Unit 4. Coding and Capacity on Fading Channels

ECE-GY 6023. WIRELESS COMMUNICATIONS

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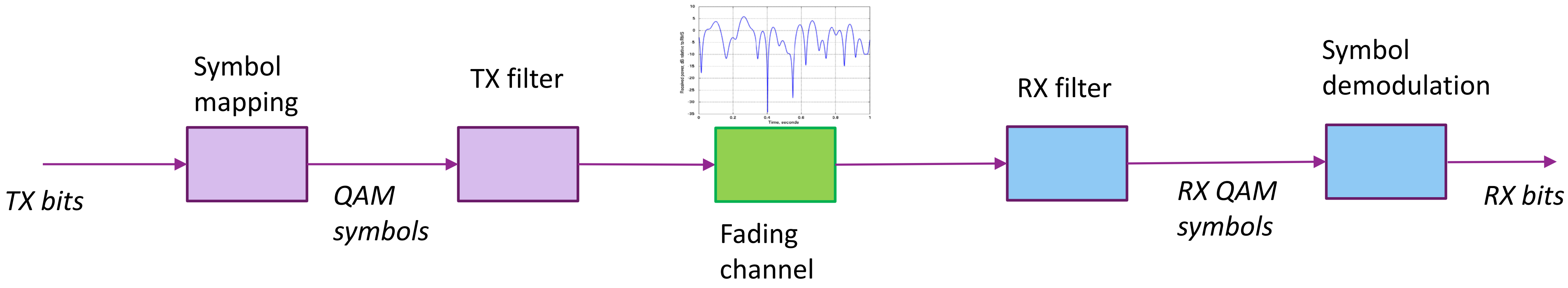
Learning Objectives

- ❑ Describe symbol mapping for QAM constellations
- ❑ Implement symbol detection for faded symbols
  - Compute average BER and SER on AWGN and flat channels and compare
- ❑ Identify if a system can be modeled as slow vs. fast and frequency-selective vs. flat fading
- ❑ For slow and flat fading, compute outage probability and capacity under a fading model
- ❑ For IID fading, compute the ergodic capacity
- ❑ Create a TX and RX chain for flat and fading channels with given components
  - Symbol equalization, soft symbol detection, interleaving, channel decoder
- ❑ Use MATLAB tools for common channel encoders and decoders
  - Convolutional, turbo codes and LDPC codes

Outline

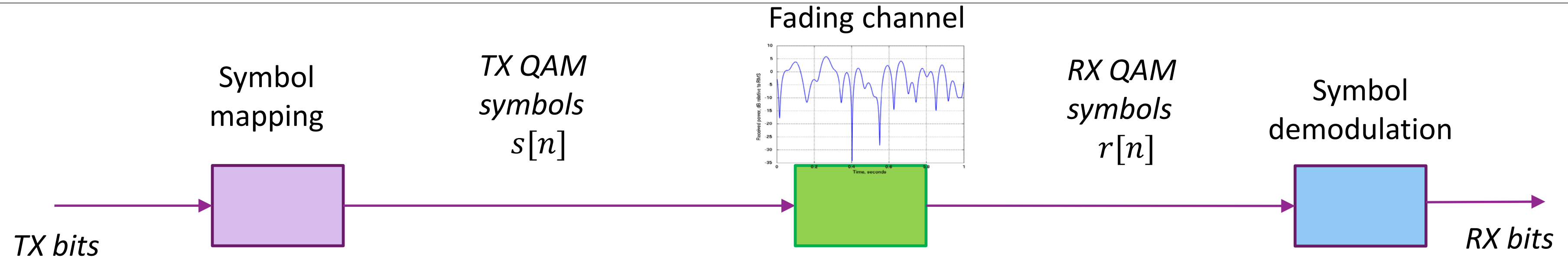
- ❑ Uncoded Modulation over Fading Channels
- ❑ Capacity with Coding over Fading Channels: Outage Capacity
- ❑ Capacity with Coding over Fading Channels: Ergodic Capacity
- ❑ Review: Coding over an AWGN Channel
- ❑ Coding over Fading Channels
- ❑ Capacity with Bit-Interleaved Coded Modulation

Uncoded Modulation



- This section: Uncoded modulation over fading channels
  - That is, communication with no channel encoding and decoding
- We will show uncoded modulation works very poorly
- Virtually all practical wireless systems use coding of some form

Mathematical Model



- **Simple memoryless model**:

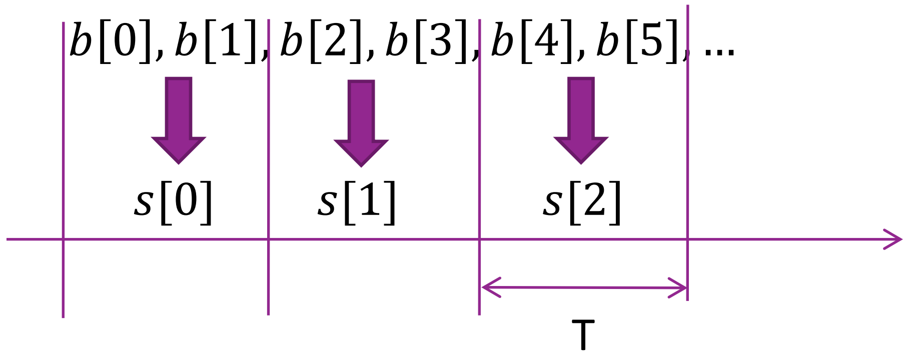
$$r[n] = h[n]s[n] + w[n]$$

  - $s[n]$ and $r[n]$: TX and RX QAM symbols
  - $h[n]$: Fading channel gain, $w[n]$ Noise

- **Assumptions**:
  - Perfect synchronization
  - No ISI in the channel (or the equalizer has removed the effect of the ISI, more on this later)
  - We can look at one symbol at a time

Review: Bit to Symbol Mapping

- $b[k] \in \{0,1\}$ = sequence of bits.
- $s[n] \in \{s_1, \ldots, s_M\}$ = sequence of complex symbols
  - Each symbol has one of M possible values
- Modulation rate: $R_{mod} = \log_2 M$ bits per symbol
  - Each R_{mod} bits gets mapped to one symbol
- Symbol period: One symbol every T seconds.
- Bit rate of $R = R_{mod}/T$ bits per second

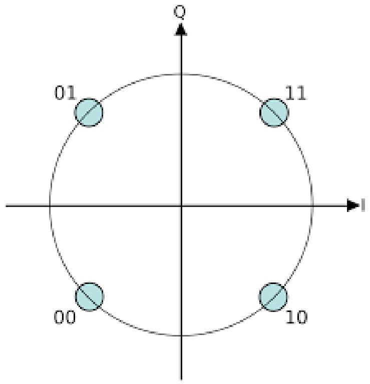


Ex. with M=4 symbols
R_{mod}=2 bits per symbol

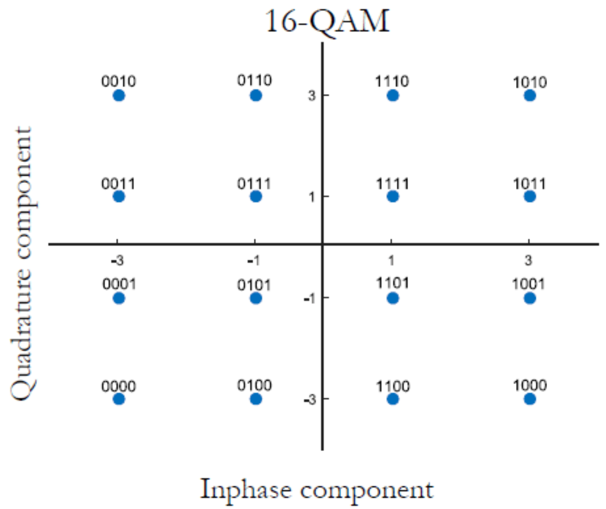
Review: QAM Modulation

- M −QAM: Most common bit to symbol mapping in wireless system
  - $R/2$ bits mapped to I and $R\ /2$ bits mapped to Q
  - Each dimension is mapped uniformly

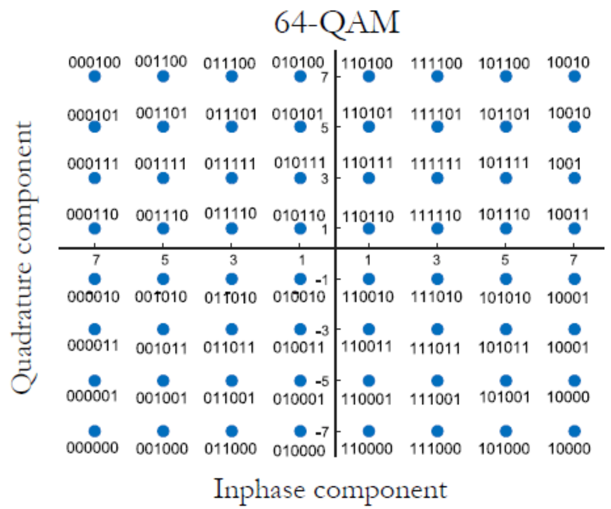
QPSK



$R\ =2$ bits/sym



$R\ =4$ bits/sym



$R\ =6$ bits/sym

ML Estimation for Symbol Demodulation

- Consider single symbol: $r = hs + w,\ w \sim CN(0, N_0),\ s \in \{s_1, \dots, s_M\}$
  - Drop the sample index n
  - s is a QAM symbol
- Maximum likelihood estimation:

$$\hat{s} = \arg \max_{s=s_1,\dots s_M} p(r|s = s_m)$$

- Given s and h: $r \sim CN(hs, N_0)$
- Hence,

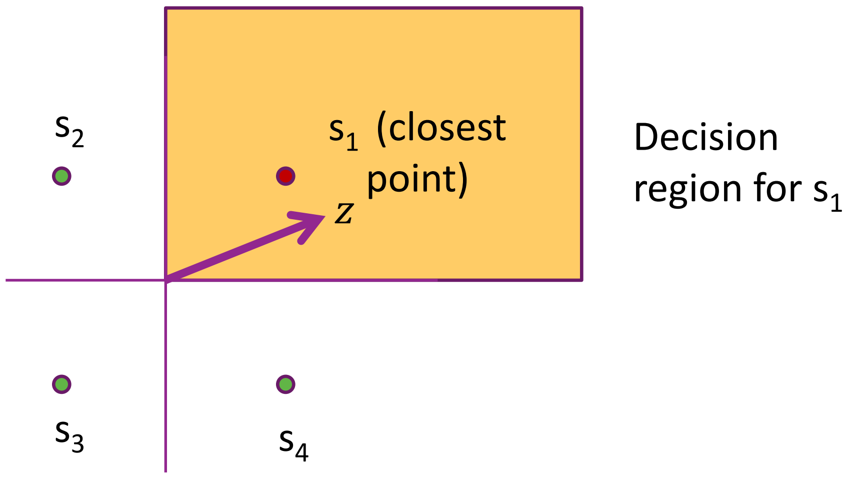
$$p(r|s) = \frac{1}{\pi\, N_0} \exp\left(-\frac{|r - hs|^2}{N_0}\right)$$

Equalization and Nearest Symbol Detection

- Likelihood: $p(r|s) = \frac{1}{\pi N_0} \exp\left(-\frac{|r-hs|^2}{N_0}\right)$
- MLE is: $\hat{s} = \arg\max_s p(r|s) = \arg\min_s |r - hs|^2 = \arg\min_s |z - s|^2$
- Here, $z = \frac{r}{h}$ = equalized symbol.

- Procedure:
  - Step 1: Equalize the symbol: $z = \frac{r}{h}$
  - Step 2: Find $s = s_1, \ldots, s_M$ closest to z in the complex plane

Decision Regions



Example: Decision region in QPSK

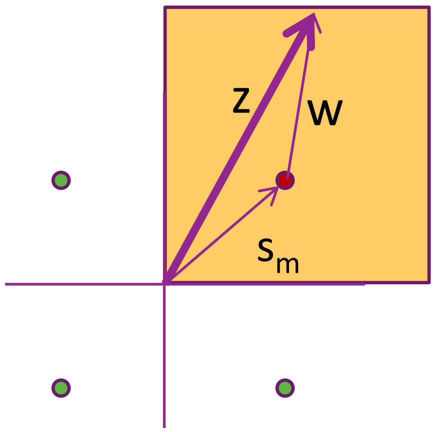
- ML estimate is closest point in constellation to z: $\hat{s} = \arg\min_i \|z - s_i\|$
- Decision region for a point s_m:
  - set of points r where s_m is the closest point: $D_m = \{\boldsymbol{r} | \hat{s} = s_m\}$

Error Probabilities on an AWGN Channel

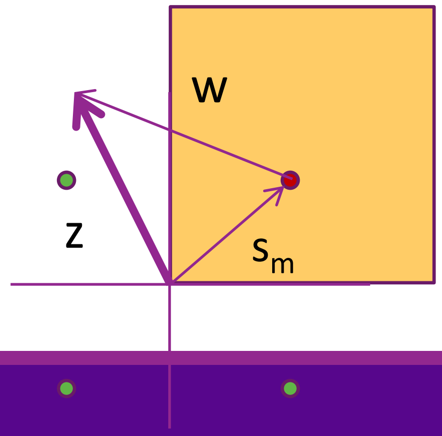
- Error probabilities:
  - Symbol error rate, SER: Prob symbol is misdetected
  - Bit error rate, BER: Probability of a bit is in error
  - Assume TX symbols are uniformly distributed
- First consider AWGN model: $z = s + v$
  - No fading
- SER for QPSK can be shown to be:

$$SER = 1 - \left(1 - Q(\sqrt{\gamma_s})\right)^2 \approx 2Q(\sqrt{\gamma_s})$$

  - SNR = $\gamma_s = \frac{E_s}{N_0} = \frac{E|s|^2}{E|v|^2}$



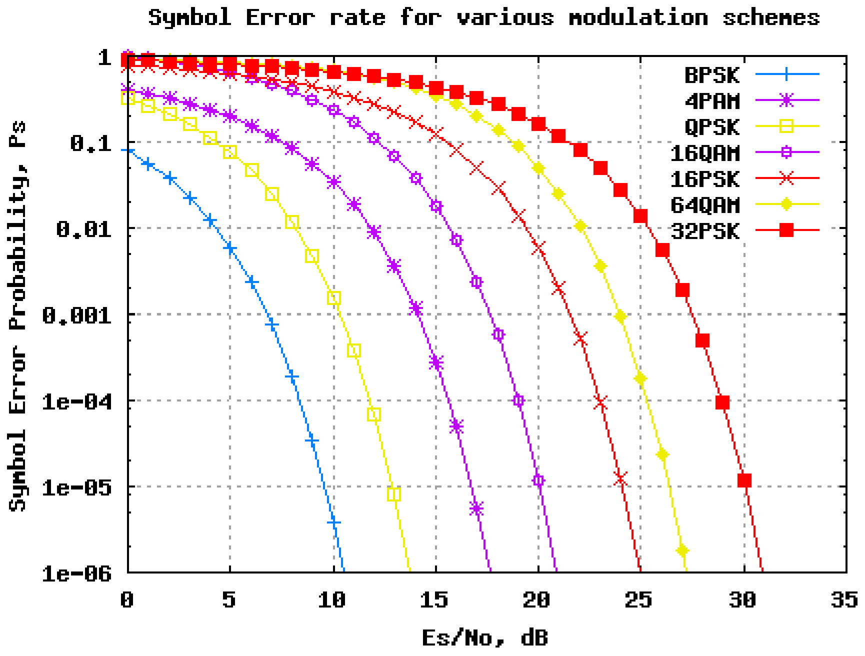
No error
z in correct decision region



Error
z not in
correct decision region

SER for AWGN Modulation

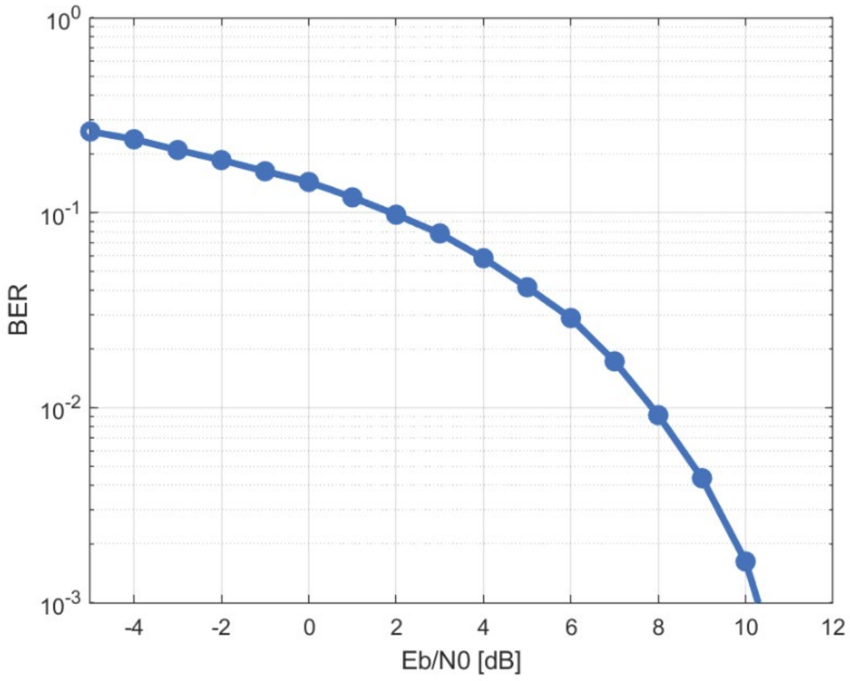
- Error formula can be derived for most QAM mappings
  - See, e.g., Proakis
- For an AWGN channel:
  - SER typically decays exponentially with SNR
  - Ex: for QPSK



