Discrete Structures Spring 2024 – Week 7

Basics Of Counting

Lecture 1

Motivation

- Suppose that a password on a computer system consists of six, seven, or eight characters. Each of these characters must be a digit or a letter of the alphabet. Each password must contain at least one digit. How many such passwords are there?

Introduction

- Combinatorics is the study of arrangements of objects – an important part of discrete mathematics
- An important part of Combinatorics is Enumeration – the counting of objects with certain properties
- Counting problems arise throughout mathematics and computer science. Counting successful outcomes of experiments and all the possible outcomes of these experiments to determine probabilities of discrete events. Or counting number of operations used by an algorithm to study its time complexity.

Basic Counting Principle – The Product Rule

• The Product Rule – One of the two basic counting principles • Used to solve many different counting problems.

• The product rule applies when a procedure is made up of separate tasks.

Basic Counting Principle – The Product Rule

- Suppose that a procedure can be broken down into a sequence of two tasks
- If there are n_1 ways to do the first task and for each of these ways of doing the first task, there are n_2 ways to do the second task, then there are $n_1 n_2$ ways to do the procedure

Basic Counting Principle – The Product Rule

• An extended version of the product rule is often useful. • Suppose that a procedure is carried out by performing the tasks T_1, T_2,…, T_m in sequence.

• If each task T_i, $i = 1, 2, \ldots, n$, can be done in n_i ways, regardless of how the previous tasks were done, then there are $n_1 \cdot n_2 \cdot \cdots \cdot n_m$ ways to carry out the procedure

Sanchez → 12 office
Patel → 12 − 1 which is Sanchez's
12 × 11

Basic Counting Principle – The Product Rule

- **Ex 1.** A new company with just two employees, Sanchez and Patel, rents a floor of a building with 12 offices. How many ways are there to assign different offices to these two employees?
- **Solution:** The procedure of assigning offices to these two employees

consists of assigning an office to Sanchez, which can be done in **12 ways**, then assigning an office to Patel different from the office assigned to Sanchez, which can be done in **11 ways**. By the product rule, there are **12 · 11 = 132** ways to assign offices to these two employees.

Basic Counting Principle – The Product Rule

- **Ex 2.** The chairs of an auditorium are to be labeled with an uppercase English letter followed by a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently? 26×100

- **Solution:** The procedure of labeling a chair consists of two tasks, namely, assigning to the seat one of the **26 uppercase English letters**, and then assigning to it one of the **100 possible integers**. The product rule shows that there are **26 · 100 = 2600** different ways that a chair can be labeled. Therefore, the largest number of chairs that can be labeled differently is 2600.

Basic Counting Principle – The Product Rule

- **Ex 3.** There are 32 computers in a data center in the cloud. Each of these computers has 24 ports. How many different computer ports are there in this data center?

- **Solution:** The procedure of choosing a port consists of two tasks, first picking a computer and then picking a port on this computer. Because there are **32 ways to choose the computer** and **24 ways to choose the port** no matter which computer has been selected, the product rule shows that there are **32 · 24 = 768 ports**

Basic Counting Principle – The Product Rule

• **Ex 4.** How many different bit strings of length seven are there? • **Solution:** Each of the seven bits can be chosen in two ways, because each bit is either 0 or 1. Therefore, the product rule shows there are a total of **2. 2. 2. 2. 2. 2. 2 = 128** different bit strings of length seven.

Basic Counting Principle –

The Product Rule

- **Ex 5.** How many different license plates can be made if each plate contains a sequence of three uppercase English letters followed by three digits?
- **Solution:** There are 26 choices for each of the three uppercase English letters and 10 choices for each of the three digits. Hence, by the product rule there are a total of $\mathbf{26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17{,}576{,}000}$ possible license plates.

Basic Counting Principle – The Product Rule

• The product rule is phrased in terms of sets in the following way: • If A_1, A_2,…, A_m are finite sets, then the number of elements in the Cartesian product of these sets is the product of the number of elements in each set

$$|A_1 \times A_2 \times \cdots \times A_m| = |A_1| \cdot |A_2| \cdot \cdots \cdot |A_m|.$$

Basic Counting Principle – The Sum Rule

- If a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the task.

Basic Counting Principle – The Sum Rule

- **Ex 6.** Suppose that either a member of the mathematics faculty or a student who is a mathematics major is chosen as a representative to a university committee. How many different choices are there for this representative if there are 37 members of the mathematics faculty and 83 mathematics majors and no one is both a faculty member and a student?

- **Solution:** There are 37 ways to choose a member of the mathematics faculty and there are 83 ways to choose a student who is a mathematics major. Choosing a member of the mathematics faculty is never the same as choosing a student who is a mathematics major because no one is both a faculty member and a student. By the sum rule it follows that there are 37 + 83 = 120 possible ways to pick this representative.

Basic Counting Principle – The Sum Rule

- **Ex 7.** A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. No project is on more than one list. How many possible projects are there to choose from?
- **Solution:** The student can choose a project by selecting a project from the first list, the second list, or the third list. Because no project is on more than one list, by the sum rule there are $23 + 15 + 19 = 57$ ways to choose a project.

