ModExp 4096
Definitions & Algorithms

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$$c = m^e \pmod{n}$$

ModExp 4096

- These slides describe the architecture, input and output definitions, and algorithms, and analysis of a 4096-bit full ModExp code
- The 4096-bit numbers are organized as sw bits, where s is the number of words and w is the word length, such that $sw = 4096$
- We will take $s = 64$ and $w = 64$ bits in this implementation
- Additionally the exponent length k is available, as $k \leq 4096$

ModExp 4096 Inputs and Output

- The **primary** input parameters are the basis m and the exponent e, each of which are s-word (sw-bits: 4096 bits) integers
- The **secondary** input parameters are n, $n0'$, r and t, such that n is the s-word (sw-bit: 4096-bit) modulus, $n0'$ is a 1-word (w-bit) integer, while r and t are s-word (sw-bit: 4096-bit) integers
- The output c is a s-word (sw-bit: 4096-bit) integer

Primary Input Parameter: m

- The code assumes that m is a 4096-bit **signed** integer:

$$m = (m_{4095} m_{4094} \cdots m_2 m_1 m_0)$$

  such that $m_{4095} = 0$ for $m > 0$ due to 2s-complement representation, and organized as a s-word by w-bit number: $sw = 4096$
- The LSB is on the top right and the MSB (the sign bit) is on the bottom left corner, for example, for $s = 4$ and $w = 6$, we have

LSB: m_0

m_5	m_4	m_3	m_2	m_1	m_0
m_{11}	m_{10}	m_9	m_8	m_7	m_6
m_{17}	m_{16}	m_{15}	m_{14}	m_{13}	m_{12}
m_{23}	m_{22}	m_{21}	m_{20}	m_{19}	m_{18}

MSB: $m_{23} = 0$ for $m > 0$

- If $m > 0$ then $m_{23} = 0$; LSB: m_0 and MSB: m_{23} (sign bit)

Primary Input Parameter: e

- The exponent e is k-bit **unsigned** binary number for $k \leq 4096$

$$e = (e_{k-1}e_{k-2}\cdots e_2 e_1 e_0)$$

- Many practical exponents are fewer than 4096 bits; we assume e is k bits for $k \leq 4096$
- The LSB is on the top right and the MSB (the sign bit) is on the bottom left corner, for example, for $s = 4$ and $w = 6$, we have

LSB: e_0

e_5	e_4	e_3	e_2	e_1	e_0
e_{11}	e_{10}	e_9	e_8	e_7	e_6
e_{17}	e_{16}	e_{15}	e_{14}	e_{13}	e_{12}
e_{23}	e_{22}	e_{21}	e_{20}	e_{19}	e_{18}

$e_{23} = e_{22} = \ldots = e_k = 0$

- The MSBs $e_{23}, e_{22}, \ldots, e_k$ are zero

Secondary Input Parameter: n

- The code assumes that n is a 4096-bit **signed** binary number:

$$n = (n_{4095} n_{4094} \cdots n_2 n_1 n_0)$$

  such that $n_{4095} = 0$ for $n > 0$ due to 2s-complement representation
- We always have $n_{4095} = 0$ ($n > 0$) and $n_0 = 1$ (odd modulus)
- The LSB is on the top right and the MSB (the sign bit) is on the bottom left corner, for example, for $s = 4$ and $w = 6$, we have

LSB: $n_0 = 1$

n_5	n_4	n_3	n_2	n_1	n_0
n_{11}	n_{10}	n_9	n_8	n_7	n_6
n_{17}	n_{16}	n_{15}	n_{14}	n_{13}	n_{12}
n_{23}	n_{22}	n_{21}	n_{20}	n_{19}	n_{18}

MSB: $n_{23} = 0$

- Note that $n_{23} = 0$ and $n_0 = 1$

Secondary Input Parameters: $n0'$, r and t

- The parameters $n0'$, r and t are derived from the modulus n:

$$\begin{aligned} n0' &= -n^{-1} \pmod{2^w} \\ r &= 2^{sw} \pmod{n} \\ t &= 2^{2sw} \pmod{n} \end{aligned}$$

