CHAPTER 7

REDUCTION FORMULAE

4.1Reduction formulae for $\sin^n x$ and $\cos^n x$:

Let $I_n = \int sin^n x dx = \int sin^{n-1} x sin\, x\, dx$

Integrating by parts by taking $sin^{n-1}x$ as first function and sin x as second function.

$I_n = \sin^{n-1}x(-\cos x) - \int (n-1)\, sin^{n-2}x \,.\cos x\,(-\cos x)\, dx$

$= -\sin^{n-1} x\cos x + (n-1) \int sin^{n-2}x cos^2\; x\, dx$

$= -\; \sin^{n-1} x \cos x \;+\; (n-1) \int sin^{n-2}x\,(1-sin^2 x)\, dx$

$= -\; \sin^{n-1} x \cos x \;+\; (n-1) \int sin^{n-2}x\, dx - (n-1) \int sin^n x\, dx$

$I_n \;= -\; \sin^{n-1} x\cos x \;+\; (n-1)\, I_{n-2} - \;(n-1)\, I_n$

$\Rightarrow\; I_n(1+(n-1)) = -\sin^{n-1}x\; \cos x + \;(n-1)\, I_{n-2}$

$\Rightarrow I_n \;=\; \dfrac{-Sin^{n-1}x\cos x}{n} \;+\; \dfrac{(n-1)}{n}\; I_{n-2}$

is the required reduction formula for $\int sin^n x\, dx$

Similary $\int cos^n x\, dx = \dfrac{cos^{n-1}x\; Sin\, x}{n} \;+\; \dfrac{(n-1)}{n} I_{n-2}$

Derivation of formula for $\int_0^{\pi/2} \sin^n x\, d\, x$

$\int sin^n\; x\, dx \;\;=\; -\frac{1}{n}\left(sin^{n-1}xcosx\right) + \left(\frac{n-1}{n}\right) I_{n-2}$ (By reduction formula for $\int sin^n x\, dx$)

$\therefore \int_0^{\pi/2} Sin^n\, x\; d\; x = -\frac{1}{n}\left[sin^{n-1}x\cos x\right]_0^{\pi/2} + \left(\frac{n-1}{n}\right) \int_0^{\pi/2} Sin^n\; x\; d\; x$

$= 0 + \left(\frac{n-1}{n}\right) \int_0^{\pi/2} Sin^{n-2}\; x\; d\; x$

$\therefore I_n = \left(\frac{n-1}{n}\right) I_{n-2}$ (where $I_n = \int_0^{\pi/2} Sin^n\; x\; d\; x$)

Changing n to n–2, n–4, n–6,…..in successive steps, we get

$I_{n-2} = \left(\frac{n-3}{n-2}\right) I_{n-4}$

$I_{n-4} = \left(\frac{n-5}{n-4}\right) I_{n-6}$ and so on.

$\therefore I_n = \frac{n-1}{n}\; . \frac{n-3}{n-2} . \frac{n-5}{n-4}\; I_{n-6}$

Case (i) If n is an even positive integer, then

$I_n = \frac{n-1}{n} \quad \frac{n-3}{n-2} \ldots\ldots \frac{5}{6}, \frac{3}{4}, \frac{1}{2} \quad \int_0^{\pi/2} 1\; d\; x$

$$\int_0^{\pi/2} Sin^n x\, d\, x = \frac{n-1}{n} \quad \frac{n-3}{n-2} \ldots\ldots \frac{5}{6}. \frac{3}{4}. \frac{1}{2} . \frac{\pi}{2}, \text{ if n is even}$$

Case (ii) If n is an odd positive integer, then

$$I_n = \frac{n-1}{n}. \frac{n-3}{n-2} . \frac{n-5}{n-4} \ldots\ldots \frac{4}{5}. \frac{2}{3} \int_0^{\pi/2} \sin x \;\, d\, x$$

$$= \frac{n-1}{n}. \frac{n-3}{n-2}. \frac{n-5}{n-4} \ldots\ldots \frac{4}{5}. \frac{2}{3} \left[- cosx\right]_0^{\pi/2}$$

$$\therefore \int_0^{\pi/2} sin^n x\, d\, x = \frac{n-1}{n}. \frac{n-3}{n-2}. \frac{n-5}{n-4} \ldots\ldots \frac{4}{5}. \frac{2}{3}.1 \quad \text{, if n is odd}$$

