Discrete Structures

Spring 2024 – Week 1

Course Introduction

Lecture 1

Course Details

- Instructor – Furqan Hussain Essani
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- Counseling Hours
- Monday 14:30 – 16:00
- Wednesday 14:30 – 16:00
- Or, with prior appointment via email

Class Details

• Three sessions a week – 15 week semester including the mid-term week • Course Learning Outcomes

1. Comprehend the key concepts of discrete structures **2. Apply** logical reasoning to real-world computing problems **3. Analyze** discrete structures in the context of computer science

Resources

- **Text Book**
  - Discrete Mathematics and Its Applications, Kenneth H. Rosen, 8/e, McGraw Hill Publication
- **References**
  - Discrete Mathematics 8th Edition by Richard Johnsonbaugh
  - Essential Discrete Mathematics for Computer Science by Harry Lewis and Rachel Zax
  - Discrete Mathematics for Computer Science 1st Edition by Jon Pierre Fortney
- **Google Classroom**
  - Lecture Slides, Supplementary Material

Marks Distribution

- Sessional 40%
  - Quizzes – 10%
  - Assignments – 10%
  - Mid-Term Examination – 20%
- Final Examination 60%

What is a Discrete Structure?

- A discrete structure is a set of distinct elements, where the elements can be enumerated and counted
- Discrete structures are used to model and solve problems that involve discrete objects or events

- Study of discrete structures is formally known as Discrete Mathematics, opposed is Calculus that deals with continuous objects
- Examples of discrete objects: integers, steps taken by a computer program, distinct paths to travel from point A to point B on a map along a road network, ways to pick a winning set of numbers in a lottery

Kinds of Problems being solved using Discrete Mathematics

- How many ways can a password be chosen following specific rules?
- How many valid Internet addresses are there?

• What is the probability of winning a particular lottery?
• Is there a link between two computers in a network?
• How can I identify spam email messages?
• How can I encrypt a message so that no unintended recipient can read it? • How can we build a circuit that adds two integers?

Kinds of Problems being solved using Discrete Mathematics

• What is the shortest path between two cities using a transportation system? •

Find the shortest tour that visits each of a group of cities only once and then ends in the starting city.

• How can we represent English sentences so that a computer can reason with them?

• How can we prove that there are infinitely many prime numbers? • How can a list of integers be sorted so that the integers are in increasing order?

• How many steps are required to do such a sorting?

• How can it be proved that a sorting algorithm always correctly sorts a list? Spring

Learning Outcome

• **Discrete Structures:** Abstract mathematical structures that represent objects and the relationships between them. Examples are sets, permutations, relations, graphs, trees, and finite state machines

- **Combinatorial Analysis:** Techniques for counting objects of different kinds
- **Mathematical Reasoning:** Ability to read, understand, and construct mathematical arguments and proofs

Sets And Functions

Lecture 2

Outline

- Introduction – Set Definition
- Describing Sets – Roster Notation, Set-Builder Notation, Venn Diagram • Subsets, Power Sets, Cartesian Products
- Set Operations
- Function
- Describing Function

Introduction

- Sets are one of the basic building blocks for the types of objects considered in discrete mathematics
- Set theory is an important branch of mathematics
- Many important discrete structures are built using sets
- Among them are combinations (unordered collections of objects used extensively in counting), relations (sets of ordered pairs that represent

relationships between objects) graphs (sets of vertices and edges that connect vertices), and finite state machines (used to model computing machines)

Set – Definition

- A *set* is an unordered collection of objects.
  - the students in this class
  - the chairs in this room
- The objects in a set are called the *elements*, or *members* of the set. A set is said to *contain* its elements.

- The notation $a \in A$ denotes that a is an element of the set A.
- If a is not a member of A, write $a \notin A$

Set Description – Roster Method

- S = {a,b,c,d}
- Order not important

    S = {a,b,c,d} = {b,c,a,d}

- Each distinct object is either a member or not; listing more than once does not change the set.

$$S = \{a,b,c,d\} = \{a,b,c,b,c,d\}$$

- Elipses (…) may be used to describe a set without listing all of the members when the pattern is clear.

$$S = \{a,b,c,d, \ldots\ldots,z\}$$

Set Description – Roster Method

- Set of all vowels in the English alphabet:

$$V = \{a,e,i,o,u\}$$

- Set of all odd positive integers less than 10:

$$O = \{1,3,5,7,9\}$$

- Set of all positive integers less than 100:

$S = \{1,2,3,\ldots\ldots\ldots,99\}$

- Set of all integers less than 0:

$S = \{\ldots., -3,-2,-1\}$

Some Important Sets

- N = *natural numbers* = {0,1,2,3….}
- Z = *integers* = {…,-3,-2,-1,0,1,2,3,…}
- Z^+ = *positive integers* = {1,2,3,…..}
- R = set of *real numbers*

- R^+ = set of *positive real numbers*
- C = set of *complex numbers.*
- **Q** = set of rational numbers

Set Description – Set Builder Notation

- Specify the property or properties that all members must satisfy: $S = \{x \mid x \text{ is a positive integer less than } 100\}$

  $O = \{x \mid x \text{ is an odd positive integer less than } 10\}$

  $O = \{x \in Z^+ \mid x \text{ is odd and } x < 10\}$

- A predicate may be used:

$$S = \{x \mid P(x)\}$$

- Example: $S = \{x \mid \text{Prime}(x)\}$

Set Description – Interval Notation

- $[a,b] = \{x \mid a \leq x \leq b\}$
- $[a,b) = \{x \mid a \leq x < b\}$
- $(a,b] = \{x \mid a < x \leq b\}$
- $(a,b) = \{x \mid a < x < b\}$

- *closed interval* [a,b]
- *open interval* (a,b)

Universal Set and Empty Set

- The *universal set* U is the set containing everything currently under consideration.
  - Sometimes implicit
- Sometimes explicitly stated.
- Contents depend on the context.

- The empty set is the set with no

elements.

Venn Diagram *U*

V

a e i

o u

- Symbolized ∅, but { } also used.

Set Equality

- **Definition**: Two sets are *equal* if and only if they have the same elements.

- Therefore if A and B are sets, then A and B are equal if and only if

$$\forall x(x \in A \leftrightarrow x \in B)$$

- We write $A = B$ if A and B are equal sets.

$$\{1,3,5\} = \{3, 5, 1\}$$

$$\{1,5,5,5,3,3,1\} = \{1,3,5\}$$

Subsets

- **Definition**: The set A is a *subset* of B, if and only if every element of A is also an element of B.

- The notation $A \subseteq B$ is used to indicate that A is a subset of the set B.
- $A \subseteq B$ holds if and only if $\forall x(x \in A \rightarrow x \in B)$ is true. • Every non-empty set S is guaranteed to have at least two subsets, the empty set and the set itself
- $\emptyset \subseteq S$,for every set S.
- $S \subseteq S$, for every set S.

Showing a Set is or is not a Subset of Another Set

- **Showing that A is a Subset of B**: To show that $A \subseteq B$, show that if x belongs to A, then x also belongs to B.
- **Showing that A is not a Subset of B**: To show that A is not a subset of B, $A \nsubseteq B$, find an element $x \in A$ with $x \notin B$. (Such an x is a counterexample to the claim that $x \in A$ implies $x \in B$.) **Examples**:

1. The set of all computer science majors at your school is a subset of all students at your school.
2. The set of integers with squares less than 100 is not a subset of the set of nonnegative integers.

Proper Subsets

- **Definition**: If $A \subseteq B$, but $A \neq B$, then we say A is a *proper subset* of B, denoted by $A \subset B$. If $A \subset B$, then

$$\forall x(x \in A \rightarrow x \in B) \wedge \exists x(x \in B \wedge x \notin A)$$

is true.

Venn Diagram

U

B

A

Set Cardinality

- **Definition**: If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is *finite*. Otherwise it is *infinite*. • **Definition**: The *cardinality* of a finite set A, denoted by $|A|$, is the number of (distinct) elements of A.

Examples:

1. $|\varnothing| = 0$
2. Let S be the letters of the English alphabet. Then $|S| = 26$
3. $|\{1,2,3\}| = 3$
4. $|\{\varnothing\}| = 1$
5. The set of integers is infinite.

Power Sets

- **Definition**: The set of all subsets of a set A, denoted $\mathcal{P}(A)$, is called the *power set* of A.
- **Example**: If $A = \{a,b\}$ then

$\mathcal{P}(A) = \{\emptyset, \{a\},\{b\},\{a,b\}\}$

- If a set has n elements, then the cardinality of the power set is 2^n.

