Unit 3. Multi-Path Fading

ECE-GY 6023. WIRELESS COMMUNICATIONS

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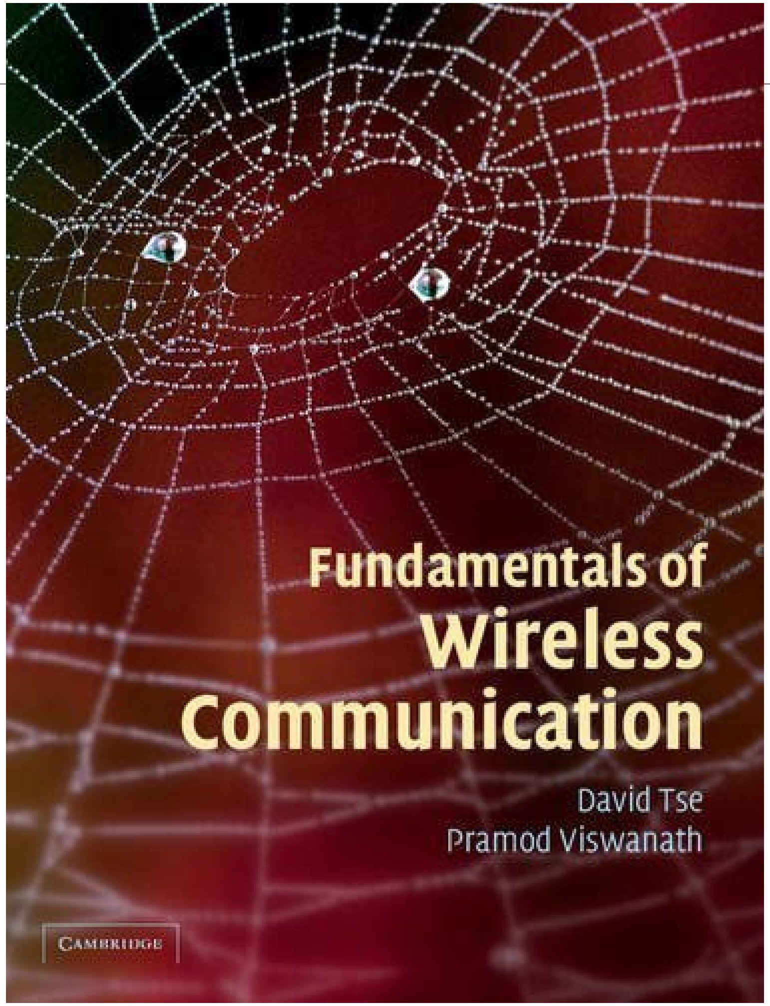
Fading

From the Introduction of a classic text:

There are two fundamental aspects of wireless communication that make the problem challenging and interesting.

…First is the phenomenon of fading …

…Second …there is significant interference …



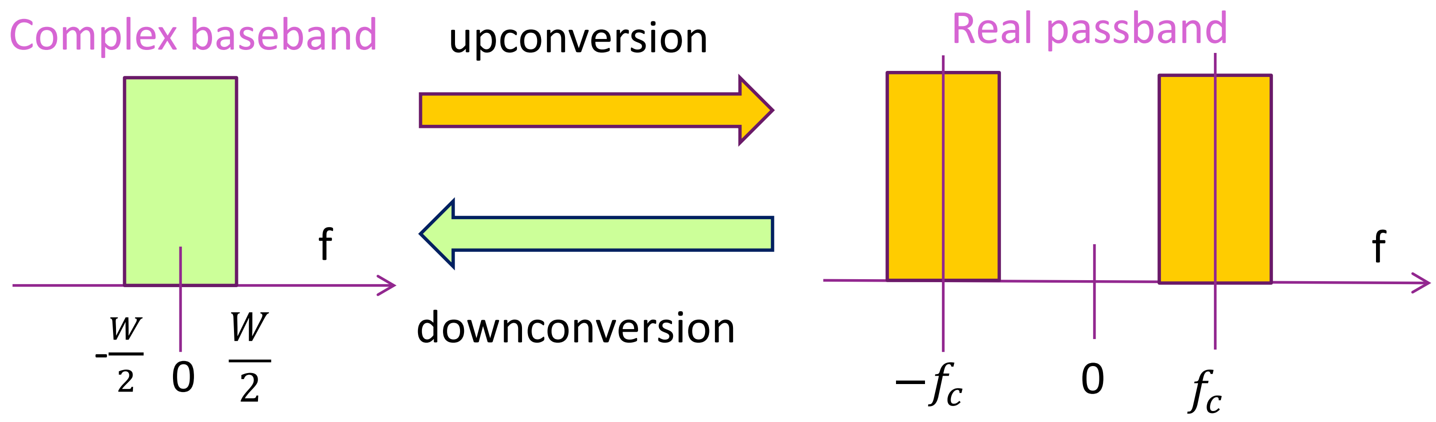
Learning Objectives

- ❑Describe up and down-conversion in time- and frequency-domain
- ❑Describe the steps in the DAC and ADC including the filtering
- ❑Compute a discrete-time and continuous-time base equivalent channels from the passband
- ❑Simulate fractional delays and gains in the sampled data
- ❑Describe and simulate a deterministic multi-path wireless channel
- ❑Compute the time-varying frequency response given the path parameters
- ❑Describe a statistical model for multi-path fading
- ❑Approximately compute the coherence time and bandwidth given a channel

Outline

- Review of Up- and Downconversion
- Review of TX and RX Sampling
- Doppler and Multi-Path Fading
- Statistical Descriptions of Fading

Up- and Downconversion



- RF communication systems:
  - Information occurs and is processed in complex baseband
  - Transmitted and received in real passband
- Up and down-conversion: Shift center frequency of signals
- Also called mixing

Up and Down-Conversion in Time Domain

- Complex baseband:
  - Two real signals, $u_I(t), u_Q(t)$
  - Or one complex signal:

$$u(t) = u_I(t) + ju_Q(t)$$

upconversion →

$$u_p(t) = Re(u(t)e^{j\omega_c t})$$

- Real passband: $u_p(t)$

← downconversion

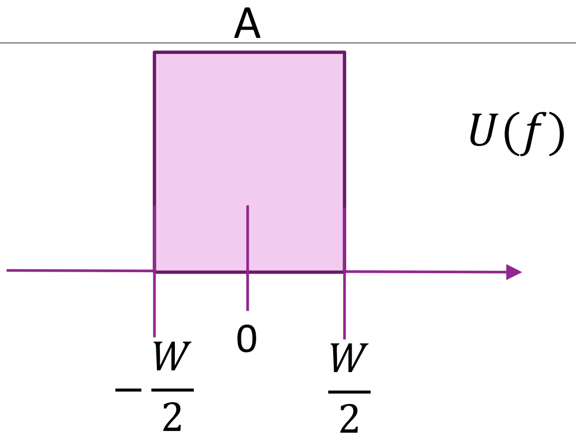
$$v(t) = 2u_p(t)e^{-j\omega_c t}$$
$$u(t) = h_{LPF}(t) * v(t)$$

Note: downconversion needs:

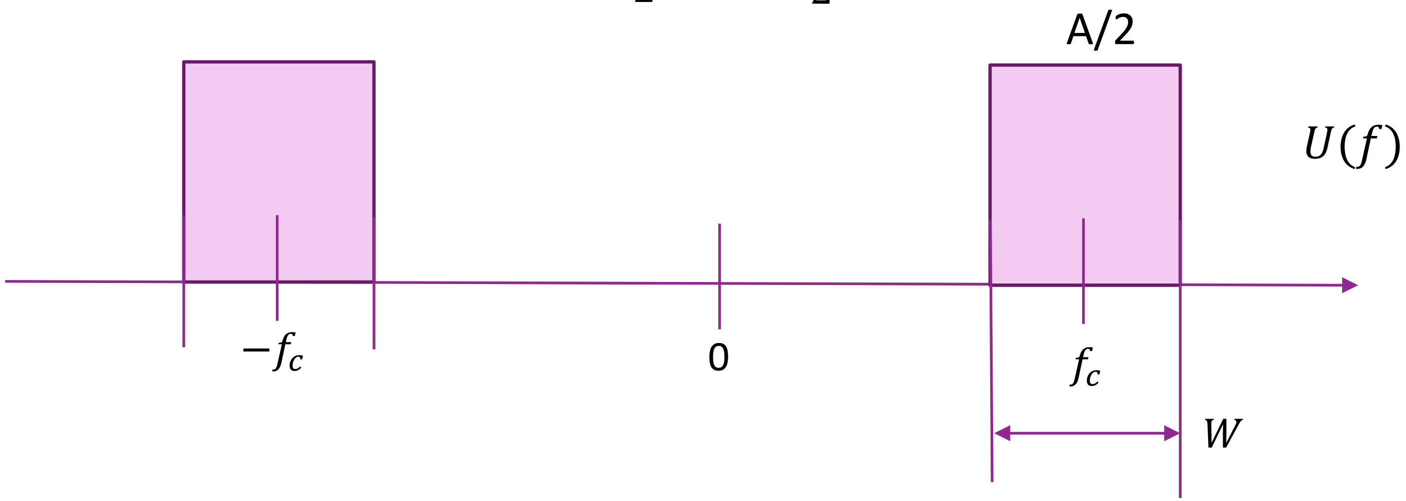
- Multiplication by 2
- Low pass filtering

Mixing in Frequency Domain

- Baseband signals
  - Centered around $f = 0$, complex
  - $\frac{W}{2}$ = single sided bandwidth
  - W = two sided bandwidth
  - Band-limited to $|f| \leq \frac{W}{2}$



- Passband signals
  - Centered around $f = f_c$, real
  - W = bandwidth (per side or image)
  - Band-limited to $|f - f_c| \leq \frac{W}{2}$



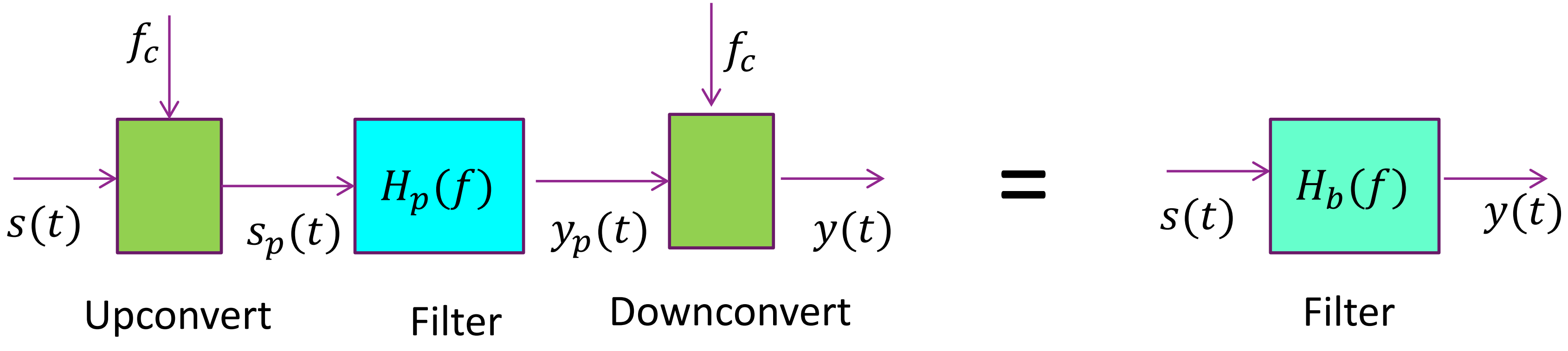
Discrete IQ Mixer

http://www.markimicrowave.com/Mixers/IQ_Quadrature-IF_Double-Balanced/IQ-0318.aspx

- LO = "local oscillator" = square or sine wave at f_c
- I1, I2 = I and Q inputs.
  - Generally, lowpass
- RF = passband output centered at f_c

Datasheet	RF [GHz]	LO [GHz]	IF [MHz]	Conversion Loss [dB]	Image Rejection [dB]	Amplitude Deviation [dB]	Phase Deviation [Degrees]	Isolations L-R [dB]	Isolations L-I [dB]
IQ-0318	3 to 18	3 to 18	DC to 500	7	22	0.75	10	40	20

Baseband Equivalent Channel



- Filtering at passband equivalent to complex baseband filter
- Assuming downconversion filter is ideal:

$$H_b(f) = H_p(f + f_c) \quad \text{for } |f| \le \frac{W}{2}$$

  - Simply shift $H_p(f)$ to the left by f_c.