Example 1 Find $I_n = \int_0^{\pi/2} cos^n x\, d\, x$

Solution: $I_n = \int_0^{\pi/2} cos^n(\frac{\pi}{2} - x)\, d\, x$ $\quad$ ($\because \int_0^a f\,(x)dx = \int_0^a f\,\left(a - x\right)dx, if$ f is continuous function on [0,a])

$$= \int_0^{\pi/2} sin^n x\, d\, x = \begin{cases} \frac{n-1}{n} \;\; \frac{n-3}{n-2} \;\; \frac{n-5}{n-4} \ldots\ldots \frac{4}{5} . \frac{2}{3}. 1, if\ n\ is\ odd \\ \frac{n-1}{n} \;\; \frac{n-3}{n-2} \ldots\ldots \frac{5}{8}. \frac{3}{4}. \frac{1}{2} \frac{\pi}{2}, if\ n\ is\ even \end{cases}$$

Example 2 Evaluate $\int_0^{\pi/2} sin^4 x\, d\, x$

Solution: $\int_0^{\pi/2} sin^4 x\, d\, x = \frac{(4-1)\,(4-3)}{4\,(4-2)} \frac{\pi}{2}$ ($\because$ n = 4 is even) $= \frac{3\pi}{16}$

Example 3 Evaluate $\int_0^{\infty} \frac{dx}{(1+x^2)^4}$

Solution: Put x = $tan\theta$ $\Rightarrow$ dx = $\sec^2 \theta$ dθ

When x $\rightarrow$ 0, $\theta \rightarrow 0$ and when x $\rightarrow \infty$, $\theta \rightarrow \frac{\pi}{2}$

$\therefore$ Given integral becomes

$$\int_0^{\pi/2} \frac{sec^2\theta d\,\theta}{(1 + tan^2\theta)^4} = \int_0^{\pi/2} \frac{sec^2\theta}{(sec^2\theta)^4} = \int_0^{\pi/2} \frac{Sec^2\theta}{Sec^8\theta}\, d\theta$$

$$= \int_0^{\pi/2} \frac{1}{Sec^6\theta} = \int_0^{\pi/2} cos^6\ \theta d\theta$$

$$= \frac{(6-1)\,(6-3)\,(6-5)}{6\,(6-2)(6-4)} \frac{\pi}{2} = \frac{15\pi}{32}$$

Example 4 Obtain the reduction formula for $\int sin^m\, x\, cos^n\, x\, dx$

Solution: Let $I_{m,n} = \int sin^m\, x\, cos^n\, x\, dx$

$$= \int sin^m\, x\, cos^{n-1}\, x cos\, x\, dx$$

$$= \int cos^{n-1}\, x\, (sin^m x \cos x)\, dx$$

$$= cos^{n-1} x\, \frac{sin^{m+1} x}{m+1} - \int (n-1)\, cos^{n-2} x\, (-\sin x). \frac{sin^{m+1} x}{m+1}\, dx$$

(Integrating by parts) $\left(\because \int Sin^m x cos\ x\, dx = \frac{sin^{m+1}x}{m+1}\right)$

$$= \frac{sin^{m+1}x\, cos^{n-1}x}{m+1} + \frac{n-1}{m+1}\int cos^{n-2}x\, sin^{m+2}x\, dx$$

$$= \frac{sin^{m+1}x\, cos^{n-1}x}{m+1} + \frac{n-1}{m+1}\int cos^{n-2}x\, sin^{m}x\, sin^2 x\, dx$$

$$= \frac{sin^{m+1}x\, cos^{n-1}x}{m+1} + \frac{n-1}{m+1}\int cos^{n-2}x\, sin^{m}x(1 - cos^2 x)\, dx$$

$$= \frac{sin^{m+1}x\, cos^{n-1}x}{m+1} + \frac{n-1}{m+1}\int cos^{n-2}x\, sin^{m}x\, dx - \frac{(n-1)}{m+1}\int cos^{n}x\, sin^{m}x\, dx$$

$$I_{m,n} = \frac{sin^{m+1}x\, cos^{n-1}x}{m+1} + \frac{n-1}{m+1} I_{m,n-2} - \frac{(n-1)}{m+1} I_{m,n}$$

