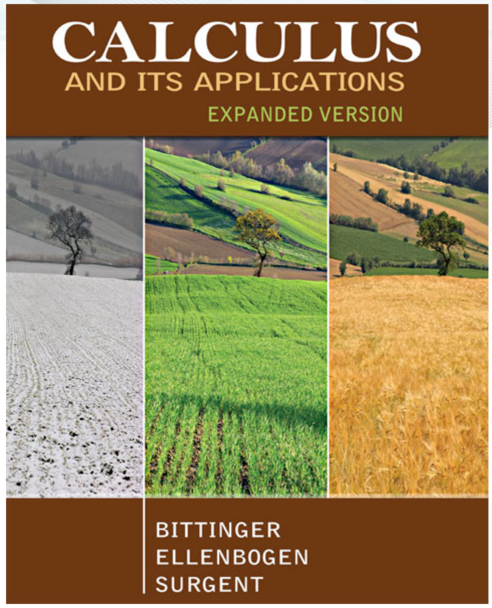
Section 2.3



Graph Sketching: Asymptotes and Rational Functions

2.3

OBJECTIVE

- Find limits involving infinity.
- Determine the asymptotes of a function's graph.
- Graph rational functions.

2.3 Graph Sketching: Asymptotes and Rational Functions

DEFINITION:

A **rational function** is a function f that can be described by

$$f(x) = \frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are polynomials, with $Q(x)$ not the zero polynomial. The domain of f consists of all inputs x for which $Q(x) \neq 0$.

2.3 Graph Sketching: Asymptotes and Rational Functions

DEFINITION:

The line $x = a$ is a **vertical asymptote** if any of the following limit statements are true:

$$\lim_{x \to a^-} f(x) = \infty \quad \text{or} \quad \lim_{x \to a^-} f(x) = -\infty \quad \text{or}$$

$$\lim_{x \to a^+} f(x) = \infty \quad \text{or} \quad \lim_{x \to a^+} f(x) = -\infty.$$

2.3 Graph Sketching: Asymptotes and Rational Functions

DEFINITION (continued):

The graph of a rational function *never* crosses a vertical asymptote. If the expression that defines the rational function f is simplified, meaning that it has no common factor other that –1 or 1, then if a is an input that makes the denominator 0, the line $x = a$ is a vertical asymptote.

2.3 Graph Sketching: Asymptotes and Rational Functions

Example 1: Determine the vertical asymptotes of the function given by

$$f(x)=\frac{P(x)}{Q(x)}$$

$$f(x)=\frac{x(x-2)}{x(x-1)(x+1)}$$

$$f(x)=\frac{(x-2)}{(x-1)(x+1)}$$

Since $x = 1$ and $x = -1$ make the denominator 0, $x = 1$ and $x = -1$ are vertical asymptotes.

2.3 Graph Sketching: Asymptotes and Rational Functions

Quick Check 1

Determine the vertical asymptotes: $f(x)=\frac{1}{x\left(x^2-16\right)}$

$$f(x)=\frac{1}{x\left(x^2-16\right)}$$

$$f(x)=\frac{1}{x\left(x+4\right)\left(x-4\right)}$$

After factoring out the denominator, we see that $x = 0$, $x = 4$, and $x = –4$ make the denominator 0. Thus, there are vertical asymptotes at $x = 0$, $x = 4$, and $x = –4$.

2.3 Graph Sketching: Asymptotes and Rational Functions

DEFINITION:

The line $y = b$ is a **horizontal asymptote** if either or both of the following limit statements are true:

$$\lim_{x\to-\infty} f\left(x\right)=b \qquad \text{or} \qquad \lim_{x\to\infty} f\left(x\right)=b.$$

2.3 Graph Sketching: Asymptotes and Rational Functions

DEFINITION (continued):

The graph of a rational function may or may not cross a horizontal asymptote. Horizontal asymptotes occur when the degree of the numerator is less than or equal to the degree of the denominator. (The degree of a polynomial in one variable is the highest power of that variable.)

