Assignment 2

Given Information
Consider the following boundary-value problem:

$$-\frac{d^2u_i}{dx^2} = f_i,\ x \in (-1,1); u_i(-1) = -6,\ u_i(1) = -2,\ i = 1,2; \tag{1}$$

with the source functions

$$f_1(x) = -6x + 2,\ f_2(x) = -12x^2 - 6x + 2,\ x \in [-1,1]. \tag{2}$$

Problem 1
Find the exact solutions ${u_1}^{ex}(x)$ and ${u_2}^{ex}(x)$ of the problem (1) corresponding to $f_1(x)$ and $f_2(x)$ given in (2).

Solution
To solve the second-order equation analytically, we need to integrate $f_i(x)$ twice to obtain the ${u_i}^{ex}$

$$u_i = \int \left[\int -f_i(x)dx\right] dx$$

$$u_1 = \int \left[\int -(-6x+2)dx\right] dx$$

Upon solving and applying boundary conditions, ${u_1}^{ex} = x^3 - x^2 + x - 3$

$$u_2 = \int \left[\int -(-12x^2 - 6x + 2)dx\right] dx$$

Upon solving and applying boundary conditions, ${u_2}^{ex} = x^4 + x^3 - x^2 + x - 4$

Problem 2
Discretize the problem (1) using the Finite-Difference Method (FDM) on a uniform grid obtained by dividing the [-1,1] interval into n=5 sub-intervals of equal length.

Solution
(a) The step size $h = \frac{x_N - x_0}{n} = \frac{1-(-1)}{5} = 0.4$

(b) Dividing into 5 intervals, we get in total 6 points. x_0 and x_5 are the 2 boundary points, x_1, x_2, x_3, x_4 are the 4 internal points.

(c) The value of the function u_i at the 4 internal points are the unknown, as we have the value of the function at points x_0 and x_5.

(d) The FD Approximation of the second derivative of $-u_{i,j}$ is:

$$-u''_{i,j} = \frac{-u_{i,j+1} + 2u_{i,j} - u_{i,j-1}}{h^2} + \mathcal{O}\left(h^2\right)$$

Where $i = 1,2$ and $j = 1,2,3,4$

(e) The FD Approximation Equations for the points $j = 1,2,3,4$:

$$-u''_{i,1} = \frac{-u_{i,2} + 2u_{i,1} - u_{i,0}}{h^2} = f_i(x_1 = -0.6)$$

$$-u''_{i,2} = \frac{-u_{i,3} + 2u_{i,2} - u_{i,1}}{h^2} = f_i(x_2 = -0.2)$$

$$-u''_{i,3} = \frac{-u_{i,4} + 2u_{i,3} - u_{i,2}}{h^2} = f_i(x_3 = 0.2)$$

$$-u''_{i,4} = \frac{-u_{i,5} + 2u_{i,4} - u_{i,3}}{h^2} = f_i(x_4 = 0.6)$$

Where $i = 1, 2$ for the two source functions f_1 and f_2. For the function f_1, the set of equations are:

$$\begin{aligned}
\frac{-u_{i,2} + 2u_{i,1}}{h^2} &= f_i(x_1 = -0.6) + \frac{u_{i,0}}{h^2} \\
\frac{-u_{i,3} + 2u_{i,2} - u_{i,1}}{h^2} &= f_i(x_2 = -0.2) \\
\frac{-u_{i,4} + 2u_{i,3} - u_{i,2}}{h^2} &= f_i(x_3 = 0.2) \\
\frac{2u_{i,4} - u_{i,3}}{h^2} &= f_i(x_4 = 0.6) + \frac{u_{i,5}}{h^2}
\end{aligned}$$

(f) In matrix form A$\mathbf{u} = \mathbf{f}$:

$$\frac{1}{(0.4)^2} \cdot \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} u_{1,1} \\ u_{1,2} \\ u_{1,3} \\ u_{1,4} \end{bmatrix} = \begin{bmatrix} f_1(-0.6) + \frac{u_{1,0}}{h^2} \\ f_1(-0.2) \\ f_1(+0.2) \\ f_1(+0.6) + \frac{u_{1,5}}{h^2} \end{bmatrix}$$

Similarly, for the function f_2:

$$\frac{1}{(0.4)^2} \cdot \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} u_{2,1} \\ u_{2,2} \\ u_{2,3} \\ u_{2,4} \end{bmatrix} = \begin{bmatrix} f_2(-0.6) + \frac{u_{2,0}}{h^2} \\ f_2(-0.2) \\ f_2(+0.2) \\ f_2(+0.6) + \frac{u_{2,5}}{h^2} \end{bmatrix}$$

Matrix $A = \frac{1}{(0.4)^2} \cdot \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 12.5 & -6.25 & 0 & 0 \\ -6.25 & 12.5 & -6.25 & 0 \\ 0 & -6.25 & 12.5 & -6.25 \\ 0 & 0 & -6.25 & 12.5 \end{bmatrix}$.

