Gamma, Beta Functions, Differentiation Under the Integral Sign

21.1 GAMMA FUNCTION

$$\int_0^\infty e^{-x} x^{n-1}\, dx \qquad (n > 0)$$

is called gamma function of n. It is also written as $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx$.

Example 1. *Prove that* $\Gamma(1) = 1$

Solution. $$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx$$

Put $n = 1$, $$\Gamma(1) = \int_0^\infty e^{-x}\, dx = \left[\frac{e^{-x}}{-1}\right]_0^\infty = 1$$ **Proved**

Example 2. *Prove that*

(*i*) $\Gamma(n+1) = n\,\Gamma(n)$ (*ii*) $\Gamma(n+1) = \underline{|n}$ (*Reduction formula*)

Solution.

(*i*) $$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx \qquad \text{...(1)}$$

Integrating by parts, we have

$$= \left[x^{n-1}\frac{e^{-x}}{-1}\right]_0^\infty - (n-1)\int_0^\infty x^{n-2}\frac{e^{-x}}{-1}\, dx$$

$$= \left[\lim_{x\to 0}\frac{x^{n-1}}{e^x} = \lim_{x\to 0} 1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \ldots + \frac{x^n}{\lfloor n} + \ldots + x^{n-1}\right] = 0$$

$$= (n-1)\int_0^\infty x^{n-2} e^{-x}\, dx$$

$\therefore$ $$\Gamma(n) = (n-1)\,\Gamma(n-1) \qquad \text{...(2)}$$

$$\Gamma(n+1) = n\,\Gamma(n)$$ Replacing n by $(n+1)$ **Proved**

(*ii*) Replace n by $n-1$ in (2), we get

$$\Gamma n - 1 = (n-2)\,\Gamma n - 2$$

Putting the value $\Gamma n - 1$ in (2), we get

$$\Gamma n = (n-1)(n-2)\,\Gamma n - 2$$

Similarly $$\Gamma n = (n-1)(n-2)\ldots 3.2.1\,\Gamma 1 \qquad \ldots (3)$$

Putting the value of $\Gamma 1$ in (3), we have

$$\Gamma n = (n-1)(n-2)\ldots 3.2.1.1$$

$$\Gamma n = \lfloor n-1$$

Replacing n by $n + 1$, we have

$$\Gamma n+1 = \lfloor n \qquad \textbf{Proved}$$

Example 3. *Evaluate* $\int_0^\infty \sqrt[4]{x}\, e^{-\sqrt{x}}\, dx$

Solution. Let $I = \int_0^\infty x^{1/4}\, e^{-\sqrt{x}}\, dx \qquad \ldots(1)$

Putting $\sqrt{x} = t$ or $x = t^2$ or $dx = 2\,t\,dt$ in (1), we get

$$I = \int_0^\infty t^{1/2}\, e^{-t}\, 2\,t\,dt = 2\int_0^\infty t^{3/2}\, e^{-t}\, dt$$

$$= 2\,\Gamma\frac{5}{2} \qquad \text{By definition}$$

$$= 2\,.\,\frac{3}{2}\,\Gamma\frac{3}{2} = 2\,.\,\frac{3}{2}\,.\,\frac{1}{2}\,\Gamma\frac{1}{2} = \frac{3}{2}\sqrt{\pi} \qquad \textbf{Ans.}$$

Example 4. *Evaluate* $\int_0^\infty \sqrt{x}\, e^{-\sqrt[3]{x}}\, dx.$

Solution. Let $I = \int_0^\infty \sqrt{x}\, e^{-\sqrt[3]{x}}\, dx \qquad \ldots(1)$

Putting $\sqrt[3]{x} = t$ or $x = t^3$ or $dx = 3\,t^2\,dt$ in (1) we get

$$I = \int_0^\infty t^{3/2}\, e^{-t}\, 3\,t^2\,dt = 3\int_0^\infty t^{7/2}\, e^{-t}\, dt = 3\,\Gamma\frac{9}{2} = 3\,.\,\frac{7}{2}\,.\,\frac{5}{2}\,.\,\frac{3}{2}\,.\,\frac{1}{2}\,\Gamma\frac{1}{2} = \frac{315}{16}\sqrt{\pi} \qquad \textbf{Ans.}$$

Example 5. *Evaluate* $\int_0^\infty x^{n-1}\, e^{-h^2x^2}\, dx.$

Solution. Let $I = \int_0^\infty x^{n-1}\, e^{-h^2x^2}\, dx \qquad \ldots(1)$

Putting $t = h^2x^2$ or $x = \dfrac{\sqrt{t}}{h}$ or $dx = \dfrac{dt}{2\,h\sqrt{t}}$,