Important Special Case: Delay

Passband channel

$$h_p(t) = A\delta(t - \tau)$$

A = gain
τ = delay

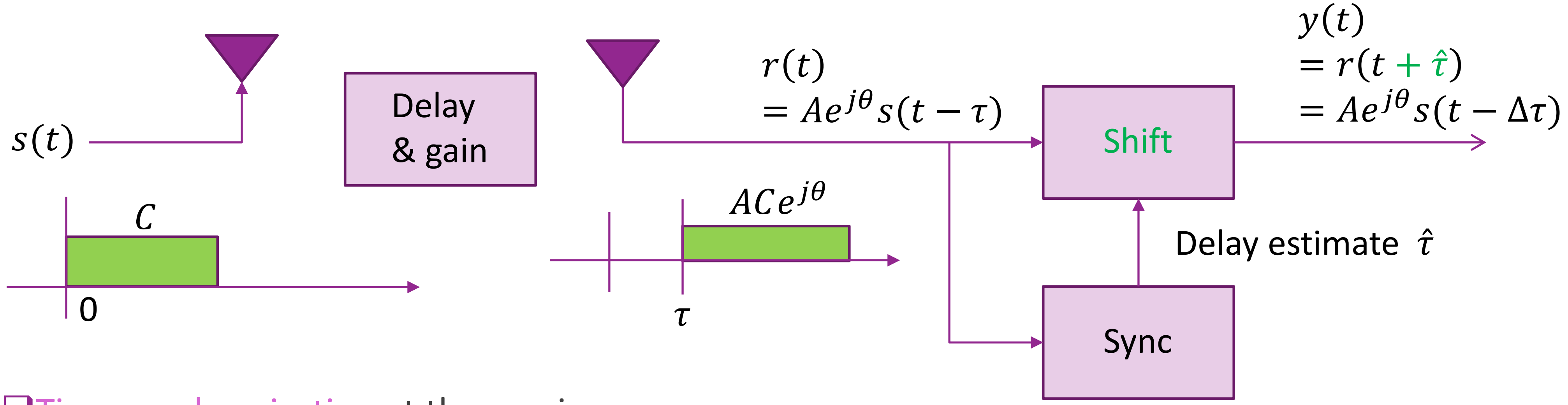
Baseband equivalent channel

$$h_b(t) = Ae^{-2\pi j f_c \tau}\delta(t - \tau)$$

A = gain
τ = delay
$\theta = 2\pi f_c \tau$ = phase rotation

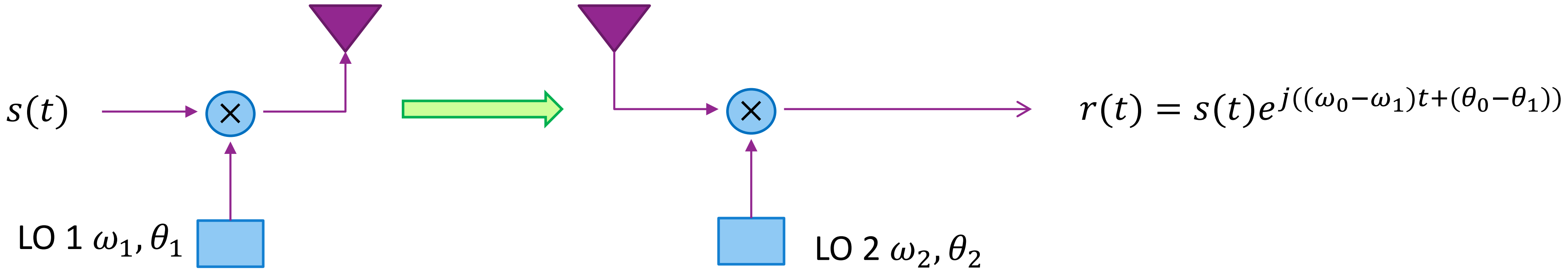
- Delay, gain in passband ⇒ delay, gain and phase rotation in baseband
- Proof:
  - Passband frequency response is: $H_p(f) = Ae^{-2\pi j f \tau}$
  - Baseband frequency response: $H_b(f) = H_p(f + f_c) = Ae^{-2\pi j (f_c + f)\tau}$
  - Equivalent impulse response: $h_b(t) = Ae^{-2\pi j f_c \tau}\delta(t - \tau)$

Synchronization and Delay Errors



- Time synchronization at the receiver:
  - Estimate the arrival time of the signal $\hat{\tau}$
  - Starts processing remainder of signal starting at $\hat{\tau}$
  - Equivalent to shifting received signal ahead in time by $\hat{\tau}$: $y(t) = r(t+\hat{\tau})$
  - Remaining time error: $\Delta\tau = \tau - \hat{\tau}$
- Later, we will discuss:
  - How to estimate τ (synchronization) and how to correct for gain and phase error (equalization)

Frequency Errors



- Oscillators at TX and RX always have some mismatch. To analyze, suppose:
  - Upconversion: $s_p(t) = Re\left(s(t)e^{j\omega_1 t+\theta_1}\right)$,
  - Downcoversion: $r(t) = LPF(\ 2s_p(t)e^{-(j\omega_2 t+\theta_2)}\ )$
- LO error leads to time-varying gain: $r(t) = g(t)s(t), \quad g(t) = e^{j((\omega_0-\omega_1)t+(\theta_0-\theta_1))}$
  - Frequency and phase shift

In-Class Problems

Problem 1: Plotting a Frequency Response of a Delay Channel

Consider a system with the following parameters

```
bw = 20e6;     % bandwdith
pl = 80;       % path loss (dB)
tau = 200e-9;    % timing error
```

Plot the real component of the frequency response at 1024 points over the bandwidth. Assume phase = 0 at DC. Use 1024 frequency points

Problem 2: Plotting the Channel Response from Frequency Error

Suppose a link has the following parameters:

```
fc = 37e9;  % carrier freq
loppm = 1;  % LO error in ppm
```

Plot the relative change of the gain:

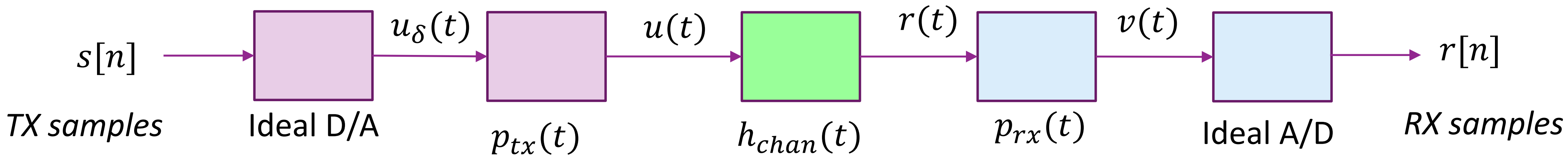
```
E(t) = |g(t)-g(t+tau)|^2 / |g(t)|^2
```

as a function of tau from 0 to 5 us.

Outline

- ❑Review of Up- and Downconversion
- ❑Review of TX and RX Sampling
- ❑Doppler and Multi-Path Fading
- ❑Statistical Descriptions of Fading

Typical Digital Communication Path



- All modern communication systems TX and RX digital samples
- Transmitter: DAC + filtering with $p_{tx}(t)$. Filtering used to:
  - Suppress out of band emissions
- Receiver: Filters with $p_{rx}(t)$ then performs ADC. Filtering plays two roles:
  - Reduces noise
  - Remove out-of-band signals before ADC. (i.e. Anti-aliasing)
- Filter design discussed in digital communications class

Review of DTFT

- Given discrete-time sequence $s[n]$
  - Real or complex
- Discrete-time Fourier Transform: $S(\Omega) = \sum_n s[n]e^{-j\Omega n}$
- Inverse DTFT: $s[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} S(\Omega)e^{j\Omega n}d\Omega$
- Note $S(\Omega)$ is always a 2π periodic signal
  - Can take integral for inverse DTFT on any period of 2π
- Ω is the discrete frequency. Units is radians per sample.
- For finite length signals and finite number of Ω, can be computed via FFT
- Review in your signals and systems class

Common DTFT Pairs

Time domain $x[n]$	Frequency domain $X_{2\pi}(\omega)$
$\delta[n]$	$X_{2\pi}(\omega) = 1$
$\delta[n-M]$	$X_{2\pi}(\omega) = e^{-i\omega M}$
$\sum_{m=-\infty}^{\infty} \delta[n-Mm]$	$X_{2\pi}(\omega) = \sum_{m=-\infty}^{\infty} e^{-i\omega Mm} = \frac{2\pi}{M}\sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{M}\right)$ $X_o(\omega) = \frac{2\pi}{M}\sum_{k=-(M-1)/2}^{(M-1)/2} \delta\left(\omega - \frac{2\pi k}{M}\right)$ odd M $X_o(\omega) = \frac{2\pi}{M}\sum_{k=-M/2+1}^{M/2} \delta\left(\omega - \frac{2\pi k}{M}\right)$ even M
$u[n]$	$X_{2\pi}(\omega) = \frac{1}{1-e^{-i\omega}} + \pi\sum_{k=-\infty}^{\infty}\delta(\omega - 2\pi k)$ $X_o(\omega) = \frac{1}{1-e^{-i\omega}} + \pi\cdot\delta(\omega)$
$a^n u[n]$	$X_{2\pi}(\omega) = \frac{1}{1-ae^{-i\omega}}$
e^{-ian}	$X_o(\omega) = 2\pi\cdot\delta(\omega+a),\quad -\pi < a < \pi$ $X_{2\pi}(\omega) = 2\pi\sum_{k=-\infty}^{\infty}\delta(\omega + a - 2\pi k)$

- See Wikipedia

$\cos(a\cdot n)$	$X_o(\omega) = \pi\left[\delta(\omega-a) + \delta(\omega+a)\right],$ $X_{2\pi}(\omega) \triangleq \sum_{k=-\infty}^{\infty} X_o(\omega - 2\pi k)$
$\sin(a\cdot n)$	$X_o(\omega) = \frac{\pi}{i}\left[\delta(\omega-a) - \delta(\omega+a)\right]$
$\operatorname{rect}\left[\frac{n-M/2}{M}\right]$	$X_o(\omega) = \frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)}\, e^{-\frac{i\omega M}{2}}$
$\operatorname{sinc}(W(n+a))$	$X_o(\omega) = \frac{1}{W}\operatorname{rect}\left(\frac{\omega}{2\pi W}\right)e^{ia\omega}$
$\operatorname{sinc}^2(Wn)$	$X_o(\omega) = \frac{1}{W}\operatorname{tri}\left(\frac{\omega}{2\pi W}\right)$

Discrete-Time Systems

$$s[n] \longrightarrow \boxed{H(\Omega)} \longrightarrow r[n]$$

- Consider discrete-time LTI system
- Time-domain: Characterized by impulse response $h[n]$

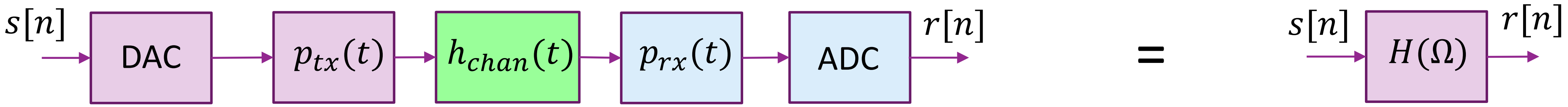
$$r[n] = h[n] * s[n] = \sum_k h[k]s[n-k]$$

- Frequency-domain: Characterized by frequency response $H(\Omega)$

$$R(\Omega) = H(\Omega)S(\Omega)$$

  - $R(\Omega) = \sum r[n]e^{-j\Omega n},\ r[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} R(\Omega)e^{j\Omega n}d\Omega$

