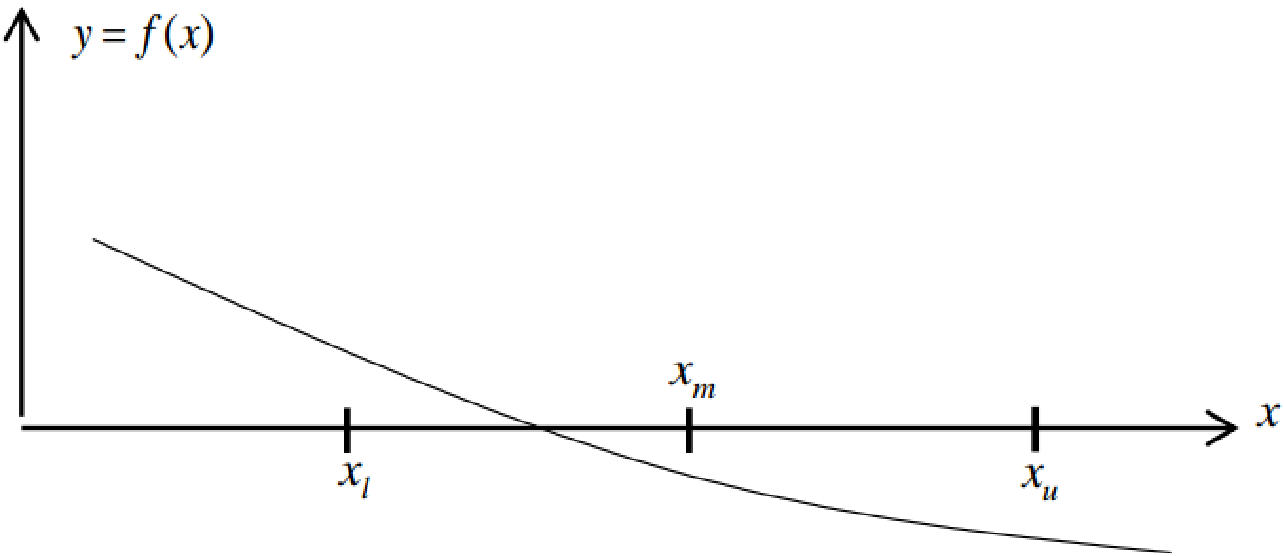
A function $f(x)$ is defined as $f(x)=x^2-4x-5$. For $y=f(x)=0$ estimate a root of this function using Bisection method. Use $[0,48]$ as initial estimation range points and use absolute relative approximate error notation for error of estimated root at each iteration.

Recall



$$x_m=\frac{x_l+x_u}{2}$$

$$|E|=\left|\frac{x_m^{new}-x_m^{old}}{x_m^{new}}\right|*100$$

$f(x_l)*f(x_m)<0$ ***the root lies between*** x_l ***and*** x_m

$f(x_l)*f(x_m)>0$ ***the root lies between*** x_m ***and*** x_u

$f(x_l)*f(x_m)=0$ x_m ***is root***

$x_l = 0 \qquad f(x_l) = f(0) = 0 - 0 - 5 = -5$

$x_u = 48 \qquad f(x_u) = f(48) = 2304 - 192 - 5 = 2107 \qquad\qquad x_m = (0+48)/2 = 24$

$x_m = 24 \qquad f(x_m) = f(24) = 576 - 96 - 5 = 475$

$f(x_l) * f(x_m) < 0$ *the root lies between* x_l *and* x_m *so new estimation range* $[0, 24]$

$x_l = 0 \qquad f(x_l) = f(0) = 0 - 0 - 5 = -5$

$x_u = 24 \qquad f(x_u) = f(24) = 576 - 96 - 5 = 475 \qquad\qquad x_m = (0+24)/2 = 12$

$x_m = 12 \qquad f(x_m) = f(12) = 144 - 48 - 5 = 91$

$f(x_l) * f(x_m) < 0$ *the root lies between* x_l *and* x_m *so new estimation range* $[0, 12]$

$x_l = 0 \qquad f(x_l) = f(0) = 0 - 0 - 5 = -5$

$x_u = 12 \qquad f(x_u) = f(12) = 144 - 48 - 5 = 91 \qquad\qquad x_m = (0+12)/2 = 6$