- Since it is often the case that the modulus is fixed, these parameters are computed at once and saved in the memory
- The ModExp code assumes that these parameters are available at the same time with the modulus n
- We will later give code for computing these parameters outside the ModExp code

Secondary Input Parameters: $n0'$, r and t

- $n0'$ is a 1-word (w-bit) unsigned integer: $n0' = (u_{w-1} \dots u_1 u_0)$
- r and t are s-word (sw-bit: 4096-bit) signed integers

r

r_5	r_4	r_3	r_2	r_1	r_0
r_{11}	r_{10}	r_9	r_8	r_7	r_6
r_{17}	r_{16}	r_{15}	r_{14}	r_{13}	r_{12}
r_{23}	r_{22}	r_{21}	r_{20}	r_{19}	r_{18}

t

t_5	t_4	t_3	t_2	t_1	t_0
t_{11}	t_{10}	t_9	t_8	t_7	t_6
t_{17}	t_{16}	t_{15}	t_{14}	t_{13}	t_{12}
t_{23}	t_{22}	t_{21}	t_{20}	t_{19}	t_{18}

n0'

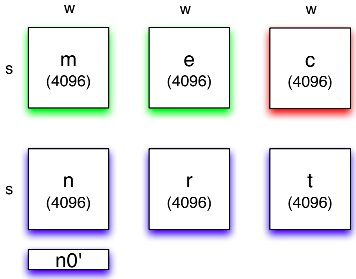
u_5	u_4	u_3	u_2	u_1	u_0

Target Device & Properties

- The target device is Altera Cyclone II EP2C50
- This device has 129 $4k$ memory blocks
- Each $4k$ block can be addressed at different configurations: $4k \times 1$, $2k \times 2$, up to 128×32
- The memory supports byte writes with port data width of $1, 2, 4, 8, 16, 32, 36$ bits
- The device has more than sufficient memory of the input and output data, as well as for temporary values
- This allow us to implement several different optimizations on the exponentiation level: 1-bit, 2-bit and 4-bit window sizes
- Some configuration decisions will be made after the coding of the MonPro block is completed

Input and Output

- The device assumes the primary inputs m and e are written into selected memory blocks (GREEN)
- The device also assumes the secondary input values n, $n0'$, r, and t are already written into selected memory blocks (BLUE)
- It is often the case that the primary values will change after each computation, while the secondary values will remain in place
- The device computes c and writes into a memory block (RED)



The ModExp Algorithm

- The algorithm of choice is the Montgomery-transformed exponentiation using s-word integers with a word size of w bits and the Montgomery constant as $r = 2^{sw}$
- We will describe the 1-bit (binary) and 2-bit (quaternary) exponentiation, however, we are experimenting with the 4-bit (hex) method, and make our final choice at the last phase of the project
- At the multiplication level, we are using the MonPro CIOS algorithm, whose details are given later on
- Primary Inputs: m and e
- Secondary Inputs: n, $n0'$, r, and t
- Output: c
- Functions: MonPro