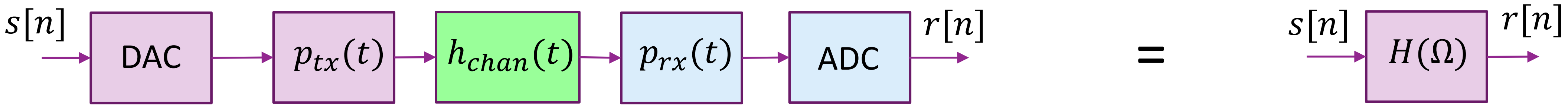
DT Equivalent Channel



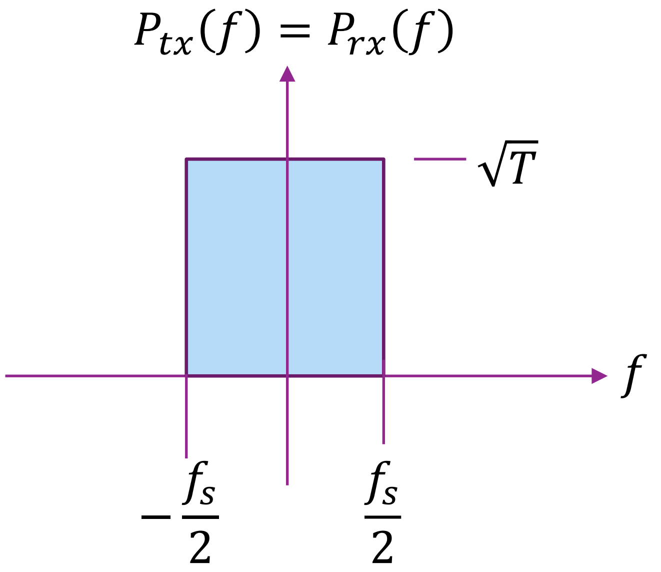
- Discrete-time baseband equivalent channel:
  - Describes equivalent mapping from $s[n]$ to $r[n]$
  - Includes effects of TX and RX filtering and continuous-time baseband channel
- Band-limited filters:
  - Suppose one of P_{rx}, P_{tx} is bandlimited to $|f| < \frac{1}{2T}$ (no out-of-band emissions or aliasing)
  - Then, discrete-time equivalent channel reduces to:

$$\boxed{H(\Omega) = \frac{1}{T} P_{rx}\left(\frac{\Omega}{2\pi T}\right) P_{tx}\left(\frac{\Omega}{2\pi T}\right) H_{chan}\left(\frac{\Omega}{2\pi T}\right) \quad \text{for } |\Omega| < \pi}$$

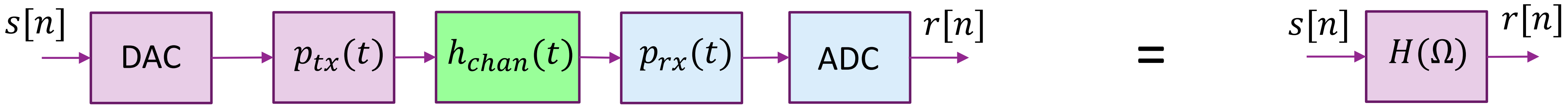
Ideal Filtering



- Suppose sample rate $f_s = \frac{1}{T}$
- "Ideal" TX and RX filter :
  - $p_{tx}(t) = p_{rx}(t) = \frac{1}{\sqrt{T}} \text{Sinc}\left(\frac{t}{T}\right)$
  - In frequency domain: $P_{rx}(f) = P_{tx}(f) = \sqrt{T}\text{Rect}(fT)$
  - Also called "brick wall" filter
- Most practical filters match this well
  - Up to gain and delay



Ideal Filtering



- Assume TX and RX filters are ideal
- Theorem: DT equivalent channel is the re-scaled continuous-time channel

$$H(\Omega) = H_{chan}\left(\frac{\Omega}{2\pi T}\right)$$

  - Frequency f mapped to $\Omega = 2\pi T f$

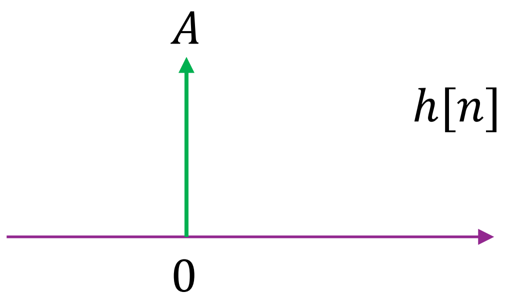
Special Case: Delay

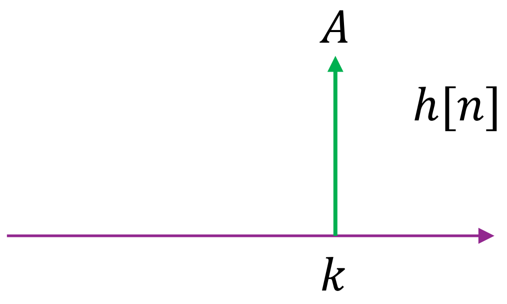
	Passband	Continuous-Time Baseband	Discrete-Time Baseband
Impulse response	$h_p(t) = A\delta(t-\tau)$	$h_{chan}(t) = Ae^{-j\omega_c\tau}\delta(t-\tau)$	$h[n] = Ae^{-j\omega_c\tau} sinc\left(\frac{\tau n}{T}\right)$
Frequency response	$H_p(f) = Ae^{-j2\pi f\tau}$	$H_{chan}(f) = Ae^{-j\omega_c\tau}e^{-j2\pi f\tau}$	$H(\Omega) = Ae^{-j\omega_c\tau}e^{-j\Omega\tau/T}$

- Suppose passband has a gain and delay.
- Then discrete-time frequency-domain: gain and linear phase rotation over frequency
  - Rotates $2\pi\ \tau/T$ radians every period
- In discrete-time time-domain: gain, constant phase rotation and sinc filter with delay

Sinc Filter with Integer Delays

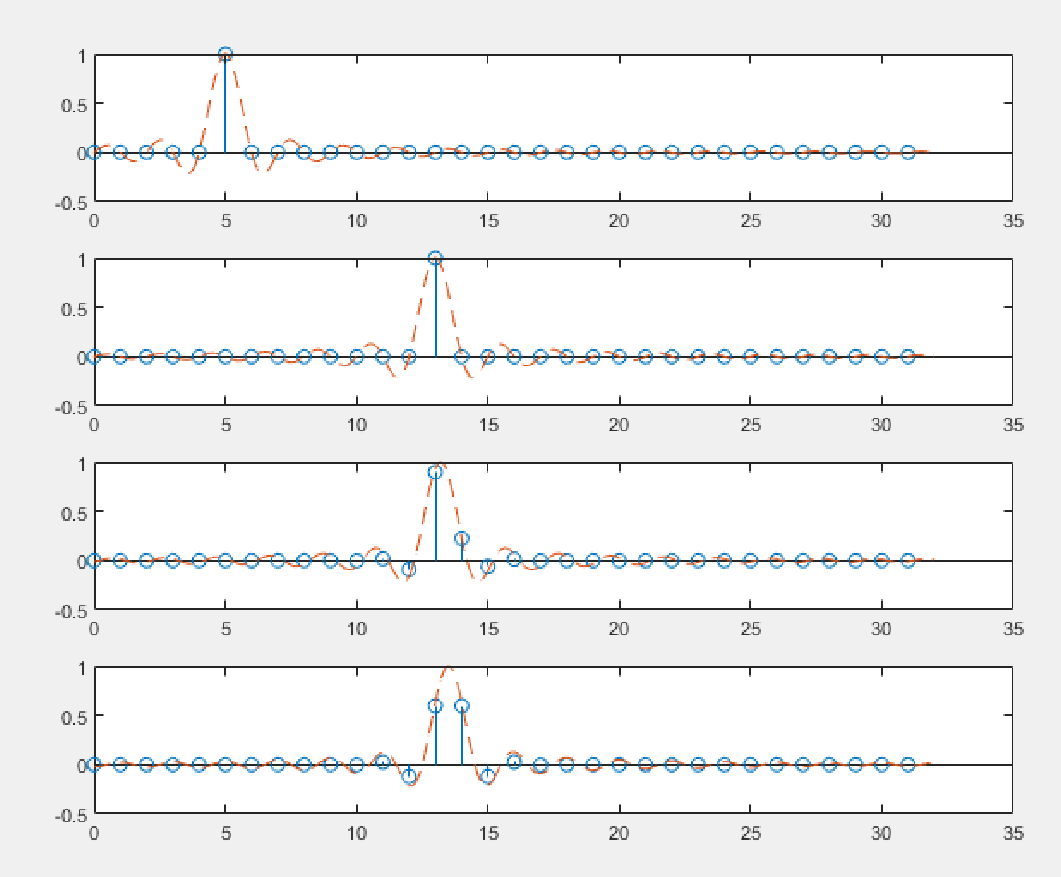
- Suppose we have ideal filtering and passband has delay and gain
- From previous slide, $r[n] = h[n] * s[n],\ h[n] = Ae^{-j\omega_c\tau} sinc\left(\frac{\tau n}{T}\right)$
- Special case 1: No delay $\tau = 0$:
  - $h[n] = A\delta[n] \Rightarrow\ r[n] = As[n]$
  - Baseband channel introduces only gain
- Special case 2: Integer delays $\tau = kT$:
  - $h[n] = A\delta[n-k] \Rightarrow\ r[n] = As[n-k]$
  - Baseband channel introduces gain and integer shift
- Ex: Suppose sample rate is 20 MHz and signal is delayed by 400 ns.
  - Integer delay in discrete-time signal is $20(0.4) = 8$ samples





Sinc Pulses with Fractional Delay

- $h[n] = Ae^{-j\omega_c\tau}sinc\left(\frac{\tau n}{T}\right)$
- Causes blurring over multiple samples
- Inter-symbol interference
- Will need equalization to correct
  - More on this later



$\tau = 5$

$\tau = 13$

$\tau = 13.2$

$\tau = 13.5$

Simulating Fractional Delays in MATLAB

- Code on previous slide was create with DSP toolbox

```
tau = [0,8,8.2,8.5];  % Delays in fractions of a sample
```

```
% Create a fractional delay object from the DSP toolbox
% We select the Farrow interpolation, which is a fast
% and accurate method.  It is important to select the options
% correctly
dly = dsp.VariableFractionalDelay(...
    'InterpolationMethod', 'Farrow','FilterLength',8,...
    'FarrowSmallDelayAction','Use off-centered kernel');
```

```
% Create delays of the sequence
y = dly.step(x,tau);
```

Creates T x D matrix
Row i is delayed by $\tau(i)$

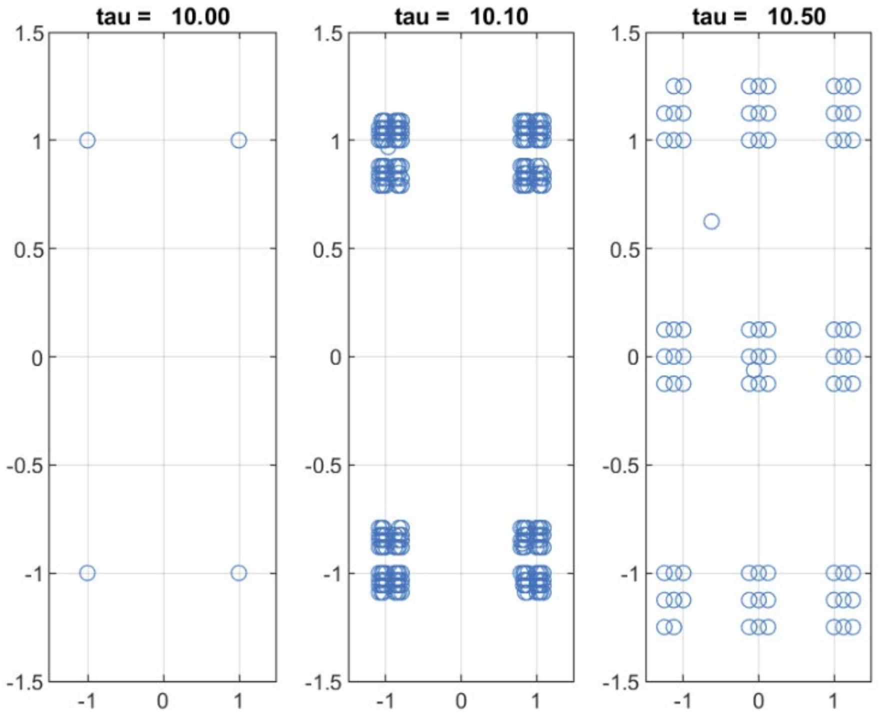
In-Class Problem:
Fractional Delays on Constellations

Problem 3: Effect of Delay on a QAM Constellation

In this problem, we will show the effect of fractional delays on a TX constellation.

