Assignment 4

Given Information
Consider the following boundary value problem:

$$-\nabla.(k\Delta u) = f,\ (x, y) \in \Omega = (0, 10) \times (0, 5),$$

$$u(x, y) = 0,\ (x, y) \in \partial\Omega$$

$$k(x, y) = 1 + 0.1(x + y + xy),\ (x, y) \in \vec{\Omega} \tag{1}$$

$$f(x, y) = \sum_{i=1}^{9}\sum_{i=1}^{4} e^{-\alpha(x-i)^2-\alpha(y-j)^2}, \alpha = 40,\ (x, y) \in \vec{\Omega} \tag{2}$$

where $\vec{\Omega} = [0, 10] \times [0, 5]$ is the rectangle with the corner (0,0), (10,0), (10,5), and (0,5).

Problem 1
Finite-Volume discretization.

Solution
(a) Considering the grid with the numbers of sub-intervals in the x and y-directions as $N_x = N_y = 4$, this is a uniform grid, as the number of sub-intervals is same in both x and y axes, but the step size is different when considering the rectangular grid in our problem ($\vec{\Omega} = [0, 10] \times [0, 5]$). We are given the value for the boundary points, so the **9 inner points are the unknowns**.

(b) Vertex-centered FVM equations for the 9 inner points is given by:

$$\left(\frac{k_{0.5,1}}{h_x^2} + \frac{k_{1,0.5}}{h_y^2} + \frac{k_{1.5,1}}{h_x^2} + \frac{k_{1,1.5}}{h_y^2}\right) u_{1,1} - \frac{k_{1.5,1}}{h_x^2} u_{2,1} - \frac{k_{1,1.5}}{h_y^2} u_{1,2} = f_{1,1}$$

$$-\frac{k_{1,1.5}}{h_y^2} u_{1,1} + \left(\frac{k_{0.5,2}}{h_x^2} + \frac{k_{1,1.5}}{h_y^2} + \frac{k_{1.5,2}}{h_x^2} + \frac{k_{1,2.5}}{h_y^2}\right) u_{1,2} - \frac{k_{1.5,2}}{h_x^2} u_{2,2} - \frac{k_{1,2.5}}{h_y^2} u_{1,3} = f_{1,2}$$

$$-\frac{k_{1,2.5}}{h_y^2} u_{1,2} + \left(\frac{k_{0.5,3}}{h_x^2} + \frac{k_{1,2.5}}{h_y^2} + \frac{k_{1.5,3}}{h_x^2} + \frac{k_{1,3.5}}{h_y^2}\right) u_{1,3} - \frac{k_{1.5,3}}{h_x^2} u_{2,3} = f_{1,3}$$

$$-\frac{k_{1.5,1}}{h_x^2} u_{1,1} + \left(\frac{k_{1.5,1}}{h_x^2} + \frac{k_{2,0.5}}{h_y^2} + \frac{k_{2.5,1}}{h_x^2} + \frac{k_{2,1.5}}{h_y^2}\right) u_{2,1} - \frac{k_{2.5,1}}{h_x^2} u_{3,1} - \frac{k_{2,1.5}}{h_y^2} u_{2,2} = f_{2,1}$$

$$-\frac{k_{1.5,2}}{h_x^2} u_{1,2} - \frac{k_{2,1.5}}{h_y^2} u_{2,1} + \left(\frac{k_{1.5,2}}{h_x^2} + \frac{k_{2,1.5}}{h_y^2} + \frac{k_{2.5,2}}{h_x^2} + \frac{k_{2,2.5}}{h_y^2}\right) u_{2,2} - \frac{k_{2.5,2}}{h_x^2} u_{3,2} - \frac{k_{2,2.5}}{h_y^2} u_{2,3} = f_{2,2}$$

$$-\frac{k_{1.5,3}}{h_x^2} u_{1,3} - \frac{k_{2,2.5}}{h_y^2} u_{2,2} + \left(\frac{k_{1.5,3}}{h_x^2} + \frac{k_{2,2.5}}{h_y^2} + \frac{k_{2.5,3}}{h_x^2} + \frac{k_{2,3.5}}{h_y^2}\right) u_{2,3} - \frac{k_{2.5,3}}{h_x^2} u_{3,3} = f_{2,3}$$

