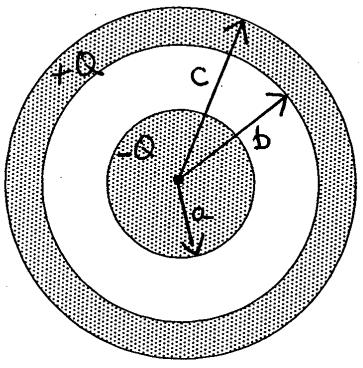
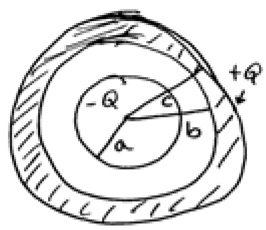
I. A spherical capacitor consists of an inner solid conducting sphere of radius a surrounded by a spherical conducting shell of inner radius b and outer radius c. The capacitor is charged with $-Q$ on the inner sphere and $+Q$ on the outer spherical shell. The coordinate r measures the distance from the center of the solid sphere.

(a) Determine the electric field $\vec{E}(r)$ everywhere in space due to this charge configuration, indicating both direction and magnitude. Sketch $E(r)$ vs. r for all r.
(b) In part (c) you will calculate the potential difference ΔV between the inner and outer conductors. Before you calculate the value, indicate what the *sign* (positive or negative) of $\Delta V = V_b - V_a$ must be and give a clear explanation as to why.
(c) Determine the potential difference $\Delta V = V_b - V_a$ between the inner and outer conductors.
(d) From the result in (c), determine the capacitance of this device and the energy stored in it.
(e) Taking the electrostatic potential to be zero at infinity, determine $V(r)$ for all r and sketch $V(r)$ vs. r.
(f) Determine the energy density in the electric field in the region between a and b. Integrate the energy density over the volume between a and b and compare to your answer in (d).



I (a)



(a) Since we have two conductors, the electric field inside the conductors is zero. All charge resides on surface of conductors.

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\varepsilon_0}$$

$r < a$ $\quad E = 0$ $\quad$ No charge enclosed

$b < r < c$ $\quad E = 0$

$a < r < b$ $\quad \oint \vec{E} \cdot d\vec{A} = -E \cdot 4\pi r^2 = \frac{Q_{enc}}{\varepsilon_0} = -\frac{Q}{\varepsilon_0} \Rightarrow \vec{E} = \frac{-Q}{4\pi\varepsilon_0 r^2}\hat{r}$

$r > c$ $\quad E = 0$ since $Q_{enc} = 0$

$E = 0$ for all space except for $a < r < b$ where $\vec{E} = \frac{-Q}{4\pi\varepsilon_0 r^2}\hat{r}$

(b) $\Delta V = V_b - V_a$ $\quad$ Since $\vec{E}$ points from high potential to low potential $\Rightarrow V_b > V_a$ so $\boxed{\Delta V = V_b - V_a > 0 \quad \text{positive}}$

(c) $\Delta V = -\int_a^b \vec{E} \cdot d\vec{l} = -\int_a^b \left(\frac{-Q}{4\pi\varepsilon_0 r^2}\right) dr = \int_a^b \frac{Q}{4\pi\varepsilon_0 r^2}\, dr$

$$\Delta V = \frac{Q}{4\pi\varepsilon_0}\int_a^b \frac{dr}{r^2} = \frac{-Q}{4\pi\varepsilon_0}\left(\frac{1}{r}\Big|_a^b\right) = \frac{-Q}{4\pi\varepsilon_0}\left(\frac{1}{b} - \frac{1}{a}\right) = \underline{\frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)}$$