First, we generate random QPSK symbols.

```
% TODO:  Generate nb=1024 bits using the randi command.
%   bits = ...
nb = 1024;
bits = randi(2,nb,1)-1;
```



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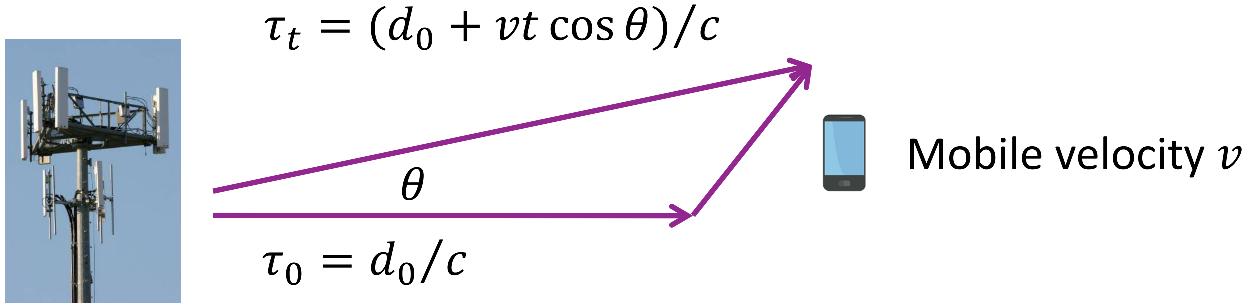
Receiver with Local Motion

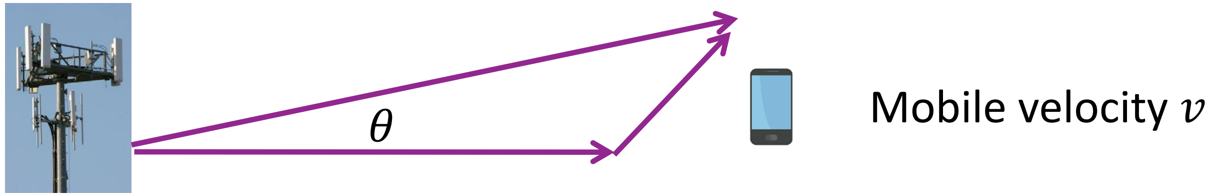


- With the RX in motion, the propagation delay changes with time.
- In complex baseband signal:

$$r(t) = \alpha e^{-j\omega_c \tau(t)} u(t-\tau_t) = \alpha e^{j2\pi\left(-\frac{tv}{c}\cos\theta - d_0 f_c\right)} u(t-\tau_t)$$

- Local motion assumption: $u(t-\tau_t) \approx u(t-\tau_0)$ for t small
  - Effect of the change in propagation delay is only in the complex exponential
- Then: $r(t) = \alpha e^{j2\pi\left(-\frac{tv}{c}\cos\theta - d_0 f_c\right)} u(t-\tau_0)$

Doppler Shift



- Single path with local motion:

$$\boxed{r(t) = g_0\, e^{j2\pi f_d t}\; u(t-\tau_0)}$$

  - Complex gain: $g_0 = \alpha e^{-2\pi j \tau_0 f_c}$
  - Doppler shift: $f_d = -\frac{v f_c \cos\theta}{c}$
  - Delay: τ_0
- For a single path: Local motion causes a phase rotation, but no change in amplitude

Example: Computing Doppler Shift

- Suppose: carrier frequency is f_c=2.1GHz Car moves towards a base station at 100 km/h.
- What is the Doppler shift?
- Answer: v=100km/h= 27.7 m/s, c= $3(10)^8$ m/s, $\theta = 180$:

$$\Delta f = -\frac{v f_c \cos\theta}{c} = -\frac{(27.7)(2.1)(10)^9(-1)}{3(10)^8} \approx 194 \text{ Hz}$$

- If the angle away from BS at $\theta = 45$:

$$\Delta f = \frac{v f_c \cos\theta}{c} = \frac{(27.7)(2.1)(10)^9 \cos(45)}{3(10)^8} \approx -138 \text{ Hz}$$

Multi-Path Models

- Most channel consists of many paths
  - Direct paths
  - Reflections, transmissions, diffraction, …
  - LOS and NLOS paths
- Wideband time-domain baseband model:

$$r(t) = \sum_{\ell=1}^{L} g_\ell e^{j\omega_\ell t} x(t - \tau_\ell)$$

  - g_ℓ: Complex path gain
  - $\omega_\ell = -\frac{2\pi v f_c}{c} \cos \theta_\ell$: Doppler shift of path
  - τ_ℓ: Delay of the path

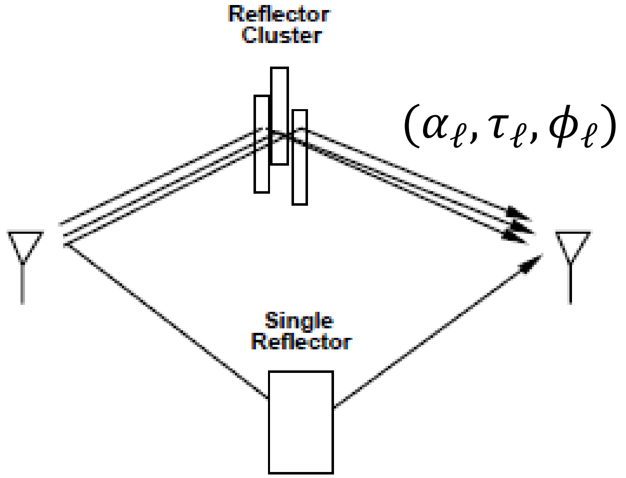


Figure 3.1: A Single Reflector and A Reflector Cluster.

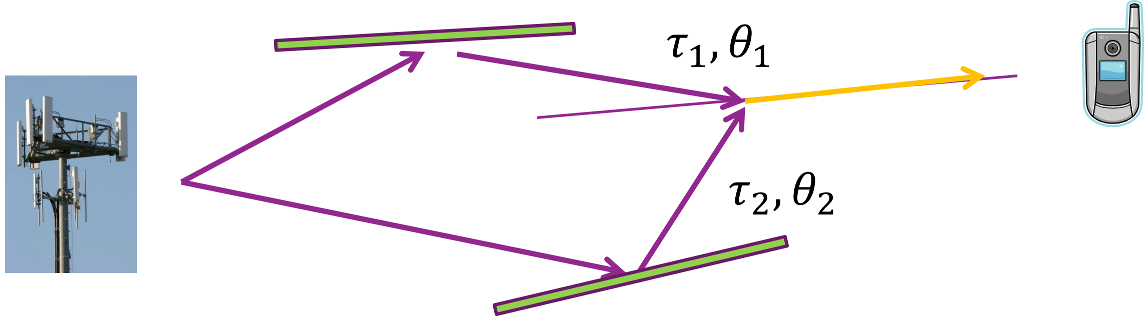
Time-Varying Frequency Response

- Multipath channel: $y(t) = \sum_{\ell=1}^{L} g_\ell e^{j\omega_\ell t} x(t - \tau_\ell)$
- Consider exponential input: $x(t) = e^{j\omega t}$
- Output is: $y(t) = H(t, \omega) x(t)$
- Time-varying frequency response

$$\boxed{H(t,\omega) = \sum_{\ell=1}^{L} g_\ell e^{j(\omega_\ell t - \omega \tau_\ell)}}$$

- May also write: $H(t, f) = H(t, 2\pi f)$

Example with Two Paths



- To simplify understanding, consider two path model

$$r(t) = h_1 e^{j\omega_1 t} u(t - \tau_1) + h_2 e^{j\omega_2 t} u(t - \tau_2)$$

$$\omega_i = -2\,\pi f_{dmax} \cos\theta_i$$

- Time-varying response:

$$H(t, \omega) = h_1 e^{j(\omega_1 t - \omega\tau_1)} + h_2 e^{j(\omega_2 t - \omega\tau_2)}$$

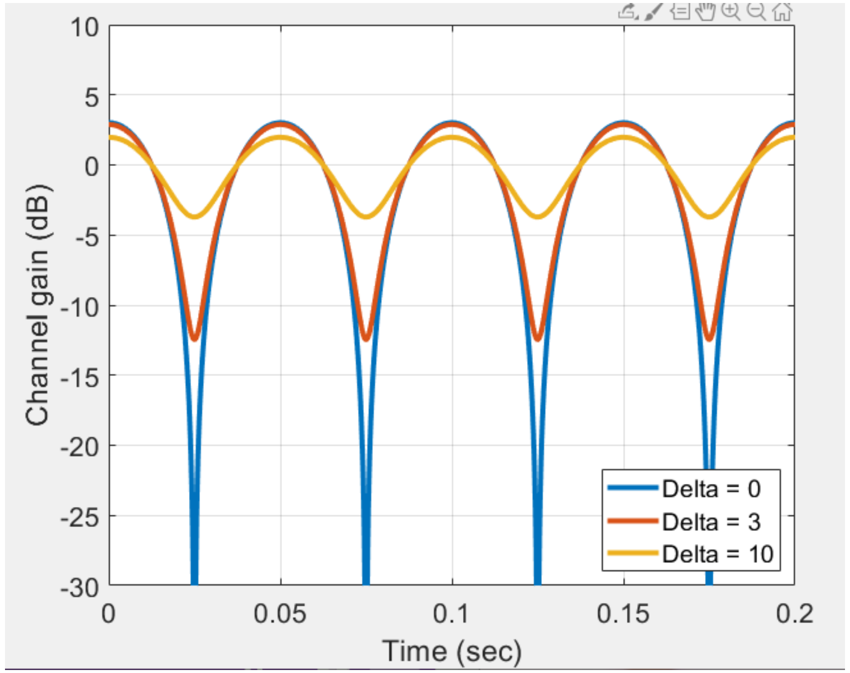
- Power gain:

$$P(t, \omega) = |H(t, \omega)|^2 = \left|h_1 e^{j(\omega_1 t - \omega\tau_1)} + h_2 e^{j(\omega_2 t - \omega\tau_2)}\right|^2$$

Variation in Time

$$P(t, \omega_0) = \left| h_1 e^{j(\omega_1 t + \phi_1)} + h_2 e^{j(\omega_2 t + \phi_2)} \right|^2$$

- Fixed frequency ω_0
- Look at time variations $P(t, \omega_0)$
- Rate of variation depends on Doppler spread:
$$\Delta f = f_{max}(\cos\theta_1 - \cos\theta_2)$$
- Size of variation depends on spread of gains:
  - Avg: $|h_1|^2 + |h_2|^2$
  - Min: $(|h_1| - |h_2|)^2$: Destructive interference
  - Max: $(|h_1| + |h_2|)^2$: : Constructive interference
- With equal path gains, there are nulls



Plot shows $f_{\max}$ =10 Hz,
$\theta_1 = 0, \theta_2 = 180,$
$h_2 = 10^{-0.05\Delta} h_2,\ |h_1|^2 + |h_2|^2 = 1$

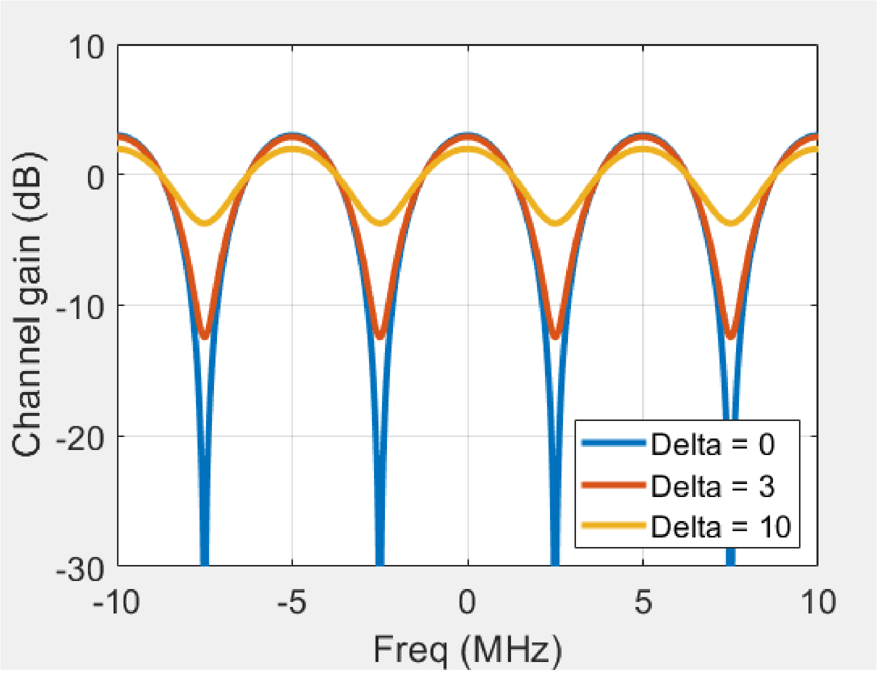
Variation in Frequency

$$P(t_0, \omega) = \left| h_1 e^{j(\omega\tau_1 + \phi_1)} + h_2 e^{j(\omega\tau_2 + \phi_2)} \right|^2$$

