COMPOSITION OF FUNCTIONS

We now consider an operation on functions, called *composition*, which has no direct analog in ordinary arithmetic. Informally stated, the operation of composition is performed by substituting some function for the independent variable of another function. For example, suppose that

$$f(x) = x^2 \quad \text{and} \quad g(x) = x + 1$$

If we substitute $g(x)$ for x in the formula for f, we obtain a new function

$$f(g(x)) = (g(x))^2 = (x + 1)^2$$

which we denote by $f \circ g$. Thus,

$$(f \circ g)(x) = f(g(x)) = (g(x))^2 = (x + 1)^2$$

In general, we make the following definition.

0.2.2 DEFINITION Given functions f and g, the ***composition*** of f with g, denoted by $f \circ g$, is the function defined by

$$(f \circ g)(x) = f(g(x))$$

The domain of $f \circ g$ is defined to consist of all x in the domain of g for which $g(x)$ is in the domain of f.

Example 3 Let $f(x) = x^2 + 3$ and $g(x) = \sqrt{x}$. Find

(a) $(f \circ g)(x)$ (b) $(g \circ f)(x)$

Solution (a). The formula for $f(g(x))$ is

$$f(g(x)) = [g(x)]^2 + 3 = (\sqrt{x}\,)^2 + 3 = x + 3$$

Since the domain of g is $[0, +\infty)$ and the domain of f is $(-\infty, +\infty)$, the domain of $f \circ g$ consists of all x in $[0, +\infty)$ such that $g(x) = \sqrt{x}$ lies in $(-\infty, +\infty)$; thus, the domain of $f \circ g$ is $[0, +\infty)$. Therefore,

$$(f \circ g)(x) = x + 3, \quad x \geq 0$$

Solution (b). The formula for $g(f(x))$ is

$$g(f(x)) = \sqrt{f(x)} = \sqrt{x^2 + 3}$$

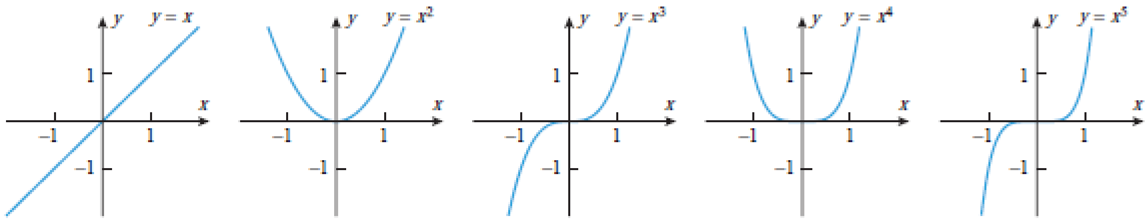
Since the domain of f is $(-\infty, +\infty)$ and the domain of g is $[0, +\infty)$, the domain of $g \circ f$ consists of all x in $(-\infty, +\infty)$ such that $f(x) = x^2 + 3$ lies in $[0, +\infty)$. Thus, the domain of $g \circ f$ is $(-\infty, +\infty)$. Therefore,

$$(g \circ f)(x) = \sqrt{x^2 + 3}$$

There is no need to indicate that the domain is $(-\infty, +\infty)$, since this is the natural domain of $\sqrt{x^2 + 3}$. ◄

POWER FUNCTIONS; THE FAMILY $y = x^n$

A function of the form $f(x) = x^p$, where p is constant, is called a ***power function***. For the moment, let us consider the case where p is a positive integer, say $p = n$. The graphs of the curves $y = x^n$ for $n = 1, 2, 3, 4,$ and 5 are shown in Figure 0.3.3. The first graph is the line with slope 1 that passes through the origin, and the second is a parabola that opens up and has its vertex at the origin (see Web Appendix H).



▲ **Figure 0.3.3**

POLYNOMIALS

A ***polynomial in x*** is a function that is expressible as a sum of finitely many terms of the form cx^n, where c is a constant and n is a nonnegative integer. Some examples of polynomials are

$$2x + 1, \quad 3x^2 + 5x - \sqrt{2}, \quad x^3, \quad 4\,(= 4x^0), \quad 5x^7 - x^4 + 3$$

The function $(x^2 - 4)^3$ is also a polynomial because it can be expanded by the binomial formula (see the inside front cover) and expressed as a sum of terms of the form cx^n:

$$(x^2 - 4)^3 = (x^2)^3 - 3(x^2)^2(4) + 3(x^2)(4^2) - (4^3) = x^6 - 12x^4 + 48x^2 - 64 \tag{3}$$

A more detailed review of polynomials appears in Appendix C.