(d) from (c)

$$\Delta V = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right) \qquad \underline{\text{Note } \Delta V > 0} \qquad C \equiv \frac{Q}{\Delta V} = 4\pi\varepsilon_0\left(\frac{1}{a}-\frac{1}{b}\right)^{-1}$$

$$\frac{1}{a}-\frac{1}{b} = \frac{b-a}{ab} \Rightarrow \boxed{C = \frac{4\pi\varepsilon_0\, ab}{b-a}} \qquad \boxed{U = \tfrac{1}{2}QV = \tfrac{1}{2}Q^2\left(\frac{b-a}{4\pi\varepsilon_0 ab}\right)} \qquad \boxed{U = \frac{Q^2(b-a)}{8\pi\varepsilon_0 ab}}$$

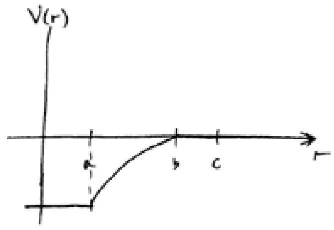
(e) $\Delta V_{\infty,r} \equiv -\int_\infty^r \vec{E}\cdot d\vec{l} \Rightarrow \underline{\Delta V_{\infty,r} = V(r) - V(\infty)}$ $\qquad \Delta V_{\infty,r}(r>c) = -\int_\infty^r 0\, dl = 0 \Rightarrow V(r) = 0$

$$\Delta V_{\infty,r}(b<r<c) = -\int_\infty^r \vec{E}\cdot d\vec{l} = -\int_\infty^r 0\, dl = 0 \Rightarrow V(r) = 0$$

$$\Delta V_{\infty,r}(a<r<b) = -\int_\infty^r \vec{E}\cdot d\vec{l} = -\int_\infty^b 0\, dr + -\int_b^r\left(\frac{-Q}{4\pi\varepsilon_0 r^2}\right)dr \qquad \Delta V_{\infty,r}(a<r<b) = +\int_b^r \frac{Q}{4\pi\varepsilon_0 r^2}\,dr = \frac{-Q}{4\pi\varepsilon_0}\left(\frac{1}{r}-\frac{1}{b}\right)$$

$$\Delta V_{\infty,r}(r<a) = \frac{-Q}{4\pi\varepsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)$$

(Note: @ $r=b$ $\Delta V = 0$)



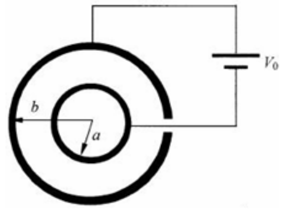
(f) $u_E = \frac{1}{2}\varepsilon_0|E|^2 = \frac{1}{2}\varepsilon_0\left|\frac{-Q}{4\pi\varepsilon_0 r^2}\right|^2 \qquad a<r<b \qquad u_E = \frac{1}{2}\varepsilon_0\frac{Q^2}{16\pi^2\varepsilon_0^2 r^4} = \frac{Q^2}{32\pi^2\varepsilon_0 r^4}$

$U = \int_a^b u_E\, dV \qquad dV \equiv$ infinitesimal volume of sphere $\qquad dV = 4\pi r^2\, dr$

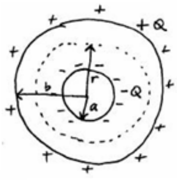
$$\underline{U = \frac{Q^2}{32\pi^2\varepsilon_0}\cdot 4\pi\int_a^b \frac{dr}{r^2}} = \underline{\frac{Q^2}{8\pi\varepsilon_0}\left(-\frac{1}{r}\Big|_a^b\right)} \qquad \boxed{U = \frac{Q^2}{8\pi\varepsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right) = \frac{Q^2}{8\pi\varepsilon_0}\left(\frac{b-a}{ab}\right)}$$

<u>Same as in (d).</u>

QUESTION 6 A spherical capacitor is formed from two concentric spherical conducting shells separated by air as shown in the figure. Inner sphere has radius a = 5 cm and outer has radius b = 10 cm. The capacitor is charged to a potential difference V_0 = 90 V.



(a) What is the capacitance of the capacitor?