Ex: BER Simulation for 16-QAM

```
% SNR levels to test
EbN0Test = (-5:11)';
ntest = length(EbN0Test);

% TX symbol energy
Es = mean(abs(s).^2);
ber = zeros(ntest,1);

for i = 1:ntest

    % Add the noise
    EbN0 = EbN0Test(i);
    chan = comm.AWGNChannel("BitsPerSymbol", bitsPerSym, 'EbNo', EbN0, ...
        'SignalPower', Es);
    r = chan.step(s);

    % Demodulate
    bitsEst = qamdemod(r,M,'UnitAveragePower',true,'Output','bit');

    % Measure the BER
    ber(i) = mean(bitsEst ~= bits);
    fprintf(1, 'EbN0=%7.2f BER=%12.4e\n', EbN0, ber(i));

    % Break if zero since higher SNRs also be zero
    if (ber(i) == 0)
        break
    end
end
```

- See demo
- Easy to do in MATLAB



SNR on a Fading Channel

- Now return to a fading channel:

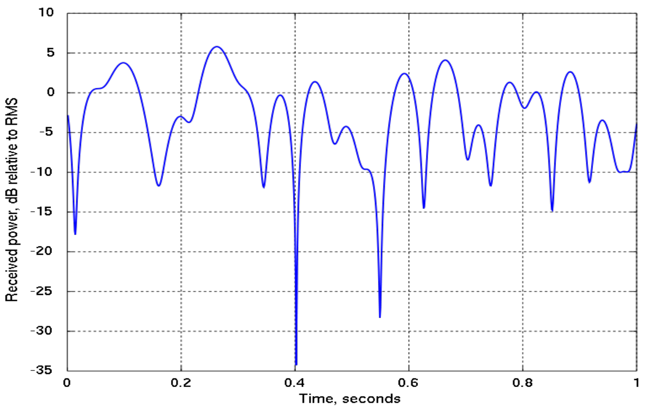
$$r = hs + w, \qquad w \sim CN(0, N_0),$$

- Equalization: $z = \frac{r}{h} = s + v,$
  - $v = \frac{w}{h}$ Effective noise after equalization
- SNR after equalization:
  - Noise energy after equalization:

$$E|v|^2 = \frac{1}{|h|^2} E|w|^2 = \frac{N_0}{|h|^2}$$

  - SNR is $\gamma_s = \frac{E|s|^2}{E|v|^2} = |h|^2 \frac{E_s}{N_0}$
  - SNR varies with the fading h
- Average SNR is: $\bar{\gamma}_s = E[\gamma_s] = E|h|^2 \frac{E_s}{N_0}$

SNR: $\gamma_s = \frac{|h|^2 E_s}{N_0}$



High SNR

Low SNR

Average SER on a Fading Channel

- Fading channel: $r = hs + w$
- With fading, SNR is random ,. SNR is $\gamma_s = |h|^2 \frac{E_s}{N_0}$
- Define the average SER:

$$\overline{SER}(\bar{\gamma}_s) = E[SER(\gamma_s)] = \int_0^\infty p(\gamma_s)\, SER(\gamma_s) d\gamma_s$$

  - A function of the average SER
  - Represents the average over independent channel realizations

- If h is Rayleigh distributed, γ_s is exponential with $\bar{\gamma}_s = E[\gamma_s] = E|h|^2 \frac{E_s}{N_0}$

$$\overline{SER}(\bar{\gamma}_s) = \frac{1}{\bar{\gamma}_s} \int_0^\infty e^{-\gamma_s/\bar{\gamma}_s}\, SER(\gamma_s) d\gamma_s$$

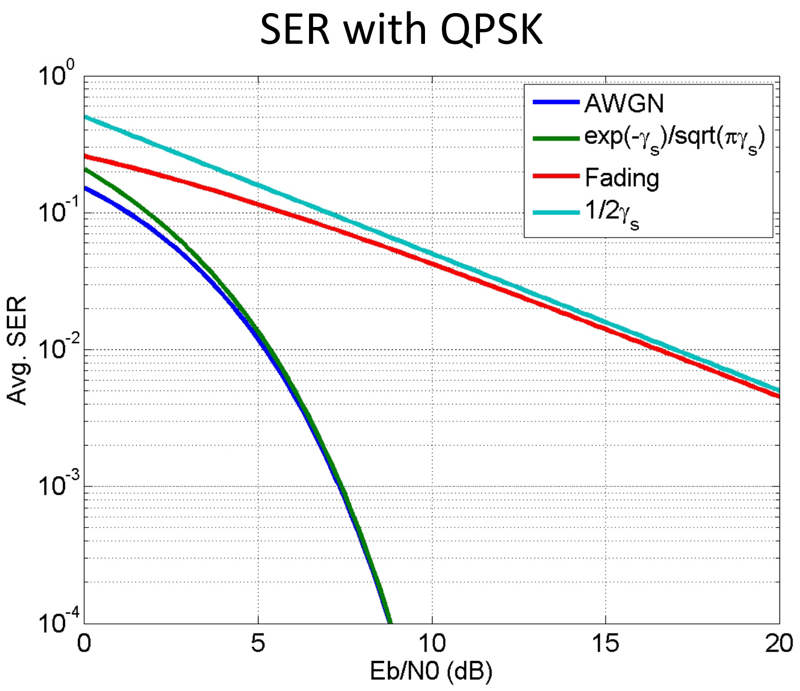
Example: SER on QPSK with Rayleigh Fading

- Rayleigh fading: γ_s is exponential $E(\gamma_s) = \bar{\gamma}_s$
- QPSK: $SER(\gamma_s) \approx 2Q\left(\sqrt{2\gamma_s}\right)$ for large γ_s
- Lemma: Suppose that γ is exponential $E(\gamma) = \bar{\gamma},\ E\left(Q(\sqrt{\alpha\gamma})\right) = \frac{1}{2}\left[1 - \sqrt{\frac{\alpha\bar{\gamma}}{2+\alpha\bar{\gamma}}}\right] \approx \frac{1}{2\alpha\bar{\gamma}}$
  - Detailed proof below. Write
- Average SER: From Lemma

$$\overline{SER} = E[SER(\gamma_s)] \approx \frac{2}{2(2)\bar{\gamma}} = \frac{1}{2\bar{\gamma}}$$

- Average SER decays as $\propto 1/\bar{\gamma}_s$
- In AWGN channel, SER decays as $Q\left(\sqrt{2\gamma_s}\right) \propto e^{-\gamma_s}$
- Much slower decay

Comparison of Fading vs. AWGN



- Error rate with fading is dramatically higher.
- Ex. for QPSK:
  - No fading, SER decays exponentially
  - With fading, SER decays with inverse SNR
- Similar relations for most other constellations
- Need much higher SNR

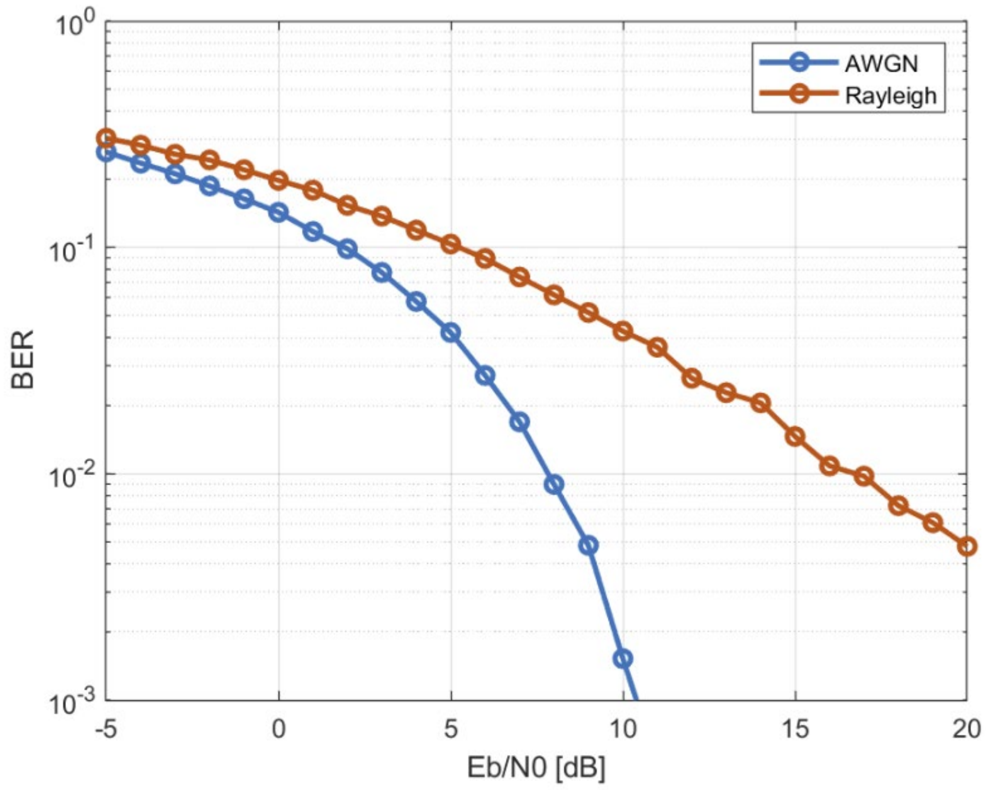
No fading

$$\overline{SER}(\bar{\gamma}_s) \approx \frac{e^{-\bar{\gamma}_s}}{\sqrt{\pi\bar{\gamma}_s}}$$

Rayleigh fading

$$\overline{SER}(\bar{\gamma}_s) \approx \frac{1}{2\bar{\gamma}_s}$$

16-QAM Example



- See demo
- Large gap between AWGN and Rayleigh

Lemma for Average of Q function

- Lemma: Suppose that γ is exponential $E(\gamma) = \bar{\gamma}$.

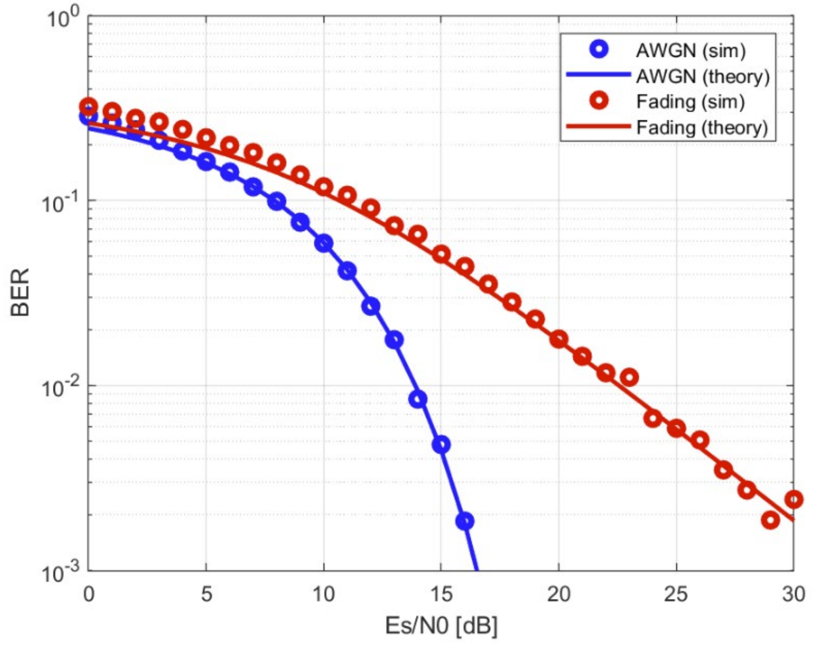
$$E\left(Q(\sqrt{\alpha\gamma})\right) = \frac{1}{2}\left[1 - \sqrt{\frac{\alpha\bar{\gamma}}{2 + \alpha\bar{\gamma}}}\right] \approx \frac{1}{2\alpha\bar{\gamma}}$$

- Proof:
  - $E\left(Q(\sqrt{\alpha\gamma})\right) = \frac{1}{\bar{\gamma}} \int_0^\infty Q(\sqrt{\alpha\gamma}) e^{-\gamma/\bar{\gamma}} d\gamma$
  - $Q(\sqrt{\alpha\gamma}) = \frac{1}{\sqrt{2\pi}} \int_{\sqrt{\alpha\gamma}}^\infty e^{-u^2/2} du$
  - Change order of integral

In-Class Exercise

Problem 1: Theoretical Error Rate Probability

Modify the code in the demo to compute the `BER` vs. `Es/N0` for 16-QAM for AWGN and fading channel. (Recall the demo measured the `BER` vs. `Eb/N0`).



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- ❑Coding over Fading Channels
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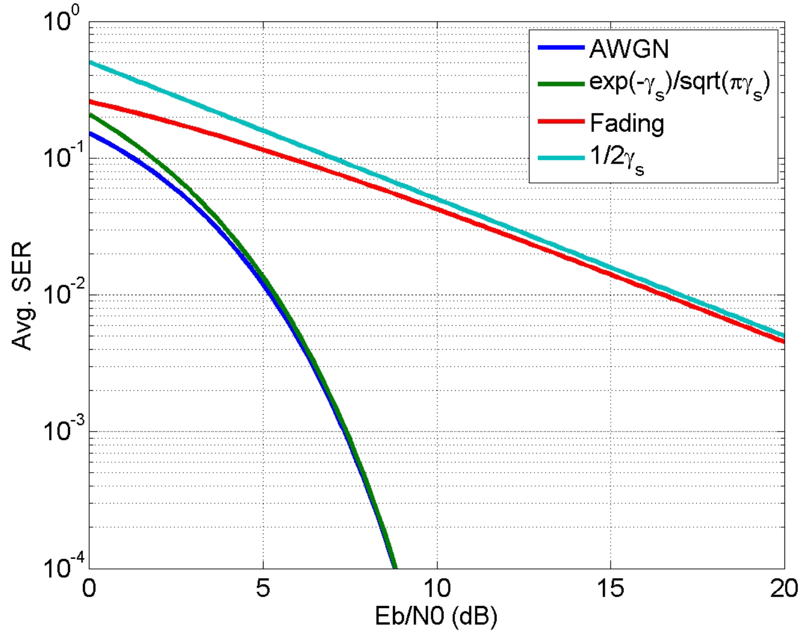
Coding Over Fading Channels

- Lesson from previous section:
  - With fading, uncoded modulation cannot provided sufficient reliability
  - Error rate decays slowly with SNR

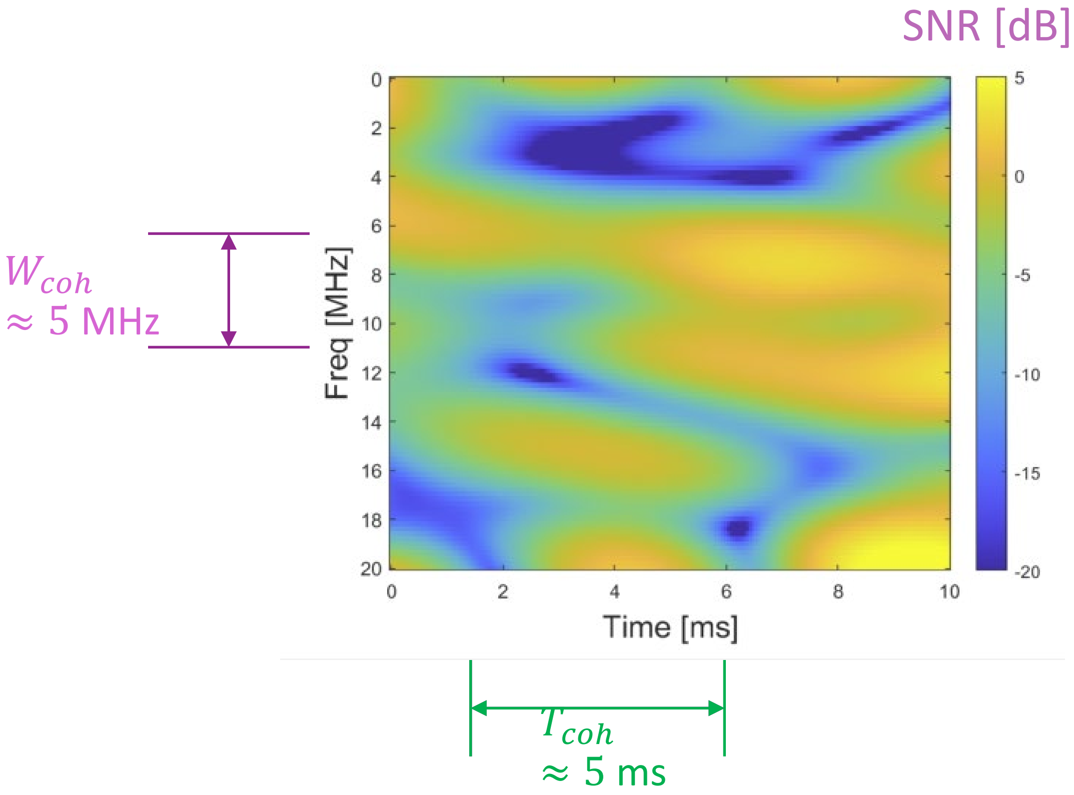


- Channel coding:
  - Send data in blocks
  - Block contains redundancy
  - If some parts fade, can still recover block

- All commercial wireless systems use coding!

