CS 536: Estimation Problems

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Uniform Estimators

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1)

① $MSE(\hat{L}) = E[(\hat{L} - L)^2]$

given $bias(\hat{\theta}) = \theta - E[\hat{\theta}]$

show $MSE(\hat{\theta}) = bias(\hat{\theta})^2 + Var(\hat{\theta})$

$$
\begin{aligned}
① \quad MSE[\hat{\theta}] &= E[\hat{\theta}^2 - 2\theta\hat{\theta} + \theta^2] \\
&= E[\hat{\theta}^2] - 2\theta E[\hat{\theta}] + \theta^2 \quad \text{---- ①} \\
bias(\hat{\theta})^2 &= (\theta - E[\hat{\theta}])^2 \\
&= \theta^2 - 2\theta E[\hat{\theta}] + E[\hat{\theta}]^2 \quad \text{---- ②} \\
Var(\hat{\theta}) &= E[\hat{\theta}^2] - E[\theta]^2
\end{aligned}
$$

adding ① and ②

$$
\begin{aligned}
MSE[\hat{\theta}] &= \theta^2 - 2\theta E[\hat{\theta}] + \cancel{E[\hat{\theta}]^2} + E[\hat{\theta}^2] - \cancel{E[\hat{\theta}]^2} \\
&= E[\hat{\theta}^2] - 2\theta E[\hat{\theta}] + \theta^2 \quad ✓ \\
&= bias(\hat{\theta})^2 + Var(\hat{\theta})
\end{aligned}
$$

2) Bias(LMOM) = 0

Bias(LMLE) = L/(n+1)

Pdf for LMLE =f(y) = $\frac{n*(y^{n-1})}{L^n}$

More in the proof below on why L MLE underestimate L

$$
\begin{aligned}
② \quad bias(\hat{L}_{MOM}) &= L - E[\hat{L}] \\
&= L - E[2\bar{X}_n] \\
&= L - \frac{2}{n} E\left[\sum_{i=1}^{n} x_i\right] \\
&= L - \frac{2}{n} * n\, E[x] \\
&= L - 2 * \frac{L}{2} = 0
\end{aligned}
$$

$bias(\hat{L}_{MLE}) = L - E[\max x_i]$

find $E[\max x_i]$: $Y = \max x_i$

$$E[y] = \int_{-\infty}^{\infty} y\, f(y)\, dy$$

$$
\begin{aligned}
F_Y(y) &= P(\max_{i=1..n} x_i \le y) \\
&= P(x_1 \le y, x_2 \le y, \dots x_n \le y) \\
&= P(x_1 \le y) * P(x_2 \le y) * \dots \\
&= \left(\frac{y}{L}\right)^N
\end{aligned}
$$

$$\boxed{f(y) = \frac{dF_y}{dy} = \frac{N}{L^N} y^{N-1}}$$

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$$* \; E[y] = \frac{N}{L^N} \int_{-\infty}^{\infty} y * y^{N-1} = \frac{N}{L^N} \int_0^L y = \frac{N}{L^N} \left. \frac{y^{N+1}}{N+1} \right]_0^L$$

$$= \frac{N}{L^N} * \frac{L^{N+1}}{N+1} = \frac{N}{N+1} * L$$

$$* \; bias(\hat{L}_{MLE}) = L - \frac{N}{N+1} L$$

we have $\frac{N}{N+1} * L < L$ and that's how $\hat{L}_{MLE}$ always underestimate L

3) var(Lmom) = $\frac{L^2}{3*n}$

var(Lmle) = $\frac{(n*L^2)}{((n+1)^2*(n+2))}$

③ $Var(\hat{L}_{mom}) = E[\hat{L}^2] - E[\hat{L}]^2$

$$Var(2\bar{x}_n) = \frac{Var(2x)}{n}$$

$$= \frac{4\,var(x)}{n} = 4\frac{L^2}{12n}$$

$$= \frac{L^2}{3*n}$$

$$E[\bar{x}_n] = \frac{L}{2}$$

$$E[\bar{x}_n^2] = \frac{1}{n^2} E[(\Sigma x)^2]$$

$$= E[x^2]$$

$$E[x^2] = \int_0^L x^2 * \frac{1}{L}\,dx$$

$$E[x^2] = \frac{L^2}{3}$$

$$Var(\hat{L}_{MLE}) = E[\hat{L}^2] - E[\hat{L}]^2$$

$$= \frac{n}{n+2} L^2 - \left(\frac{n}{n+1} L\right)^2$$

$$= L^2\left(\frac{n}{n+2} - \left(\frac{n}{n+1}\right)^2\right)$$

4)