Gauss law: $\varepsilon_0 \oint \vec{E} \cdot d\vec{s} = q_{enc}$

$\varepsilon_0 E(4\pi r^2) = Q \rightarrow E = \frac{kQ}{r^2}$

Potential diff. between conductors: $V_{ab} = -\int_a^b \vec{E} \cdot d\vec{l} = -\int_a^b E\,dr = -\int_a^b \frac{kQ}{r^2}dr$

$= kQ\left(\frac{b-a}{ab}\right)$

Capacitance:

$$C = \frac{Q}{V_{ab}} = \frac{Q}{kQ\left(\frac{b-a}{ab}\right)} = \frac{ab}{k(b-a)} = \frac{(0.05)(0.1)}{9\times10^9(0.1-0.05)} = 1.1\times10^{-11}F$$

$\boxed{C = 1.1\times10^{-11}F}$

(b) What charge is collected in outer surface?

$$Q = CV_0 = (1.1\times10^{-11})(90) = 1\times10^{-9}C = 1nC$$

$\boxed{Q = 1\times10^{-9}C}$

(c) What is the stored energy in the capacitor?

$$U = \frac{1}{2}CV_0^2 = \frac{1}{2}(1.1\times10^{-11})(90)^2 = 4.5\times10^{-8}J$$

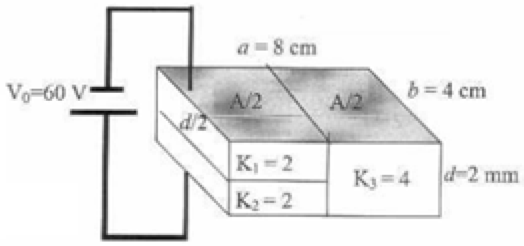
$\boxed{U = 4.5\times10^{-8}J}$

(d) What is the average energy density (stored energy per unit volume) in the capacitor?

$$u = \frac{Energy}{Volume} = \frac{U}{\frac{4}{3}\pi(b^3-a^3)} = \frac{4.5\times10^{-8}}{\frac{4}{3}\pi([0.1]^3-[0.05]^3)} = 1.2\times10^{-7}\frac{J}{m^3}$$

$\boxed{u = 1.2\times10^{-7}J/m^3}$

QUESTION 7
A rectangular parallel plate capacitor having sides a and b and separation d is filled with three dielectric materials as seen in the Figure.



(a) Find the equivalent capacitance value of the system.

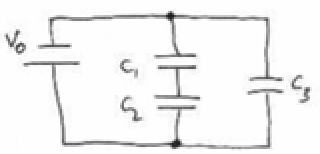


Plate area : $A = ab = (0.08)(0.04) = 3.2\times10^{-3}m^2$

$$C_1 = \frac{\varepsilon_0 K_1 (A/2)}{d/2} = \frac{\varepsilon_0 K_1 A}{d} = \frac{(8.85\times10^{-12})(2)(3.2\times10^{-3})}{2\times10^{-3}} = 2.8\times10^{-11}F$$

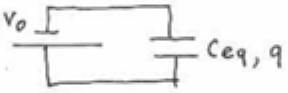
$$C_2 = \frac{\varepsilon_0 K_2 (A/2)}{d/2} = \frac{\varepsilon_0 K_2 A}{d} = \frac{(8.85\times10^{-12})(2)(3.2\times10^{-3})}{2\times10^{-3}} = 2.8\times10^{-11}F$$

$$C_3 = \frac{\varepsilon_0 K_3 (A/2)}{d} = \frac{\varepsilon_0 K_3 A}{2d} = \frac{(8.85\times10^{-12})(4)(3.2\times10^{-3})}{(2)(2\times10^{-3})} = 2.8\times10^{-11}F$$

$$C_{eq} = \frac{C_1 C_2}{C_1+C_2} + C_3 = \frac{C_1}{2} + C_3 = \frac{3C_1}{2} = \frac{3}{2}(2.8\times10^{-11}F) = 4.2\times10^{-11}F$$

$\boxed{C_{eq} = 4.2\times10^{-11}F}$

(b) Find the charge on each capacitor (dielectric).



