LECTURE 6 – DISJOINT SET UNION (UNION-FIND) AND MINIMUM SPANNING TREES (KRUSKAL / PRIM)

Olzhas Aimukhambetov
olzhas.aimukhambetov@astanait.edu.kz

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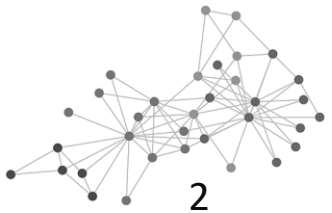

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GRAPH CONNECTIVITY PROBLEMS

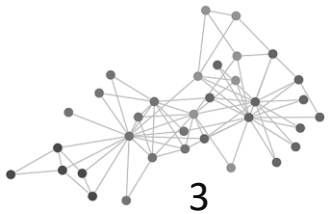
Many graph algorithms rely on quickly answering: “Are these two vertices connected?”

Examples:

- Detecting cycles in undirected graphs
- Building Minimum Spanning Trees (MST)
- Tracking connected components in dynamic networks
- Clustering, percolation, union operations in simulations

Naive approach: recompute connectivity by BFS/DFS after each change → O(V+E) per query.

Better: maintain dynamic partition of vertices (DSU) → near O(1) per query.



DISJOINT SET UNION: CONCEPT AND OPERATIONS

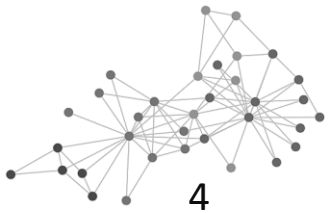
Disjoint Set Union (DSU) / Union-Find manages a collection of disjoint sets under two operations:

- find(x) — returns representative (root) of x's set
- union(x, y) — merges two sets if distinct

Representation:

- Each set represented by a tree; root is the leader
- Initially each vertex is its own parent
- With heuristics (union by rank, path compression) operations are extremely fast

Typical usage: dynamic connectivity queries, cycle detection, Kruskal's algorithm.



UNION BY RANK AND PATH COMPRESSION

Two heuristics ensure almost-constant performance:

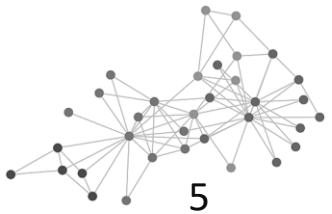
Union by Rank (or Size):

• Attach smaller tree under root of larger tree to keep depth small.

Path Compression:

• During find(x), make each node on path point directly to root → flattens tree.

Complexity: each operation ≈ **O(α(n))** where **α** is inverse Ackermann (≤ 5 for all practical n).



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JAVA IMPLEMENTATION

```java
class DSU {
    private int[] parent;
    private int[] rank;

    public DSU(int n) {
        parent = new int[n];
        rank = new int[n];
        for (int i = 0; i < n; i++) parent[i] = i;
    }

    public int find(int x) {
        if (parent[x] != x)
            parent[x] = find(parent[x]); // path compression
        return parent[x];
    }

    public void union(int x, int y) {
        int rootX = find(x), rootY = find(y);
        if (rootX == rootY) return;

        if (rank[rootX] < rank[rootY]) parent[rootX] = rootY;
        else if (rank[rootX] > rank[rootY]) parent[rootY] = rootX;
        else {
            parent[rootY] = rootX;
            rank[rootX]++;
        }
    }
}
```

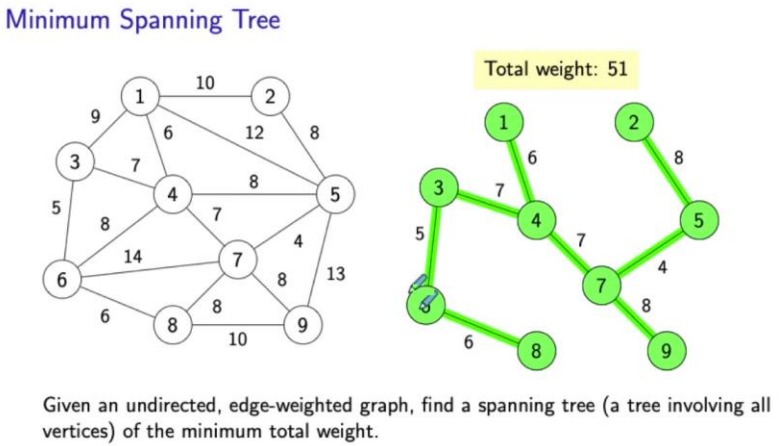
MINIMUM SPANNING TREE: DEFINITION AND INTUITION

Minimum Spanning Tree (MST): a subset of edges connecting all vertices with minimal total weight.

