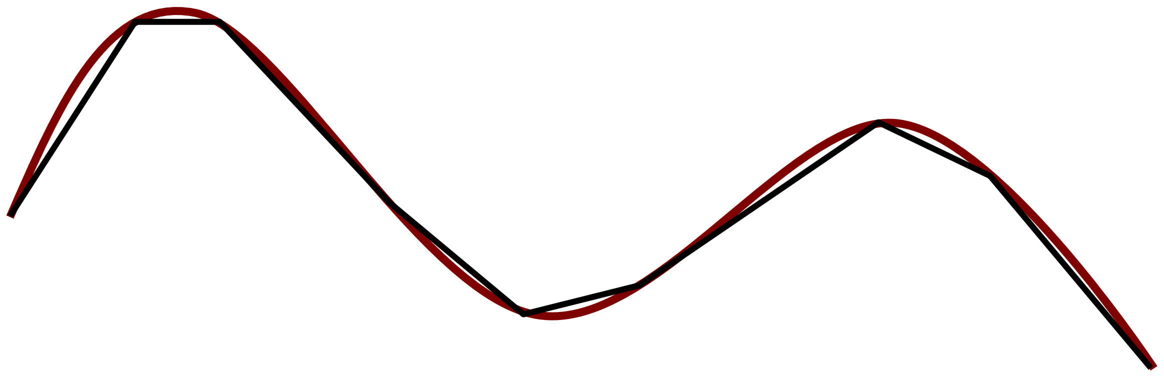
CENG 477
Introduction to Computer Graphics

Representing Curves and Surfaces

Introduction

- There are no perfectly straight lines or flat faces in nature!
- Therefore, representing and generating smooth shapes is a requirement in many CG applications
- Rendering will still use lines and triangles but their vertices will be sampled from a curve or surface



Curves

- There are many ways to represent curves:
  - must be practical (easy to manage and render)
  - must be flexible (general enough to be used in various modeling tasks)
- A good compromise is cubic polynomials
- Each x, y, z coordinate is expressed as a cubic polynomial with potentially different coefficients (t is the parameter)

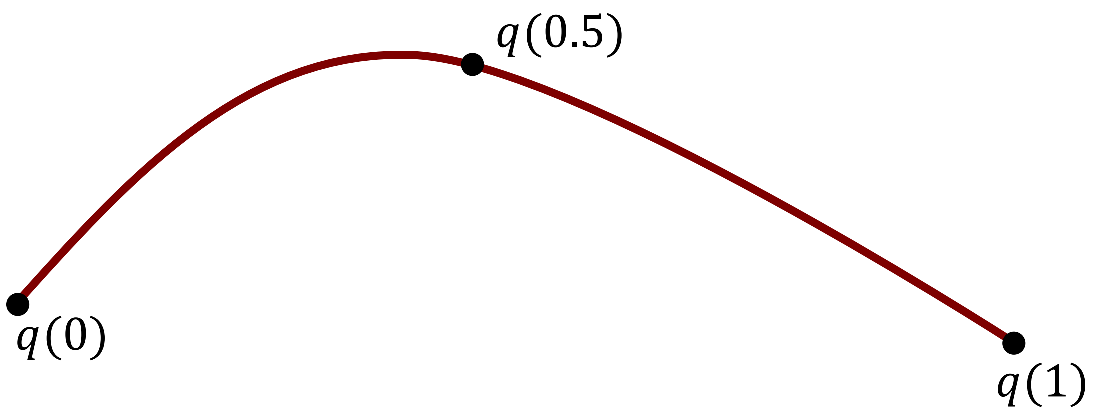
$$x(t) = a_x t^3 + b_x t^2 + c_x t + d_x$$

$$y(t) = a_y t^3 + b_y t^2 + c_y t + d_y$$

$$z(t) = a_z t^3 + b_z t^2 + c_z t + d_z$$

Curves

- For a given t value, $q(t) = [x(t)\ y(t)\ z(t)]$represents the 3D position along the curve
- Similar to rays in ray tracing except that it may follow a curvy path instead of a straight one!
- The t parameter is taken to be in range $[0, 1]$



Cubic Polynomials

- As cubic polynomials have 4 unknowns (per component), we need 4 constraints to find them
- Different curves are distinguished by different constraints
  - **Hermite curves:** 2 end points + 2 tangent vectors
  - **Bezier curves:** 2 end points + 2 control points
  - **Splines:** 4 control points (for each piece of the curve)

Matrix Form

- Cubic polynomials are conveniently expressed in matrix form:

$$x(t) = a_x t^3 + b_x t^2 + c_x t + d_x$$

$$y(t) = a_y t^3 + b_y t^2 + c_y t + d_y$$

$$z(t) = a_z t^3 + b_z t^2 + c_z t + d_z$$

$$Q(t) = [x(t)\ y(t)\ z(t)] = [t^3\ t^2\ t\ 1]\begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \\ d_x & d_y & d_z \end{bmatrix}$$

$$Q(t) = TC$$

Matrix Form

- We will also need the derivative of this curve to specify tangent vectors

$$\frac{dQ(t)}{dt} = Q'(t) = \frac{dT}{dt}C = [3t^2\ 2t\ 1\ 0]C$$

Constraints

- Imagine that we want to specify certain geometrical constraints such as:
  - Start point
  - End point
  - Start direction (i.e. tangent vector at start point)
  - End direction (i.e. tangent vector at end point)
- We need to split the matrix C into two to allow embedding of these constraints

Constraints

- Rewrite $C = MG$, where G represents the geometry constraints

$$C = \begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \\ d_x & d_y & d_z \end{bmatrix} = \underbrace{\begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} g_{1x} & g_{1y} & g_{1z} \\ g_{2x} & g_{2y} & g_{2z} \\ g_{3x} & g_{3y} & g_{3z} \\ g_{4x} & g_{4y} & g_{4z} \end{bmatrix}}_{G}$$

- Here, M is called the basis matrix
- G is called the geometry or the constraints matrix
- Different types of curves differ in M and G

Constraints

- Note that $Q(t) = TMG$
- For instance:

$$x(t) = [t^3\ t^2\ t\ 1]\left(g_{1x}\begin{bmatrix} m_{11} \\ m_{21} \\ m_{31} \\ m_{41} \end{bmatrix} + g_{2x}\begin{bmatrix} m_{12} \\ m_{22} \\ m_{32} \\ m_{42} \end{bmatrix} + g_{3x}\begin{bmatrix} m_{13} \\ m_{23} \\ m_{33} \\ m_{43} \end{bmatrix} + g_{4x}\begin{bmatrix} m_{14} \\ m_{24} \\ m_{34} \\ m_{44} \end{bmatrix} \right)$$

Blending Functions

- Rewriting this gives us:

$$x(t) = [t^3\ t^2\ t\ 1]\left(g_{1x}\begin{bmatrix} m_{11} \\ m_{21} \\ m_{31} \\ m_{41} \end{bmatrix} + g_{2x}\begin{bmatrix} m_{12} \\ m_{22} \\ m_{32} \\ m_{42} \end{bmatrix} + g_{3x}\begin{bmatrix} m_{13} \\ m_{23} \\ m_{33} \\ m_{43} \end{bmatrix} + g_{4x}\begin{bmatrix} m_{14} \\ m_{24} \\ m_{34} \\ m_{44} \end{bmatrix} \right)$$

$$\Downarrow$$

$$\begin{aligned} x(t) = {} & (t^3 m_{11} + t^2 m_{21} + t m_{31} + m_{41}) g_{1x} + \\ & (t^3 m_{12} + t^2 m_{22} + t m_{32} + m_{42}) g_{2x} + \\ & (t^3 m_{13} + t^2 m_{23} + t m_{33} + m_{43}) g_{3x} + \\ & (t^3 m_{14} + t^2 m_{24} + t m_{34} + m_{44}) g_{4x} \end{aligned}$$

- That is, the curve is a weighted sum of the elements of the geometry matrix

Blending Functions

- The weights are each cubic polynomials of t
- These polynomials are called blending functions

$$\begin{aligned} x(t) = {} & (t^3 m_{11} + t^2 m_{21} + t m_{31} + m_{41}) g_{1x} + \\ & (t^3 m_{12} + t^2 m_{22} + t m_{32} + m_{42}) g_{2x} + \\ & (t^3 m_{13} + t^2 m_{23} + t m_{33} + m_{43}) g_{3x} + \\ & (t^3 m_{14} + t^2 m_{24} + t m_{34} + m_{44}) g_{4x} \end{aligned}$$

$$\Downarrow$$

$$B_{1x4} = T_{1x4} M_{4x4}$$

Vector of
blending functions

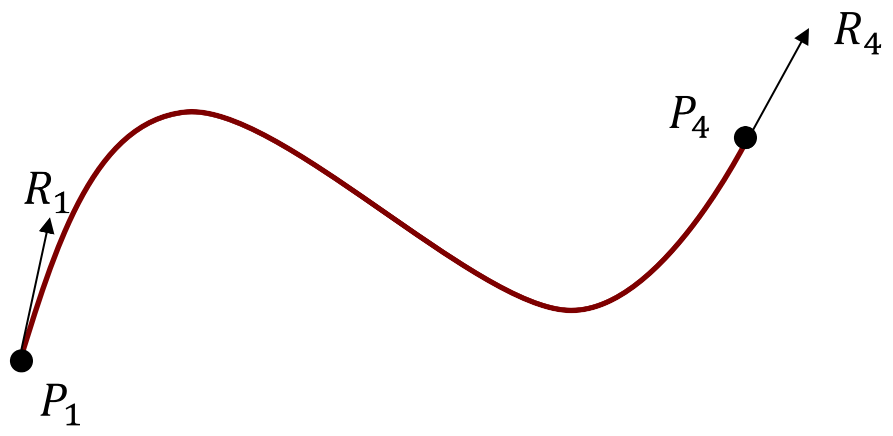
Two Important Curves

- Now we will see how this background can be used to define two types of curves:
  - Hermine curves
  - Bezier curves
- Both curves can draw the same curves (they are equally powerful) but they have different geometry constraints

hermine, bezier, and natural splice curves differ in how they specify the additional two constraints (known 2 constraints are the start and endpoints of the curves (t=0 and t=1))