$$\left(1 + \frac{n-1}{m+1}\right) I_{m,n} = \frac{Sin^{m+1}x\, cos^{n-1}x}{m+1} + \frac{n-1}{m+n} I_{m,n-2}$$

$$\Rightarrow I_{m,n}\,(m+n) = \sin^{m+1}x\cos^{n-1}x + (n-1)I_{m,n-2}$$

$$\Rightarrow I_{m,n} = \frac{Sin^{m+1}x\, cos^{n-1}x}{m+n} + \frac{n-1}{m+n}\int sin^{m}x\, cos^{n-2}x\, dx$$

$$\Rightarrow \int sin^{m}x\, cos^{n}x\, dx = \frac{Sin^{m+1}x\, cos^{n-1}x}{m+n} + \frac{n-1}{m+n}\int sin^{m}x\, cos^{n-2}x\, dx$$

Example 5 If $U_n = \int_0^{\pi/2} x^n\, sinx\, dx$ and n > 1 prove that

$$U_n + n(n-1)U_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}$$

Solution: $U_n = \int_0^{\pi/2} x^n\, sinx\, dx$

$$= x^n \int \sin x\, dx - \int_2^{\pi/2}\left\{\frac{d}{dx}(x^n)\left[\int \sin x\, dx\right]\right\} dx$$

$$= \left[x^n(-\cos x)\right]_0^{\pi/2} - \int_0^{\pi/2} n\, x^{n-1}(-\cos x)\, dx$$

$$= -\left[\left(\frac{\pi}{2}\right)^n \cos\frac{\pi}{2} - 0\right] + \int_0^{\pi/2} n\, x^{n-1}\cos x\, dx$$

$$= n\int_0^{\pi/2} x^{n-1}\cos x\, dx$$

$$= n\left\{[x^{n-1}sinx]_0^{\pi/2} - \int_0^{\pi/2}(n-1)\, x^{n-2}\sin x\, d\,x\right\}$$

$$= n\left[x^{n-1}\sin x\right]_0^{\pi/2} - n\,(n-1)\int_0^{\pi/2} x^{n-2}\sin x\, dx$$

$$\Rightarrow U_n = n\left[\left(\frac{\pi}{2}\right)^{n-1} sin\frac{\pi}{2} - 0\right] - n(n-1)U_{n-2}$$

$$\Rightarrow U_n + n(n-1)U_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}$$

Example 6 Evaluate $\int_0^{\pi/2} x^4 \sin x\, dx$

Solution: $U_n = \int_0^{\pi/2} x^4 \sin x\, dx$

Now $U_n + \mathrm{n}(\mathrm{n}-1)U_{n-2} - 2 = \mathrm{n}\left(\frac{\pi}{2}\right)^{\mathrm{n}-1}$ ………(1)

Putting n = 4 in (1), we get

$$\mathrm{U}_4 + 4(4-1)\mathrm{U}_{4-2} = 4\left(\frac{\pi}{2}\right)^{4-1}$$

$$\Rightarrow \mathrm{U}_4 + 12\mathrm{U}_2 = \frac{\pi^3}{2} \qquad \text{………(2)}$$

Putting n = 2 in (1) , we get

$$\mathrm{U}_2 + 2(2-1)\mathrm{U}_{2-2} = 2\left(\frac{\pi}{2}\right)^{2-1}$$

$$\mathrm{U}_2 + 2\mathrm{U}_0 = \pi \qquad \text{………..(3)}$$

Now $U_0 = \int_0^{\pi/2} x^0 \sin x\, dx = \int_0^{\pi/2} \sin x\, dx$

$$= \left[-\cos x\right]_0^{\pi/2} = -\cos\frac{\pi}{2} + \cos 0 = 1$$

Hence equation (3) becomes

$$U_2 + 2\,(1) = \pi$$

$$\Rightarrow U_2 = \pi - 2$$

$\therefore$ (2) becomes $U_4 + 12\,(\pi - 2) = \frac{\pi^3}{2}$

$$U_4 = \frac{\pi^3}{2} - 12\,\pi + 24$$

$$\Rightarrow \int_0^{\pi/2} x^4 \sin x\, dx = \frac{\pi^3}{2} - 12\,\pi + 24$$

Example 7 If $I_{m,n} = \int_0^{\pi/2} sin^m cos^n x\, dx$ then prove that

$$I_{m,n} = \frac{\mathrm{m}-1}{\mathrm{m}+\mathrm{n}} \cdot \frac{\mathrm{m}-3}{\mathrm{m}+\mathrm{n}-2} \cdot \frac{\mathrm{m}-5}{\mathrm{m}+\mathrm{n}-4} \cdots\cdots\cdots \frac{2}{3+\mathrm{n}} \cdot \frac{1}{1+\mathrm{n}}$$

where m is an odd positive integer and n is a positive integer, even or odd.

