Discrete Structures Spring 2024 – Week 3

Predicates

Lecture 1

Introduction

- Propositional logic cannot adequately express the meaning of all statements in mathematics and in natural language
- Suppose if we have:"All men are mortal.", there is no rule of propositional logic allow to conclude the truth of the statement, "Socrates is a man."
- Need a language that talks about objects, their properties, and their relations

Predicate Logic

- Predicate logic (First Order Logic) – more powerful type of logic
- Used to express the meaning of a wide range of statements in mathematics and computer science in ways that permit us to **reason** and **explore relationships** between objects
- Predicate logic uses the following new features:
- Variables: x, y, z

- Predicates: P(x), M(x)
- Quantifiers (to be covered later)

Predicates

- Statements involving variables, such as, **x > 3** are neither true nor false when the values of the variables are not specified
- The statement "x is greater than 3" has two parts

1. Variable x, the subject of the statement
2. "is greater than 3" – property that the subject of the statement can have
   – a **predicate**

Predicates

- The statement “x is greater than 3” can be denoted by $P(x)$, where P denotes the predicate “is greater than 3” and x is the variable
- The statement $P(x)$ is also said to be the value of the **propositional function** P at x

Predicate / Propositional Functions • Predicate /

Propositional Function are a generalization of propositions

- Propositional functions become propositions (and have truth values) when their variables are each replaced by a value from the domain (or bound by a quantifier)

Predicate / Propositional Functions

- Let P(x) denote "$x > 0$" and the domain be the integers. Then find:
- P(-3)
- P(0)

- P(3)

Predicate / Propositional Functions

- Let P(x) denote “x > 0” and the domain be the integers. Then:
- P(-3) false
- P(0) false

- P(3) true

- Often the domain is denoted by **U**.
- In this example **U** is the integers

Example I

- Let P(x) denote the statement “x > 3.” What are the truth values of P(4) and P(2)?
- Solution:

- Obtain the statement P(4) by setting x = 4 in the statement “x > 3.” Hence, P(4), which is the statement “4 > 3,” is true.
- However, P(2), which is the statement “2 > 3,” is false

Example II

- Let “x + y = z” be denoted by R(x, y, z) and U (for all three variables) be the integers. Find the following truth values:

1. R(2,-1,5)

2. R(3,4,7)

3. R(x, 3, z)

Example II

- Let "x + y = z" be denoted by R(x, y, z) and U (for all three variables) be the integers. Find the following truth values:

1. R(2,-1,5) Solution: F

2. R(3,4,7) Solution: T

3. R(x, 3, z) Solution: Not a Proposition

Example III

- Let "x - y = z" be denoted by Q(x, y, z), with U as the integers. Find these truth values:

1. Q(2,-1,3)

2. Q(3,4,7)

3. Q(x, 3, z)

Compound Expressions

- Connectives from propositional logic carry over to predicate logic.
- Disjunction $\vee$
- Conjunction $\wedge$

- Implication →
- Bi-Conditioal ↔

Compound Expressions

- Connectives from propositional logic carry over to predicate logic.
- If P(x) denotes "x > 0," find these truth values:
- $P(3) \lor P(-1)$

- $P(3) \wedge P(-1)$
- $P(3) \rightarrow P(-1)$
- $P(3) \rightarrow \neg P(-1)$

Compound Expressions

- If $P(x)$ denotes "$x > 0$," find these truth values:

- $P(3) \vee P(-1)$ Solution: T

- $P(3) \land P(-1)$ Solution: F
- $P(3) \rightarrow P(-1)$ Solution: F
- $P(3) \rightarrow \neg P(-1)$ Solution: T

Quantifiers

Lecture 2

Quantifiers

- We need quantifiers to express the meaning of English words including *all* and *some*
  - "All men are Mortal."
  - "Some cats do not have fur."

- The two most important quantifiers are:
    - *Universal Quantifier*, “For all,” symbol: $\forall$
    - *Existential Quantifier*, “There exists,” symbol: $\exists$

Quantifiers

- Quantifiers are used to write as in $\forall x\ P(x)$ and $\exists x\ P(x)$

- $\forall x\ P(x)$ asserts $P(x)$ is true for <u>every</u> x in the *domain*.

- $\exists x\, P(x)$ asserts $P(x)$ is true for <u>some</u> x in the *domain*.

- The quantifiers are said to bind the variable x in these expressions.

Universal Quantifier

- $\forall x\, P(x)$ is read as "For all x, P(x)" or "For every x, P(x)"

- *Examples:*
- If $P(x)$ denotes "$x > 0$" and U is integers, then $\forall x\, P(x)$ is false • If $P(x)$ denotes "$x > 0$" and U is positive integers, then $\forall x\, P(x)$ is true. • If $P(x)$ denotes "x is even" and U is integers, then $\forall x\, P(x)$ is false

Existential Quantifier

- $\exists x\, P(x)$ is read as "For some x, $P(x)$", or as "There is an x such that

P(x)," or "For at least one x, P(x)."