(g) Calculating the eigenvalues and eigenvectors: $det(A - \lambda I) = 0$. Upon solving:

$$\lambda^4 - 50\lambda^3 + 820.313\lambda^2 - 4882.81\lambda + 7629.39 = 0$$

Table 1: Eigenvalues of Matrix A

	Eigenvalue
λ_1	$3.125(3 - \sqrt{5}) = 2.3872875703$
λ_2	$3.125(5 - \sqrt{5}) = 8.6372875703$
λ_3	$3.125(3 + \sqrt{5}) = 16.36271243$
λ_4	$3.125(5 + \sqrt{5}) = 22.61271243$

(h) The eigenvalue of the negative second derivative can be found from a similar treatment we performed in the previous matrix, which gives us the equation:

$$-\frac{d^2u_i}{dx^2} = \lambda u_i$$

Where $x \in [0, L]$ with $u_i(0) = u_i(L) = 0$, which is the boundary condition for u_i.

The eigenvalues of the Discrete Finite Difference Laplacian matrix A of the order $(n-1)x(n-1)$ can be given by:

$$\lambda_i = \frac{4}{h^2} sin^2 \left(\frac{\pi i}{2n} \right)$$

Where $i = 1, 2, 3, ..., n-1$. When $h \to 0$

$$\lambda_i = \left(\frac{\pi i}{L} \right)^2$$

The 4 smallest eigenvalues can be found by substituting i = 1, 2, 3, 4. The domain varies from -1 to 1, so $L = 1 - (-1) = 1$. Populating the table after calculating the eigenvalues:

Table 2: Eigenvalues of Matrix A and Negative Second Derivative Operator

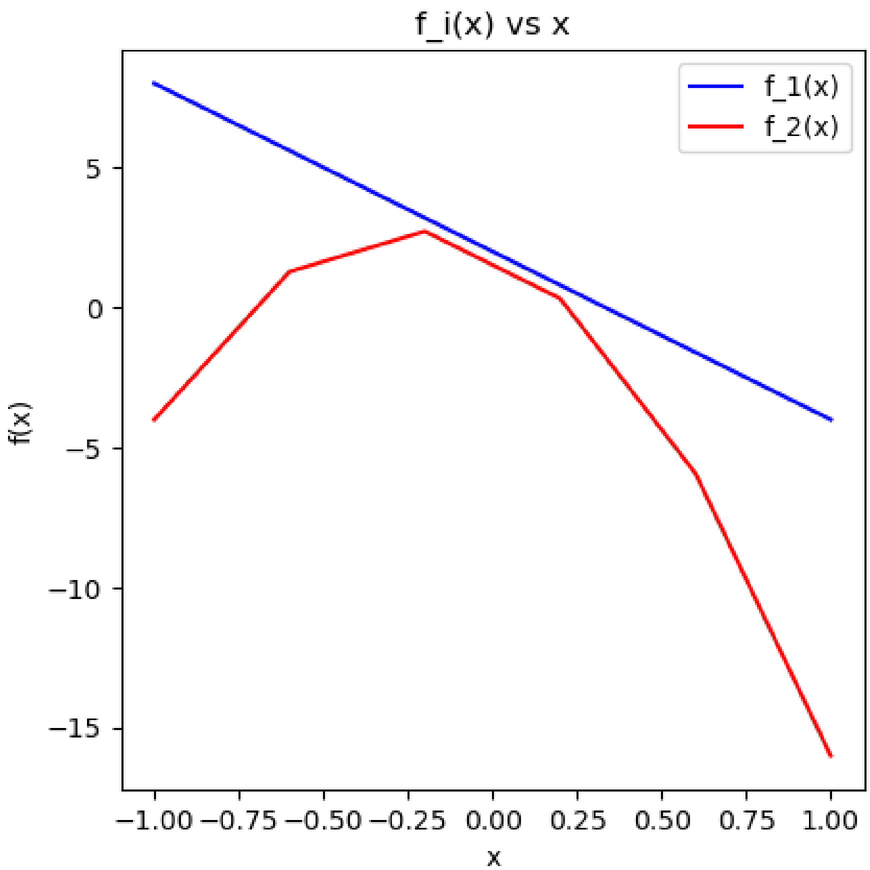
	Eigenvalue of A	Eigenvalue of Operator
λ_1	2.3872875703	2.4674011002
λ_2	8.6372875703	9.8696044010
λ_3	16.36271243	22.2066099024
λ_4	22.61271243	39.4784176043

