39. Show that the volume of the solid bounded by the coordinate planes and a plane tangent to the portion of the surface $xyz = k$, $k > 0$, in the first octant does not depend on the point of tangency.

40. **Writing** Discuss the role of the chain rule in defining a tangent plane to a level surface.

41. **Writing** Discuss the relationship between tangent planes and local linear approximations for functions of two variables.

✔QUICK CHECK ANSWERS 13.7

1. $2(x-1)+y+(z+1)=0$; $x=1+2t$; $y=t$; $z=-1+t$
2. $z=4+2(x-3)-3(y-1)$; $x=3+2t$; $y=1-3t$; $z=4-t$
3. $z=8+8(x-2)+(y-4)$; $x=2+8t$; $y=4+t$; $z=8-t$ **4.** $x=2+t$; $y=1$; $z=2-t$

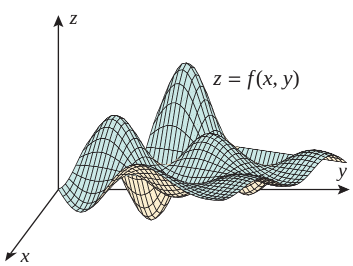
13.8 MAXIMA AND MINIMA OF FUNCTIONS OF TWO VARIABLES

Earlier in this text we learned how to find maximum and minimum values of a function of one variable. In this section we will develop similar techniques for functions of two variables.

EXTREMA

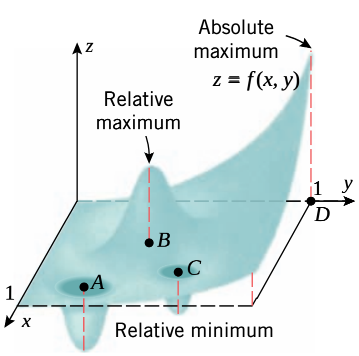
If we imagine the graph of a function f of two variables to be a mountain range (Figure 13.8.1), then the mountaintops, which are the high points in their immediate vicinity, are called *relative maxima* of f, and the valley bottoms, which are the low points in their immediate vicinity, are called *relative minima* of f.

Just as a geologist might be interested in finding the highest mountain and deepest valley in an entire mountain range, so a mathematician might be interested in finding the largest and smallest values of $f(x, y)$ over the *entire* domain of f. These are called the *absolute maximum* and *absolute minimum values* of f. The following definitions make these informal ideas precise.



▲ **Figure 13.8.1**

13.8.1 DEFINITION A function f of two variables is said to have a ***relative maximum*** at a point (x_0, y_0) if there is a disk centered at (x_0, y_0) such that $f(x_0, y_0) \geq f(x, y)$ for all points (x, y) that lie inside the disk, and f is said to have an ***absolute maximum*** at (x_0, y_0) if $f(x_0, y_0) \geq f(x, y)$ for all points (x, y) in the domain of f.



▲ **Figure 13.8.2**

13.8.2 DEFINITION A function f of two variables is said to have a ***relative minimum*** at a point (x_0, y_0) if there is a disk centered at (x_0, y_0) such that $f(x_0, y_0) \leq f(x, y)$ for all points (x, y) that lie inside the disk, and f is said to have an ***absolute minimum*** at (x_0, y_0) if $f(x_0, y_0) \leq f(x, y)$ for all points (x, y) in the domain of f.

If f has a relative maximum or a relative minimum at (x_0, y_0), then we say that f has a ***relative extremum*** at (x_0, y_0), and if f has an absolute maximum or absolute minimum at (x_0, y_0), then we say that f has an ***absolute extremum*** at (x_0, y_0).

Figure 13.8.2 shows the graph of a function f whose domain is the square region in the xy-plane whose points satisfy the inequalities $0 \leq x \leq 1$, $0 \leq y \leq 1$. The function f has relative minima at the points A and C and a relative maximum at B. There is an absolute minimum at A and an absolute maximum at D.

For functions of two variables we will be concerned with two important questions:

- Are there any relative or absolute extrema?
- If so, where are they located?

BOUNDED SETS

Just as we distinguished between finite intervals and infinite intervals on the real line, so we will want to distinguish between regions of "finite extent" and regions of "infinite extent" in 2-space and 3-space. A set of points in 2-space is called ***bounded*** if the entire set can be contained within some rectangle, and is called ***unbounded*** if there is no rectangle that contains all the points of the set. Similarly, a set of points in 3-space is ***bounded*** if the entire set can be contained within some box, and is unbounded otherwise (Figure 13.8.3).

Explain why any subset of a bounded set is also bounded.

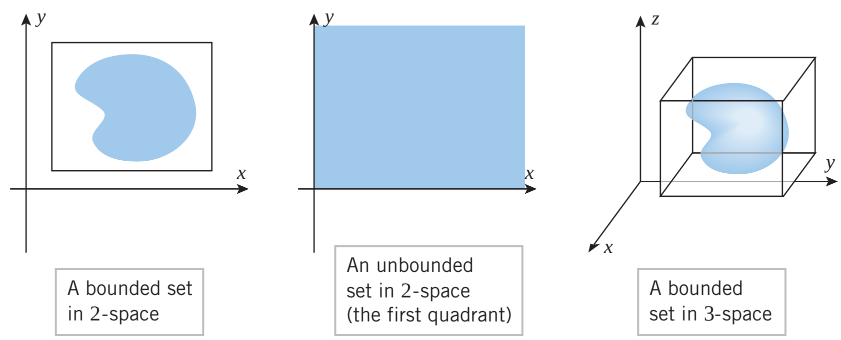


Figure 13.8.3

THE EXTREME-VALUE THEOREM

For functions of one variable that are continuous on a closed interval, the Extreme-Value Theorem (Theorem 4.4.2) answered the existence question for absolute extrema. The following theorem, which we state without proof, is the corresponding result for functions of two variables.

> **13.8.3 THEOREM (*Extreme-Value Theorem*)** *If $f(x, y)$ is continuous on a closed and bounded set R, then f has both an absolute maximum and an absolute minimum on R.*

Example 1 The square region R whose points satisfy the inequalities

$$0 \le x \le 1 \quad \text{and} \quad 0 \le y \le 1$$

is a closed and bounded set in the xy-plane. The function f whose graph is shown in Figure 13.8.2 is continuous on R; thus, it is guaranteed to have an absolute maximum and minimum on R by the last theorem. These occur at points D and A that are shown in the figure. ◀

REMARK If any of the conditions in the Extreme-Value Theorem fail to hold, then there is no guarantee that an absolute maximum or absolute minimum exists on the region R. Thus, a discontinuous function on a closed and bounded set need not have any absolute extrema, and a continuous function on a set that is not closed and bounded also need not have any absolute extrema.