Tuples

- The *ordered n-tuple* $(a_1,a_2,.....,a_n)$ is the ordered collection that has a_1 as its first element and a_2 as its second element and so on until a_n as its last element.
- Two n-tuples are equal if and only if their corresponding elements are equal.
- 2-tuples are called *ordered pairs*.
- The ordered pairs (a,b) and (c,d) are equal if and only if $a = c$ and $b = d$.

Cartesian Product

- **Definition**: The *Cartesian Product* of two sets A and B, denoted by $A \times B$ is the set of ordered pairs (a,b) where $a \in A$ and $b \in B$.

Example: $A \times B = \{(a, b) | a \in A \wedge b \in B\}$

$A = \{a,b\}$ $B = \{1,2,3\}$

$A \times B = \{(a,1),(a,2),(a,3), (b,1),(b,2),(b,3)\}$

Cartesian Product

Definition: The cartesian products of the sets $A_1, A_2, \ldots\ldots, A_n$, denoted by $A_1 \times A_2 \times \ldots\ldots \times A_n$, is the set of ordered *n*-tuples $(a_1, a_2, \ldots\ldots, a_n)$ where a_i belongs to A_i for $i = 1, \ldots n$.

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) | a_i \in A_i \text{ for } i = 1, 2, \ldots n\}$$

Example: What is $A \times B \times C$ where $A = \{0,1\}$, $B = \{1,2\}$ and $C = \{0,1,2\}$

Solution: $A \times B \times C = \{(0,1,0), (0,1,1), (0,1,2),(0,2,0), (0,2,1), (0,2,2),(1,1,0), (1,1,1), (1,1,2), (1,2,0), (1,2,1), (1,2,2)\}$

Set Operation – Union

- **Definition:** Let A and B be sets. The union of the sets A and B, denoted by A ∪ B, is the set:

$$\{x|x \in A \vee x \in B\}$$

- Example: What is

{1,2,3} ∪ {3, 4, 5}?

Solution: {1,2,3,4,5}

U

Venn Diagram for A ∪ B A B

Set Operation – Intersection

- **Definition**: The *intersection* of sets A and B, denoted by $A \cap B$, is

$$\{x | x \in A \land x \in B\}$$

- Note if the intersection is empty, then A and B are said to be *disjoint*.
- **Example**: What is? $\{1,2,3\} \cap \{3,4,5\}$?

Solution: $\{3\}$

- **Example:** What is? $\{1,2,3\} \cap \{4,5,6\}$? Solution: $\emptyset$

U

A B

Venn Diagram for $A \cap B$

Set Operation – Complement

Definition: If A is a set, then the complement of the A (with respect to U), denoted by $\bar{A}$ is the set U - A

$$\bar{A} = \{x \in U \mid x \notin A\}$$

(The complement of A is sometimes denoted by A^c.)

Example: If U is the positive integers less than 100, what is the complement of $\{x \mid x > 70\}$

Solution: $\{x \mid x \leq 70\}$

U

A

Venn Diagram for Complement $\bar{A}$

Set Operation – Difference

- **Definition**: Let A and B be sets. The *difference* of A and B, denoted by $A - B$, is the set containing the elements of A that are not in B. The difference of A and B is also called the complement of B with respect to A.

$$A - B = \{x \mid x \in A \land x \notin B\} = A \cap \square B$$

Venn Diagram for A – B

U

A

B

Set Operation – Symmetric Difference **Definition:**

The *symmetric difference* of **A** and **B**, denoted by $A \oplus B$ is the set

$$(A - B) \cup (B - A)$$

Example:

$U = \{0,1,2,3,4,5,6,7,8,9,10\}$

$A = \{1,2,3,4,5\}$ $B = \{4,5,6,7,8\}$

What is $A \oplus B$:

- **Solution:** {1,2,3,6,7,8}

U

A B

Venn Diagram

$A \oplus B$

Review Questions

Example: $U = \{0,1,2,3,4,5,6,7,8,9,10\}$ $A = \{1,2,3,4,5\}$, $B = \{4,5,6,7,8\}$

1. $A \cup B$

   Solution: {1,2,3,4,5,6,7,8}

2. $A \cap B$

Solution: $\{4,5\}$

3. $\bar{A}$

Solution: $\{0,6,7,8,9,10\}$

Review Questions

Example: $U = \{0,1,2,3,4,5,6,7,8,9,10\}$ $A = \{1,2,3,4,5\}$, $B = \{4,5,6,7,8\}$

4. $\bar{B}$

Solution: $\{0,1,2,3,9,10\}$

5. $A - B$

Solution: $\{1,2,3\}$

6. $B - A$

Solution: $\{6,7,8\}$

Review Questions

- Determine whether each of these statements is true or false.