Problem 3
Check your Python environment. Enter the following lines in an ASCII text called `assignment-2.py`

Solution
No errors

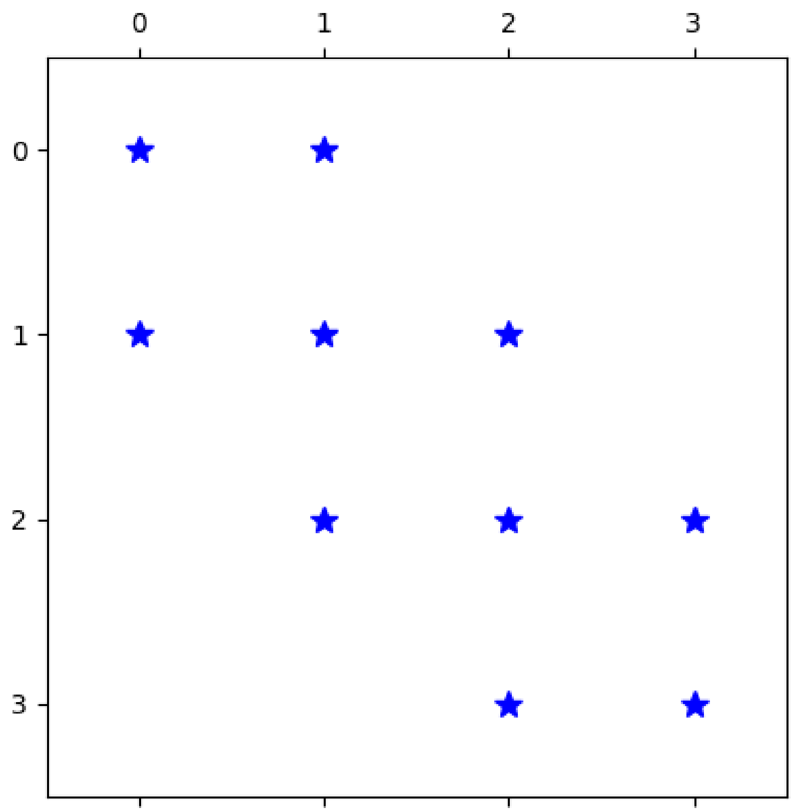
Problem 4
Construct a uniform grid and display the source functions and the exact solutions.

Solution
Implementing the steps mentioned in (a), (b), (c), (d), (e), (f):



Problem 5
Assemble the Finite-Difference (negative) Laplacian matrix.

Solution
(a) For $n = 5$, we got a 4x4 A matrix. For any n, we will have $n + 1$ points, out of which $n - 1$ points are internal points. Therefore, dimensions of Matrix A is $(n - 1)$x$(n - 1)$
Implementing the steps mentioned in (b), (c):