FINDING RELATIVE EXTREMA

Recall that if a function g of one variable has a relative extremum at a point x_0 where g is differentiable, then $g'(x_0) = 0$. To obtain the analog of this result for functions of two variables, suppose that $f(x, y)$ has a relative maximum at a point (x_0, y_0) and that the partial derivatives of f exist at (x_0, y_0). It seems plausible geometrically that the traces of the surface $z = f(x, y)$ on the planes $x = x_0$ and $y = y_0$ have horizontal tangent lines at (x_0, y_0) (Figure 13.8.4), so

$$f_x(x_0, y_0) = 0 \quad \text{and} \quad f_y(x_0, y_0) = 0$$

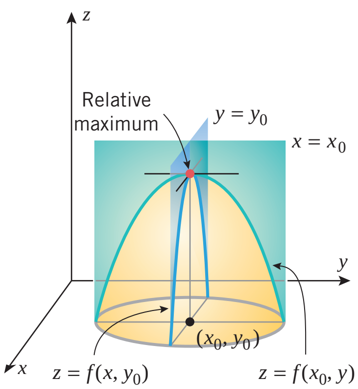


Figure 13.8.4

The same conclusion holds if f has a relative minimum at (x_0, y_0), all of which suggests the following result, which we state without formal proof.

13.8.4 THEOREM *If f has a relative extremum at a point (x_0, y_0), and if the first-order partial derivatives of f exist at this point, then*

$$f_x(x_0, y_0) = 0 \quad \textit{and} \quad f_y(x_0, y_0) = 0$$

Recall that the *critical points* of a function f of one variable are those values of x in the domain of f at which $f'(x) = 0$ or f is not differentiable. The following definition is the analog for functions of two variables.

Explain why

$$D_{\mathbf{u}} f(x_0, y_0) = 0$$

for all $\mathbf{u}$ if (x_0, y_0) is a critical point of f and f is differentiable at (x_0, y_0).

13.8.5 DEFINITION A point (x_0, y_0) in the domain of a function $f(x, y)$ is called a ***critical point*** of the function if $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$ or if one or both partial derivatives do not exist at (x_0, y_0).

It follows from this definition and Theorem 13.8.4 that relative extrema occur at critical points, just as for a function of one variable. However, recall that for a function of one variable a relative extremum need not occur at *every* critical point. For example, the function might have an inflection point with a horizontal tangent line at the critical point (see Figure 4.2.6). Similarly, a function of two variables need not have a relative extremum at every critical point. For example, consider the function

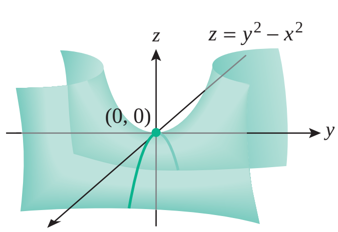
$$f(x, y) = y^2 - x^2$$

This function, whose graph is the hyperbolic paraboloid shown in Figure 13.8.5, has a critical point at $(0, 0)$, since

$$f_x(x, y) = -2x \quad \text{and} \quad f_y(x, y) = 2y$$

from which it follows that

$$f_x(0, 0) = 0 \quad \text{and} \quad f_y(0, 0) = 0$$

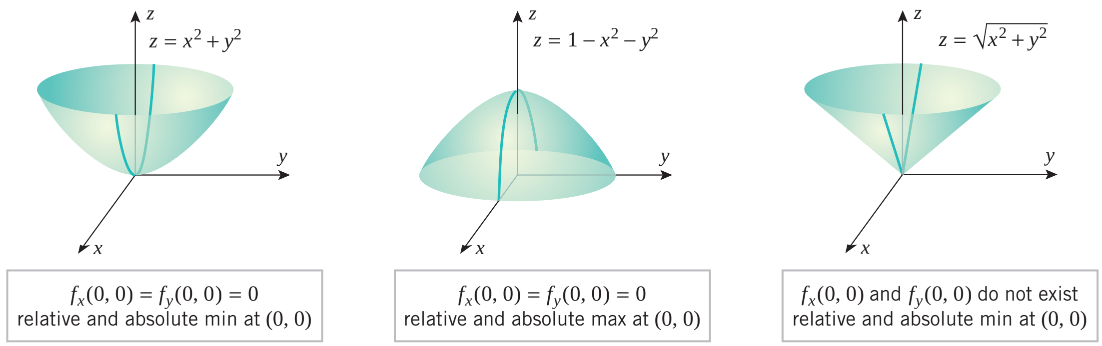


The function $f(x, y) = y^2 - x^2$ has neither a relative maximum nor a relative minimum at the critical point (0, 0).

Figure 13.8.5

However, the function f has neither a relative maximum nor a relative minimum at $(0, 0)$. For obvious reasons, the point $(0, 0)$ is called a *saddle point* of f. In general, we will say that a surface $z = f(x, y)$ has a ***saddle point*** at (x_0, y_0) if there are two distinct vertical planes through this point such that the trace of the surface in one of the planes has a relative maximum at (x_0, y_0) and the trace in the other has a relative minimum at (x_0, y_0).

Example 2 The three functions graphed in Figure 13.8.6 all have critical points at $(0, 0)$. For the paraboloids, the partial derivatives at the origin are zero. You can check this algebraically by evaluating the partial derivatives at (0, 0), but you can see it geometrically by observing that the traces in the xz-plane and yz-plane have horizontal tangent lines at (0, 0). For the cone neither partial derivative exists at the origin because the traces in the xz-plane and the yz-plane have corners there. The paraboloid in part (*a*) and the cone in part (*c*) have a relative minimum and absolute minimum at the origin, and the paraboloid in part (*b*) has a relative maximum and an absolute maximum at the origin. ◀



(*a*) (*b*) (*c*)

▲ **Figure 13.8.6**

THE SECOND PARTIALS TEST

For functions of one variable the second derivative test (Theorem 4.2.4) was used to determine the behavior of a function at a critical point. The following theorem, which is usually proved in advanced calculus, is the analog of that theorem for functions of two variables.