$x_m = 6 \qquad f(x_m) = f(6) = 36 - 24 - 5 = 7$

$f(x_l) * f(x_m) < 0$ *the root lies between* x_l *and* x_m *so new estimation range* $[0, 6]$

$x_l = 0 \qquad f(x_l) = f(0) = 0 - 0 - 5 = -5$

$x_u = 6 \qquad f(x_u) = f(6) = 36 - 24 - 5 = 7 \qquad\qquad x_m = (0+6)/2 = 3$

$x_m = 3 \qquad f(x_m) = f(3) = 9 - 12 - 5 = -8$

$f(x_l) * f(x_m) > 0$ *the root lies between* x_m *and* x_u *so new estimation range* $[3,6]$

$x_l = 3 \qquad f(x_l) = f(3) = 9 - 12 - 5 = -8$

$x_u = 6 \qquad f(x_u) = f(6) = 36 - 24 - 5 = 7 \qquad\qquad x_m = (3+6)/2 = 4,5$

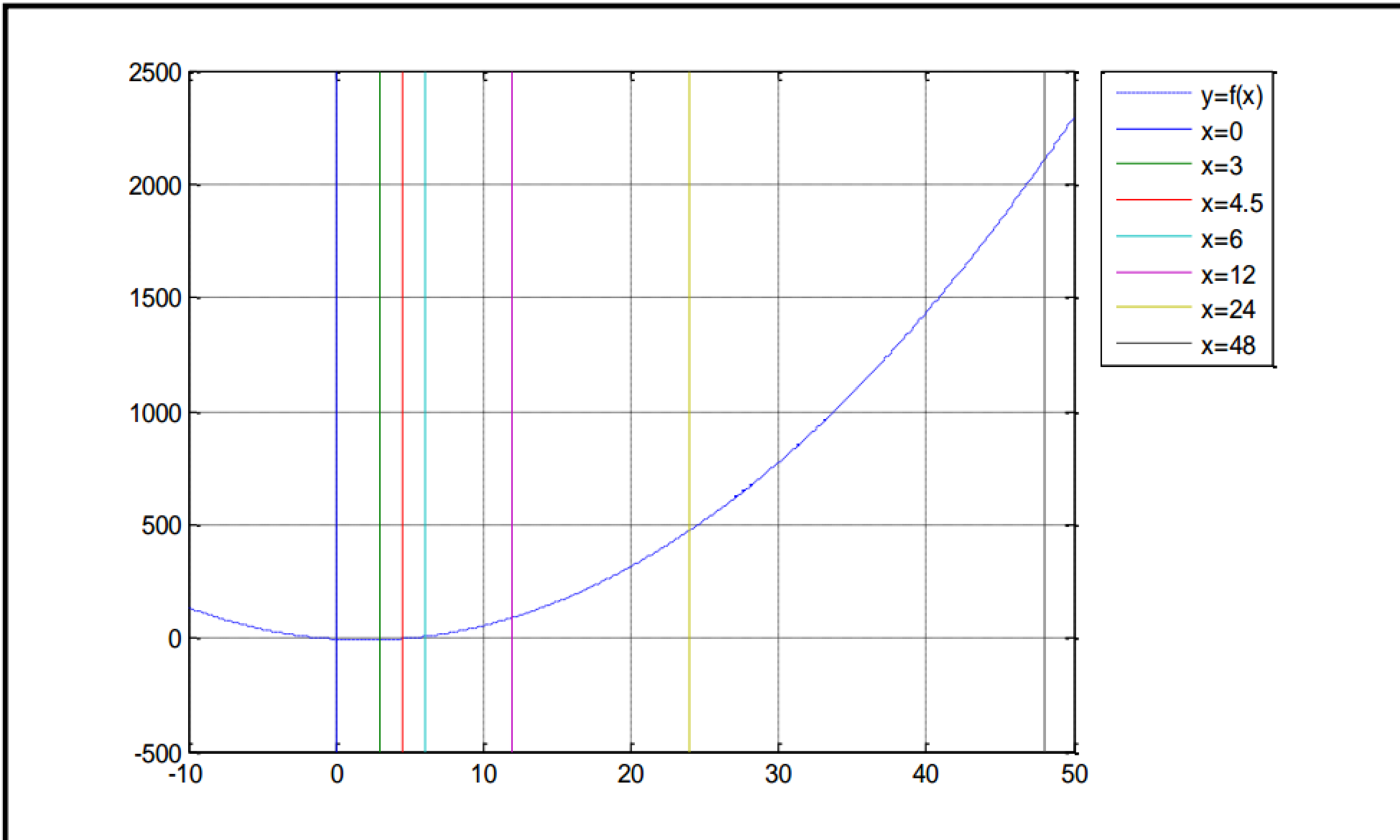
$x_m = 4,5 \qquad f(x_m) = f(4,5) = 20,25 - 18 - 5 = -2,75$