- Fixed frequency t_0
- Look at time variations $P(t, \omega_0)$
- Period of variation depends on delay spread:

$$\Delta f = \frac{1}{\tau_2 - \tau_1}$$

- Size of variation depends on spread of gains:
  - Avg: $|h_1|^2 + |h_2|^2$
  - Min: $(|h_1| - |h_2|)^2$
  - Max: $(|h_1| + |h_2|)^2$



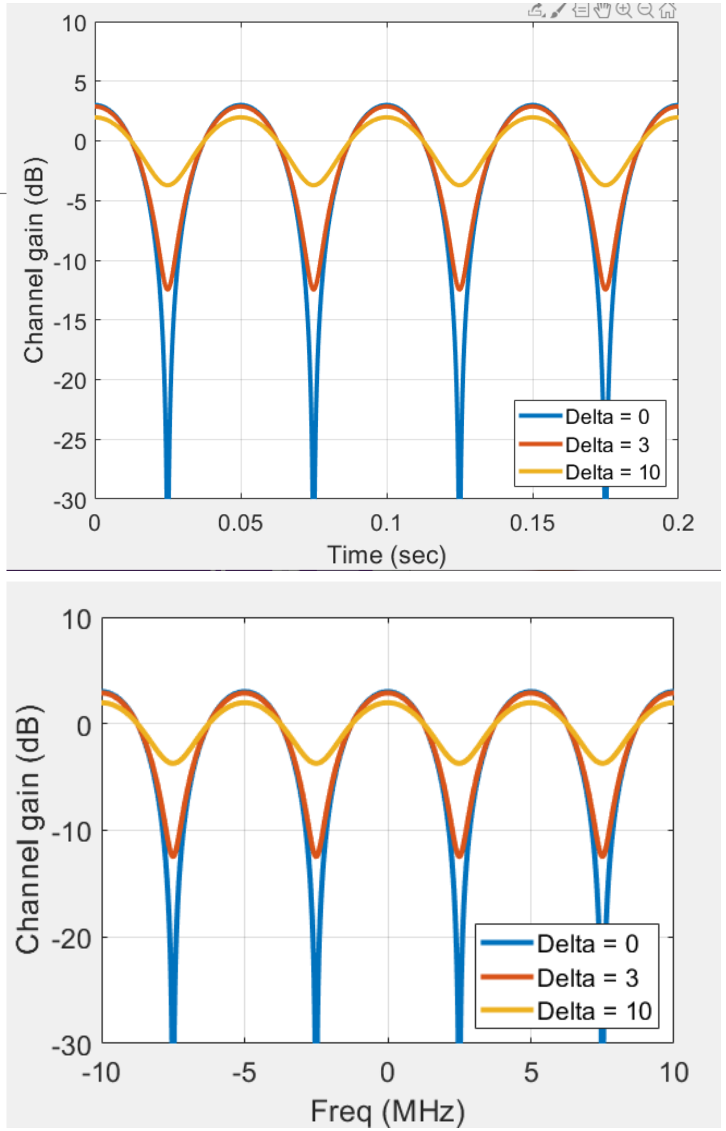
Plot shows
$\tau_1 = 0, \tau_2 = 200$ ns,
$h_2 = 10^{-0.05\Delta} h_2,\ |h_1|^2 + |h_2|^2 = 1$

Fading

- Over time and frequency, paths can either
  - Constructively interfere ⇒ Peaks
  - Destructively interfere ⇒ Nulls

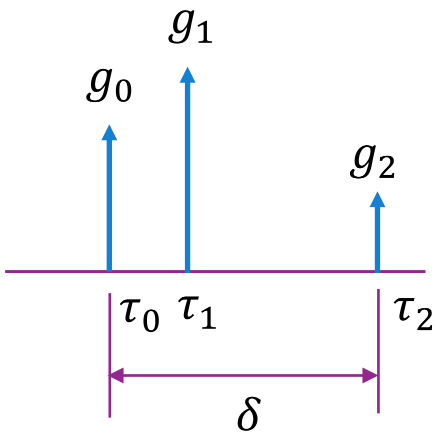
- Process is called fading
  - Intermittent channel quality

- One of the most significant challenges in wireless
- Later, we will discuss how to overcome fading



Narrowband Approximation

- Multi-path channel: $r(t) \quad = \sum_{\ell=1}^{L} g_\ell e^{j\omega_\ell t} x(t-\tau_\ell)$
- Define delay spread: $\delta = \max_\ell |\tau_\ell - \tau_0|$
  - Max path difference in seconds

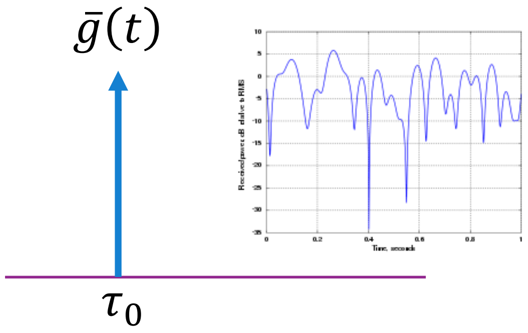


- Narrowband approximation: $X(f)$ is band-limited to $|f-f^0| \ll \frac{1}{2\delta}$ then

$$\boxed{r(t) \approx H(t,f^0)x(t-\tau_0)}$$

  - Proof below
  - Coherence bandwidth $= \frac{1}{2\delta}$

- Effective single path gain: $\bar{g}(t) = H(t,f^0) = \sum_{\ell=1}^{L} g_\ell e^{j\omega_\ell t - j\omega^0\tau_\ell}$:
  - Channel appears as a single path channel with time-varying gain
  - Channel gain $\bar{g}(t)$ is band-limited to max Doppler $\max_\ell \omega_\ell$

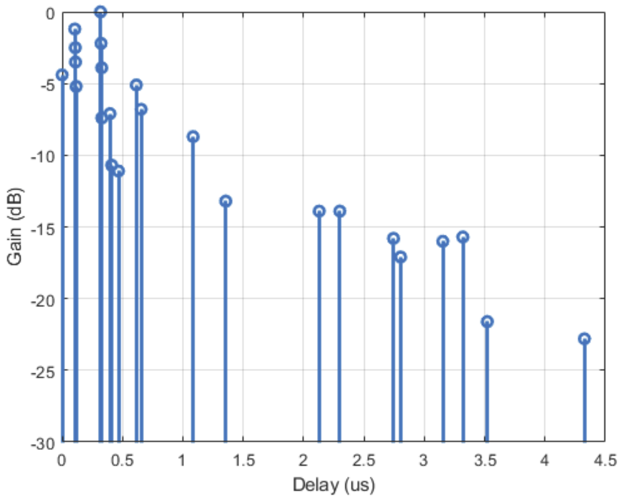


Example: 3GPP Cluster Delay Line Model

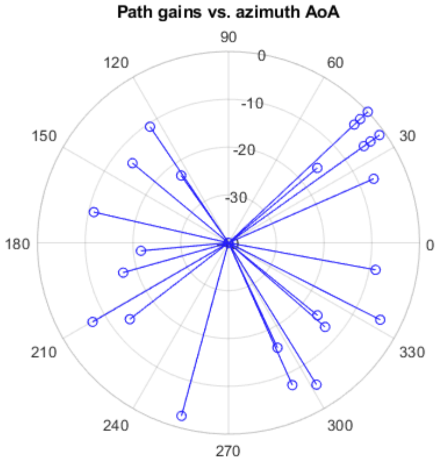
```
fc = 2.3e9;            % carrier in Hz
dlySpread = 0.5e-6;    % delay spread in seconds

% Get NR channel object and data
chan = nrCDLChannel('DelayProfile','CDL-C',...
    'DelaySpread',dlySpread, 'CarrierFrequency', fc, ...
    'NormalizePathGains', true);
chaninfo = info(chan);
```

- 3GPP has several deterministic multi-path models
- Called Cluster-Delay Line (CDL)
- Can be downloaded in MATLAB 5G Toolbox
  - Gives the gain, delay and angles of each path
  - This ex: CDL-C with 24 paths

Path delay profile



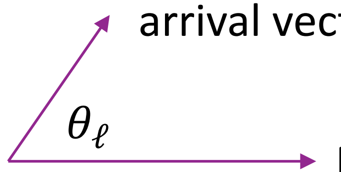
Path AoA azimith profile



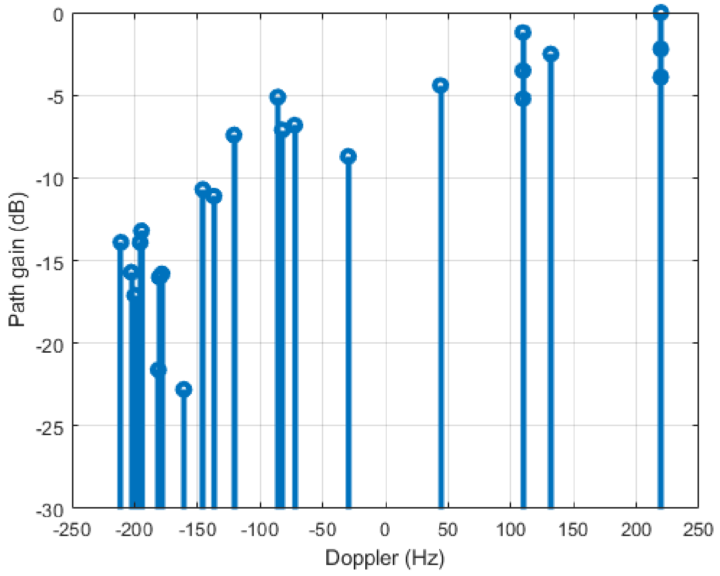

```
gain = chaninfo.AveragePathGains';
aoaAz  = chaninfo.AnglesAoA';
aoaEl = 90-chaninfo.AnglesZoA';
dly = chaninfo.PathDelays';
```

Computing the Doppler of Each Path

- Computing the Doppler shift of each path
  - Suppose that RX has velocity vector $\boldsymbol{v} = (v_x, v_y, v_z)$
  - Doppler shift of path ℓ is: $f_\ell = -\frac{f_c}{c}\boldsymbol{v}^T\boldsymbol{u}_\ell = -\frac{f_c}{c}\|\boldsymbol{v}\|\cos\theta_\ell$
- In this simulation: $v = 30$ m/s in x-axis

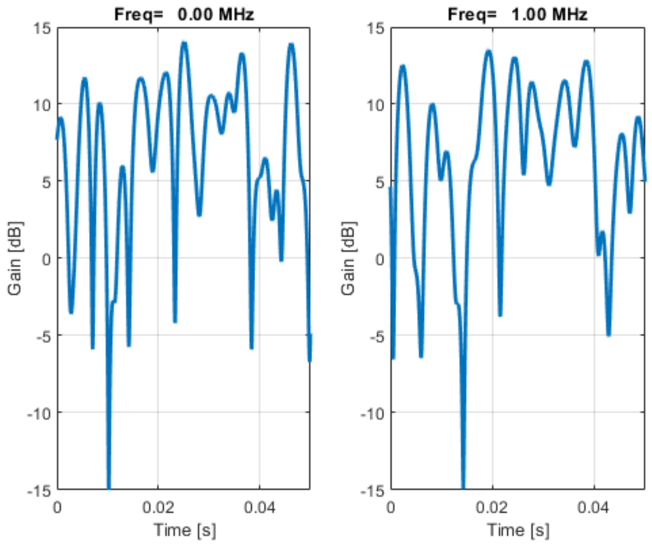



```
% UE velocity vector in m/s
ueVel = [30; 0; 0];

% Create a unit vectors in the direction of the path
[ux, uy, uz] = sph2cart(deg2rad(aoaAz), deg2rad(aoaEl), 1);
U = [ux uy uz];

% Compute Doppler shift
vc = physconst('Lightspeed');
fd = -fc/vc*U*ueVel;
```



Computing the Narrowband Response

- Narrowband response $H(t, f^0) = \sum_{\ell=1}^{L} g_\ell e^{j\omega_\ell t - j\omega^0 \tau_\ell}$
- Plot at $f^0 = 0$ and 1 MHz
- Max Doppler $f_{dmax} \approx 200$ Hz
  - See fast variation on order $\frac{1}{f_{dmax}} = 5$ ms
- Can see deep fades