Time and Frequency Fading

- Recall: Channels vary over time and frequency
- Variation in time:
  - Due to Doppler spread, δf
  - Coherence time, $T_{coh} \approx \frac{1}{2\delta f}$
  - Time over which channel changes significantly
- Variation in frequency
  - Due to delay spread $\delta\tau$
  - Coherence bandwidth, $W_{coh} \approx \frac{1}{2\delta\tau}$
  - Frequency over which channel changes significantly



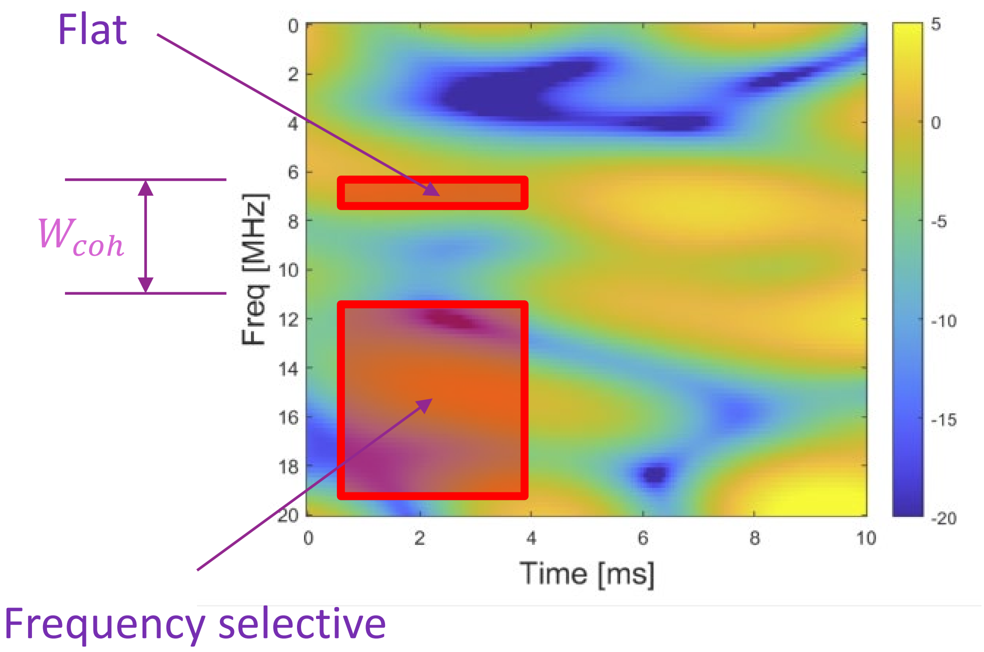
20 path random channel with
$\delta f = 100$ Hz, $\delta\tau = 100$ ns

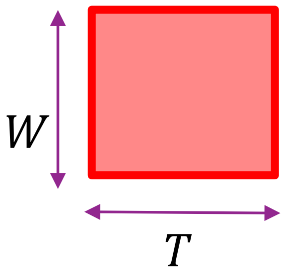
Flat vs. Frequency-Selective Fading

- Suppose we transmit a coding block
  - T in time and W in bandwidth
  - $T \times W$ region in time and frequency

- Flat fading: $W \ll W_{coh}$
  - Channel does not vary in frequency over coding block

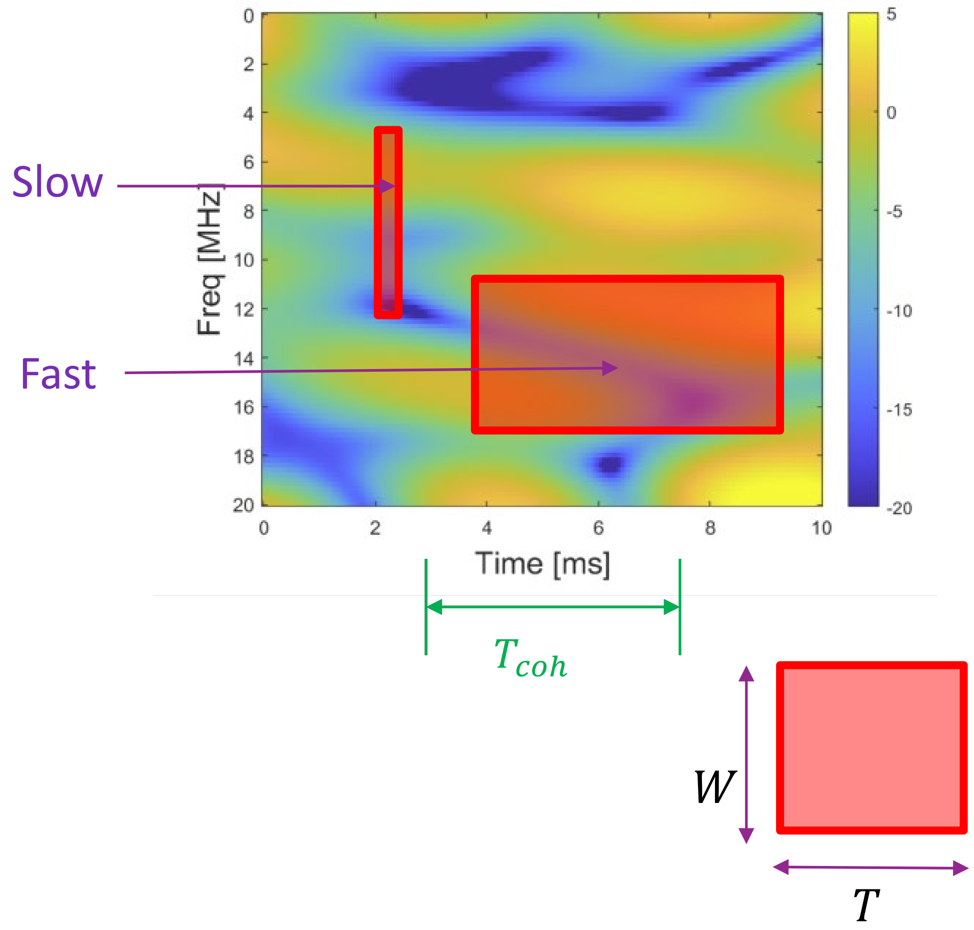
- Frequency selective fading: $W \geq W_{coh}$
  - Channel varies over frequency





Slow vs. Fast Fading

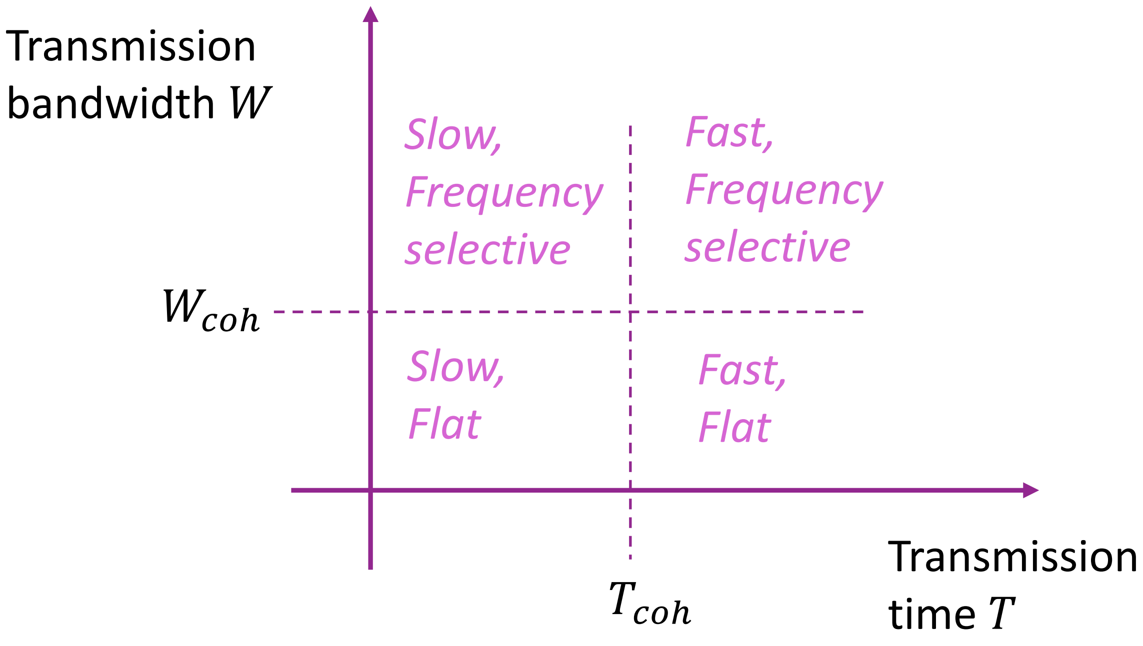
- Suppose we transmit a coding block
  - T in time and W in bandwidth
  - $T \times W$ region in time and frequency

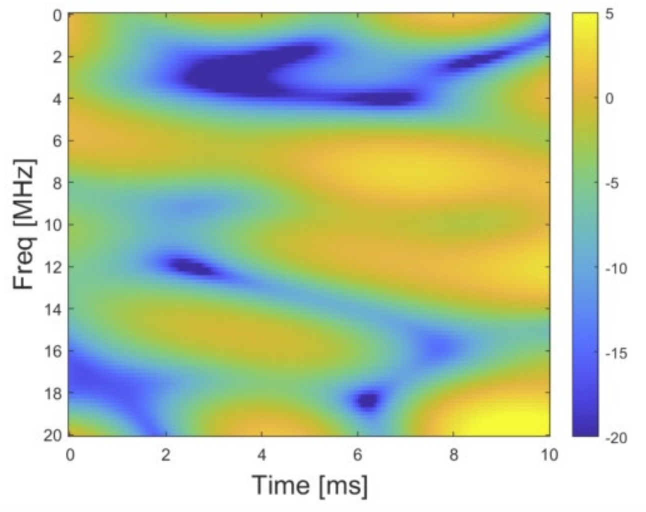


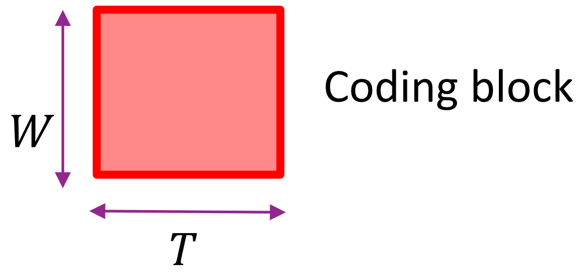
- Slow fading: $T \ll T_{coh}$
  - Channel does not vary in time over coding block

- Fast fading: $T \geq T_{coh}$
  - Channel varies over frequency

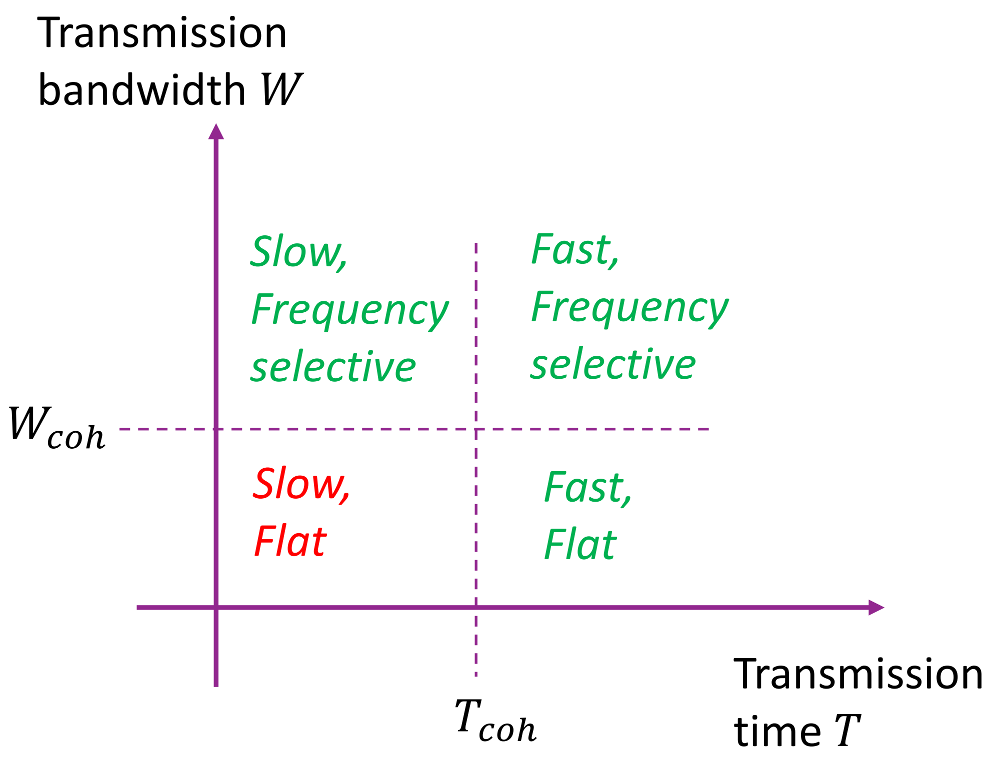
Summary: Four Regimes







Regimes to Model Coding



- To analyze fading, consider two extreme cases

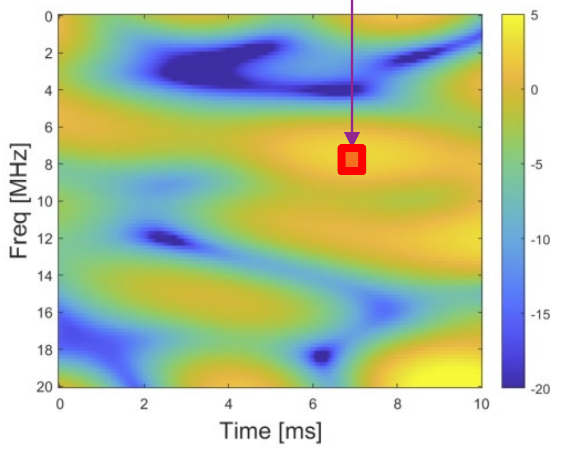
- Flat and slow fading over coding block:
  - Channel is flat and slow fading over coding block
  - All symbols see approximately same fading

- IID fading in coding block:
  - Channel fades in time and/or frequency over block
  - Fast and/or frequency selective
  - Model as large number of independent fades

Analysis of Coding with Flat and Slow Fading

- Coding block sees an SNR γ
- SNR γ varies but is constant over each block
  - Transmission time $\ll$ Coherence time
  - Transmission bandwidth $\ll$ Coherence bandwidth
- Suppose code has some target SNR γ_{tgt}
  - Target could be based on some block error probability
- Assume γ has some distribution
- Outage probability: $\mathrm{P}_{out} = P(\gamma \le \gamma_{tgt})$
  - The fraction of time target is not met
  - Can be computed from the distribution of γ