Hermite Curves

- The constraints of Hermite curves are:
  - **Two end points:** P_1 and P_4
  - **Two tangent vectors:** R_1 and R_4



Hermite Curves

- The geometry matrix then becomes:

$$G = \begin{bmatrix} P_1 \\ P_4 \\ R_1 \\ R_4 \end{bmatrix} = \begin{bmatrix} P_{1x} & P_{1y} & P_{1z} \\ P_{4x} & P_{4y} & P_{4z} \\ R_{1x} & R_{1y} & R_{1z} \\ R_{4x} & R_{4y} & R_{4z} \end{bmatrix}$$

- To find M remember that the curve in matrix form we have:

$$Q(t) = TMG$$

- And for derivative: $Q'(t) = T'MG$

Hermite Curves

- We can now plug in values for the t parameter
  - Compute $Q(0), Q(1), Q'(0), Q'(1)$

$$Q(0) = [0\ 0\ \ 0\ 1]MG$$
$$Q(1) = [1\ 1\ \ 1\ 1]MG$$
$$Q'(0) = [0\ 0\ \ 1\ 0]MG$$
$$Q'(1) = [3\ 2\ \ 1\ 0]MG$$

- This is the same as:

$$\begin{bmatrix} Q(0) \\ Q(1) \\ Q'(0) \\ Q'(1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} MG$$

Hermite Curves

- Remember that G was equal to:

$$G = \begin{bmatrix} P_1 \\ P_4 \\ R_1 \\ R_4 \end{bmatrix} = \begin{bmatrix} Q(0) \\ Q(1) \\ Q'(0) \\ Q'(1) \end{bmatrix}$$

- So we have:

$$\begin{bmatrix} Q(0) \\ Q(1) \\ Q'(0) \\ Q'(1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} M \begin{bmatrix} Q(0) \\ Q(1) \\ Q'(0) \\ Q'(1) \end{bmatrix}$$

Only possible
if M is the inverse
of the matrix

Same

Hermite Curves

- Therefore, Hermite curves have the following basis matrix:

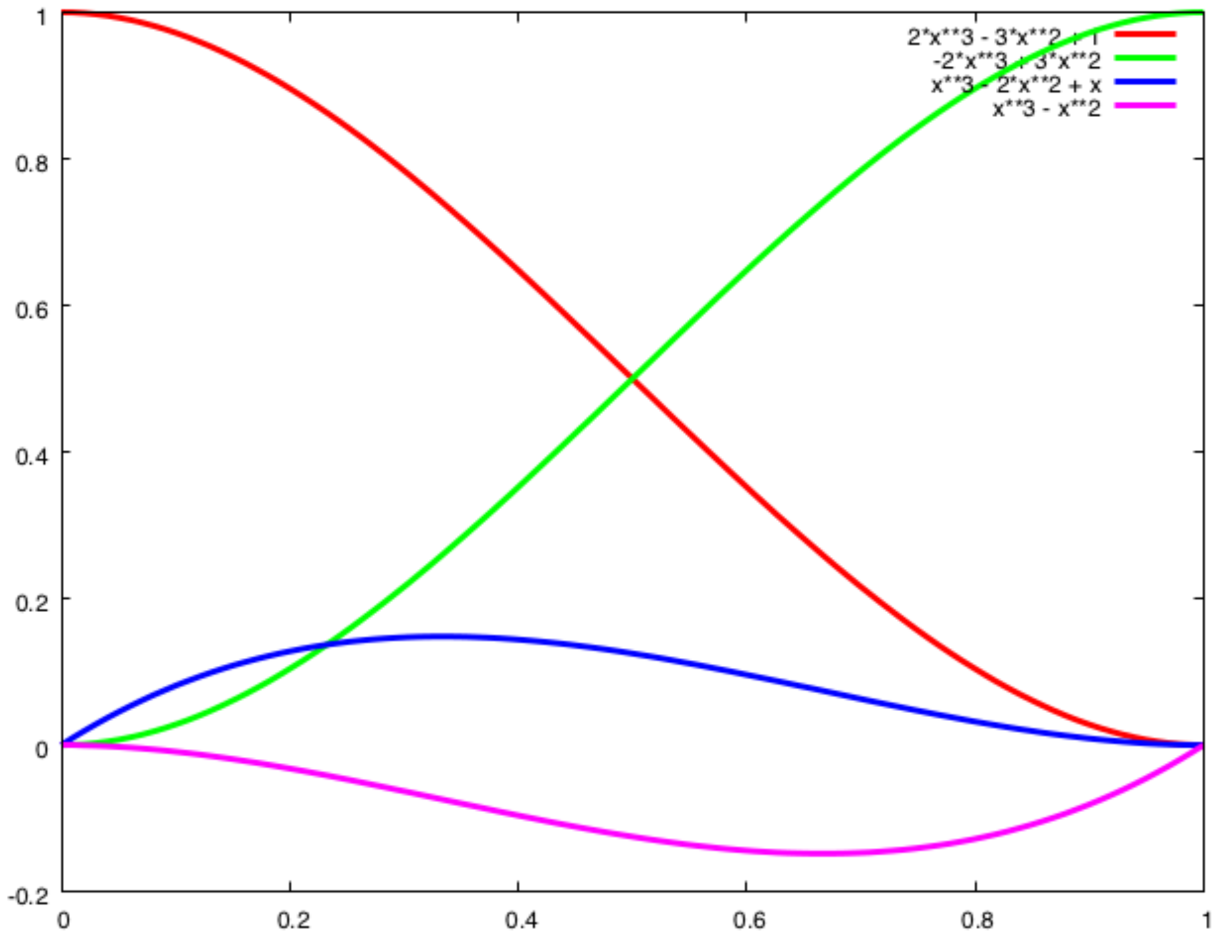
$$M = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

- This yields the following blending functions:

$$B = \begin{bmatrix} 2t^3 - 3t^2 + 1 \\ -2t^3 + 3t^2 \\ t^3 - 2t^2 + t \\ t^3 - t^2 \end{bmatrix}^T$$

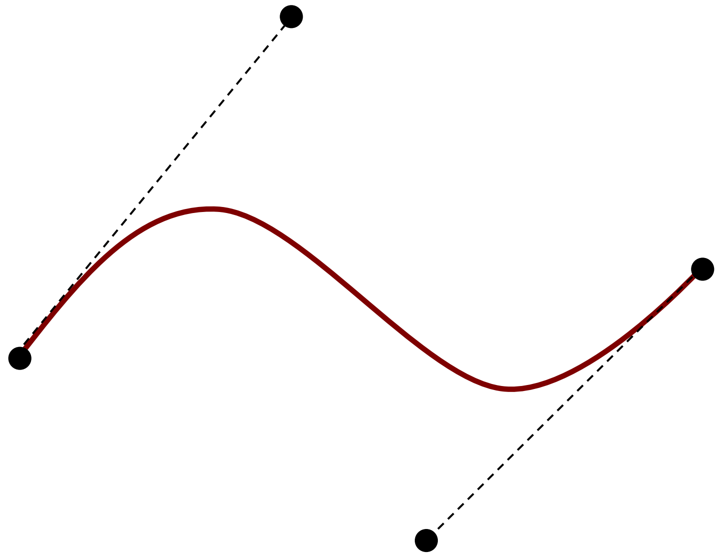
Let's plot these with gnuplot and do some experimentation with Matlab!