13.8.6 THEOREM (*The Second Partials Test*) *Let f be a function of two variables with continuous second-order partial derivatives in some disk centered at a critical point (x_0, y_0), and let*

$$D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}^2(x_0, y_0)$$

(*a*) *If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then f has a relative minimum at (x_0, y_0).*

(*b*) *If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then f has a relative maximum at (x_0, y_0).*

(*c*) *If $D < 0$, then f has a saddle point at (x_0, y_0).*

(*d*) *If $D = 0$, then no conclusion can be drawn.*

With the notation of Theorem 13.8.6, show that if $D > 0$, then $f_{xx}(x_0, y_0)$ and $f_{yy}(x_0, y_0)$ have the same sign. Thus, we can replace $f_{xx}(x_0, y_0)$ by $f_{yy}(x_0, y_0)$ in parts (*a*) and (*b*) of the theorem.

▶ **Example 3** Locate all relative extrema and saddle points of

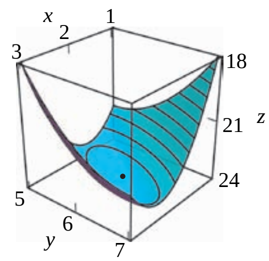
$$f(x, y) = 3x^2 - 2xy + y^2 - 8y$$

Solution. Since $f_x(x, y) = 6x - 2y$ and $f_y(x, y) = -2x + 2y - 8$, the critical points of f satisfy the equations

$$6x - 2y = 0$$
$$-2x + 2y - 8 = 0$$

Solving these for x and y yields $x = 2$, $y = 6$ (verify), so (2, 6) is the only critical point. To apply Theorem 13.8.6 we need the second-order partial derivatives

$$f_{xx}(x, y) = 6, \quad f_{yy}(x, y) = 2, \quad f_{xy}(x, y) = -2$$



$f(x, y) = 3x^2 - 2xy + y^2 - 8y$

▲ **Figure 13.8.7**

At the point (2, 6) we have

$$D = f_{xx}(2, 6)f_{yy}(2, 6) - f_{xy}^2(2, 6) = (6)(2) - (-2)^2 = 8 > 0$$

and

$$f_{xx}(2, 6) = 6 > 0$$

so f has a relative minimum at (2, 6) by part (a) of the second partials test. Figure 13.8.7 shows a graph of f in the vicinity of the relative minimum. ◄

▶ **Example 4** Locate all relative extrema and saddle points of

$$f(x, y) = 4xy - x^4 - y^4$$

Solution. Since

$$\begin{aligned} f_x(x, y) &= 4y - 4x^3 \\ f_y(x, y) &= 4x - 4y^3 \end{aligned} \tag{1}$$

the critical points of f have coordinates satisfying the equations

$$\begin{aligned} 4y - 4x^3 &= 0 \\ 4x - 4y^3 &= 0 \end{aligned} \quad \text{or} \quad \begin{aligned} y &= x^3 \\ x &= y^3 \end{aligned} \tag{2}$$

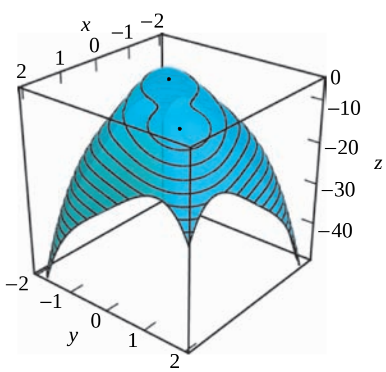
Substituting the top equation in the bottom yields $x = (x^3)^3$ or, equivalently, $x^9 - x = 0$ or $x(x^8 - 1) = 0$, which has solutions $x = 0$, $x = 1$, $x = -1$. Substituting these values in the top equation of (2), we obtain the corresponding y-values $y = 0$, $y = 1$, $y = -1$. Thus, the critical points of f are (0, 0), (1, 1), and (−1, −1).

From (1),

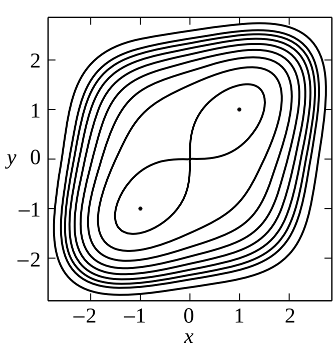
$$f_{xx}(x, y) = -12x^2, \quad f_{yy}(x, y) = -12y^2, \quad f_{xy}(x, y) = 4$$

which yields the following table:

CRITICAL POINT (x_0, y_0)	$f_{xx}(x_0, y_0)$	$f_{yy}(x_0, y_0)$	$f_{xy}(x_0, y_0)$	$D = f_{xx}f_{yy} - f_{xy}^2$
(0, 0)	0	0	4	−16
(1, 1)	−12	−12	4	128
(−1, −1)	−12	−12	4	128



$f(x, y) = 4xy - x^4 - y^4$



▲ **Figure 13.8.8**

At the points (1, 1) and (−1, −1), we have $D > 0$ and $f_{xx} < 0$, so relative maxima occur at these critical points. At (0, 0) there is a saddle point since $D < 0$. The surface and a contour plot are shown in Figure 13.8.8. ◄

The "figure eight" pattern at (0, 0) in the contour plot for the surface in Figure 13.8.8 is typical for level curves that pass through a saddle point. If a bug starts at the point (0, 0, 0) on the surface, in how many directions can it walk and remain in the xy-plane?

The following theorem, which is the analog for functions of two variables of Theorem 4.4.3, will lead to an important method for finding absolute extrema.