Solution: $\int sin^m x\, cos^n x\, d\,x = \int sin^{m-1} x\, (\sin x\ cos^n)\ d\,x$

$$= \frac{-cos^{n+1} x\, sin^{m-1} x}{n+1} + \frac{m-1}{n+1}\int cos^{n+1} x\, sin^{m-2} x \cos x\, dx$$

(Integrating using by parts)

$$= \frac{-cos^{n+1} x\, sin^{m-1} x}{n+1} + \frac{m-1}{n+1}\int cos^{m-2} x\, cos^n x\, \left(1 - sin^2 x\right) dx$$

$$\Rightarrow \left(1 + \frac{m-1}{n+1}\right)\int sin^m x\, cos^n x\, dx = -\frac{cos^{n+1} x\, Sin^{m-1} x}{n+1} + \frac{m-1}{n+1}\int sin^{m-2} x\, cos^n x\, dx$$

$$\Rightarrow \int sin^m x\, cos^n x\, dx = -\frac{cos^{n+1} x\, Sin^{m-1} x}{n+1} + \frac{m-1}{n+1}\int sin^{m-2} x\, cos^n x\, dx$$

Now $I_{m,n} = \int_0^{\pi/2} sin^m x\, cos^n x\, dx$

$$= \left[\frac{-\cos^{n+1}x\, \sin^{m-1}x}{m+n}\right]_0^{\pi/2} + \frac{m-1}{m+n} \int_0^{\pi/2} \sin^{m-2}x\, \cos^n x\, dx$$

$$\Rightarrow \int_0^{\pi/2} \sin^m x\, \cos^n x\, dx = \frac{m-1}{m+n} \int_0^{\pi/2} \sin^{m-2}x\, \cos^n x\, dx$$

Hence, $I_{m,n} = \frac{m-1}{m+n} I_{m-2,n}$

Replacing m by m – 2, m – 4, …….,3, 2, we obtain

$$I_{m-2,n} = \frac{m-3}{m+n-2} I_{m-4,n}$$

$$I_{m-4,n} = \frac{m-5}{m+n-4} I_{m-6,n}$$

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$$I_{3,n} = \frac{2}{3+n} I_{1,n}$$

$$I_{2,n} = \frac{1}{2+n} I_{0,n}$$

From these relations, we obtain

$$I_{m,n} = \begin{cases} \frac{m-1}{m+n} \frac{m-3}{m+n-2} \frac{m-5}{m+n-4} \dots \dots \frac{2}{3+n} I_{1,n}, \text{ if m is odd} \\ \frac{m-1}{m+n} \frac{m-3}{m+n-2} \frac{m-5}{m+n-4} \dots \dots \frac{1}{2+n} I_{0,n}, \text{ if m is even} \end{cases}$$

Now, we have

$$I_{1,n} = \int_0^{\pi/2} \sin x\, \cos^n x\, dx = -\left[\frac{\cos^{n+1}x}{n+1}\right]_0^{\pi/2} = \frac{1}{n+1}$$

And $I_{0,n} = \int_0^{\pi/2} \cos^n x\, dx = \begin{cases} \frac{n-1}{n} \frac{n-3}{n-2} \dots \dots \frac{2}{3}.1, if\ n\ is\ odd \\ \frac{n-1}{n} \frac{n-3}{n-2} \dots \dots \frac{1}{2}.\frac{\pi}{2}\ if\ n\ is\ even \end{cases}$

$$\therefore I_{m,n} = \begin{cases} \frac{m-1}{m+n} \frac{m-3}{m+n-2} \dots .. \frac{2}{3+n} . \frac{1}{1+n} \\ if\ m\ is\ odd\ \ and\ n\ may\ be\ even\ or\ odd \\ \frac{m-1}{m+n} . \frac{m-3}{m+n-2} \dots \dots \frac{1}{2+n} \frac{n-1}{n} - \frac{n-3}{n-2} \dots \dots . \frac{2}{3} \\ if\ m\ is\ even\ and\ n\ is\ odd \\ \frac{m-1}{m+n} \frac{m-3}{m+n-2} \dots \dots \frac{1}{2+n} \frac{n-1}{n} - \frac{n-3}{n-2} \dots \dots \frac{1}{2} . \frac{\pi}{2} \\ if\ m\ is\ even\ \&\ n\ is\ even \end{cases}$$

