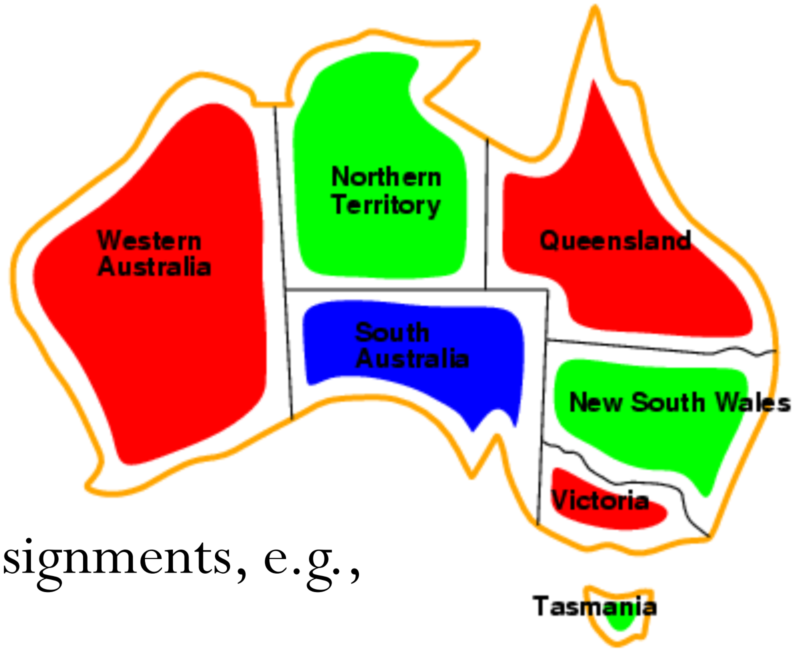
$$(x_1 \vee x_2 \vee x_4) \wedge$$
$$(x_2 \vee x_4 \vee \neg x_5) \wedge$$
$$(x_3 \vee \neg x_4 \vee \neg x_5)$$

Example2: Map-Coloring Problem



- Variables *WA, NT, Q, NSW, V, SA, T*
- Domain D_i = {red, green, blue}
- Constraints: adjacent regions must have different colors
  - e.g., Color(WA) ≠ Color(NT) or in short WA ≠ NT
  - (WA, NT) ∈ {(red, green), (red, blue), (green, red), (green, blue), (blue, red), (blue, green)} OR
  - (WA, NT) ∉ {(red, red), (blue, blue), (green, green)}
  - Graph Coloring Problem (more general)!

Example: Map-Coloring



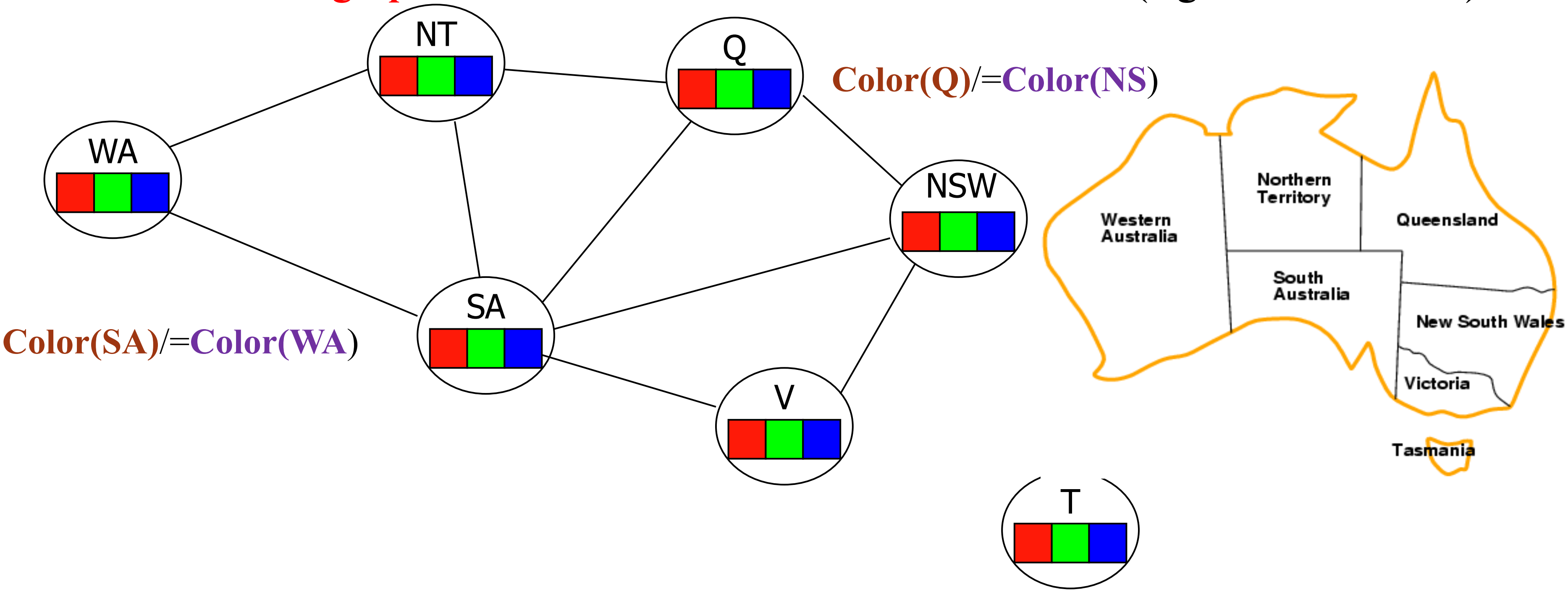
- Solutions are complete and consistent assignments, e.g.,

  WA = red, NT = **green**, Q = red, NSW = **green**, V = red, SA = **blue**, T = **green**

- Complete: all are assigned, consistent: obeys the constraints.
- A state may be incomplete e.g., just WA=red

Constraint graph

- It is helpful to visualize a CSP as a **constraint graph**
  - **Binary CSP:** each constraint relates two variables [here states]
  - **Constraint graph:** nodes are variables, arcs are constraints (e.g. color different)



Varieties of CSPs

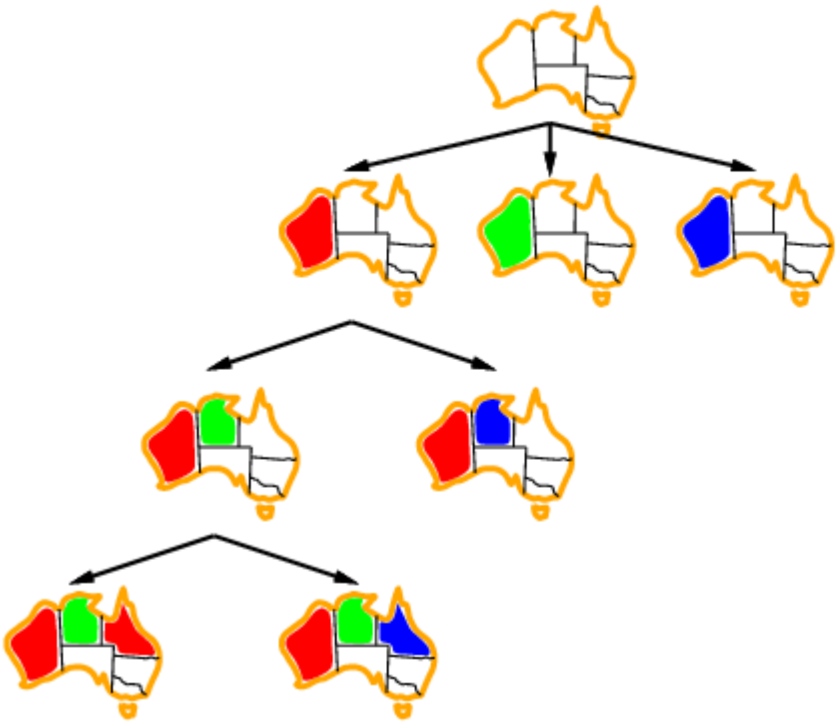
- Discrete variables
  - finite domains:
    - n variables, domain size d , O(dn) complete assignments
    - e.g., Boolean CSPs, incl.~Boolean satisfiability (NP-complete)
  - infinite domains:
    - integers, strings, etc.
    - e.g., job scheduling, variables are start/end days for each job
    - need a constraint language, e.g., StartJob1 + 5 ≤ StartJob3

- Continuous variables
  - e.g., Time: start/end times for Hubble Space Telescope observations
  - linear constraints solvable in polynomial time by linear programming

CSP as Incremental Search Problem

- initial state
  - all (or at least some) variables UNassigned
- successor function
  - assign a value to an UNassigned variable
  - must not conflict with previously assigned variables
- goal test
  - all variables have values assigned
  - no conflicts exist (in the assignments)
- path cost
  - e.g. constant for each step [some colors may be expensive]
  - may be problem-specific

Example



CSPs and Search

In principle, any search algorithm can be used to solve a CSP, but:

- awful branching factor
  - $n*d$ for n variables with d values at the top level, $(n-1)*d$ at the next level, etc.
- not very efficient, since they neglect some CSP properties
  - commutativity: the order in which values are assigned to variables is irrelevant, since the outcome is the same

Backtracking Search for CSPs

A variation of depth-first search that is often used for CSPs

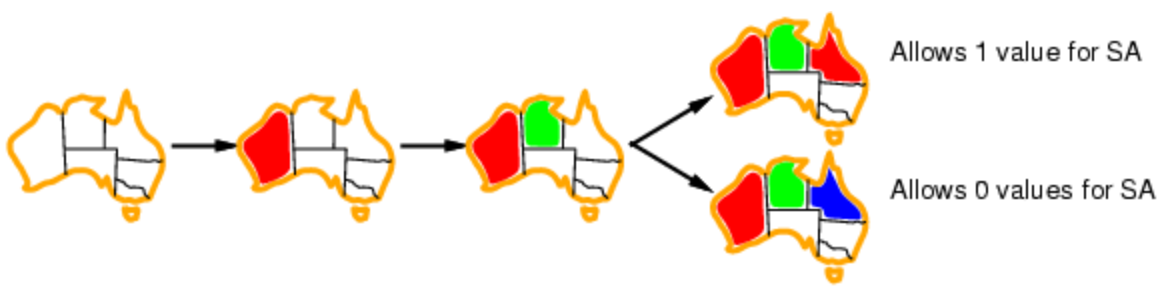
- values are chosen for one variable at a time
- if no legal values are left, the algorithm backs up and changes a **previous assignment**
- very easy to implement
  - initial state, successor function, goal test are standardized
- not very efficient
  - can be improved by trying to select more **suitable unassigned** variables **first**

Improving backtracking efficiency

- General-purpose methods can give huge gains in speed:

1. Which variable should be assigned next? { *WA, NT, Q, NSW,V, SA, T* }

2. In what order should its values be tried? [R,B,G], [R, G, B],…
3. Can we detect inevitable failure early? Case 2 below



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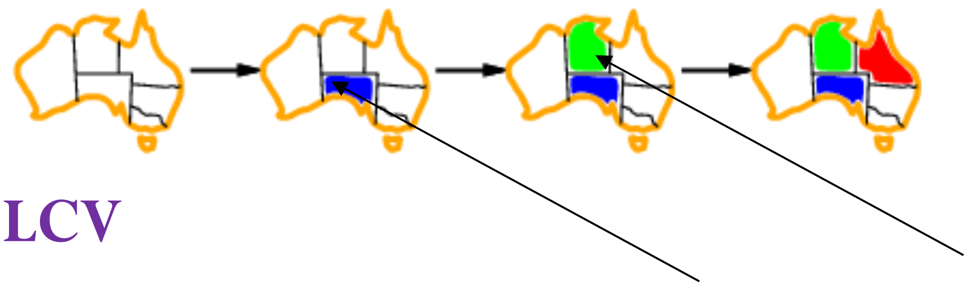
Heuristics for CSP

1. most-constrained variable (**M**inimum **R**emaining **V**alues: **MRV**, "fail-first")

- variable with the **fewest** possible values is selected
- tends to minimize the branching factor

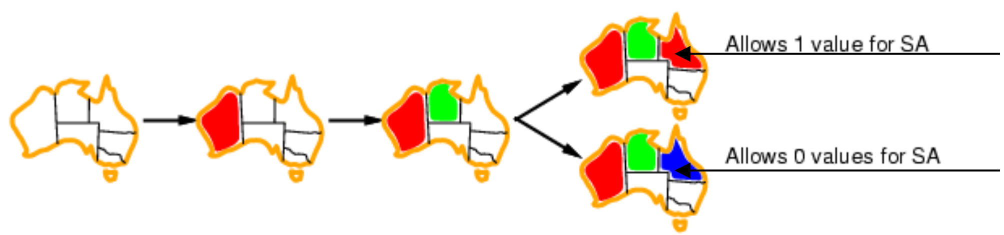
2. most-constraining variable **MCV**

- variable with the **largest** number of constraints **on other** unassigned variables



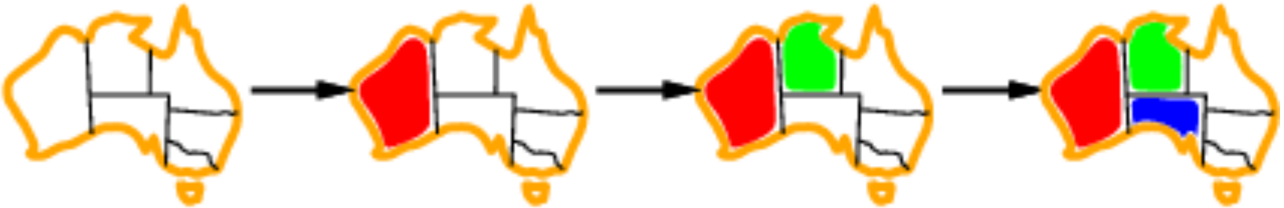
3. least-constraining value **LCV**

- for a selected variable, choose the value that leaves more freedom for future choices



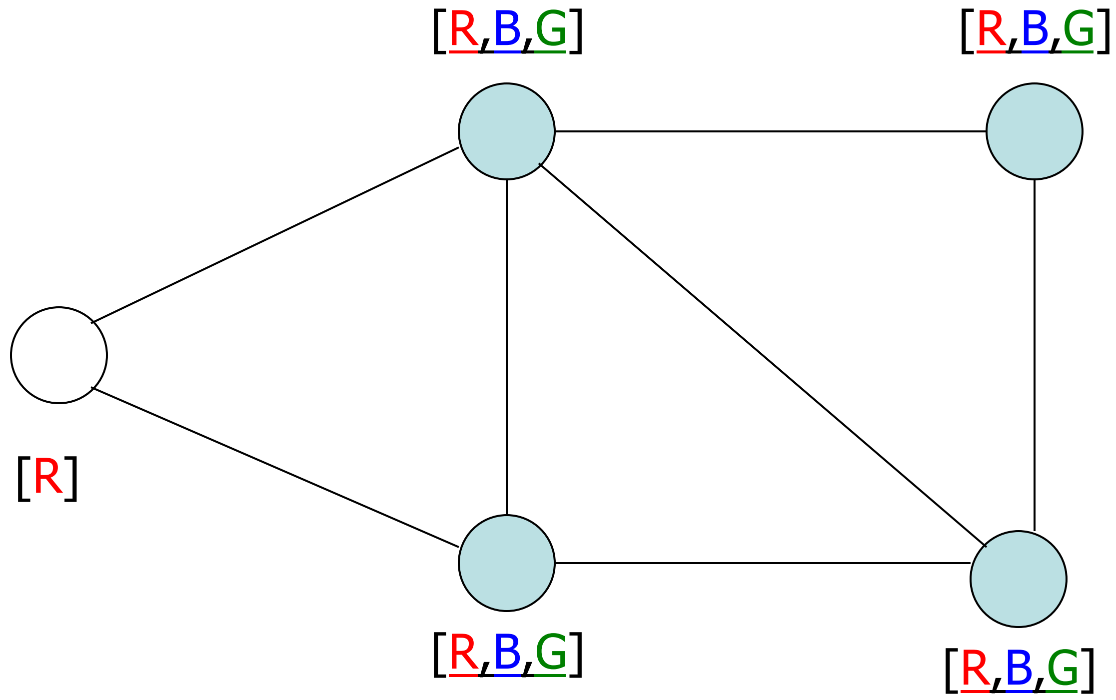
Most constrained variable
Minimum Remaining Values (MRV)

- Most constrained variable:
  choose the variable with the **fewest legal values**

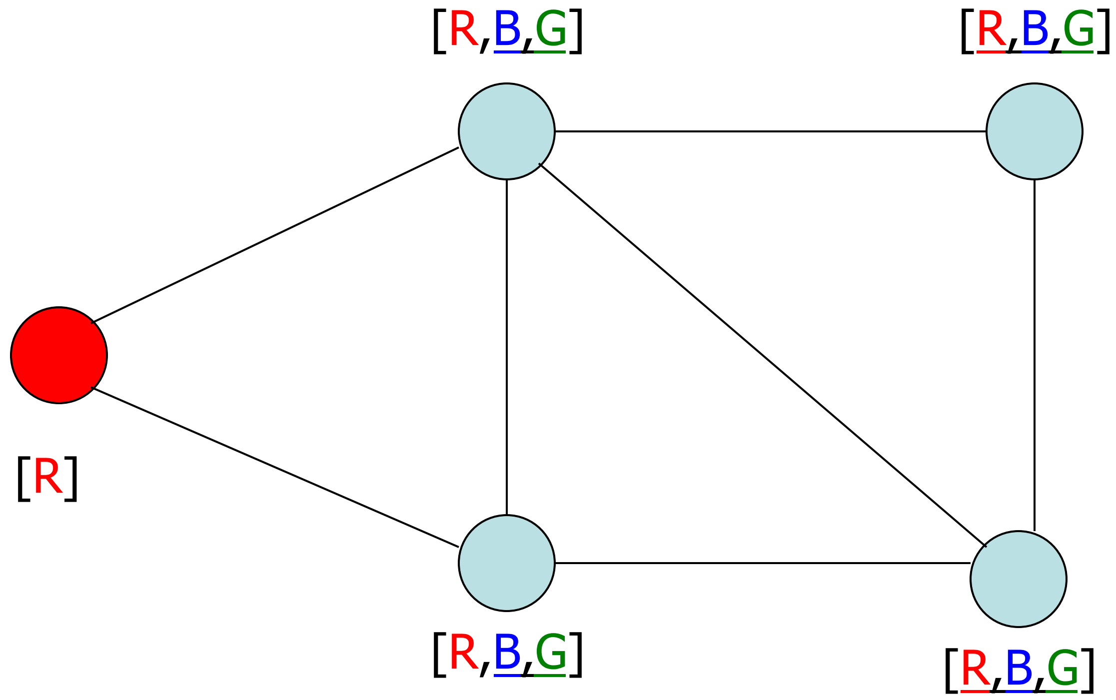


- Called minimum remaining values (MRV) heuristic
- “fail-first” heuristic: Picks a variable which will cause failure as soon as possible, allowing the tree to be pruned.

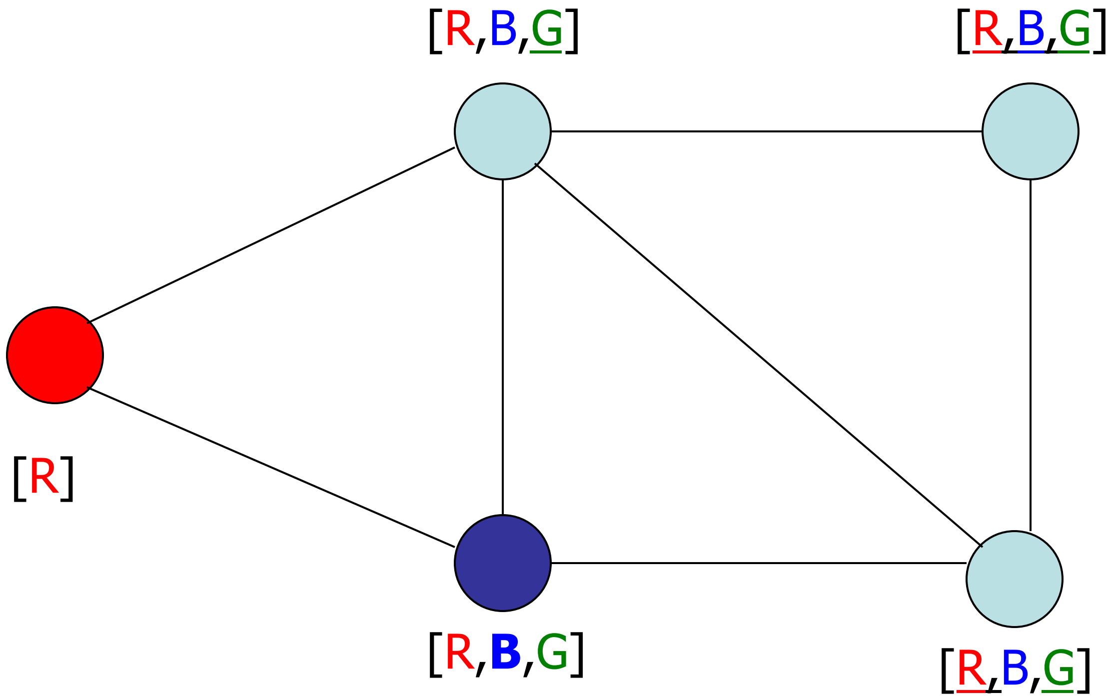
Backpropagation - MRV



Backpropagation – MRV minimum remaining values



Backpropagation - MRV**m**inimum **r**emaining **v**alues



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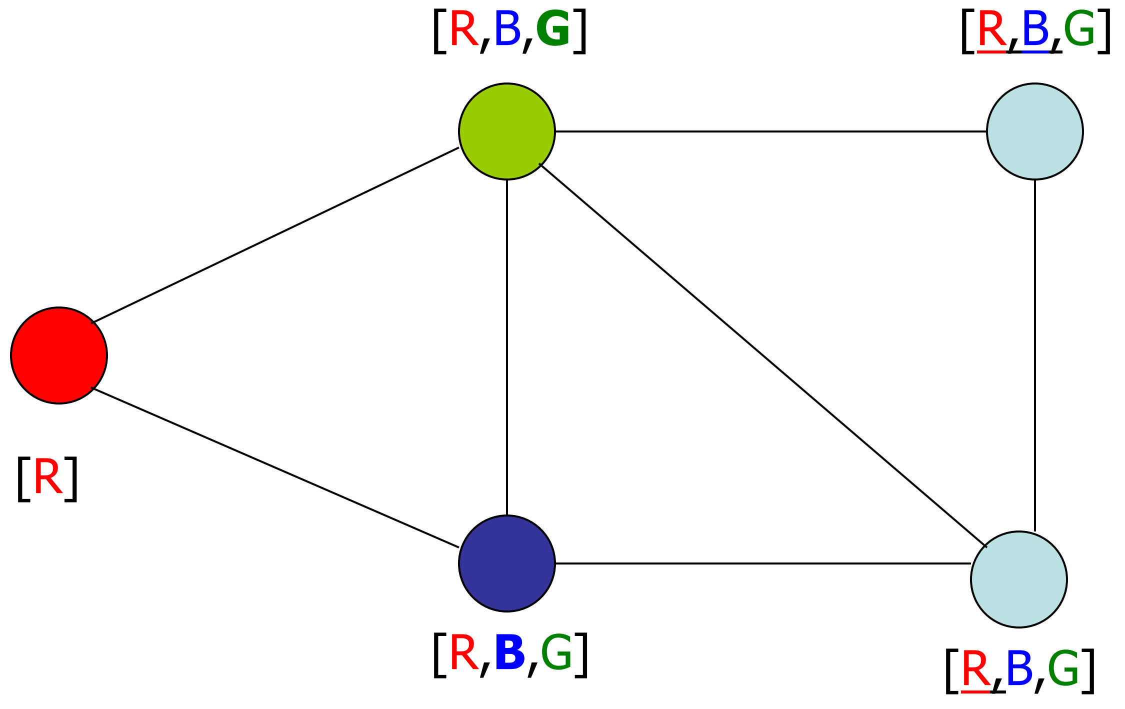

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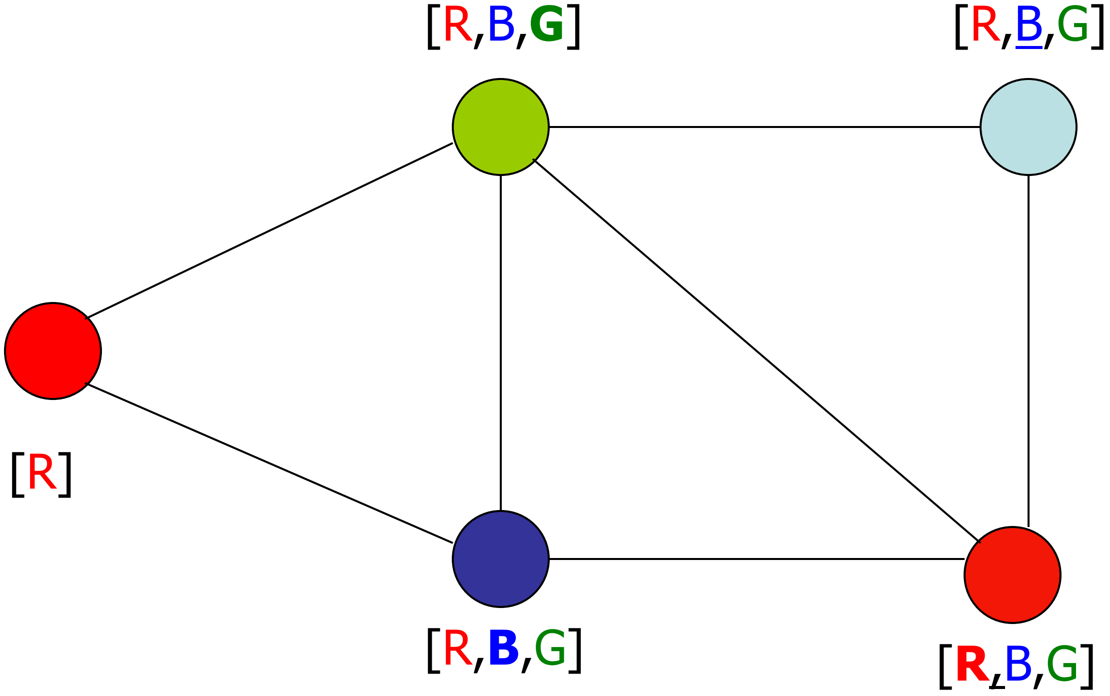

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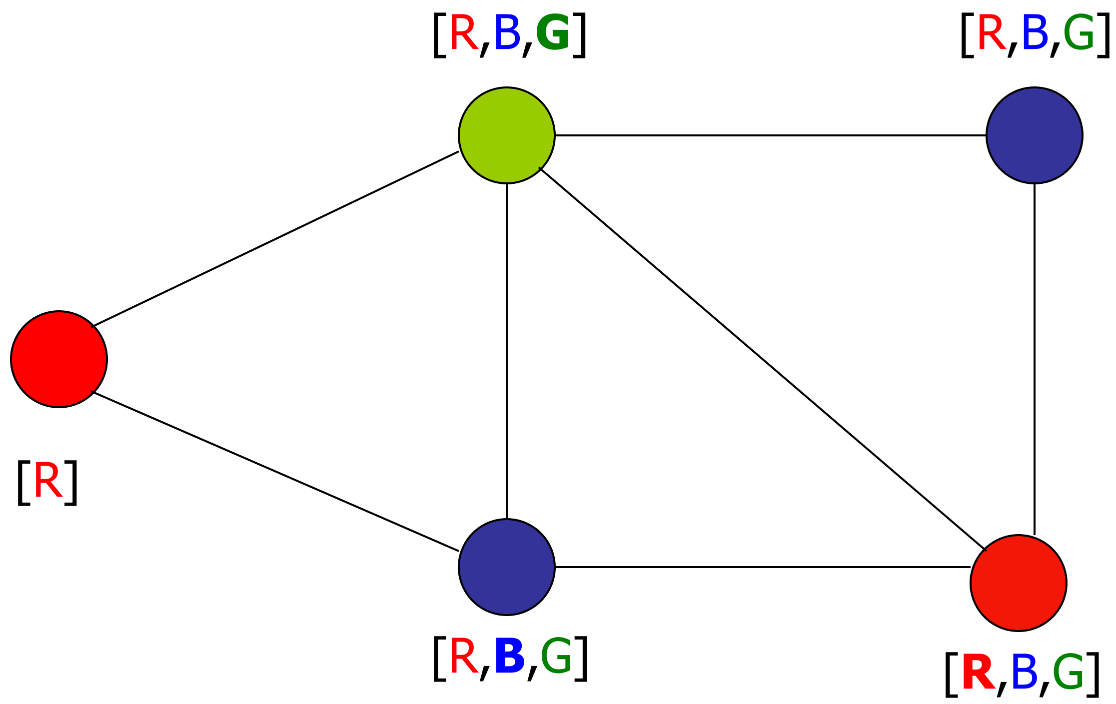
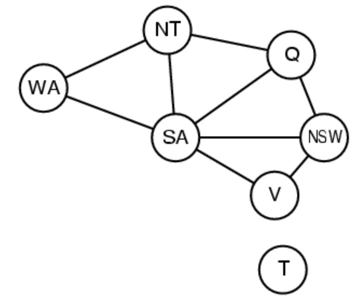
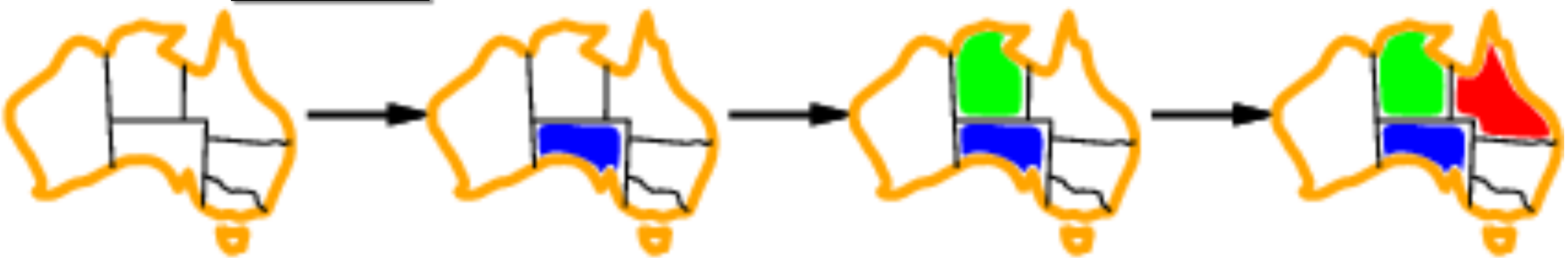


Solution !!!