Coding block



Outage Probability for Rayleigh Fading

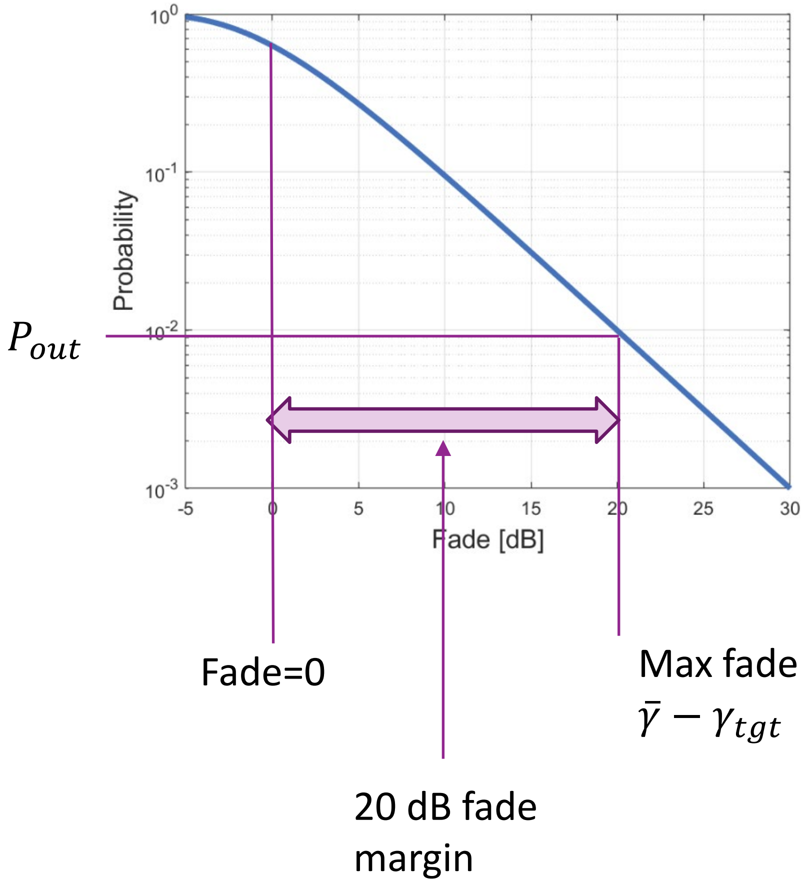
- Suppose a channel is Rayleigh fading
- SNR γ is exponentially distributed with some mean $\bar{\gamma}$
- Outage probability: $P_{out} = P(\gamma < \gamma_{tgt}) = 1 - e^{-\frac{\gamma_{tgt}}{\bar{\gamma}}}$
- Average SNR for a given outage probability: $\bar{\gamma} = -\frac{\gamma_{tgt}}{\ln(1-P_{out})}$
- Fade margin: Additional SNR needed above target for a given outage probability:
  - In linear scale: $\frac{\bar{\gamma}}{\gamma_{tgt}} = -\frac{\gamma_{tgt}}{\ln(1-P_{out})} \approx \frac{1}{P_{out}}$
  - In dB: $\bar{\gamma} \approx \gamma_{tgt} - 10\log_{10}(P_{out})$

Fade Margin Example

- Example:
  - Target SNR is $\gamma_{tgt} = 10$ dB
  - Outage probability: $P_{out} = 0.01$
- From previous slide, necessary average SNR is:

$$\bar{\gamma} \approx \gamma_{tgt} - 10\log_{10}(P_{out})$$
$$= 10 - 10\log_{10}(0.01) = 30 \text{ dB}$$

- The average SNR needs to be 20 dB above target!
- Plot: Fade margin vs. outage
- Fade margins with Rayleigh fading can be enormous!



Outage Capacity

- Suppose we can achieve some rate $R(\gamma)$ as a function of SNR γ
- When SNR γ is random, so is the rate $R(\gamma)$
- Outage capacity: Rate, R_{out}, we can achieve with a probability P_{out}

$$P_{out} = P(R(\gamma) \le R_{out})$$

- Example:
  - Suppose system has 20 MHz bandwidth and the rate is 60% of Shannon capacity
  - The average SNR is 20 dB.
  - What is the outage capacity for 1% outage assuming Rayleigh fading?
- Solution:
  - From earlier, for Rayleigh fading, the SNR achievable at the outage probability is

$$\gamma \approx \bar{\gamma} + 10\log_{10}(P_{out}) = 20 + 10\log_{10}(0.01) = 20 - 20 = 0$$

  - In linear scale, $\gamma = 1$
  - Outage capacity: $R_{out} = 0.6(20)\log_2(1+1) = 12$ Mbps
  - At the average SNR the rate is $R = 0.6(20)\log_2(1+100) = 80$ Mbps

System Implications for Outage

- With flat and slow Rayleigh fading, need to add large fade margin
- Channel coding does not mitigate fading
  - Fading causes all bits to fail
  - Still may be useful to use channel coding (e.g., for noise across the symbols)
- Possible solutions?
  - If there is motion, perhaps we can retransmit later
  - Go to a lower rate (needs less SNR)
  - Just accept that some locations are in outage
- Some of these solutions are discussed in the next unit

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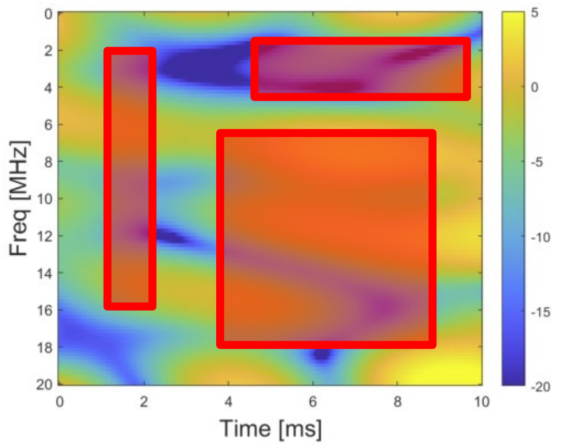
IID Fading Model

- Coding block with fast and/or frequency selective fading
- Simple mathematical model:

$$r[n] = h[n]s[n] + w[n], \qquad n = 1, \ldots, N$$

  - Each $r[n]$ is a symbol in time and frequency
  - Assume channel gains $h[n]$ are i.i.d. with some distribution
  - $w[n] \sim CN(0, N_0)$ and $E|s[n]|^2 = E_s$
  - Each symbol experiences an SNR $\gamma_s[n] = \frac{|h[n]|^2 E_s}{N_0}$
  - Number of symbols $N \to \infty$
- Assumption implicitly assumes:
  - We have a very large coding blocks in time or frequency
  - Can experience many independent fades

Coding blocks with
fast and/or frequency selective fading

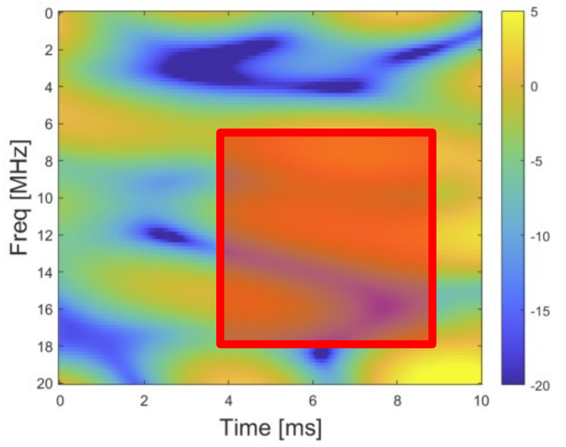


Ergodic Capacity

- IID fading model: $r[n] = h[n]s[n] + w[n],\ \ w[n] \sim CN(0, N_0)$
  - Channel gains $h[n]$ are i.i.d. with some distribution
- Ergodic capacity: Theoretical maximum rate per symbol
  - Assume average transmit power limit $E|s[n]|^2 = E_s$
  - Maximum taken over all codes and blocklength
  - No computational limits
- Theorem: Ergodic capacity of an IID fading channel is:

$$\boxed{C = E[\log(1+\gamma)], \qquad \gamma = \frac{|h|^2 E_s}{N_0}}$$

  - Value is in bits per symbol
  - Expectation is over channel distribution h

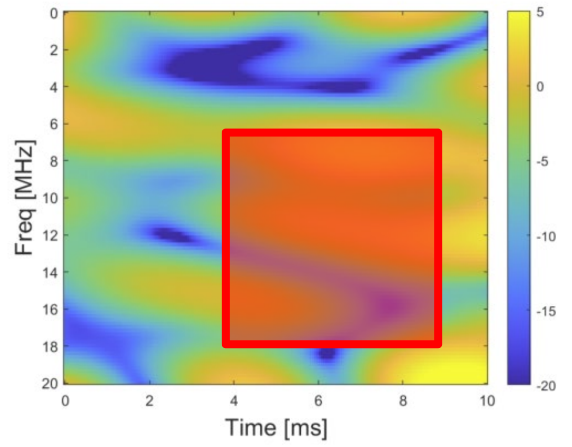


Shannon Ergodic Capacity Key Remarks

- From previous slide, ergodic capacity is:

$$C = E[\log(1+\gamma)], \qquad \gamma = \frac{|h|^2 E_s}{N_0}$$

- Theoretical result: Needs infinite computation and delay
  - We will look at performance of real codes next
- TX does not need to know channel h!
  - But RX must estimate and use this channel.
  - We will see RX design is critical
- If TX knew the channel, it could get theoretically get slightly higher rate
  - Uses a method called water-filling
  - Place more power on symbols with better SNR.
  - Gain is not typically large and rarely used in practical wireless systems

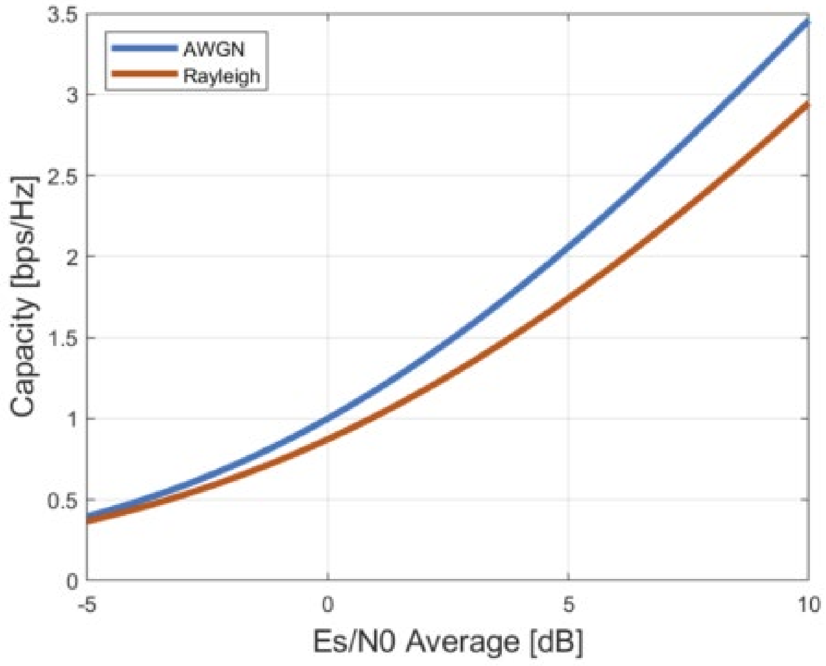


Comparing Ergodic and Flat Capacity

- Fading capacity is always lower than flat fading
  - Keeping the same average SNR the same
  - This fact follows from Jensen's inequality:

$$C = E[\log(1+\gamma)] \leq \log\big(1+E(\gamma)\big) = C_{AWGN}$$

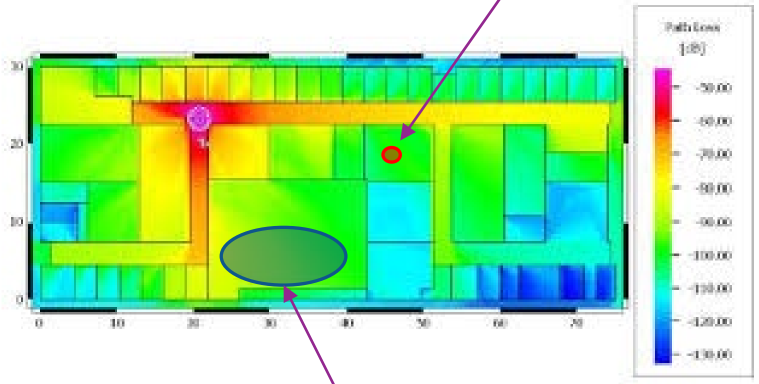
- But gap is not that large at low to moderate SNRs
  - See graph to the right. Loss of only 1-2 dB in
- Conclusions:
  - We should try to code over large number of fading realizations
  - In this case, the capacity loss is theoretically small
  - Much better than the case of uncoded modulation
- We will look at practical codes next



Small-Scale and Large-Scale Fading

- Up to now we have considered variations due to small scale fading
  - Variations from constructive or destructive interference of multipath components
  - May or may not cause variations within a coding block
  - Ex: Variations within a few wavelength in one location in the office area

Small scale region



Large scale region

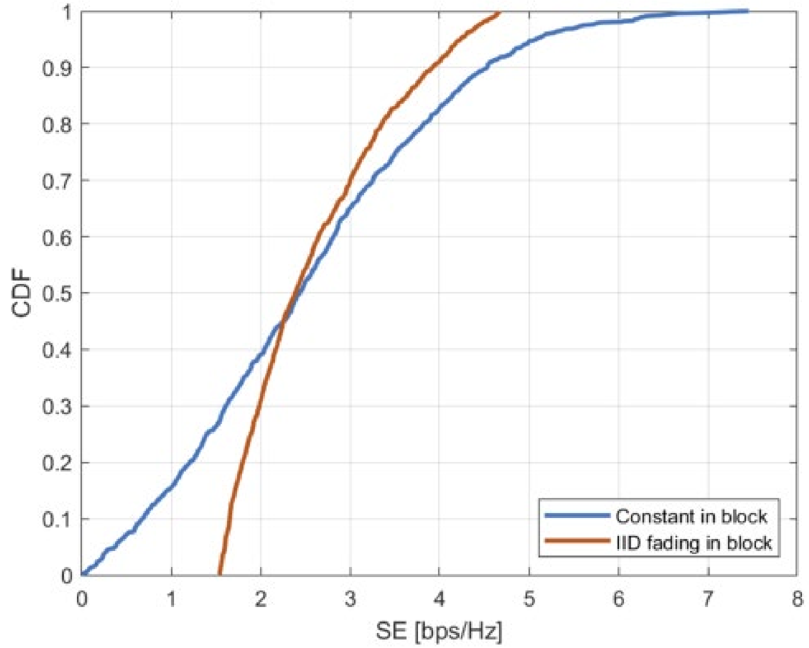
- Most scenarios also have variations due to large scale fading:
  - Changes in distance-based path loss, shadowing, angles, …
  - Typically occur at slower time scales (100s of ms or more)
  - Rarely causes variation over a coding block (typically 10s of ms or less)
  - Ex: Moving within the office space to the right

Analysis with Small- and Large-Scale Fading

- Suppose SNR varies as $\gamma(u, v)$:
  - u: Vector of large-scale parameters, e.g., distance, angles, shadowing, etc.
  - v: Small-scale parameters, e.g., time-frequency location of a degree of freedom
- If fading in each coding block is slow and flat:
  - Each coding block has an SNR $\gamma(u, v)$
  - Can compute outage probability: $P(\gamma(u, v) \leq \gamma_{tgt})$
  - Probabilities computed over small-scale and large-scale parameters
- If fading in coding block can be modeled as large number of i.i.d. samples of v:
  - Ergodic capacity is $R(u) = E_v(\log(1 + \gamma(u, v)))$
  - Take average over small-scale, but NOT large scale
  - Rate is a function of u
  - Then outage probability is: $P(R(u) \leq R_{tgt})$

Example: Rate CDF Calculation

- Large-scale model:
  - SNR due to large scale variations $\bar{\gamma}(d) = \gamma_0 \left(\frac{d_0}{d}\right)^2$ [Simple model just for exercise]
  - γ_0= 10 dB and d_0 =100 m
  - Distances vary d uniformly in [50,200]m
- Small scale model:
  - Variation within a location is Rayleigh
  - SNR at a particular time-freq DoF will be $\gamma = \bar{\gamma}(d)v$
  - v can be modeled as exponential
- Plotted:
  - SE under a constant model (slow and flat fading)
  - SE under IID fading at each distance d
- See demo



In-Class Exercise

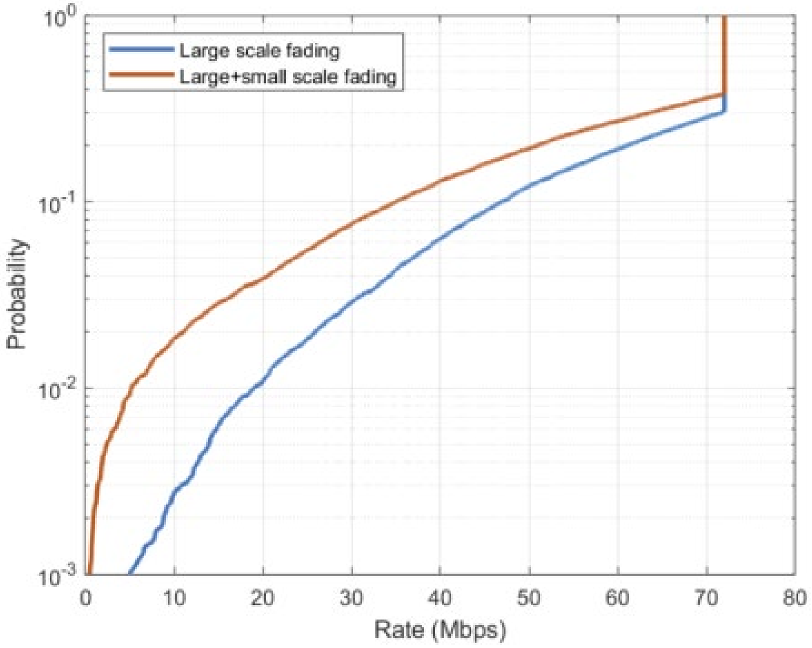
Problem 2: Outage Capacity in an Indoor Environment

In this problem, we will estimate the outage capacity in an indoor setting. Our goal is to look at the effects of both large-scale and small-scale fading.