2.3 Graph Sketching: Asymptotes and Rational Functions

Example 2: Determine the horizontal asymptote of the function given by

$$f(x)=\frac{3x^2+2x-4}{2x^2-x+1}.$$

First, divide the numerator and denominator by x^2.

$$f(x)=\frac{3+\frac{2}{x}-\frac{4}{x^2}}{2-\frac{1}{x}+\frac{1}{x^2}}$$

2.3 Graph Sketching: Asymptotes and Rational Functions

Example 2 (continued):

Second, find the limit as $|x|$ gets larger and larger.

$$\lim_{x\to-\infty}\frac{3+\frac{2}{x}-\frac{4}{x^2}}{2-\frac{1}{x}+\frac{1}{x^2}}=\frac{3}{2} \quad \text{and} \quad \lim_{x\to\infty}\frac{3+\frac{2}{x}-\frac{4}{x^2}}{2-\frac{1}{x}+\frac{1}{x^2}}=\frac{3}{2}$$

Thus, the line $y=\frac{3}{2}$ is a horizontal asymptote.

2.3 Graph Sketching: Asymptotes and Rational Functions

Quick Check 2

Determine the horizontal asymptote of the function given by

$$f(x)=\frac{(2x-1)(x+1)}{(3x+2)(5x+6)}.$$

First we should multiply both the numerator and denominator out:

$$f(x)=\frac{(2x-1)(x+1)}{(3x+2)(5x+6)}=\frac{2x^2+x-1}{15x^2+28x+12}$$

2.3 Graph Sketching: Asymptotes and Rational Functions

Quick Check 2 Concluded

Since both the numerator and denominator have the same power of x, we can divide both by that power:

$$f(x) = \frac{2x^2 + x - 1}{15x^2 + 28x + 12} = \frac{2 + \frac{1}{x} - \frac{1}{x^2}}{15 + \frac{28}{x} + \frac{12}{x^2}}$$

Now we can see that as $|x|$ gets very large, the numerator approaches 2 and the denominator approaches 15. Therefore the value of the function gets very close to $\frac{2}{15}$. Thus, $\lim_{x\to-\infty} f(x) = \frac{2}{15}$ and $\lim_{x\to\infty} f(x) = \frac{2}{15}$.

Therefore there is a horizontal asymptote at $y = \frac{2}{15}$.

2.3 Graph Sketching: Asymptotes and Rational Functions

DEFINITION:

A linear asymptote that is neither vertical nor horizontal is called a **slant**, or **oblique**, **asymptote**. For any rational function of the form $f(x) = p(x)/q(x)$, a slant asymptote occurs when the degree of $p(x)$ is exactly 1 more than the degree of $q(x)$. A graph can cross a slant asymptote.

2.3 Graph Sketching: Asymptotes and Rational Functions

Example 3: Find the slant asymptote:

$$f(x)=\frac{x^2-4}{x-1}$$

First, divide the numerator by the denominator.

$$\begin{array}{r|l} & x+1 \\ \hline x-1 & x^2 \quad -4 \\ & \underline{x^2-x} \\ & \quad x-4 \\ & \quad \underline{x-1} \\ & \quad -3 \end{array}$$

$$\Rightarrow \quad f(x)=\frac{x^2-4}{x-1}=(x+1)+\frac{-3}{x-1}$$

2.3 Graph Sketching: Asymptotes and Rational Functions

Example 3 (concluded):

Second, now we can see that as $|x|$ gets very large, $-3/(x-1)$ approaches 0. Thus, for very large $|x|$, the expression $x + 1$ is the dominant part of

$$(x+1)+\frac{-3}{x-1}$$

thus $y = x + 1$ is the slant asymptote.

2.3 Graph Sketching: Asymptotes and Rational Functions

Quick Check 3

Find the slant asymptote: $g(x)=\dfrac{2x^2+x-1}{x-3}$

Use polynomial division to solve for this:

$$\begin{array}{r}
2x \;\; +7 \\
x-3\,\overline{)\,2x^2+x-1}\\
-(2x^2-6x)\downarrow\\
\hline
7x \;\; -1\\
-(\;7x-21\;)\\
\hline
20
\end{array}$$

Since we have a remainder of 20, we can see that as $|x|$ gets very large, the remainder approaches 0. Thus the dominant part of $2x+7+\dfrac{20}{x-3}$ is $2x = 7$.