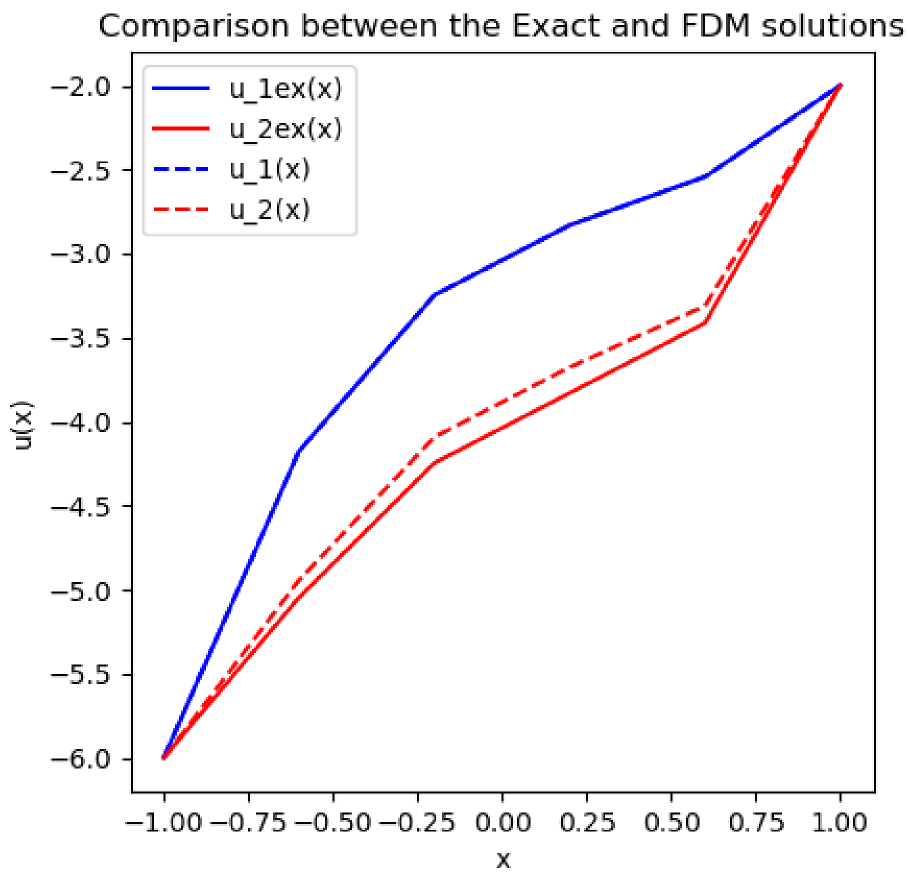
(d) After computing eigenvalues using *np.linalg.eig()*:

Table 3: Eigenvalues of Matrix A with the newly computed values and the operator values

	Eigenvalues (Analytical)	Eigenvalue of Operator	Eigenvalues (Python)
λ_1	2.3872875703	2.4674011002	2.38728757
λ_2	8.6372875703	9.8696044010	8.63728757
λ_3	16.36271243	22.2066099024	16.36271243
λ_4	22.61271243	39.4784176043	22.61271243

Problem 6
Solve the linear algebraic problem.

Solution
The graph below shows the plots of the exact solution and FDM solution of both u_1 and u_2. Since the two solutions for u_1 almost coincide, the dotted blue line is not visible.



Problem 7
Analyze your results.

Solution
(a) Global error in the case of FDM is calculated using Root Mean Squared Error (RMSE)

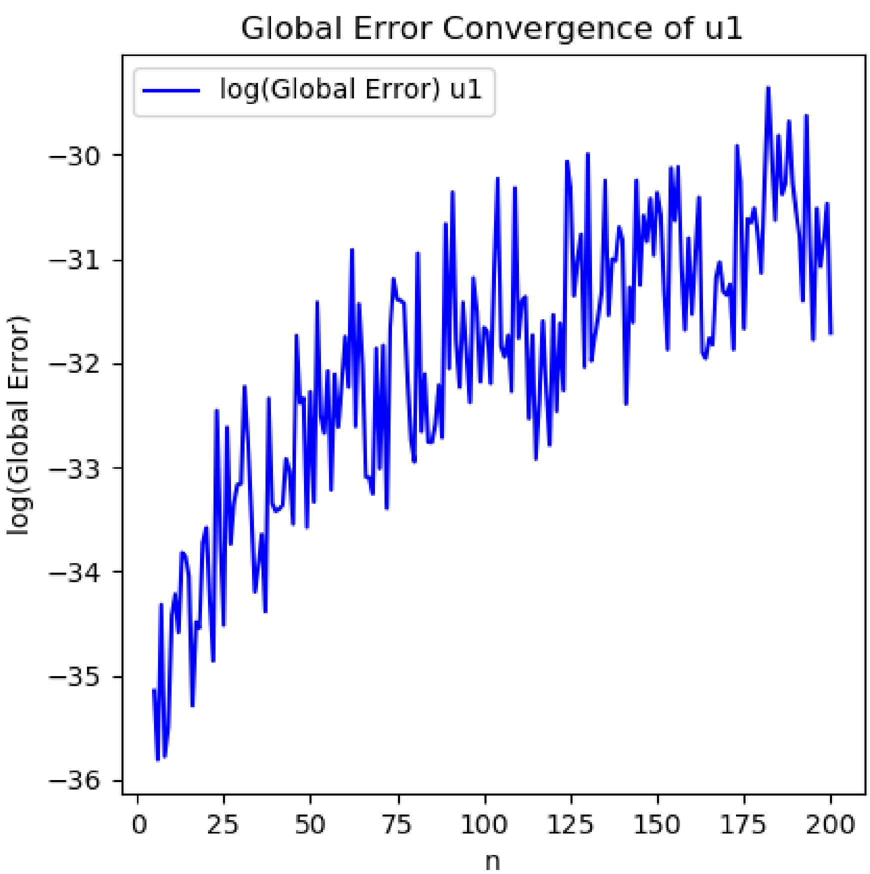
$$\epsilon = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N-1} |u_i - {u_i}^{ex}|^2}$$

