CS 211: Discrete Structures
Midterm 2 Solutions
Fall 2018

QUESTION 1: Prove by mathematical induction: $n^3 + 2n$ is divisible by 3, $\forall n(n \in Z \wedge n \geq 1)$. Show all steps**. (Marks: 10)**

SOLUTION

Let P(n): $3 \mid (n^3 + 2n)$ (note here | is the divides symbol)

Base case: P(1) : $3 \mid (1)^3 + 2(1)$
$3 \mid 3$ (true)

Our inductive hypothesis is: P(k): $3 \mid (k^3+2k)$ is true for an arbitrary integer $k \geq 1$
Inductive step: Prove $P(k) \rightarrow P(k+1)$ (we have to show that if P(k) is true then P(k+1) is also true)

P(k+1): $3 \mid ((k+1)^3+2(k+1))$

$(k+1)^3+2(k+1) = (k^3+3k^2+3k+1) + 2k + 2 = (k^3+2k) + 3(k^2+k+1)$
If P(k) is true then $(k^3+2k) + 3(k^2+k+1)$ is also divisible by 3 as $3 \mid (k^3+2k)$ is true (from our inductive hypothesis) and $3 \mid 3(k^2+k+1)$.

As our base case is true and $P(k) \rightarrow P(k+1)$ for any integer k, hence $\forall n(n \in Z \wedge n \geq 1)\ 3 \mid (n^3 + 2n)$

QUESTION 2: What are the solutions of the following system of congruences? Specify all values such that $10 \leq z \leq 400$
$z \equiv 3 \bmod 7$ $z \equiv 6 \bmod 5$ $z \equiv 8 \bmod 3$ **(Marks: 10)**

SOLUTION

(do the working yourself)
$z \equiv 731 \bmod 105$
Possible values of z in the given range are: {101,206,311}

QUESTION 3: Show that the set of integral multiples of 3 are in one-to-one correspondence with the set of integral multiples of 7. (**Marks: 10**)

SOLUTION

Let

$X_3 = \{3k \mid k \in Z\}$
$X_7 = \{7k \mid k \in Z\}$

Define f, f: $X_3 \rightarrow X_7$
$f(x) = (7/3)\ x$
(we know that x is a multiple of 3 so the codomain of f is an integer which is a member of X_7)

To show that f defines a one-to-one correspondence, we have to show that it is both one-to-one and onto.
To show that f is one-to-one we prove $\forall a, \forall b\ (f(a) = f(b) \rightarrow a = b)$
$f(a) = (7/3)\ a$
$f(b) = (7/3)\ b$

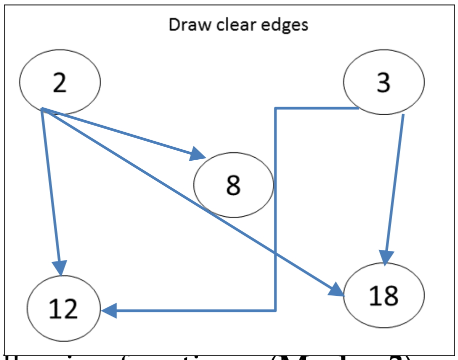
f(a) = f(b) is only possible if
$(7/3)a=(7/3)b$
$a=b$

hence f is injective.
f is onto because for every $y \in X_7$ there is an $x \in X_3$ such that $f(x) = y$.
As $f(x) = y$ if and only if $7/3*x = y$ or $x = 3y/7$, hence every value $y \in X_7$ is being mapped to by some value x. There is no y which is not mapped to by a corresponding member of X_3.

As f is one-to-one and onto, therefore, f is a one-to-one correspondence. Which means that the set of integral multiples of 3 are in one-to-one correspondence with the set of integral multiples of 7.

QUESTION 4a: Let A = { 2,3,8,12,18}. Define a relation R on A as R= { (x,y): x divides y, for all x,y in A}. Draw R and check (by stating 'yes' or 'no') whether R is
(Marks: 2+5)

(i) Reflexive ____yes___________
(ii) Symmetric _no______________
(iii) Antisymmetric ___yes__________
(iv) Equivalence ______no________
(v) Transitive_______yes________



QUESTION 4b: Tick all properties that hold for the following functions. (**Marks: 3**)

i. $f(x) = \lvert x \rvert$	f: $(-1,1) \rightarrow (0,1)$	√ Surjective(onto)	☐ Injective(1-to-1)	☐ Bijective
ii. $f(x) = x^2$	f: $(1,2) \rightarrow (1,4)$	√ Surjective(onto)	√ Injective(1-to-1)	√ Bijective
iii. $f(x) = 2x+1$	f: $(-1,1) \rightarrow (-1,4)$	☐ Surjective(onto)	√ Injective(1-to-1)	☐ Bijective

QUESTION 4c: Let R = {(a,b),(a,c),(a,a)}, R defined on {a,b,c,d} (**Marks: 3**)

i. What is the reflexive closure for R? {(a,a), (a,b), (a,c), (b,b), (c,c), (d,d)}
ii. What is the symmetric closure for R? {(a,a), (a,b), (b,a), (a,c), (c,a) }
iii. What is the transitive closure for R? {(a,a), (a,b), (a,c) }

QUESTION 4d (**Marks: 2**)
Give the recurrence relation for the number of bit strings of length n that begin with 1. Also write the initial conditions.
SOLUTION
$a_n = 2a_{n-1}$
with $a_1 = 1$