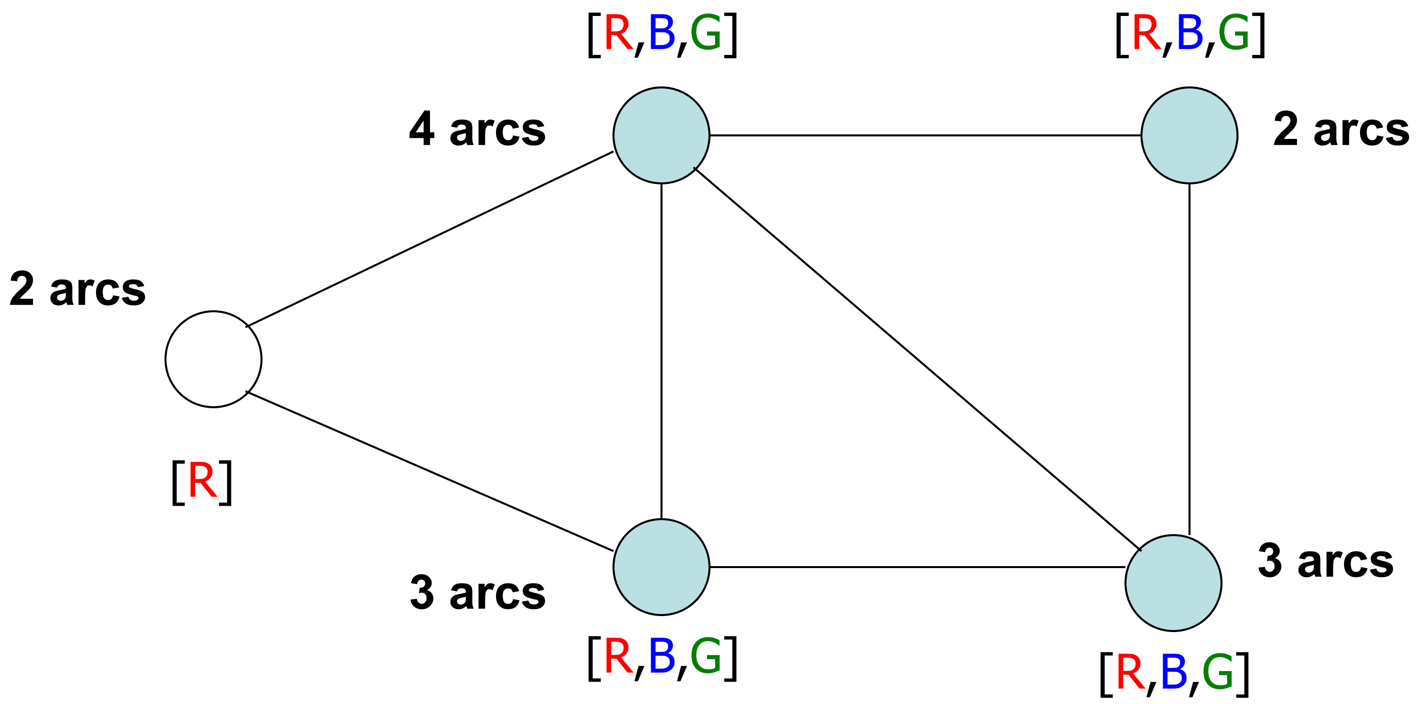
Most constraining variable - MCV



- **Tie-breaker** among most constrain**ed** variables (**MRV**)
- Most constraining variable:
  – choose the variable **with the most constraints on remaining variables** (select variable that is involved in the largest number of constraints - edges in graph on other unassigned variables: **SA:5**, WA:2, NT:3, Q:3, NSW:3, V:2 then:
  – WA:1, NT:2, Q:2, NSW:2, V:1 then
  – **Q:1**, NSW:2, V:1 then WA:0, NSW:1, V:1 ?? Which?



Backpropagation - MCV Most constraining variable



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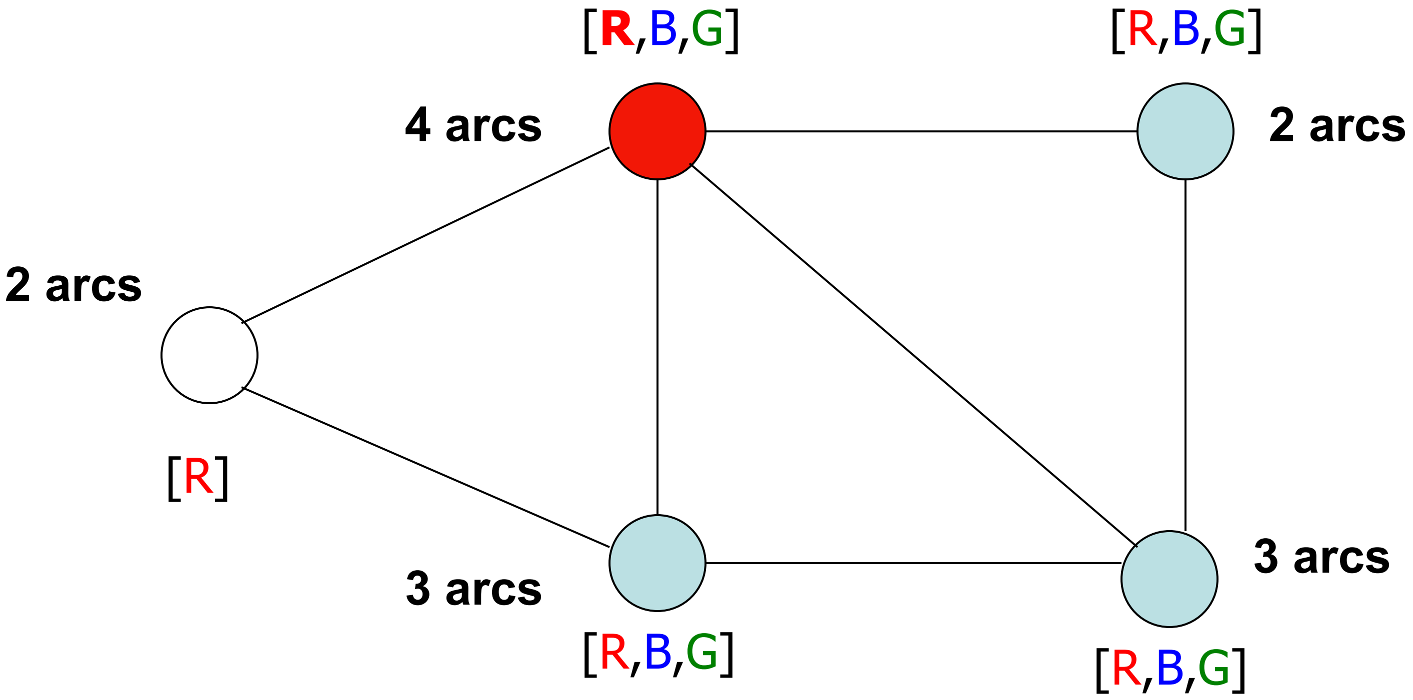

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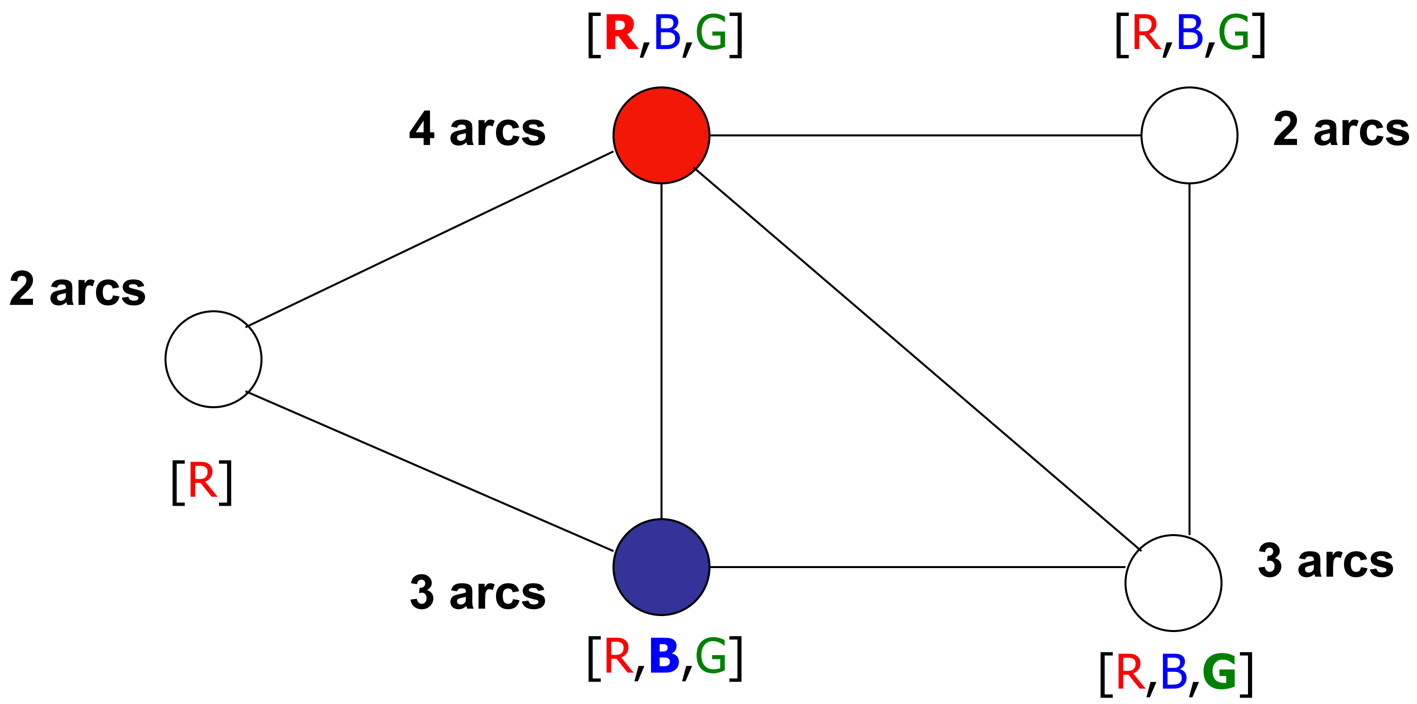

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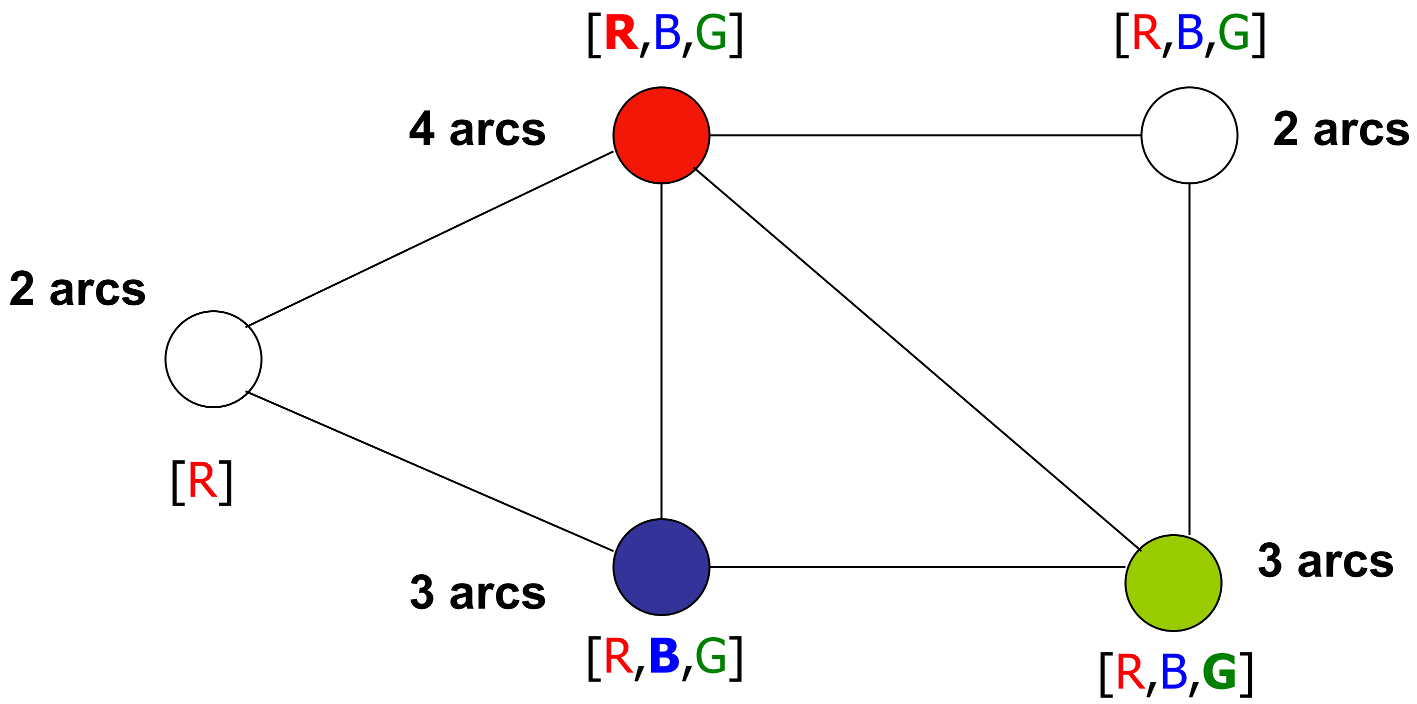

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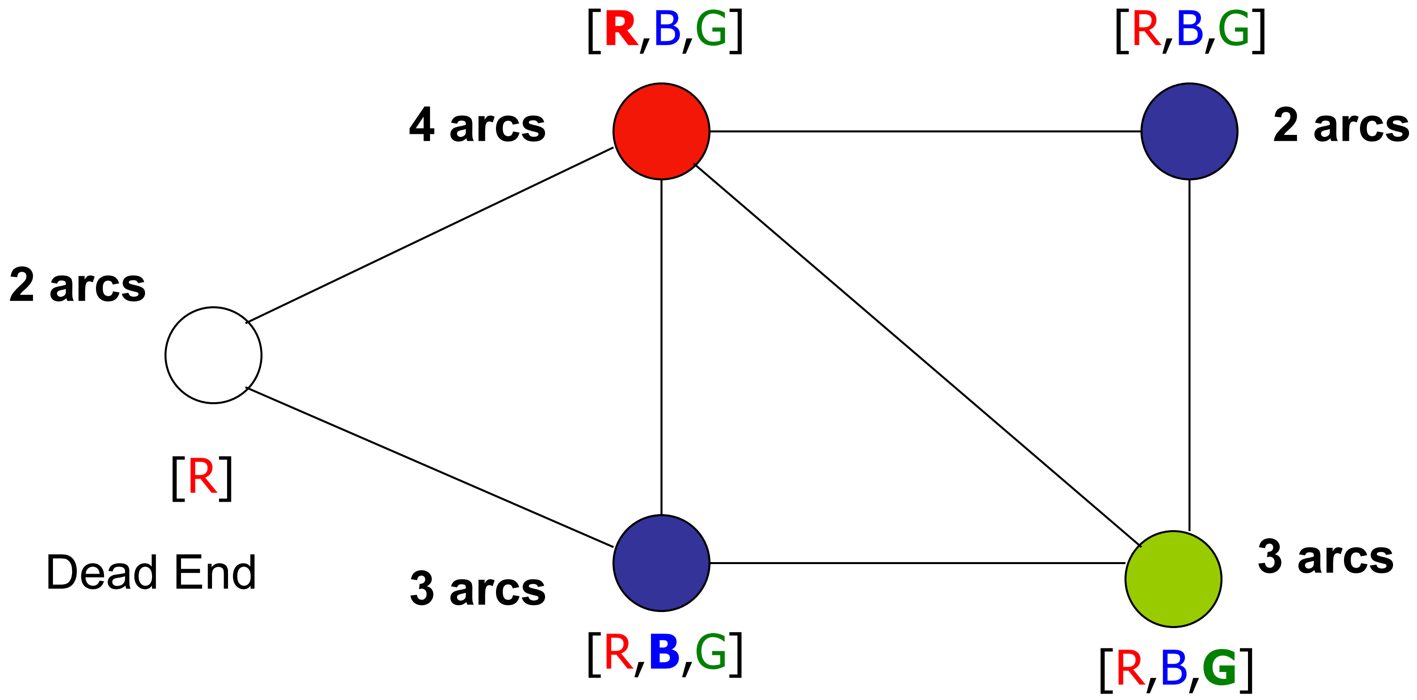

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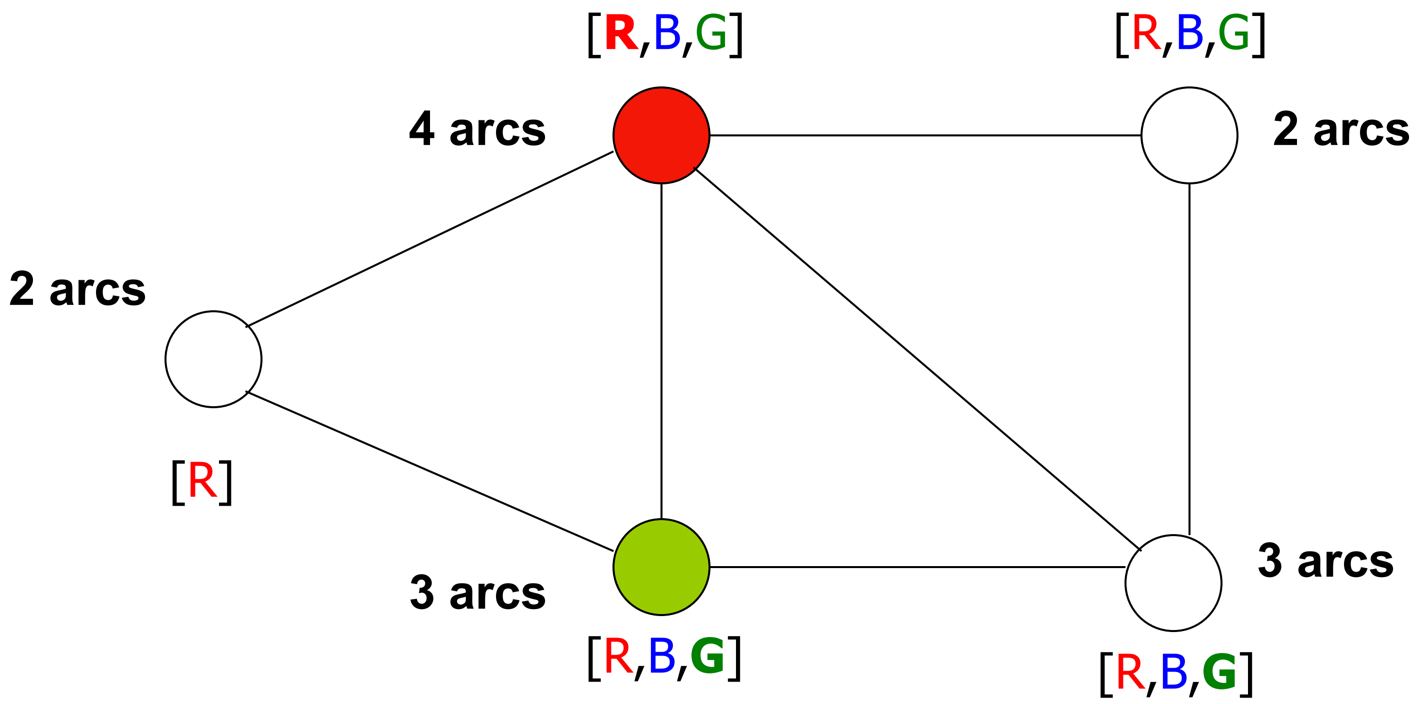

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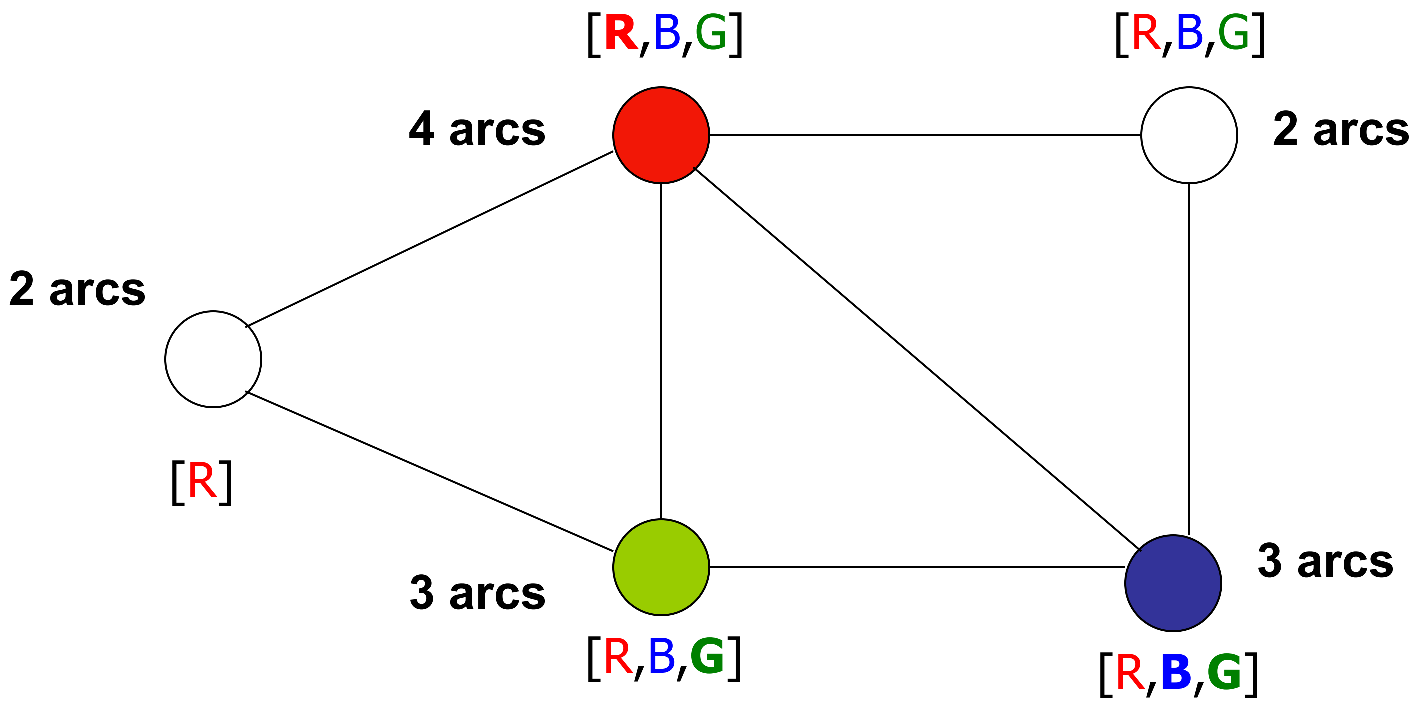

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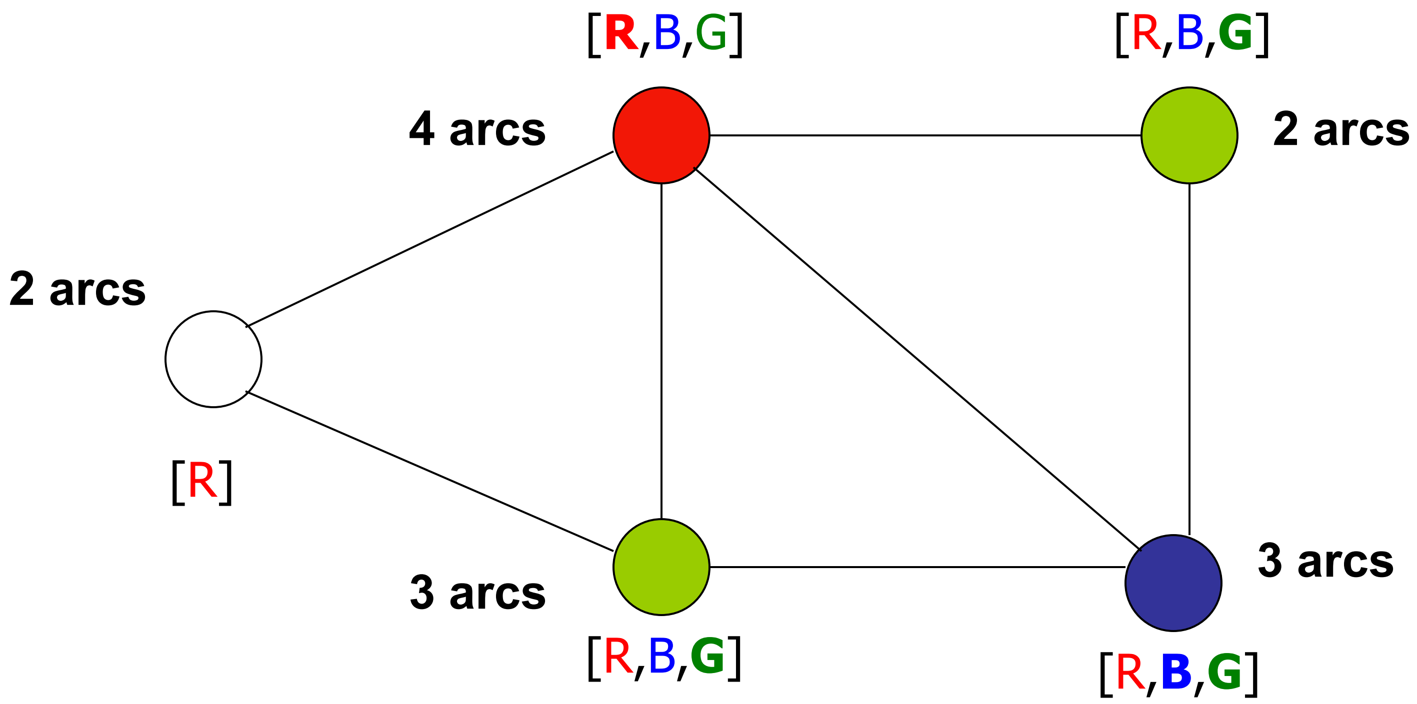

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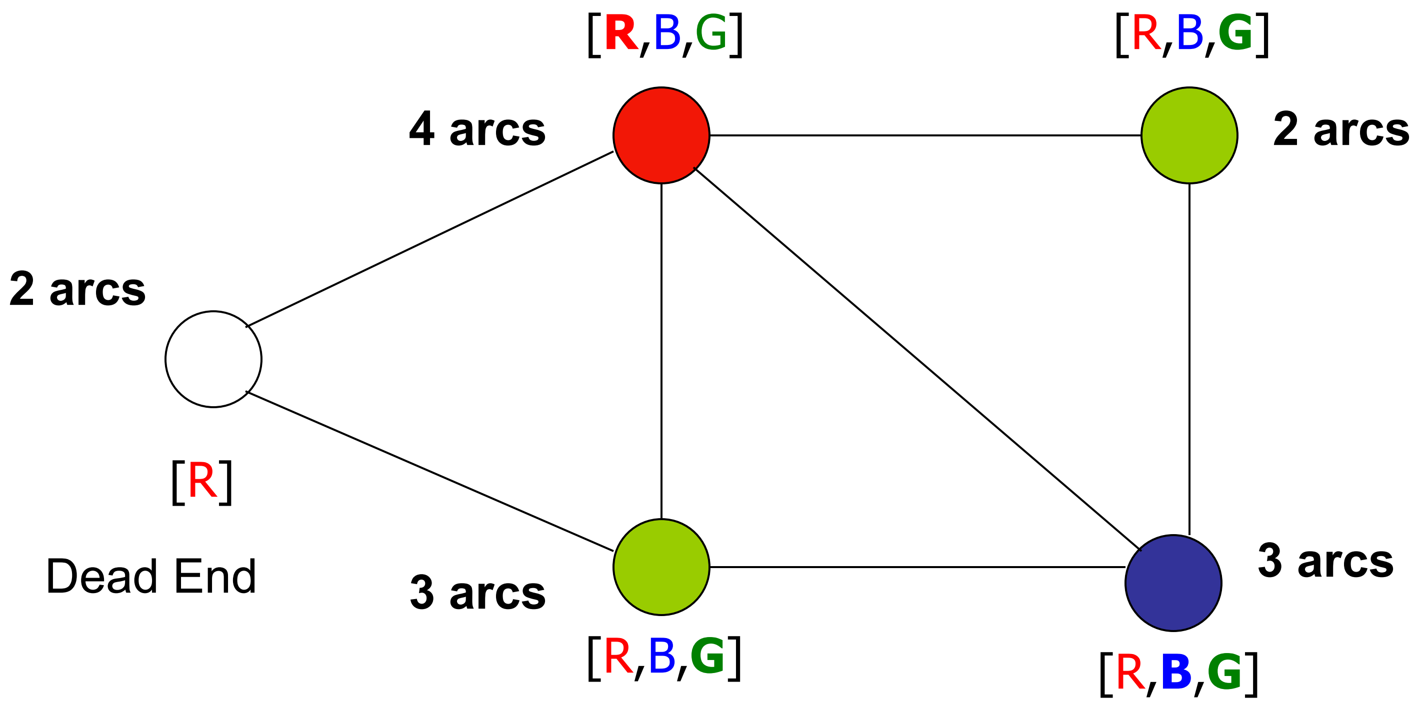