13.8.7 THEOREM *If a function f of two variables has an absolute extremum (either an absolute maximum or an absolute minimum) at an interior point of its domain, then this extremum occurs at a critical point.*

PROOF If f has an absolute maximum at the point (x_0, y_0) in the interior of the domain of f, then f has a relative maximum at (x_0, y_0). If both partial derivatives exist at (x_0, y_0), then

$$f_x(x_0, y_0) = 0 \quad \text{and} \quad f_y(x_0, y_0) = 0$$

by Theorem 13.8.4, so (x_0, y_0) is a critical point of f. If either partial derivative does not exist, then again (x_0, y_0) is a critical point, so (x_0, y_0) is a critical point in all cases. The proof for an absolute minimum is similar. ■

FINDING ABSOLUTE EXTREMA ON CLOSED AND BOUNDED SETS

If $f(x, y)$ is continuous on a closed and bounded set R, then the Extreme-Value Theorem (Theorem 13.8.3) guarantees the existence of an absolute maximum and an absolute minimum of f on R. These absolute extrema can occur either on the boundary of R or in the interior of R, but if an absolute extremum occurs in the interior, then it occurs at a critical point by Theorem 13.8.7. Thus, we are led to the following procedure for finding absolute extrema:

Compare this procedure with that in Section 4.4 for finding the extreme values of $f(x)$ on a closed interval.

How to Find the Absolute Extrema of a Continuous Function f of Two Variables on a Closed and Bounded Set R

Step 1. Find the critical points of f that lie in the interior of R.

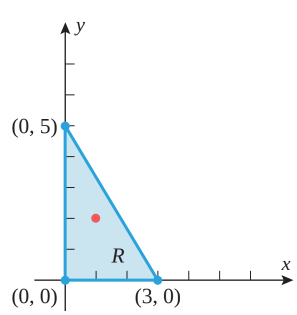
Step 2. Find all boundary points at which the absolute extrema can occur.

Step 3. Evaluate $f(x, y)$ at the points obtained in the preceding steps. The largest of these values is the absolute maximum and the smallest the absolute minimum.

Example 5 Find the absolute maximum and minimum values of

$$f(x, y) = 3xy - 6x - 3y + 7 \tag{3}$$

on the closed triangular region R with vertices (0, 0), (3, 0), and (0, 5).



▲ Figure 13.8.9

Solution. The region R is shown in Figure 13.8.9. We have

$$\frac{\partial f}{\partial x} = 3y - 6 \quad \text{and} \quad \frac{\partial f}{\partial y} = 3x - 3$$

so all critical points occur where

$$3y - 6 = 0 \quad \text{and} \quad 3x - 3 = 0$$

Solving these equations yields $x = 1$ and $y = 2$, so $(1, 2)$ is the only critical point. As shown in Figure 13.8.9, this critical point is in the interior of R.

Next we want to determine the locations of the points on the boundary of R at which the absolute extrema might occur. The boundary of R consists of three line segments, each of which we will treat separately:

The line segment between (0, 0) *and* (3, 0): On this line segment we have $y = 0$, so (3) simplifies to a function of the single variable x,

$$u(x) = f(x, 0) = -6x + 7, \quad 0 \le x \le 3$$

This function has no critical points because $u'(x) = -6$ is nonzero for all x. Thus the extreme values of $u(x)$ occur at the endpoints $x = 0$ and $x = 3$, which correspond to the points (0, 0) and (3, 0) of R.

The line segment between $(0, 0)$ *and* $(0, 5)$: On this line segment we have $x = 0$, so (3) simplifies to a function of the single variable y,

$$v(y) = f(0, y) = -3y + 7, \quad 0 \le y \le 5$$

This function has no critical points because $v'(y) = -3$ is nonzero for all y. Thus, the extreme values of $v(y)$ occur at the endpoints $y = 0$ and $y = 5$, which correspond to the points $(0, 0)$ and $(0, 5)$ of R.

The line segment between $(3, 0)$ *and* $(0, 5)$: In the xy-plane, an equation for this line segment is

$$y = -\tfrac{5}{3}x + 5, \quad 0 \le x \le 3 \tag{4}$$

so (3) simplifies to a function of the single variable x,

$$\begin{aligned} w(x) &= f\left(x, -\tfrac{5}{3}x + 5\right) = 3x\left(-\tfrac{5}{3}x + 5\right) - 6x - 3\left(-\tfrac{5}{3}x + 5\right) + 7 \\ &= -5x^2 + 14x - 8, \quad 0 \le x \le 3 \end{aligned}$$

Since $w'(x) = -10x + 14$, the equation $w'(x) = 0$ yields $x = \frac{7}{5}$ as the only critical point of w. Thus, the extreme values of w occur either at the critical point $x = \frac{7}{5}$ or at the endpoints $x = 0$ and $x = 3$. The endpoints correspond to the points $(0, 5)$ and $(3, 0)$ of R, and from (4) the critical point corresponds to $\left(\frac{7}{5}, \frac{8}{3}\right)$.

Finally, Table 13.8.1 lists the values of $f(x, y)$ at the interior critical point and at the points on the boundary where an absolute extremum can occur. From the table we conclude that the absolute maximum value of f is $f(0, 0) = 7$ and the absolute minimum value is $f(3, 0) = -11$. ◀

Table 13.8.1

(x, y)	$(0, 0)$	$(3, 0)$	$(0, 5)$	$\left(\frac{7}{5}, \frac{8}{3}\right)$	$(1, 2)$
$f(x, y)$	7	−11	−8	$\frac{9}{5}$	1

▶ **Example 6** Determine the dimensions of a rectangular box, open at the top, having a volume of 32 ft^3, and requiring the least amount of material for its construction.

Solution. Let

x = length of the box (in feet)
y = width of the box (in feet)
z = height of the box (in feet)
S = surface area of the box (in square feet)

We may reasonably assume that the box with least surface area requires the least amount of material, so our objective is to minimize the surface area

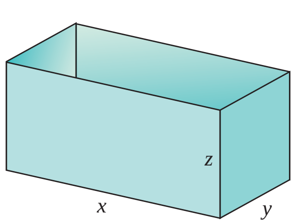
$$S = xy + 2xz + 2yz \tag{5}$$

(Figure 13.8.10) subject to the volume requirement

$$xyz = 32 \tag{6}$$

From (6) we obtain $z = 32/xy$, so (5) can be rewritten as

$$S = xy + \frac{64}{y} + \frac{64}{x} \tag{7}$$



Two sides each have area xz.
Two sides each have area yz.
The base has area xy.

▲ **Figure 13.8.10**

which expresses S as a function of two variables. The dimensions x and y in this formula must be positive, but otherwise have no limitation, so our problem reduces to finding the absolute minimum value of S over the open first quadrant: $x > 0$, $y > 0$. Because this region is neither closed nor bounded, we have no mathematical guarantee at this stage that an absolute minimum exists. However, if S has an absolute minimum value in the open first quadrant, then it must occur at a critical point of S. Thus, our next step is to find the critical points of S.