As Shown below MLE is a better estimator and has smaller MSE

MOM MSE = $\frac{L^2}{3*n}$

MLE MSE = $\frac{2*L^2}{(n+1)*(n+2)}$

$\frac{2*L^2}{(n+1)*(n+2)} < \frac{L^2}{3*n}$

④ $MSE(\hat{L}_{Mom}) = bias(L_{mom})^2 + Var(L_{mom})$

$= 0 + \frac{L^2}{3n} = \frac{L^2}{3n}$

$MSE(\hat{L}_{mLE}) = \left(L\left(1-\frac{n}{n+1}\right)\right)^2 + L^2\left(\frac{n}{n+2} - \left(\frac{n}{n+1}\right)^2\right)$

$= \frac{1}{(n+1)^2}L^2 + \frac{n}{(n+1)^2(n+2)}L^2$

$= L^2\left(\frac{(n+2)+n}{(n+1)^2(n+2)}\right) = L^2\left(\frac{2(n+1)}{(n+1)^2(n+2)}\right)$

$= L^2\left(\frac{2}{(n+1)(n+2)}\right)$

assume $n = 10$

$MSE_{mom} = \frac{L^2}{30}$

$MSE_{mLE} = \frac{L^2}{66}$

$MSE_{mom} > MSE_{mLE}$

So we have mLE as better estimator.

5) The code as text is in the last page

```
#Fatima AlSaadeh (fya7)
# CS 536 : Estimation Problems
# Question #5
n=100
l = 10
dist = runif(n, min =0,max = l)
lmom = 2*mean(dist)
lmle = max(dist)
print(lmle)
print(lmom)
mle <- c()
mom <- c()
for (j in 1:1000){
   dist = runif(n, min =0,max = l)
   mle[j]=max(dist)
   mom[j]=2*mean(dist)
}
biasmle = mean(mle) - l
msemle =(biasmle)^2 +var(mle)
biasmom = mean(mom) - l
msemom = (biasmom)^2 +var(mom)
print(msemom)
print(msemle)
```

	Estimated
L mom	10.62278
L mle	9.902543

	Theoretically	Experimentally
MSE mom	0.333	0.3419466
MSE mle	0.0194	0.01878703

From this we have that the estimated MSEs are very close to the theoretical ones.

6) From the numbers in the experiment, MLE estimator always underestimate L which is a better case than overestimating it, making the MSE less. Also having MLE MSE = $\frac{2*L^2}{(n+1)*(n+2)}$, the MSE has an inverse relation with the square of n making it much smaller as n increase.

7)

7) $P(\hat{L}_{MLE} < L-\epsilon)$

$$= P(\max_{i=1\ldots n} x_i < L-\epsilon) = \prod_{i=1}^{n} P(x_i < L-\epsilon)$$

$$= \left(\frac{L-\epsilon}{L}\right)^n \quad \ldots \quad \text{having } F(y) = \left(\frac{y}{L}\right)^n$$

--> The number of samples

$$P(\hat{L}_{MLE} \geq L-\epsilon) \leq \delta$$

$$1-\left(\frac{L-\epsilon}{L}\right)^n \leq \delta$$

$$1-\delta \leq \left(\frac{L-\epsilon}{L}\right)^n \dashrightarrow \ln(1-\delta) \leq n \ln\left(\frac{L-\epsilon}{L}\right)$$

$$n \geq \frac{\ln(1-\delta)}{\ln(L-\epsilon)-\ln(L)}$$

8)

(2) $\hat{L} = \left(\frac{n+1}{n}\right) \max_{i=1..n} X_i$

$\hat{L} = \left(\frac{n+1}{n}\right) \hat{L}_{MLE}$

$\rightarrow E[\hat{L}] = \left(\frac{n+1}{n}\right) * E[\hat{L}_{MLE}]$

$= \left(\frac{n+1}{n}\right) * \frac{n}{n+1} * L$

$= L$

$\text{bias} = L - E[\hat{L}] \rightarrow$ unbiased

$= 0$

$\rightarrow var(\hat{L}) = Var\left(\frac{n+1}{n} \hat{L}_{MLE}\right)$

$= \left(\frac{n+1}{n}\right)^2 Var(\hat{L}_{MLE})$

$= \frac{\cancel{(n+1)^2}}{n^{\cancel{2}}} * \frac{\cancel{n}}{\cancel{(n+1)^2}(n+2)} L^2$

$Var(\hat{L}) = \frac{L^2}{n(n+2)}$

$\rightarrow MSE = \frac{L^2}{n(n+2)}$

$n=10$

$= \frac{L^2}{120}$

* still smaller than what we calculated before.

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mle <- c()
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for (j in 1:1000){
  dist = runif(n, min =0,max = l)
  mle[j]=max(dist)
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}
biasmle = mean(mle) - l
msemle =(biasmle)^2 +var(mle)
biasmom = mean(mom) - l
msemom = (biasmom)^2 +var(mom)
print(msemom)
print(msemle)
```