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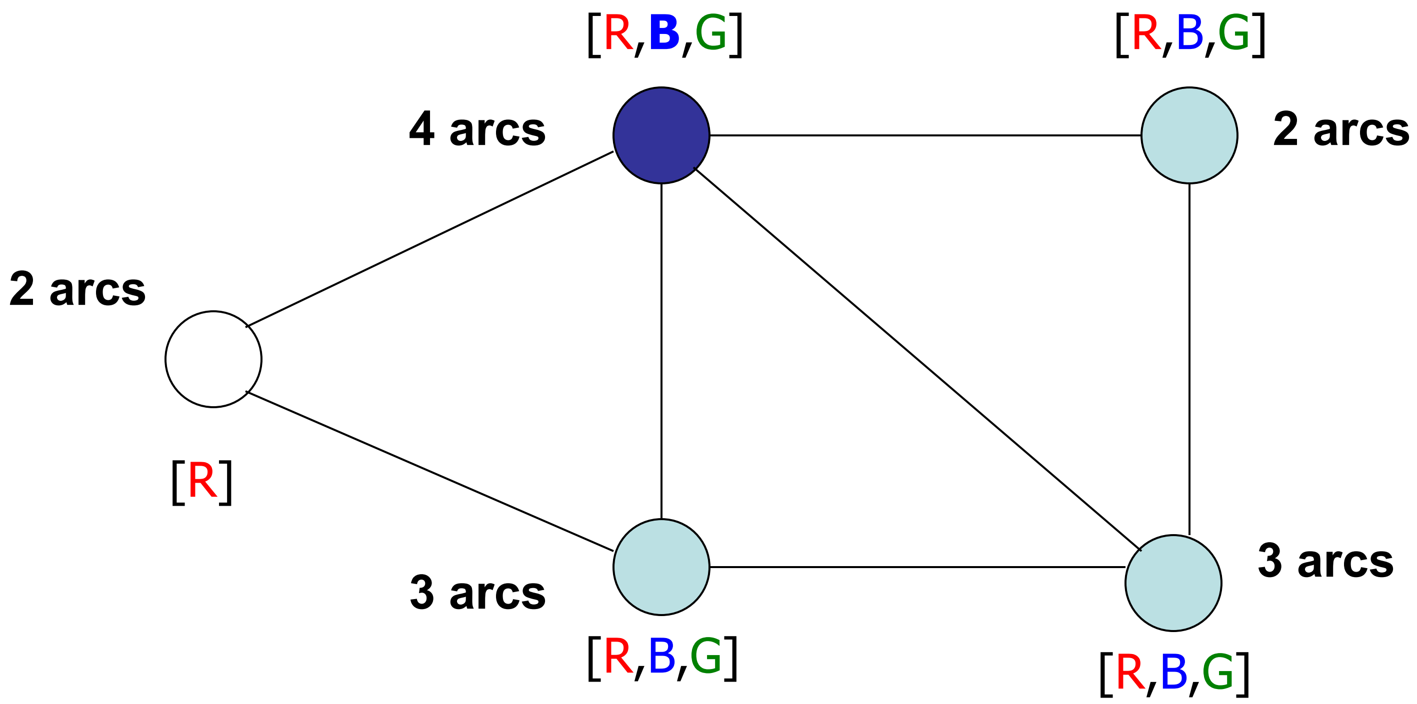

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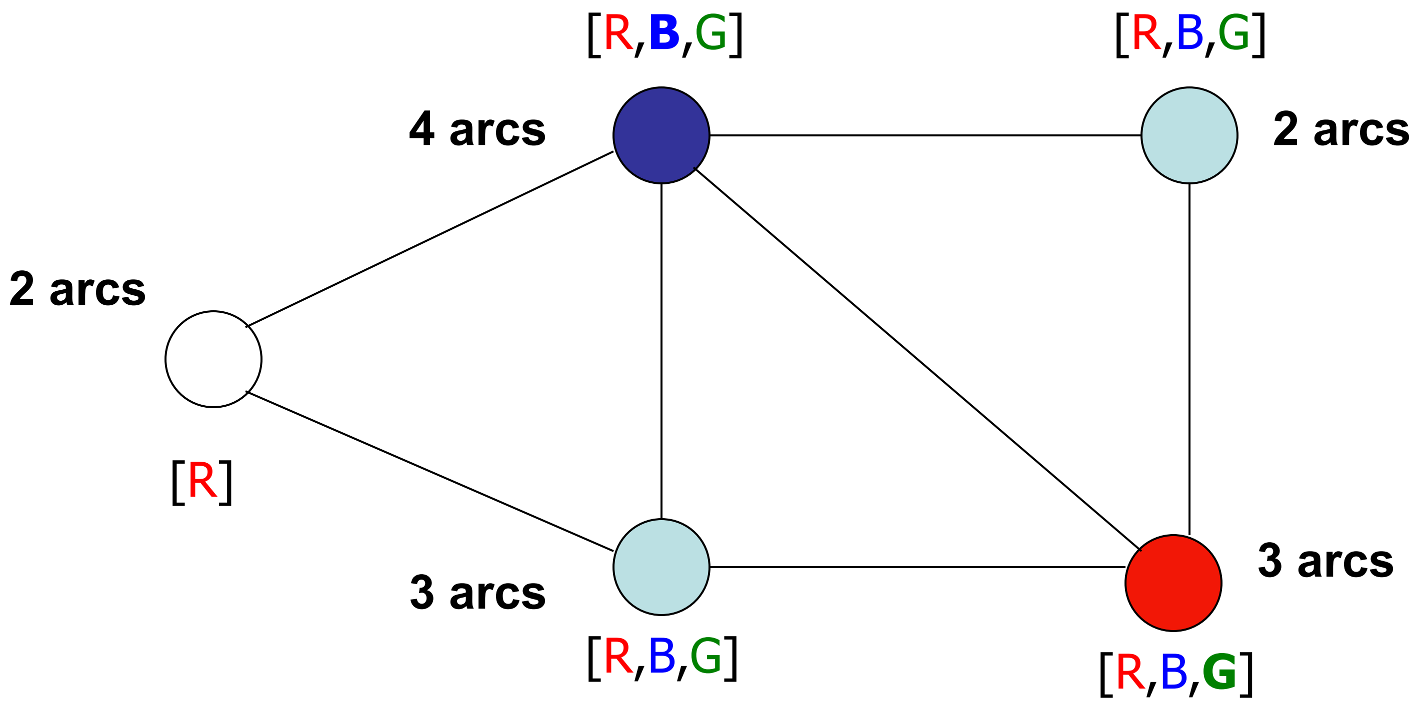

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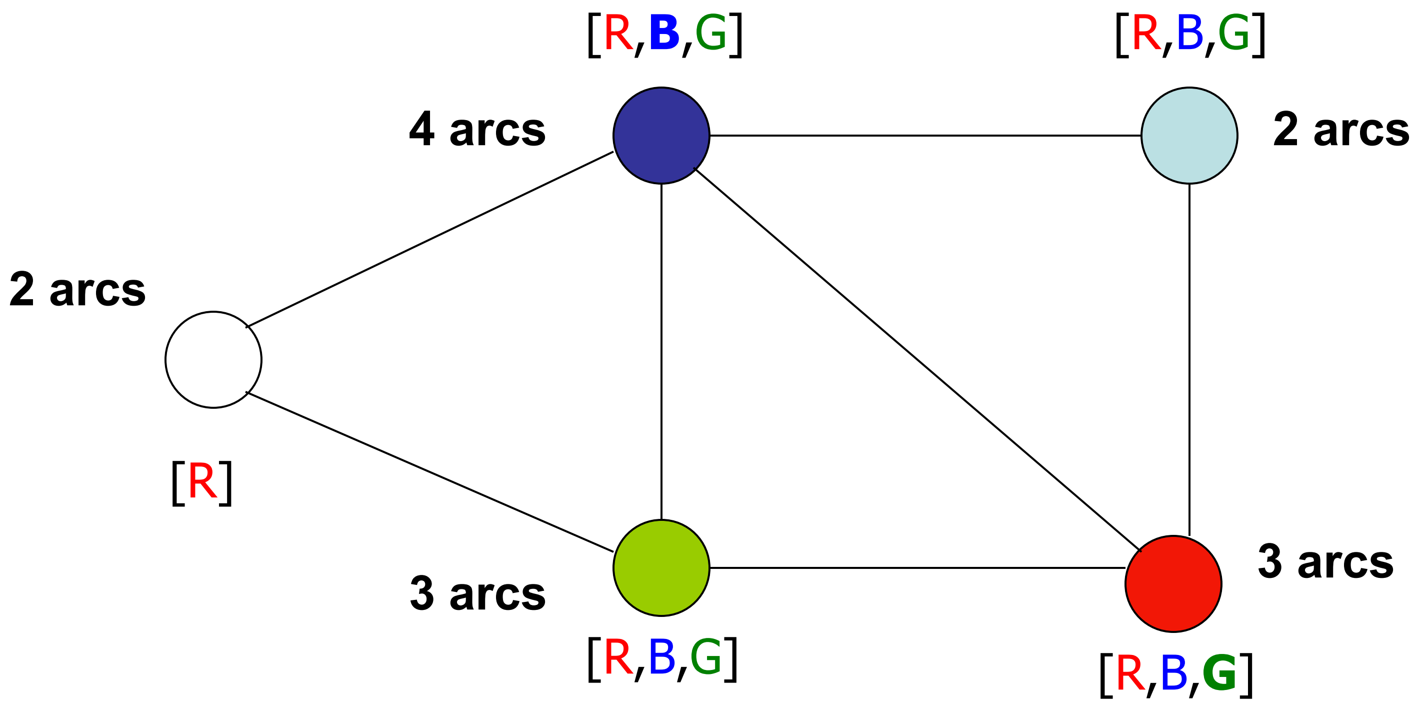

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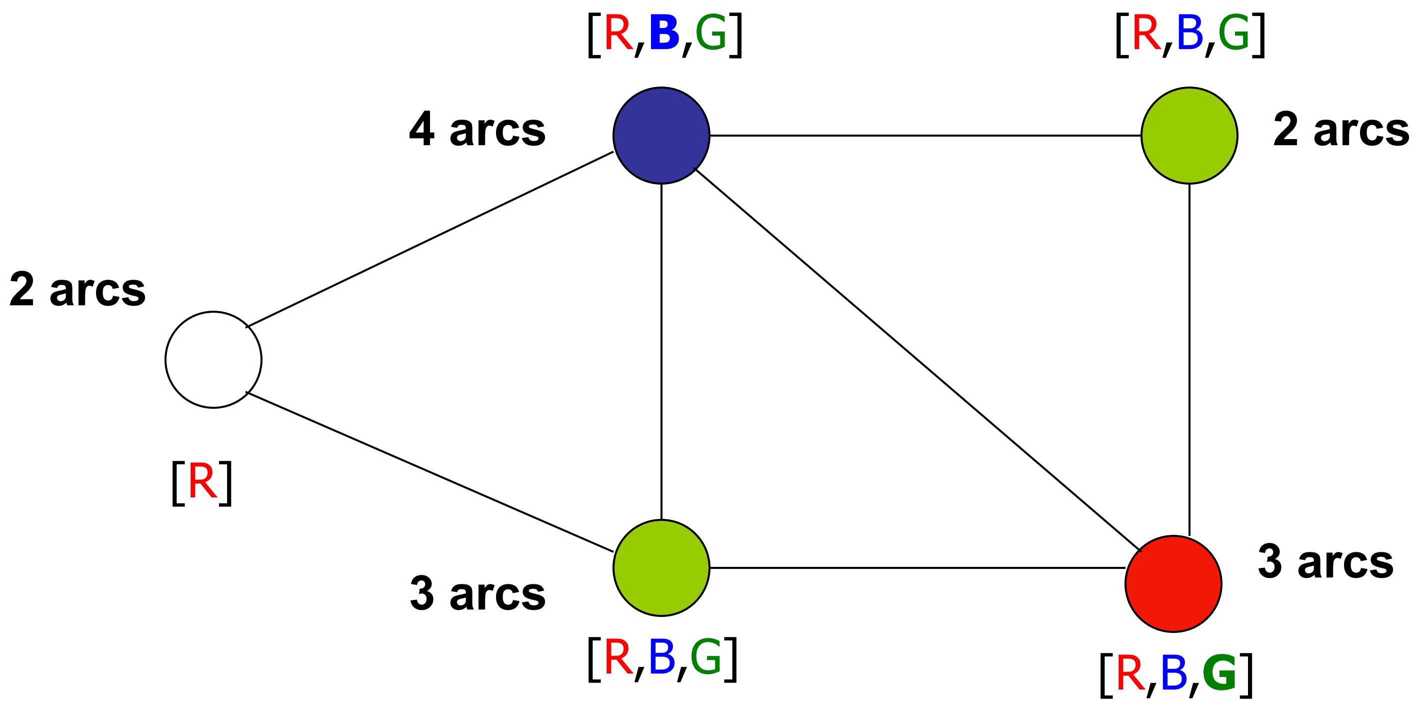

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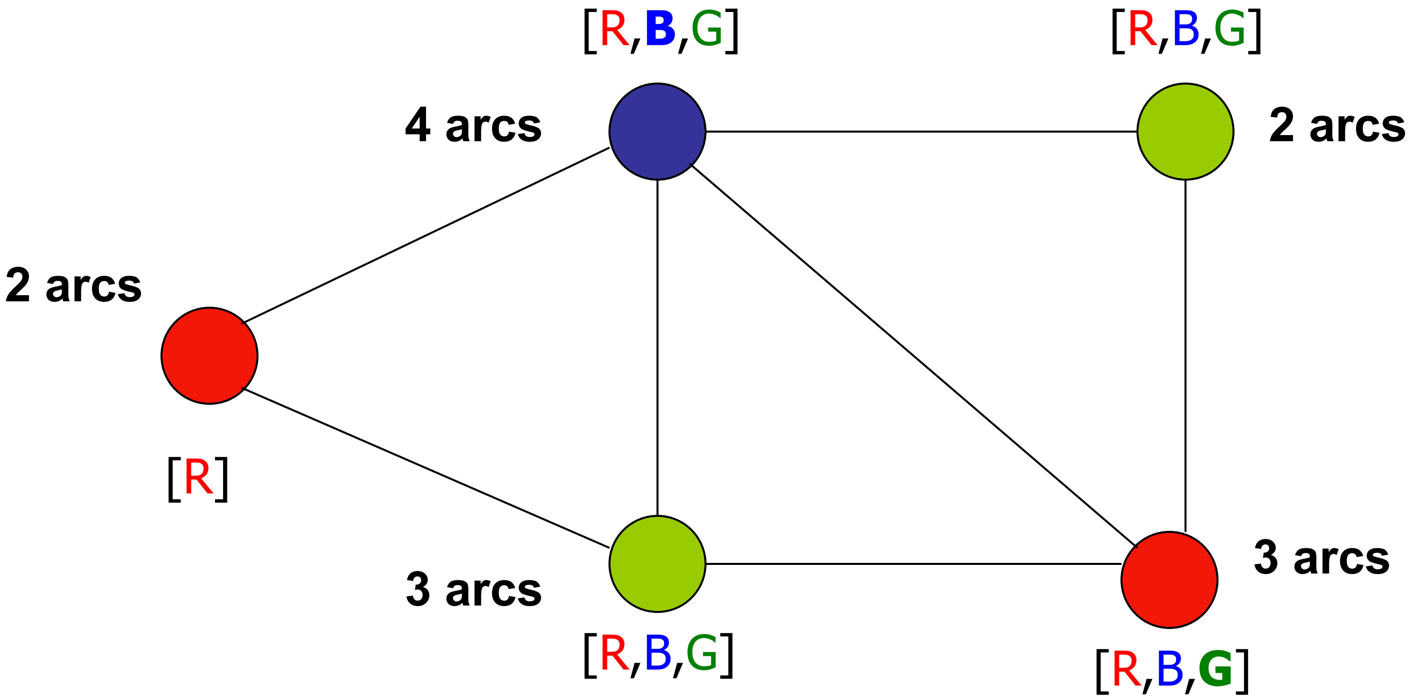
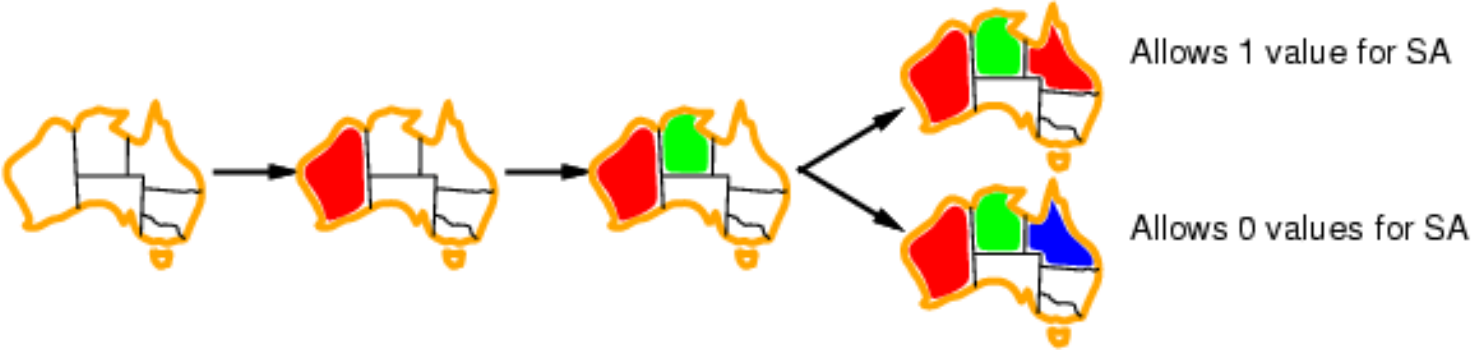


Solution !!!

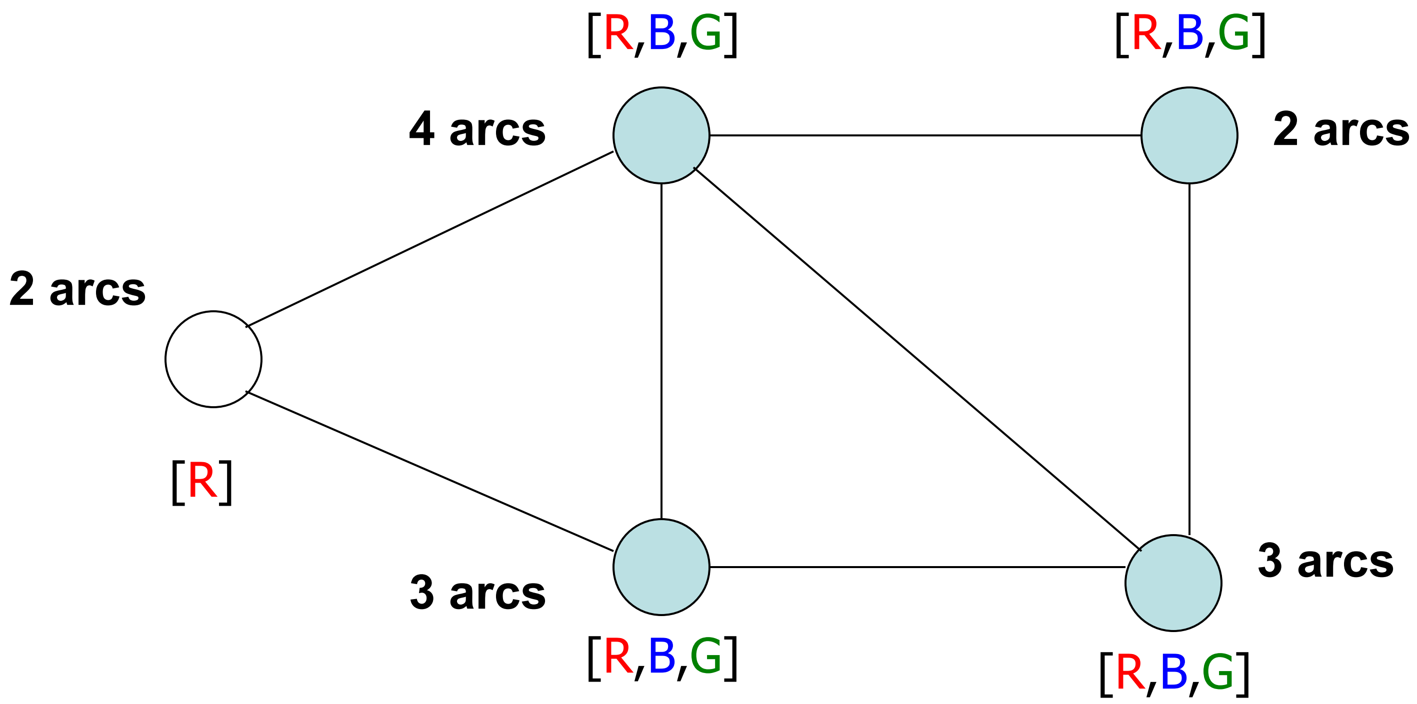
Least constraining value - LCV

- Given a variable, choose the least constraining value:
  - the one that rules out/eliminates the fewest values in the remaining variables (keeps the most)



- Combining these heuristics makes 1000 queens feasible

Backpropagation – LCV Least constraining value



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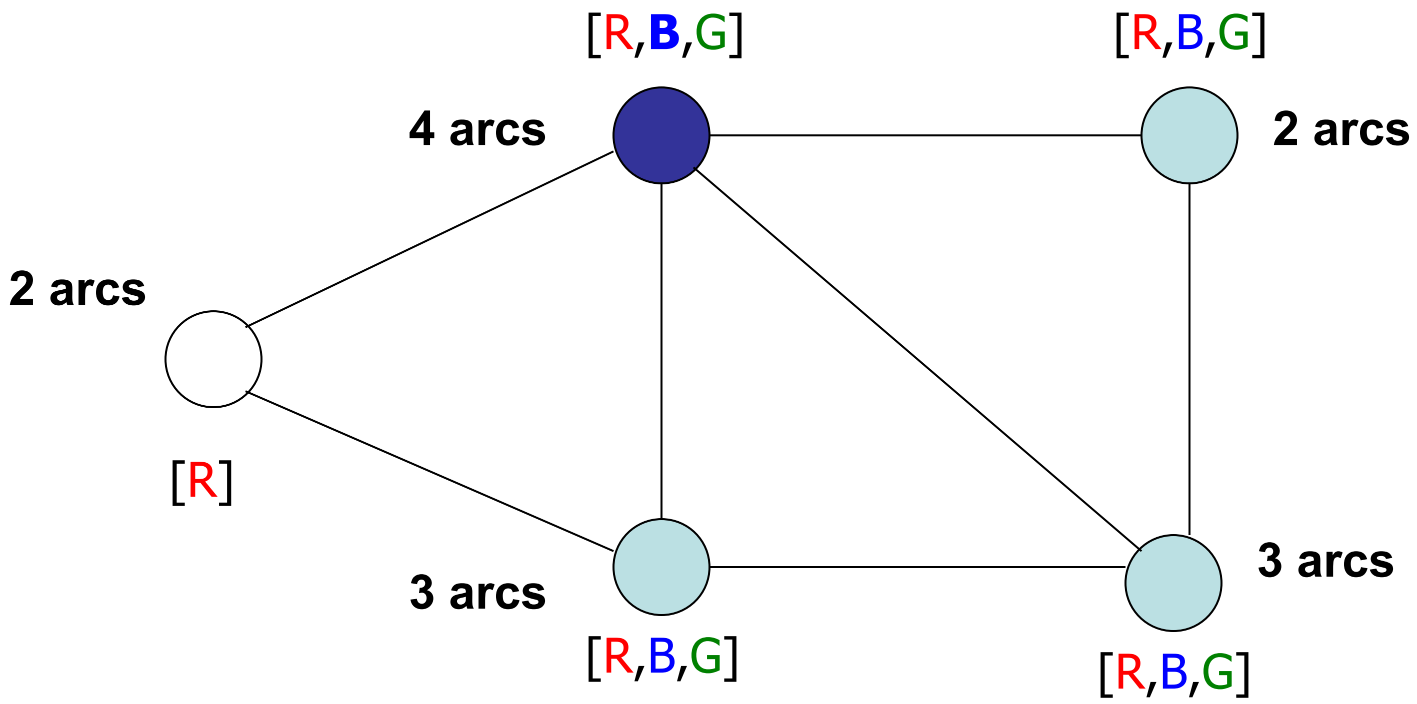

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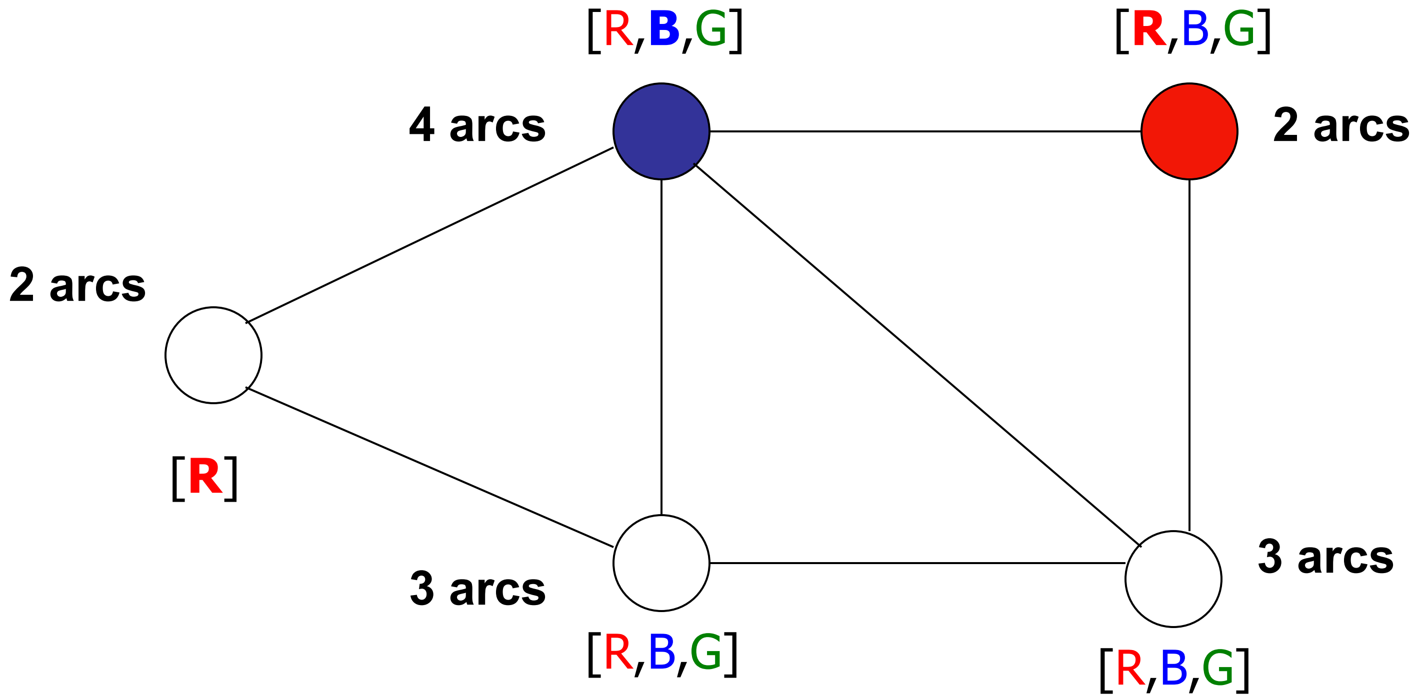

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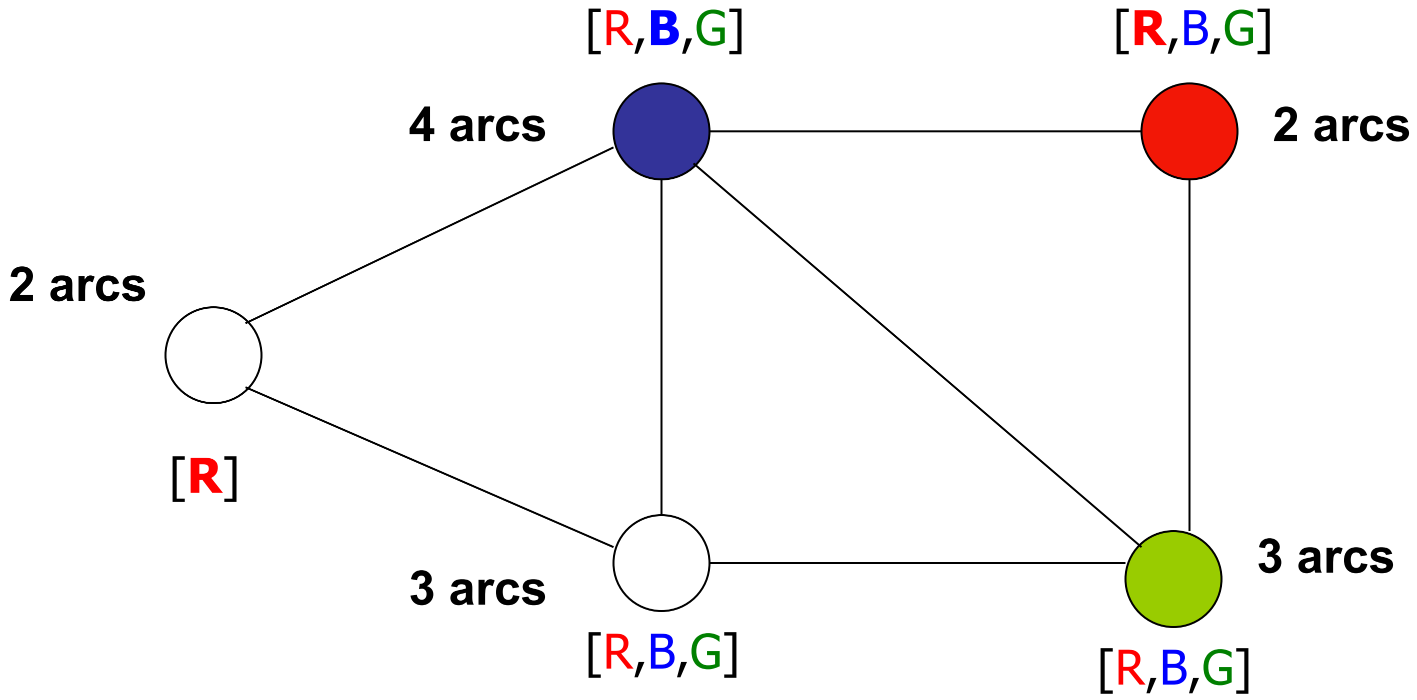