The 1-bit ModExp Algorithm

Find the index $k < 4095$ of the leftmost 1 in e

$\bar{m} = \text{MonPro}(m, t)$

$\bar{c} = r$

for $i = k - 1$ down to 0

  $\bar{c} = \text{MonPro}(\bar{c}, \bar{c})$

  if $e_i = 1$ then $\bar{c} = \text{MonPro}(\bar{c}, \bar{m})$

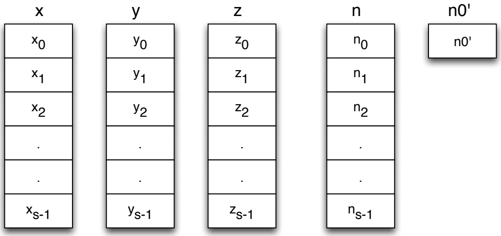
$c = \text{MonPro}(\bar{c}, 1)$

The 2-bit ModExp Algorithm

Express e in 2-bit blocks $e = (E_{2047} \cdots E_1 E_0)$ with $E_j \in \{0, 1, 2, 3\}$
Find the index $k < 2047$ of the leftmost nonzero in e
$\bar{m}1 = \text{MonPro}(m, t)$
$\bar{m}2 = \text{MonPro}(\bar{m}1, \bar{m}1)$
$\bar{m}3 = \text{MonPro}(\bar{m}2, \bar{m}1)$
$\bar{c} = \text{MonPro}(1, t)$
for $i = k$ down to 0
  $\bar{c} = \text{MonPro}(\bar{c}, \bar{c})$
  $\bar{c} = \text{MonPro}(\bar{c}, \bar{c})$
  if $E_i = 1$ then $\bar{c} = \text{MonPro}(\bar{c}, \bar{m}1)$
    else if $E_i = 2$ then $\bar{c} = \text{MonPro}(\bar{c}, \bar{m}2)$
      else if $E_i = 3$ then $\bar{c} = \text{MonPro}(\bar{c}, \bar{m}3)$
$c = \text{MonPro}(\bar{c}, 1)$

The MonPro Algorithm

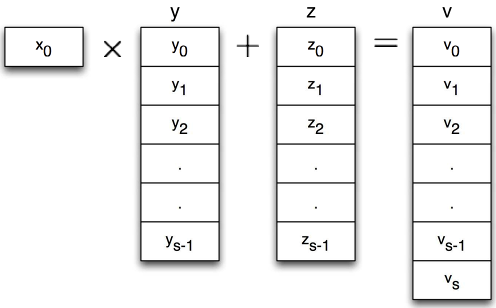
- The MonPro algorithm takes two inputs: x and y, and computes the output z such that each variable holds an s-word (sw-bit: 4096-bit) signed integer: $x = (x_{s-1}x_{s-2}\cdots x_1x_0)$, $y = (y_{s-1}y_{s-2}\cdots y_1y_0)$, and $z = (z_{s-1}z_{s-2}\cdots z_1z_0)$ with x_i, y_i, z_i as w-bit for $s-1 \geq i \geq 0$, and the initial value of z is zero
- The secondary inputs n, $n0'$, r and t are also available; we particularly need $n0'$ and n, which are 1-word and s-word integers



The MonPro Algorithm

1: We take the LSW of x, namely x_0 and multiply by the s-word y, and add it to the s-word partial product z (which is now all zero) to obtain the $(s+1)$-word temporary result v as

$$v = x_0 \cdot \sum_{i=0}^{s-1} y_i 2^{wi} + \sum_{i=0}^{s-1} z_i 2^{wi}$$



1a: The computation in Step 1 is accomplished using a Multiply-Add block that multiplies two 1-word numbers (x_0 and y_0), adds the previous higher word (C_0), and adds another 1-word number (z_0), producing a 2-word number (C_1, S_0); the lower word word (S_0) is assigned to (v_0), while the higher word (C_1) is kept for the next Multiply-Add step, as follows:

$$\begin{aligned}(C_1, S_0) &= x_0 \cdot y_0 + C_0 + z_0 \\ v_0 &= S_0 \\ (C_2, S_1) &= x_0 \cdot y_1 + C_1 + z_1 \\ v_1 &= S_1 \\ (C_3, S_2) &= x_0 \cdot y_2 + C_2 + z_2 \\ v_2 &= S_2 \\ &\cdots\end{aligned}$$

such that the initial value $C_0 = 0$

The MonPro Algorithm

2: Then, we take the LSW of v, namely v_0 an multiply by the 1-word $n0'$ modulo 2^w and obtain the 1-word integer m as