Differentiating (7) we obtain

$$\frac{\partial S}{\partial x} = y - \frac{64}{x^2}, \quad \frac{\partial S}{\partial y} = x - \frac{64}{y^2} \tag{8}$$

so the coordinates of the critical points of S satisfy

$$y - \frac{64}{x^2} = 0, \quad x - \frac{64}{y^2} = 0$$

Solving the first equation for y yields

$$y = \frac{64}{x^2} \tag{9}$$

and substituting this expression in the second equation yields

$$x - \frac{64}{(64/x^2)^2} = 0$$

which can be rewritten as

$$x\left(1 - \frac{x^3}{64}\right) = 0$$

The solutions of this equation are $x = 0$ and $x = 4$. Since we require $x > 0$, the only solution of significance is $x = 4$. Substituting this value into (9) yields $y = 4$. We conclude that the point $(x, y) = (4, 4)$ is the only critical point of S in the first quadrant. Since $S = 48$ if $x = y = 4$, this suggests we try to show that the minimum value of S on the open first quadrant is 48.

It immediately follows from Equation (7) that $48 < S$ at any point in the first quadrant for which at least one of the inequalities

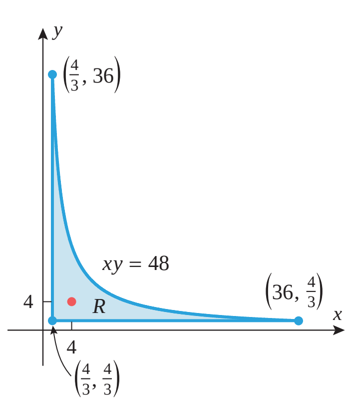
$$xy > 48, \quad \frac{64}{y} > 48, \quad \frac{64}{x} > 48$$

is satisfied. Therefore, to prove that $48 \le S$, we can restrict attention to the set of points in the first quadrant that satisfy the three inequalities

$$xy \le 48, \quad \frac{64}{y} \le 48, \quad \frac{64}{x} \le 48$$

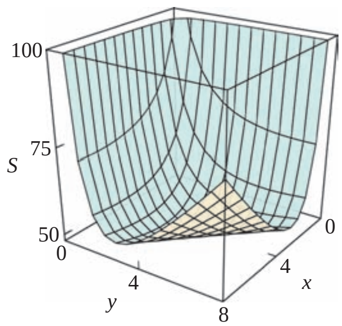
These inequalities can be rewritten as

$$xy \le 48, \quad y \ge \frac{4}{3}, \quad x \ge \frac{4}{3}$$



▲ **Figure 13.8.11**

and they define a closed and bounded region R within the first quadrant (Figure 13.8.11). The function S is continuous on R, so Theorem 13.8.3 guarantees that S has an absolute minimum value somewhere on R. Since the point $(4, 4)$ lies within R, and $48 < S$ on the boundary of R (why?), the minimum value of S on R must occur at an interior point. It then follows from Theorem 13.8.7 that the mimimum value of S on R must occur at a critical point of S. Hence, the absolute minimum of S on R (and therefore on the entire open first quadrant) is $S = 48$ at the point $(4, 4)$. Substituting $x = 4$ and $y = 4$ into (6) yields $z = 2$, so the box using the least material has a height of 2 ft and a square base whose edges are 4 ft long. ◄



▲ **Figure 13.8.12**

REMARK

Fortunately, in our solution to Example 6 we were able to prove the existence of an absolute minimum of S in the first quadrant. The general problem of finding the absolute extrema of a function on an unbounded region, or on a region that is not closed, can be difficult and will not be considered in this text. However, in applied problems we can sometimes use physical considerations to deduce that an absolute extremum has been found. For example, the graph of Equation (7) in Figure 13.8.12 strongly suggests that the relative minimum at $x = 4$ and $y = 4$ is also an absolute minimum.

✔QUICK CHECK EXERCISES 13.8 (See page 989 for answers.)

1. The critical points of the function $f(x, y) = x^3 + xy + y^2$ are ________.
2. Suppose that $f(x, y)$ has continuous second-order partial derivatives everywhere and that the origin is a critical point for f. State what information (if any) is provided by the second partials test if
   (a) $f_{xx}(0, 0) = 2,\ f_{xy}(0, 0) = 2,\ f_{yy}(0, 0) = 2$
   (b) $f_{xx}(0, 0) = -2,\ f_{xy}(0, 0) = 2,\ f_{yy}(0, 0) = 2$
   (c) $f_{xx}(0, 0) = 3,\ f_{xy}(0, 0) = 2,\ f_{yy}(0, 0) = 2$
   (d) $f_{xx}(0, 0) = -3,\ f_{xy}(0, 0) = 2,\ f_{yy}(0, 0) = -2$.
3. For the function $f(x, y) = x^3 - 3xy + y^3$, state what information (if any) is provided by the second partials test at the point
   (a) $(0, 0)$ (b) $(-1, -1)$ (c) $(1, 1)$.
4. A rectangular box has total surface area of 2 ft^2. Express the volume of the box as a function of the dimensions x and y of the base of the box.

EXERCISE SET 13.8 Graphing Utility C CAS

1–2 Locate all absolute maxima and minima, if any, by inspection. Then check your answers using calculus. ■

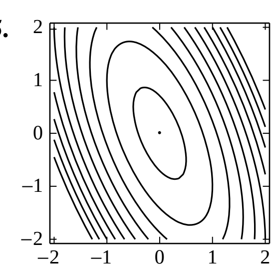
1. (a) $f(x, y) = (x - 2)^2 + (y + 1)^2$
   (b) $f(x, y) = 1 - x^2 - y^2$
   (c) $f(x, y) = x + 2y - 5$
2. (a) $f(x, y) = 1 - (x + 1)^2 - (y - 5)^2$
   (b) $f(x, y) = e^{xy}$
   (c) $f(x, y) = x^2 - y^2$

3–4 Complete the squares and locate all absolute maxima and minima, if any, by inspection. Then check your answers using calculus. ■

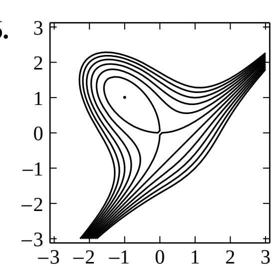
3. $f(x, y) = 13 - 6x + x^2 + 4y + y^2$
4. $f(x, y) = 1 - 2x - x^2 + 4y - 2y^2$

FOCUS ON CONCEPTS

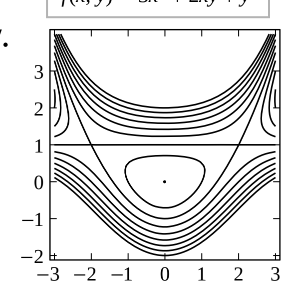
5–8 The contour plots show all significant features of the function. Make a conjecture about the number and the location of all relative extrema and saddle points, and then use calculus to check your conjecture. ■

5.