First generate `nx` locations in a box of size 30 x 40 m representing locations in some large indoor environment. Assume an access point is located at the origin and compute a vector, `dist`, representing the distance in meters from the AP to each location.

```
% Parameters
nx = 10000;
xmax = [30,40];
d = 2;
```

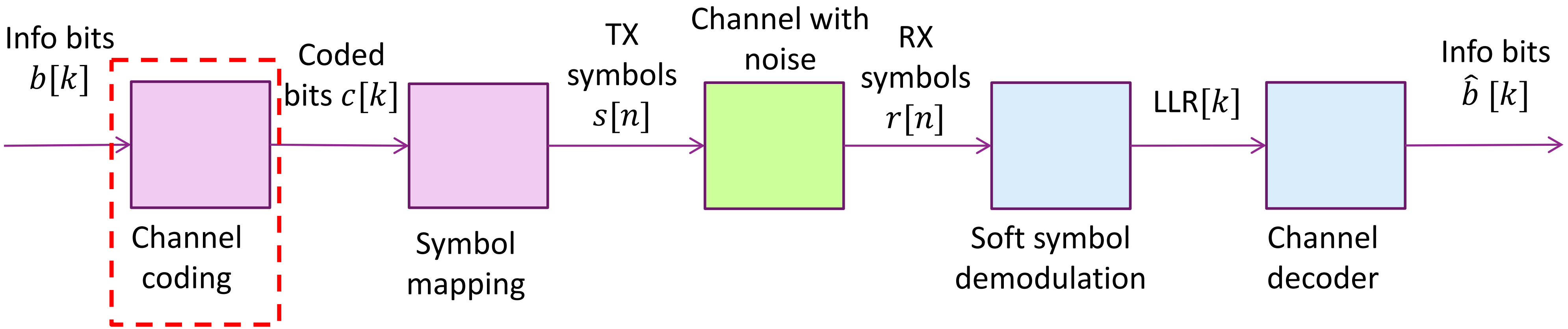
- Indoor environment
- Look at large scale and small-scale fading



Outline

- ❑Uncoded Modulation over Fading Channels
- ❑Capacity with Coding over Fading Channels: Outage Capacity
- ❑Capacity with Coding over Fading Channels: Ergodic Capacity
- ❑Review: Coding over an AWGN Channel
- ❑Coding over Fading Channels
- ❑Capacity with Bit-Interleaved Coded Modulation

Coded Communication on an AWGN Channel

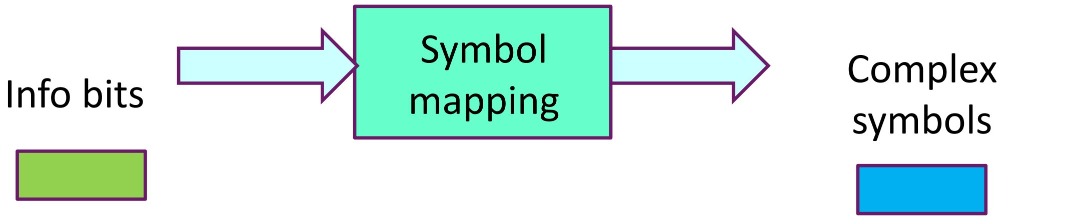


- We first review channel coding on a flat channel:

$$r[n] = s[n] + w[n], \qquad w[n] \sim CN(0, N_0)$$

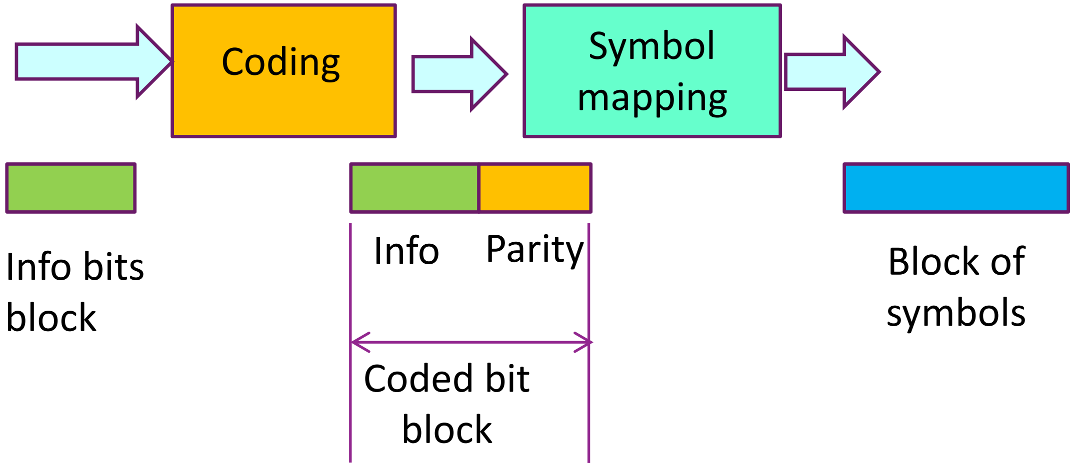
- All details can be found in the digital communications class

Uncoded vs. Coded Modulation



Uncoded Modulation:

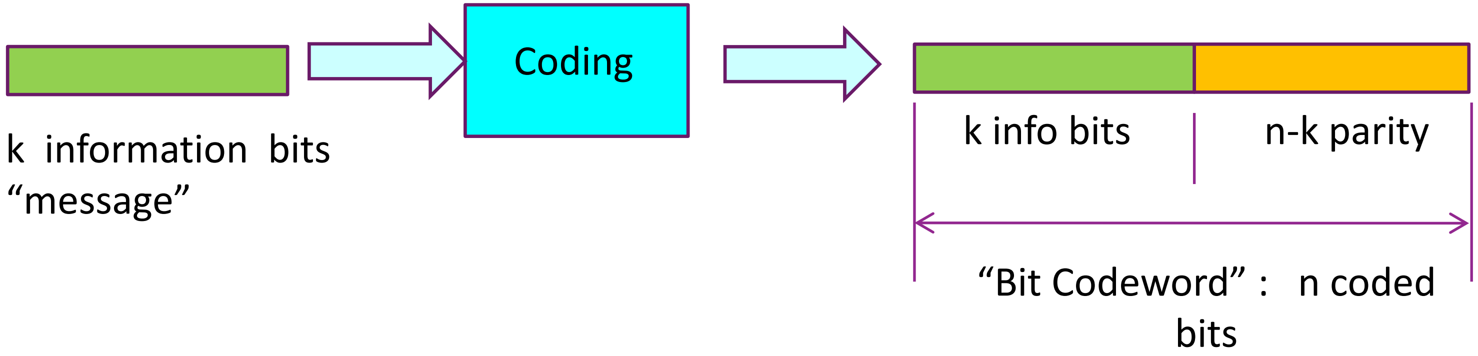
- Modulate raw information bits
- One symbol at a time.
- Any symbol is in error,   data packet is lost!



Coded modulation:

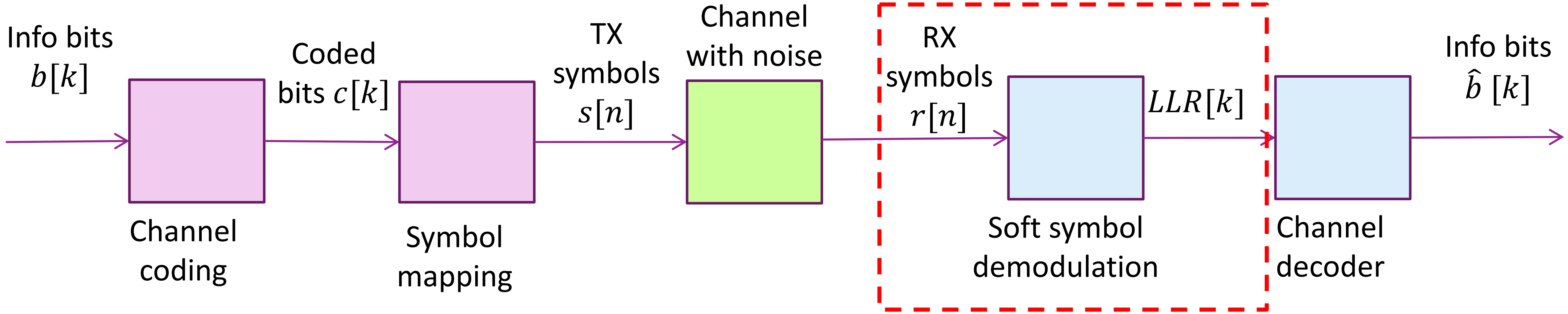
- Transmit in blocks (also called frames)
- Add extra parity bits to each block for reliability
- Decode entire block together

Key Parameters of Block Codes



- An (n, k) block code has:
  - k = number of information bits (input block size)
  - n = number of coded bits (output block size)
  - $n - k$ = number of additional bits, typically parity
  - R_{cod} = coding rate = k/n.
- Typical values in wireless:
  - Block size: $k = 100$ to 10000
  - Code rate: $\frac{1}{3}$ to $\frac{5}{6}$

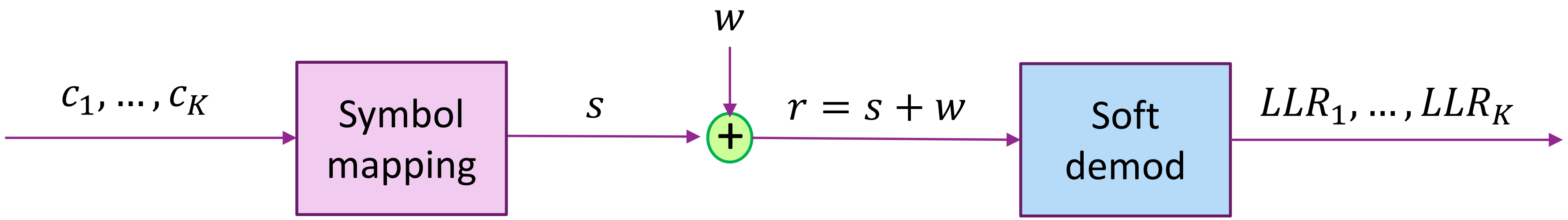
Coded Communication on an AWGN Channel



- We first review channel coding on a flat channel:

$$r[n] = s[n] + w[n], \qquad w[n] \sim CN(0, N_0)$$

Soft Symbol Demodulation

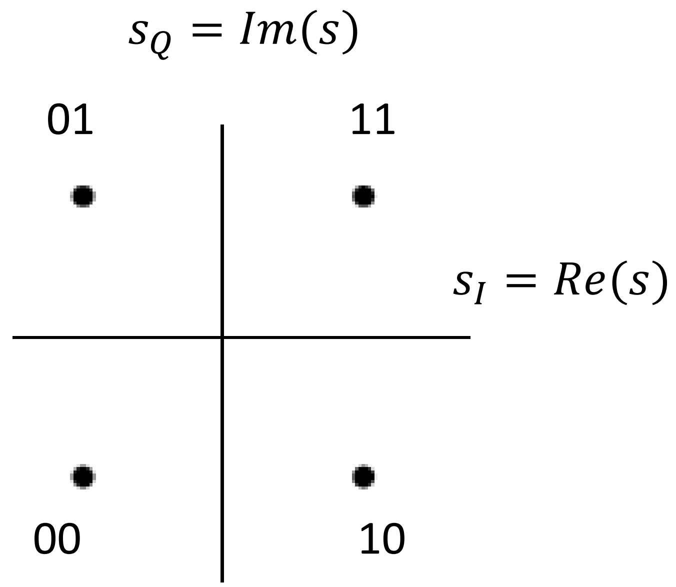


- Set-up: Coded bits $(c_1, \dots, c_K)$ get mapped to symbol s
  - Receive $r = s + w, \quad w \sim CN(0, N_0)$
- Uncoded systems use hard decision detection:
  - Estimate bits $(\hat{c}_1, \dots, \hat{c}_K)$ from symbol s
  - Makes a discrete decision.
- Coded systems generally use soft decision demodulation:
  - Output log likelihood ratios: $LLR_k = \ln \frac{P(r|c_k=1)}{P(r|c_k=0)}$
  - LLR_k positive $\Rightarrow c_k = 1$ more likely
  - LLR_k negative $\Rightarrow c_k = 0$ more likely

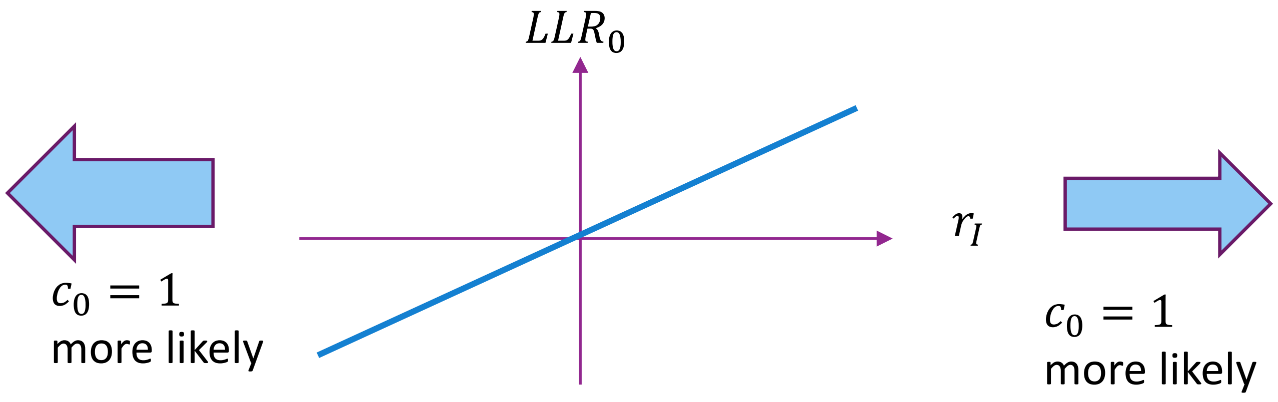
LLR for QPSK

- TX symbol: $s = \pm A \pm iA,\ A = \sqrt{\frac{E_s}{2}}$
- RX symbol: $r = s + w,\ \ w \sim CN(0, N_0)$
- LLR for bit c_0
  - $s_I = Re(s) = \begin{cases} A & c_0 = 1 \\ -A & c_0 = 0 \end{cases}$
  - $r_I = s_I + w_I,\ \ w_I \sim N(0, \frac{N_0}{2})$ [Each dim has $\frac{N_0}{2}$]
  - Likelihood: $p(r_I|s_I) = \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{1}{N_0}(r_I - s_I)^2\right]$
  - $LLR_0 = \ln\frac{p(r_I|c_0 = 1)}{p(r_I|c_0 = 0)} = \ln\frac{p(r_I|s_I = A)}{p(r_I|s_I = -A)}$
  - With some algebra: $LLR_0 = \frac{4Ar_I}{N_0} = \frac{4}{N_0}\sqrt{\frac{E_s}{2}}\, r_I$

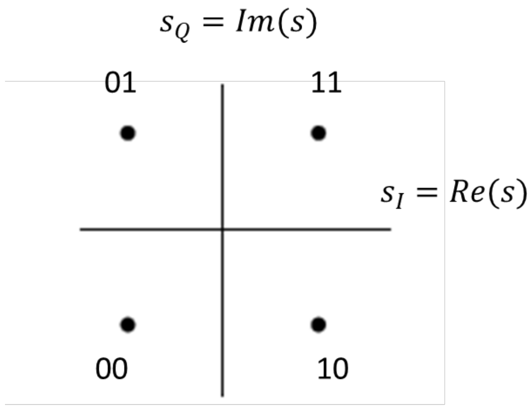
Mapping of bits (c_0, c_1)



QPSK LLR Visualized



Mapping of bits (c_0, c_1)