Therefore, there is slant asymptote at $y = 2x + 7$

2.3 Graph Sketching: Asymptotes and Rational Functions

Strategy for Sketching Graphs:

a) *Intercepts*. Find the x-intercept(s) and the y-intercept of the graph.

b) *Asymptotes*. Find any vertical, horizontal, or slant asymptotes.

c) *Derivatives and Domain*. Find $f'(x)$ and $f''(x)$. Find the domain of f.

d) *Critical Values of f*. Find any inputs for which $f'(x)$ is not defined or for which $f'(x)=0$.

2.3 Graph Sketching: Asymptotes and Rational Functions

Strategy for Sketching Graphs (continued):

e) *Increasing and/or decreasing; relative extrema*. Substitute each critical value, x_0, from step (d) into f''(x), and apply the Second Derivative Test. If no critical value exists, use f' and test values to find where f is increasing or decreasing.

f) *Inflection Points*. Determine candidates for inflection points by finding x-values for which $f''(x)$ does not exist or for which $f''(x) = 0$. Find the function values at these points.

2.3 Graph Sketching: Asymptotes and Rational Functions

Strategy for Sketching Graphs (concluded):

g) *Concavity*. Use the values c from step (f) as endpoints of intervals. Determine the concavity by checking to see where f' is increasing – that is, $f''(x) > 0$ – and where f' is decreasing – that is, $f''(x) < 0$. Do this by selecting test points and substituting into $f''(x)$. Use the results of step (d).

h) *Sketch the graph*. Use the information from steps (a) – (g) to sketch the graph, plotting extra points as needed.

2.3 Graph Sketching: Asymptotes and Rational Functions

Example 4: Sketch the graph of $f(x) = \dfrac{8}{x^2 - 4}$.

a) *Intercepts*. The x-intercepts occur at values for which the numerator equals 0. Since $8 \neq 0$, there are no x-intercepts. To find the y-intercept, we find $f(0)$.

$$f(0) = \frac{8}{0^2 - 4} = \frac{8}{-4} = -2$$

Thus, we have the point (0, –2).

2.3 Graph Sketching: Asymptotes and Rational Functions

Example 4 (continued):

b) *Asymptotes*.

Vertical:
$$\begin{aligned} x^2 - 4 &= 0 \\ (x-2)(x+2) &= 0 \end{aligned}$$

So, $x = 2$ and $x = -2$ are vertical asymptotes.

Horizontal: The degree of the numerator is less than the degree of the denominator. So, the x-axis, $y = 0$ is the horizontal asymptote.

2.3 Graph Sketching: Asymptotes and Rational Functions

Example 4 (continued):

Slant: There is no slant asymptote since the degree of the numerator is not 1 more than the degree of the denominator.

c) *Derivatives and Domain*. Using the Quotient Rule, we get

$$f'(x) = \frac{-16x}{(x^2-4)^2} \quad \text{and} \quad f''(x) = \frac{16(3x^2+4)}{(x^2-4)^3}.$$

The domain of f is all real numbers, $x \neq 2$ and $x \neq -2$.

2.3 Graph Sketching: Asymptotes and Rational Functions

Example 4 (continued):

d) *Critical Values of f*. $f'(x)$ equals 0 where the numerator equals 0 and does not exist where the denominator equals 0.

$$\begin{aligned} -16x &= 0 \\ x &= 0 \end{aligned}$$

$$\begin{aligned} (x^2-4)^2 &= 0 \\ x^2-4 &= 0 \\ (x-2)(x+2) &= 0 \end{aligned}$$

$$x = 2 \quad or \quad x = -2$$

However, since f does not exist at $x = 2$ or $x = -2$, $x = 0$ is the only critical value.