$f(x_l) * f(x_m) > 0$ *the root lies between* x_m *and* x_u *so new estimation range* $[4.5\ ,\ 6]$

Table - Estimation of root for initial range given at each iteration. Iteration will be continued until root is estimated with an acceptable error.

Iteration	x_l	x_u	x_m	$f(x_l)$	$f(x_u)$	$f(x_m)$	E
1	**0**	**48**	**24**	**-5**	**2107**	**475**	**-**
2	**0**	**24**	**12**	**-5**	**475**	**91**	**%100**
3	**0**	**12**	**6**	**-5**	**91**	**7**	**%100**
4	**0**	**6**	**3**	**-5**	**7**	**-8**	**%100**
5	**3**	**6**	**4,5**	**-8**	**7**	**-2,75**	**%33,33**

Figure - Bisection lines for given non-linear equation.



Fixed point theorem

To answer the first question above, suppose that there are two values $a < b$ such that $g(a) \geq a$ and $g(b) \leq b$. If $g(a) = a$ or $g(b) = b$, then a fixed point has been found, so now assume $g(a) > a$ and $g(b) < b$. Then for the continuous function

$$\phi(x) = g(x) - x$$

we have $\phi(a) > 0$, $\phi(b) < 0$. (Note that ϕ *does not* have to coincide with the function f that we have started with; there are lots of ϕ's for a given f.) Hence, by the Intermediate Value Theorem given on page 10, just as before, there is a root $a < x^* < b$ such that $\phi(x^*) = 0$. Thus, $g(x^*) = x^*$, so x^* is a fixed point. We have established the existence of a root: $f(x^*) = 0$.

Next, suppose that g is not only continuous but also differentiable and that there is a positive number $\rho < 1$ such that

$$|g'(x)| \leq \rho, \quad a < x < b.$$

Then the root x^* is unique in the interval $[a,b]$, for if there is also y^* which satisfies $y^* = g(y^*)$, then

$$|x^* - y^*| = |g(x^*) - g(y^*)| = |g'(\xi)(x^* - y^*)| \leq \rho|x^* - y^*|,$$

where ξ is an intermediate value between x^* and y^*. Obviously, this inequality can hold with $\rho < 1$ only if $y^* = x^*$.

Convergence of the fixed point iteration

Turning to the fixed point iteration and the third question on the preceding page, similar arguments establish convergence (now that we know that there is a unique solution): under the same assumptions we have