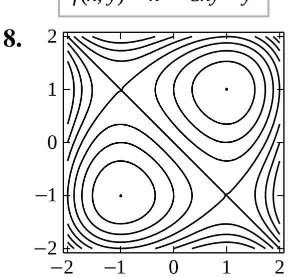
$f(x, y) = 3x^2 + 2xy + y^2$

6.


$f(x, y) = x^3 - 3xy - y^3$

7.


$f(x, y) = x^2 + 2y^2 - x^2y$

8.


$f(x, y) = x^3 + y^3 - 3x - 3y$

9–20 Locate all relative maxima, relative minima, and saddle points, if any. ■

9. $f(x, y) = y^2 + xy + 3y + 2x + 3$

10. $f(x, y) = x^2 + xy - 2y - 2x + 1$

11. $f(x, y) = x^2 + xy + y^2 - 3x$

12. $f(x, y) = xy - x^3 - y^2$

13. $f(x, y) = x^2 + y^2 + \dfrac{2}{xy}$

14. $f(x, y) = xe^y$

15. $f(x, y) = x^2 + y - e^y$

16. $f(x, y) = xy + \dfrac{2}{x} + \dfrac{4}{y}$

17. $f(x, y) = e^x \sin y$

18. $f(x, y) = y \sin x$

19. $f(x, y) = e^{-(x^2+y^2+2x)}$

20. $f(x, y) = xy + \dfrac{a^3}{x} + \dfrac{b^3}{y} \quad (a \neq 0, b \neq 0)$

C **21.** Use a CAS to generate a contour plot of
$$f(x, y) = 2x^2 - 4xy + y^4 + 2$$
for $-2 \leq x \leq 2$ and $-2 \leq y \leq 2$, and use the plot to approximate the locations of all relative extrema and saddle points in the region. Check your answer using calculus, and identify the relative extrema as relative maxima or minima.

C **22.** Use a CAS to generate a contour plot of
$$f(x, y) = 2y^2x - yx^2 + 4xy$$
for $-5 \leq x \leq 5$ and $-5 \leq y \leq 5$, and use the plot to approximate the locations of all relative extrema and saddle points in the region. Check your answer using calculus, and identify the relative extrema as relative maxima or minima.

23–26 True–False Determine whether the statement is true or false. Explain your answer. In these exercises, assume that $f(x, y)$ has continuous second-order partial derivatives and that
$$D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - f_{xy}^2(x, y)$$ ■

23. If the function f is defined on the disk $x^2 + y^2 \leq 1$, then f has a critical point somewhere on this disk.

24. If the function f is defined on the disk $x^2 + y^2 \leq 1$, and if f is not a constant function, then f has a finite number of critical points on this disk.

25. If $P(x_0, y_0)$ is a critical point of f, and if f is defined on a disk centered at P with $D(x_0, y_0) > 0$, then f has a relative extremum at P.

26. If $P(x_0, y_0)$ is a critical point of f with $f(x_0, y_0) = 0$, and if f is defined on a disk centered at P with $D(x_0, y_0) < 0$, then f has both positive and negative values on this disk.

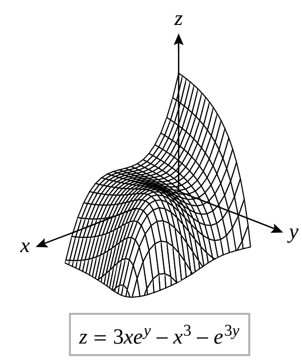
FOCUS ON CONCEPTS

27. (a) Show that the second partials test provides no information about the critical points of the function $f(x, y) = x^4 + y^4$.
(b) Classify all critical points of f as relative maxima, relative minima, or saddle points.

28. (a) Show that the second partials test provides no information about the critical points of the function $f(x, y) = x^4 - y^4$.
(b) Classify all critical points of f as relative maxima, relative minima, or saddle points.

29. Recall from Theorem 4.4.4 that if a continuous function of one variable has exactly one relative extremum on an interval, then that relative extremum is an absolute extremum on the interval. This exercise shows that this result does not extend to functions of two variables.
(a) Show that $f(x, y) = 3xe^y - x^3 - e^{3y}$ has only one critical point and that a relative maximum occurs there. (See the accompanying figure.)
(b) Show that f does not have an absolute maximum.

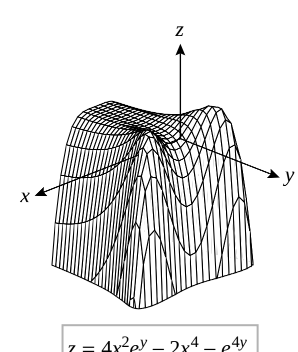
Source: This exercise is based on the article "The Only Critical Point in Town Test" by Ira Rosenholtz and Lowell Smylie, *Mathematics Magazine*, Vol. 58, No. 3, May 1985, pp. 149–150.



◀ **Figure Ex-29**

30. If f is a continuous function of one variable with two relative maxima on an interval, then there must be a relative minimum between the relative maxima. (Convince yourself of this by drawing some pictures.) The purpose of this exercise is to show that this result does not extend to functions of two variables. Show that $f(x, y) = 4x^2e^y - 2x^4 - e^{4y}$ has two relative maxima but no other critical points (see Figure Ex-30).

Source: This exercise is based on the problem "Two Mountains Without a Valley" proposed and solved by Ira Rosenholtz, *Mathematics Magazine*, Vol. 60, No. 1, February 1987, p. 48.