2.3 Graph Sketching: Asymptotes and Rational Functions

Example 4 (continued):

e) *Increasing and/or decreasing; relative extrema.*

$$f''(0)=\frac{16(3\cdot 0^2+4)}{(0^2-4)^3}=\frac{64}{-64}=-1<0$$

Thus, $x = 0$ is a relative maximum and f is increasing on (–2, 0) and decreasing on (0, 2).

Since f'' does not exist at $x = 2$ and $x = –2$, we use f' and test values to see if f is increasing or decreasing on (–∞, 2) and (2, ∞).

2.3 Graph Sketching: Asymptotes and Rational Functions

Example 4 (continued):

$$f'(-3)=\frac{-16(-3)}{\left((-3)^2-4\right)^2}=\frac{48}{25}>0$$

So, f is increasing on (–∞, 2).

$$f'(3)=\frac{-16(3)}{\left((3)^2-4\right)^2}=\frac{-48}{25}<0$$

So, f is decreasing on (2, ∞).

2.3 Graph Sketching: Asymptotes and Rational Functions

Example 4 (continued):

f) *Inflection points*. f'' does not exist $x = 2$ and $x = -2$. However, neither does f. Thus we consider where f'' equals 0.

$$16(3x^2 + 4) = 0$$

Note that $16(3x^2 + 4) > 0$ for all real numbers x, so there are no points of inflection.

2.3 Graph Sketching: Asymptotes and Rational Functions

Example 4 (continued):

g) *Concavity*. Since there are no points of inflection, the only places where f could change concavity would be on either side of the vertical asymptotes.

Note that we already know from step (e) that f is concave down at $x = 0$. So we need only test a point in $(-\infty, 2)$ and a point in $(2, \infty)$.

2.3 Graph Sketching: Asymptotes and Rational Functions

Example 4 (continued):

$$f''(-3) = \frac{16(3\cdot(-3)^2+4)}{((-3)^2-4)^3} = \frac{496}{125} > 0$$

Thus, f is concave up on $(-\infty, 2)$.

$$f''(3) = \frac{16(3\cdot(3)^2+4)}{((3)^2-4)^3} = \frac{496}{125} > 0$$

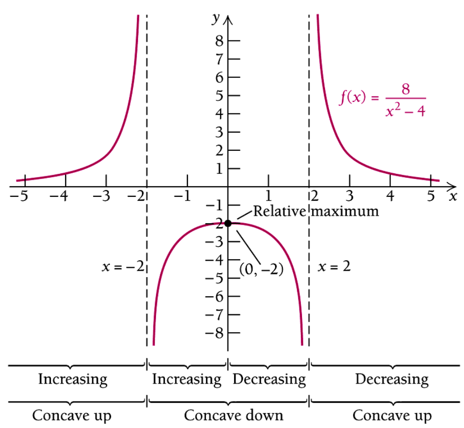
Thus, f is concave up on $(2, \infty)$.

2.3 Graph Sketching: Asymptotes and Rational Functions

Example 4 (continued):

h) *Sketch the graph*. Using the information in steps (a) – (g), the graph follows.

x	$f(x)$ approximately
−5	0.38
−4	0.67
−3	1.6
−1	−2.67
0	−2
1	−2.67
3	1.6
4	0.67
5	0.38



2.3 Graph Sketching: Asymptotes and Rational Functions

Section Summary

- A line $x = a$ is a *vertical asymptote* if $\lim_{x \to a^-} f(x) = \pm\infty$ or $\lim_{x \to a^+} f(x) = \pm\infty$
- A line $y = b$ is a *horizontal asymptote* if $\lim_{x \to \infty} f(x) = b$ or $\lim_{x \to -\infty} f(x) = b$
- A graph may cross a horizontal asymptote but never a vertical asymptote.
- A *slant asymptote* occurs when the degree of the numerator is 1 greater than the degree of the denominator. Long division of polynomials can be used to determine the equation of the slant asymptote.
- Vertical, horizontal, and slant asymptotes can be used as guides for accurate curve sketching. Asymptotes are not a part of a graph but are visual guides only.

