T.R.

GEBZE TECHNICAL UNIVERSITY

FACULTY OF ENGINEERING

DEPARTMENT OF COMPUTER ENGINEERING

CERTIFICATION OF REAL SOLUTIONS OF POLYNOMIAL ROOTS

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GEBZE TECHNICAL UNIVERSITY	GRADUATION PROJECT JURY APPROVAL FORM

This study has been accepted as an Undergraduate Graduation Project in the Department of Computer Engineering on 22.06.2023 by the following jury.

JURY

Member
(Supervisor) : DR. Tülay Ayyıldız

Member : Assoc. Prof. Didem Gözüpek

ABSTRACT

Polynomial equations can have multiple roots, some of which may be real while others are complex. The challenge lies in developing efficient and reliable methods to certify which roots are real and providing mathematical guarantees for their validity.

ACKNOWLEDGEMENT

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Fatih Doğaç

CONTENTS

1. INTRODUCTION

Certification of real solutions of polynomial roots is a fundamental problem in mathematics with numerous applications in various fields, including engineering, computer science, and finance. Given a polynomial equation, determining whether it possesses real solutions and providing a rigorous certification for these solutions is of utmost importance for ensuring the validity and reliability of mathematical models and computational algorithms.

2. POLYNOMIALS

$$p(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 \in \mathbb{R}[x]$$

A polynomial is a mathematical expression consisting of variables and coefficients, combined using addition, subtraction, and multiplication operations. It is a finite sum of terms, where each term is a product of a coefficient and a power of the variable.

3. COMPANION MATRIX

In linear algebra, a companion matrix is a square matrix that represents a polynomial.

The companion matrix has a distinctive structure, with ones in the first sub-diagonal, and the last row containing the negative ratios of the coefficients of the polynomial divided by the coefficient of the highest degree term. The remaining entries of the matrix are all zero. Often, some authors use the transpose of this matrix.

$$C(p) = \begin{bmatrix} 0 & 0 & \dots & 0 & -c_0 \\ 1 & 0 & \dots & 0 & -c_1 \\ 0 & 1 & \dots & 0 & -c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -c_{n-1} \end{bmatrix} C^T(p) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -c_0 & -c_1 & -c_2 & \cdots & -c_{n-1} \end{bmatrix}$$

We will later see the companion matrices in eigenvalues of companion matrix section.

The Algorithm 1 shows the pseudo code of how to crate a companion matrix.

Algorithm 1 Comparison Matrix Creator

Require: Polynomial array which includes the coefficients of the polynomial
Ensure: Companion Matrix of The Polynomial

```
for each x in Array do
    if x is not a real number then
        Print "Polynomial is not real."
        return []
    end if
end for
degree ← length of polynomial − 1
matrix ← create a degree × degree matrix with all elements initialized to 0
for x in range(degree) do
    if x equals 0 then
        matrix[x, degree − 1] ← −polynomial[degree] / polynomial[0]      ▷ First row
        continue
    end if
    matrix[x, x − 1] ← 1                 ▷ Placing 1's starting from the second row
    matrix[x, degree − 1] ← −polynomial[degree−x] / polynomial[0]
end for
return matrix
```

4. EIGENVALUES

Eigenvalues are a fundamental concept in linear algebra that play a crucial role in the study of matrices.
Although there are several algorithms to find the eigenvalues. We will use the QR algorithm with numpy library

4.1. Eigenvalues of Companion Matrix

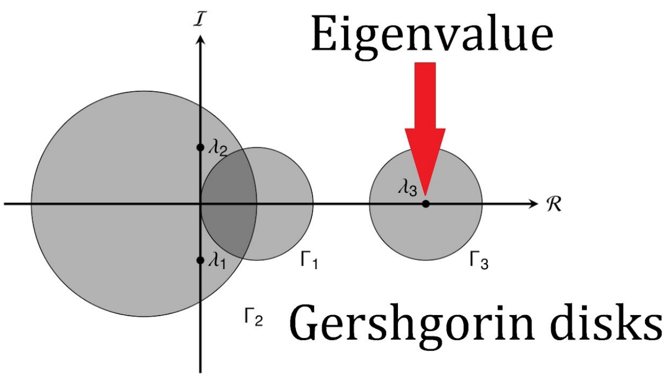
By computing the eigenvalues of the companion matrix, we can determine the roots of the polynomial. This connection between eigenvalues and polynomial roots is a powerful tool for analyzing and solving polynomial equations.

5. GERSHGORIN CIRCLES

Gershgorin circles, are a set of circular regions in the complex plane associated with the eigenvalues of a square matrix. They provide useful bounds on the possible locations of eigenvalues based on the matrix's entries.

Let A be a complex $n \times n$ matrix, and let R_i be the sum of the absolute values of the non-diagonal entries in the i-th row:

$$R_i = \sum_{j \neq i} |a_{i_j}|$$



6. CONCLUSIONS

6.1. The research question

Polynomial equations can have multiple roots, some of which may be real while others are complex. The challenge lies in developing efficient and reliable methods to certify which roots are real and providing mathematical guarantees for their validity.

6.2. Our approach

In this project, gershgorin circles of a given polynomial will be shown in a coordinate plane and the roots of the polynomial will be shown in that plane as well.

Of course we expect the roots to be inside the Gerschgorin circles. By that method, the roots will be certified to be true and if the calculated roots are not inside the Gerschgorin circles, then it means something went wrong and the calculated roots are not relatively correct.