$$-\frac{k_{2.5,1}}{h_x^2} u_{2,1} + \left(\frac{k_{2.5,1}}{h_x^2} + \frac{k_{3,0.5}}{h_y^2} + \frac{k_{3.5,1}}{h_x^2} + \frac{k_{3,1.5}}{h_y^2}\right) u_{3,1} - \frac{k_{3,1.5}}{h_y^2} u_{3,2} = f_{3,1}$$

$$-\frac{k_{2.5,2}}{h_x^2} u_{2,2} - \frac{k_{3,1.5}}{h_y^2} u_{3,1} + \left(\frac{k_{2.5,2}}{h_x^2} + \frac{k_{3,1.5}}{h_y^2} + \frac{k_{3.5,2}}{h_x^2} + \frac{k_{3,2.5}}{h_y^2}\right) u_{3,2} - \frac{k_{3,2.5}}{h_y^2} u_{3,3} = f_{3,2}$$

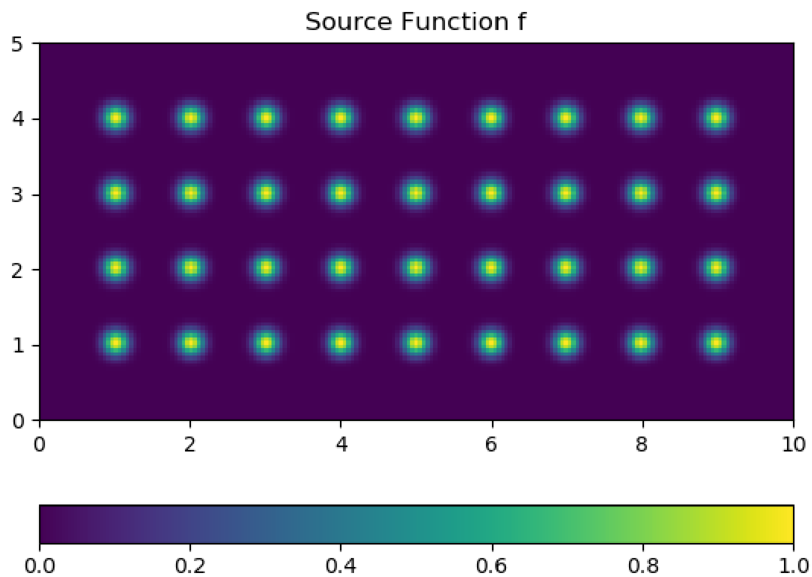
$$-\frac{k_{2.5,3}}{h_x^2} u_{2,3} - \frac{k_{3,2.5}}{h_y^2} u_{3,2} + \left(\frac{k_{2.5,3}}{h_x^2} + \frac{k_{3,2.5}}{h_y^2} + \frac{k_{3.5,3}}{h_x^2} + \frac{k_{3,3.5}}{h_y^2}\right) u_{3,3} = f_{3,3}$$

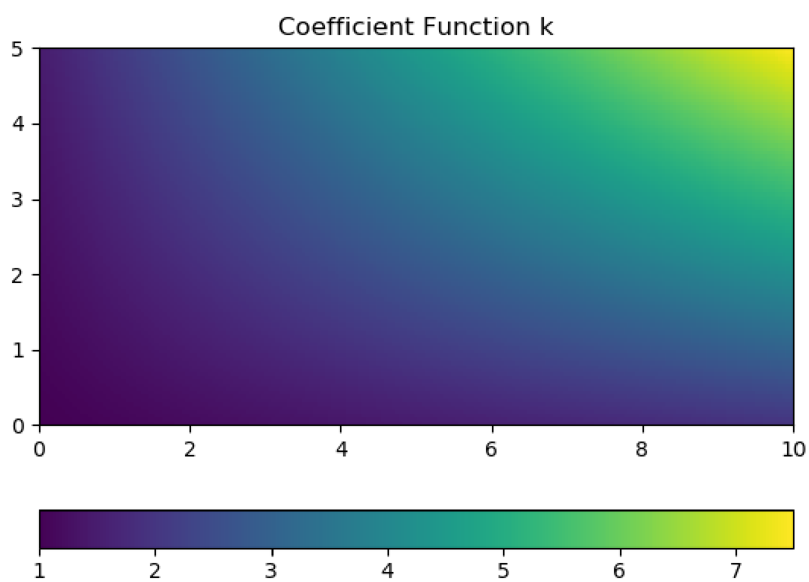
(c) Setting $k(x,y) = 1$, we obtain the following system matrix for the solution vector u and source function vector in a lexicographic ordering:

$$\begin{bmatrix} 1.6 & -0.64 & 0 & -0.16 & 0 & 0 & 0 & 0 & 0 \\ -0.64 & 1.6 & -0.64 & 0 & -0.16 & 0 & 0 & 0 & 0 \\ 0 & -0.64 & 1.6 & 0 & 0 & -0.16 & 0 & 0 & 0 \\ -0.16 & 0 & 0 & 1.6 & -0.64 & 0 & -0.16 & 0 & 0 \\ 0 & -0.16 & 0 & -0.64 & 1.6 & -0.64 & 0 & -0.16 & 0 \\ 0 & 0 & -0.16 & 0 & -0.64 & 1.6 & 0 & 0 & -0.16 \\ 0 & 0 & 0 & -0.16 & 0 & 0 & 1.6 & -0.64 & 0 \\ 0 & 0 & 0 & 0 & -0.16 & 0 & -0.64 & 1.6 & -0.64 \\ 0 & 0 & 0 & 0 & 0 & -0.16 & 0 & -0.64 & 1.6 \end{bmatrix}$$

Problem 2
System matrix and right-hand side assembly. In this Assignment we study a different assembly method, common when programming in general (compiled) languages, such as C and FORTRAN. The method involves nested loops and careful approach to array indexing.

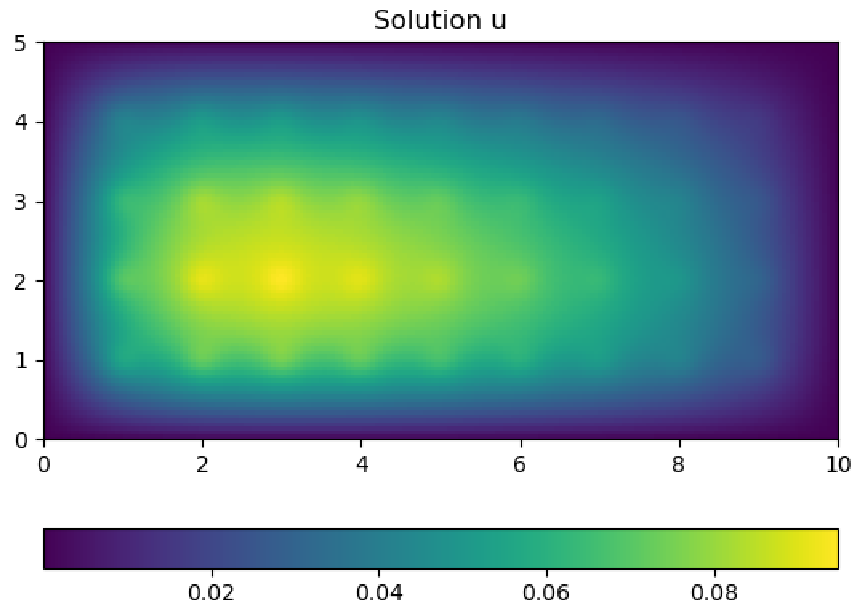
Solution
Implementing the steps mentioned in (a), (b), (c), we receive the plots for the functions f and k:





Problem 3
Applying a linear solver and analyzing the numerical solution.

Solution
(a) Following the guidelines from Assignment 3, we solve the linear algebra problem and obtain the solution array $u(x, y)$. Reshaping the array and plotting, we obtain:



(b) We can consider a heat transfer system to understand the physical meaning of our solution. Consider a rectangular block of material, with several cylindrical heating elements placed within it. These element provide a flux source, which is given by the source function $f(x, y)$. Materials fabricated in real life are not strictly isotropic to heat transfer via conduction, and hence we deduce a function for the spatially varying heat transfer coefficient given by the function $k(x, y)$.

As heat transfer coefficient increases, the temperature gradient decreases (which is given by Fourier's Law of Conduction). This can be seen in our solution $u(x, y)$ which is the temperature distribution in the 2D plane (assuming there is no significant heat transfer in the z-axis, as the length scale is smaller in z-axis). Lower k(x,y) sections in our domain have the highest temperature gradient and vice versa.

Mathematically, we can perform a scaling analysis on the equation to prove that our solution is indeed correct, as it is inversely varying with the plot of coefficient function $k(x, y)$.