Hermite Blending Functions



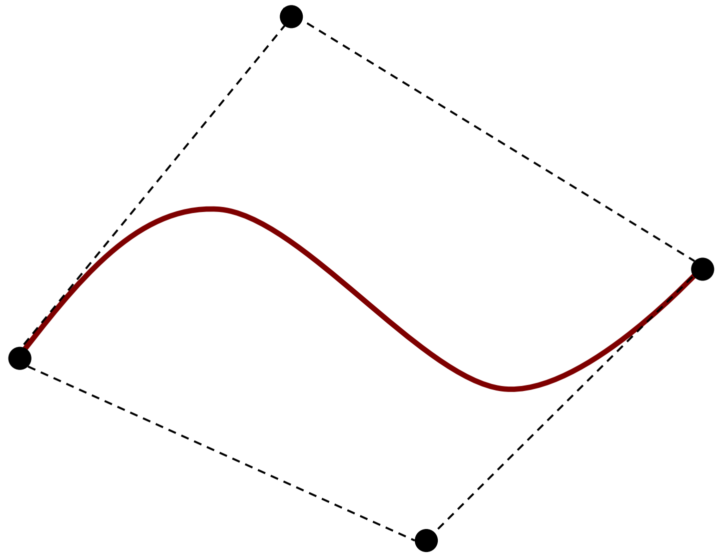
Bezier Curves

- Bezier curves can draw the same curves as Hermite curves
- They are defined using control points instead of derivatives
- Two control points are interpolated and two control points are approximated



Bezier Curves

- The curve lies entirely within the convex-hull of these four points
- This is useful for clipping, culling, and intersection tests as the convex-hull can be tested first instead of each line segment



Bezier Curves

- Bezier curves are related to the Hermite curves as:

$$R_1 = 3(P_2 - P_1)$$

$$R_4 = 3(P_4 - P_3)$$

- The factor 3 ensures that P_2 has the highest weight at $t = 1/3$ and P_3 has the highest weight at $t = 2/3$, logically dividing the curve into 3 pieces

Bezier Curves

- In matrix form, this relationship can be express as:

$$G_H = \begin{bmatrix} P_1 \\ P_4 \\ R_1 \\ R_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 3 & 0 & 0 \\ 0 & 0 & -3 & 3 \end{bmatrix} \underbrace{\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix}}_{G_B}$$

This matrix translates Bezier geometry matrix to the Hermite geometry matrix (let’s call this matrix as M_{BH})

Bezier Curves

- Then Bezier curves can be defined as:

$$Q(t) = TM_H G_H = TM_H M_{BH} G_B$$

$$Q(t) = TM_B G_B$$

where $M_B = M_H M_{BH}$

$$Q(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix}$$

Bezier Curves

- This is equivalent to:

$$Q(t) = (-t^3 + 3t^2 - 3t + 1)P_1 + (3t^3 - 6t^2 + 3t)P_2 + (-3t^3 + 3t^2)P_3 + (t^3)P_4 \quad = \quad (1-t)^3 P_1 + 3t(1-t)^2 P_2 + 3t^2(1-t)P_3 + t^3 P_4 +$$

- These are called Bernstein polynomials
  - Their sum is always 1
  - They are always non-negative when $t \in [0,1]$
  - That's why the resulting curve is in the convex-hull of P_1, P_2, P_3, P_4
  - Bernstein polynomial of degree n is $B_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i}$

Bezier Curves

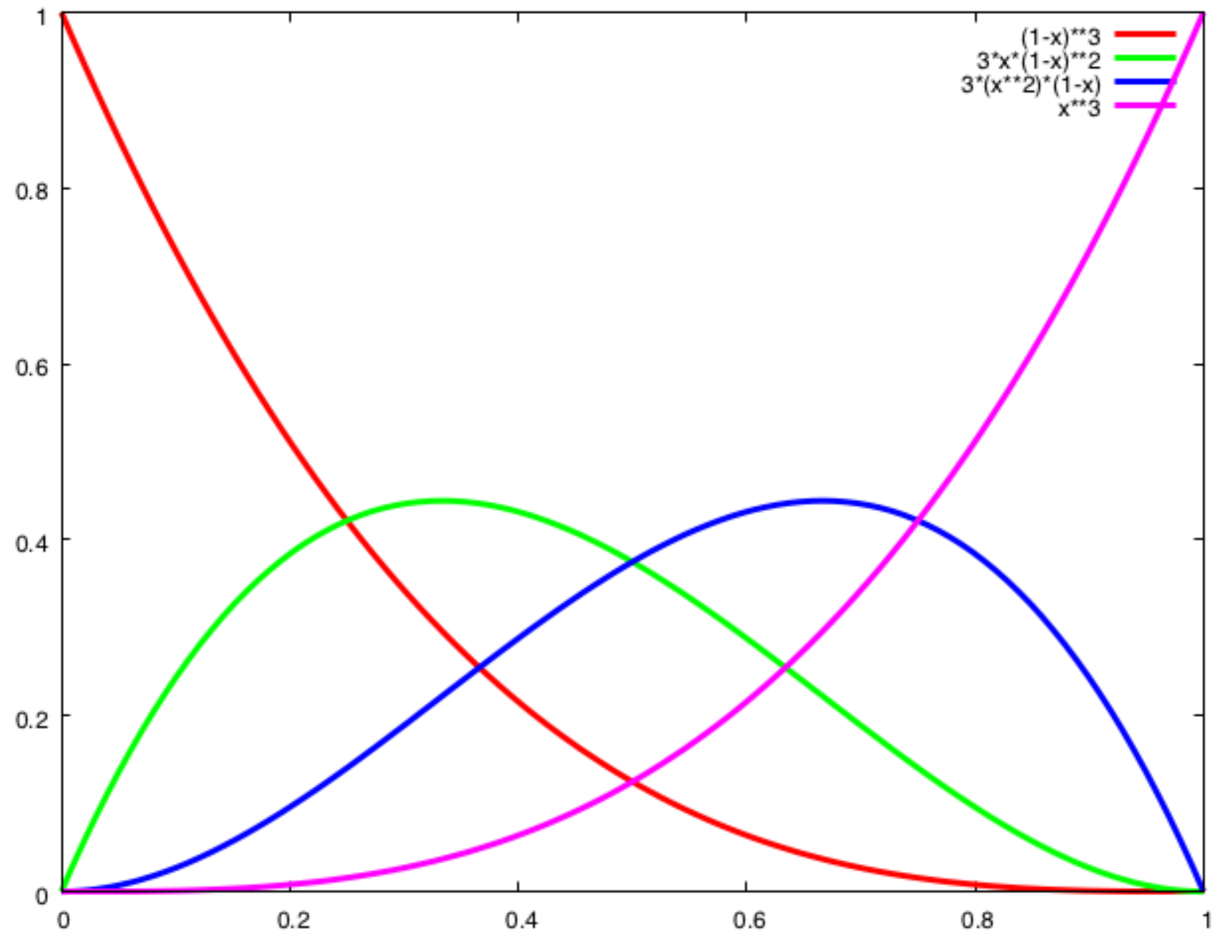
- Bezier curve is the sum of Bernstein polynomials

$$Q(t) = (1-t)^3 P_1 + 3t(1-t)^2 P_2 + 3t^2(1-t) P_3 + t^3 P_4$$

$$Q(t) = \sum_{i=0}^{n} B_{i,n}(t) P_{i+1}$$

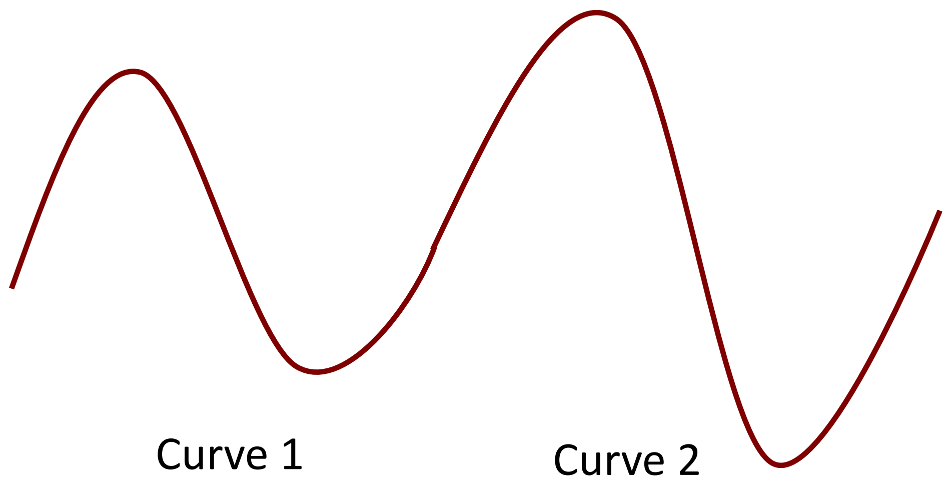
$$B_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

Bernstein Polynomials



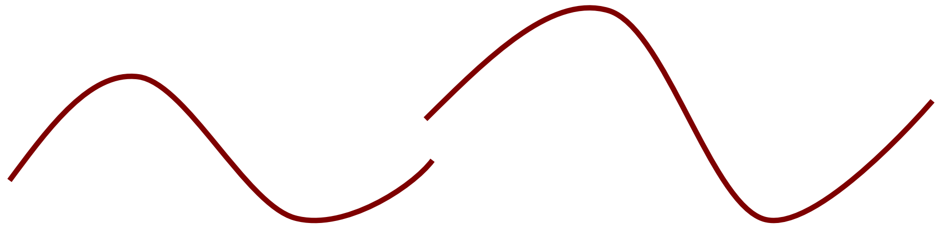
Continuity

- Until now, we learned to draw a single curve segment
- If we want to combine multiple curve segments, we must ensure maintaining continuity