A general polynomial can be written in either of the following forms, depending on whether one wants the powers of x in ascending or descending order:

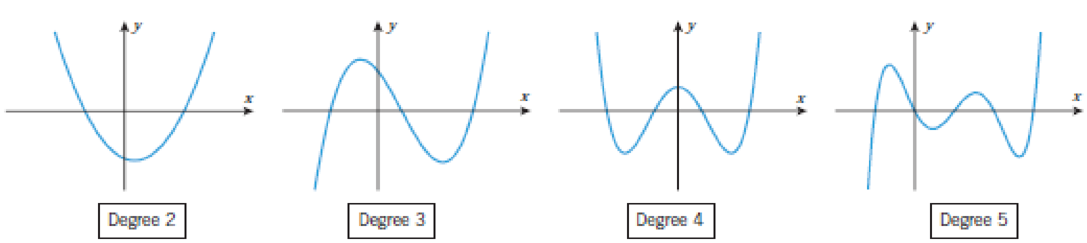
$$c_0 + c_1x + c_2x^2 + \cdots + c_nx^n$$
$$c_nx^n + c_{n-1}x^{n-1} + \cdots + c_1x + c_0$$

The constants $c_0, c_1, \ldots, c_n$ are called the ***coefficients*** of the polynomial. When a polynomial is expressed in one of these forms, the highest power of x that occurs with a nonzero coefficient is called the ***degree*** of the polynomial. Nonzero constant polynomials are considered to have degree 0, since we can write $c = cx^0$. Polynomials of degree 1, 2, 3, 4, and 5 are described as ***linear***, ***quadratic***, ***cubic***, ***quartic***, and ***quintic***, respectively. For example,

The constant 0 is a polynomial called the ***zero polynomial***. In this text we will take the degree of the zero polynomial to be undefined. Other texts may use different conventions for the degree of the zero polynomial.

$3 + 5x$	$x^2 - 3x + 1$	$2x^3 - 7$
Has degree 1 (linear)	Has degree 2 (quadratic)	Has degree 3 (cubic)
$8x^4 - 9x^3 + 5x - 3$	$\sqrt{3} + x^3 + x^5$	$(x^2 - 4)^3$
Has degree 4 (quartic)	Has degree 5 (quintic)	Has degree 6 [see (3)]

The natural domain of a polynomial in x is $(-\infty, +\infty)$, since the only operations involved are multiplication and addition; the range depends on the particular polynomial. We already know that the graphs of polynomials of degree 0 and 1 are lines and that the graphs of polynomials of degree 2 are parabolas. Figure 0.3.10 shows the graphs of some typical polynomials of higher degree. Later, we will discuss polynomial graphs in detail, but for now it suffices to observe that graphs of polynomials are very well behaved in the sense that they have no discontinuities or sharp corners. As illustrated in Figure 0.3.10, the graphs of polynomials wander up and down for awhile in a roller-coaster fashion, but eventually that behavior stops and the graphs steadily rise or fall indefinitely as one travels along the curve in either the positive or negative direction. We will see later that the number of peaks and valleys is less than the degree of the polynomial.



RATIONAL FUNCTIONS

A function that can be expressed as a ratio of two polynomials is called a ***rational function***. If $P(x)$ and $Q(x)$ are polynomials, then the domain of the rational function

$$f(x) = \frac{P(x)}{Q(x)}$$

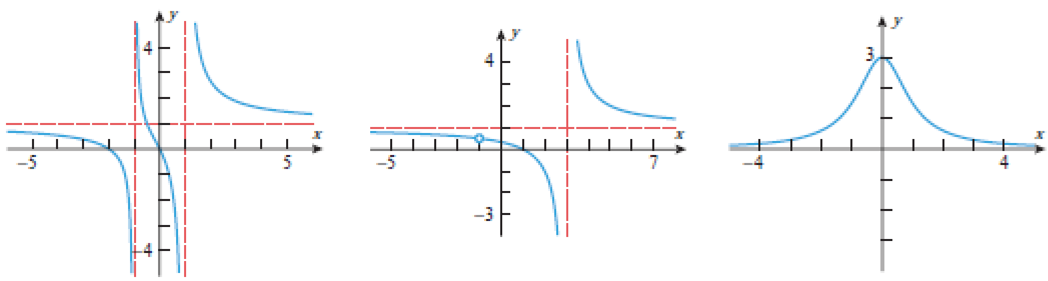
consists of all values of x such that $Q(x) \neq 0$. For example, the domain of the rational function

$$f(x) = \frac{x^2 + 2x}{x^2 - 1}$$

consists of all values of x, except $x = 1$ and $x = -1$. Its graph is shown in Figure 0.3.11 along with the graphs of two other typical rational functions.