Formal: For a connected weighted undirected graph G(V,E), MST $\subseteq$ E connects all vertices with minimum $\sum$ w(e).

Properties:

- A tree (no cycles) that spans all vertices
- Unique if all edge weights are distinct
- Fundamental in network design, clustering, approximation algorithms



Key insight: repeatedly add the minimum safe edge that does not form a cycle.

KRUSKAL'S ALGORITHM

Kruskal's algorithm — greedy, edge-based:

1. Sort all edges by weight (ascending).

2. Initialize DSU with each vertex separate.

3. For each edge (u,v,w) in sorted order:

   • if find(u) != find(v): add edge to MST; union(u,v)

   • else skip (would form cycle)

Correctness: by cut and cycle properties — smallest edge across any cut is safe.

Complexity:

• Sorting: O(E log E)

• DSU ops: O(E α(V)) ≈ O(E)

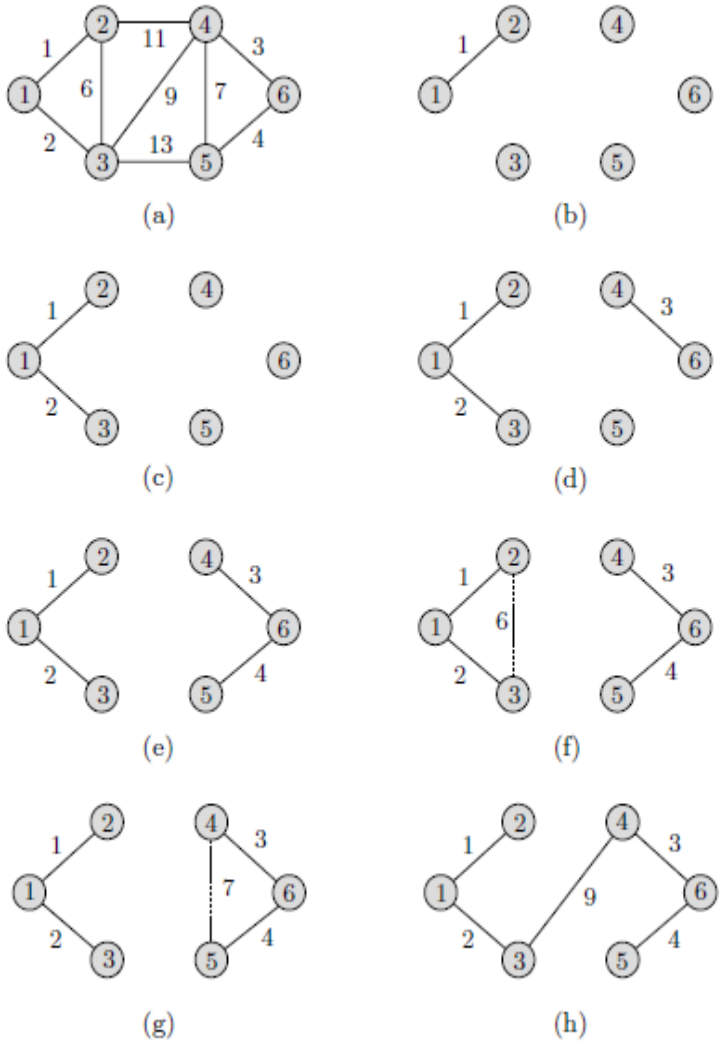


Fig. 7.4. An example of Kruskal Algorithm.

KRUSKAL'S ALGORITHM (WITH DSU)

Input: A weighted connected undirected graph $G = (V,E)$ with n vertices.

Output: The set of edges T of a minimum cost spanning tree for G.

- 1. Sort the edges in E by nondecreasing weight.
- 2. **for** each vertex $v \in V$
- 3. makeset({v})
- 4. **end for**
- 5. $T = \{\}$
- 6. **while** $|T| < n - 1$
- 7. Let (x, y) be the next edge in E.
- 8. **if** find(x) = find(y) **then**
- 9. Add (x, y) to T
- 10. union(x, y)
- 11. **end if**
- 12. **end while**

PRIM'S ALGORITHM (PRIORITY QUEUE)

Prim's algorithm — grow-from-vertex approach:

- Start from any vertex.
- Maintain min-priority queue of edges that cross from current MST to non-tree vertices.
- Repeatedly extract min edge and add the adjacent vertex if it's not in MST.

Implementation notes:

- Use adjacency list + PriorityQueue<Edge>
- Track visited[] to avoid adding same vertex twice

Complexity: O(E log V) with binary heap (improvable with Fibonacci heap).

