ISTANBUL**TECHNICAL**UNIVERSITY

BLG202E
Recitation I

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Outline

- MATLAB Tutorial
- Floating Point Systems
  - Decimal to Binary
  - The IEEE Standard
- Taylor’s Theorem

Floating Point Systems

Example #1

Convert $(11.1875)_{10}$ to binary representation $(?.?)_2$

$$
\begin{aligned}
(11)_{10} &= 2^3 + 3 \\
&= 2^3 + 2^1 + 1 \\
&= 2^3 + 2^1 + 2^0 \\
&= 1\times 2^3 + 0\times 2^2 + 1\times 2^1 + 1\times 2^0 \\
&= (1011)_2
\end{aligned}
$$

$2^0 = 1$
$2^1 = 2$
$2^2 = 4$
$2^3 = 8$
.
.
.

Floating Point Systems

Example #1

Convert $(11.1875)_{10}$ to binary representation $(?.?)_2$

$$
\begin{aligned}
(0.1875)_{10} &= 2^{-3} + 0.0625 \\
&= 2^{-3} + 2^{-4} \\
&= 0\times 2^{-1} + 0\times 2^{-2} + 1\times 2^{-3} + 1\times 2^{-4} \\
&= (.0011)_2
\end{aligned}
$$

$2^{-1} = 0.5$
$2^{-2} = 0.25$
$2^{-3} = 0.125$
$2^{-4} = 0.0625$
.
.
.

$$(11.1875)_{10} = (1011.0011)_2$$

Floating Point Systems

Example #2

Sign (s)	Exponent (e’)	Mantissa (m)
1	1 0 1 0 0 0 1 0	1 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0

Floating Point Systems

Example #2

1	1	0	1	0	0	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Sign (s) Exponent (e') Mantissa (m)

Normalize the exponent by subtracting 127

$$\text{Value} = (-1)^{s} \times 2^{(e'-127)} \times (1.m)$$

Floating Point Systems

Example #2

1	1	0	1	0	0	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Sign (s) Exponent (e’) Mantissa (m)

$$\text{Value} = (-1)^{s} \times (1.m)_2 \times 2^{e'-127}$$

Floating Point Systems

Example #2

1	1	0	1	0	0	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Sign (s) Exponent (e’) Mantissa (m)

$$\text{Value} = (-1)^s \times (1.m)_2 \times 2^{e'-127}$$

$$= (-1)^1 \times (1.10100000)_2 \times 2^{(10100010)_2 - 127}$$

Floating Point Systems

Example #2

$$\underbrace{1}_{\text{Sign (s)}}\ \underbrace{1\,0\,1\,0\,0\,0\,1\,0}_{\text{Exponent (e')}}\ \underbrace{1\,0\,1\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0}_{\text{Mantissa (m)}}$$

$$\begin{aligned}
\text{Value } &= (-1)^{s} \times (1.m)_2 \times 2^{e'-127} \\
&= (-1)^{1} \times (1.10100000)_2 \times 2^{(10100010)_2-127} \\
&= (-1) \times (1.625) \times 2^{162-127}
\end{aligned}$$

Floating Point Systems

Example #2

1	1	0	1	0	0	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Sign (s) Exponent (e’) Mantissa (m)

$$\text{Value} = (-1)^{s} \times (1.m)_2 \times 2^{e'-127}$$

$$= (-1)^{1} \times (1.10100000)_2 \times 2^{(10100010)_2 - 127}$$

$$= (-1) \times (1.625) \times 2^{162-127}$$

$$= (-1) \times (1.625) \times 2^{35} = -5.5834 \times 10^{10}$$

Taylor's Theorem

Importance of Taylor's Theorem: This theorem can be used to approximate functions like *sin*, *exp* and *log* by polynomials (which are easy to compute) and provides an estimate of the error involved in the approximation.

Taylor's Theorem. *Let f be an $(n+1)$ times differentiable function on an open interval containing the points a and x. Then*

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

Error

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some number c between a and x.

Taylor's Theorem

The remainder term is given by

$$R_n(x) = f(x) - f(a) - f'(a)(x-a) - \frac{f''(a)}{2!}(x-a)^2 - \cdots - \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Do not confuse f(x) with the approximation. It is the original function that we want to find its estimation !

Taylor’s Theorem

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Fix x and a. For t between x and a set

$$F(t) = f(x) - f(t) - f'(t)(x-t) - \frac{f''(t)}{2!}(x-t)^2 - \cdots - \frac{f^{(n)}(t)}{n!}(x-t)^n,$$

so that $F(a) = R_n(x)$. Then

F(t) can be seen as a general function and $R_n(x)$ is a sample from this function

Taylor’s Theorem

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so that $F(a) = R_n(x)$. Then

$$\begin{aligned} F'(t) &= -f'(t) - f''(t)(x-t) + f'(t) - \frac{f'''(t)}{2!}(x-t)^2 + f''(t)(x-t) \\ &\quad - \cdots - \frac{f^{(n+1)}(t)}{n!}(x-t)^n + \frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} \\ &= -\frac{f^{(n+1)}(t)}{n!}(x-t)^n. \end{aligned}$$

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Now defining

$$G(t) = F(t) - \left(\frac{x-t}{x-a}\right)^{n+1} F(a),$$

Taylor’s Theorem

Rolle’s Theorem: Let f be continuous on a closed interval [a, b] and differentiable on the open interval (a, b). If f(a) = f(b), then there is at least one point c in (a, b) where f'(c) = 0.

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we have $G(a) = 0$ and $G(x) = F(x) = 0$. Applying Rolle's theorem to the function G shows that there is a c between a and x with $G'(c) = 0$. Now

$$\begin{aligned} 0 = G'(c) &= F'(c) + (n+1)\frac{(x-c)^n}{(x-a)^{n+1}}F(a) \\ &= -\frac{f^{(n+1)}(c)}{n!}(x-c)^n + (n+1)\frac{(x-c)^n}{(x-a)^{n+1}}F(a). \end{aligned}$$

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$$\begin{aligned} 0 = G'(c) &= F'(c) + (n+1)\frac{(x-c)^n}{(x-a)^{n+1}}F(a) \\ &= -\frac{f^{(n+1)}(c)}{n!}(x-c)^n + (n+1)\frac{(x-c)^n}{(x-a)^{n+1}}F(a). \end{aligned}$$

But $F(a) = R_n(x)$ and rearranging the last equation gives

$$R_n(x) = F(a) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

Tips

- Bounding Taylor Error
  - https://www.khanacademy.org/math/ap-calculus-bc/bc-series-new/bc-10-12/v/error-or-remainder-of-a-taylor-polynomial-approximation
- Taylor & Maclaurin Series Intuition
  - https://www.khanacademy.org/math/ap-calculus-bc/bc-series-new/bc-10-11/v/maclaurin-and-taylor-series-intuition