The graphs of rational functions with nonconstant denominators differ from the graphs of polynomials in some essential ways:

- Unlike polynomials whose graphs are continuous (unbroken) curves, the graphs of rational functions have discontinuities at the points where the denominator is zero.
- Unlike polynomials, rational functions may have numbers at which they are not defined. Near such points, many rational functions have graphs that closely approximate a vertical line, called a ***vertical asymptote***. These are represented by the dashed vertical lines in Figure 0.3.11.
- Unlike the graphs of nonconstant polynomials, which eventually rise or fall indefinitely, the graphs of many rational functions eventually get closer and closer to some horizontal line, called a ***horizontal asymptote***, as one traverses the curve in either the positive or negative direction. The horizontal asymptotes are represented by the dashed horizontal lines in the first two parts of Figure 0.3.11. In the third part of the figure the x-axis is a horizontal asymptote.



$y = \dfrac{x^2 + 2x}{x^2 - 1}$ $\quad$ $y = \dfrac{x^2 - 1}{x^2 - 2x - 3}$ $\quad$ $y = \dfrac{3}{x^2 + 1}$

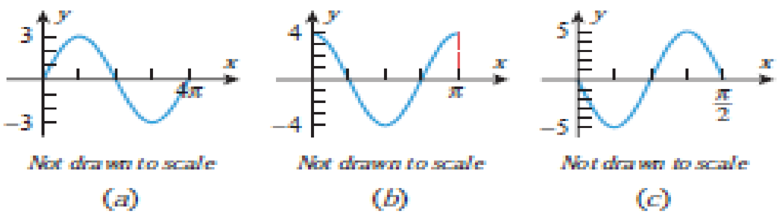
▲ Figure 0.3.11

Practice Problems

19. Sketch the graph of $y = x^2 + 2x$ by completing the square and making appropriate transformations to the graph of $y = x^2$.

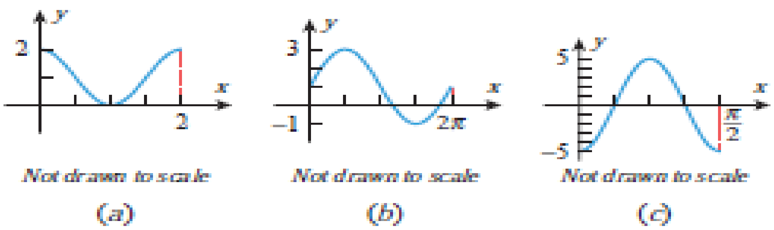
31–32 Find an equation of the form $y = D + A \sin Bx$ or $y = D + A \cos Bx$ for each graph.

31.



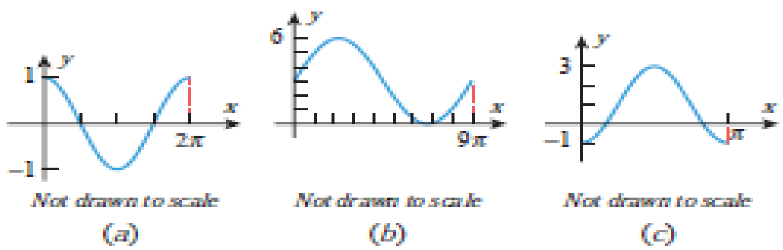
▲ **Figure Ex-31**

32.



▲ **Figure Ex-32**

33. In each part, find an equation for the graph that has the form $y = y_0 + A \sin(Bx - C)$.



▲ **Figure Ex-33**

35–36 Find the amplitude and period, and sketch at least two periods of the graph by hand. If you have a graphing utility, use it to check your work.

35. (a) $y = 3 \sin 4x$ (b) $y = -2 \cos \pi x$
(c) $y = 2 + \cos\left(\dfrac{x}{2}\right)$

36. (a) $y = -1 - 4 \sin 2x$ (b) $y = \frac{1}{2} \cos(3x - \pi)$
(c) $y = -4 \sin\left(\dfrac{x}{3} + 2\pi\right)$

LIMITS

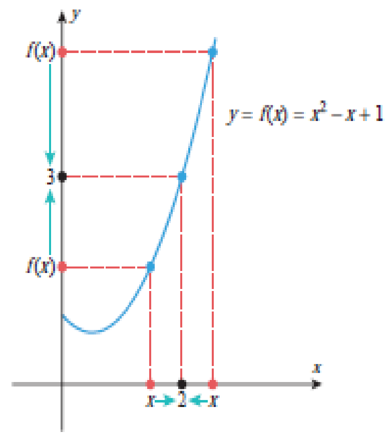
Now that we have seen how limits arise in various ways, let us focus on the limit concept itself.