- *Examples*
- If $P(x)$ denotes "$x > 0$" and U is the integers, then $\exists x\, P(x)$ is true. It is also true if U is the positive integers
- If $P(x)$ denotes "$x < 0$" and U is the positive integers, then $\exists x\, P(x)$ is false
- If $P(x)$ denotes "x is even" and U is the integers, then $\exists x\, P(x)$ is true

Thinking About Quantifiers

- When the domain of discourse is finite, we can think of quantification as looping through the elements of the domain
- To evaluate $\forall x\, P(x)$ loop through all x in the domain
  - If at every step P(x) is true, then $\forall x\, P(x)$ is true.
  - If at a step P(x) is false, then $\forall x\, P(x)$ is false and the loop terminates
- To evaluate $\exists x\, P(x)$ loop through all x in the domain
  - If at some step, P(x) is true, then $\exists x\, P(x)$ is true and the loop terminates. • If the loop ends without finding an x for which P(x) is true, then $\exists x\, P(x)$ is false

Properties of Quantifiers

- The truth value of $\exists x\, P(x)$ and $\forall x\, P(x)$ depend on both the propositional function $P(x)$ and on the domain U

1. If U is the positive integers and $P(x)$ is the statement "$x < 2$", then $\exists x\, P(x)$ is true, but $\forall x\, P(x)$ is false
2. If U is the negative integers and $P(x)$ is the statement "$x < 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are true
3. If U consists of 3, 4, and 5, and $P(x)$ is the statement "x > 2", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are true. But if $P(x)$ is the statement "x < 2", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are false

Precedence of Quantifiers

- The quantifiers $\forall$ and $\exists$ have higher precedence than all the logical operators
- For example, $\forall x\, P(x) \lor Q(x)$ means $(\forall x\, P(x)) \lor Q(x)$
- $\forall x\, (P(x) \lor Q(x))$ means something different
- Unfortunately, often people write $\forall x\, P(x) \lor Q(x)$ when they mean $\forall x\, (P(x) \lor Q(x))$.

Equivalences in Predicate Logic

- Statements involving predicates and quantifiers are *logically equivalent* if and only if they have the same truth value
- For every predicate substituted into these statements, and • For every domain of discourse used for the variables in the expressions. • The notation $S \equiv T$ indicates that S and T are logically equivalent

Quantifiers as Conjunctions and Disjunctions

- If the domain is finite, *U* consists of 1, 2, and 3
- Universally quantified proposition is equivalent to a conjunction of propositions without quantifiers

$$\forall x P(x) \equiv P(1) \wedge P(2) \wedge P(3)$$

- Existentially quantified proposition is equivalent to a disjunction of propositions without quantifiers

$$\exists x P(x) \equiv P(1) \vee P(2) \vee P(3)$$

Negating Quantified Expressions – Universal Quantifier

- Consider $\forall x\, J(x)$ "Every student in your class has taken a course in Java." Here $J(x)$ is "x has taken a course in Java"
- Negating the original statement gives
- "It is not the case that every student in your class has taken Java."
- This implies that "There is a student in your class who has not taken Java."

- Symbolically $\neg\forall x\, J(x)$ and $\exists x\, \neg J(x)$ are equivalent

Negating Quantified Expressions – Existential Quantifier

- Consider $\exists x\, J(x)$ "There is a student in this class who has taken a course in Java." Where $J(x)$ is "x has taken a course in Java."
- Negating the original statement gives
- "It is not the case that there is a student in this class who has taken Java."
- This implies that "Every student in this class has not taken Java" •

Symbolically $\neg\exists\, x\, J(x)$ and $\forall\, x\, \neg J(x)$ are equivalent

De Morgan's Law for Quantifiers

- The rules for negating quantifiers are:

TABLE 2 De Morgan's Laws for Quantifiers.

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

- The reasoning in the table shows that:

$$\neg \exists x P(x) \equiv \forall x \neg P(x)$$

$$\neg \forall x P(x) \equiv \exists x \neg P(x)$$

Using Predicate Logic

Lecture 3

Review Question I

• Let $P(x)$ denote the statement "$x \leq 4$." What are these truth values? • **a)** $P(0)$

• **b)** $P(4)$

• **c)** $P(6)$

Review Question II

- Let $P(x)$ be the statement "x spends more than five hours every weekday in class," where the domain for x consists of all students. Express each of these quantifications in English.
- **a)** $\exists x\, P(x)$
- **b)** $\forall x\, P(x)$

Review Question III

- Determine the truth value of each of these statements if the domain for all variables consists of all integers.
- **a)** $\forall n\,(n^2 \geq 0)$
- **b)** $\exists n\,(n^2 == 2)$
- **c)** $\forall n\,(n^2 \geq n)$
- **d)** $\exists n\,(n^2 < 0)$

Translating from English to Predicate Logic

1. Decide on U (Domain)
2. Define Propositional Function $P(x)$
3. Use Appropriate Quantifiers – Universal / Existential

Translating from English to Predicate Logic – I

- “Every student in this class has taken a course in Java.”