Basic Counting Principle –

The Sum Rule

- The sum rule can be phrased in terms of sets as:
- If A_1, A_2,…, A_m are pairwise disjoint finite sets, then the number of elements in the union of these sets is the sum of the numbers of elements in the sets.

$$|A_1 \cup A_2 \cup \cdots \cup A_m| = |A_1| + |A_2| + \cdots + |A_m| \text{ when } A_i \cap A_j = \quad \text{for all } i, j.$$

Complex Counting

Problems Lecture 2

Complex Counting Problems

- Many counting problems cannot be solved using just the sum rule or just the product rule.
- However, many complicated counting problems can be solved using both of these rules in combination

Complex Counting Problems – Total Variables

- **Ex 1.** Suppose statement labels in a programming language can be either a single letter or a letter followed by a digit. Find the number of possible labels

- **Solution:** Combining sum and the product rule
- $26 + 26 \cdot 10 = 286$

Complex Counting Problems – Total No. of Passwords

- **Ex 2.** Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

- **Solution**: Let P be the total number of passwords, and let P_6, P_7, and P_8 be the passwords of length 6, 7, and 8. By the sum rule $P = P_6 + P_7 + P_8$. To find each of P_6, P_7, and P_8, we find the number of passwords of the specified length composed of letters and digits and subtract the number composed only of letters

Complex Counting Problems – Total No. of Passwords

- $P_6 = 36^6 - 26^6 = 2{,}176{,}782{,}336 - 308{,}915{,}776 = 1{,}867{,}866{,}560.$
- $P_7 = 36^7 - 26^7 = 78{,}364{,}164{,}096 - 8{,}031{,}810{,}176 = 70{,}332{,}353{,}920.$
- $P_8 = 36^8 - 26^8 = 2{,}821{,}109{,}907{,}456 - 208{,}827{,}064{,}576 = 2{,}612{,}282{,}842{,}880.$
- Consequently, $P = P_6 + P_7 + P_8 = 2{,}684{,}483{,}063{,}360.$ 22

Complex Counting Problems – Total Internet Addresses

- Version 4 of the Internet Protocol (IPv4) uses 32 bits.
- Class A Addresses: used for the largest networks, a 0, followed by a 7-bit netid and a 24-bit hostid
- Class B Addresses: used for the medium-sized networks, a 10, followed by a 14-bit netid and a 16-bit hostid
- Class C Addresses: used for the smallest networks, a 110, followed by a 21-bit netid and a 8-bit hostid

Complex Counting Problems – Total Internet Addresses

Bit Number	0	1	2	3	4	8	16	24	31
Class A	0	netid				hostid			
Class B	1	0	netid				hostid		
Class C	1	1	0	netid				hostid	
Class D	1	1	1	0	Multicast Address				
Class E	1	1	1	1	0	Address			

- Neither Class D nor Class E addresses are assigned as the address of a computer on the internet. Only Classes A, B, and C are available.
- 1111111 is not available as the netid of a Class A network. • Hostids consisting of all 0s and all 1s are not available in any network.

Complex Counting Problems – Total Internet Addresses

- Let x be the number of available addresses, and let x_A, x_B, and x_C denote the number of addresses for the respective classes.

- To find, x_A:
- $2^7 - 1 = 127$ netids.
- $2^{24} - 2 = 16{,}777{,}214$ hostids.
- $x_A = 127 \cdot 16{,}777{,}214 = 2{,}130{,}706{,}178$.

Complex Counting Problems – Total Internet Addresses

- To find, x_B:
- $2^{14} = 16{,}384$ netids.
- $2^{16} - 2 = 16{,}534$ hostids.
- $x_B = 16{,}384 \cdot 16{,}534 = 1{,}073{,}709{,}056$.

Complex Counting Problems –

Total Internet Addresses

- To find, x_{C}:
- $2^{21} = 2{,}097{,}152$ netids.
- $2^{8} - 2 = 254$ hostids.
- $x_{\mathrm{C}} = 2{,}097{,}152 \cdot 254 = 532{,}676{,}608$.

Complex Counting Problems – Total Internet Addresses

- Hence, the total number of available IPv4 addresses is
- $x = x_A + x_B + x_C$
- $= 2{,}130{,}706{,}178 + 1{,}073{,}709{,}056 + 532{,}676{,}608$
- $= 3{,}737{,}091{,}842.$

Basic Counting Principle – Subtraction Rule

- If a task can be done either in one of n_1 ways or in one of n_2 ways, then the total number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways. • Also known as, the *principle of inclusion-exclusion*

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Subtraction Rule – Counting Bit Strings

- **Ex 4.** How many bit strings of length eight either start with a **1** bit or end with the two bits **00**?

$$1\ \underbrace{\_\ \_\ \_\ \_\ \_\ \_\ \_}_{2^7 = 128 \text{ ways}}$$

$$\underbrace{\_\ \_\ \_\ \_\ \_\ \_}_{2^6 = 64 \text{ ways}}\ 0\ 0$$

$$1\ \underbrace{\_\ \_\ \_\ \_\ \_}_{2^5 = 32 \text{ ways}}\ 0\ 0$$

Subtraction Rule – Counting Bit Strings

- **Solution:**

- Number of bit strings of length eight that start with a 1 bit: $2^7 = 128$ • Number of bit strings of length eight that end with bits 00: $2^6 = 64$
- Number of bit strings of length eight that start with a 1 bit and end with bits 00 : $2^5 = 32$
- Therefore, the number is
- $128 + 64 – 32 = 160$

Basic Counting Principle – Division Rule

- The division rule comes in handy when it appears that a task can be done in n different ways, but it turns out that for each way of doing the task, there are d equivalent ways of doing it.
- Under these circumstances, we can conclude that there are n/d equivalent ways of doing the task.