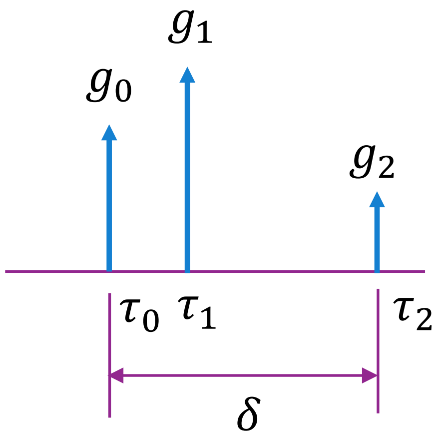

```
% Compute the fading gain over time
t = linspace(0,0.05,1000)';
npath = length(gain);

% Random initial phase on each path at freq = 0
phi0 = 2*pi*rand(npath,1);

% Number of frequencies to test
ftest = [0,1e6];
nfreq = length(ftest);

for i=1:nfreq
    % Initial phase
    phi = phi0 - 2*pi*ftest(i)*dly;

    % Gains on each path
    G = db2mag(gain).*exp(1i*(2*pi*fd*t' + phi) );

    % Sum over paths
    g = sum(G,1).';

    % Get power gain
    gpow = 20*log10(abs(g));

    subplot(1,nfreq,i);
    plot(t, gpow, 'Linewidth', 2);
    grid on;
    xlabel('Time [s]');
    ylabel('Gain [dB]');
    title(sprintf('Freq=%7.2f MHz', ftest(i)/1e6));
    ylim([-15,15]);
    xlim([0,0.05]);

end
```

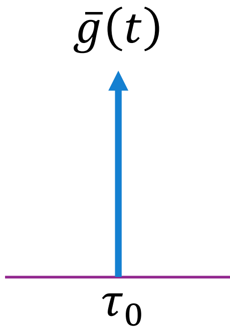
Narrowband Approximation Proof

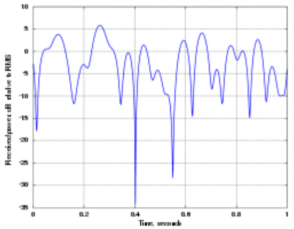
- Want to show: If $X(f)$ band-limited to $|f - f^0| \ll \frac{1}{2\delta}$ then $r(t) \approx H(t, f^0)x(t - \tau_0)$
- Prove this for $f^0 = 0$. Other frequencies are similar.
- Thus, $X(f)$ is bandlimited to $|f| \le \frac{1}{2\delta}$.
- Therefore, $x(s_1) \approx x(s_2) \quad for \ \ |s_1 - s_2| \le \delta$
- In particular $x(t - \tau_i) \approx x(t - \tau_0)$ since $|\tau - \tau_0| \le \delta$
- Hence:

$$r(t) = \sum_{\ell=1}^{L} g_\ell e^{j\omega_\ell t)} x(t - \tau_\ell) \approx \sum_{\ell=1}^{L} g_\ell e^{j\omega_\ell t} x(t - \tau_0)$$

$$= H(t, 0)x(t - \tau_0)$$



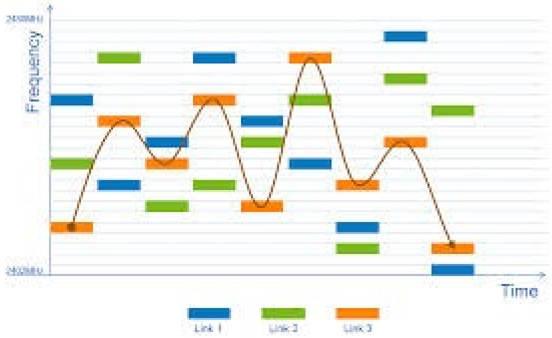




Examples: When is Narrowband Valid?

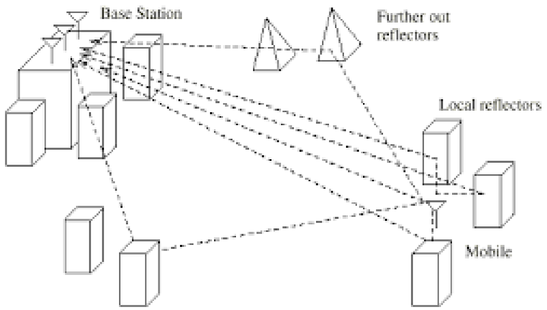
❑ Bluetooth

- Bluetooth hops over channels of bandwidth B =1 MHz each.
- Indoor delay spread typically $\ll 50$ ns
- Coherence bandwidth $\frac{1}{\delta} = \frac{1}{50} = 20$ MHz $\gg$ B
- Narrowband approximation valid

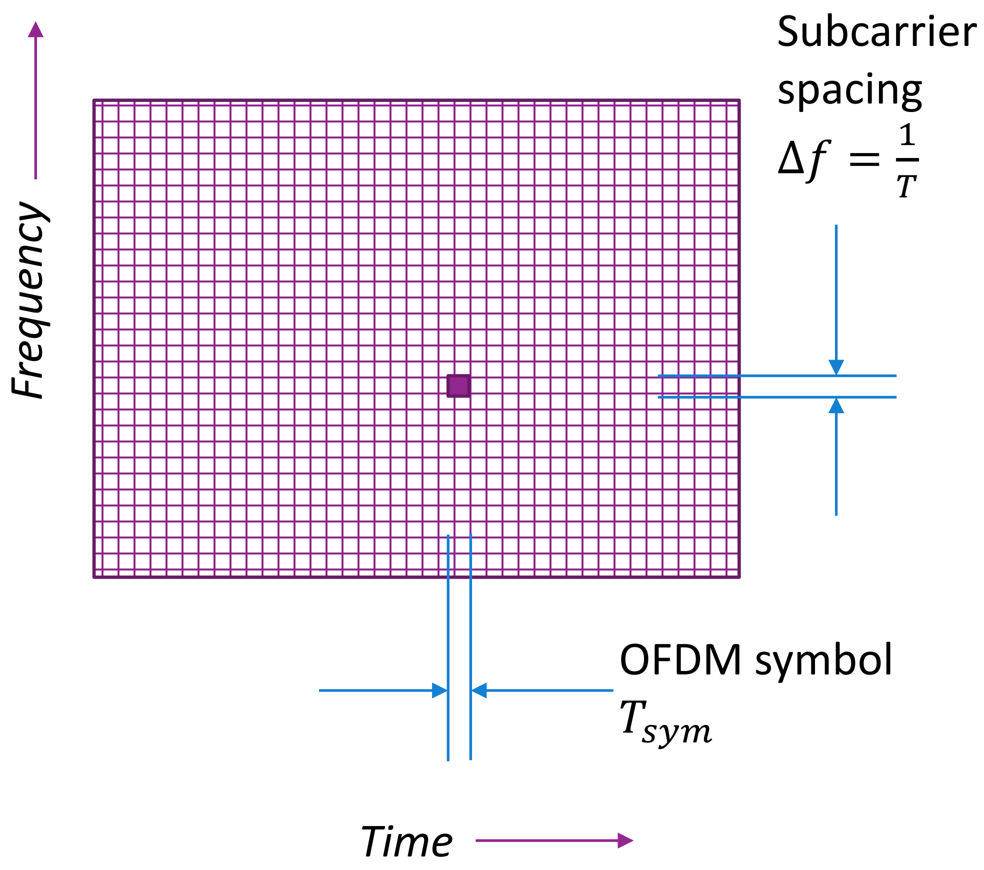


❑ LTE outdoor cellular system

- A typical channel is B =20 MHz
- Outdoor delay spread $\delta \approx 1\ \mu s$
- Coherence bandwidth $\frac{1}{\delta} = \frac{1}{1} = 1$ MHz $\ll$ B
- Narrowband approximation not valid



OFDM Time-Frequency Grid



- OFDM modulation: Widely-used method
  - 4G and 5G cellular systems
  - Many 802.11 standards
- Divide channel into sub-carriers and OFDM symbols
  - Resource element: One time-frequency point
- Data is transmitted is an array: $X[n, k]$
  - k = OFDM symbol index
  - n = subcarrier index
  - One complex value per RE.
  - Called a modulation symbol
- See digital communication class
  - We will also review again when we discuss equalization

OFDM Channel with Fading

- OFDM channel acts as multiplication:
  Under normal operation (delay spread is contained in CP):

$$\underbrace{Y[k,n]}_{\text{RX symbols}} = \underbrace{H[k,n]}_{\text{Channel}} \; \underbrace{X[k,n]}_{\text{TX symbols}}$$

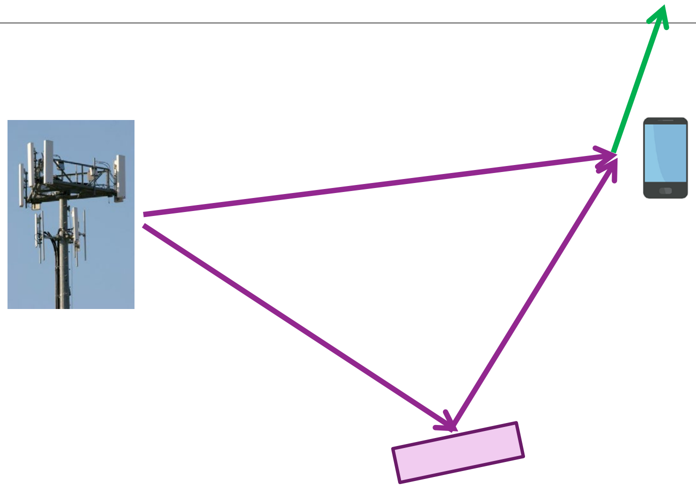
- OFDM channel gains can be computed from the multi-path components

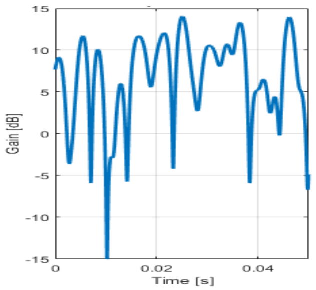
$$\boxed{H[k,n] = \sum_{\ell=1}^{L} \sqrt{E_\ell} e^{-2\pi j\,(Tkf_\ell + Sn\tau_\ell + \phi_\ell)}}$$

  - T = OFDM symbol time, S = sub-carrier spacing
  - For each path: f_ℓ =Doppler shift, τ_ℓ =Delay, ϕ_ℓ = phase of path, E_ℓ = path received energy

Summary

- Single path with no motion:
  - Delay and constant phase shift
- Local motion in single path causes Doppler
  - A time-varying phase rotation
  - But channel gain is constant
- Multiple paths causes fading
  - Constructive and destructive interference of paths
  - Variation in gain over time
- Described by a time-varying frequency response $H(t, f)$
  - Variations is time due to Doppler spread
  - Variations in frequency due to delay spread



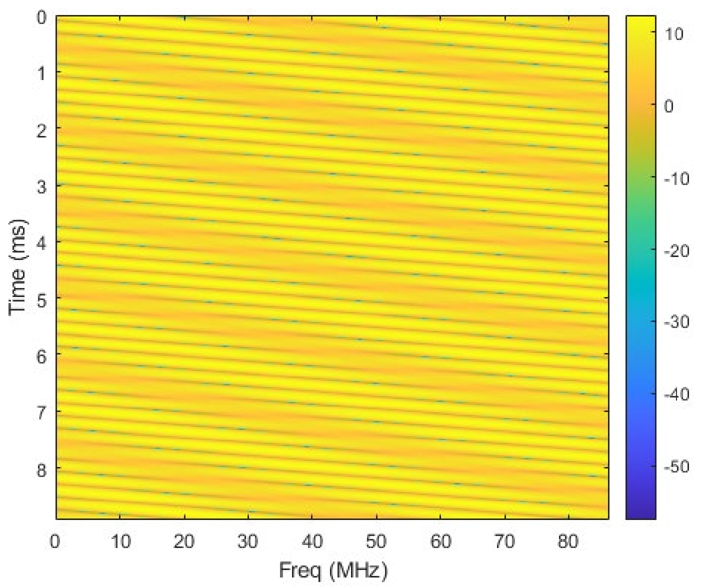


In-Class Exercise: OFDM Channel Response

Problem 4: Computing an OFDM Frequency Response

Consider a system with the following parameters. These paramters are similar to common configuration for a 5G NR system used in the mmWave system

```
scs = 120e3;  % sub-carrier spacing
nsc = 12*60;  % number of sub-carriers
tsym = 1e-3/14/8;       % OFDM symbol period
nsym = 1000;  % number of symbols to plot

% Channel parameters
fc = 73e9;  % carrier frequency
v = 10;     % RX velocity in m/s
dly = [0,20,50]'*1e-9;   % Delay in sec of the paths
theta = [0,pi/4,pi]';    % Path AoA relative to motion
gaindB = [0,-3,-5]';     % gain of each path in dB

% Random initial phase of the gains
npath = length(dly);
phi = rand(npath,1)*2*pi;
```