$$|x_{k+1} - x^*| = |g(x_k) - g(x^*)| \leq \rho |x_k - x^*|.$$

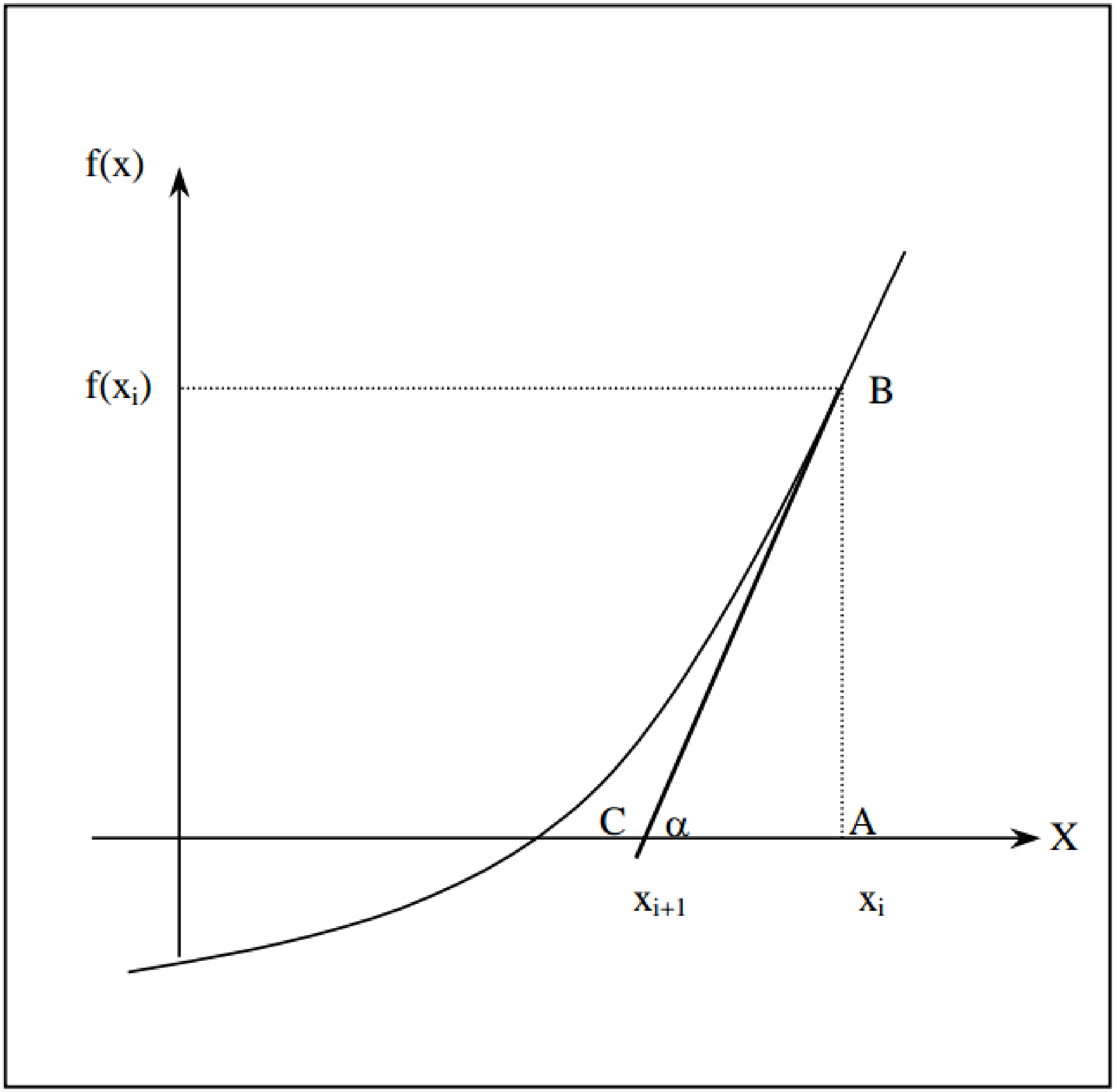
This is a **contraction** by the factor ρ. So

$$|x_{k+1} - x^*| \leq \rho |x_k - x^*| \leq \rho^2 |x_{k-1} - x^*| \leq \cdots \leq \rho^{k+1} |x_0 - x^*|.$$

Since $\rho < 1$, we have $\rho^k \to 0$ as $k \to \infty$. This establishes convergence of the fixed point iteration.

A function $f(x)$ is defined as $f(x)=x^3-10x^2+100$. For $y=f(x)=0$ estimate a root of this function using Newton-Raphson method. Initial guess value of root x_0 is 15 and use absolute relative approximate error notation for error of estimated root.