$q = C_{eq} V_o = (4.2 \times 10^{-11})(60) = 2.5 \times 10^{-9}\ C$

$q_3 = C_3 V_o = (2.8 \times 10^{-11})(60) = 1.7 \times 10^{-9}\ C$

$q_1 = q_2 = q_{12}$

$q = q_{12} + q_3$

$q_{12} = q - q_3 = (2.5 - 1.7) \times 10^{-9} = 0.8 \times 10^{-9}\ C$

$q_1 = 0.8 \times 10^{-9} C$
$q_2 = 0.8 \times 10^{-9} C$
$q_3 = 1.7 \times 10^{-9} C$

5. A coaxial cable of length l has an inner conductor of radius of a which carries a charge Q. The surrounding conductor has an inner radius b and a charge of $-Q$. Assume the electrical properties of the material between the conductors are the same as empty space.

(a) Find the electric field as a function of r between the conductors, for $a < r < b$.

This is one of the geometries where the electric field can be found using Gauss's Law. Construct a Gaussian cylinder of radius r and length L between the two conductors of the cable. The electric flux through this cylinder is $\Phi = EA = 2\pi r l E$. Gausses law relates this to the charge inside:

$2\pi r l\, E = \Phi = \dfrac{Q}{\epsilon_0}$ for $\lambda = Q/l$. Therefore, $E(r) = \dfrac{\boldsymbol{Q}}{\boldsymbol{2\pi\epsilon_0 r l}}$

(b) Find the potential difference $V_a - V_b$ between the conductors if $Q = 25\ \mathrm{nC}$, $a = 1.2\ \mathrm{mm}$ $b = 2.4\ \mathrm{mm}$, and $L = 30.0\ \mathrm{m}$.

$$V_a - V_b = \int_a^b \frac{Q\,dr}{2\pi\epsilon_0 r l} = \frac{Q}{2\pi\epsilon_0 l}\ln\left(\frac{b}{a}\right) = \frac{25 \times 10^{-9}\ \ln 2}{2\pi(8.85 \times 10^{-12})(30)}\mathrm{V} = \mathbf{10.4\ V}.$$

(c) Find the capacitance of this length of cable. [If you don't have a result for the previous question, assume $\Delta V = 12.0$ V for (c) and (d).]

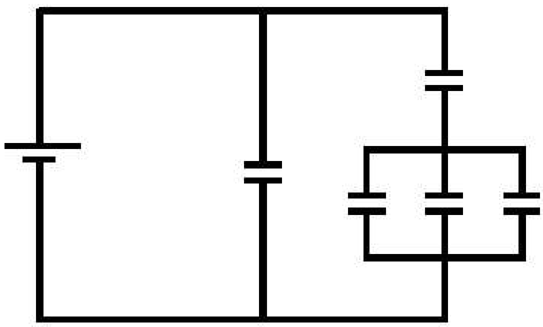
$$C = \frac{Q}{\Delta V} = \mathbf{2.40\ nF}.$$

(d What electrical energy is stored in the cable?

$$U_e = \frac{1}{2}C(\Delta V)^2 = \frac{1}{2}(3.38 \times 10^{-9})(10.4)^2\,\mathrm{J} = \mathbf{366\ nJ}.$$

3. Consider the circuit shown below. Assume all capacitors are 1 mF and the battery is 10 V

a) What is the equivalent capacitance.



$C_{3p} = C + C + C = 3mF$ $\quad \dfrac{1}{C_s} = \dfrac{1}{C_{3p}} + \dfrac{1}{C} \Rightarrow C_s = \dfrac{3}{4}mF$

$C_{eq} = C_s + C = \dfrac{7}{4}mF$

b) What is the charge on each capacitor?

The charge on the left capacitor is the easiest.

$q_L = CV = 1mF \cdot 10V = 10mC$

Now compute the charge on the equivalent capacitance for the array of charges on the right.