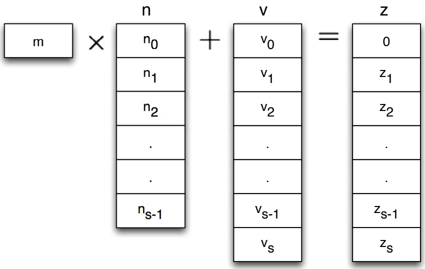
$$m = n0' \cdot v_0 \pmod{2^w}$$

$$\boxed{v_0} \times \boxed{n0'} = \boxed{m} \pmod{2^w}$$

The MonPro Algorithm

3: Then, we take the 1-word m and multiply by the s-word n, and add it to the $(s+1)$-word temporary value v to obtain the new partial product which is $(s+1)$-word

$$z = m \cdot \sum_{i=0}^{s-1} n_i 2^{wi} + \sum_{i=0}^{s} v_i 2^{wi}$$



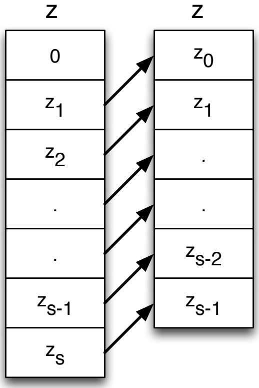
The MonPro Algorithm

3a: The computation in Step 3 is accomplished using a Multiply-Add block that multiplies two 1-word numbers (m and n_0), adds the previous higher word(C_0), adds another 1-word number (v_0), producing a 2-word number (C_1, S_0); assigning S_0 to z_0, while keeping the higher word (C_1) for the next Multiply-Add step, as follows:

$$
\begin{aligned}
(C_1, S_0) &= m \cdot n_0 + C_0 + v_0 \\
z_0 &= S_0 \\
(C_2, S_1) &= m \cdot n_1 + C_1 + v_1 \\
z_1 &= S_1 \\
(C_3, S_2) &= m \cdot n_2 + C_2 + v_2 \\
z_2 &= S_2 \\
&\cdots
\end{aligned}
$$

such that the initial value $C_0 = 0$

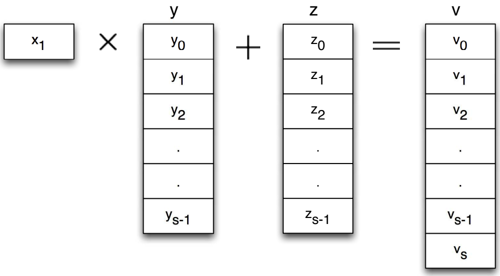
4: The resulting partial product z has its LSW as zero, due to the Montgomery property, and therefore, we shift up z to obtain the new s-word partial product



The MonPro Algorithm

5: In the next step the next word of x, namely x_1, is taken and multiplied with the s-word y, and added to the partial product z to obtain the new temporary result v as

$$v = x_1 \cdot \sum_{i=0}^{s-1} y_i 2^{wi} + \sum_{i=0}^{s-1} z_i 2^{wi}$$



The MonPro Algorithm

6: This is followed up by computing the new m value

$$m = n0' \cdot v_0 \pmod{2^w}$$

and then the multiplication of m by n, and then the addition of the result to v to obtain the new $(s+1)$-word partial product z

$$z = m \cdot \sum_{i=0}^{s-1} n_i 2^{wi} + \sum_{i=0}^{s} v_i 2^{wi}$$

and finally shifting up z by one word (since $z_0 = 0$) to obtain the new s-word z value

7: Proceeding in this way, we multiply all x_i values by the multiplicand y and reduce it modulo n, for $i = 0, 1, 2, \ldots, s-1$

Resources and Components

- The ModExp requires implementation of several blocks
- A finite state machine to scan the bits of the exponent, 1-bit, 2-bit and 4-bit at a time
- A w-by-w bit integer multiplier producing $2w$-bit products
- A w-by-w bit integer adder producing $(w+1)$-bit sums
- A finite state machine to control the flow of the MonPro algorithm from $i = 0$ to $i = s - 1$
- Several registers for the primary and secondary inputs, for the output, and for the temporary values