These formulae can be expressed as a single formula

$\int_0^{\pi/2} \sin^m x\ \cos^n x\, dx = \frac{(m-1)(m-3)\ldots\ldots\ldots(n-1)(n-3)\ldots\ldots}{(m+n)(m+n-2)(m+n-4)\ldots\ldots}$

to be multiplied by $\frac{\pi}{2}$ when m & n both are even integers.

Example 8 Find $\int_0^{\pi/2} \sin^6 x\ cos^5 x\, dx$

Solution: Here m = 6 and n = 5

$$\int_0^{\pi/2} \sin^6 x\ cos^5 x\, dx = \frac{(6-1)(6-3)(6-5)(5-1)(5-3)}{(6+5)(6+5-2)(6+5-4)(6+5-6)(6+5-8)(6+5-10)} = \frac{8}{693}$$

Example 9 Evaluate $\int_0^{\pi} x\, sin^7 x\, cos^4 x\, dx$

Solution: Let I = $\int_0^{\pi} x\, sin^7 x\, cos^4 x\, dx$

$$= \int_0^{\pi} (\pi - x)\, sin^7 (\pi - x)\, cos^4 (\pi - x)\, dx \quad \left(\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right)$$

$$= \int_0^{\pi} (\pi - x) sin^7 x\, cos^4 x\, dx$$

$$= \pi \int_0^{\pi} sin^7 x\, cos^4 dx - \int_0^{\pi} x\, sin^7 x\, cos^4 x\, dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} sin^7 x\, cos^4 dx$$

$$= 2\int_0^{\pi/2} sin^7 x\, cos^4 x dx \quad \because \int_0^{2a} f(x) dx = \begin{cases} 2\int_0^a f(x)dx, & if\ f(2a-x) = f(x) \\ 0, & if\ f(2a-x) = -f(x) \end{cases}$$

$$\Rightarrow I = \pi \int_0^{\pi/2} sin^7 x\, cos^4 dx$$

$$= \frac{\pi (7-1)(7-3)(7-5)(4-1)(4-3)}{(7+4)(7+4-2)(7+4-4)(7+4-6)(7+4-8)} = \frac{16\pi}{385}$$

$$\left(using \int_0^{\pi/2} sin^m x\, cos^n dx = \frac{(m-1)(m-3)\ldots\ldots(n-1)(n-3)\ldots\ldots}{(m+n)(m+n-2)(m+n-4)\ldots\ldots\ldots\ldots} \right)$$

Example 10 Evaluate $\int_0^4 x^3 \sqrt{4x - x^2}\, dx$

Solution: Let I = $\int_0^4 x^3 \sqrt{4x - x^2}\, dx = \int_0^4 x^3 \sqrt{x(4-x)}\, dx$

$$= \int_0^4 x^3 \sqrt{x} \sqrt{(4-x)}\, dx$$

$$= \int_0^4 x^{7/2} (4-x)^{1/2}\, dx$$

Putting x = 4 $sin^2\theta$ $\Rightarrow$ dx = 8sin $\theta cos\,\theta$ d θ

Hence I = $\int_0^{\pi/2} 4^{7/2} sin^7\ \theta (4 - 4\, sin^2\theta)^{1/2}\, 8 \sin\theta \cos\theta\, d\theta$

$$= \int_0^{\pi/2} 4^{7/2}\, 4^{1/2}\, 8 sin^7\ \theta\, (1 - sin^2\theta)^{1/2} \sin\theta \cos\theta\, d\theta$$

$$= 8.\ 4^4 \int_0^{\pi/2} sin^8 x\, cos^2 dx$$

$$= 8.4^4 \frac{(8-1)(8-3)(8-5)(8-7)}{(8+2)(8+2-2)(8+2-4)(8+2-6)(8+2-8)} \frac{\pi}{2} = \frac{8.4^4.7.5.3}{10.8.6.4.2} \frac{\pi}{2} = 28\pi$$