$q = C_{series}V = \dfrac{3}{4}mF \cdot 10V = \dfrac{30}{4}mC$

The charge on this equivalent capacitor is the same as the charge on the series capacitors that were used to create it. So the charge on the upper right capacitor has this value We are left with the three capacitors in parallel. The voltage across the three is the same as the voltage across the equivalent capacitor. We can calculate this voltage, since we know the charge on the equivalent capacitor--it is the charge we just computed.

$V = \dfrac{q}{C_{3p}} = \dfrac{30/4\ mC}{3mF} = 2.5V$

Now that we know the voltage across each of these capacitors, we can find the charge on each.

$$q = CV = 1mF \cdot 2.5V = 2.5mC$$

We repeat this for each capacitor in parallel, but in this case, they are all the same...

c) How much energy is stored in the capacitors?

The easiest way to do this is to consider the equivalent capacitance

$$U = \frac{1}{2}CV^2 = \frac{1}{2}\cdot\frac{7}{4}mF\cdot(10V)^2 = 87.5J$$

Problem 1: In lecture, you saw a demo with a parallel plate capacitor with capacitance C_0. The capacitor was connected to an ideal power supply with an output voltage ΔV and charged to a charge Q.The capacitor was disconnected from the power supply, still carrying the same charge Q. Then the distance between the plates was increased by a factor of 2.

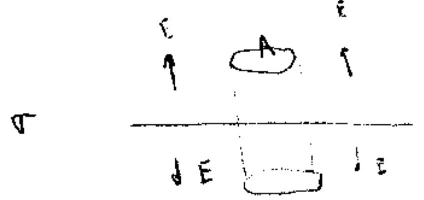
(a) How big is the potential difference between the capacitor plates after they have been moved apart?

(b) How big is the stored electrical energy in the capacitor after the plates have been moved apart?

(c) Explain how energy was conserved when moving the plates apart (1 sentence).

(d) Suppose we had not disconnected the power supply before moving the plates apart. How big would the stored energy be after the plates have been moved apart in this case?

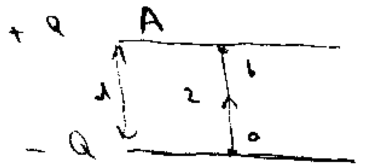
1 -



$$\oint E\cdot da = Q_{in}/\epsilon_0$$

$$E\cdot 2A = \sigma A/\epsilon_0$$

$$E = \frac{\sigma}{2\epsilon_0}$$



$\uparrow E_+ \downarrow E_- \Rightarrow \vec{E} = 0$

$\downarrow E_+ \downarrow E_- \Rightarrow E = \frac{\sigma}{\epsilon}$

$\downarrow E_+ \uparrow E_- \Rightarrow E = 0$

$$V_{ab} = -\int_a^b E\cdot ds = -\int_0^b \frac{\sigma}{\epsilon_0}(-\hat{z})\cdot d\hat{z} = \frac{\sigma}{\epsilon_0}d = \frac{Q}{A\epsilon_0}d$$

$Q = CV \Rightarrow C = \frac{A\epsilon_0}{d}$

a) $d \to 2d$ $\quad Q \to Q$

$C \to C_0/2$ $\quad V \to 2\Delta V$

b) $P = IV = \frac{dQ}{dt} V$

$$U = \int P\,dV = \int_0^Q \frac{dQ'}{dt} \frac{Q'}{C} = \frac{1}{2}\frac{Q^2}{C}$$

$Q \to Q$ $\quad C \to C/2$

$U \to 2U = \frac{Q^2}{C_0}$

c) Energy was added to the sytem by doing work against the electrical force to move the plates.

d) $U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CV^2$

$V \to V_0$ $\quad C \to C_0/2$ $\quad U = U_0/2 = \frac{Q^2}{4C_0}$

Notice: E remains the same in parts a-c but the volume it is affecting is doubled. In part d, E reduces to half its value.