- **a)** $x \in \{x\}$
- **b)** $\{x\} \subseteq \{x\}$
- **c)** $\{x\} \in \{x\}$
- **d)** $\{x\} \in \{\{x\}\}$
- **e)** $\emptyset \subseteq \{x\}$
- **f)** $\emptyset \in \{x\}$

False • **d)** True

• **e)** True • **f)** False

• **a)** True • **b)** True • **c)**

Review Questions

- Let A be the set of students who live within one mile of school and let B be the set of students who walk to classes. Describe the students in each of these sets.

- **a)** $A \cap B$
- **Solution:** Set of students who live within one mile of school AND who walk to classes
- **b)** $A - B$
- **Solution:** Set of students who live within one mile of school AND DO NOT walk to classes

Review Questions

- Suppose that A is the set of sophomores at your school and B is the set of students in discrete mathematics at your school. Express each of these sets in terms of A and B.

- **a)** the set of students at your school who either are sophomores or are taking discrete mathematics
- **b)** the set of students at your school who either are not sophomores or are not taking discrete mathematics

Functions

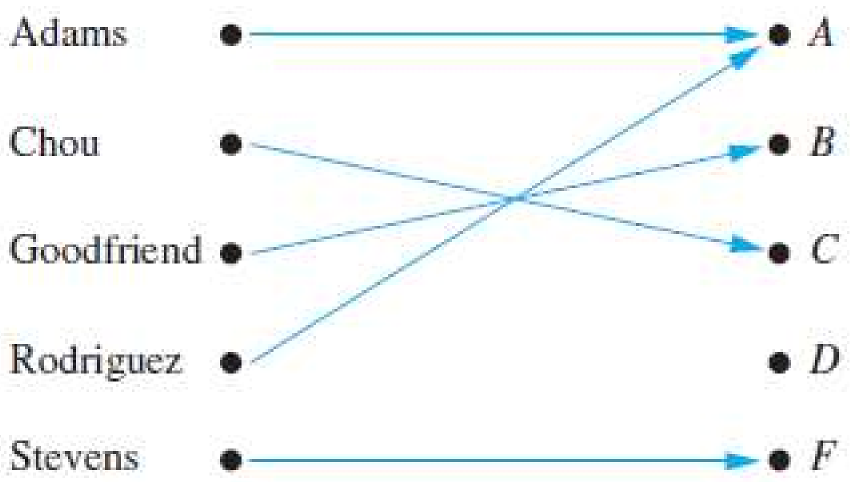
Lecture 3

Functions

- **Definition:** Let A and B be nonempty sets. A *function* f from A to B, denoted f: $A \rightarrow B$ is an assignment of each element of A to exactly one element of B.
- We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A.

Functions

- Functions are sometimes called *mappings* or *transformations*. • Assignment of grades in a discrete mathematics class

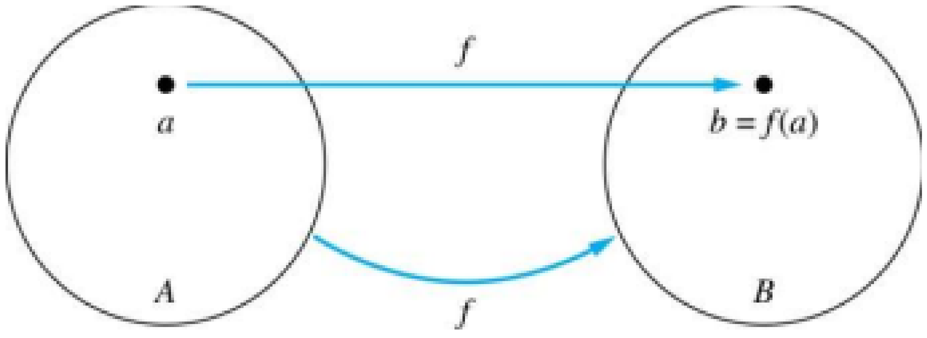


Functions

Given a function f: $A \to B$:

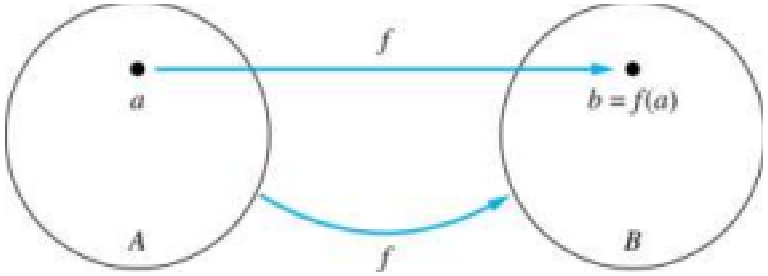
- We say f *maps* A to B *or* f is a *mapping* from A to B.
- A is called the *domain* of f and B is called the *codomain* of f.

- If $f(a) = b$,
  - then b is called the *image* of a under f.
  - a is called the *preimage* of **b.**



Functions

- The range of f is the set of all images of points in **A** under f
- We denote it by f(**A**)

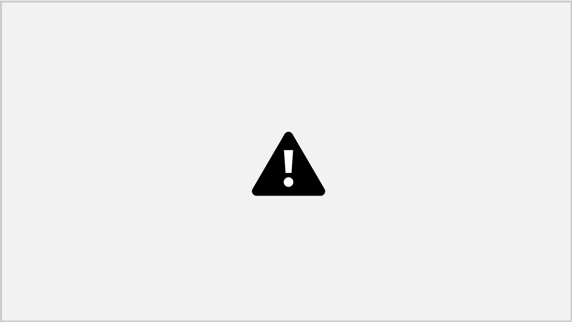


- Two functions are *equal* when they have the same domain, the same codomain and map each element of the domain to the same element of the codomain.

Review Question

- What are the domain, codomain, and range of the function that assigns grades to students?

Answer!



- Let F be the function mapping students to grades
- The domain of F is the set {Adams, Chou, Goodfriend, Rodriguez, Stevens}
- Codomain is the set $\{A, B, C, D, F\}$
- Range of F is the set $\{A, B, C, F\}$

Review Questions

z

f(a) = ? A B

The image of d is ? z

a

x

The domain of f is ? A

b

y

The codomain of f is ? B

c

The preimage of y is ? b

z

d

{y,z}

$f(A) = ?$

The preimage(s) of z is (are) ? {a,c,d}

Representing Functions

- Functions may be specified in different ways:

1. An explicit statement of the assignment: Students and grades example
2. A formula: $f(x) = x + 1$
3. A computer program: A C++/Java program that when given an integer n, produces the nth Fibonacci Number

One-To-One Function – Injections

- **Definition**: A function f is said to be *one-to-one* , or *injective*, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f.
- A function is said to be an *injection* if it is one-to-one.

Onto Function – Surjections

- **Definition:** A function *f* from *A* to *B* is called *onto* or *surjective*, if and only if for every element there is an element with . • A function *f* is called a *surjection* if it is *onto*.

Bijections

- **Definition:** A function f is a *one-to-one correspondence*, or a *bijection*, if it is both one-to-one and onto (surjective and injective).

A B

Illustration I

a

w

b

c

d
x

y

z

Illustration II

Showing that f is one-to-one or onto

Showing that *f* is one-to-one or onto

- **Example 1**: Let f be the function from $\{a,b,c,d\}$ to $\{1,2,3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f an onto function?

- **Solution**: Yes, f is onto since all three elements of the codomain are images of elements in the domain. If the codomain were changed to $\{1,2,3,4\}$, f would not be onto.

Showing that f is one-to-one or onto

- **Example 2**: Is the function $f(x) = x^2$ from the set of integers to the set of integers onto?

- **Solution**: No, f is not onto because there is no integer x with $x^2 = -1$, for example.

Practice Questions

- Section 2.1 Exercises – Page 131

- Questions 5, 11, 34

- Section 2.2 Exercises – Page 144
- Questions 1, 2, 3

- Section 2.3 Exercises – Page 161
- Questions 5, 10, 11, 17