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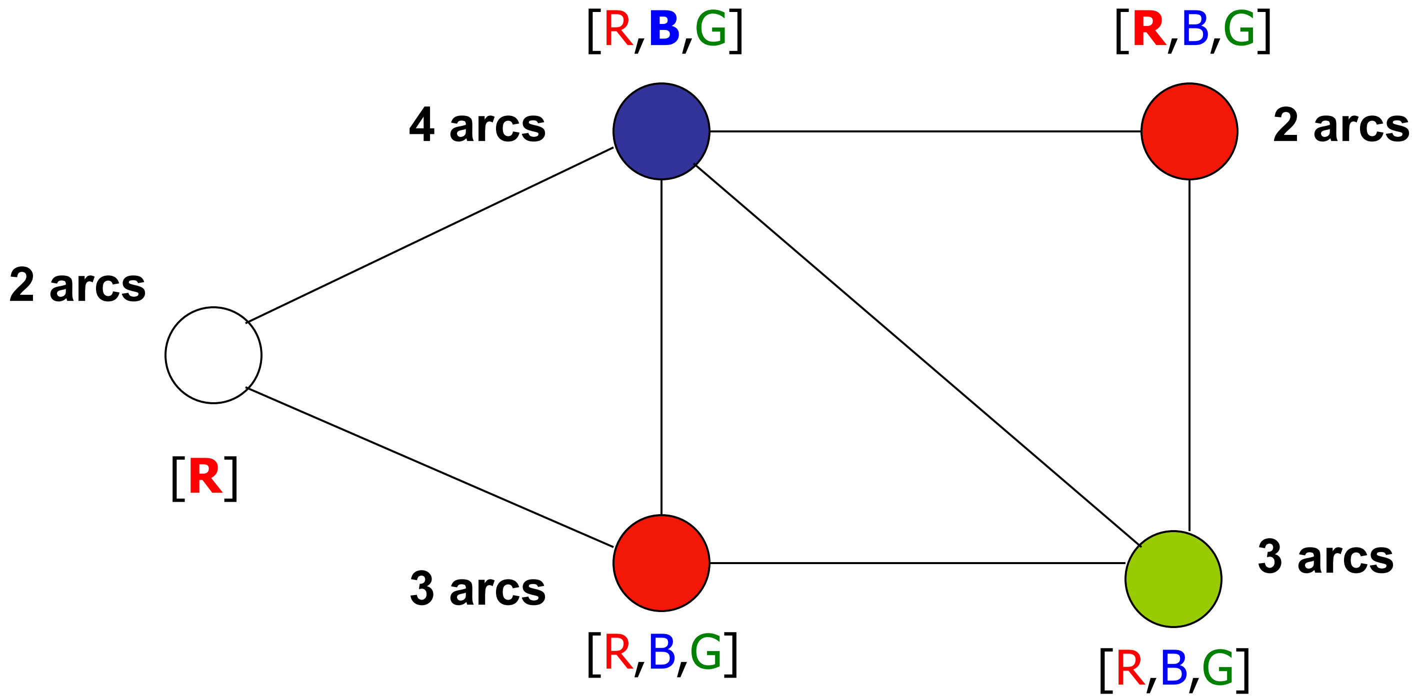

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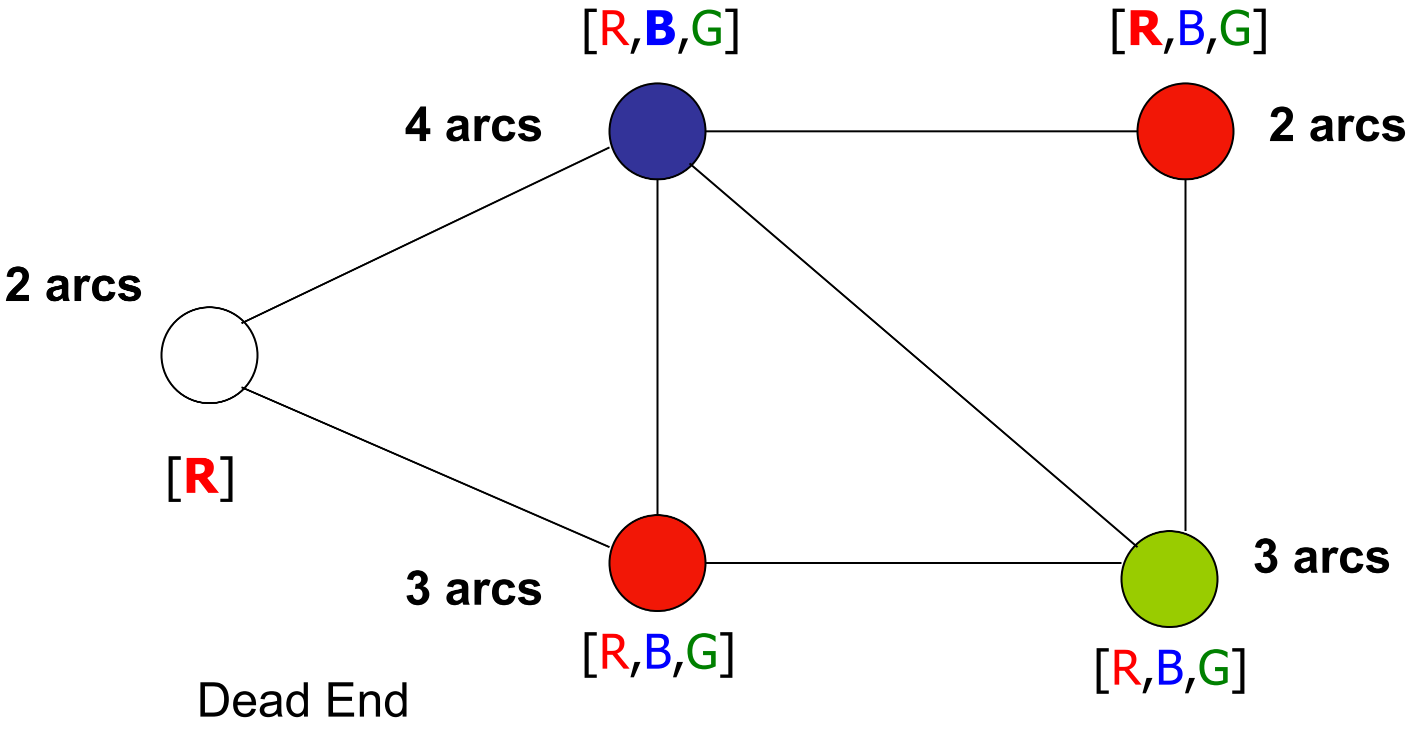

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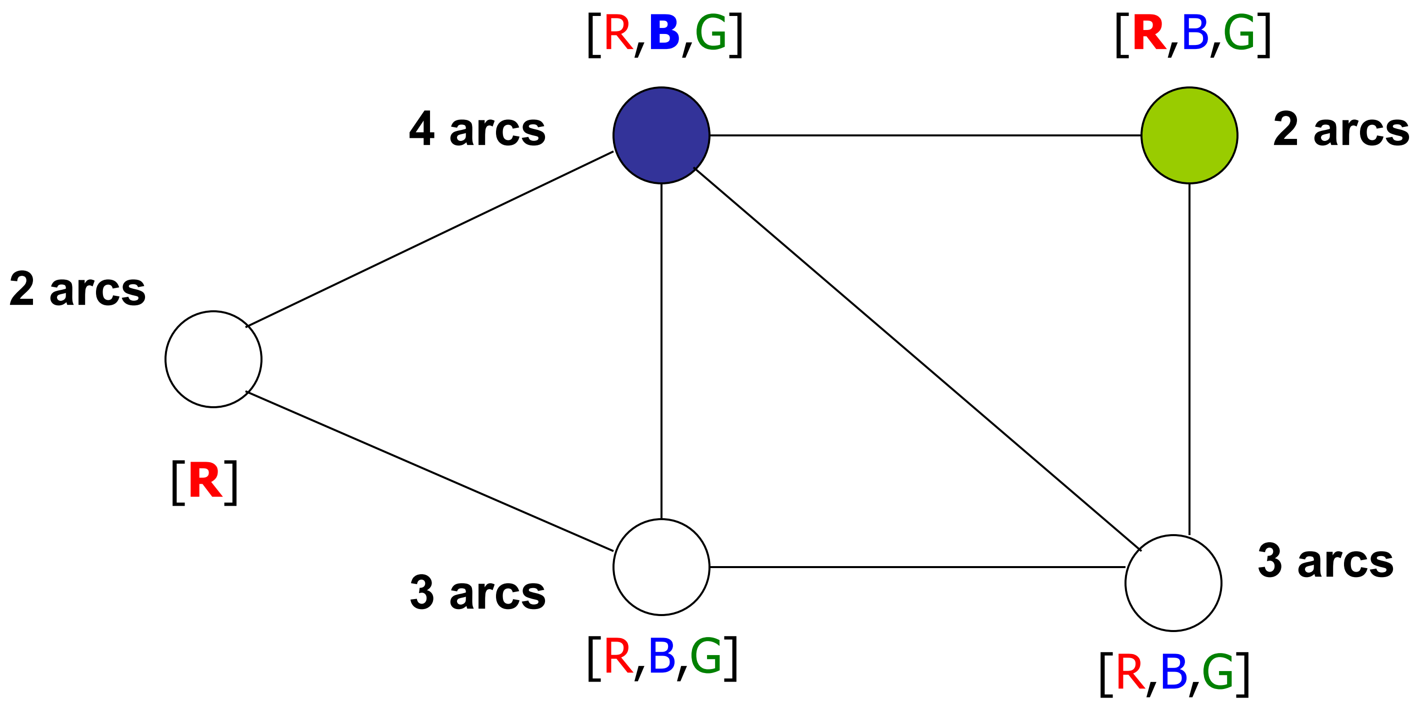

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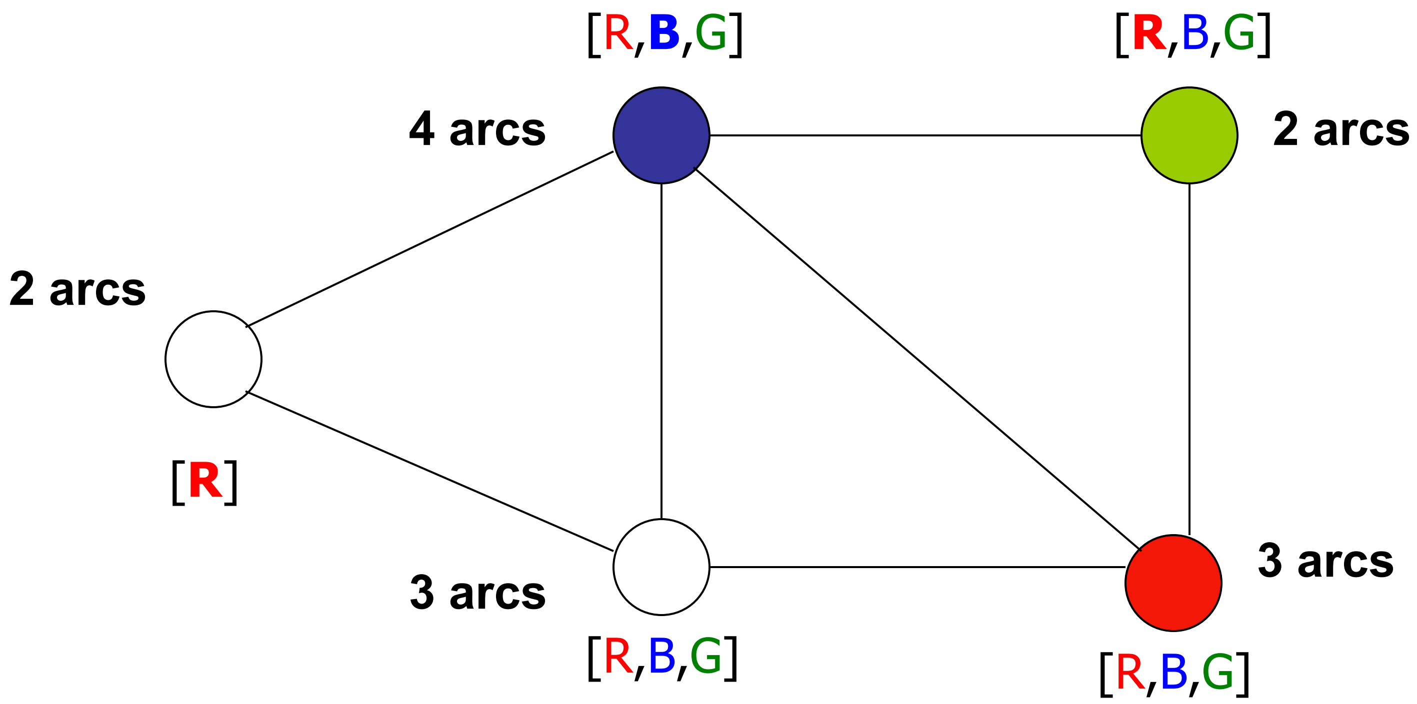

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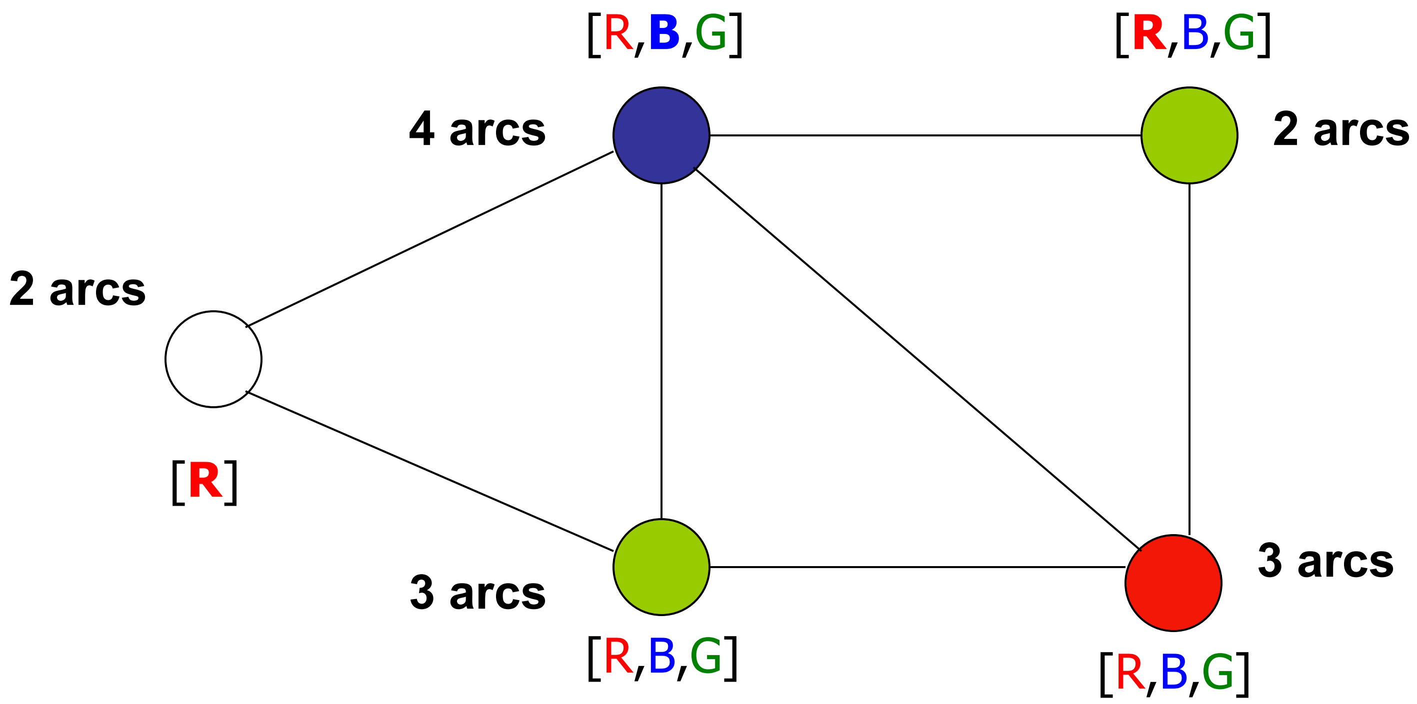

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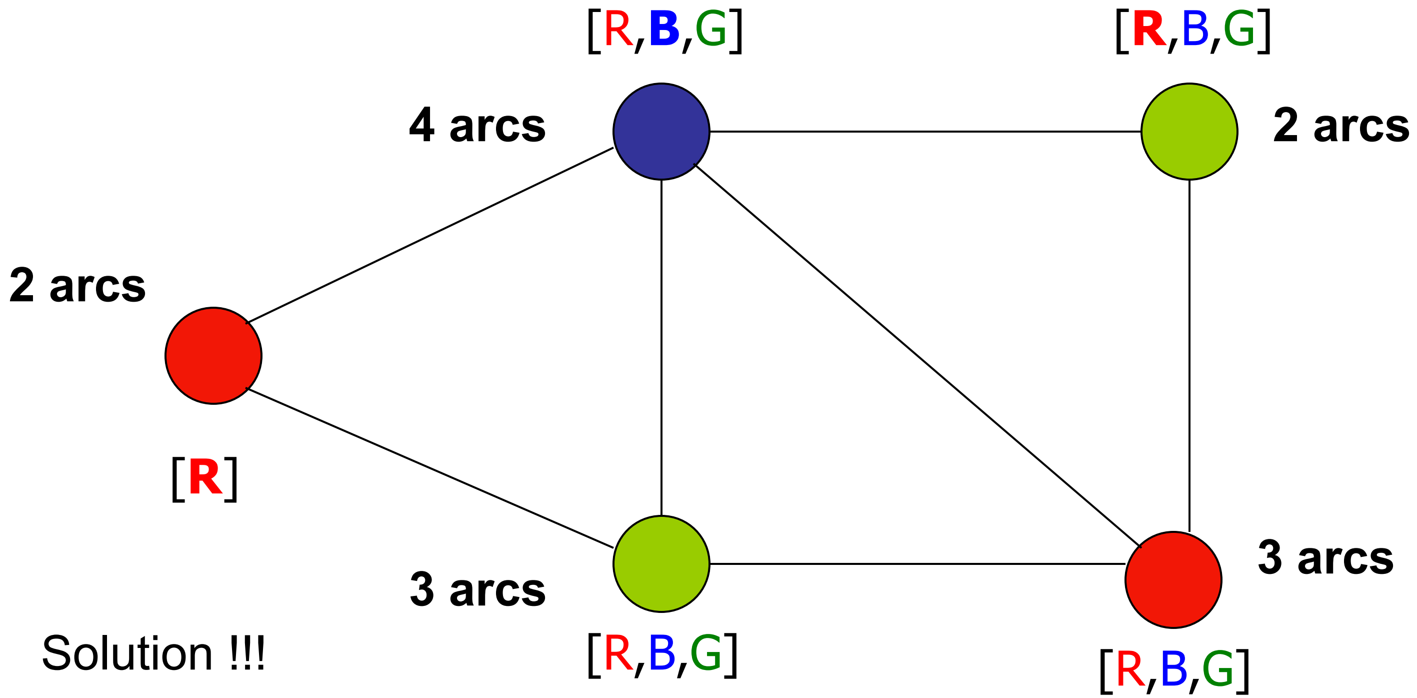

Solution !!!

Analyzing Constraints

- forward checking
  - when a value X is assigned to a variable, inconsistent values are eliminated for all variables connected to X [remove conflicting values]
    - identifies "dead" branches of the tree before they are visited
- constraint propagation
  - analyses interdependencies between variable assignments via arc consistency
    - an arc between X and Y is consistent if for every possible value x of X, there is some value y of Y that is consistent with x
    - more powerful than forward checking, but still reasonably efficient
    - but does not reveal every possible inconsistency

Forward checking

- Idea:
  - Keep track of remaining legal values for unassigned variables
  - Terminate search when any variable has no legal values

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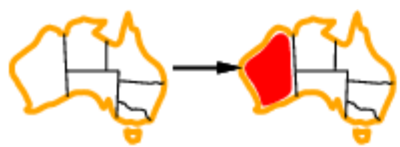

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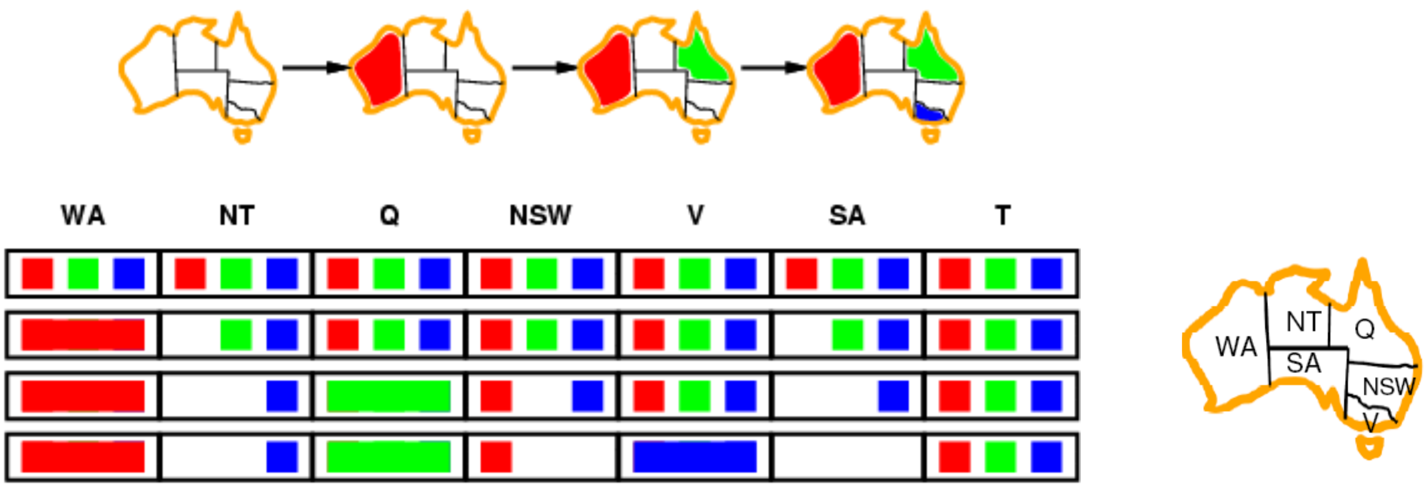

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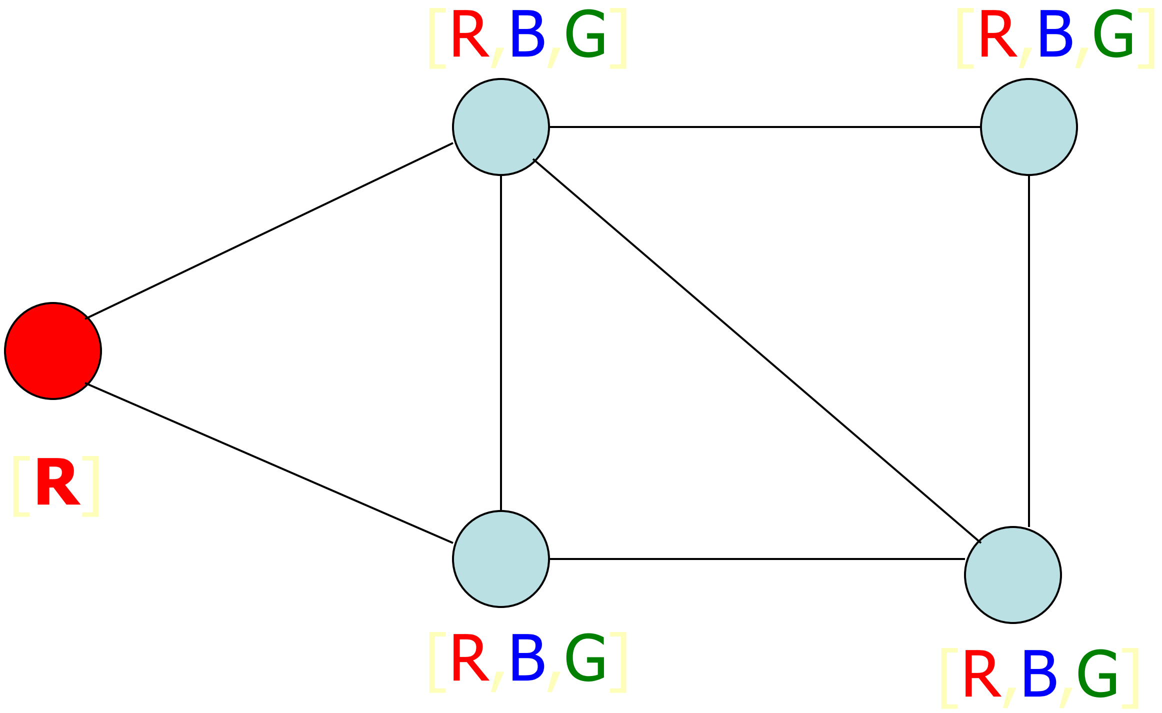

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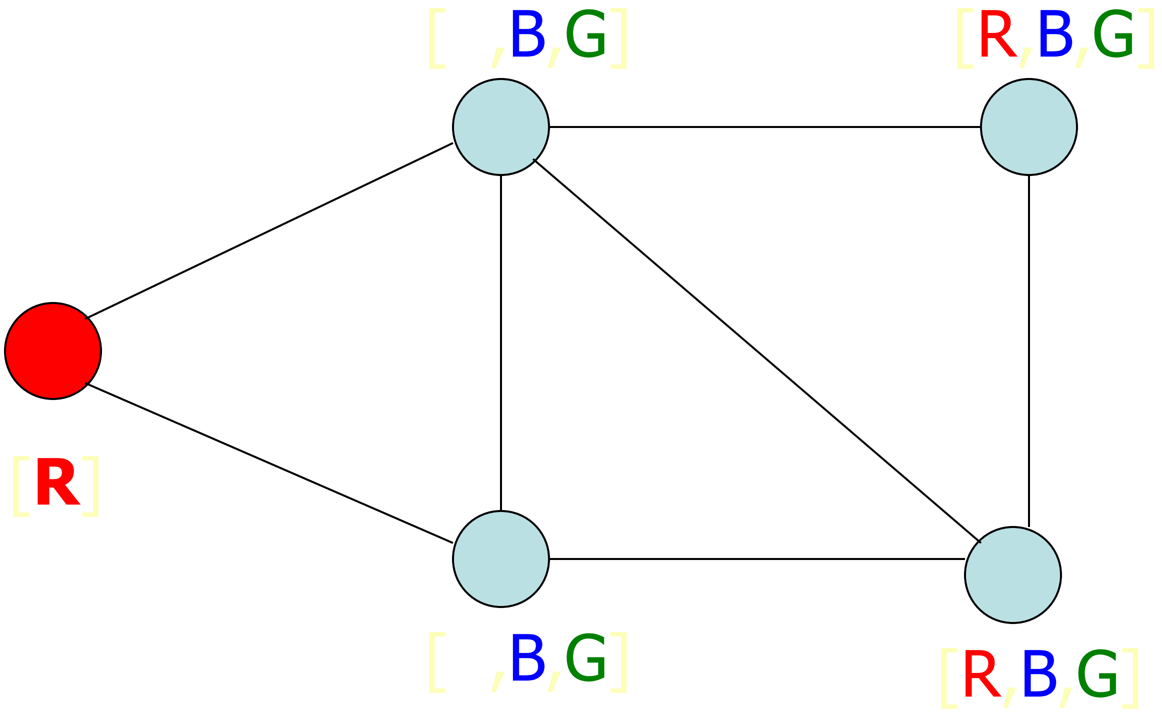