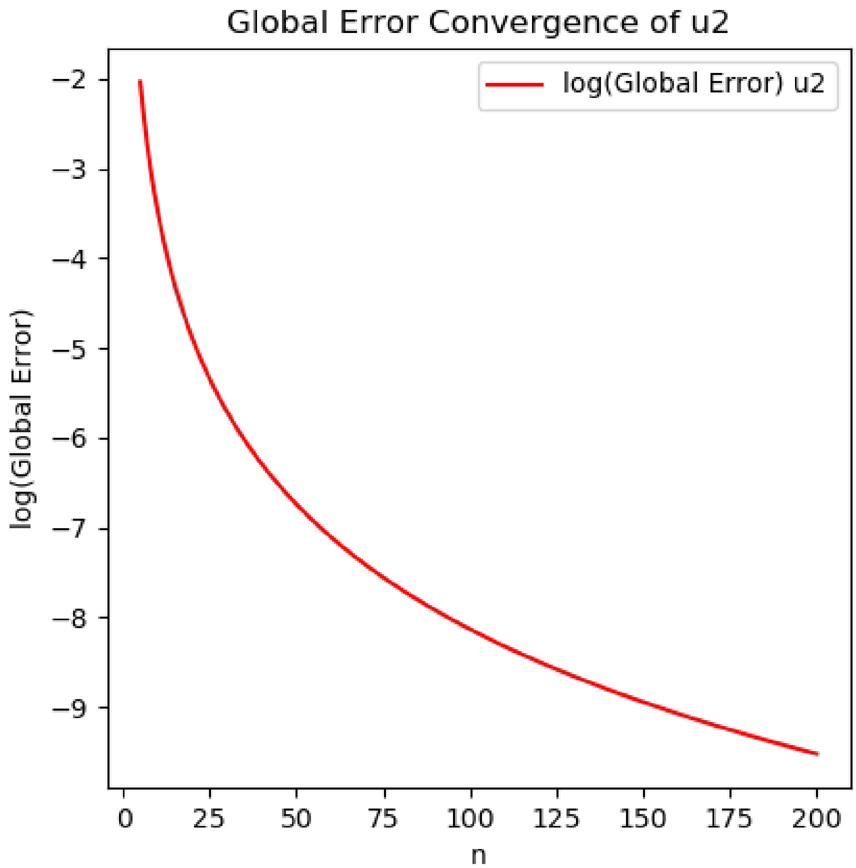
Upon computing for our case where N=5, Global Error for u_1 and u_2 is:

$$\epsilon_1 = 5.438959822042073 * 10^{-16}$$
$$\epsilon_2 = 0.13053489954797415$$

The error in the case of u_1 is very small because the source function is linear and the approximation performed in FDM is also linear. Whereas, in the case of u_2, the source function is a second-degree polynomial and hence we observe a higher error. As the order of the actual function of u_2 is higher than that of u_1, the error due to linearization rises.

(b) After iterating till n = 200, the following are the plots for the Global Error of u1 and u2 in logarithmic scale:





For u_1, we observe that the global error oscillates although it's a small value. We can also see that there is no convergence. This is because of the way python works. The python compiler stores small errors in the order of -16. This causes a floating point error which is responsible for the oscillations. As this is accumulated for increasing n, the error increases.

For u_2, we see a decay in the global error as n increases. This is an expected behavior as we are decreasing the step size, the linear approximation is made smoother and starts getting closer to the exact values of the function.