Recall



$$\tan(\alpha)=\frac{AB}{AC}$$

$$f'(x_i)=\frac{f(x_i)}{x_i-x_{i+1}}$$

$$x_{i+1}=x_i-\frac{f(x_i)}{f'(x_i)}$$

Gist of prediction of x_{i+1}

$$|E|=\left|\frac{x_{i+1}-x_i}{x_{i+1}}\right|*100$$

Function : $f(x) = x^3 - 10x^2 + 100$

Derivative of function : $f'(x) = 3x^2 - 20x$

$$x_1 = 15 - \frac{f(15)}{f'(15)} = 15 - \frac{3375 - 2250 + 100}{675 - 300} = 15 - \frac{1225}{375} = 15 - 3,26 = 11,74$$

$$x_2 = 11,74 - \frac{f(11,74)}{f'(11,74)} = 11,74 - \frac{1618,09 - 1378,27 + 100}{413,48 - 234,8} = 11,74 - \frac{339,82}{178,68} = 11,74 - 1,89 = 9,85$$

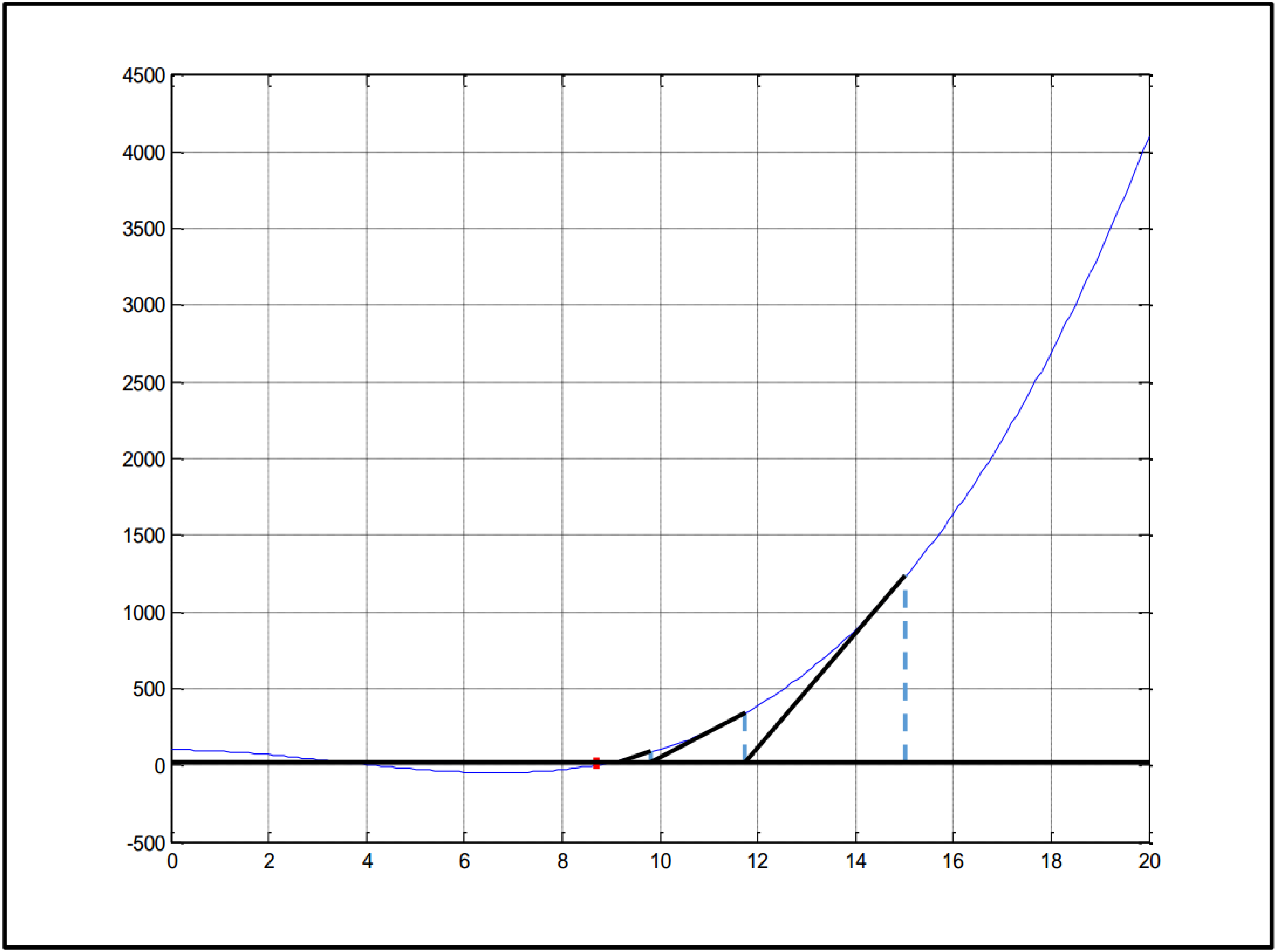
$$x_3 = 9,85 - \frac{f(9,85)}{f'(9,85)} = 9,85 - \frac{955,67 - 970,22 + 100}{291,06 - 197} = 9,85 - \frac{85,45}{94,06} = 9,85 - 1,89 = 8,94$$