Example 11 Evaluate $\int_0^{\infty} \frac{x^6 - x^3}{(1+x^3)^5} x^2 \ \mathrm{dx}$

Solution: Let I = $\int_0^{\infty} \frac{(x^6 - x^3)}{(1+x^3)^5} x^2 \ \mathrm{dx}$

Put $x^3 = \tan^2 \theta \Rightarrow 3x^2 \, dx = 2 \tan \theta \sec^2 \theta d \theta$

Then I = $\int_0^{\pi/2} \frac{(tan^4\theta - tan^2\theta)}{(1+tan^2\theta)^5} \ \frac{2}{3} \tan \theta \sec^2 \theta d\theta$

$$= \frac{2}{3} \int_0^{\pi/2} \frac{tan^5\theta}{(sec^2\theta)^5} \quad Sec^2\theta \, d\theta - \frac{2}{3} \int_0^{\pi/2} \frac{tan^3\theta}{(sec^2\theta)^5} sec^2\theta \, \mathrm{d}\,\theta$$

$$= \frac{2}{3} \int_0^{\pi/2} \frac{tan^5\theta}{sec^8\theta} \mathrm{d}\theta \quad - \frac{2}{3} \int_0^{\pi/2} \frac{tan^3\theta}{sec^8\theta} \mathrm{d}\,\theta$$

$$= \frac{2}{3} \int_0^{\pi/2} sin^5\,\theta\; cos^3\theta d\,\theta - \frac{2}{3} \int_0^{\pi/2} sin^3\,\theta\; cos^5\theta d\,\theta$$

$$= \frac{2}{3} \int_0^{\pi/2} sin^5\,\theta\; cos^3\theta d\,\theta - \frac{2}{3} \int_0^{\pi/2} sin^3 \left(\frac{\pi}{2} - \theta\right)\; cos^5 \left(\frac{\pi}{2} - \theta\right) d\,\theta$$

$$= \frac{2}{3} \int_0^{\frac{\pi}{2}} sin^5\,\theta\; cos^3\theta d\,\theta - \frac{2}{3} \int_0^{\frac{\pi}{2}} cos^3\,\theta\; sin^5\,\theta d$$

$$\left[\because \int_0^a f\,(x) d\,x = \int_0^a f\,(a - x) d\,x\right]$$

$= 0$

Example 12 Evaluate $\int_0^{\pi/2} sin^5\, x\; d\,x$

Solution: We know $\int sin^n x\; d\,x = -\frac{sin^{n-1}x \cos x}{n} + \frac{n-1}{n} \int sin^{n-2}\, x\; dx$

$$\therefore \int_0^{\pi/4} sin^5\, x\; d\,x = \left[\frac{-sin^{5-1}x \cos x}{5}\right]_0^{\pi/4} + \frac{5-1}{5} \int_0^{\pi/4} sin^{5-2}\, x\; d\,x$$

$$= \frac{-1}{5} \left[sin^4 x \cos x\right]_0^{\pi/4} + \frac{4}{5} \int_0^{\pi/4} sin^3\, x\; d\,x$$

$$= \frac{-1}{5} \left[\left(\frac{1}{\sqrt{2}}\right)^4 \left(\frac{1}{\sqrt{2}}\right)\right] + \frac{4}{5} \int_0^{\pi/4} sin^3\, x\; d\,x \qquad \ldots\ldots..(1)$$

Now $\int_0^{\pi/4} sin^3\, x\; d\,x = \left[-\frac{Sin^{3-1}x \cos x}{3}\right]_0^{\pi/4} + \frac{3-1}{3} \int_0^{\pi/4} sin^{3-2}\, x\; d\,x$

$$= -\frac{1}{3} \left[sin^2 x \cos x\right]_0^{\pi/4} + \frac{2}{3} \int_0^{\pi/4} sin^3\, x\; d\,x$$

$$= -\frac{1}{3} \left[\left(\frac{1}{\sqrt{2}}\right)^2 \frac{1}{\sqrt{2}}\right] + \frac{2}{4} \left(-\cos x\right)^{\pi/4}{}_0$$

$$= \frac{-1}{3.2\sqrt{2}} \quad \frac{-2}{4} \quad \left(\frac{1}{\sqrt{2}} - 1\right)$$