(1) becomes $$I = \int_0^\infty \left(\frac{\sqrt{t}}{h}\right)^{n-1} e^{-t}\,\frac{dt}{2\,h\,\sqrt{t}}$$

$$= \frac{1}{2\,h^n}\int_0^\infty t^{\frac{n-1}{2}}\, e^{-t}\,\frac{dt}{\sqrt{t}} = \frac{1}{2\,h^n}\int_0^\infty t^{\frac{n-2}{2}}\, e^{-t}\, dt$$

$$= \frac{1}{2\,h^n}\,\Gamma\frac{n}{2} \qquad \textbf{Ans.}$$

Exercise 21.1

Evaluate :

1. (*i*) $\Gamma\left(-\frac{1}{2}\right)$ (*ii*) $\Gamma\left(\frac{-3}{2}\right)$ (*iii*) $\Gamma\left(\frac{-15}{2}\right)$ (*iv*) $\Gamma\left(\frac{7}{2}\right)$ (*v*) $\Gamma 0$

   Ans. (*i*) $-2\sqrt{\pi}$ (*ii*) $\frac{4}{3}\sqrt{\pi}$ (*iii*) $\frac{2^8\sqrt{\pi}}{15\times 13\times 11\times 9\times 7\times 5\times 3}$ (*iv*) $\frac{15\sqrt{\pi}}{8}$ (*v*) ∞

2. $\int_0^\infty \sqrt{x}\, e^{-x}\, dx$ **Ans.** $\Gamma\left(\frac{3}{2}\right)$

3. $\int_0^\infty x^4 e^{-x^2}\, dx$ **Ans.** $\frac{3\sqrt{\pi}}{8}$.

4. $\int_0^\infty e^{-h^2x^2}\, dx$ **Ans.** $\frac{\sqrt{\pi}}{2h}$

21.3 BETA FUNCTION

$$\int_0^\infty x^{l-1}(1-x)^{m-1}\, dx$$

is called the Beta function of *l, m*. It is also written as

$$\beta(l, m) = \int_0^1 x^{l-1}(1-x)^{m-1}\, dx.$$

21.4 EVALUATION OF BETA FUNCTION

$$\beta(l, m) = \frac{\Gamma l\, \Gamma m}{\Gamma(l+m)}$$

21.5 A PROPERTY OF BETA FUNCTION

$$\beta(l, m) = \beta(m, l)$$

Example 8. *Evaluate* $\int_0^1 x^4 (1-\sqrt{x})^5\,dx$

Solution. Let $\sqrt{x} = t$ or $x = t^2$ or $dx = 2\,tdt$

$$\int_0^1 x^4 (1-\sqrt{x})^5\,dx = \int_0^1 (t^2)^4 (1-t)^5 (2\,tdt)$$

$$= 2\int_0^1 t^9 (1-t)^5\,dt = 2\,\beta(10, 6) = 2\frac{\Gamma 10\,\Gamma 6}{\Gamma 16} = 2\frac{\lfloor 9 \lfloor 5}{\lfloor 15}$$

$$= 2\,.\,\frac{\lfloor 5}{10\times 11\times 12\times 13\times 14\times 15} = \frac{2\times 1\times 2\times 3\times 4\times 5}{10\times 11\times 12\times 13\times 14\times 15}$$

$$= \frac{1}{11\times 13\times 7\times 15} = \frac{1}{15015}$$

Example 9. *Evaluate* $\int_0^1 (1-x^3)^{-\frac{1}{2}}\,dx$

Solution. Let $x^3 = y$ or $x = y^{1/3}$ or $dx = \frac{1}{3}y^{-\frac{2}{3}}\,dy$

$$\int_0^1 (1-x^3)^{-\frac{1}{2}}\,dx = \int_0^1 (1-y)^{-\frac{1}{2}}\left(\frac{1}{3}y^{-\frac{2}{3}}\,dy\right)$$

$$= \frac{1}{3}\int_0^1 y^{-\frac{2}{3}}(1-y)^{-\frac{1}{2}}\,dy = \frac{1}{3}\beta\left(\frac{1}{3}, \frac{1}{2}\right) = \frac{1}{3}\frac{\Gamma\frac{1}{3}\,\Gamma\frac{1}{2}}{\Gamma\frac{5}{6}}$$

The Reduction Formulas:

$$\int \sin^n x\,dx = \frac{-1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$$

$$\int \tan^n x\,dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx \ (n \neq 1)$$

$$\int \cot^n x\,dx = \frac{-\cot^{n-1} x}{n-1} - \int \cot^{n-2} x\,dx \ (n \neq 1)$$

$$\int \sec^n x\,dx = \frac{\sec^{n-1} x\tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx \quad (n \neq 1)$$

$$\int \csc^n x\,dx = \frac{-\csc^{n-1} x\cot x}{n-1} + \frac{n-2}{n-1}\int \csc^{n-2} x\,dx \quad (n \neq 1)$$