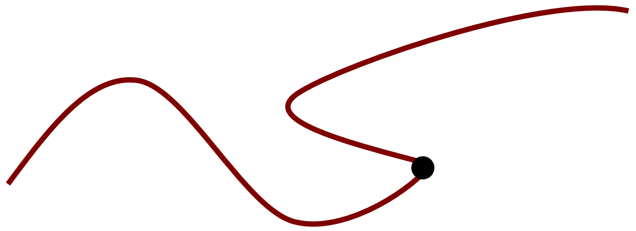


Types of Continuity

- **No continuity:**
  - The curves do not meet

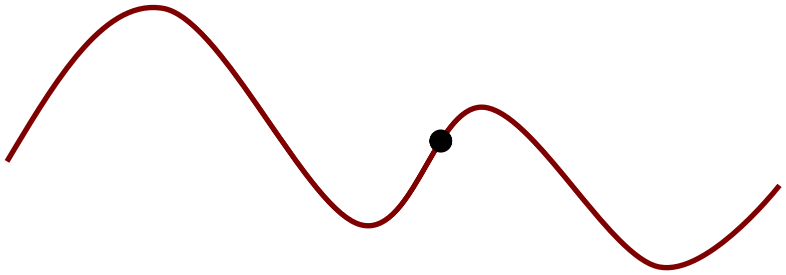


- **C0 continuity:**
  - The end points meet, also know as positional continuity

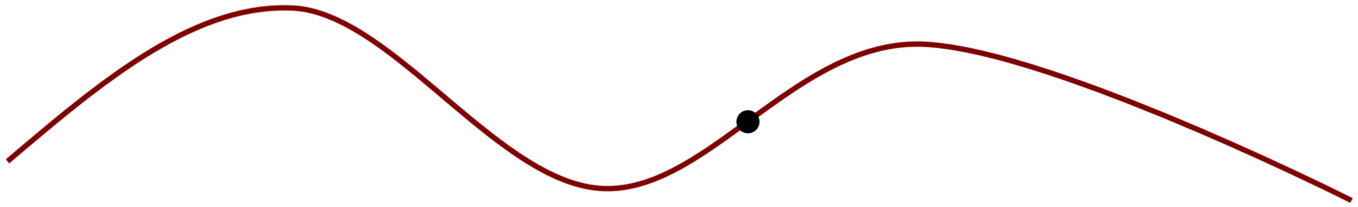


Types of Continuity

- **C1 continuity:**
  - The curves meet and have identical tangent vectors at the connection



- **C2 continuity:**
  - The curves meet and have identical curvature at the connection
  - The curvature is defined as the rate of change of tangents



Types of Continuity

- Imagine a camera moving along a curve with multiple segments
  - **No continuity:** camera will make jumps between segments
  - **C0 continuity:** camera velocity may suddenly change
  - **C1 continuity:** camera acceleration may suddenly change
  - **C2 continuity:** camera motion will appear smooth
- In general, maintaining C2 continuity is desired

Maintaining Continuity

- Imagine having two Hermite curves:

$$G_l = \begin{bmatrix} P_1 \\ P_4 \\ R_1 \\ R_4 \end{bmatrix} \qquad G_r = \begin{bmatrix} Q_1 \\ Q_4 \\ T_1 \\ T_4 \end{bmatrix}$$

- C1 continuity can be maintained if $P_4 = Q_1$ and $R_4 = T_1$

Maintaining Continuity

- Similarly for two Bezier curves:

$$G_l = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix} \qquad G_r = \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix}$$

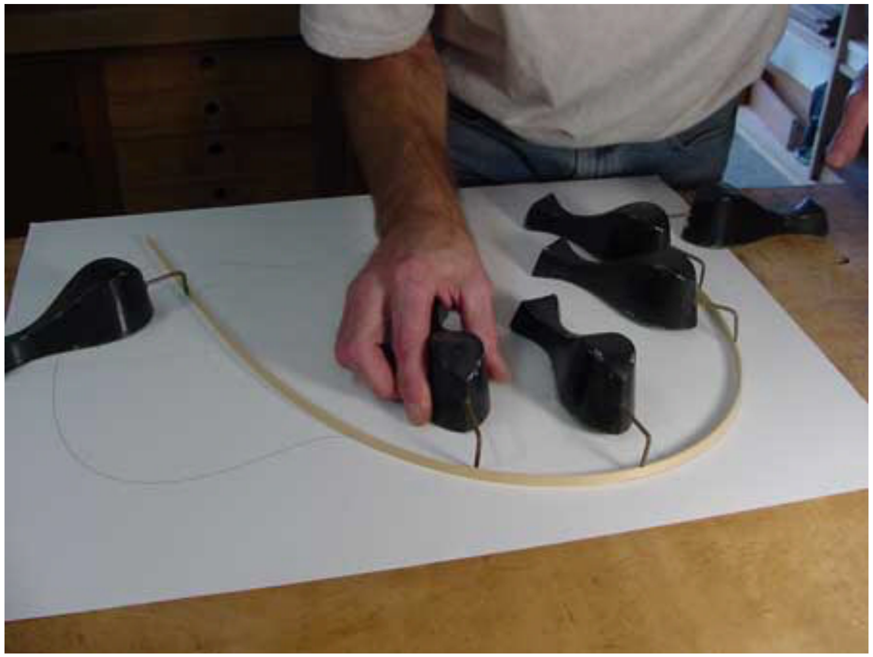
- C1 is ensured if $P_4 = Q_1$ and $P_4 - P_3 = Q_2 - Q_1$

Maintaining Continuity

- What if we want to maintain C2 continuity?
- Unfortunately, neither Hermite nor Bezier curves can guarantee C2 continuity
- For this we have a new type of curve called splines

Splines

- The term spline was used to refer to flexible metal strips used by draftspersons to design the surfaces of airplanes, cars, and ships, etc.



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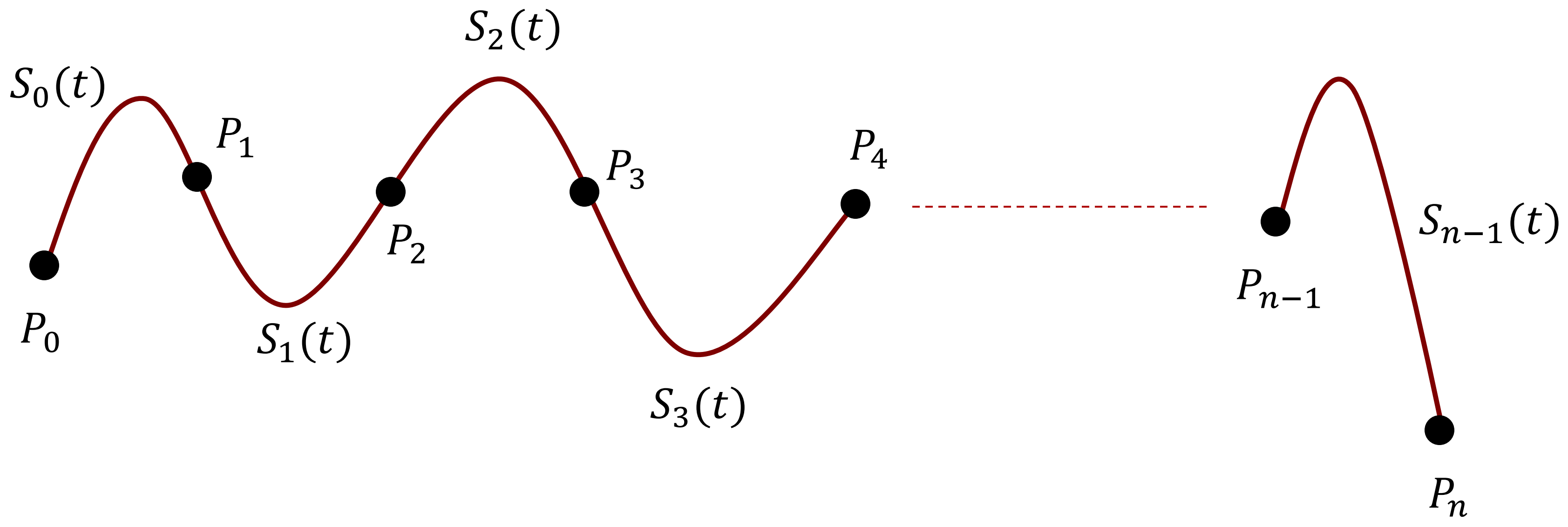
Splines

- The splines, due to physical properties of the metal strips, had second order (C2) continuity
- Its mathematical equivalent is natural cubic splines
- Splines have one more degree of continuity than that is afforded by Hermite and Bezier curves
- There are other types of splines:
  - B-Splines
  - Uniform Nonrational B-Splines
  - Nonuniform Nonrational B-Splines
  - Nonuniform Rational B-Splines
  - Beta-Splines
  - V-Splines

Natural Cubic Splines

- Defined by *n+1* control points
- The spline, consisting of *n* curves, interpolates all of these points



Natural Cubic Splines

- The spline is defined as:

$$S(t) = \begin{cases} S_0(t), & t_0 \leq t \leq t_1 \\ S_1(t), & t_1 \leq t \leq t_2 \\ \vdots & \\ S_{n-1}(t), & t_{n-1} \leq t \leq t_n \end{cases}$$

- Each curve is a cubic polynomial:

$$S_0(t) = a_0 t^3 + b_0 t^2 + c_0 t + d_0$$

$$\vdots$$

$$S_{n-1}(t) = a_{n-1} t^3 + b_{n-1} t^2 + c_{n-1} t + d_{n-1}$$

There are a total of *4n* unknowns!