Interface

- Our analysis shows that $w = 32$ and $s = 128$ values will produce an implementation that will satisfy the specified requirements
- The target device Altera Cyclone II EP2C50 has 129 $4k$ memory blocks which can be configured as 128×32
- At this configuration these memory blocks are not accessible in the true dual-port mode, however, we will not need that
- It is therefore assumed that the initiating processor uploads the 128-word primary inputs m and e, and 128-word secondary inputs n, r, and t, and 1-word secondary input $n0'$ to selected memory blocks
- The ModExp code will run and produce the 128-word output c in a selected memory block

Interface

- The ModExp code will not destroy the input values and all of them can be reused for subsequent executions
- It is generally the case the primary inputs m and e will be refreshed by the initiating processor, leaving the secondary input values n, r, and t, and $n0'$ intact for subsequent execution(s)

Computation of Secondary Input Parameters: $n0'$, r and t

- As we have said, the parameters $n0'$, r and t are derived from the modulus n:

$$\begin{aligned} n0' &= -n^{-1} \pmod{2^w} \\ r &= 2^{sw} \pmod{n} \\ t &= 2^{2sw} \pmod{n} \end{aligned}$$

- Since it is often the case that the modulus is fixed, these parameters are computed at once and saved in the memory
- The ModExp code assumes that these parameters are available at the same time with the modulus n
- We now give code for computing these parameters

Computation of $n0'$

- $n0'$ is a one-word integer: $1 \leq n0' \leq 2^w - 1$
- It is equal to $-(n_0)^{-1} \pmod{2^w}$, i.e., the negative of the multiplicative inverse of the least significant word of n modulo 2^w
- There is a simple code for computing $n0'$

```
Input n_0, of the least significant word of n
Output n0'
y_1 = 1
for i = 2 to w
    if 2^(i-1) < n_0 · y_(i-1) (mod 2^i)
        then y_i = y_(i-1) + 2^(i-1)
        else y_i = y_(i-1)
return -y_w
```

Computation of r

- Since n is assumed up to 4096-bit (s-word) integer, the definition of r is as follows:

$$r = 2^{sw} \pmod{n}$$

- 2^{sw} is an integer with a single 1 in the most significant bit position, and then sw zeros, for example, for $s = 5$ and $w = 4$, we have $2^{sw} = 2^{20}$ as follows:

1 0000 0000 0000 0000 0000

Computation of r

- $r = 2^{sw} \pmod{n}$ can be computed by aligning the most significant (nonzero) bit of n to the $(se-1)$th position, and then doing a single multi-precision (s-word) subtraction
- Following on the previous example, Let's take the modulus as:

$$0100\ 1111\ 1100\ 0011\ 1011$$

- We align n with the $(se-1)$th position, and make a single multi-precision (s-word) subtraction:

$$\begin{array}{l} 1\ 0000\ 0000\ 0000\ 0000\ 0000 \\ 0\ 1001\ 1111\ 1000\ 0111\ 011 \end{array}$$

Computation of t

- Since n is assumed up to 4096-bit (s-word) integer, the definition of t is as follows:

$$t \quad = \quad 2^{2sw} \pmod{n}$$

- 2^{2sw} is an integer with a single 1 in the most significant bit position, and then $2sw$ zeros, for example, for $s = 4$ and $w = 3$, we have $2^{2sw} = 2^{24}$ as follows:

1 000 000 000 000 000 000 000 000

Computation of t

- $t = 2^{sw} \pmod{n}$ can be computed by aligning the most significant (nonzero) bit of n to the $(2se - 1)$th position, and then performing successive multi-precision (s-word) subtractions until all most significant positions in the resulting number until to the position $(2sw - 1)$ are zero
- Following on the previous example, Let's take the modulus as:

$$001\ 111\ 000\ 111$$

- By aligning with 2^{sw}, we make successive multi-precision (s-word) subtractions:

$$1\ 000\ 000\ 000\ 000\ 000\ 000\ 000\ 000$$
$$0\ 111\ 100\ 011\ 1$$