When to use: practical for dense graphs or when adjacency is given.

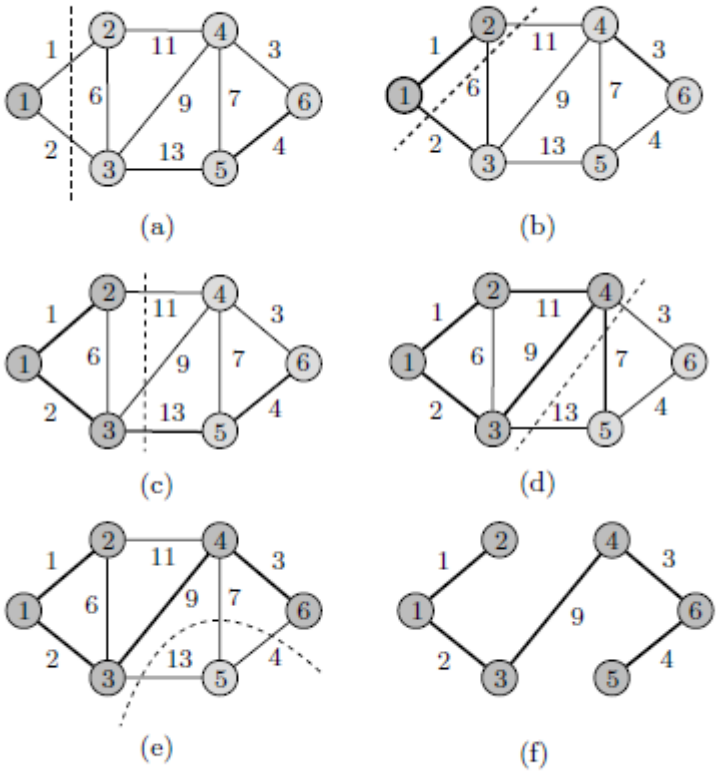


Fig. 7.5. An example of Prim's algorithm.

PRIM'S ALGORITHM (PRIORITY QUEUE)

Input: A weighted connected undirected graph $G = (V,E)$, where $V = \{1, 2, \ldots, n\}$.

Output: The set of edges T of a minimum cost spanning tree for G.

- 1. $T \leftarrow \{\}$; $X \leftarrow \{1\}$; $Y \leftarrow V - \{1\}$
- 2. **for** $y \leftarrow 2$ **to** n
- 3. **if** y adjacent to 1 **then**
- 4. $N[y] \leftarrow 1$
- 5. $C[y] \leftarrow c[1, y]$
- 6. **else** $C[y] \leftarrow \infty$
- 7. **end if**
- 8. **end for**
- 9. **for** $j \leftarrow 1$ **to** $n - 1$ {find $n - 1$ edges}
- 10. Let $y \in Y$ be such that $C[y]$ is minimum
- 11. $T \leftarrow T \cup \{(y,N[y])\}$ {add edge $(y,N[y])$ to T}
- 12. $X \leftarrow X \cup \{y\}$ {add vertex y to X}
- 13. $Y \leftarrow Y - \{y\}$ {delete vertex y from Y }
- 14. **for** each vertex $w \in Y$ that is adjacent to y
- 15. **if** $c[y,w] < C[w]$ **then**
- 16. $N[w] \leftarrow y$
- 17. $C[w] \leftarrow c[y,w]$
- 18. **end if**
- 19. **end for**

COMPARISON & PRACTICAL CONSIDERATIONS

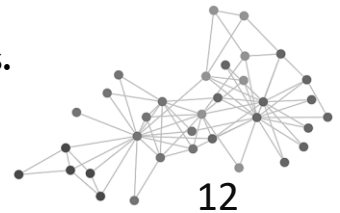
Comparison:

- Kruskal: edge-based (global), uses DSU. Complexity O(E log E). Best for sparse graphs.
- Prim: vertex-based (local), uses priority queue. Complexity O(E log V). Best for dense graphs.

Other notes:

- Both produce same total weight for MST if graph connected.
- Use Kruskal when edge-list is available or graph is sparse.
- Use Prim when adjacency lists/matrix are convenient (dense graph).
- For very large graphs, consider external sorting / parallelization.

Applications: network design, clustering (single-linkage), approximation algorithms.



COMPARISON (KRUSKAL VS. PRIM)

Aspect	Kruskal	Prim
Strategy	Edge-based (sort + union)	Vertex-based (grow MST)
Data structure	DSU	Priority Queue
Best for	Sparse graphs	Dense graphs
Complexity	O(E log E)	O(E log V)
Edge addition	Global order	Local minimum edge

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THANK YOU!