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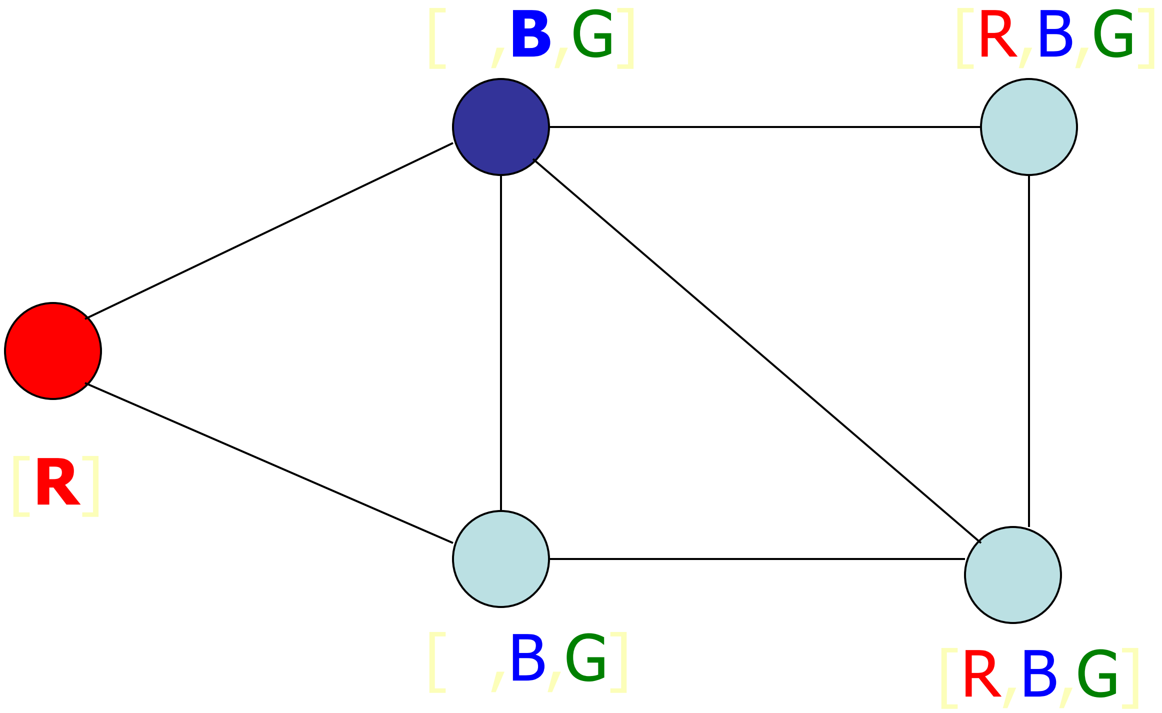

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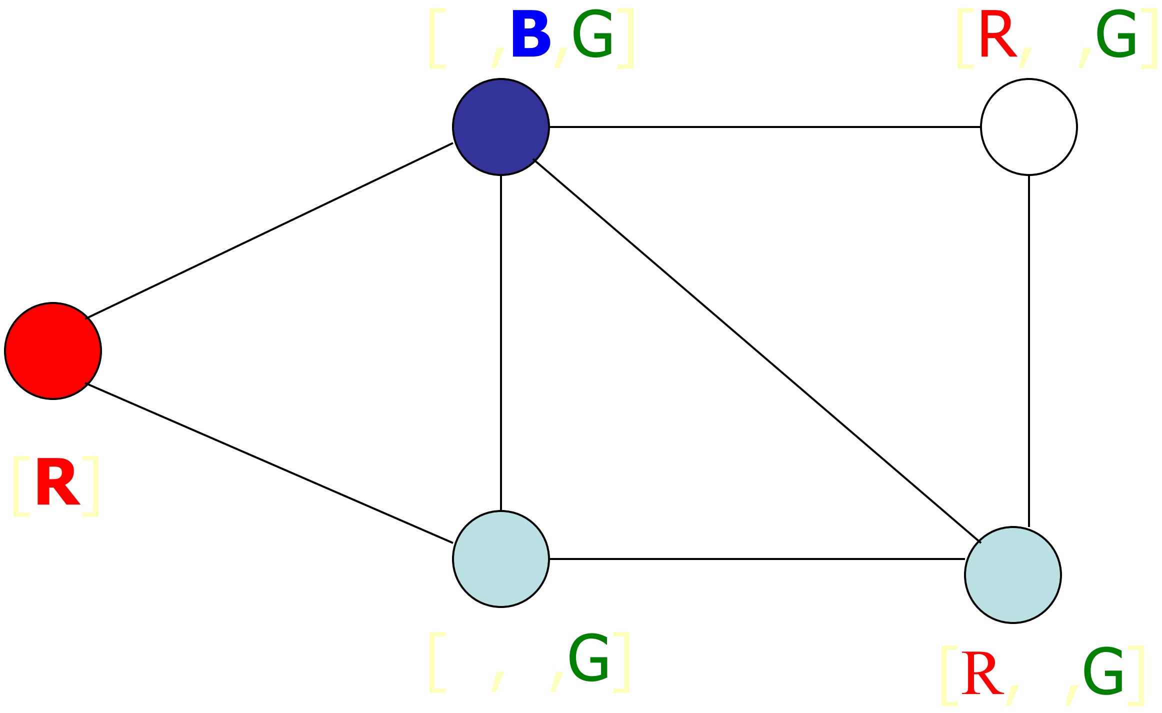

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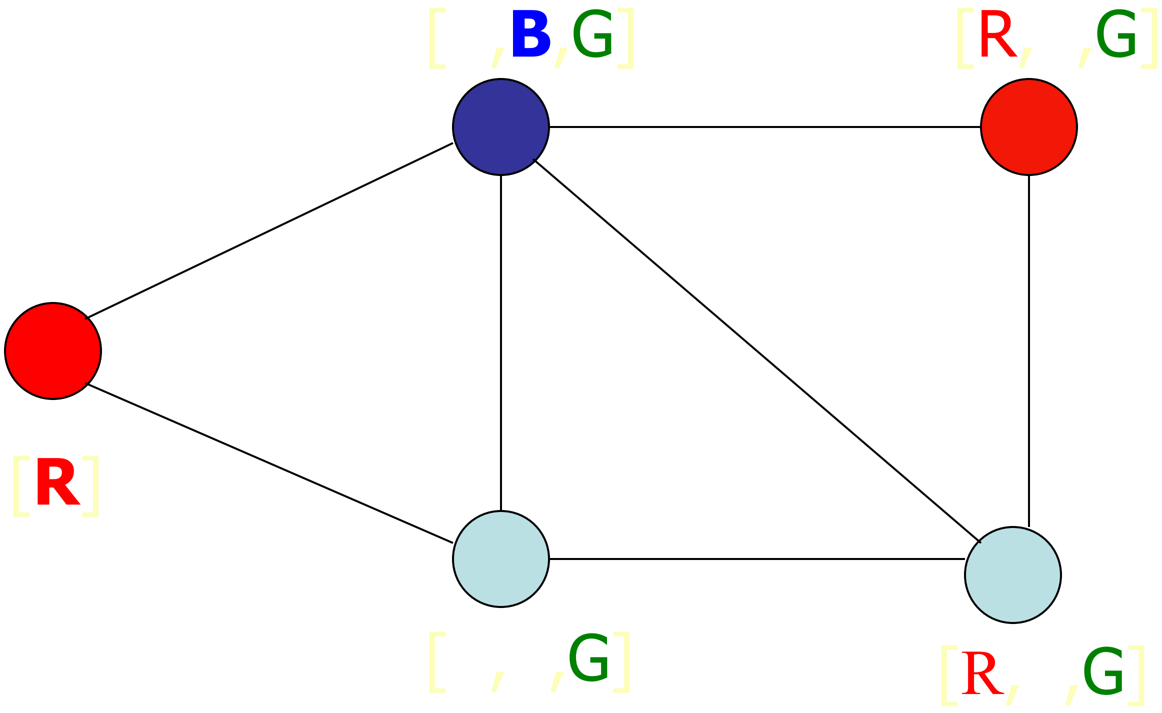

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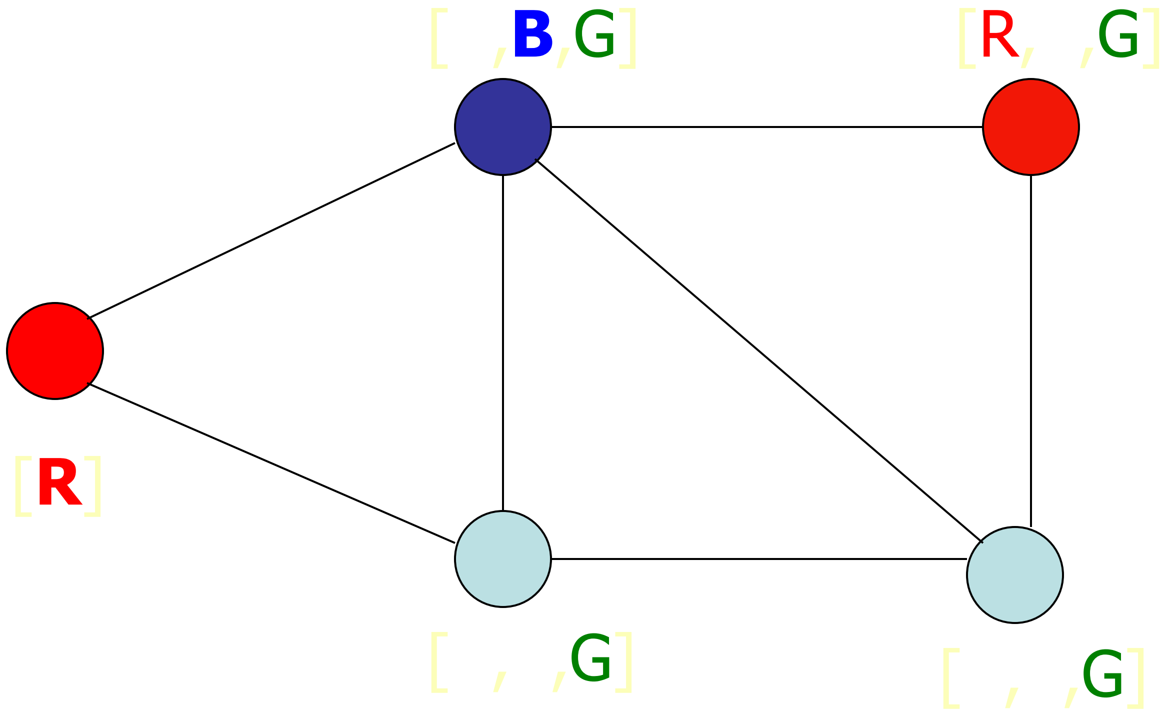

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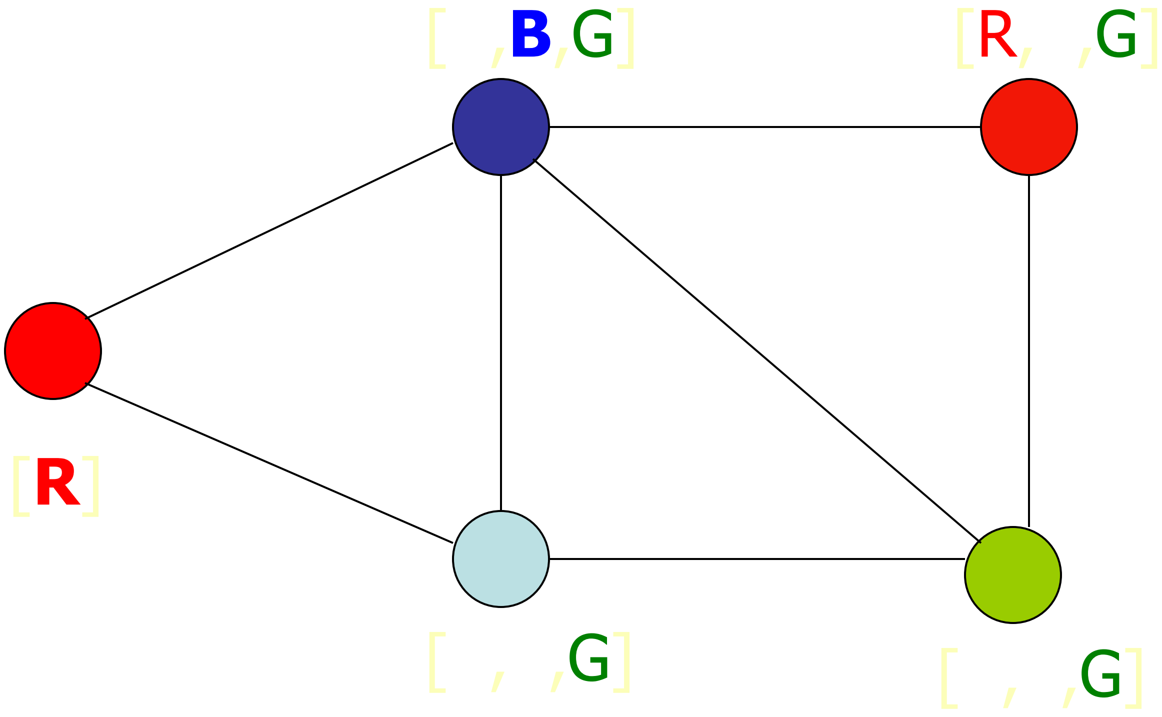

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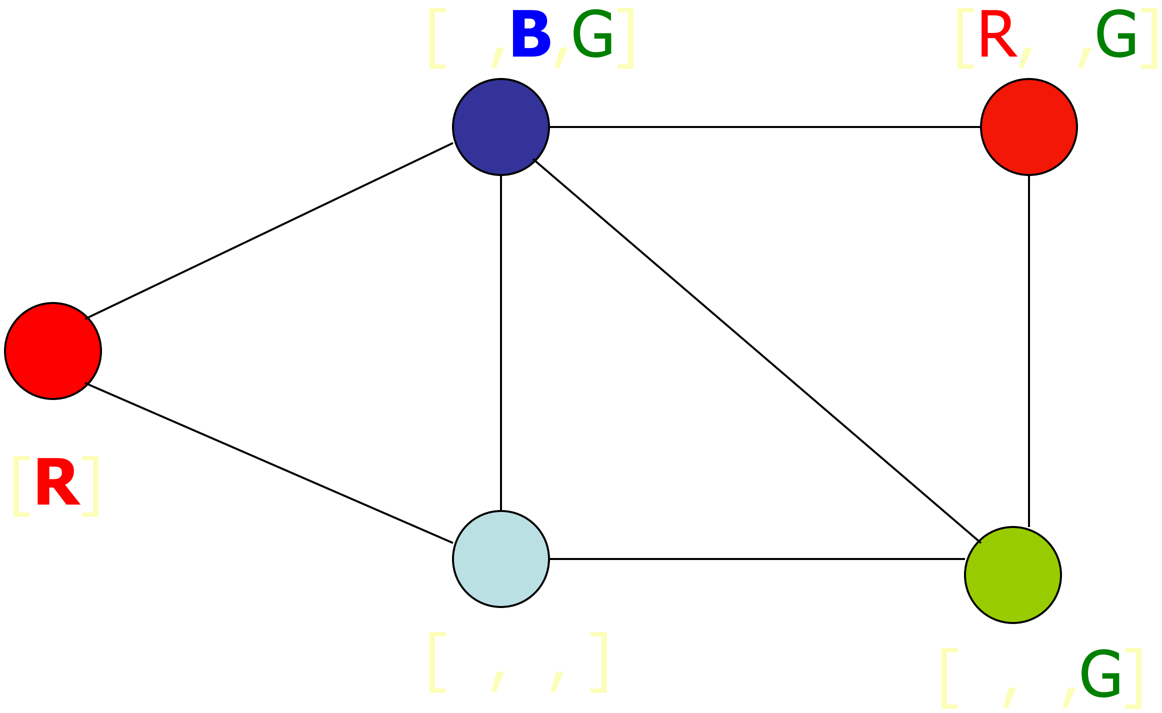

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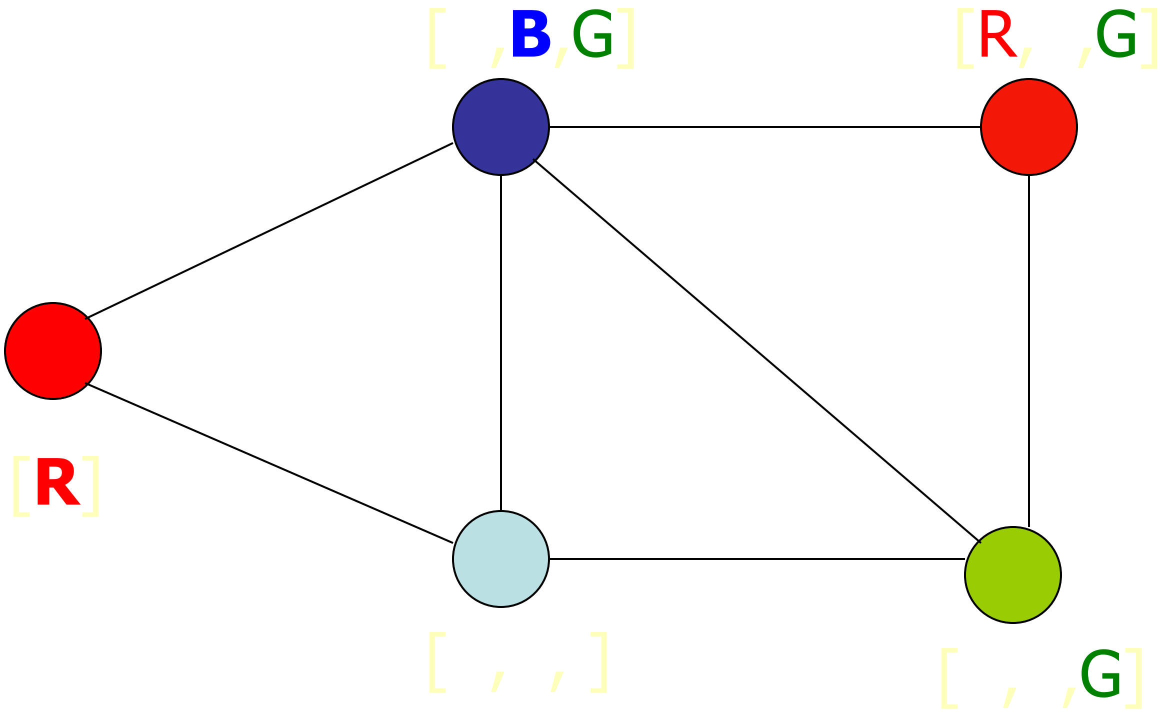
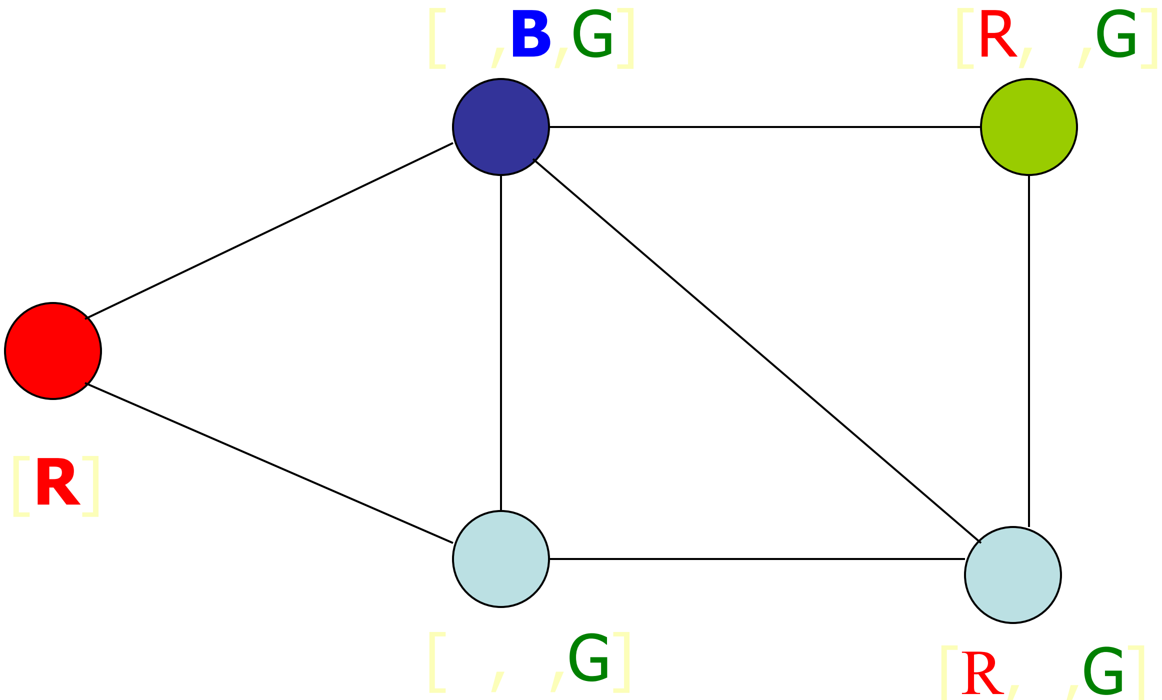


Dead End

Forward Checking



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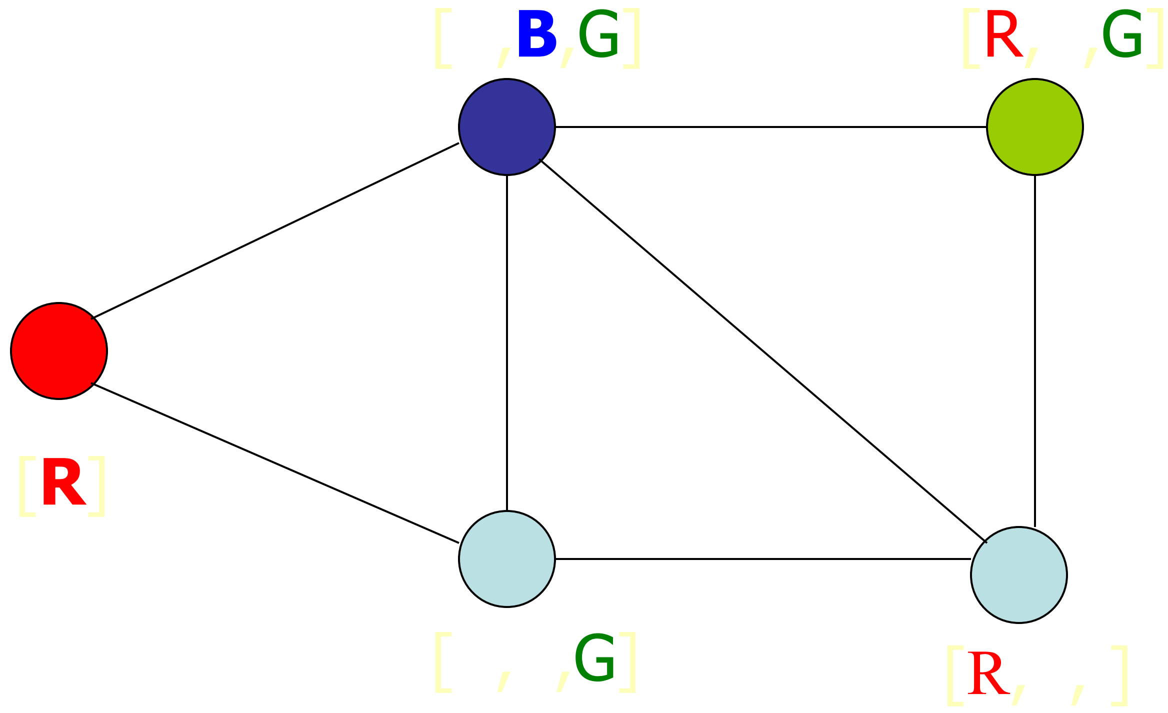
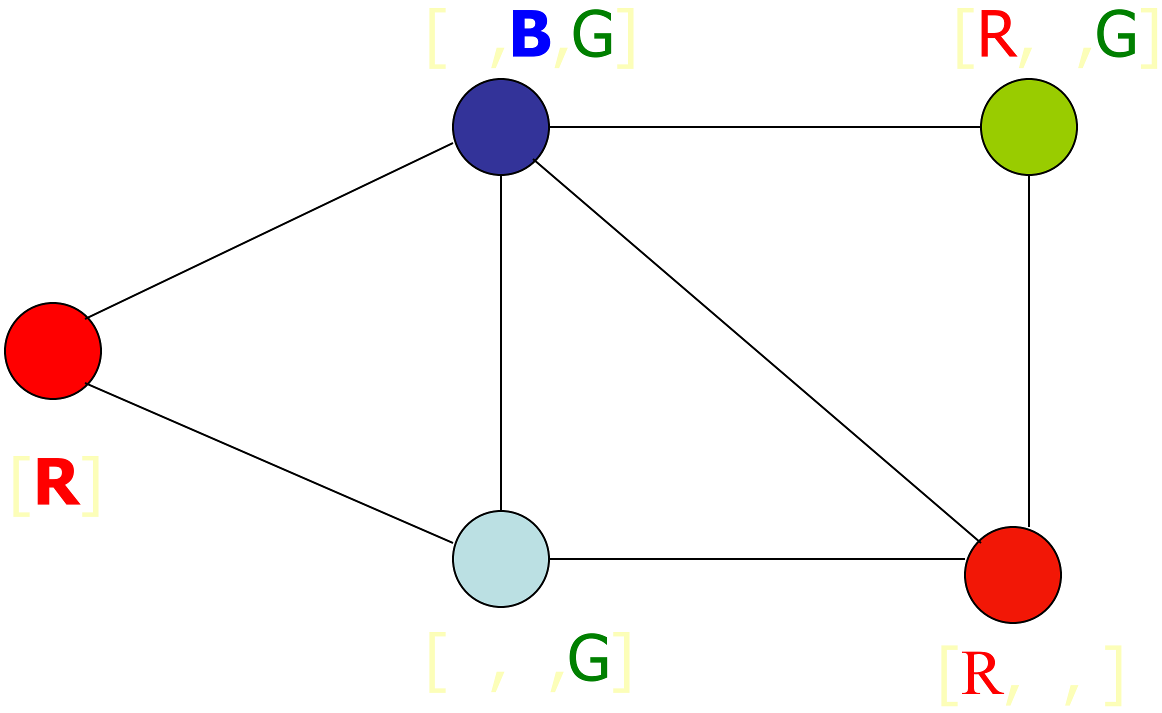


Forward Checking



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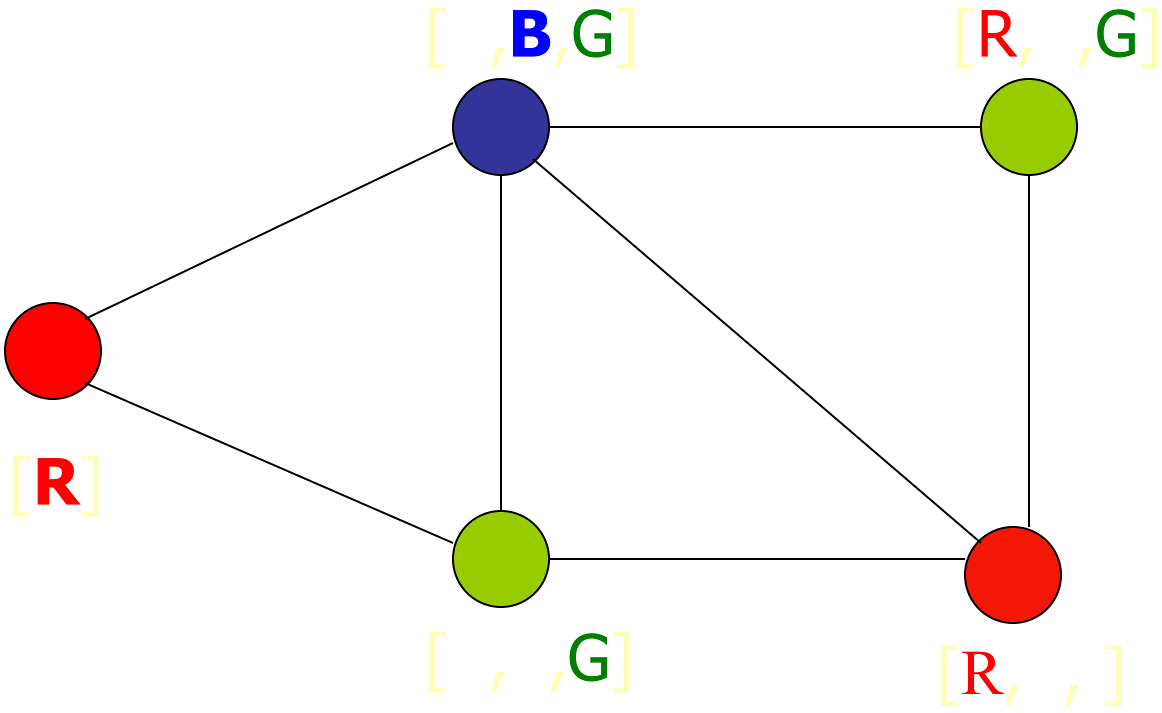
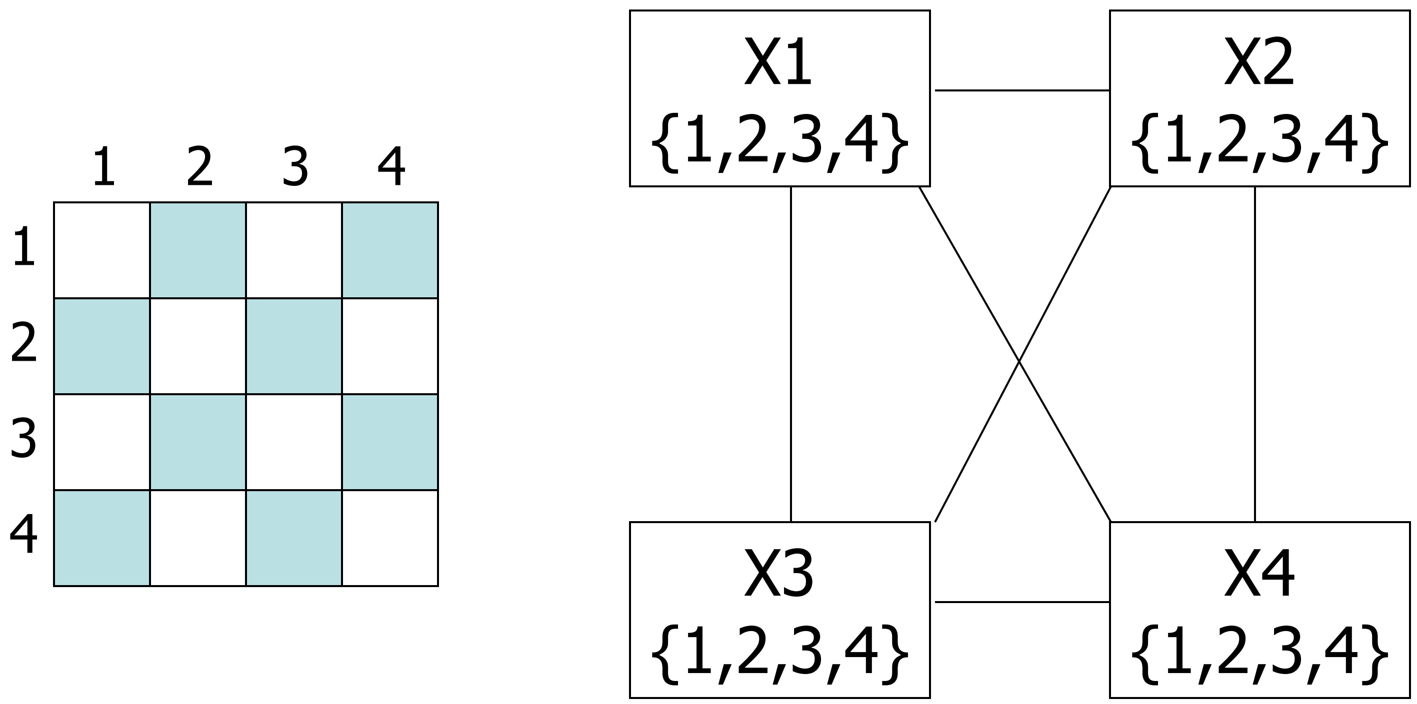


Solution !!!