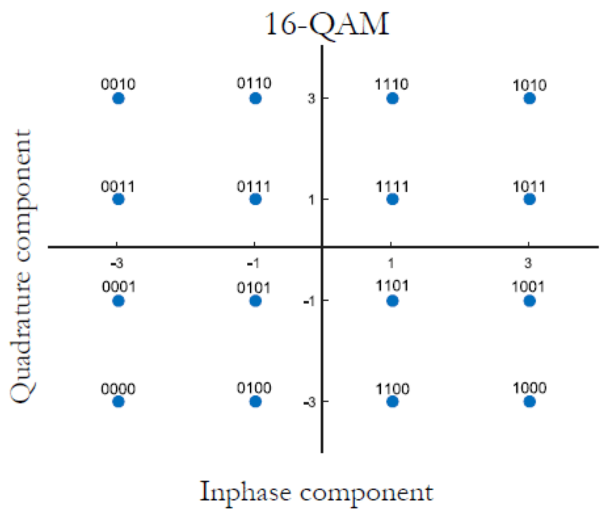


- LLR for c_0 is: $LLR_0 = \frac{4}{N_0}\sqrt{\frac{E_s}{2}}r_I$
- LLR for c_1 is: $LLR_1 = \frac{4}{N_0}\sqrt{\frac{E_s}{2}}r_Q$

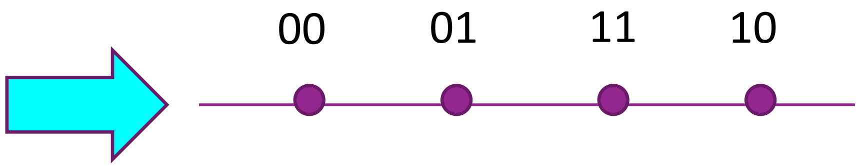
High Order Constellations

- Higher order constellations (eg. 16- or 64-QAM)
- Each constellation r is a point is a function of multiple bits.
- Example: For 16-QAM
  - In phase dimension r_I depends on bits (c_0, c_1)
- Cannot compute LLR on an individual bit directly

Mapping of bits (c_1, c_2, c_3, c_4)



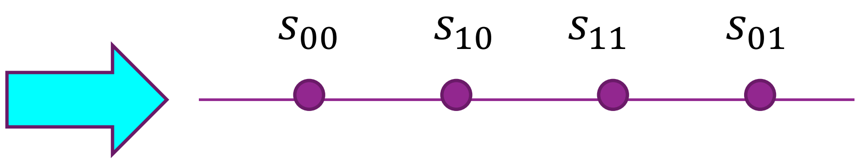
Two bits:
(c_1, c_2)



$r = s + n$

High Order Constellations

Two bits:
(c_1, c_2)



$$r = s + n$$

- To create LLRs for individual bits use total probability rule:

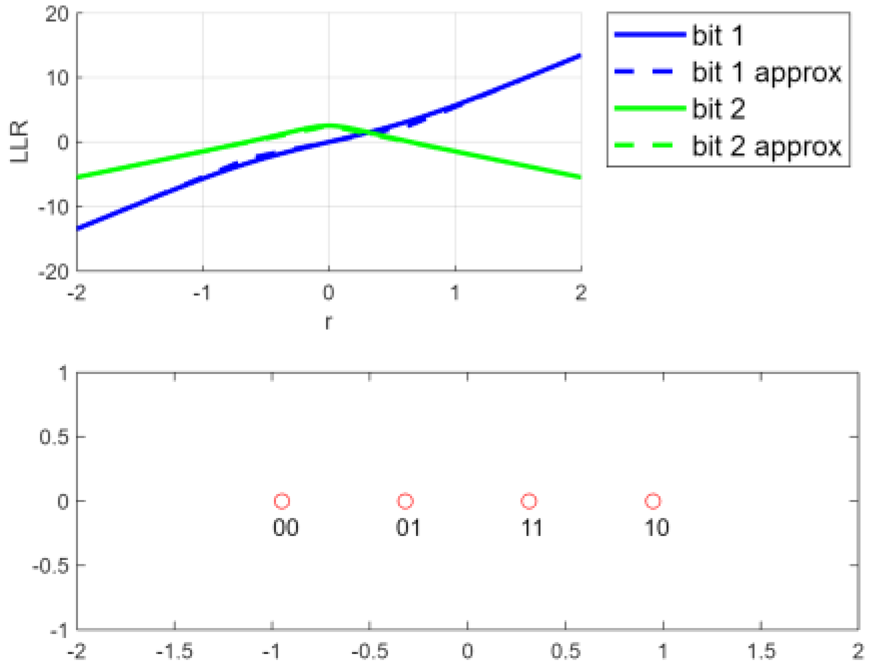
$$p(r|c_1) = \frac{1}{2}\left(p(r|c_1, c_2 = 0) + p(r|c_1, c_2 = 1)\right)$$

- Resulting bitwise LLR:

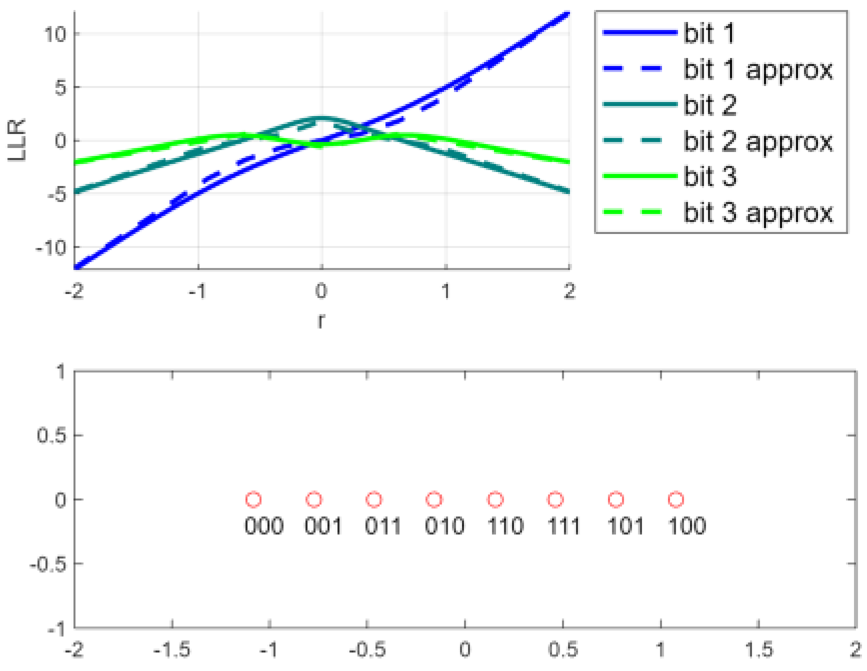
$$LLR \text{ for } c_1 = \log \frac{p(r|c_1, c_2 = 1,0) + p(r|c_1, c_2 = 1,1)}{p(r|c_1, c_2 = 0,0) + p(r|c_1, c_2 = 0,1)}$$

High Order Constellation Examples

2 bits / dim (16-QAM)



3 bits / dim (64-QAM)



Approximate Bitwise LLR

- Exact LLR:

$$LLR_k = \log \frac{P_1}{P_0}, \qquad P_1 = \sum_{s:bit\ k=1} e^{-(r-s)^2/N_0}, \quad P_0 = \sum_{s:bit\ k=0} e^{-(r-s)^2/N_0}$$

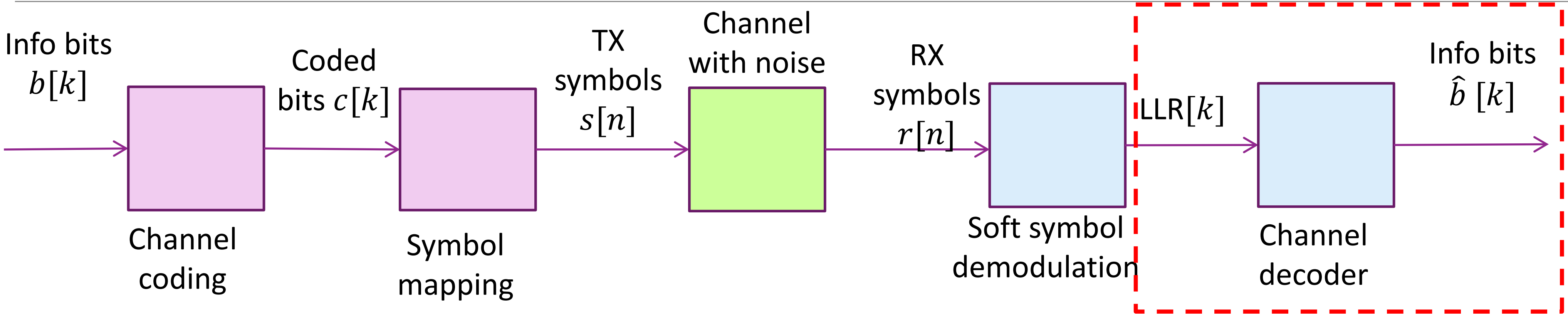
  - Can be too computationally expensive

- Approximate LLR. For N_0 small:

$$LLR_k \approx \frac{1}{N_0}(D_0 - D_1), \qquad D_1 = \min_{s:bit\ k=1} (r-s)^2, \qquad D_0 = \min_{s:bit\ k=0} (r-s)^2$$

  - Typically, very close to exact LLR
  - Computationally simpler. Avoid exponentials and logarithms

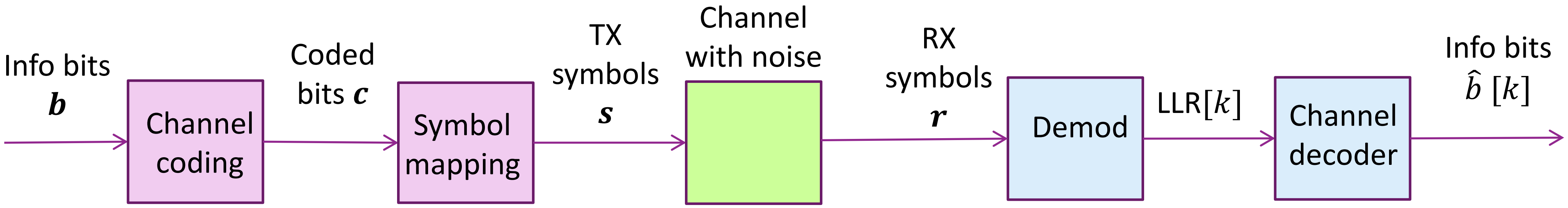
Coded Communication on an AWGN Channel



- We first review channel coding on a flat channel:

$$r[n] = s[n] + w[n], \qquad w[n] \sim CN(0, N_0)$$

Maximum Likelihood Channel Decoding

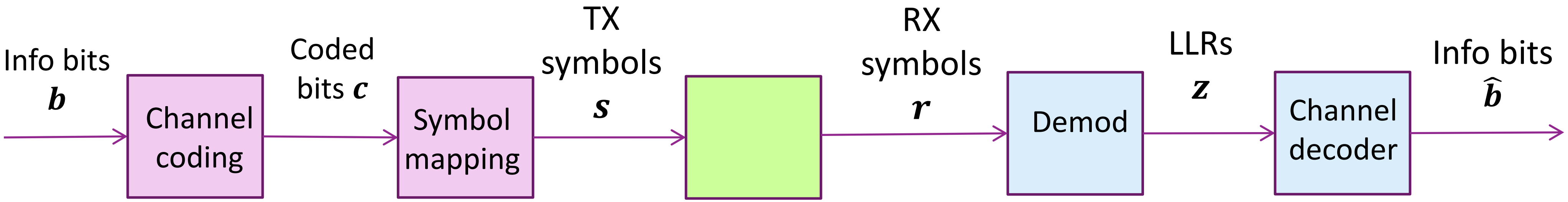


- Channel coding: Information block: $\boldsymbol{b} = (b_1, \dots, b_K)$ generates a codeword $\boldsymbol{c} = (c_1, \dots, c_N)$
- Receiver gets a vector $\boldsymbol{r} = (r_1, \dots, r_L)$, $L =$ number of complex modulation symbols
- Channel decoder: Goal is to estimate $\boldsymbol{b}$ (or equivalently $\boldsymbol{c}$) from $\boldsymbol{r}$.
- Ideally will use maximum likelihood decoding:

$$\hat{\boldsymbol{c}} = \arg\max_{\boldsymbol{c}} \log p(\boldsymbol{r}|\boldsymbol{c})$$

  - Finds the codeword that is most likely given the receive vector

Decoding from LLRs



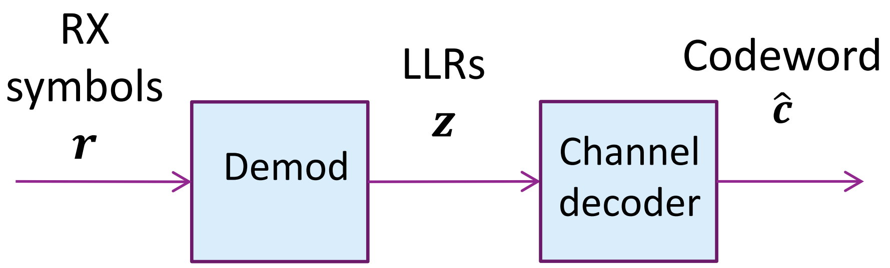
- LLRs sent to decoder: $z_i = \log \frac{P(\boldsymbol{r}|c_i = 1)}{P(\boldsymbol{r}|c_i = 0)}$
- Decoder typically approximates: $\hat{\boldsymbol{c}} = \arg\max_{\boldsymbol{c}} \log p(\boldsymbol{z}|\boldsymbol{c})$
  - The codeword most likely to produce the observed LLRs
  - Works on LLRs not RX symbols

Decoding from LLRs via Bitwise Approx

- Bitwise independent approximation: Assume

$$p(\boldsymbol{z}|\boldsymbol{c}) \approx \prod_{i=1}^{n} p(z_i|c_i)$$

  - Ignore dependencies between coded bits



- Under bitwise independent approximation, optimal decoder is:

$$\hat{\boldsymbol{c}} = \arg\max_{\boldsymbol{c}} p(\boldsymbol{z}|\boldsymbol{c}) = \arg\max_{\boldsymbol{c}} \sum_{i=1}^{n} c_i z_i$$

  - $z_i = \log \frac{p(r|c_i = 1)}{p(r|c_i = 0)}$ LLR for coded bit i
  - Proof in digital communications class

Decoding Complexity

- Channel decoding ideally selects codeword

$$\hat{\boldsymbol{c}} = \arg\max_{\boldsymbol{c}} \sum_{n=1}^{N} c_n LLR_n$$

- Brute force optimization is exponentially difficult:
  - Suppose the information block is $\boldsymbol{b} = (b_1, \ldots, b_K)$
  - Each $\boldsymbol{b}$ generates one codeword $\boldsymbol{c} = (c_1, \ldots, c_N)$
  - Optimization must, ideally, search over 2^K possible codewords $\boldsymbol{c}$
  - Computationally impossible

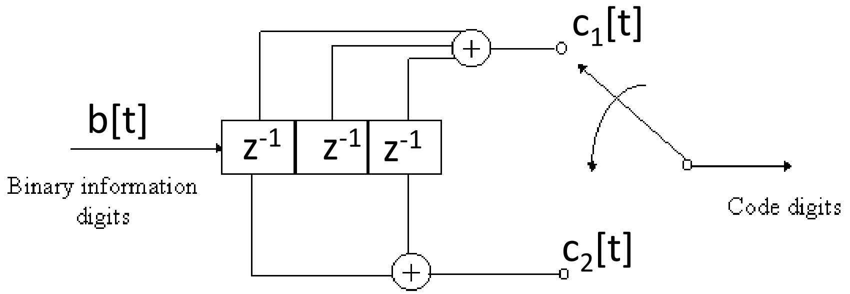
- Coding design requires searching over coding mechanisms with:
  - Computationally tractable decoding
  - But still have good performance

Quest for the Shannon Limit

- Shannon capacity formula and random codes, 1948.
  - Determines the capacity, but no practical code to achieve it.
- Hamming (7,4) code, 1950
- Reed-Solomon codes based on polynomials over finite fields, 1960
  - Used in Voyager program, 1977. CD players, 1982.
- Convolutional codes.
  - Viterbi algorithm, 1969. Widely used in cellular systems. (Viterbi later invents CDMA and founds Qualcomm)
  - Typically, within 3-4 dB of capacity
- Turbo codes, Berrou, Glavieux, Thitimajshima, 1993.
  - Able to achieve capacity within a fraction of dB.
  - Adopted as standard in all 4G and 5G cellular systems by the late 1990s.
- LDPC codes
  - Similar iterative technique as turbo codes. Re-discovered in 1996.
  - Used in 5G systems
- See digital communications class for many examples