Putting this value is (1) ,we get

$$\int_0^{\pi/4} sin^5\, x\; d\,x = -\frac{1}{5} \quad \left[\left(\frac{1}{\sqrt{2}}\right)^4 \frac{1}{\sqrt{2}}\right] + \frac{4}{5} \quad \left[\frac{-1}{6\sqrt{2}} - \frac{1}{2} \left(\frac{1}{\sqrt{2}} - 1\right)\right]$$

$$= \frac{-1}{5.4\sqrt{2}} - \frac{4}{5} \quad \left[\frac{1}{6\sqrt{2}} + \frac{1}{2\sqrt{2}} - \frac{1}{2}\right]$$

Example 13 Evaluate $\int_0^1 \frac{x^5}{2\sqrt{1-x^2}}\,dx$

Solution: Put x= Sin θ $\Rightarrow$ dx = cosθ d θ

Then the given integral becomes

$$\frac{1}{2}\int_0^{\pi/2} \frac{sin^5\theta}{\sqrt{1-sin^2\theta}} \cos\theta\, d\theta = \frac{1}{2}\int_0^{\pi/2} sin^5\,\theta\, \mathrm{d}\theta$$

$$= \frac{1}{2}\,\frac{(5-1)\ (5-3)}{5\,(5-2)\,(5-4)} = \frac{4}{15}$$

Example 14 Evaluate $\int_{-\pi/2}^{\pi/2} cos^3\,\theta\,(1+sin\theta)^2 d\theta$

Solution: $\int_{-\pi/2}^{\pi/2} cos^3\,\theta\,(1-sin\theta)^2 d\theta = \int_{-\pi/2}^{\pi/2} cos^3\,\theta\,(1+sin^2\theta+2sin\theta)d\theta$

$$= \int_{-\pi/2}^{\pi/2} cos^3\,\theta\, d\theta + \int_{-\pi/2}^{\pi/2} cos^3\,\theta sin^2\theta + 2\int_{-\pi/2}^{\pi/2} cos^3\,\theta \sin\theta\, d\theta$$

$$= 2\int_0^{\pi/2} cos^3\,\theta\, d\theta + 2\int_0^{\pi/2} cos^3\, sin^2\,\theta + 0$$

$$= \frac{2(3-1)}{3\,(3-2)} + 2\,\frac{(2-1)\,(3-1)}{(3+2)\,(3+2-2)\,(3+2-4)}$$

$$= \frac{4}{3} + \frac{4}{15} = \frac{8}{5}$$

Exercise7A

1. Evaluate $\int_0^{2a} x^3\,(2ax-x^2)^{3/2}\ dx$ $\left(\text{Ans. } \frac{9\pi a^7}{16}\right)$
2. Evaluate $\int_0^{\infty} \frac{x^3}{(1+x^2)^{9/2}}\,dx$ $\left(\text{Ans } \frac{2}{35}\right)$
3. Evaluate $\int_0^{\pi/2} (cos2\theta)^{3/2} \cos\theta \quad d\theta$ $\left(\text{Ans } \frac{3\pi}{16\sqrt{2}}\right)$
4. Evaluate $\int_0^{\pi/2} sin^4x\ cos3x\ dx$ $\left(\text{Ans } \frac{-13}{35}\right)$
5. Evaluate $\int_0^{a} x^2\,\sqrt{ax-x^2}\,dx$ $\left(\text{Ans } \frac{5\pi a^4}{128}\right)$
6. Evaluate $\int_0^{\pi/2} \frac{cos^2\theta}{cos^2\theta+4\,sin^2\theta}$ dθ $\left(\text{Ans } \frac{\pi}{6}\right)$
7. Evaluate $\int_0^{\pi} \frac{sin^4\theta\sqrt{1-\cos\theta}}{(1+\cos\theta)^2}$ dθ $\left(\text{Ans } \frac{64\sqrt{2}}{15}\right)$
8. Evaluate $\int_0^1 x^{3/2}(1-x)^{3/2}$ dx $\left(\text{Ans } \frac{3\pi}{128}\right)$