Natural Cubic Splines

- The end points of the curves must meet (C0 cont.) and their first two derivatives must be equal (C1 and C2):

$$\left.\begin{aligned} S_{i-1}(t_i) &= S_i(t_i) \\ S'_{i-1}(t_i) &= S'_i(t_i) \\ S''_{i-1}(t_i) &= S''_i(t_i) \end{aligned}\right\} \quad i = 1 \ldots n-1$$

- This gives us *3n – 3* equations

Natural Cubic Splines

- We also know the values of the spline at the control points:

$$S(t_i) = P_i, \qquad i = 0 \dots n$$

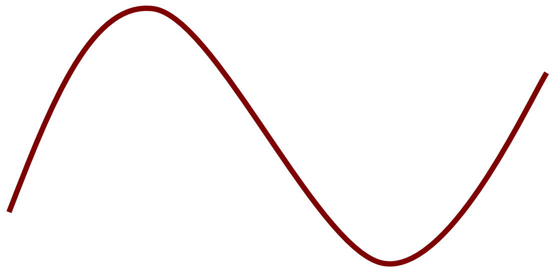
- This gives us another *n+1* equations
- We still need two more …
- $S''(t_0) = S''(t_n) = 0$ gives us natural cubic splines

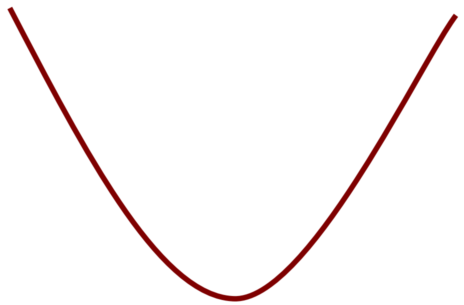
Computing Polynomials

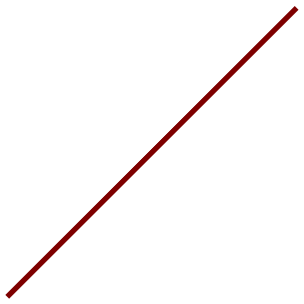
- Let’s call the second derivatives at control points as:

$$Z_i = S''(t_i)$$

- For natural cubic spline we have $Z_0 = Z_n = [0\ 0\ 0]^T$
- How does $S''(t)$ look like?


$S(t)$

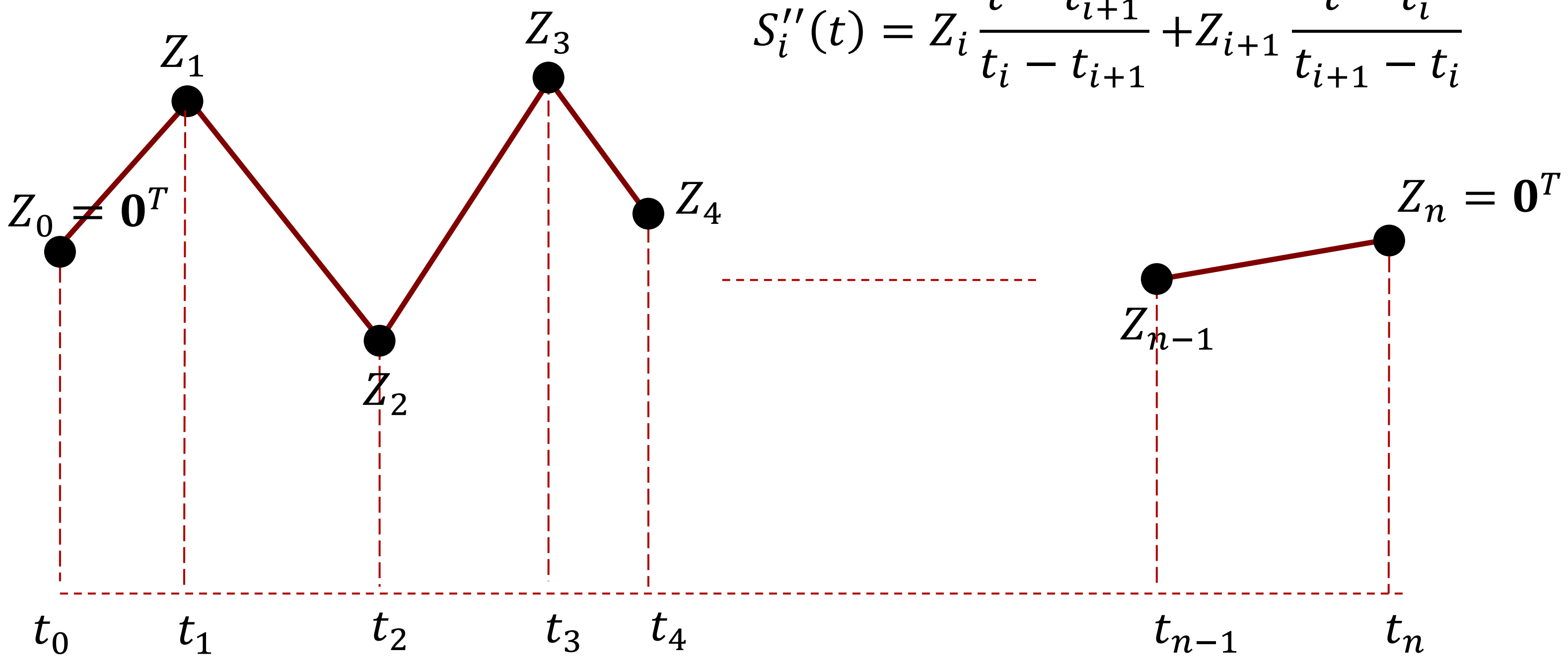

$S'(t)$


$S''(t)$

Computing Polynomials

- $S''(t)$ will be piecewise-linear

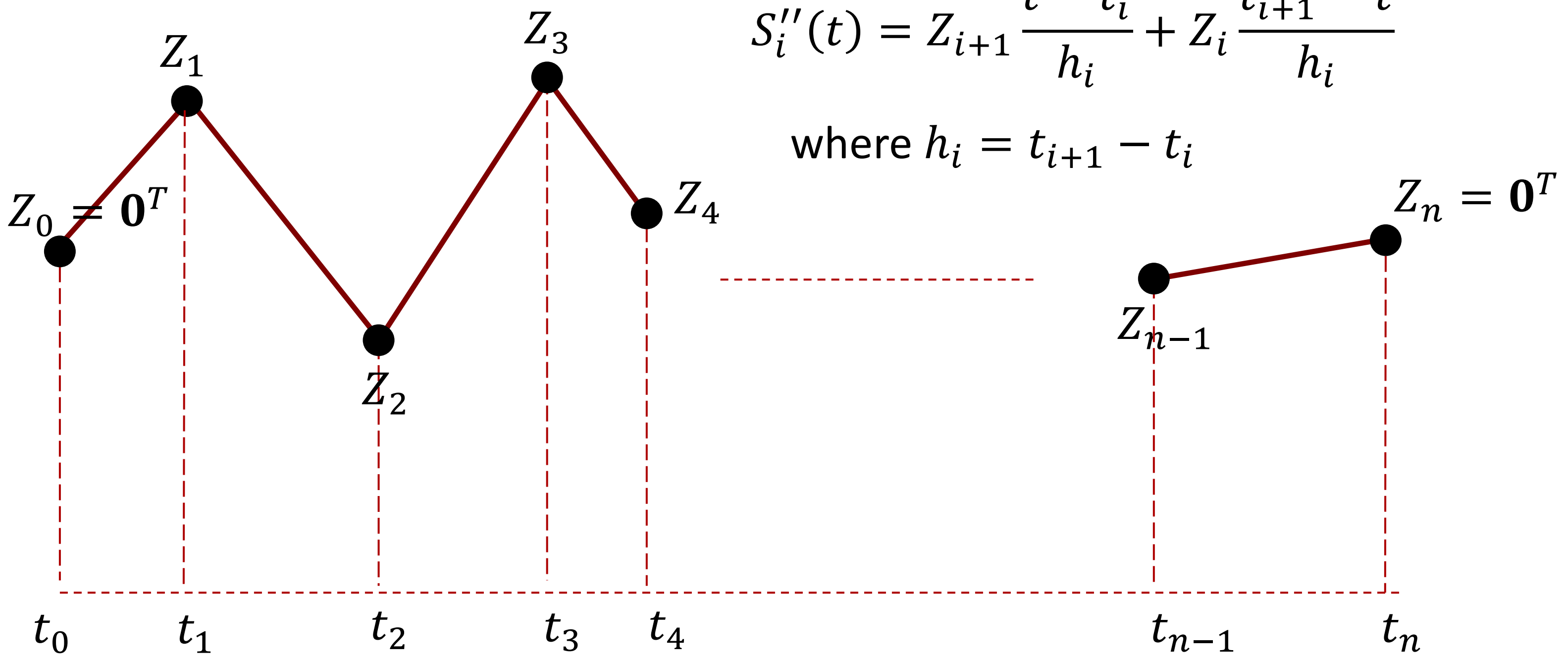
$$S_i''(t) = Z_i \frac{t - t_{i+1}}{t_i - t_{i+1}} + Z_{i+1} \frac{t - t_i}{t_{i+1} - t_i}$$



This derivation is largely inspired from Arne Morten Kvarving's slides on cubic splines

Computing Polynomials

- $S''(t)$ will be piecewise-linear



$$S_i''(t) = Z_{i+1}\frac{t - t_i}{h_i} + Z_i\frac{t_{i+1} - t}{h_i}$$

where $h_i = t_{i+1} - t_i$

Computing Polynomials

- At this point, we need to integrate twice to obtain $S(t)$

$$S_i''(t) = Z_{i+1}\frac{t - t_i}{h_i} + Z_i\frac{t_{i+1} - t}{h_i}$$

$$S_i'(t) = Z_{i+1}\frac{(t - t_i)^2}{2h_i} + Z_i\frac{(t_{i+1} - t)^2}{-2h_i} + C_i$$

$$S_i(t) = Z_{i+1}\frac{(t - t_i)^3}{6h_i} + Z_i\frac{(t_{i+1} - t)^3}{6h_i} + C_i t + D_i$$

$$S_i(t) = Z_{i+1}\frac{(t - t_i)^3}{6h_i} + Z_i\frac{(t_{i+1} - t)^3}{6h_i} + E_i(t - t_i) + F_i(t_{i+1} - t)$$

where $C_i = E_i - F_i$ and $D_i = F_i t_{i+1} - E_i t_i$

Computing Polynomials

- At this point, the only unknowns are Z_i, Z_{i+1}, E_i and F_i

$$S_i(t) = Z_{i+1}\frac{(t-t_i)^3}{6h_i} + Z_i\frac{(t_{i+1}-t)^3}{6h_i} + E_i(t-t_i) + F_i(t_{i+1}-t)$$