Example: 4-Queens Problem



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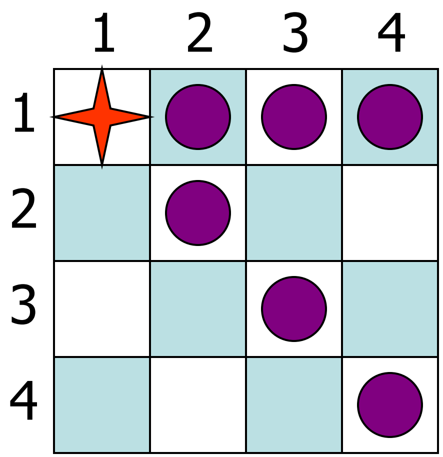

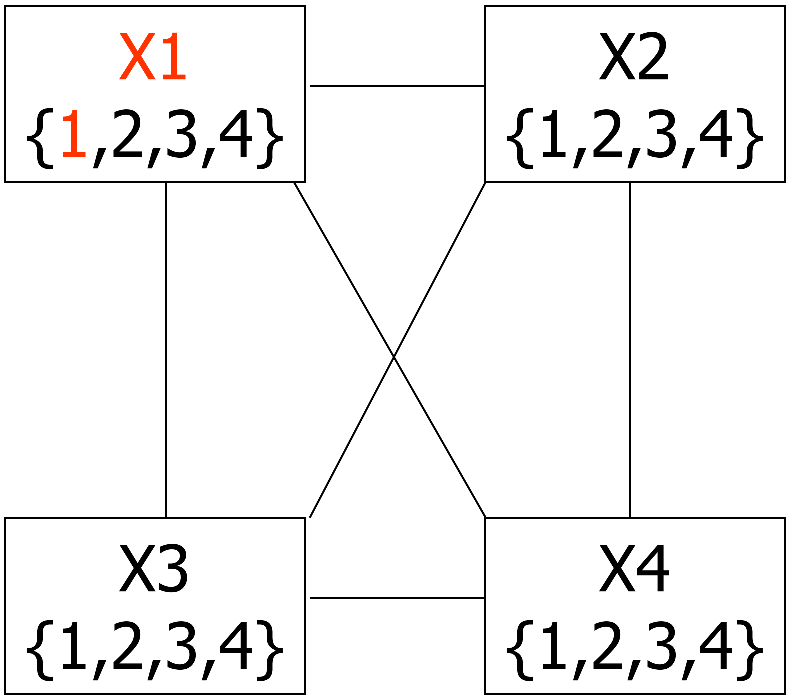

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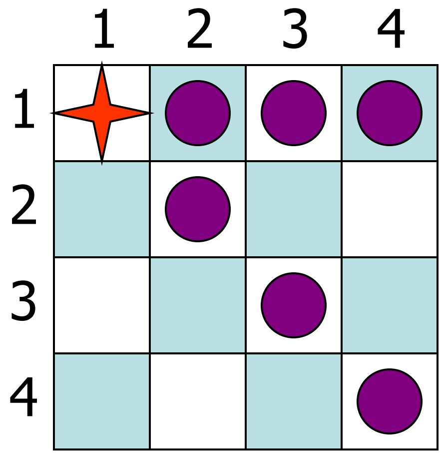

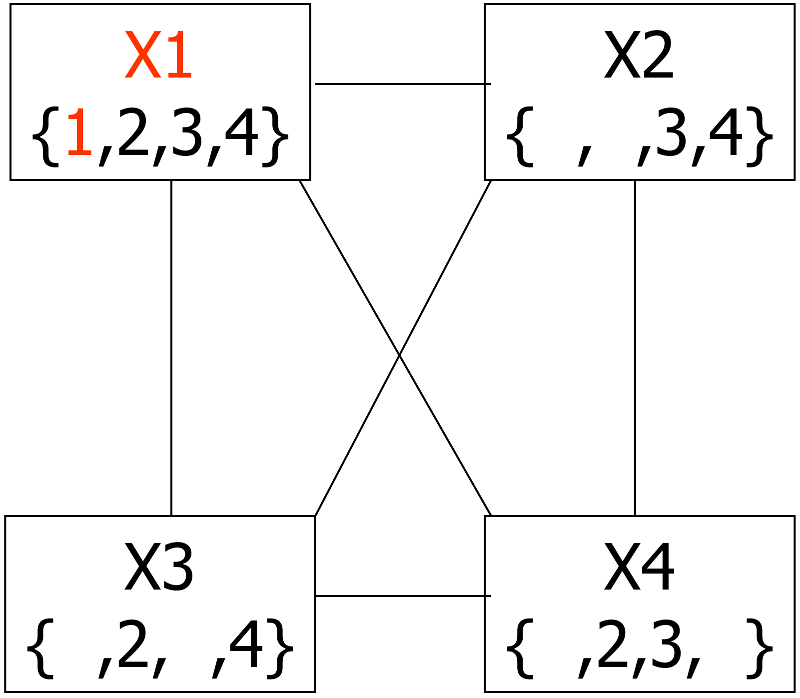

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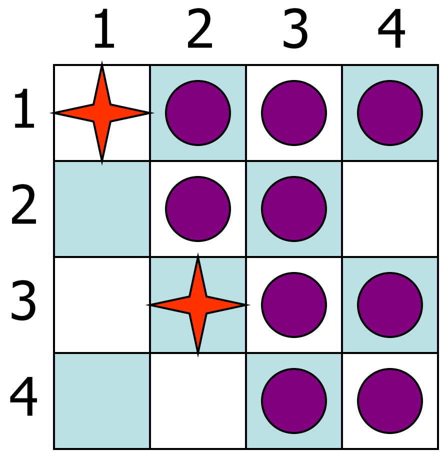

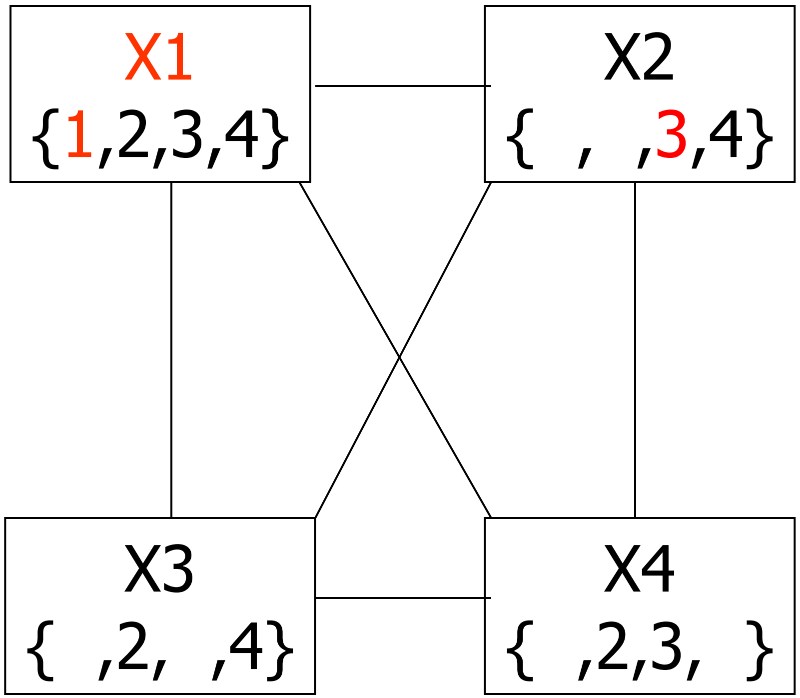

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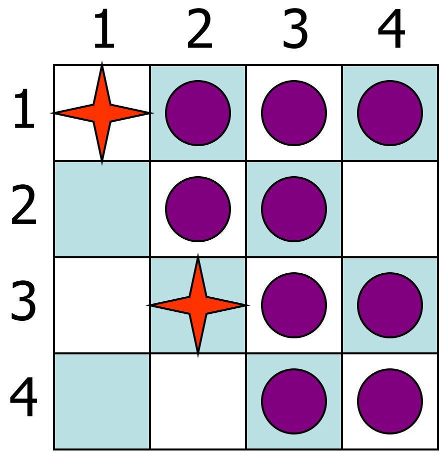

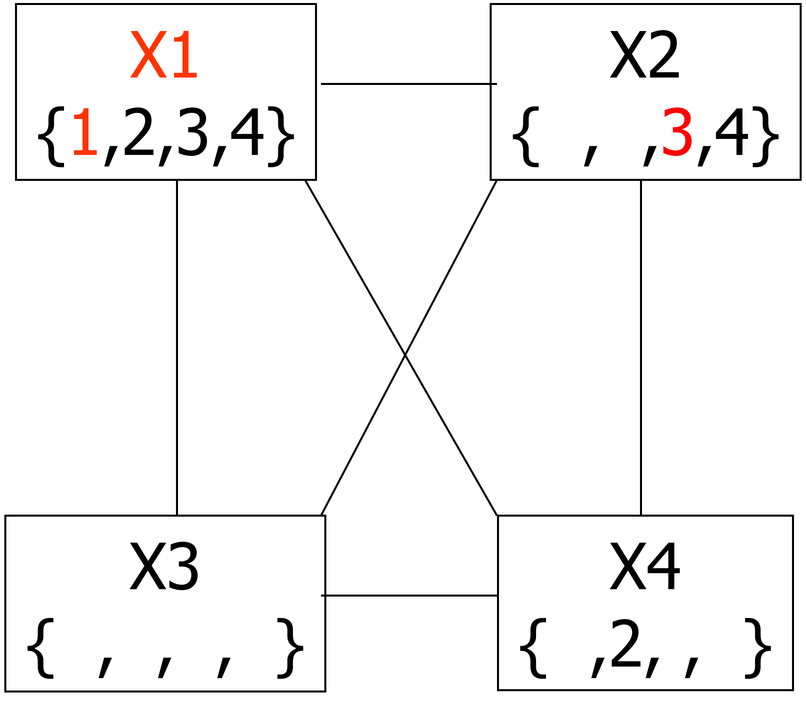

Dead End ⟶ Backtrack

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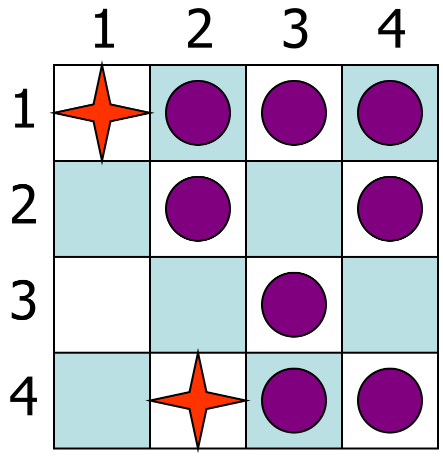

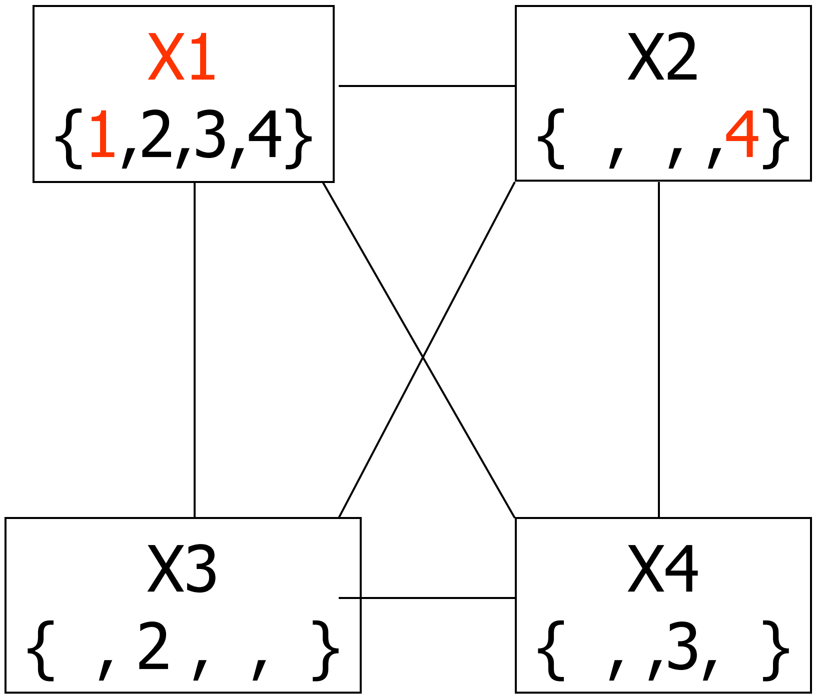

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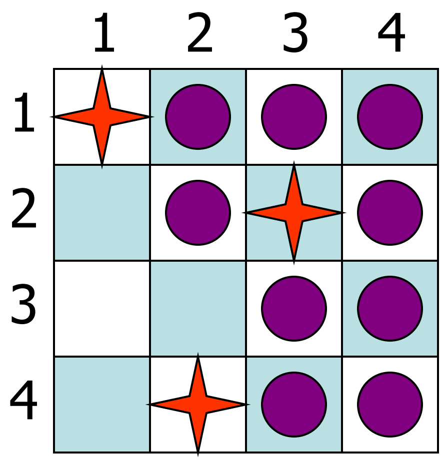

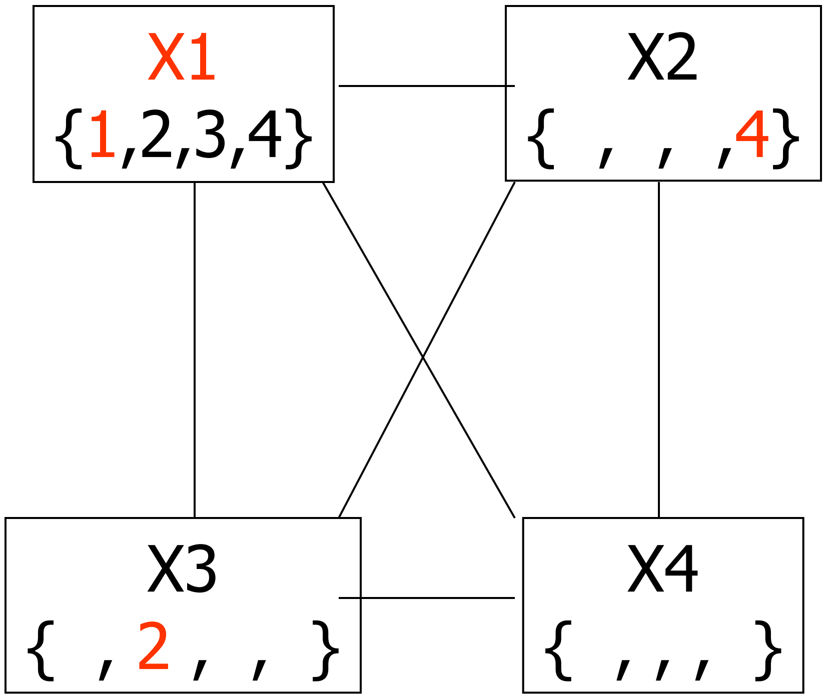

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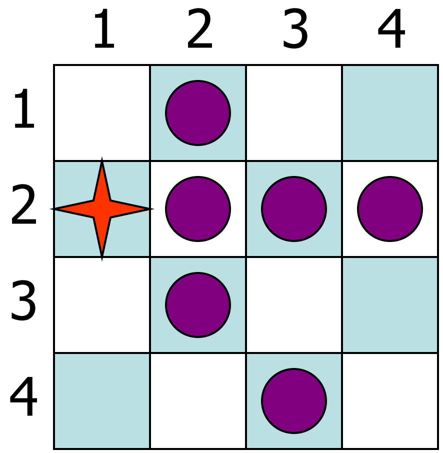

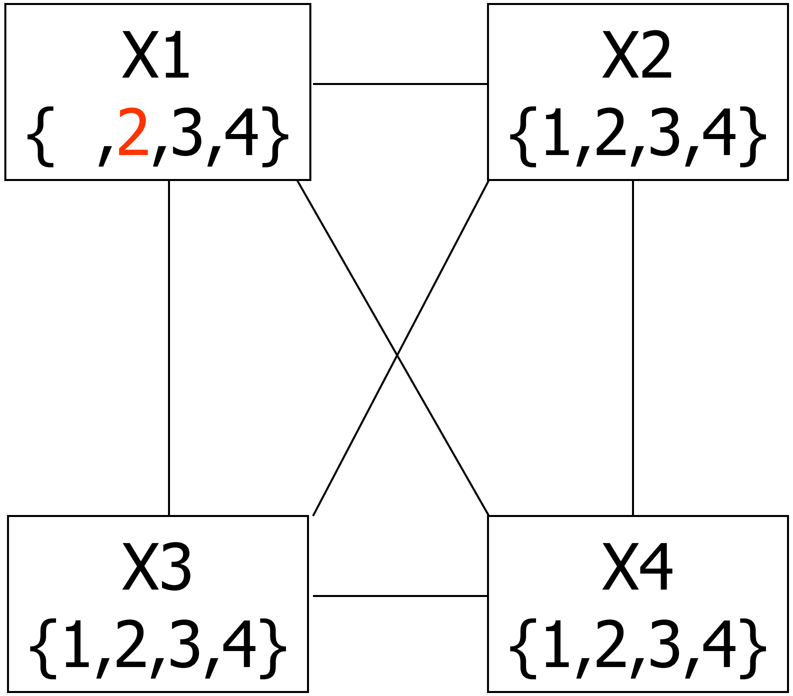

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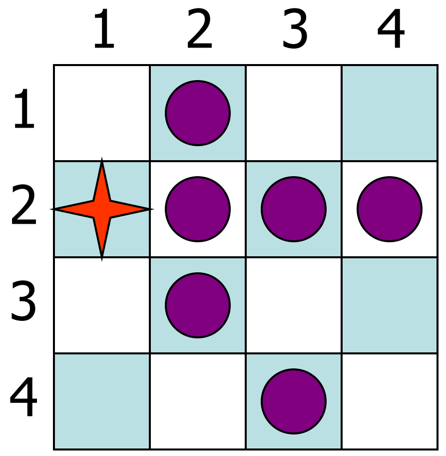

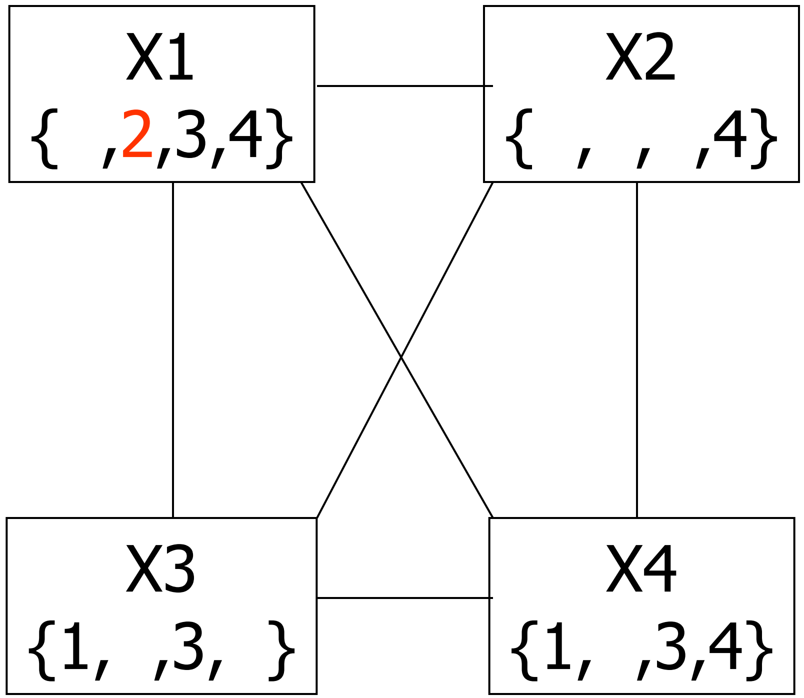

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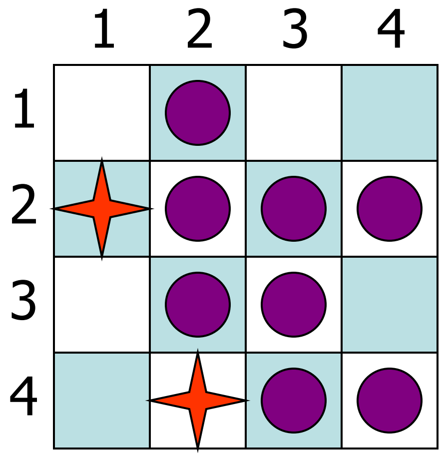

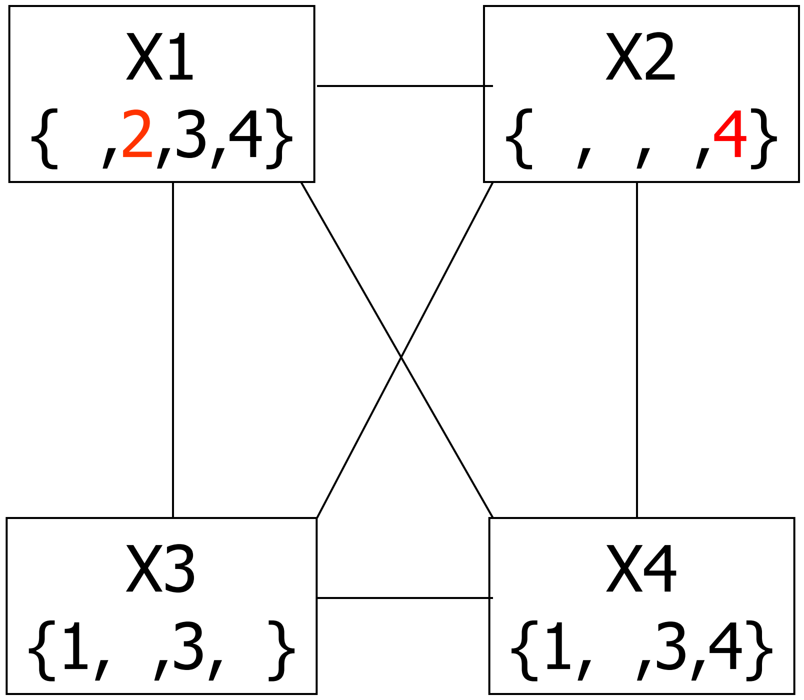

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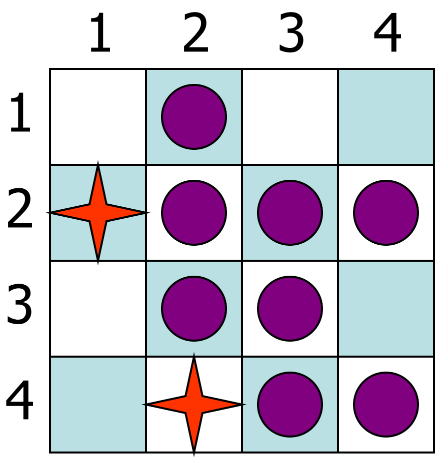

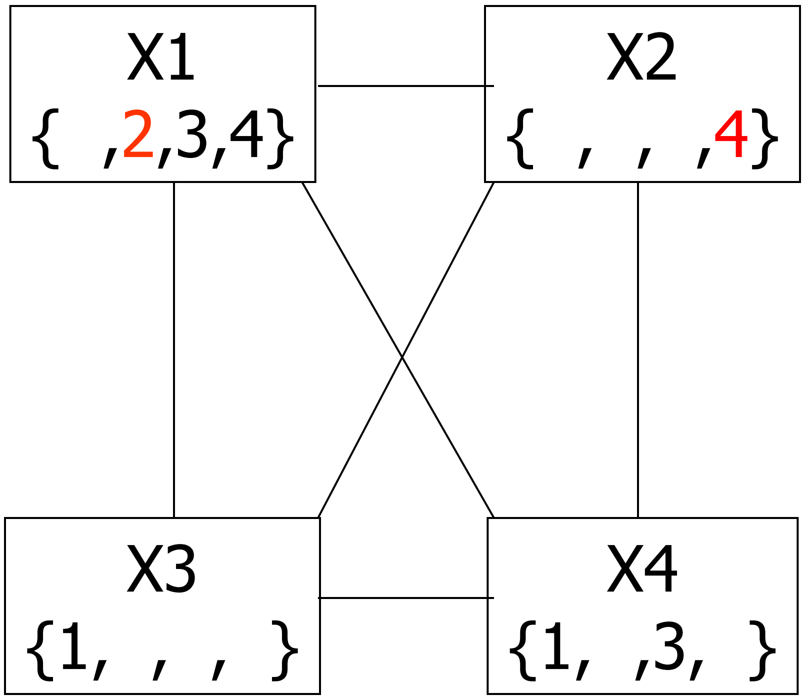

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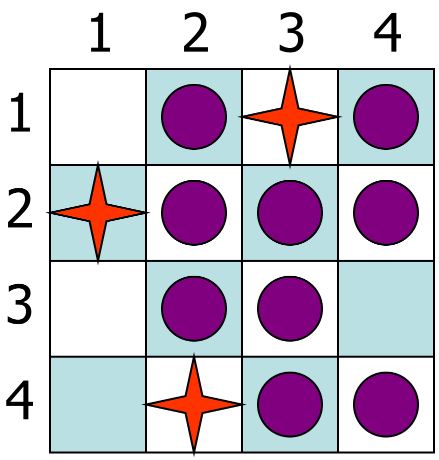

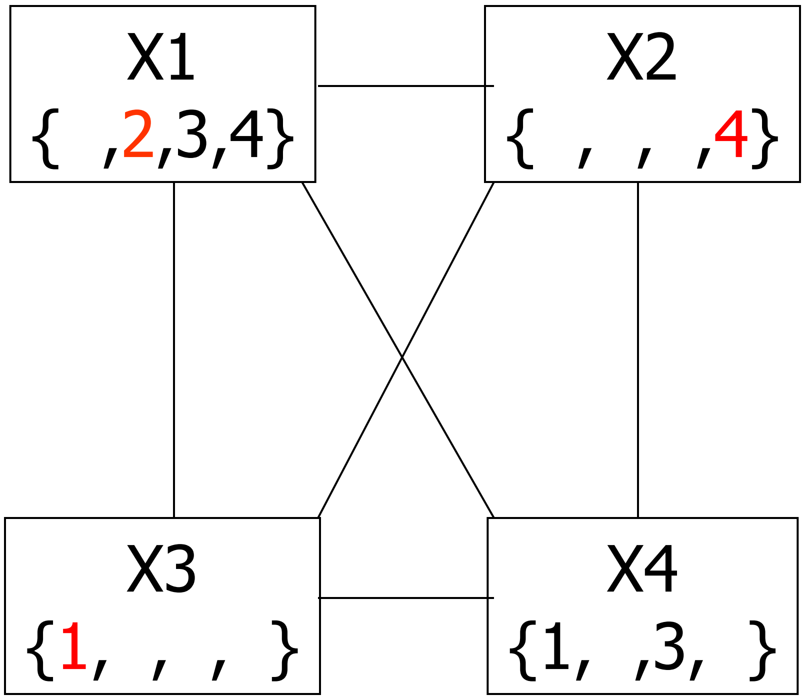

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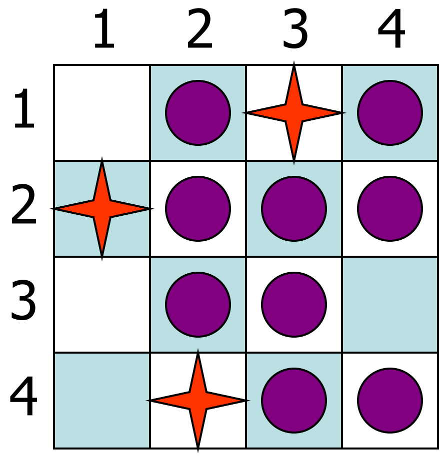