The most basic use of limits is to describe how a function behaves as the independent variable approaches a given value. For example, let us examine the behavior of the function

$$f(x) = x^2 - x + 1$$

for x-values closer and closer to 2. It is evident from the graph and table in Figure 1.1.8 that the values of $f(x)$ get closer and closer to 3 as values of x are selected closer and closer to 2 on either the left or the right side of 2. We describe this by saying that the "limit of $x^2 - x + 1$ is 3 as x approaches 2 from either side," and we write

$$\lim_{x \to 2} (x^2 - x + 1) = 3 \tag{5}$$



x	1.0	1.5	1.9	1.95	1.99	1.995	1.999	2	2.001	2.005	2.01	2.05	2.1	2.5	3.0
$f(x)$	1.000000	1.750000	2.710000	2.852500	2.970100	2.985025	2.997001		3.003001	3.015025	3.030100	3.152500	3.310000	4.750000	7.000000

Left side → ← Right side

▲ **Figure 1.1.8**

This leads us to the following general idea.

1.1.1 LIMITS (AN INFORMAL VIEW) If the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but not equal to a), then we write

$$\lim_{x \to a} f(x) = L \tag{6}$$

which is read "the limit of $f(x)$ as x approaches a is L" or "$f(x)$ approaches L as x approaches a." The expression in (6) can also be written as

$$f(x) \to L \quad \text{as} \quad x \to a \tag{7}$$

Since x is required to be different from a in (6), the value of f at a, or even whether f is defined at a, has no bearing on the limit L. The limit describes the behavior of f *close to* a but not *at* a.

▶ Example 9 Find

(a) $\lim_{x \to 3} \dfrac{x^2 - 6x + 9}{x - 3}$ (b) $\lim_{x \to -4} \dfrac{2x + 8}{x^2 + x - 12}$ (c) $\lim_{x \to 5} \dfrac{x^2 - 3x - 10}{x^2 - 10x + 25}$

Solution (a). The numerator and the denominator both have a zero at $x = 3$, so there is a common factor of $x - 3$. Then

$$\lim_{x \to 3} \frac{x^2 - 6x + 9}{x - 3} = \lim_{x \to 3} \frac{(x-3)^2}{x - 3} = \lim_{x \to 3} (x - 3) = 0$$

Solution (b). The numerator and the denominator both have a zero at $x = -4$, so there is a common factor of $x - (-4) = x + 4$. Then

$$\lim_{x \to -4} \frac{2x + 8}{x^2 + x - 12} = \lim_{x \to -4} \frac{2(x + 4)}{(x + 4)(x - 3)} = \lim_{x \to -4} \frac{2}{x - 3} = -\frac{2}{7}$$

Solution (c). The numerator and the denominator both have a zero at $x = 5$, so there is a common factor of $x - 5$. Then

$$\lim_{x \to 5} \frac{x^2 - 3x - 10}{x^2 - 10x + 25} = \lim_{x \to 5} \frac{(x - 5)(x + 2)}{(x - 5)(x - 5)} = \lim_{x \to 5} \frac{x + 2}{x - 5}$$

LIMITS OF PIECEWISE-DEFINED FUNCTIONS

For functions that are defined piecewise, a two-sided limit at a point where the formula changes is best obtained by first finding the one-sided limits at that point.

▶ Example 11 Let

$$f(x) = \begin{cases} 1/(x+2), & x < -2 \\ x^2 - 5, & -2 < x \le 3 \\ \sqrt{x+13}, & x > 3 \end{cases}$$

Find

$$\text{(a)} \lim_{x \to -2} f(x) \qquad \text{(b)} \lim_{x \to 0} f(x) \qquad \text{(c)} \lim_{x \to 3} f(x)$$

Solution (a). We will determine the stated two-sided limit by first considering the corresponding one-sided limits. For each one-sided limit, we must use that part of the formula that is applicable on the interval over which x varies. For example, as x approaches -2 from the left, the applicable part of the formula is

$$f(x) = \frac{1}{x+2}$$

and as x approaches -2 from the right, the applicable part of the formula near -2 is

$$f(x) = x^2 - 5$$

Thus,

$$\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} \frac{1}{x+2} = -\infty$$

$$\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} (x^2 - 5) = (-2)^2 - 5 = -1$$

from which it follows that $\lim_{x \to -2} f(x)$ does not exist.

Solution (b). The applicable part of the formula is $f(x) = x^2 - 5$ on both sides of 0, so there is no need to consider one-sided limits here. We see directly that

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} (x^2 - 5) = 0^2 - 5 = -5$$

Solution (c). Using the applicable parts of the formula for $f(x)$, we obtain

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (x^2 - 5) = 3^2 - 5 = 4$$

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} \sqrt{x+13} = \sqrt{\lim_{x \to 3^+} (x+13)} = \sqrt{3+13} = 4$$

Since the one-sided limits are equal, we have

$$\lim_{x \to 3} f(x) = 4$$

We note that the limit calculations in parts (a), (b), and (c) are consistent with the graph of f shown in Figure 1.2.5. ◀