- Plug-in the values at the control points:

$$S_i(t_i) = P_i = Z_i\frac{{h_i}^2}{6} + F_i h_i$$

$$S_i(t_{i+1}) = P_{i+1} = Z_{i+1}\frac{{h_i}^2}{6} + E_i h_i$$

- From here, we can determine E_i and F_i

Computing Polynomials

- This gives us:

$$S_i(t) = \boldsymbol{Z_{i+1}} \frac{(t - t_i)^3}{6h_i} + \boldsymbol{Z_i} \frac{(t_{i+1} - t)^3}{6h_i} + \left(\frac{P_{i+1}}{h_i} - \frac{Z_{i+1}h_i}{6}\right)(t - t_i) + \left(\frac{P_i}{h_i} - \frac{Z_i h_i}{6}\right)(t_{i+1} - t)$$

- Finally, we need to compute the $\boldsymbol{Z_i}$ terms
- We know that $\boldsymbol{Z_0} = \boldsymbol{Z_n} = [0\ 0\ 0]^T$

Computing Polynomials

- We did not use the constraint that the first derivatives at the control points are equal; so take the derivative

$$\boldsymbol{S}_i'(\boldsymbol{t}) = \boldsymbol{Z}_{i+1}\frac{(t-t_i)^2}{2h_i} - \boldsymbol{Z}_i\frac{(t_{i+1}-t)^2}{2h_i} + \underbrace{\frac{1}{h_i}(\boldsymbol{P}_{i+1}-\boldsymbol{P}_i)}_{\boldsymbol{B}_i} - \frac{h_i}{6}(\boldsymbol{Z}_{i+1}-\boldsymbol{Z}_i)$$

$$\boldsymbol{S}_i'(\boldsymbol{t}_i) = -\boldsymbol{Z}_i\frac{h_i}{2} + \boldsymbol{B}_i - \frac{h_i}{6}\boldsymbol{Z}_{i+1} + \frac{h_i}{6}\boldsymbol{Z}_i$$

Computing Polynomials

- Repeat this for the previous (or the next) segment:

$$\boldsymbol{S}'_{\boldsymbol{i-1}}(\boldsymbol{t}) = \boldsymbol{Z}_{\boldsymbol{i}}\frac{(t-t_{i-1})^2}{2h_{i-1}} - \boldsymbol{Z}_{\boldsymbol{i-1}}\frac{(t_i-t)^2}{2h_{i-1}} + \underbrace{\frac{1}{h_{i-1}}(\boldsymbol{P}_{\boldsymbol{i}} - \boldsymbol{P}_{\boldsymbol{i-1}})}_{\boldsymbol{B}_{\boldsymbol{i-1}}} - \frac{h_{i-1}}{6}(\boldsymbol{Z}_{\boldsymbol{i}} - \boldsymbol{Z}_{\boldsymbol{i-1}})$$

$$\boldsymbol{S}'_{\boldsymbol{i-1}}(\boldsymbol{t}_{\boldsymbol{i}}) = \boldsymbol{Z}_{\boldsymbol{i}}\frac{h_{i-1}}{2} + \boldsymbol{B}_{\boldsymbol{i-1}} - \frac{h_{i-1}}{6}\boldsymbol{Z}_{\boldsymbol{i}} + \frac{h_{i-1}}{6}\boldsymbol{Z}_{\boldsymbol{i-1}}$$

Computing Polynomials

- Now equate the segments at the control points:

$$\boldsymbol{S}_i'(\boldsymbol{t}_i) = \boldsymbol{S}_{i-1}'(\boldsymbol{t}_i)$$

$$-\boldsymbol{Z}_i \frac{h_i}{2} + \boldsymbol{B}_i - \frac{h_i}{6}\boldsymbol{Z}_{i+1} + \frac{h_i}{6}\boldsymbol{Z}_i = \boldsymbol{Z}_i \frac{h_{i-1}}{2} + \boldsymbol{B}_{i-1} - \frac{h_{i-1}}{6}\boldsymbol{Z}_i + \frac{h_{i-1}}{6}\boldsymbol{Z}_{i-1}$$

$$-3\boldsymbol{Z}_i h_i + 6\boldsymbol{B}_i - h_i\boldsymbol{Z}_{i+1} + h_i\boldsymbol{Z}_i = 3\boldsymbol{Z}_i h_{i-1} + 6\boldsymbol{B}_{i-1} - h_{i-1}\boldsymbol{Z}_i + h_{i-1}\boldsymbol{Z}_{i-1}$$

$$6(\boldsymbol{B}_i - \boldsymbol{B}_{i-1}) = h_{i-1}\boldsymbol{Z}_{i-1} + 2(h_{i-1} + h_i)\boldsymbol{Z}_i + h_i\boldsymbol{Z}_{i+1}$$

Computing Polynomials

- We can set up *n-1* equations in this form and we also have *n-1* unknowns (from Z_1 to Z_{n-1})

$$6(\boldsymbol{B_i} - \boldsymbol{B_{i-1}}) = h_{i-1}\boldsymbol{Z_{i-1}} + 2(h_{i-1} + h_i)\boldsymbol{Z_i} + h_i\boldsymbol{Z_{i+1}}$$

- In the above equation, plug $i = 1 \ldots n - 1$, and solve the resulting system

Computing Polynomials

- Setup the system such that we have the $Ax = b$ form

$$\begin{bmatrix} v_1 & h_1 & 0 & \cdots & & \\ h_1 & v_2 & h_2 & & & \\ 0 & h_2 & v_3 & h_3 & & \\ \vdots & & & \ddots & & \\ & & & & v_{n-2} & h_{n-2} \\ & & & & h_{n-2} & v_{n-1} \end{bmatrix} \begin{bmatrix} Z_{1,x} \\ Z_{2,x} \\ Z_{3,x} \\ \vdots \\ \\ Z_{n-1,x} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ \\ b_{n-1} \end{bmatrix}$$

$$h_i = t_{i+1} - t_i$$

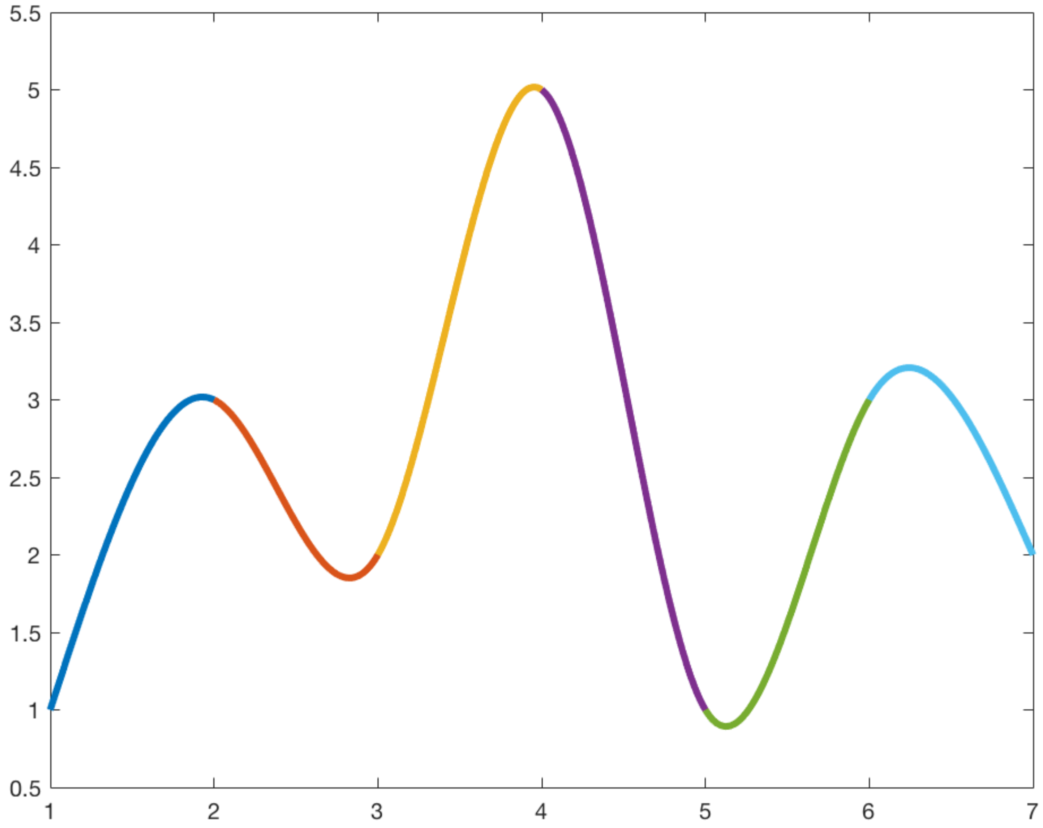
$$v_i = 2(h_{i-1} + h_i)$$

$$b_i = 6(B_{i,x} - B_{i-1,x})$$

$$B_{i,x} = \frac{1}{h_i}\left(P_{i+1,x} - P_{i,x}\right)$$

- Note that we are solving for the x-components
- We need to solve for y- and z-components if our curve is 3 dimensional