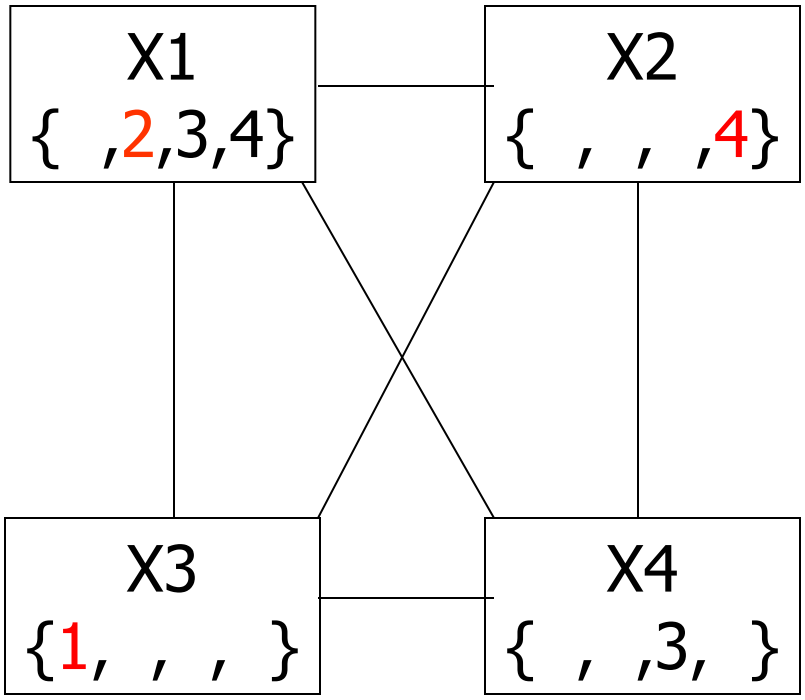

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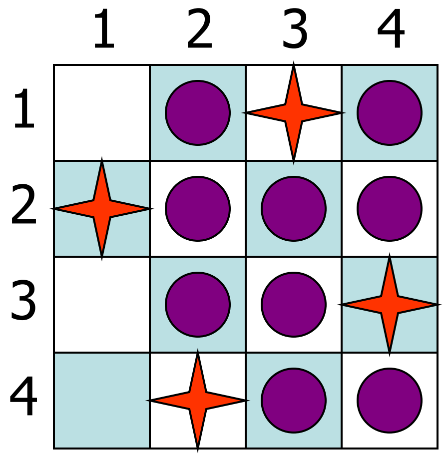
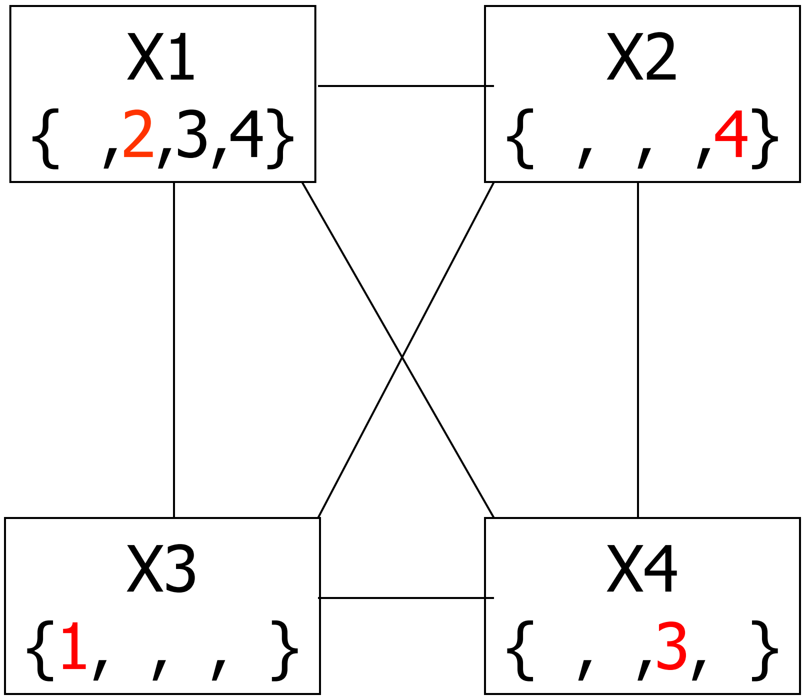
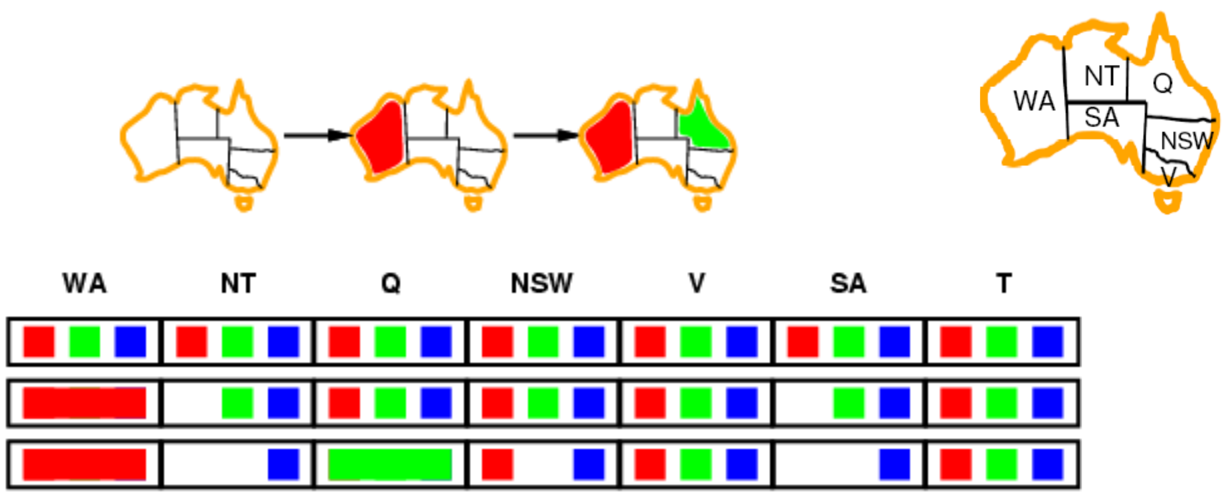




Solution !!!!

Constraint propagation

- Forward checking propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures:



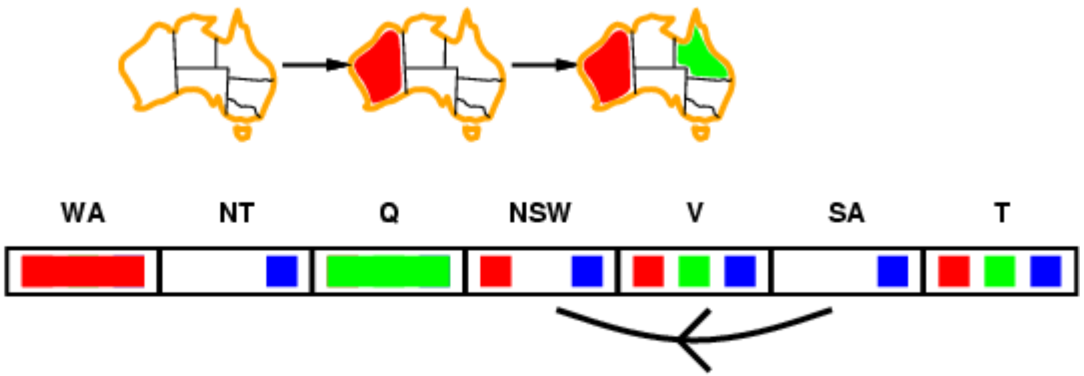
- NT and SA cannot both be blue!

- Constraint propagation repeatedly enforces constraints locally

Arc consistency

- Simplest form of propagation makes each arc consistent
- X →Y is consistent iff

for every value x of X there is some allowed y



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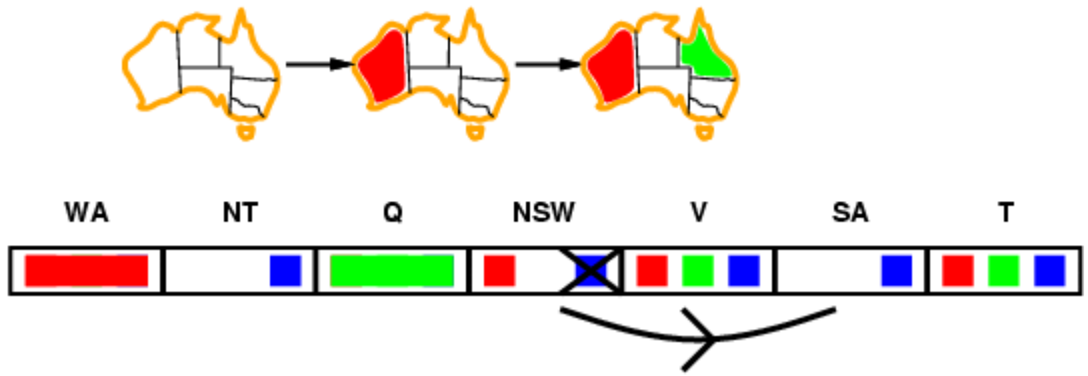

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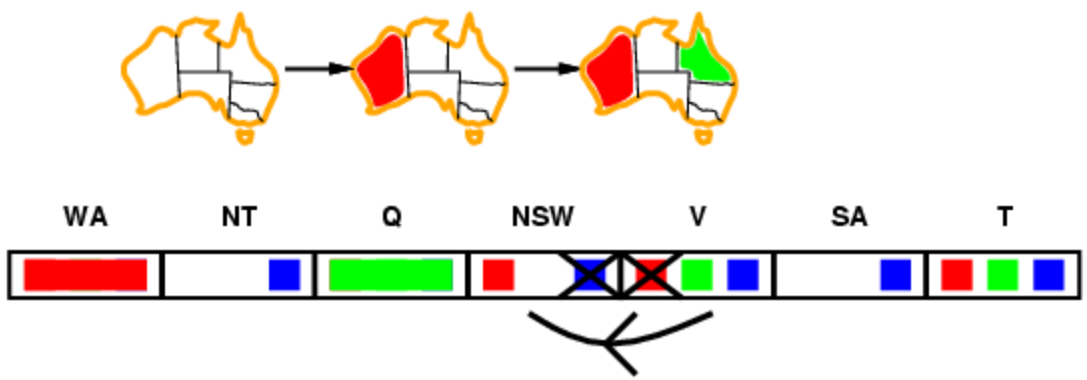

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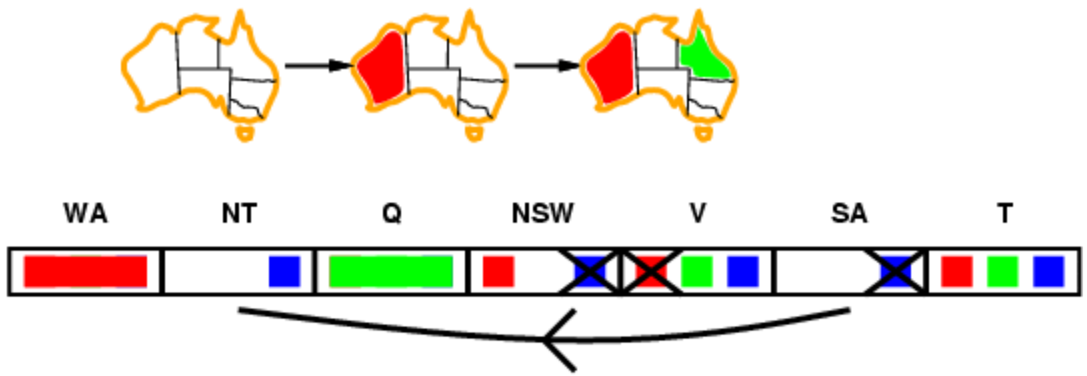
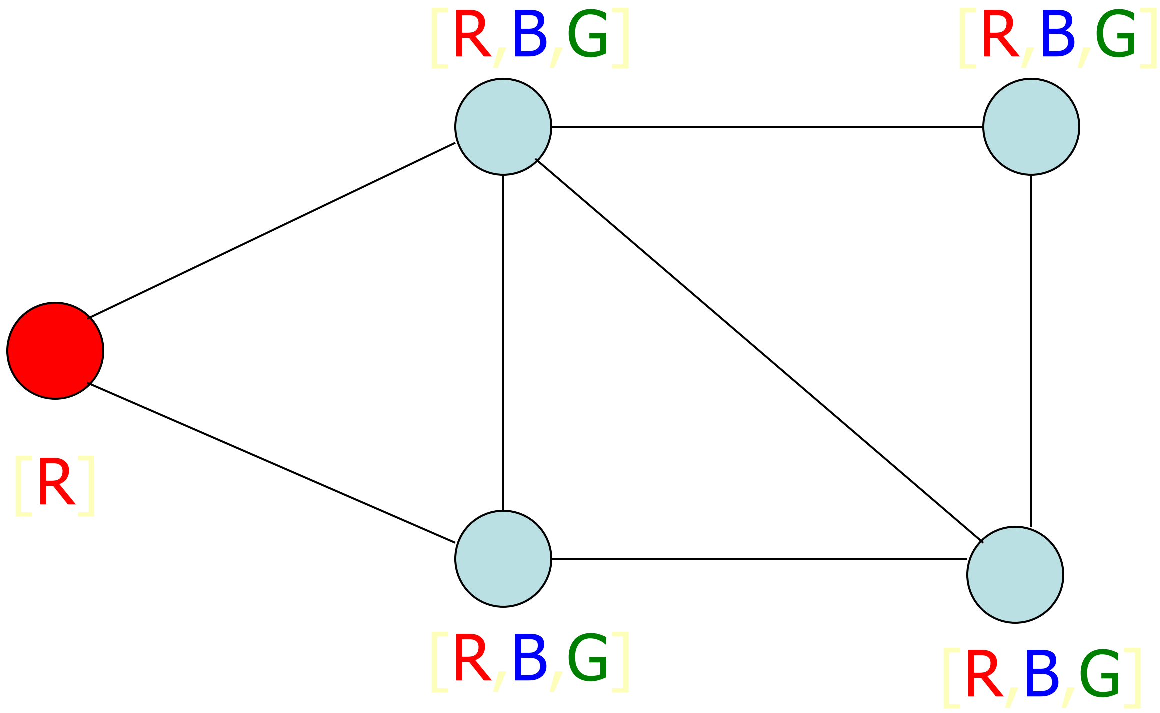


- If X loses a value, neighbors of X need to be rechecked
- Arc consistency detects failure earlier than forward checking
- Can be run as a preprocessor or after each assignment

Arc Consistency: AC3



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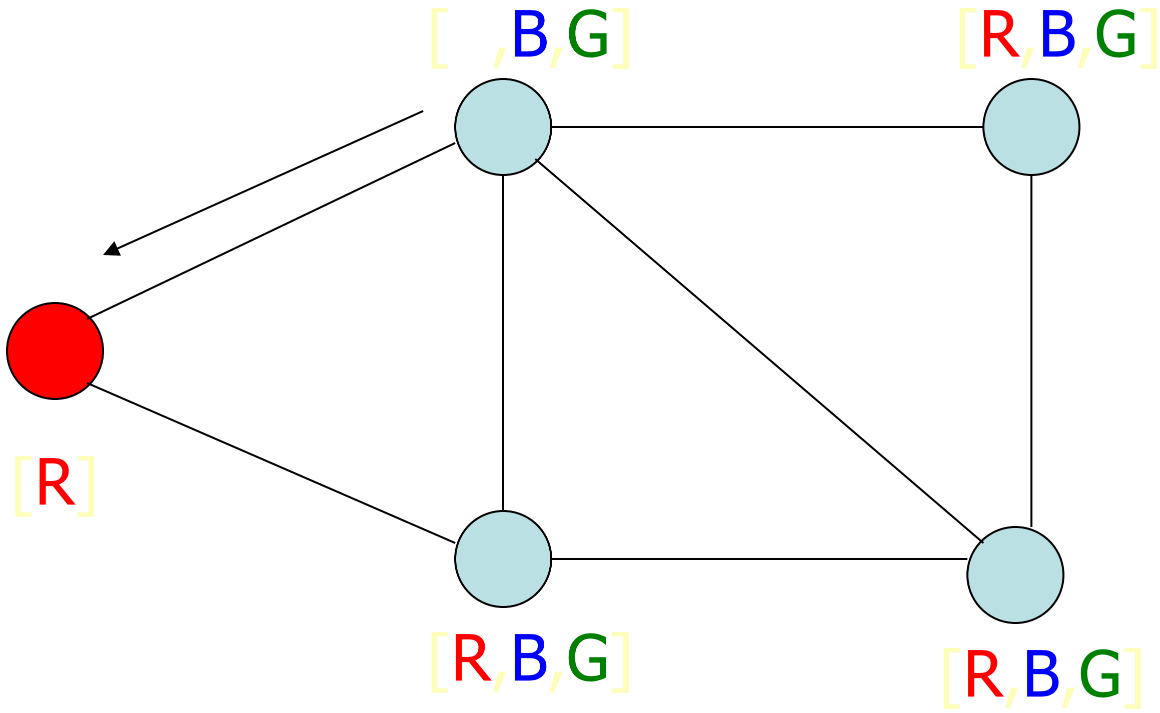

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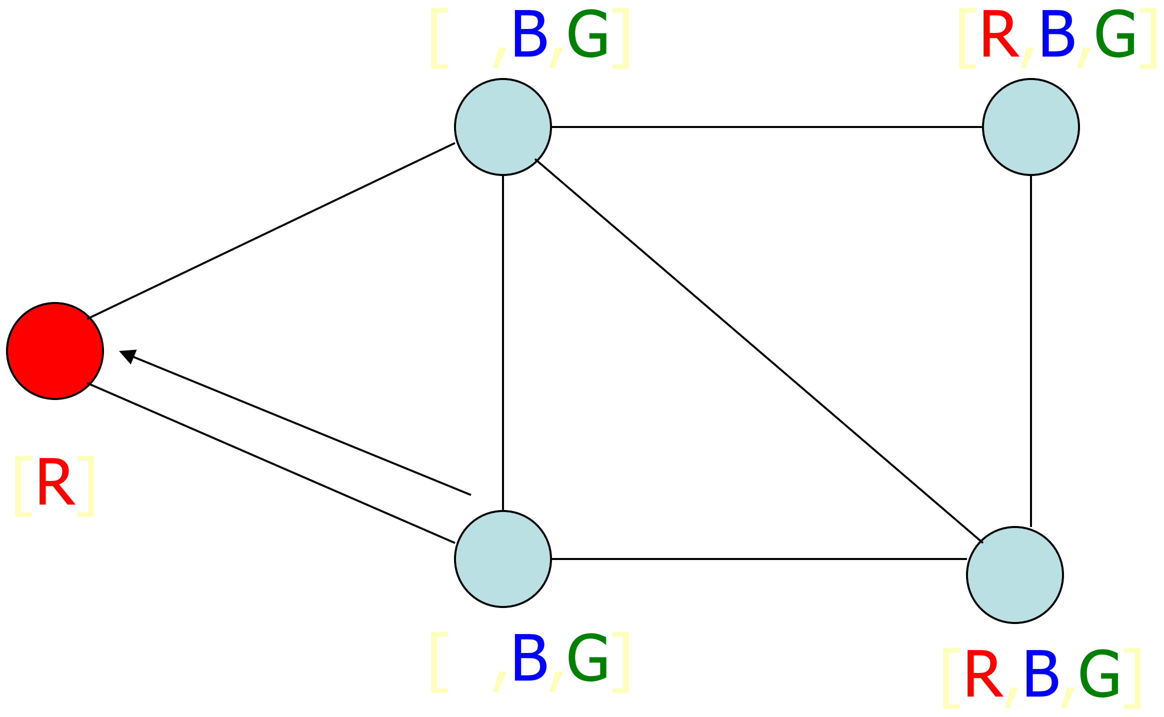

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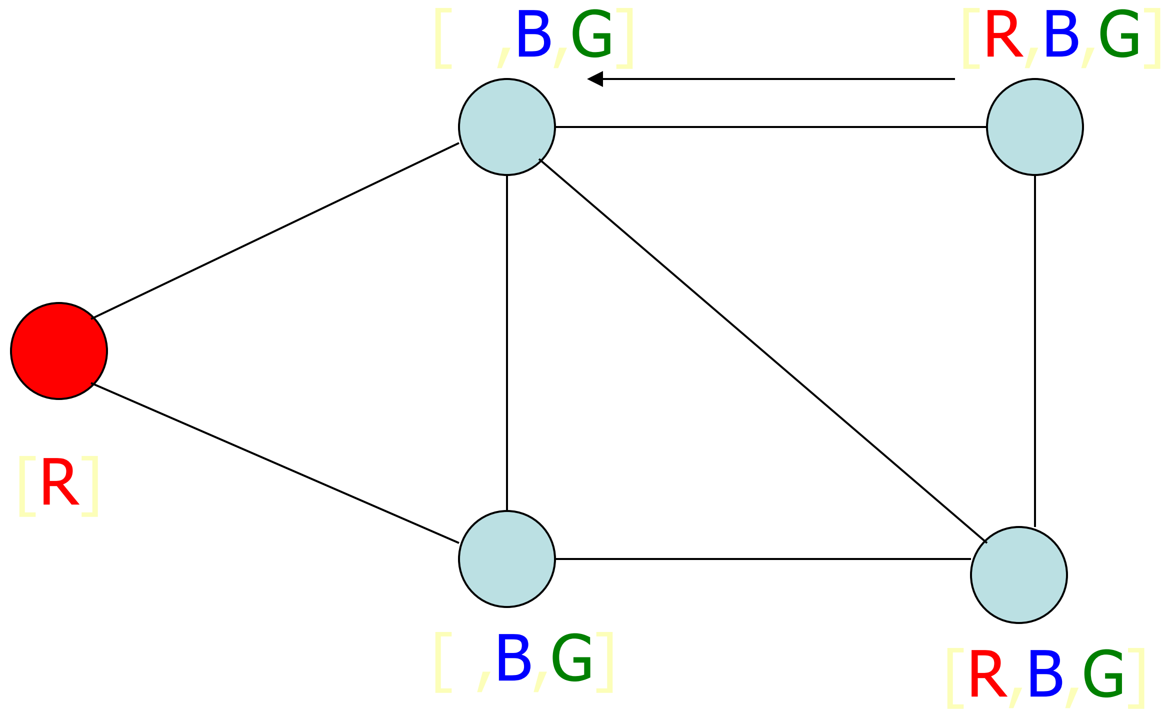

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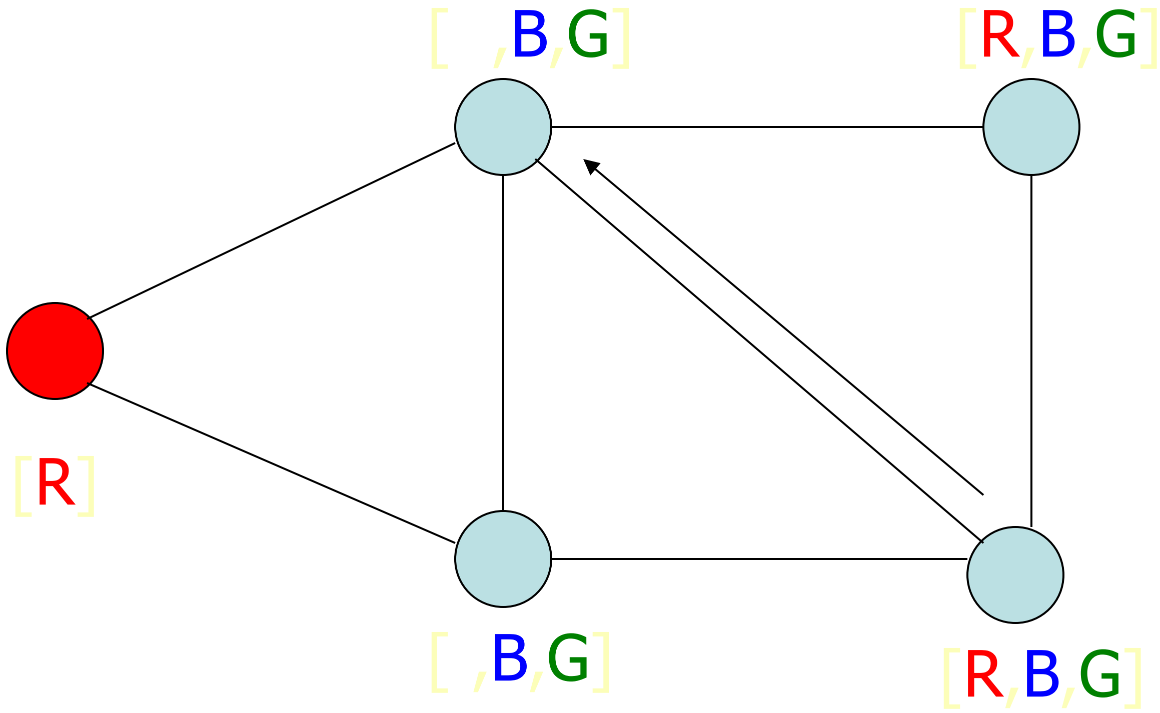

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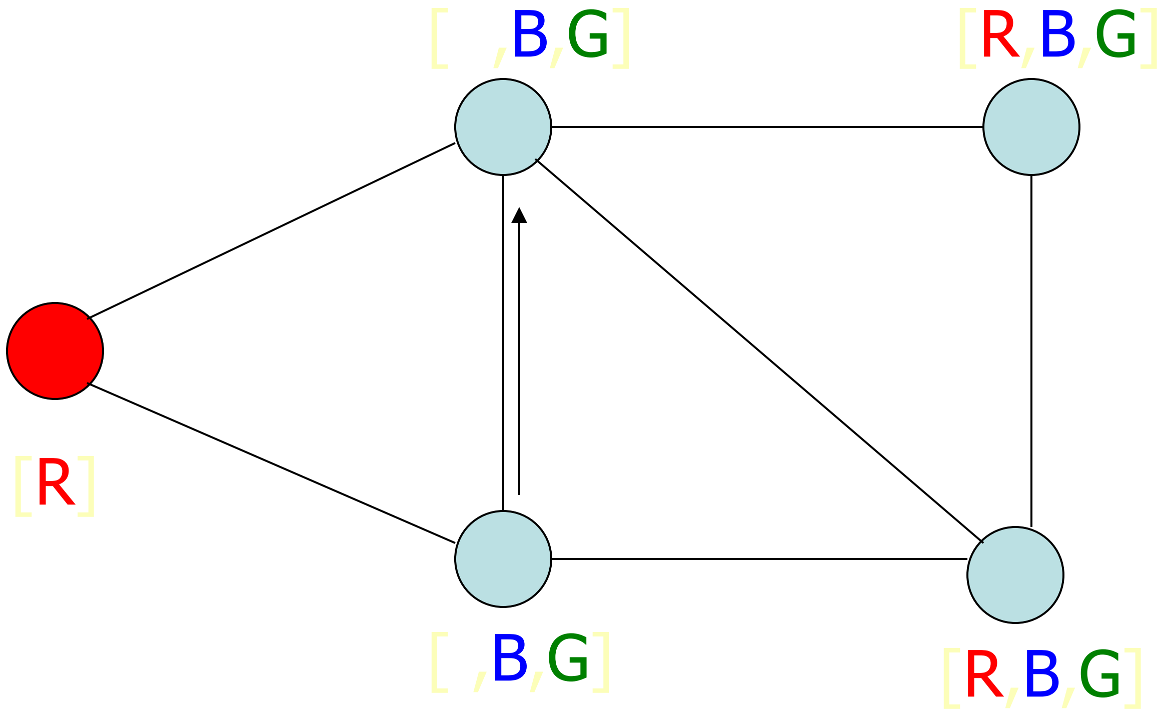

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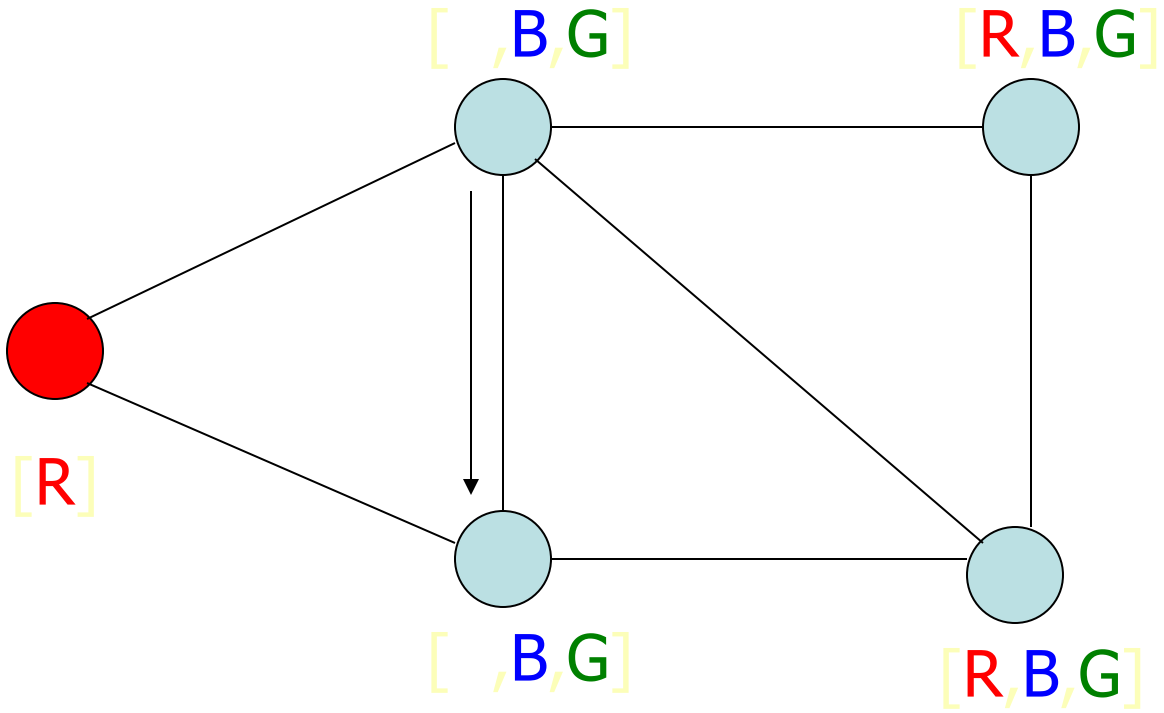

