Rateless Erasure Codes

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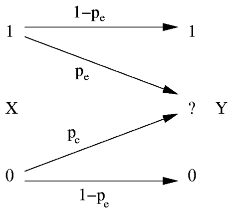
Outline

The BEC

Consider the binary erasure channel:



BEC [1]

[1]Image source: Wikipedia

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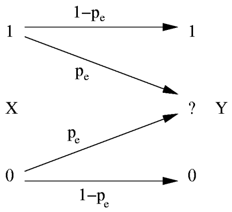

BEC [1]

For example, the Internet.

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Reliable transmission over BEC

TCP/IP

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TCP/IP

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This calls for some coding theory rescue.

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- Fast algorithms

Reed Solomon Codes

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Advantages

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Enter rateless erasure codes.

A digital fountain

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The Encoder [2]

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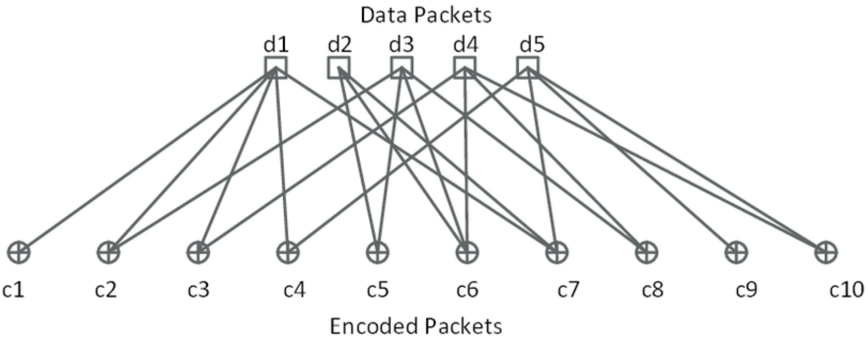
- Collect $\approx K$ drops

Fountain code encoder

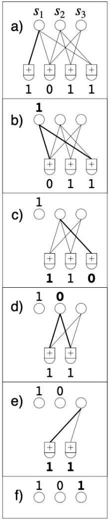


The encoding process [2]

[2] Image source: https://www.researchgate.net/figure/A-typical-encoding-process-of-LT-codes_fig2_307915267

Fountain code decoder

Complexity of decoding algorithm determined by #edges in this graph (Image source: [1])



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3. $y_i = \sum_j v_j x_j$

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Design objective: $\mathcal{D}$ should enable simple linear time decoding of $\{x_1, \ldots, x_K\}$ as soon as any $K(1+\epsilon)$ of y-s are received.

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Let the LT-Code has the degree distribution $\rho(d)$ and let $\mathcal{G}$ denotes decoding graph(a bipartite graph with K input nodes and N output nodes).

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By Taylor expansion of $ln(1 - x)$ gives,

$$ln(1 - a/K) \geq (a/K)/(1 - a/K) \implies (1 - a/K)^N \geq e^{-\alpha/(1-\alpha/n)}$$

where $\alpha = aN/K$

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$$aN \geq cKlog(K)$$

aN is the average number of edges in the graph $\mathcal{G}$. $\square$

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Ideal Soliton distribution

$\rho(d)$ for $d = \{1, 2, \cdots, K\}$

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- Is good only in an expected sense

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Robust Soliton distribution

$\mu(\cdot)$, Let $R = cln(K/\delta)\sqrt{K}$, Define $\tau(\cdot)$ as follows,

$$\tau(i) = \begin{cases} R/ik & \text{for } i = 1, \cdots, K/R - 1 \\ Rln(R/\delta)/K & \text{for} i = K/R \\ 0 & \text{for} i = K/R + 1, \cdots, K \end{cases}$$

Add the Ideal Soliton distribution $\rho(\cdot)$ to $\tau(\cdot)$ and normalize to obtain $\mu(\cdot)$:

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- δ is the allowable failure probability
- Average degree $\approx ln(K/\delta)$

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  But can recover a large fraction! How much?

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Hence can recover a large fraction of the data!

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The outer code just needs to be able to correct erasures with an erasure rate of f

Raptor codes: An example

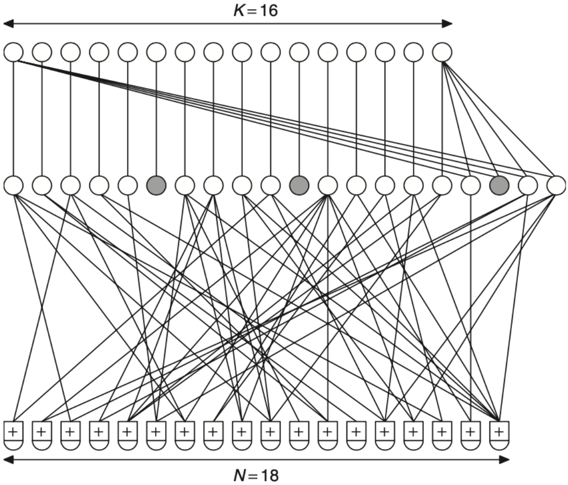
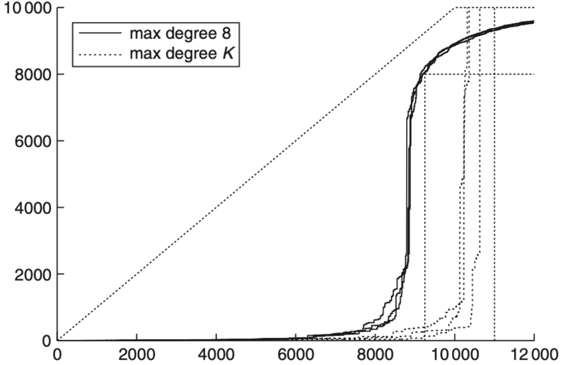


Image source: [1]

Performance comparison



#packets decoded vs #received packets [4]

[4] [1]

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- Useful in storage and broadcast applications

References

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Thank you