Sample Output



P = {(1, 1), (2, 3), (3, 2), (4, 5), (5, 1), (6, 3), (7, 2)}
T = {0, 1, 2, 3, 4, 5, 6}

Parametric Bicubic Surfaces

- Generalization of parametric cubic curves
- Recall $Q(t) = TMG$
- First replace t by s such that $Q(s) = SMG$
- Now allow the points in G to vary along a curve parametrized by t

$$Q(s,t) = SMG(t)$$

$$G_{1x}(t) \quad G_{1y}(t) \quad G_{1z}(t)$$

$$Q(s,t) = SM\begin{bmatrix} G_1(t) \\ G_2(t) \\ G_3(t) \\ G_4(t) \end{bmatrix}$$

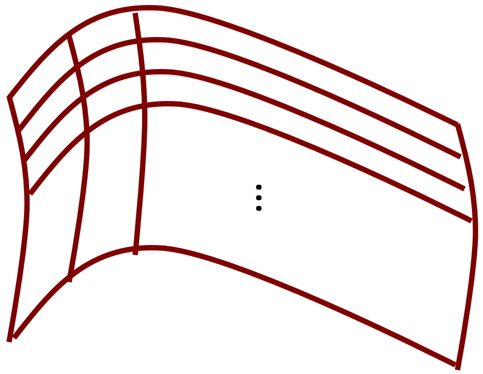
Parametric Bicubic Surfaces

- We can setup separate equations for *x*, *y*, and *z*:

$$Q_x(s,t) = SM\begin{bmatrix} G_{1x}(t) \\ G_{2x}(t) \\ G_{3x}(t) \\ G_{4x}(t) \end{bmatrix} \qquad Q_y(s,t) = SM\begin{bmatrix} G_{1y}(t) \\ G_{2y}(t) \\ G_{3y}(t) \\ G_{4y}(t) \end{bmatrix} \qquad Q_z(s,t) = SM\begin{bmatrix} G_{1z}(t) \\ G_{2z}(t) \\ G_{3z}(t) \\ G_{4z}(t) \end{bmatrix}$$

Parametric Bicubic Surfaces

- Now for a fixed $t = t_1$, $Q(s, t_1)$ is a curve because $G(t_1)$ is constant
- Allowing t to take on a different value, t_2, where $t_2 - t_1$ is very small, $Q(s, t_2)$ is a slightly different curve
- Repeating this arbitrarily many times gives you a large set of curves, which is our surface



Called bicubic if $G_i(t)$ are themselves cubic

Derivation

- Assume that $G_i(t)$ themselves are defined by:

$$G_i(t) = TMZ_i$$

$$G_i(t) = TM\begin{bmatrix} Z_{i1} \\ Z_{i2} \\ Z_{i3} \\ Z_{i4} \end{bmatrix}$$

$$\begin{tikzcd} Z_{i1} \arrow[r, bend left] & Z_{i1x} \quad Z_{i1y} \quad Z_{i1z} \end{tikzcd}$$

$$G_{ix}(t) = TM\begin{bmatrix} Z_{i1x} \\ Z_{i2x} \\ Z_{i3x} \\ Z_{i4x} \end{bmatrix} \qquad G_{iy}(t) = TM\begin{bmatrix} Z_{i1y} \\ Z_{i2y} \\ Z_{i3y} \\ Z_{i4y} \end{bmatrix} \qquad G_{iz}(t) = TM\begin{bmatrix} Z_{i1z} \\ Z_{i2z} \\ Z_{i3z} \\ Z_{i4z} \end{bmatrix}$$

Derivation

- Remember that we have:

$$Q_x(s,t) = SM\begin{bmatrix} G_{1x}(t) \\ G_{2x}(t) \\ G_{3x}(t) \\ G_{4x}(t) \end{bmatrix} \qquad G_{ix}(t) = TM\begin{bmatrix} Z_{i1x} \\ Z_{i2x} \\ Z_{i3x} \\ Z_{i4x} \end{bmatrix}$$

- To combine them into a single equation, take the transpose of $G_{ix}(t)$, which is equal to itself due to its being a scalar:

$$G_{1x}(t)^T = G_{1x}(t) = [Z_{11x} \quad Z_{11x} \quad Z_{13x} \quad Z_{14x}]M^T T^T$$
$$G_{2x}(t)^T = G_{2x}(t) = [Z_{21x} \quad Z_{21x} \quad Z_{23x} \quad Z_{24x}]M^T T^T$$
$$G_{3x}(t)^T = G_{3x}(t) = [Z_{31x} \quad Z_{31x} \quad Z_{33x} \quad Z_{34x}]M^T T^T$$
$$G_{4x}(t)^T = G_{4x}(t) = [Z_{41x} \quad Z_{41x} \quad Z_{43x} \quad Z_{44x}]M^T T^T$$

Derivation

- Remember that we have:

$$Q_x(s,t) = SM\begin{bmatrix} G_{1x}(t) \\ G_{2x}(t) \\ G_{3x}(t) \\ G_{4x}(t) \end{bmatrix} \qquad G_{ix}(t) = TM\begin{bmatrix} Z_{i1x} \\ Z_{i2x} \\ Z_{i3x} \\ Z_{i4x} \end{bmatrix}$$

- To combine them into a single equation, take the transpose of $G_{ix}(t)$, which is equal to itself due to its being a scalar:

$$\begin{bmatrix} G_{1x}(t) \\ G_{2x}(t) \\ G_{3x}(t) \\ G_{4x}(t) \end{bmatrix} = \begin{bmatrix} Z_{11x} & Z_{11x} & Z_{13x} & Z_{14x} \\ Z_{21x} & Z_{21x} & Z_{23x} & Z_{24x} \\ Z_{31x} & Z_{31x} & Z_{33x} & Z_{34x} \\ Z_{41x} & Z_{41x} & Z_{43x} & Z_{44x} \end{bmatrix} M^T T^T$$

Derivation

- This gives us:

$$Q_x(s,t) = S\underbrace{\begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix}}_{M}\underbrace{\begin{bmatrix} Z_{11x} & Z_{11x} & Z_{13x} & Z_{14x} \\ Z_{21x} & Z_{21x} & Z_{23x} & Z_{24x} \\ Z_{31x} & Z_{31x} & Z_{33x} & Z_{34x} \\ Z_{41x} & Z_{41x} & Z_{43x} & Z_{44x} \end{bmatrix}}_{G_x} M^T T^T$$

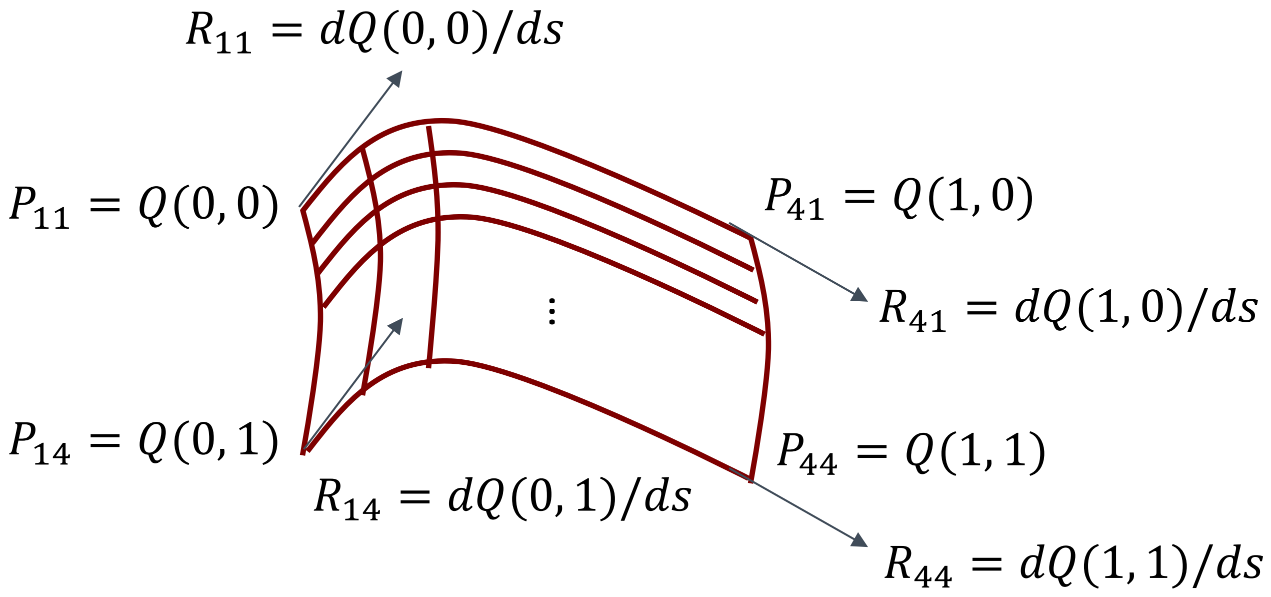
- Similarly for *y* and *z*:

$$Q_y(s,t) = SMG_yM^TT^T$$

$$Q_z(s,t) = SMG_zM^TT^T$$

Hermite Surfaces

- For Hermite surfaces, M is the Hermite basis matrix
- The elements of the geometry matrix (G_x, G_y, G_z) store how each component changes with respect to t:



Hermite Surfaces

- How the starting point (P_1) changes with respect to t:

$$G_x = \begin{bmatrix} Q_x(0,0) & Q_x(0,1) & \frac{dQ_x(0,0)}{dt} & \frac{dQ_x(0,1)}{dt} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}$$