Basic Counting Principle –

Division Rule

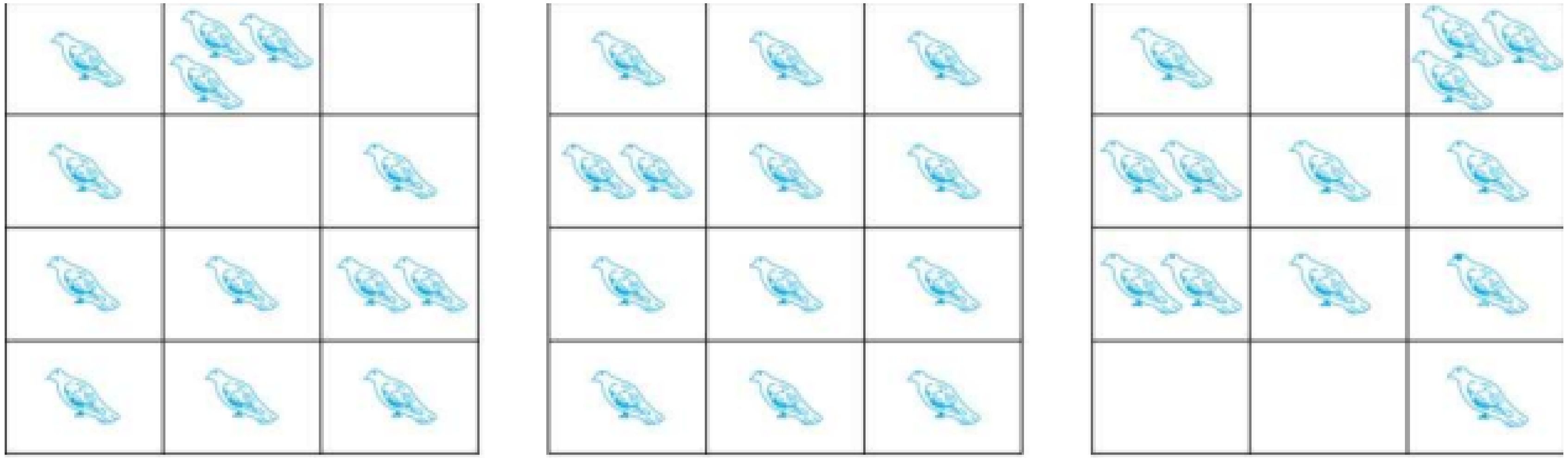
- **Example**: How many ways are there to seat four people around a circular table, where two seating are considered the same when each person has the same left and right neighbor?
- **Solution**: Number the seats around the table from 1 to 4 proceeding clockwise. There are four ways to select the person for seat 1, 3 for seat 2, 2, for seat 3, and one way for seat 4. Thus there are $4! = 24$ ways to order the four people. But since two seating are the same when each person has the same left and right neighbor, for every choice for seat 1, we get the same seating. Therefore, by the division rule, there are $24/4 = 6$ different seating arrangements.

The Pigeonhole Principle

Lecture 3

The Pigeonhole Principle

- If a flock of 20 pigeons roosts in a set of 19 pigeonholes, one of the pigeonholes must have more than 1 pigeon



(a) (b) (c)

The Pigeonhole Principle

- If k is a positive integer and $k + 1$ objects are placed into k boxes, then at least one box contains two or more objects.
- **Proof**: We use a proof by contraposition.

• Suppose none of the k boxes has more than one object. • Then the total number of objects would be at most k.

• This contradicts the statement that we have $k + 1$ objects

The Pigeonhole Principle

• **Example**: Among any group of 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays

Generalized Pigeonhole Principle

- If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

- **Example**: Among 100 people there are at least $\lceil 100/12 \rceil = 9$ who were born in the same month

Generalized Pigeonhole Principle

- **Example:** How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?

- **Solution:** We assume four boxes; one for each suit. Using the generalized pigeonhole principle, at least one box contains at least $\lceil N/4 \rceil$ cards.
- At least three cards of one suit are selected if $\lceil N/4 \rceil \geq 3$
- The smallest integer N such that $\lceil N/4 \rceil \geq 3$ is
- $N = 2 \cdot 4 + 1 = 9$.

Practice

- What is the least number of 16 pigeons occupying 5 holes?
- **Solution**
- N = 16, k = 5,

- Therefore minimum number of pigeons in a hole $\lceil 16/5 \rceil = 4$