$$x_4 = 8,94 - \frac{f(8,94)}{f'(8,94)} = 8,94 - \frac{714,51 - 799,23 + 100}{239,77 - 178,8} = 8,94 - \frac{15,28}{60,97} = 8,94 - 0,25 = 8,69$$

$$x_5 = 8,69 - \frac{f(8,69)}{f'(8,69)} = 8,69 - \frac{656,23 - 755,16 + 100}{226,54 - 173,8} = 8,69 - \frac{1,07}{52,74} = 8,69 - 0,02 = 8,67$$

Table - Estimation of root for the function described in Q2 at each iteration. Iteration will be continued until root is estimated with an acceptable error.

Iteration	x	$f(x)$	$f'(x)$	E
0	$x_0 = 15$	**1225**	**375**	**-**
1	$x_1 = 11,74$	**339,82**	**178,68**	**%27,76**
2	$x_2 = 9,85$	**85,45**	**94,06**	**%19,18**
3	$x_3 = 8,94$	**15,28**	**60,97**	**%10,17**
4	$x_4 = 8,69$	**1,07**	**52,74**	**%2,87**
5	$x_5 = 8,67$	**-**	**-**	**%0,23**


4500
4000
3500
3000
2500
2000
1500
1000
500
0
-500
0
2
4
6
8
10
12
14
16
18
20

Use Naive Gauss Elimination to solve the system of linear equations.

$x_1 + 2x_2 + x_3 = -8$

$2x_1 - 2x_2 - 3x_3 = 0$

$-x_1 + x_2 + 2x_3 = 3$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & -2 & -3 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -8 \\ 0 \\ 3 \end{bmatrix}$$

Add -2 times row1 to row2

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -6 & -5 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -8 \\ 16 \\ 3 \end{bmatrix}$$

Add -1 times row1 to row3

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -6 & -5 \\ 0 & 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -8 \\ 16 \\ -5 \end{bmatrix}$$

Add (1/2) times row2 to row3

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -6 & -5 \\ 0 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -8 \\ 16 \\ 3 \end{bmatrix}$$

Now elimination stops.

$$0.5x_3 = 3$$

$$x_3 = 6$$

$$-6x_2 - 5 * 6 = 16$$

$$x_2 = -\frac{23}{3}$$

$$x_1 + 2x_2 + x_3 = -8$$

$$x_1 = \frac{4}{3}$$

Solve same linear system using LU decomposition.

Ax=B

Apply gauss elimination **only matrix A**.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -2 & -3 \\ -1 & 1 & 2 \end{bmatrix} \qquad \text{same operation as gauss elimination } U = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -6 & -5 \\ 0 & 0 & 0.5 \end{bmatrix}$$

Now find [L] matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \qquad l_{21} = \frac{a_{21}}{a_{11}}, \quad l_{31} = \frac{a_{31}}{a_{11}}, \quad l_{32} = \frac{a_{32}}{a_{22}}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -0.5 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -0.5 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} -8 \\ 0 \\ 3 \end{bmatrix}$$

$$Z = \begin{bmatrix} -8 \\ 16 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -6 & -5 \\ 0 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -8 \\ 16 \\ 3 \end{bmatrix}$$

$$0.5x_3 = 3$$

$$x_3 = 6$$

$$-6x_2 - 5 * 6 = 16$$

$$x_2 = -\frac{23}{3}$$

$$x_1 + 2x_2 + x_3 = -8$$

$$x_1 = \frac{4}{3}$$

Details of LU decomposition.

$$A=LU$$

$$LUx=B$$

$$Ux=L^{-1}B$$

$$x=U^{-1}L^{-1}B$$

$$x=L^{-1}Z$$