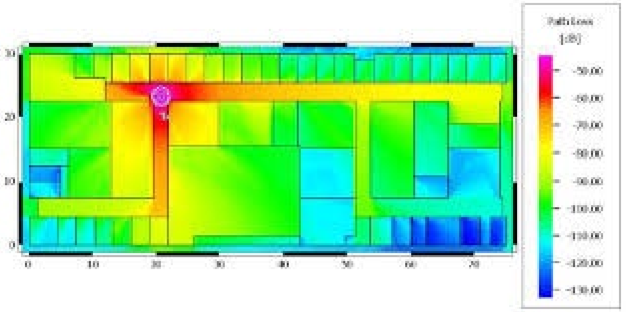


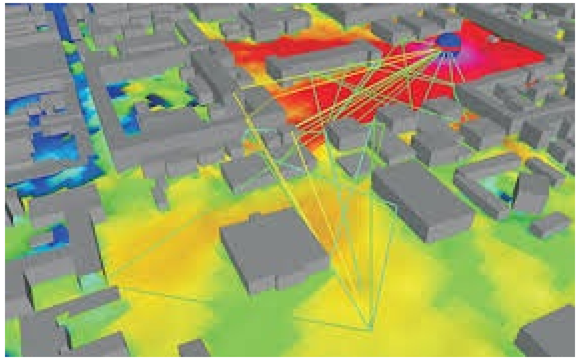
Outline

- ❑Review of Up- and Downconversion
- ❑Review of TX and RX Sampling
- ❑Doppler and Multi-Path Fading
- ❑Statistical Descriptions of Fading

Statistical Model

- Fading depends on the multipath distribution
- Multipath is site-specific
- Statistical model:
  - Describes a probability distribution of channels
  - Trained on an ensemble of channels in some environment
- Used in evaluation of communication system
- Example questions:
  - How well does a system do *on average*?
  - What is the *probability* that I will obtain sufficient coverage?



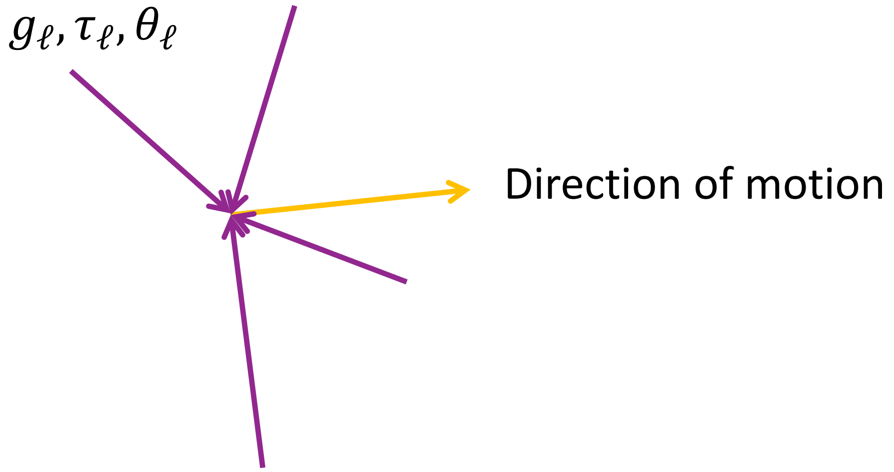


Random Path Statistical Model

- RX signal has many random, independent paths
- Time-varying frequency response:

$$h(t,f) = \frac{1}{\sqrt{L}} \sum_{\ell=1}^{L} g_\ell e^{2\pi i (t f_d \cos\theta_\ell + f\tau_\ell)}$$

  - Assume $(g_\ell, \tau_\ell, \theta_\ell)$ i.i.d.
  - Path gains: g_ℓ are zero mean and $\mathrm{E}|g_\ell|^2 = G_0$

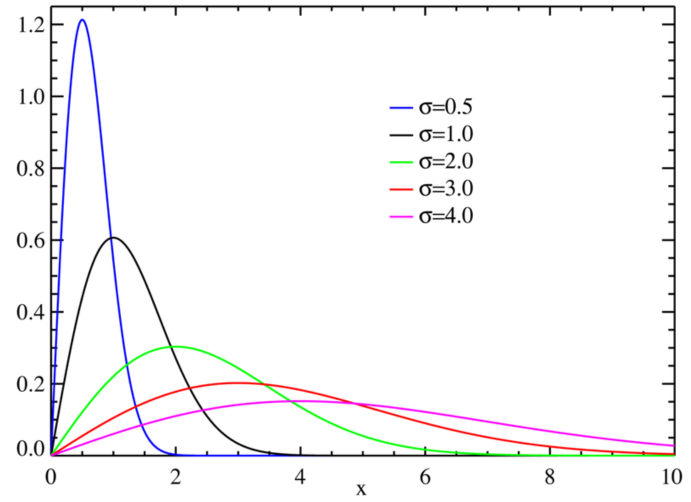


- As $L \to \infty$, $h(t,f)$ is a complex Gaussian, $h(t,f) \sim CN(0, G_0)$
  - Follows from Central Limit Theorem
  - Independent real and imaginary components
  - G_0: Average power gain
  - Variance $G_0/2$ for real and imaginary components

Rayleigh Distribution

- Rayleigh fading: Channel response is $h(t,f) \sim CN(0, G_0)$
- Let $R = |h|$ magnitude
  - Represents amplitude gain
- Has Rayleigh distribution:
  - PDF: $p(r) = \frac{2r}{P} e^{-r^2/P}$
  - CDF: $\mathrm{P}(R \leq r) = 1 - e^{-r^2/P}$
  - Mean: $E(R) = \sqrt{\frac{G_0 \pi}{2}}$
  - Second moment: $ER^2 = G_0$

Probability distribution



https://en.wikipedia.org/wiki/Rayleigh_distribution

Exponential Distribution

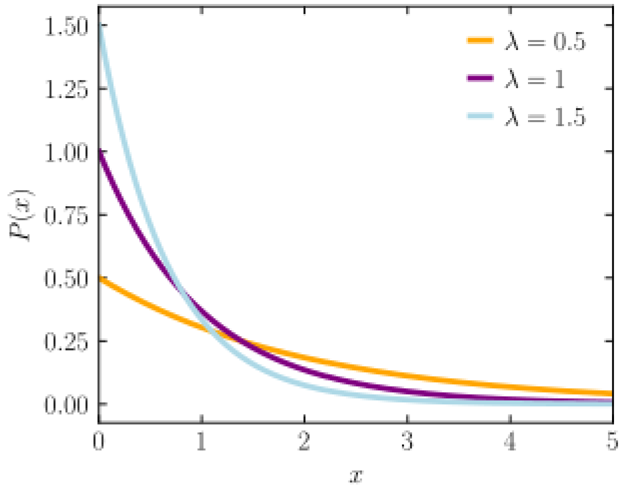
- Consider Rayleigh fading complex gain $h \sim CN(0, G_0)$
- Magnitude $R = |h|$ is Rayleigh

$$P(R \geq r) = e^{-r^2/G_0}$$

- Instantaneous gain $G = |h|^2$ has exponential distribution

$$P(G \geq g) = P(R \geq \sqrt{g}) = e^{-g/G_0}$$

  - Average gain is $E(G) = E|h|^2 = G_0$
- For channel, G represent power gain (in linear scale)



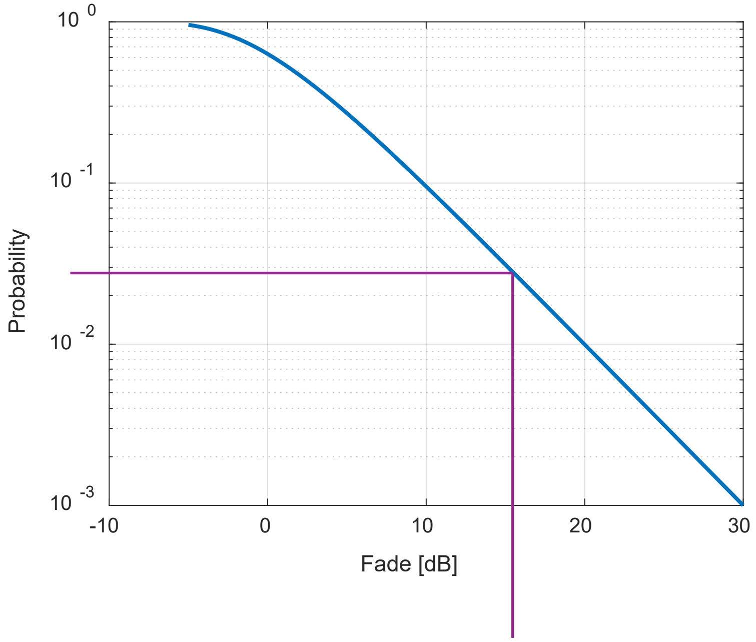
PDF for $\lambda = \frac{1}{E(G)}$

Example Calculation

- Suppose the channel experiences Rayleigh fading.
- What is probability gain will be 15 dB below the average?
  - Called a 15 dB fade.
- Answer:
  - Gain is 15 dB below average when $G \leq 10^{-0.1(15)} G_0$
  - From exponential distribution:

$$P(G \leq \beta G_0) = 1 - e^{-\beta G_0/G_0} = 1 - e^{-\beta}$$

  - For small β, $P(G \leq \beta G_0) \approx \beta$
  - For 15 dB fade, $\beta = 10^{-0.1(15)} \approx 0.032.$



Doppler Spectra

- Consider statistical model:

$$h(t,f) = \frac{1}{\sqrt{L}} \sum_{\ell=1}^{L} g_\ell e^{2\pi i (t f_d \cos\theta_\ell + f\tau_\ell)}$$

  - Paths are i.i.d. and g_ℓ are zero mean, $E|g_\ell|^2 = G_0$
  - Assume L is large
- For a given (t,f), complex gain $h(t,f) \sim CN(0, G_0)$
- As varies (t,f), $h(t,f)$ is a Gaussian random process
- Auto-correlation:

$$\begin{aligned} R(\delta t, \delta f) &= E[h(t,f) h^*(t+\delta t, f+\delta f)] \\ &= G_0\, E\left\{e^{2\pi i (\delta t f_d \cos\theta_\ell + \delta f \tau_\ell)}\right\} \end{aligned}$$

  - Describes how correlated the process is over time and frequency
  - Depends on the distribution of angles θ_ℓ and delays τ_ℓ

Jakes Model

- At a fixed frequency f_0:

$$h(t,f_0) = \frac{1}{\sqrt{L}}\sum_{\ell=1}^{L} g_\ell e^{2\pi i(tf_d\cos\theta_\ell + f_0\tau_\ell)}$$
$$= \frac{1}{\sqrt{L}}\sum_{\ell=1}^{L} g'_\ell e^{2\pi i t f_d \cos\theta_\ell}$$

- Complex Gaussian process
- Statistics depend on angular distribution

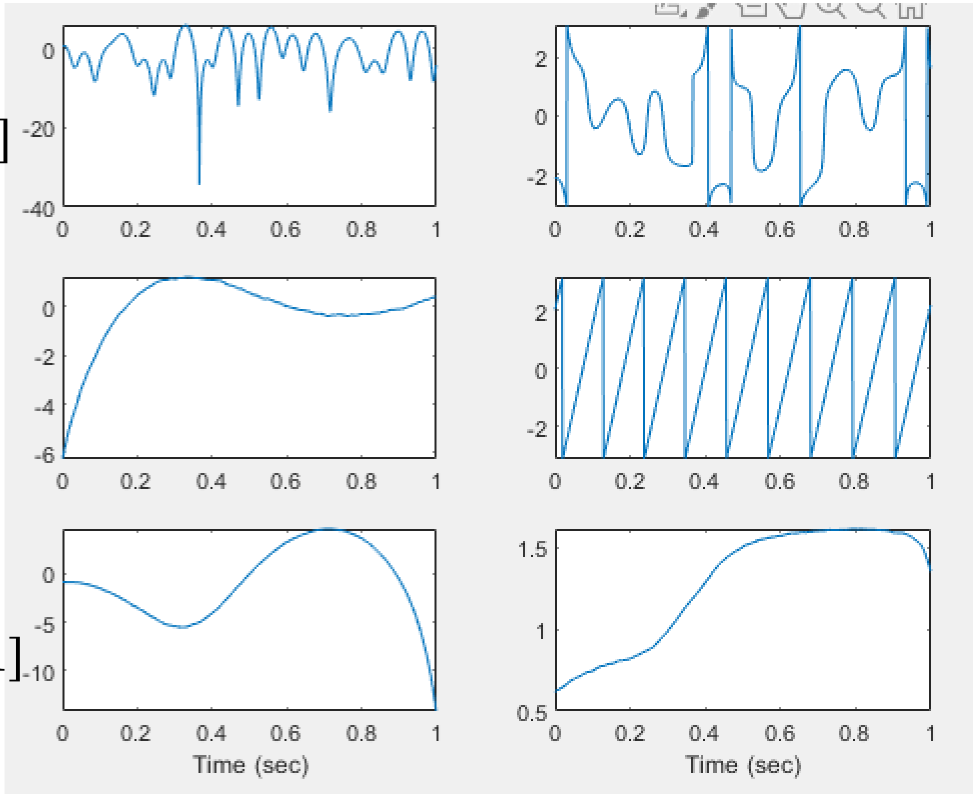
- Jakes model:
  - Angles uniform from $[0,2\pi]$
- Asymmetric Jakes:
  - $\theta \in [\theta_1,\theta_2]$ uniform
- Angular spread:
  - Arises from diffuse reflection