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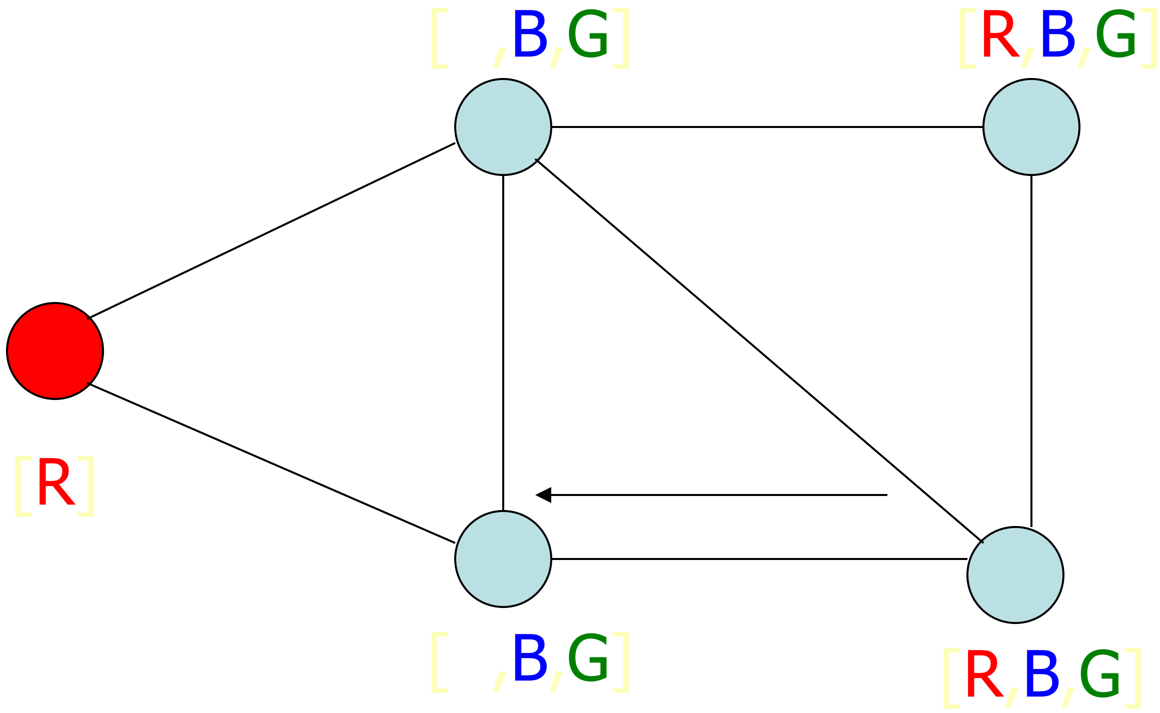

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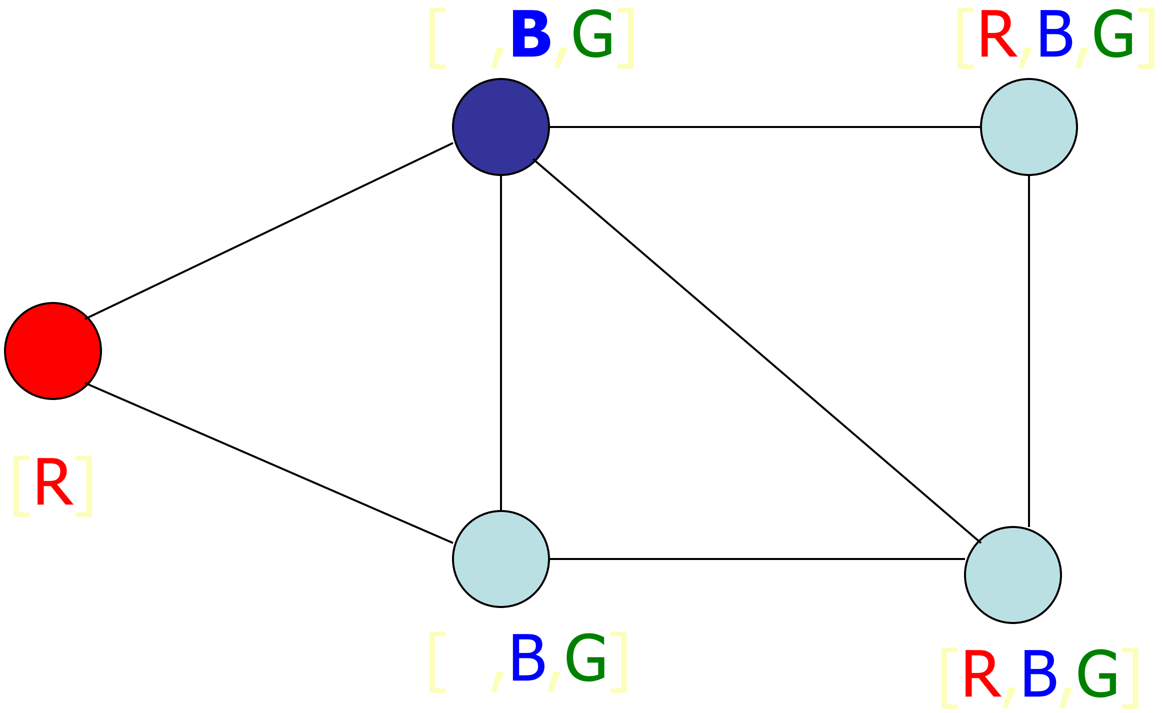

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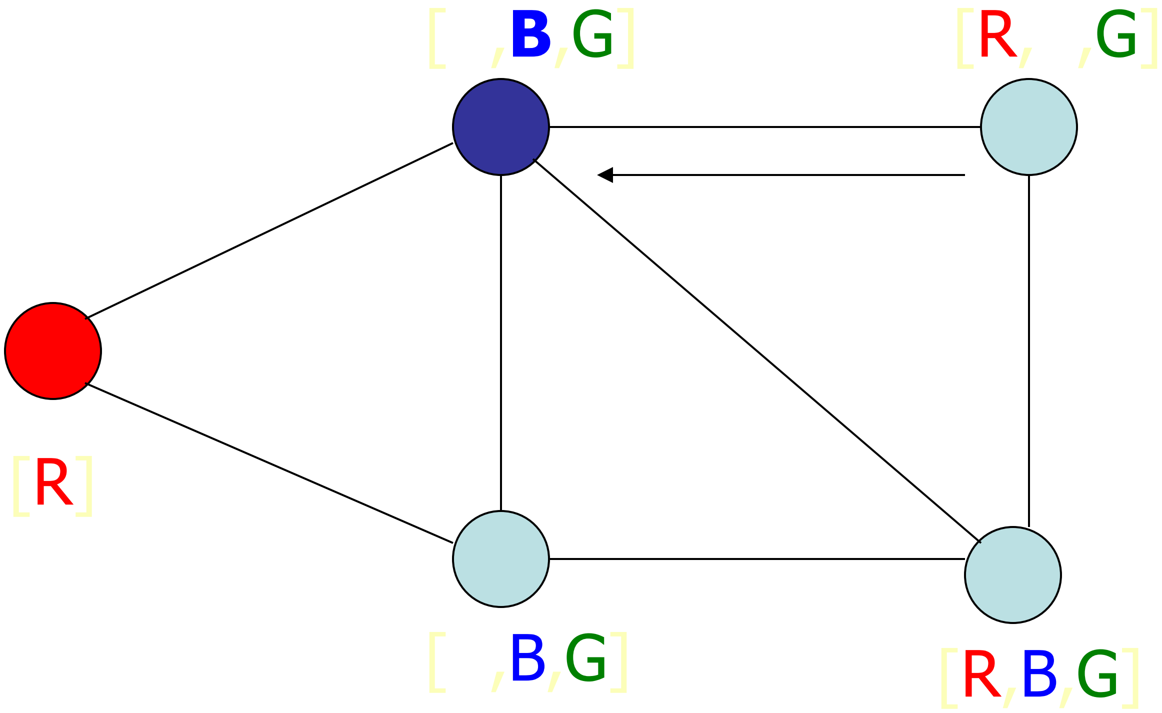

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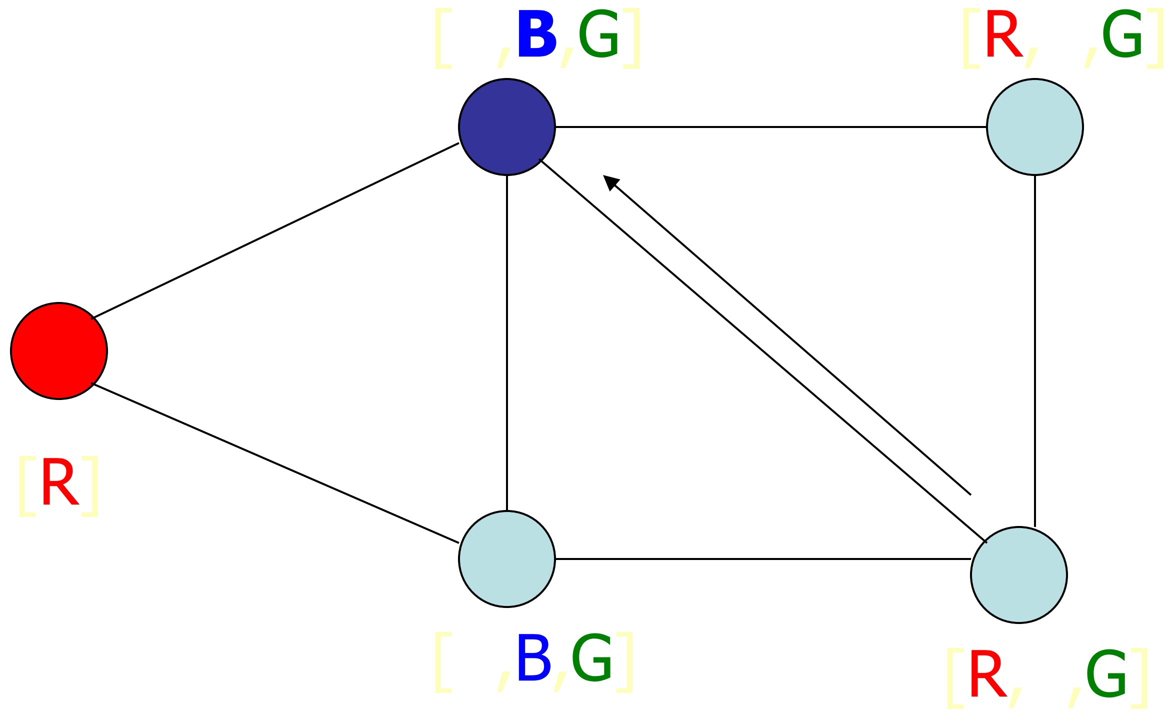

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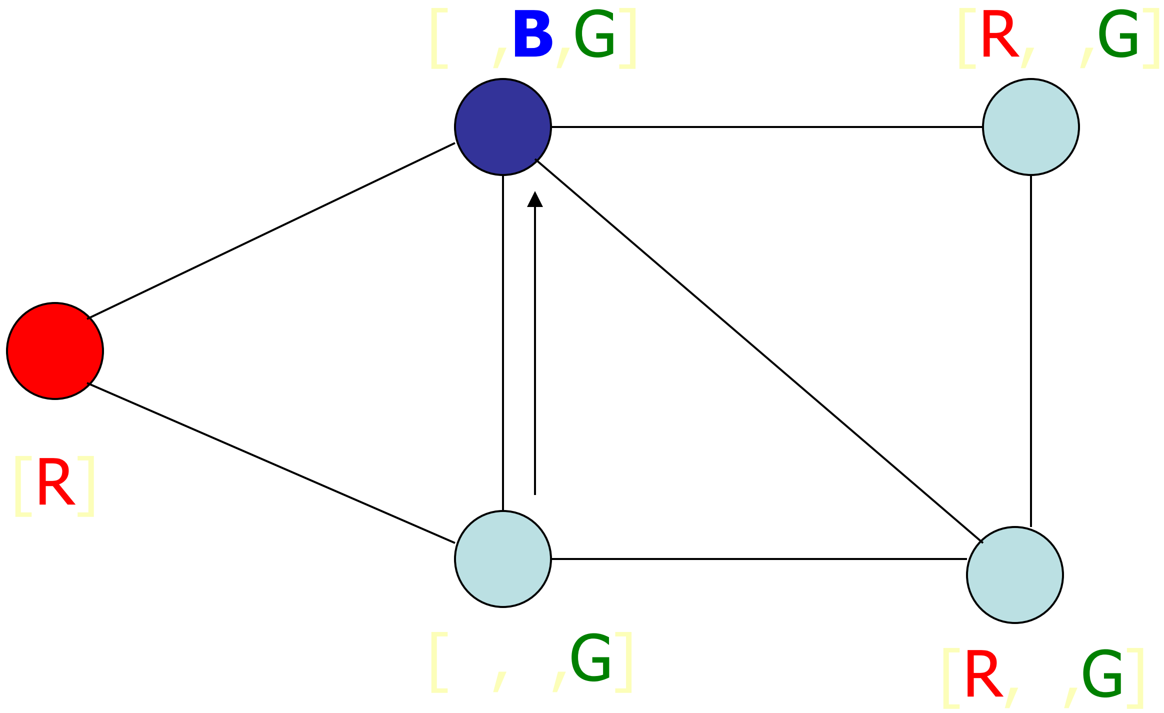

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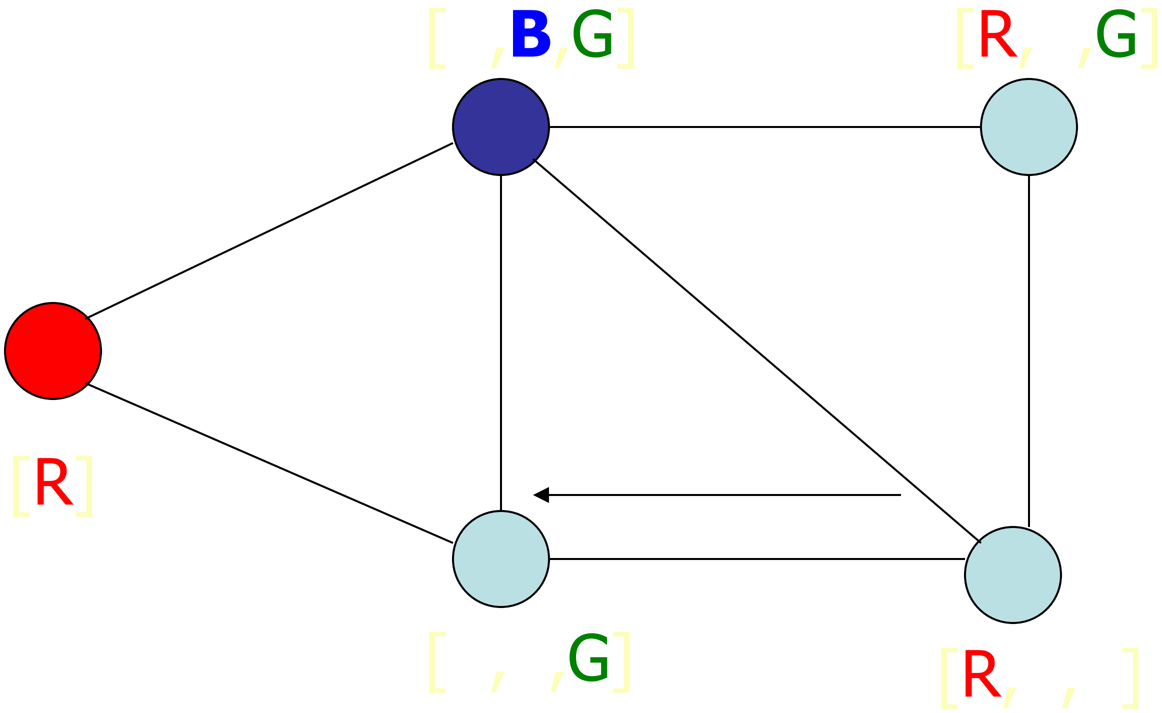

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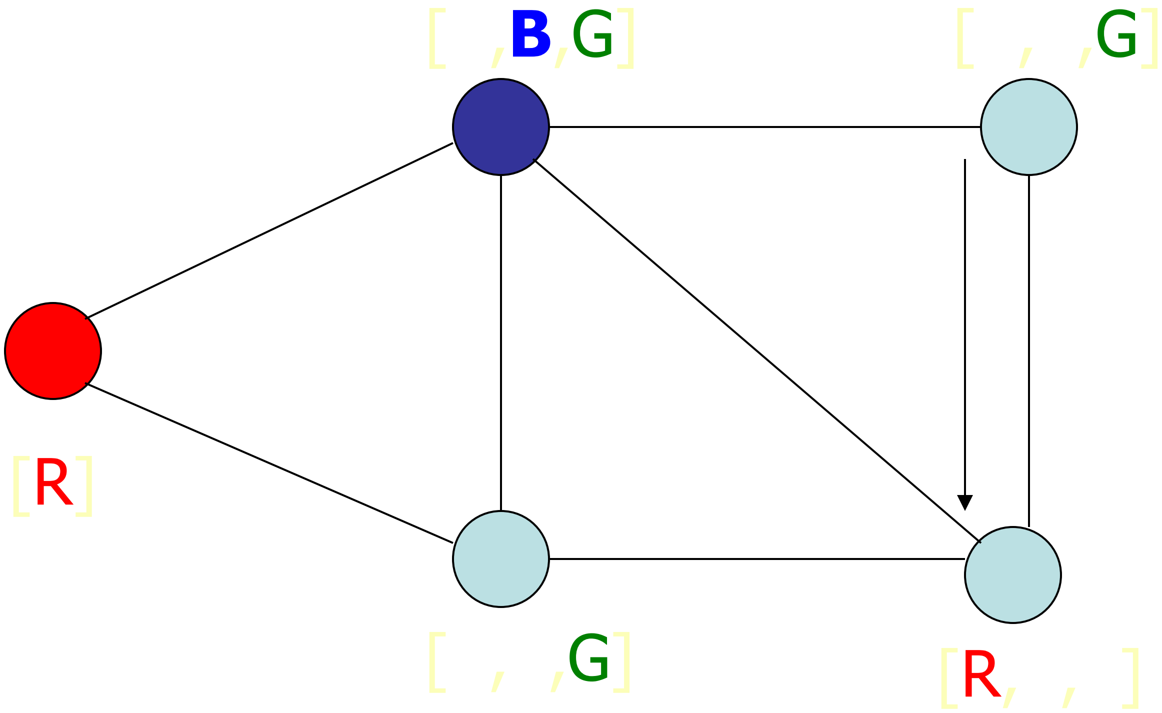

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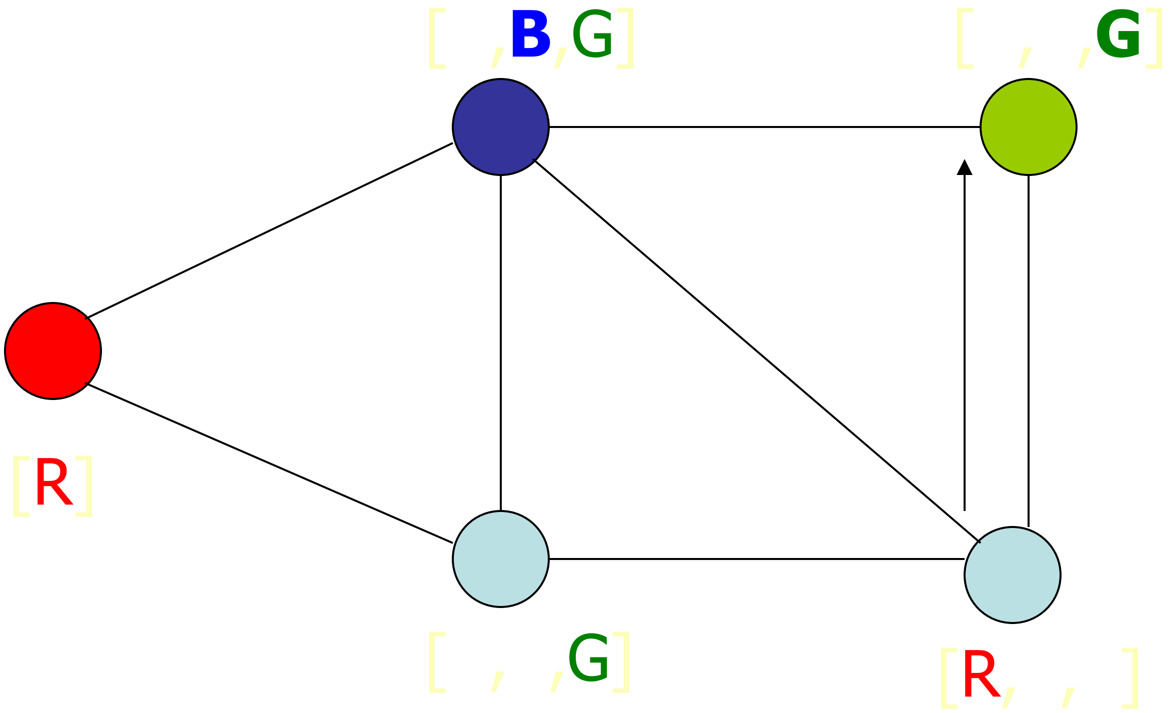

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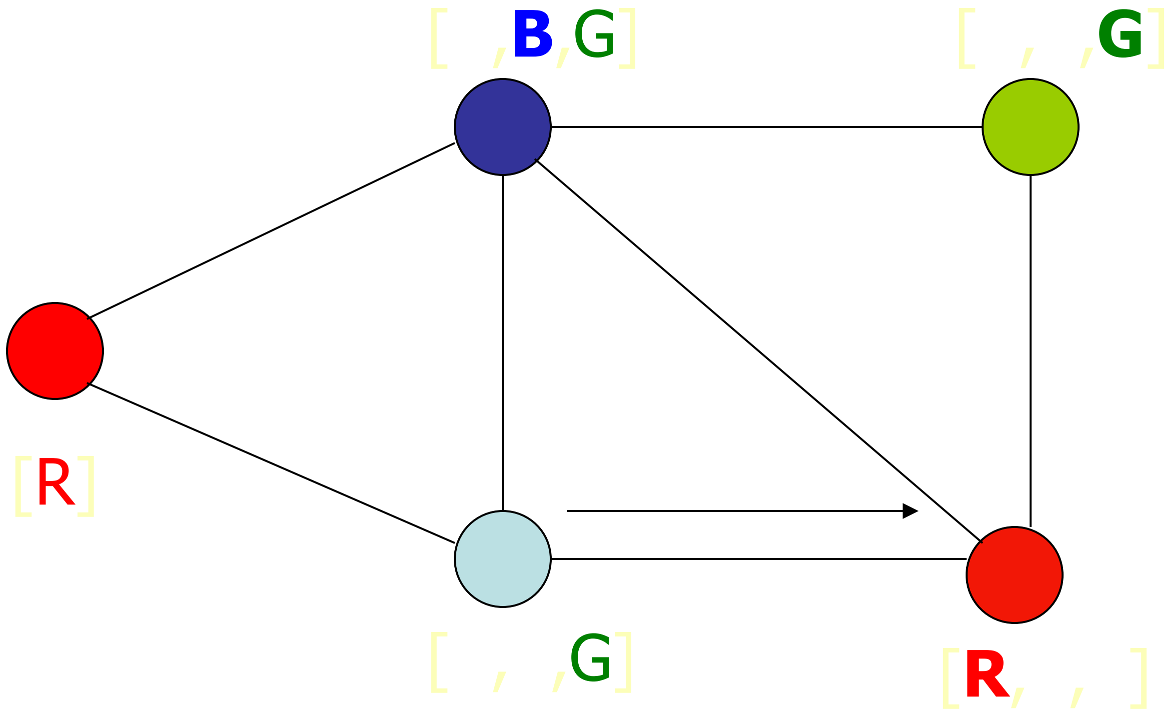

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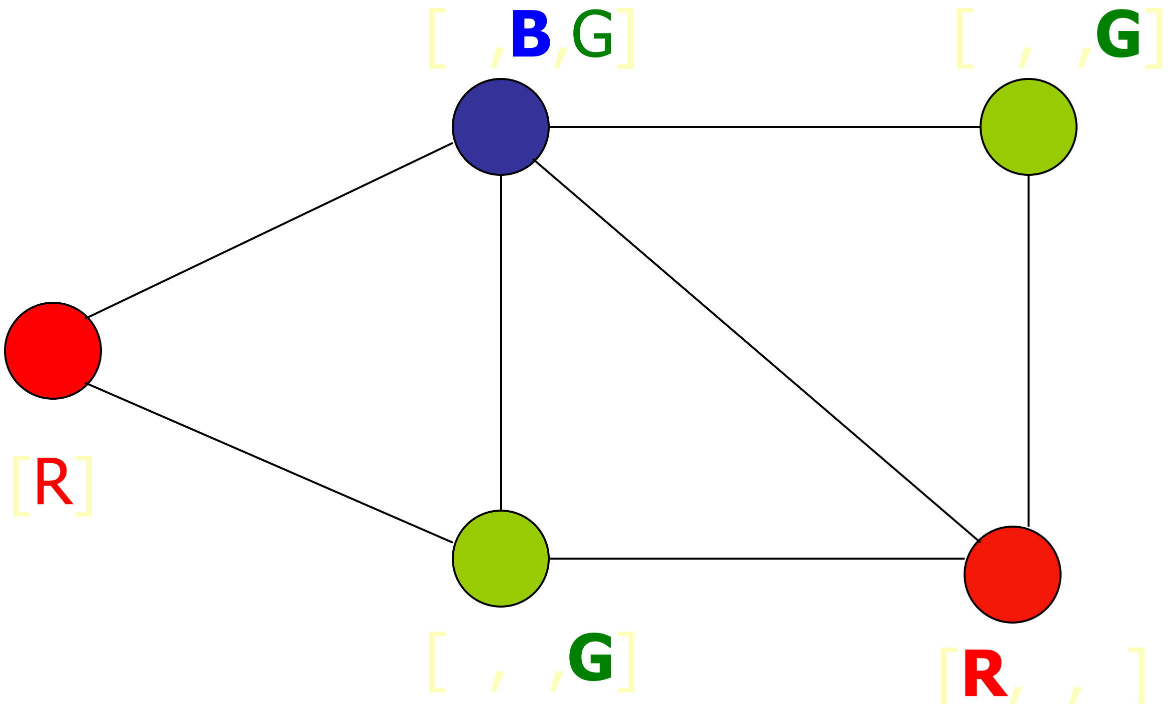


Solution !!!

Local Search and CSP

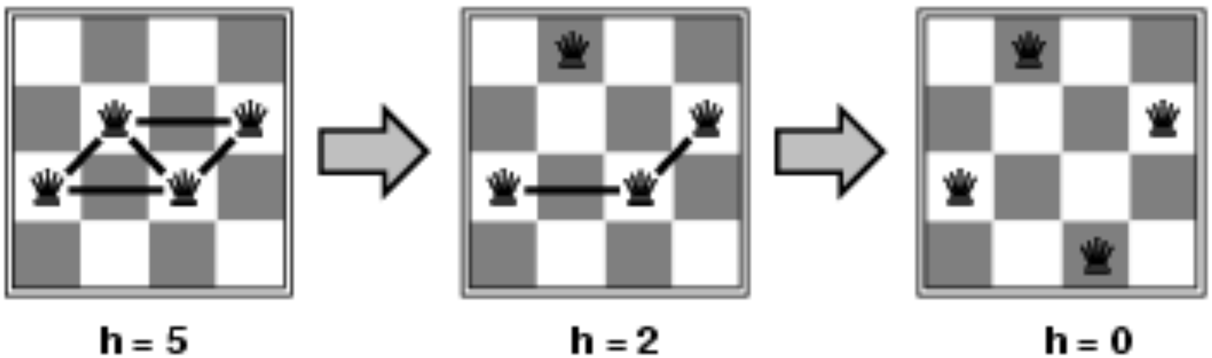
- local search (iterative improvement) is frequently used for constraint satisfaction problems
  - values are assigned to all variables
  - modification operators move the configuration towards a solution
- often called heuristic repair methods
  - repair inconsistencies in the current configuration
- simple strategy: min-conflicts
  - minimizes the number of conflicts with other variables
  - solves many problems very quickly
    - million-queens problem in less than 50 steps
- can be run as online algorithm
  - use the current state as new initial state

Local search for CSPs

- Hill-climbing, simulated annealing typically work with "complete" states, i.e., all variables assigned
- To apply to CSPs:
  - allow states with unsatisfied constraints
  - operators reassign variable values
- Variable selection: randomly select any conflicted variable
- Value selection by min-conflicts heuristic:
  - choose value that violates the fewest constraints
  - i.e., hill-climb with $h(n)$ = total number of violated constraints

Example: 4-Queens

- States: 4 queens in 4 columns ($4^4 = 256$ states)
- Actions: move queen in column
- Goal test: no attacks
- Evaluation: $h(n)$ = number of attacks



- Given random initial state, can solve n-queens in almost constant time for arbitrary n with high probability (e.g., n = 10,000,000)