◀ **Figure Ex-30**

31–36 Find the absolute extrema of the given function on the indicated closed and bounded set R. ■

31. $f(x, y) = xy - x - 3y$; R is the triangular region with vertices $(0, 0)$, $(0, 4)$, and $(5, 0)$.

32. $f(x, y) = xy - 2x$; R is the triangular region with vertices $(0, 0)$, $(0, 4)$, and $(4, 0)$.

33. $f(x, y) = x^2 - 3y^2 - 2x + 6y$; R is the region bounded by the square with vertices $(0, 0)$, $(0, 2)$, $(2, 2)$, and $(2, 0)$.

34. $f(x, y) = xe^y - x^2 - e^y$; R is the rectangular region with vertices $(0, 0)$, $(0, 1)$, $(2, 1)$, and $(2, 0)$.

35. $f(x, y) = x^2 + 2y^2 - x$; R is the disk $x^2 + y^2 \le 4$.

36. $f(x, y) = xy^2$; R is the region that satisfies the inequalities $x \ge 0$, $y \ge 0$, and $x^2 + y^2 \le 1$.

37. Find three positive numbers whose sum is 48 and such that their product is as large as possible.

38. Find three positive numbers whose sum is 27 and such that the sum of their squares is as small as possible.

39. Find all points on the portion of the plane $x + y + z = 5$ in the first octant at which $f(x, y, z) = xy^2z^2$ has a maximum value.

40. Find the points on the surface $x^2 - yz = 5$ that are closest to the origin.

41. Find the dimensions of the rectangular box of maximum volume that can be inscribed in a sphere of radius a.

42. An international airline has a regulation that each passenger can carry a suitcase having the sum of its width, length, and height less than or equal to 129 cm. Find the dimensions of the suitcase of maximum volume that a passenger can carry under this regulation.

43. A closed rectangular box with a volume of 16 ft^3 is made from two kinds of materials. The top and bottom are made of material costing 10¢ per square foot and the sides from material costing 5¢ per square foot. Find the dimensions of the box so that the cost of materials is minimized.

44. A manufacturer makes two models of an item, standard and deluxe. It costs \$40 to manufacture the standard model and \$60 for the deluxe. A market research firm estimates that if the standard model is priced at x dollars and the deluxe at y dollars, then the manufacturer will sell $500(y - x)$ of the standard items and $45{,}000 + 500(x - 2y)$ of the deluxe each year. How should the items be priced to maximize the profit?

45. Consider the function

$$f(x, y) = 4x^2 - 3y^2 + 2xy$$

over the unit square $0 \le x \le 1$, $0 \le y \le 1$.

(a) Find the maximum and minimum values of f on each edge of the square.

(b) Find the maximum and minimum values of f on each diagonal of the square.

(c) Find the maximum and minimum values of f on the entire square.

46. Show that among all parallelograms with perimeter l, a square with sides of length $l/4$ has maximum area. [*Hint:* The area of a parallelogram is given by the formula $A = ab \sin \alpha$, where a and b are the lengths of two adjacent sides and α is the angle between them.]

47. Determine the dimensions of a rectangular box, open at the top, having volume V, and requiring the least amount of material for its construction.

48. A length of sheet metal 27 inches wide is to be made into a water trough by bending up two sides as shown in the accompanying figure. Find x and ϕ so that the trapezoid-shaped cross section has a maximum area.

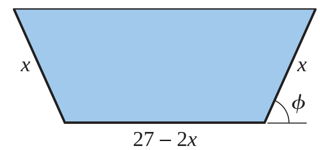


Figure Ex-48

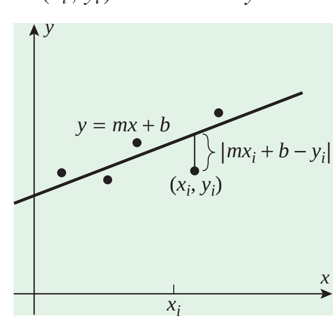
49–50 A common problem in experimental work is to obtain a mathematical relationship $y = f(x)$ between two variables x and y by "fitting" a curve to points in the plane that correspond to experimentally determined values of x and y, say

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$$

The curve $y = f(x)$ is called a ***mathematical model*** of the data. The general form of the function f is commonly determined by some underlying physical principle, but sometimes it is just determined by the pattern of the data. We are concerned with fitting a straight line $y = mx + b$ to data. Usually, the data will not lie on a line (possibly due to experimental error or variations in experimental conditions), so the problem is to find a line that fits the data "best" according to some criterion. One criterion for selecting the line of best fit is to choose m and b to minimize the function

$$g(m, b) = \sum_{i=1}^{n} (mx_i + b - y_i)^2$$

This is called the ***method of least squares***, and the resulting line is called the ***regression line*** or the ***least squares line of best fit***. Geometrically, $|mx_i + b - y_i|$ is the vertical distance between the data point (x_i, y_i) and the line $y = mx + b$.



These vertical distances are called the ***residuals*** of the data points, so the effect of minimizing $g(m, b)$ is to minimize the sum of the squares of the residuals. In these exercises, we will derive a formula for the regression line.

49. The purpose of this exercise is to find the values of m and b that produce the regression line.

(a) To minimize $g(m, b)$, we start by finding values of m and b such that $\partial g/\partial m = 0$ and $\partial g/\partial b = 0$. Show that these equations are satisfied if m and b satisfy the conditions

$$\left(\sum_{i=1}^{n} x_i^2\right) m + \left(\sum_{i=1}^{n} x_i\right) b = \sum_{i=1}^{n} x_i y_i$$

$$\left(\sum_{i=1}^{n} x_i\right) m + nb = \sum_{i=1}^{n} y_i$$

(b) Let $\bar{x} = (x_1 + x_2 + \cdots + x_n)/n$ denote the arithmetic average of $x_1, x_2, \ldots, x_n$. Use the fact that

$$\sum_{i=1}^{n} (x_i - \bar{x})^2 \geq 0$$

to show that

$$n\left(\sum_{i=1}^{n} x_i^2\right) - \left(\sum_{i=1}^{n} x_i\right)^2 \geq 0$$

with equality if and only if all the x_i's are the same.