Gain (dB) | Angle (rads)

Jakes
Angles unif $[0,2\pi]$

Asym Jakes
$\cos\theta \in [0.9,1]$

Asym Jakes
$\cos\theta \in [-0.1,0.1]$



Fading Models in MATLAB

- Comm Toolbox:
  - Efficient, general fading models
- Create a comm.RayleighChannel object
- Run the channel to get:
  - Output and gain

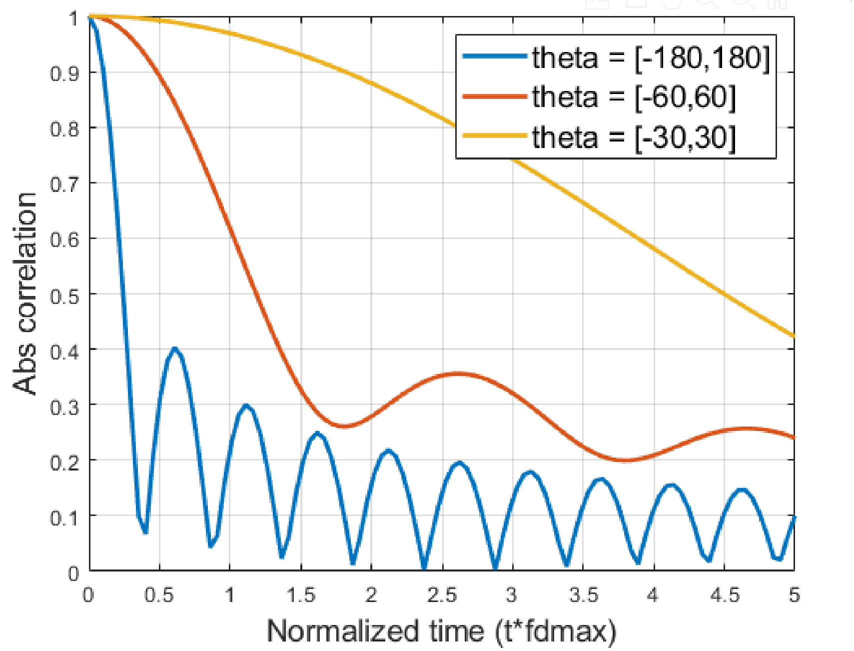
```
% Create Doppler models
nmod = 3;
dopMod = cell(nmod,1);
dopMod{1} = doppler('Jakes');
dopMod{2} = doppler('Asymmetric Jakes', [0.9 1]);
dopMod{3} = doppler('Asymmetric Jakes', [-0.1 0.1]);

% Simulate the channel gains for each model
for i = (1:nmod)
    chan = comm.RayleighChannel(...
        'SampleRate', fsym, 'AveragePathGains', 0, ...
        'MaximumDopplerShift', fdmax,...
        'DopplerSpectrum', dopMod{i}, ...
        'PathGainsOutputPort', true);

    [y, gain] = chan.step(x);
```

Auto-Correlation

- Fix a frequency f_0 and consider auto-correlation over time:

$$R(\delta t) = E[h(t, f_0) h^*(t + \delta t, f_0)] = G_0\, E\left\{e^{2\pi i(\delta t f_d \cos\theta_\ell)}\right\}$$

  - Expectation is over angle θ_ℓ
  - Depends on distribution of θ_ℓ
- Plot: $R(\delta t)$ for θ uniform in $[-\theta_m, \theta_m]$
  - Computed numerically in MATLAB (see demo)
  - Plotted vs normalized delay $f_d \delta t$
  - For $\theta_m = 180° \Rightarrow$ Jakes spectra
  - Uncorrelated at $\delta t \approx \frac{1}{2f_d}$
- As angular distribution is smaller:
  - Correlation is higher with delay
  - Highly directional channel vary slower

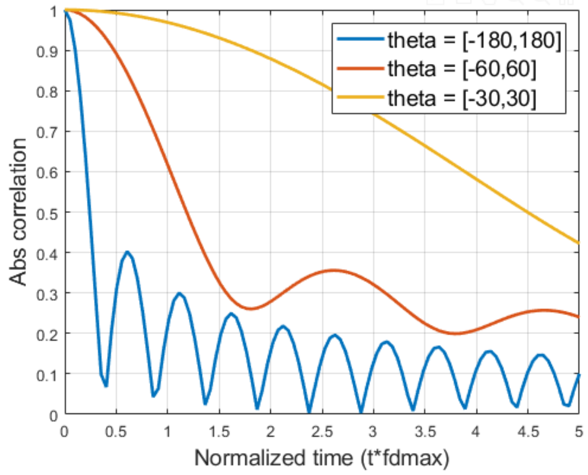


Computing Auto-Correlation Numerically

- Correlation in previous slide

$$R(\delta t) = G_0\, E\{e^{2\pi i(\delta t f_d \cos\theta_\ell)}\}$$

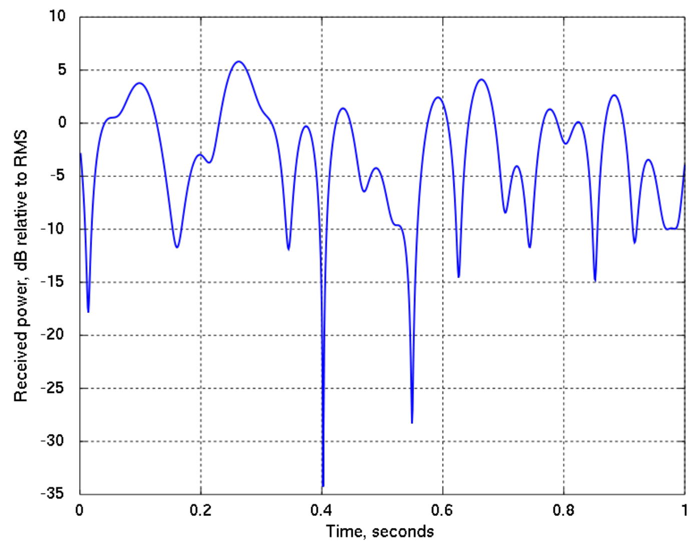
  - Expectation is over angle θ_ℓ
- Generally, no analytic solution
- Compute via numerical integration
  - See demo




```
% Function handle for the integrand
rfun = @(theta,tf) exp(1i*2*pi*cos(deg2rad(theta))*tf);

% Angle ranges to test
thetaMaxTest = [180, 60, 30];
ntheta = length(thetaMaxTest);

% Times to test:
t = linspace(0,5,100);
nt = length(t);

% Compute correlation for each angle range and time
Rcorr = zeros(nt, ntheta);
legStr = cell(ntheta,1);
for j = 1:ntheta
    thetam = thetaMaxTest(j);
    legStr{j} = sprintf('theta = [-%d,%d]', thetam, thetam);
    for i = 1:nt

        Z1 = integral(@(theta) rfun(theta,t(i)), -thetam, thetam);
        Rcorr(i,j) = abs(Z1/(2*thetam));
    end
end

plot(t, Rcorr, 'LineWidth', 2);
grid on;
xlabel('Normalized time (t*fdmax)', 'FontSize', 14);
ylabel('Abs correlation', 'FontSize', 14);
legend(legStr, 'FontSize', 14);
```

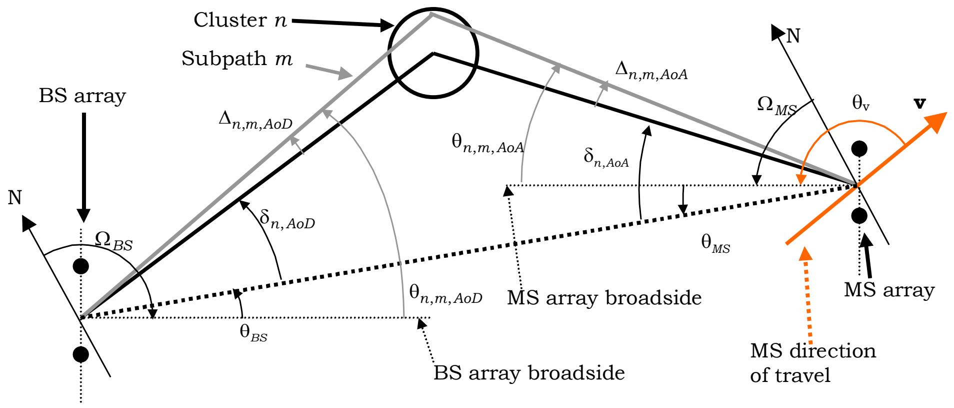
Coherence Time and Frequency

- Consider time varying frequency response $H(t, f)$
- Coherence time:
  - Max interval Δt where $H(t, f) \approx H(t + \Delta t, f)$
  - How fast channel changes in time
  - Related to Doppler spread $\approx \frac{1}{f_{max} - f_{min}}$
- Coherence bandwidth
  - Max interval Δf where $H(t, f) \approx H(t, f + +\Delta f)$
  - How fast channel changes in frequency
  - Related to delay spread $\approx \frac{1}{\tau_{max} - \tau_{min}}$
- Critical for many procedures:
  - Channel estimation, tracking, coding, ARQ, …
  - More on this later



Realization of a Jakes process with $1/f_{max}$ = 0.1 sec

Winner-3GPP-Spatial Cluster Model



From 3GPP SCM-132

- ❑Paths arrive in clusters.
- ❑Clusters have subpaths (also called rays)
- ❑Each cluster has:
  - Center angle and a statistical model for the delay and angular spread

Fading at Different Time Scales

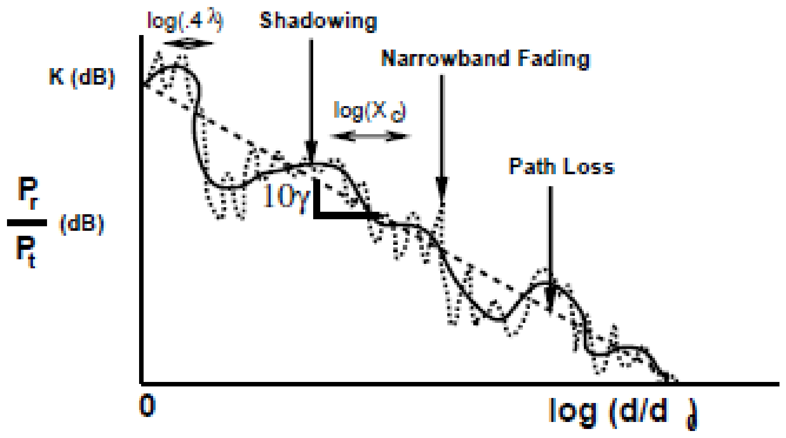


Figure 3.8: Combined Path Loss, Shadowing, and Narrowband Fading.

From Goldsmith, “Wireless Communications”

Three mechanisms for path loss variations

Distance-based path loss

Shadowing

Small-scale multi-path fading

Slower

Faster

Fading at Different Scales Models

Source of variation	Mathematical model	Typical spatial coherence	Typical temporal coherence
Small-scale fading from multi-path fading	Rayleigh or Rician distribution	~ 1 wavelength	15 ms (v=10m/s, fc=2GHz)
Large-scale fading from variations in shadowing	Lognormal distribution	10 to 100 m	1 to 10 sec
Path loss variations	Path loss exponent	100 m or larger	10 sec

- Different fading processes and variations occur at much different time / space scales
- Methods to combat these are different