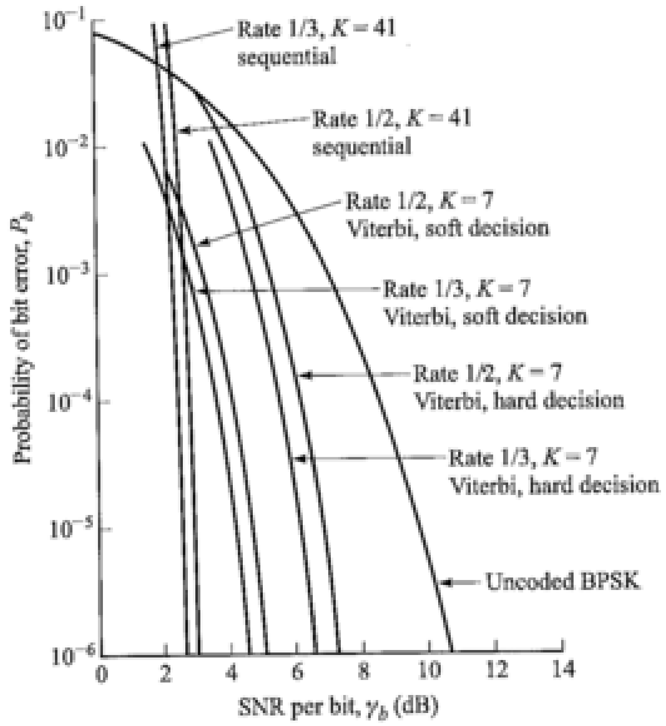
Convolutional Codes

- Encode data through parallel binary (usu. FIR) filters
- Example convolutional code:
  - Rate = ½ (two output bits ($c1[t], c2[t]$) for each input bit $b[t]$.
  - Constraint length *K=3* (size of shift register)
  - Additions are modulo two
- Benefits:
  - Easy to implement, good performance
  - Can be decoded with Viterbi algorithm
    Iterative procedure similar to dynamic programming procedure
  - See digital comm class for more details



$$c_1[t] = b[t] + b[t-1] + b[t-2]$$
$$c_2[t] = b[t] + b[t-2]$$

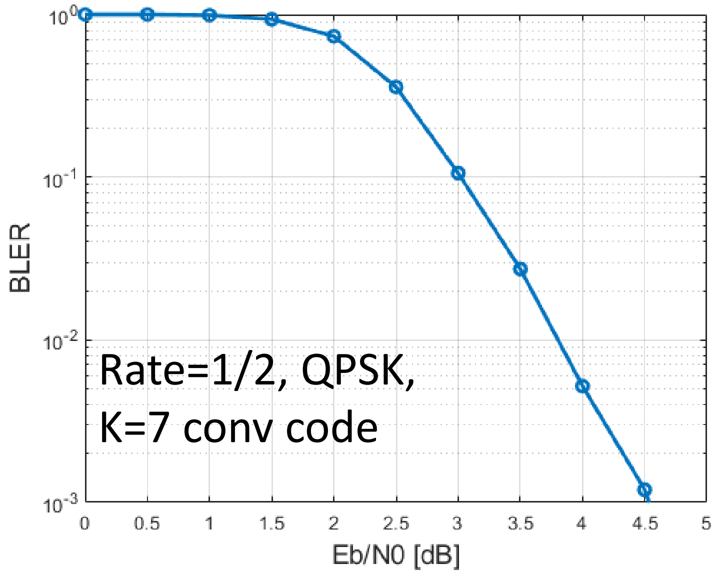
Convolutional Code Performance



- ❑ Convolutional codes performance:
  - > 5 dB better than uncoded BPSK at low BER
- ❑ Only moderate constraint length (K=7) needed
- ❑ Source: Proakis, "Digital communications"

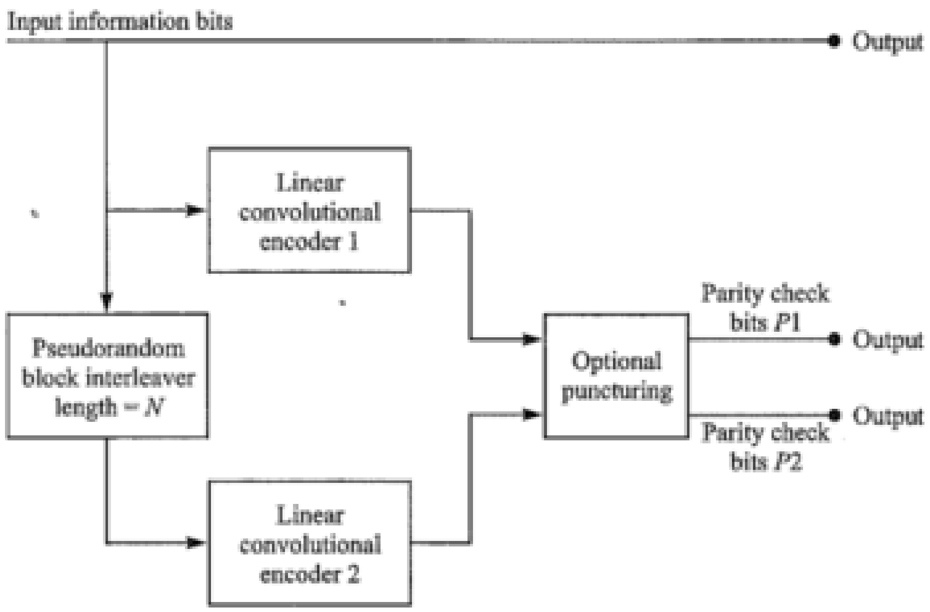
Simulation in MATLAB

- MATLAB has excellent tools
  - Conv encoder / decoder
  - LLR
- See demo



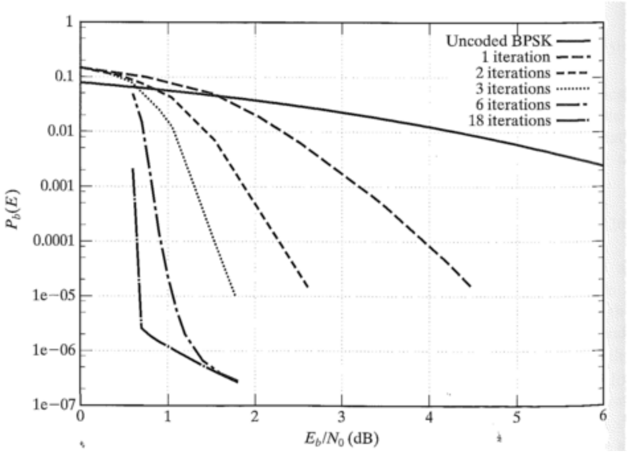

```
% Generate random bits
bitsIn = randi([0,1], nbits, 1);

% Convolutionally encode the data
bitsEnc = convEnc.step(bitsIn);

% QAM modulate
txSig = qammod(bitsEnc, M,'InputType','bit','UnitAveragePower',true);

% Add noise
rate = nbits/length(bitsEnc);
Es = mean(abs(txSig).^2);
EsN0 = EbN0 + 10*log10(rate*bitsPerSym);
chan = comm.AWGNChannel('NoiseMethod', 'Signal to noise ratio (Es/No)', ...
    'EsNo', EsN0, 'SignalPower', Es);
rxSig = chan.step(txSig);

% Compute LLRs
noiseVar = Es*db2pow(-EsN0);
llr = qamdemod(rxSig,M,'OutputType','approxllr', ...
    'UnitAveragePower',true,'NoiseVariance', noiseVar);

% Run Viterbi decoder.  We remove the tail bits
bitsOut = convDec.step(llr);
bitsOut = bitsOut(1:nbits);
```

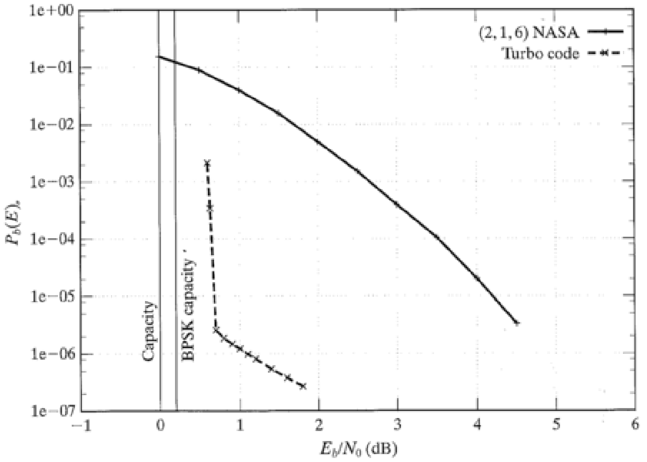
Turbo Codes



- Turbo codes:
  - Concatenation of two convolutional codes Typically IIR and short (K=3)
  - Interleaver: Randomly permutes the input bits
- Output
  - Input bit, and
  - Parity bits from each convolutional encoder
  - With no puncturing R=1/3
- Discovered in 1993, ,
  - Berrou, Glavieux, Thitimajshima, 1993.
  - Able to achieve capacity within a fraction of dB.
- Used in 3G and 4G standards

Turbo Code Iterative Decoding



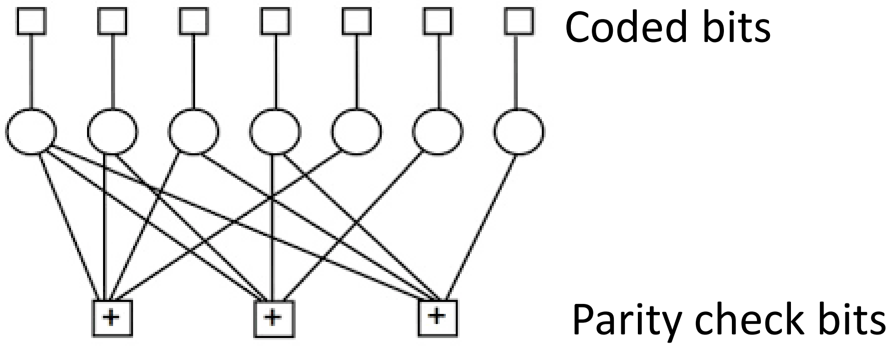


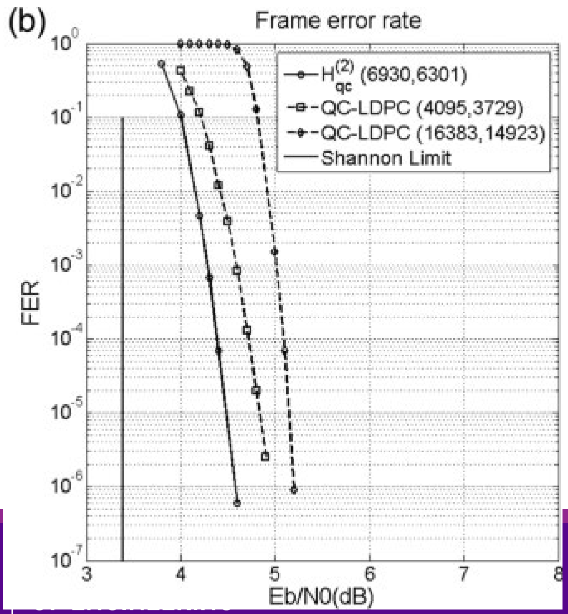
- Turbo decoder uses an iterative message passing
  - Decode each convolutional coder one at a time
  - Use posterior information of one code as prior for the other
- Good performance in small number (usu. ~8) iterations
  - Typically use short codes (K=3).
  - Complexity similar to convolutional codes
- Close to Shannon capacity
  - Much better than convolution codes

Source: Lin, Costello, “Error Control Coding”

LDPC Codes

LDPC Graph





- ❑ Code defined by a bipartite graph
  - Connects n coded bits and $n-k$ parity bits
  - Data k information bits
- ❑ Also use a message passing decoder
  - Based on graphical models
- ❑ Obtains excellent performance
  - Lower complexity than turbo decoder
  - Good for very high data rate applications
- ❑ Used in 802.11ad and 5G New Radio

In-Class Problem

Problem 3: NR LDPC Coding

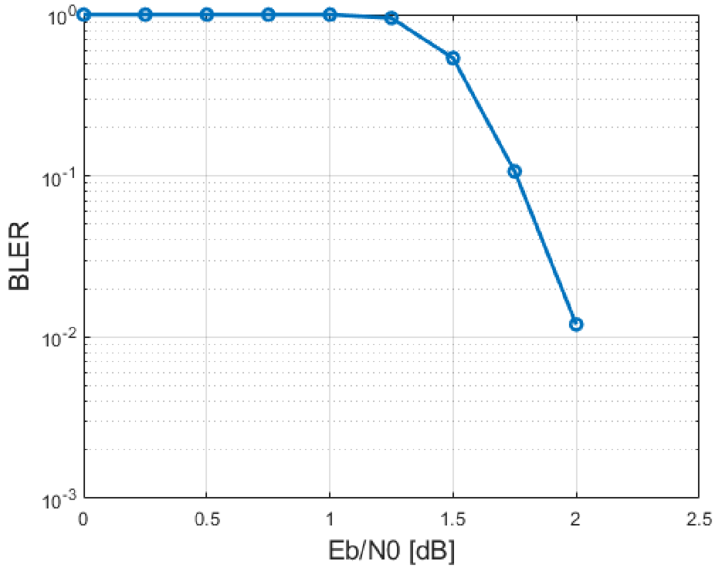
In this problem, we will simulate the LDPC encoding and decoding used in the 5G NR standard. The MATLAB 5G Toolbox has an amazingly good implementation of this code, so we can just call it. In the NR LDPC code, the number of input bits is given by:

```
nbitsIn = nrows*nlift;
```

where nrows is the number of rows in the LDPC base graph, and `nlift` is the so-called lifting factor which expands the graph to different block sizes. We will use the following parameters:

```
bgn = 1;      % LDPC base graph number (1 or 2)
nrows = 22;  % number of rows in the base graph
nlift = 128; % lifting factor
nbitsIn = nrows*nlift;   % number of input bits
maxNumIters = 8;         % max number of LDPC decode iterations
```

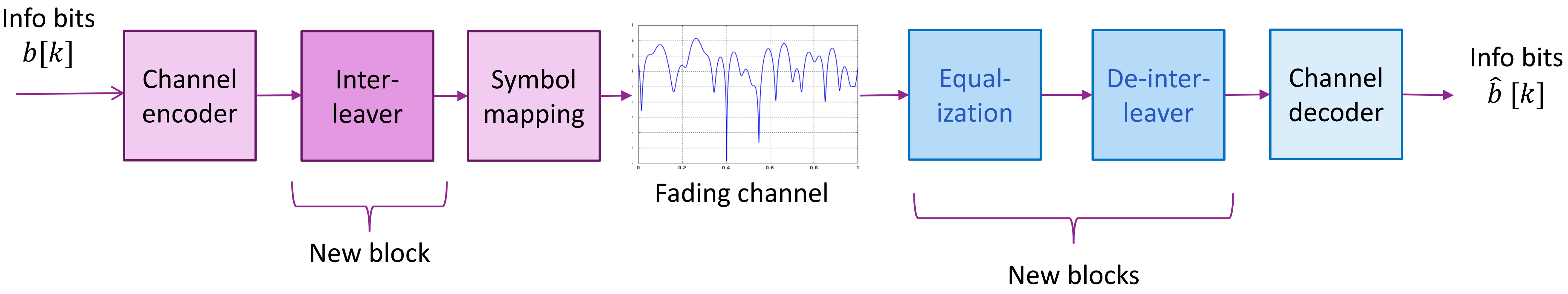
- Simulate the commercial 5G NR LDPC code
  - A rate 1/3 code
  - Much better performance than convolutional code



Outline

- Uncoded Modulation over Fading Channels
- Capacity with Coding over Fading Channels: Outage Capacity
- Capacity with Coding over Fading Channels: Ergodic Capacity
- Review: Coding over an AWGN Channel
- Coding over Fading Channels
- Capacity with Bit-Interleaved Coded Modulation

Coded Communication on a Fading Channel

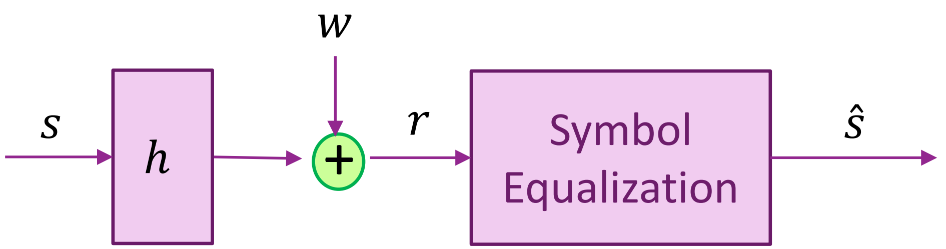


- Now consider fading channel: $r[n] = h[n]s[n] + w[n], \quad w[n] \sim CN(0, N_0)$
- To handle fading we need to introduce a few new blocks
  - Interleaving and de-interleaving
  - Equalization