Hermite Surfaces

- How the end point (P_4) changes with respect to t:

$$G_x = \begin{bmatrix} \cdots & \cdots & \cdots & \cdots \\ Q_x(1,0) & Q_x(1,1) & \dfrac{dQ_x(1,0)}{dt} & \dfrac{dQ_x(1,1)}{dt} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

Hermite Surfaces

- How the starting tangent vector (R_1), defined with respect to s, changes with respect to t:

$$G_x = \begin{bmatrix} \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{dQ_x(0,0)}{ds} & \frac{dQ_x(0,1)}{ds} & \frac{d^2Q_x(0,0)}{dsdt} & \frac{d^2Q_x(0,1)}{dsdt} \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

Hermite Surfaces

- How the ending tangent vector (R_4), defined with respect to s, changes with respect to t:

$$G_x = \begin{bmatrix} \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \frac{dQ_x(1,0)}{ds} & \frac{dQ_x(1,1)}{ds} & \frac{d^2Q_x(1,0)}{dsdt} & \frac{d^2Q_x(1,1)}{dsdt} \end{bmatrix}$$

Hermite Surfaces

- So the entire geometry matrix looks like:

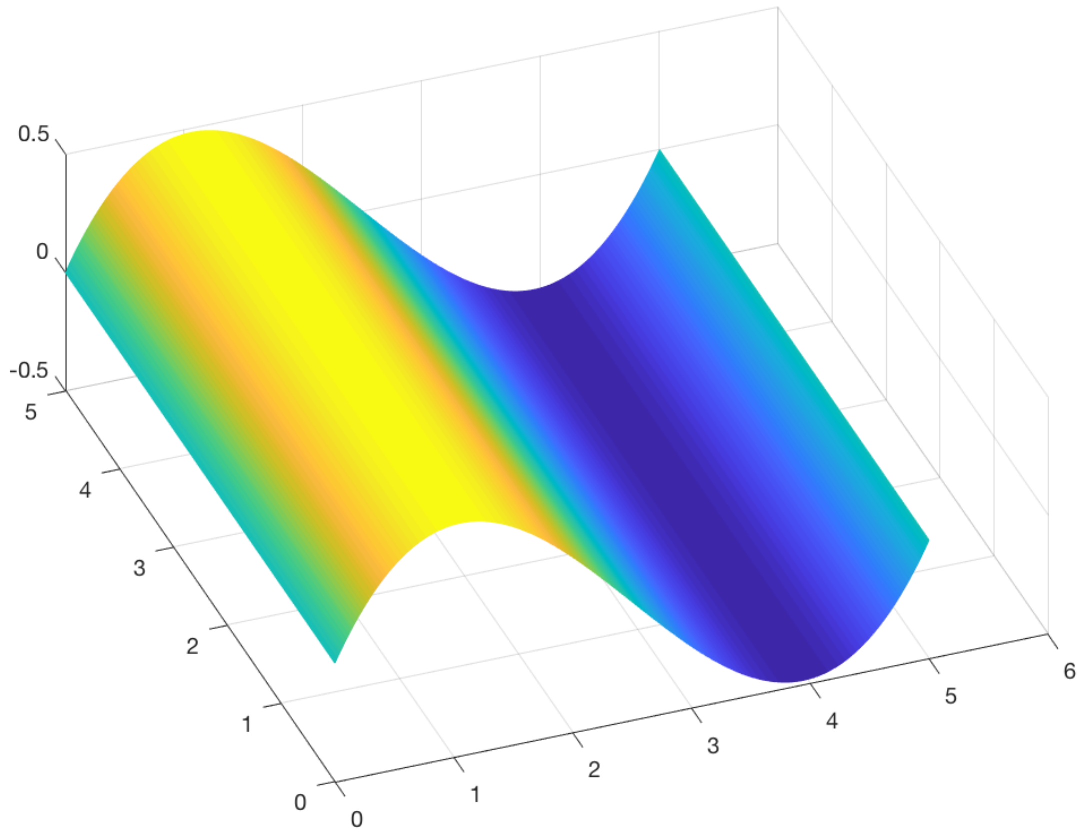
$$G_x = \begin{bmatrix} Q_x(0,0) & Q_x(0,1) & \frac{dQ_x(0,0)}{dt} & \frac{dQ_x(0,1)}{dt} \\ Q_x(1,0) & Q_x(1,1) & \frac{dQ_x(1,0)}{dt} & \frac{dQ_x(1,1)}{dt} \\ \frac{dQ_x(0,0)}{ds} & \frac{dQ_x(0,1)}{ds} & \frac{d^2Q_x(0,0)}{dsdt} & \frac{d^2Q_x(0,1)}{dsdt} \\ \frac{dQ_x(1,0)}{ds} & \frac{dQ_x(1,1)}{ds} & \frac{d^2Q_x(1,1)}{dsdt} & \frac{d^2Q_x(1,1)}{dsdt} \end{bmatrix}$$

Hermite Surfaces

$$G_x = \begin{bmatrix} 0 & 5 & 5 & 5 \\ 0 & 5 & 5 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$G_y = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 5 & 5 & 0 & 0 \\ 5 & 5 & 0 & 0 \\ 5 & 5 & 0 & 0 \end{bmatrix}$$

$$G_z = \begin{bmatrix} 0 & 0 & 5 & 5 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



Bezier Surfaces

- Remember that Bezier curves was defined using Bernstein polynomials:

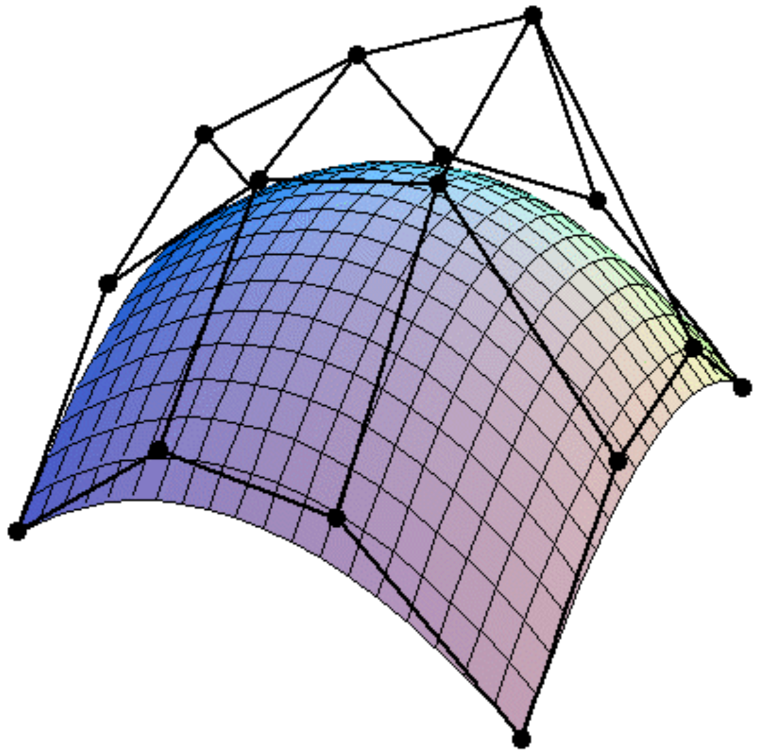
$$Q(t) = \sum_{i=0}^{n} B_{i,n}(t) P_{i+1} \qquad B_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

- Their extension to surfaces is straightforward:

$$Q(s,t) = \sum_{i=0}^{n} \sum_{j=0}^{m} B_{i,n}(s) B_{i,m}(t) P_{i+1}$$

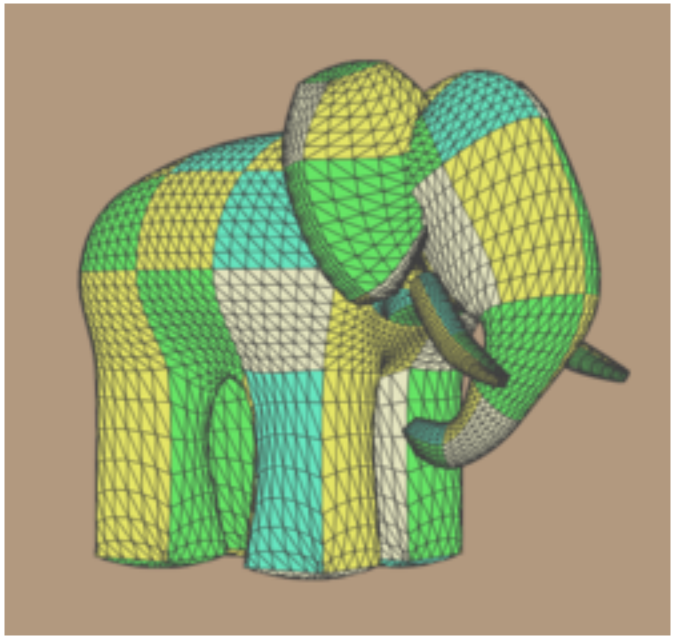
Bezier Surfaces

- Similar to a Bezier curve, a Bezier surface interpolates the end points and approximates the interior control points



Bezier Surfaces

- Complex models can be created using Bezier surfaces
- In such models, the entire surface is composed of multiple Bezier surfaces, known as patches



Gumbo Model

Bezier Surfaces

- Such patches allows tessellating a surface at the desired level of detail depending on viewing distance or other parameters
- OpenGL tessellation shaders provide hardware support for this