(c) Assuming that not all the x_i's are the same, prove that the equations in part (a) have the unique solution

$$m = \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}$$

$$b = \frac{1}{n}\left(\sum_{i=1}^{n} y_i - m\sum_{i=1}^{n} x_i\right)$$

[*Note:* We have shown that g has a critical point at these values of m and b. In the next exercise we will show that g has an absolute minimum at this critical point. Accepting this to be so, we have shown that the line $y = mx + b$ is the regression line for these values of m and b.]

50. Assume that not all the x_i's are the same, so that $g(m, b)$ has a unique critical point at the values of m and b obtained in Exercise 49(c). The purpose of this exercise is to show that g has an absolute minimum value at this point.

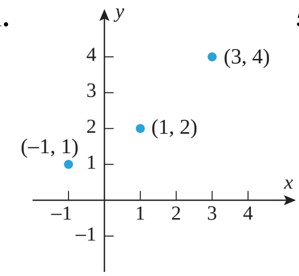
(a) Find the partial derivatives $g_{mm}(m, b)$, $g_{bb}(m, b)$, and $g_{mb}(m, b)$, and then apply the second partials test to show that g has a relative minimum at the critical point obtained in Exercise 49.

(b) Show that the graph of the equation $z = g(m, b)$ is a quadric surface. [*Hint:* See Formula (4) of Section 11.7.]

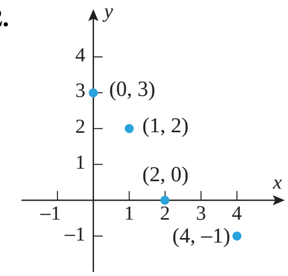
(c) It can be proved that the graph of $z = g(m, b)$ is an elliptic paraboloid. Accepting this to be so, show that this paraboloid opens in the positive z-direction, and explain how this shows that g has an absolute minimum at the critical point obtained in Exercise 49.

51–54 Use the formulas obtained in Exercise 49 to find and draw the regression line. If you have a calculating utility that can calculate regression lines, use it to check your work. ■

51.



52.



53.

x	1	2	3	4
y	1.5	1.6	2.1	3.0

54.

x	1	2	3	4	5
y	4.2	3.5	3.0	2.4	2.0

55. The following table shows the life expectancy by year of birth of females in the United States:

YEAR OF BIRTH	2000	2001	2002	2003	2004	2005	2006	2007
LIFE EXPECTANCY	79.3	79.4	79.5	79.6	79.9	79.9	80.2	80.4

Source: Data from *The 2011 Statistical Abstract*, the U.S. Census Bureau.

(a) Take $t = 0$ to be the year 1930, and let y be the life expectancy for birth year t. Use the regression capability of a calculating utility to find the regression line of y as a function of t.

(b) Use a graphing utility to make a graph that shows the data points and the regression line.

(c) Use the regression line to make a conjecture about the life expectancy of females born in the year 2015.

56. A company manager wants to establish a relationship between the sales of a certain product and the price. The company research department provides the following data:

PRICE (x) IN DOLLARS	\$35.00	\$40.00	\$45.00	\$48.00	\$50.00
DAILY SALES VOLUME (y) IN UNITS	80	75	68	66	63

(a) Use a calculating utility to find the regression line of y as a function of x.

(b) Use a graphing utility to make a graph that shows the data points and the regression line.

(c) Use the regression line to make a conjecture about the number of units that would be sold at a price of \$60.00.

57. If a gas is cooled with its volume held constant, then it follows from the ***ideal gas law*** in physics that its pressure drops proportionally to the drop in temperature. The temperature that, in theory, corresponds to a pressure of zero is called ***absolute zero***. Suppose that an experiment produces the following data for pressure P versus temperature T with the volume held constant:

(cont.)

P (KILOPASCALS)	134	142	155	160	171	184
T (°CELSIUS)	0	20	40	60	80	100

(a) Use a calculating utility to find the regression line of P as a function of T.
(b) Use a graphing utility to make a graph that shows the data points and the regression line.
(c) Use the regression line to estimate the value of absolute zero in degrees Celsius.

58. Find
(a) a continuous function $f(x, y)$ that is defined on the entire xy-plane and has no absolute extrema on the xy-plane;
(b) a function $f(x, y)$ that is defined everywhere on the rectangle $0 \le x \le 1, 0 \le y \le 1$ and has no absolute extrema on the rectangle.

59. Show that if f has a relative maximum at (x_0, y_0), then $G(x) = f(x, y_0)$ has a relative maximum at $x = x_0$ and $H(y) = f(x_0, y)$ has a relative maximum at $y = y_0$.

60. **Writing** Explain how to determine the location of relative extrema or saddle points of $f(x, y)$ by examining the contours of f.

61. **Writing** Suppose that the second partials test gives no information about a certain critical point (x_0, y_0) because $D(x_0, y_0) = 0$. Discuss what other steps you might take to determine whether there is a relative extremum at that critical point.

✔ QUICK CHECK ANSWERS 13.8

1. $(0, 0)$ and $\left(\frac{1}{6}, -\frac{1}{12}\right)$ **2.** (a) no information (b) a saddle point at $(0, 0)$ (c) a relative minimum at $(0, 0)$ (d) a relative maximum at $(0, 0)$ **3.** (a) a saddle point at $(0, 0)$ (b) no information, since $(-1, -1)$ is not a critical point (c) a relative minimum at $(1, 1)$ **4.** $V = \dfrac{xy(1 - xy)}{x + y}$

13.9 LAGRANGE MULTIPLIERS

In this section we will study a powerful new method for maximizing or minimizing a function subject to constraints on the variables. This method will help us to solve certain optimization problems that are difficult or impossible to solve using the methods studied in the last section.

EXTREMUM PROBLEMS WITH CONSTRAINTS

In Example 6 of the last section, we solved the problem of minimizing

$$S = xy + 2xz + 2yz \tag{1}$$

subject to the constraint

$$xyz - 32 = 0 \tag{2}$$

This is a special case of the following general problem:

> **13.9.1** ***Three-Variable Extremum Problem with One Constraint***
> Maximize or minimize the function $f(x, y, z)$ subject to the constraint $g(x, y, z) = 0$.

We will also be interested in the following two-variable version of this problem:

> **13.9.2** ***Two-Variable Extremum Problem with One Constraint***
> Maximize or minimize the function $f(x, y)$ subject to the constraint $g(x, y) = 0$.