Symbol Equalization via Inversion

- Received noisy symbol:

$$r = hs + w, \qquad w \sim CN(0, N_0)$$



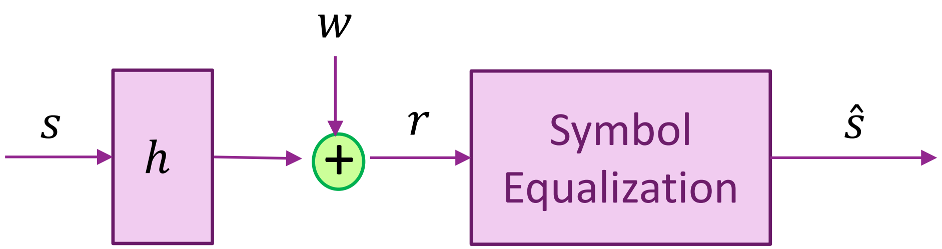
- Symbol equalization:
  - Estimate s from r
  - Also obtain a noise estimate (needed for LLR)
- Channel inversion:
  - Symbol estimate $\hat{s} = \frac{r}{h} = s + v, \;\; v = \frac{w}{h}$
  - Noise estimate: $E|v|^2 = \frac{1}{|h|^2} E|w|^2 = \frac{N_0}{|h|^2}$

MMSE Symbol Equalization

- Received noisy symbol:

$$r = hs + w, \qquad w \sim CN(0, N_0)$$

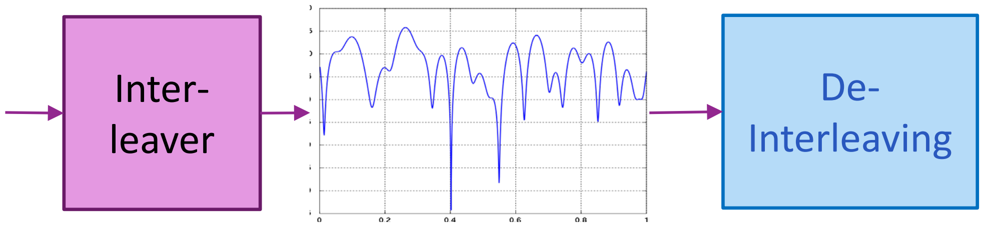
- MMSE estimation:
  - Use linear estimate $\hat{s} = \alpha r$
  - Select α to minimize $E|s - \hat{s}|^2 = E|s - \alpha r|^2$



- Resulting estimate (shown with simple algebra):
  - Estimate: $\hat{s} = \alpha r, \quad \alpha = \frac{E_s h^*}{|h|^2 E_s + N_0}$
  - Noise variance: $E|s - \hat{s}|^2 = \frac{E_s N_0}{|h|^2 E_s + N_0}$
- Provides lower noise estimate than channel inversion

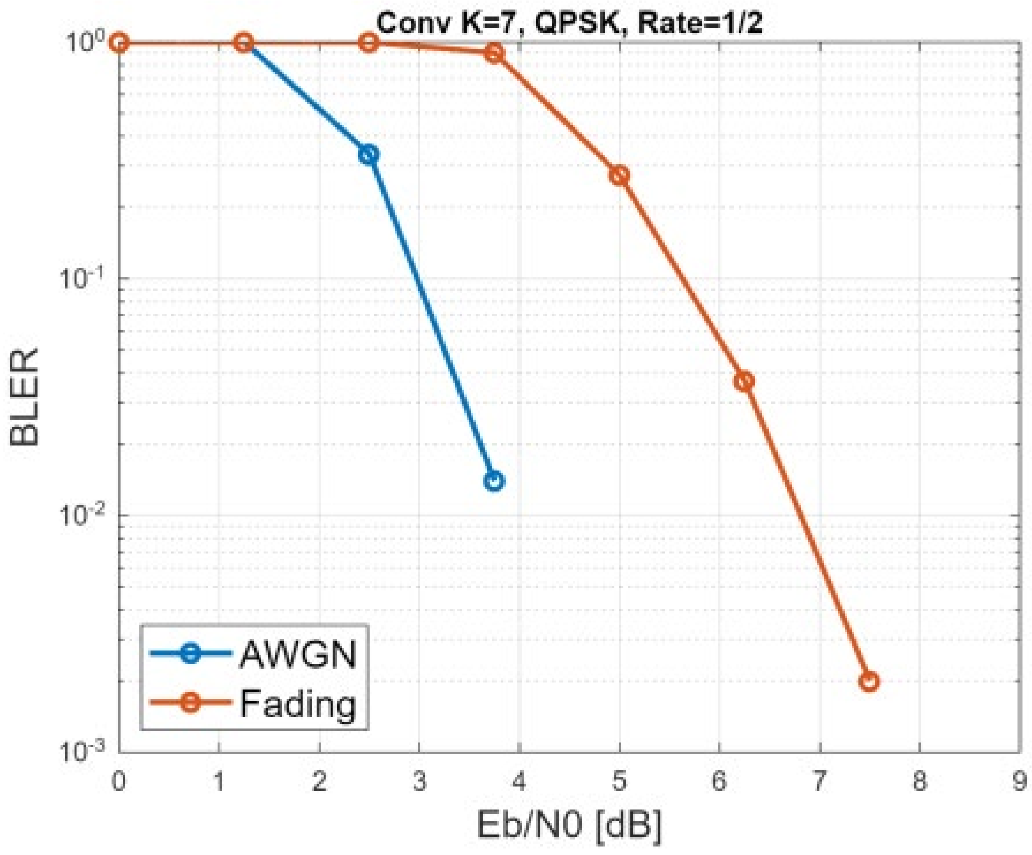
Interleaving and De-Interleaving

- Problem: Fading is correlated in time
  - Will result in many consecutive faded bits
  - Many codes perform poorly if errors are together
- Interleaver
  - Shuffles the bits before symbol mapping
  - De-interleaving is performed on LLRs
  - Randomizes locations of errors
  - Removes time correlations
- Many interleavers used in practice
  - Random interleaver (with some seed at TX and RX)
  - Row-column interleavers…



Type of Interleaver	Interleaver Function
General block interleaver	`intrlv`
Algebraic interleaver	`algintrlv`
Helical scan interleaver	`helscanintrlv`
Matrix interleaver	`matintrlv`
Random interleaver	`randintrlv`

Simulation



- Simulation:
  - Convolutional code, rate $\frac{1}{2}$ with QPSK
  - Constraint length $K = 7$
  - Plotted is block error rate (BLER) vs. $\frac{E_b}{N_0}$
- Gap between AWGN and fading:
  - Approximately 3 dB at BLER = 10^{-2}
  - Much smaller gap than uncoded modulation

Scaling the LLRs in MATLAB

- Noise variance after equalization:

$$\sigma_n^2 = \frac{E_s N_0}{|h_n|^2 E_s + N_0}$$

  - Changes with channel gain
  - Difference symbols have different σ_n^2

- Recall that for approximate LLR: $LLR \propto \frac{1}{\sigma_n^2}$

- MATLAB:
  - Compute LLRs with noise variance = 1
  - Then scale with noise variance
  - Built-in scaling doesn't appear to work

Manual noise scaling: Works

```
% Compute LLRs
llrInt = qamdemod(z,M,'OutputType','approxllr', ...
    'UnitAveragePower',true);

% Scale the LLRs with the noise variance
nsym = length(z);
Isym = repmat((1:nsym),bitsPerSym,1);
Isym = Isym(:);
llrInt = llrInt./noiseVar(Isym);
```

MATLAB built-in scaling: Does not work!

```
llrInt = qamdemod(z,M,'OutputType','approxllr', ...
    'UnitAveragePower',true, 'NoiseVariance', noiseVar);
```

Simulating in MATLAB

❑Transmitter and Channel Fading

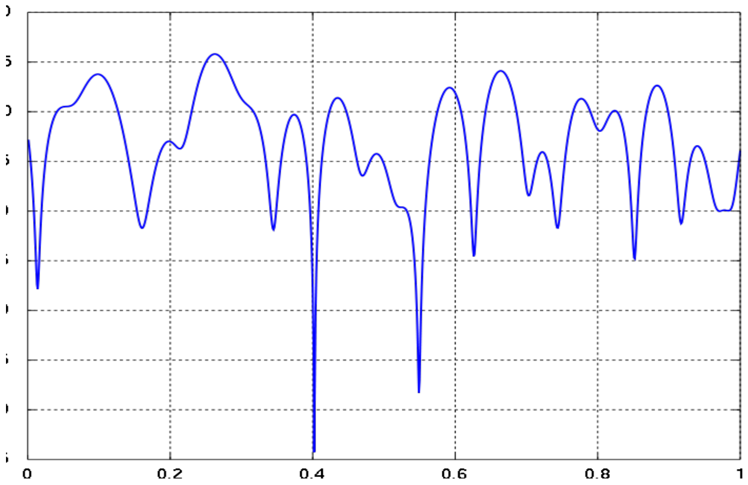
```
% Generate random bits
bitsIn = randi([0,1], nbits, 1);

% Convolutionally encode the data
bitsEnc = convEnc.step(bitsIn);

% Random interleaver
state = randi(2^16,1);
bitsInt = randintrlv(bitsEnc,state);

% QAM modulate
txSig = qammod(bitsInt,M,'InputType','bit','UnitAveragePower',true);

% Add fading
nout = length(txSig);
if fading
    h = sqrt(1/2)*(randn(nout,1) +1i*randn(nout,1));
else
    h = ones(nout,1);
end
rxSig0 = h.*txSig;
```

❑Channel Noise and Receiver

```
% Pass through AWGN channel
rxSig = chan.step(rxSig0);

% MMSE Equalize
wvar = Es*10.^(-0.1*EsN0);
z = conj(h).*rxSig ./ (abs(h).^2 + wvar);
svar = 1;
noiseVar = wvar*svar./(wvar + svar*abs(h).^2);

% Compute LLRs
llrInt = qamdemod(z,M,'OutputType','approxllr', ...
    'UnitAveragePower',true);

% Scale the LLRs with the noise variance
nsym = length(z);
Isym = repmat((1:nsym),bitsPerSym,1);
Isym = Isym(:);
llrInt = llrInt./noiseVar(Isym);

% De-interleave
llr = randdeintrlv(llrInt,state);

% Run Viterbi decoder
bitsOut = convDec.step(llr);
bitsOut = bitsOut(1:nbits);
```

Summary

- **Fading**: Causes variations in SNR
- **Uncoded modulation**:
  - Dramatically increases error rate
  - Must add significant fade margin
- **Coding with flat and slow fading**
  - All symbols are faded together
  - Fade margin still necessary
- **Coding with fast or frequency-selective fading**
  - Can greatly mitigate fading
  - Recover faded bits with redundancy
  - But needs to encoded over many independent fades
  - Transmit over many coherence or bandwidth



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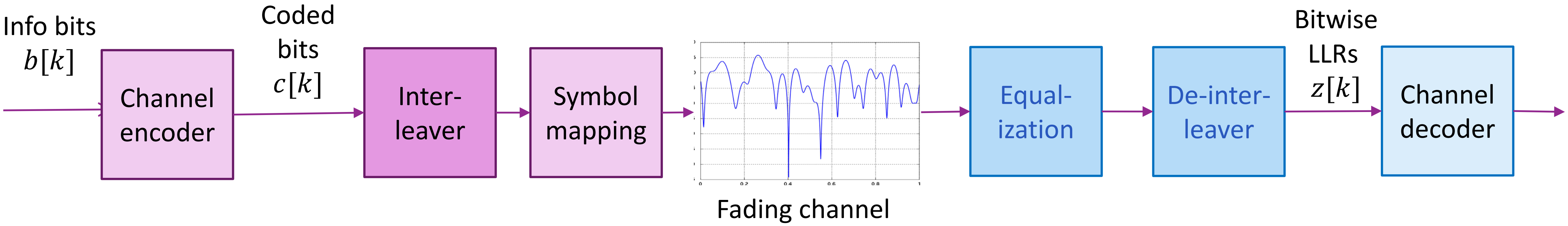
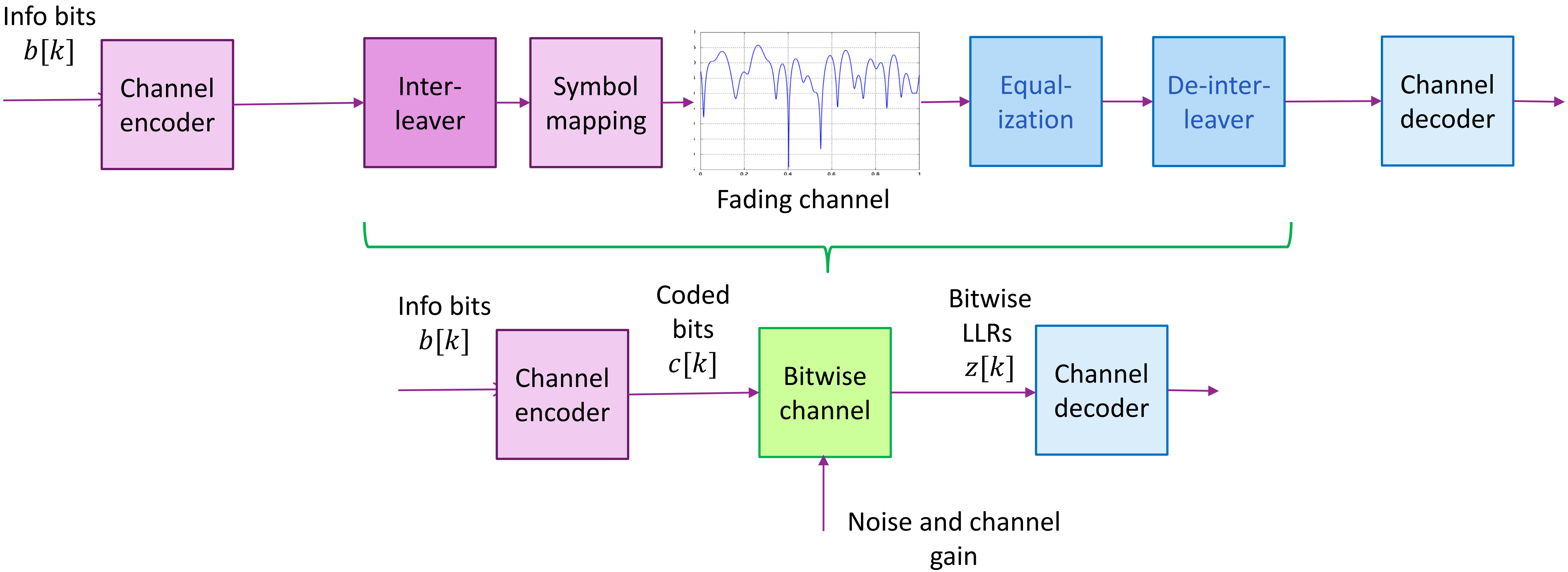
Practical Fading Channel Capacity



- Previously, we computed ergodic capacity $C = E[\log_2(1+\gamma)]$
- But achieving this capacity requires:
  - Gaussian signaling while practical systems use M-QAM
  - Ideal decoding while practical systems use bitwise LLRs
- Bitwise Interleaved Coded Modulation
  - An estimate of the capacity under constraints of using bitwise LLRs

Bitwise Channel



BICM Theorem

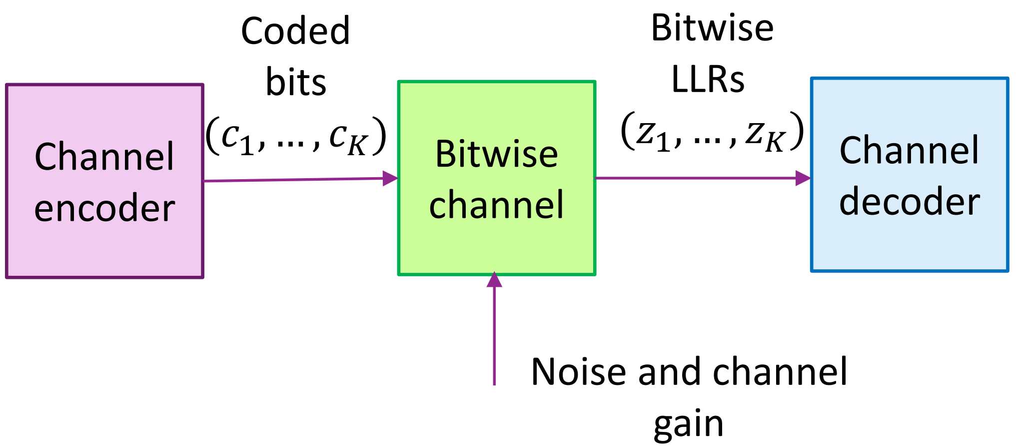
- Consider channel from coded bits to LLRs
  - Theoretical capacity = $I(c;z)$
  - Information bits / coded bit $\in [0,1]$

Theorem: Mutual information is bounded below:

$$I(c;z) \geq 1 - E[BCE(c,z)]$$

  - Binary cross entropy:
$$BCE(c,z) = \log(1+e^{-z}) - cz$$

- Bound is exact when:
  - $z_k = \log \frac{P(r_k|c_k=1)}{P(r_k|c_k=0)}$ for some received data r_k
  - z_k are independent given $\boldsymbol{c}$



BICM Capacity

- ❑Model $r = hs + w$
  - $s =$ QAM symbol
- ❑AWGN $h = 1$
- ❑Fading: h i.i.d. complex Normal
- ❑Loss is ~2 dB with optimal MCS choice