1. Decide on U (domain): **U** all students in this class
2. Define Propositional Function: **J(x)** denoting, “x has taken a course in Java”
3. Use Appropriate Quantifiers

- Answer
- $\forall \mathbf{x}\ \mathbf{J(x)}$

Translating from English to Predicate Logic – II

- "Every student in this class has taken a course in Java."

1. Decide on U (domain): **U** all people
2. Define Propositional Function: **J(x)** denoting, "x has taken a course in Java", **S(x)** denoting, "x is a student in this class"
3. Use Appropriate Quantifiers

- Answer
- $\boldsymbol{\forall x\ (S(x) \rightarrow J(x))}$

Translating from English to Predicate Logic – III

- “Some student in this class has taken a course in Java.”
- If **U** is all students in this class
- *J(x)* denoting, “*x* has taken a course in Java”
- Answer: **$\exists$ x J(x)**

- If **U** is all people
- *S(x)* denoting, “x is a student in this class”,

- Answer: $\exists x\ (S(x) \land J(x))$

Translating from English to Predicate Logic – IV

1. "Some student in this class has visited Mexico."

- Let **U** be all people, **M(x)** denote "*x* has visited Mexico" and **S(x)** denote "*x* is a student in this class"

$$\exists x\ (S(x) \land M(x))$$

2. "Every student in this class has visited Canada or Mexico."

- Add **C(x)** denoting “x has visited Canada.”

$$\forall \mathbf{x}\ (\mathbf{S(x)} \rightarrow (\mathbf{M(x)} \vee \mathbf{C(x)}))$$

Review Question IV

- Let $P(x)$ be the statement “x can speak Russian” and let $Q(x)$ be the statement “x knows the computer language C++.” Express each of these sentences in terms of $P(x)$, $Q(x)$, quantifiers, and logical connectives. The domain for quantifiers consists of all students at your school.
- **a)** There is a student at your school who can speak Russian and who knows C++.

- **b)** There is a student at your school who can speak Russian but who doesn't know C++.

Review Question IV (contd.)

- **c)** Every student at your school either can speak Russian or knows C++.
- **d)** No student at your school can speak Russian or knows C++.

Review Question V

- Translate each of these statements into logical expressions using predicates, quantifiers, and logical connectives.

- **a)** No one is perfect.
- **b)** Not everyone is perfect.
- **c)** All your friends are perfect.
- **d)** At least one of your friends is perfect

Review Question V (contd.)

- **e)** Everyone is your friend and is perfect.

- **f)** Not everybody is your friend or someone is not perfect.

Practice

- $U = \{$fleegles, snurds, thingamabobs$\}$
- $F(x)$: x is a fleegle
- $S(x)$: x is a snurd
- $T(x)$: x is a thingamabob

1. “Everything is a fleegle”

- **Solution**: $\forall x\ F(x)$

Practice

- U = {fleegles, snurds, thingamabobs}
- *F(x)*: *x* is a fleegle
- *S(x)*: *x* is a snurd
- *T(x)*: x is a thingamabob

2. “Nothing is a snurd.”

- **Solution: $\neg\exists x\ S(x)$ or $\forall x\ \neg\ S(x)$**

Practice

- $U = \{$fleegles, snurds, thingamabobs$\}$
- *F(x)*: *x* is a fleegle
- *S(x)*: *x* is a snurd
- *T(x)*: x is a thingamabob

3. "All fleegles are snurds."

- **Solution**: $\forall x\, (F(x) \rightarrow S(x))$

Practice

- U = {fleegles, snurds, thingamabobs}
- $F(x)$: x is a fleegle
- $S(x)$: x is a snurd
- $T(x)$: x is a thingamabob

4. "Some fleegles are thingamabobs."

- **Solution**: $\exists x\ (F(x) \land T(x))$

Practice

- U = {fleegles, snurds, thingamabobs}
- $F(x)$: x is a fleegle
- $S(x)$: x is a snurd
- $T(x)$: x is a thingamabob

5. "No snurd is a thingamabob."

- **Solution**: $\neg\exists x\ (S(x) \land T(x))$ *or* $\forall x\ (\neg S(x) \lor \neg T(x))$

Practice

- U = {fleegles, snurds, thingamabobs}
- *F(x)*: *x* is a fleegle
- *S(x)*: *x* is a snurd
- *T(x)*: x is a thingamabob

6. “If any fleegle is a snurd then it is also a thingamabob.” •

Solution: $\forall x\ ((F(x) \wedge S(x)) \rightarrow T(x))$