Discrete Structures Spring 2024 – Week 9

Permutations

Lecture 1

Introduction

- Many counting problems can be solved by finding the number of ways to arrange a specified number of distinct elements of a set of a particular size, where **the order of these elements matters**

Example

- **In how many ways can we select three students from a group of five students to stand in line for a picture?**
- By the product rule, there are $5 \cdot 4 \cdot 3 = 60$ ways to select three students from a group of five students to stand in line for a picture.
- **In how many ways can we arrange all five of these students in a line for a picture?**
- Consequently, there are $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ ways to arrange all five students in a line for a picture.

Permutations

- **Definition**: A *permutation* of a set of distinct objects is an ordered arrangement of these objects.
- An ordered arrangement of r elements of a set is called an *r-permuation*.
- **Example**: Let $S = \{1,2,3\}$
- The ordered arrangement 3,1,2 is a permutation of S.
- The ordered arrangement 3,2 is a 2-permutation of S.

Permutations

- The number of r-permuatations of a set with n elements is denoted by $P(n,r)$.

- The 2-permutations of $S = \{1,2,3\}$ are
- 1,2; 1,3; 2,1; 2,3; 3,1; and 3,2.

1 2 3 4 5 6

- Hence, $P(3,2) = 6$.

Formula for Number of Permutations

- **Theorem 1**: If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are
- $P(n, r) = n(n-1)(n-2) \cdots (n-(r-1))$
- r-permutations of a set with n distinct elements
- **Proof**: Use the product rule. The first element can be chosen in n ways.

The second in $n-1$ ways, and so on until there are $(n-(r-1))$ ways to choose the last element.

Formula for Number of Permutations

- Note that

$P(n,0) = 1$, since there is only one way to order zero elements

- **Corollary 1**: If n and r are integers with $1 \leq r \leq n$, then

$$P(n,r) = \frac{n!}{(n-r)!}$$

Solving Counting Problems by Counting Permutations

- **Example 1**: How many ways are there to select a first-prize winner, a second prize winner, and a third-prize winner from 100 different people who have entered a contest?

- **Solution**:
- $P(100,3) = 100 \cdot 99 \cdot 98 = 970{,}200$

Solving Counting Problems by Counting Permutations

- **Example 2**: Suppose that there are eight runners in a race. The winner receives a gold medal, the second place finisher receives a silver medal, and the third-place finisher receives a bronze medal. How many different ways are there to award these medals, if all possible outcomes of the race can occur and there are no ties?

- **Solution**:
- The number of different ways to award the medals is the number of 3-permutations of a set with eight elements. Hence, there are $P(8, 3) = 8 \cdot 7 \cdot 6 = 336$ possible ways to award the medals.

Solving Counting Problems by Counting Permutations

- **Example 3**: Suppose that a saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities?
- **Solution**: The first city is chosen, and the rest are ordered arbitrarily. Hence the orders are:
- $7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$
- If she wants to find the tour with the shortest path that visits all the cities, she must consider 5040 paths!

Solving Counting Problems by Counting

Permutations

- **Example 4**: How many permutations of the letters *ABCDEFGH* contain the string *ABC* ?
- **Solution**: We solve this problem by counting the permutations of six objects, *ABC*, *D*, *E*, *F*, *G*, and *H*.
- $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$

Combinations

Lecture 2

Introduction

• Many other counting problems can be solved by finding the number of ways to select a particular number of elements from a set of a particular size, where **the order of the elements selected does not matter**.

Example

- How many different committees of three students can be formed from a group of four students?

- To answer this question, we need only find the number of subsets with three elements from the set containing the four students. We see that there are four such subsets, one for each of the four students, because choosing three students is the same as choosing one of the four students to leave out of the group.

Combinations

- **Definition**: An *r-combination* of elements of a set is an unordered selection of *r* elements from the set.
- Thus, an *r*-combination is simply a subset of the set with *r* elements.
- The number of *r*-combinations of a set with n distinct elements is denoted by $C(n, r)$.

- The notation $\binom{n}{r}$ is also used and is called a *binomial coefficient*

Combinations

- **Example**: Let S be the set $\{a, b, c, d\}$
- Then $\{a, c, d\}$ is a 3-combination from S. It is the same as $\{d, c, a\}$ since the order listed does not matter.
- $C(4,2) = 6$ because the 2-combinations of $\{a, b, c, d\}$ are the six subsets $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, and $\{c, d\}$.

Combinations

- **Theorem 2**: The number of r-combinations of a set with n elements, where $n \geq r \geq 0$, equals

$$C(n,r) = \frac{n!}{(n-r)!r!}.$$

- **Proof:** By the product rule $P(n, r) = C(n,r) \cdot P(r,r)$.
- Therefore,

$$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{n!/(n-r)!}{r!/(r-r)!} = \frac{n!}{(n-r)!r!} .$$

Combinations

- **Example**: How many poker hands of five cards can be dealt from a standard deck of 52 cards? Also, how many ways are there to select 47 cards from a deck of 52 cards?
- **Solution**: Since the order in which the cards are dealt does not matter, the number of five card hands is:

$$C(52,5) = \frac{52!}{5!47!}$$

$$= \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 26 \cdot 17 \cdot 10 \cdot 49 \cdot 12 = 2,598,960$$

- The different ways to select 47 cards from 52 is

$$C(52,47) = \frac{52!}{47!5!} = C(52,5) = 2,598,960.$$

https://www.fgbradleys.com/et_poker.asp

Combinations

- **Corollary** 2: Let n and r be nonnegative integers with $r \leq$
n. • Then $C(n, r) = C(n, n - r)$.

- Proof: From Theorem 2, it follows that

$$C(n,r) = \frac{n!}{(n-r)!r!}$$

and

Hence, $C(n, r) = C(n, n - r)$

$$C(n,n-r) = \frac{n!}{(n-r)![n-(n-r)]!} = \frac{n!}{(n-r)!r!} \; .$$

Combinations

- **Example**: How many ways are there to select five players from a **10**-member tennis team to make a trip to a match at another school.
- **Solution**: By Theorem **2**, the number of combinations is

$$C(10,5) = \frac{10!}{5!5!} = 252.$$

$30C_6$

Combinations

- **Example**: A group of 30 people have been trained as astronauts to go on the first mission to Mars. How many ways are there to select a crew of six people to go on this mission?
- **Solution**: By Theorem 2, the number of possible crews is

$$C(30,6) = \frac{30!}{6!24!} = \frac{30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 593,775 \ .$$

Combinations

- Suppose that there are 9 faculty members in the mathematics department and 11 in the computer science department. How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consist of three faculty members from the mathematics department and four from the computer science department?

$_9C_3 \times {}_{11}C_4$

Combinations

- By the product rule, the answer is the product of the number of 3-combinations of a set with 9 elements and the number of 4-combinations of a set with 11 elements.

- By Theorem 2, the number of ways to select the committee is

$$C(9,3) \cdot C(11,4) = \frac{9!}{3!6!} \cdot \frac{11!}{4!7!} = 84 \cdot 330 = 27{,}720.$$

Binomial Coefficients

Lecture 3

Powers of Binomial Expressions

- **Definition**: A *binomial* expression is the sum of two terms, such as $x + y$.
  (More generally, these terms can be products of constants and variables.)
- We can use counting principles to find the coefficients in the expansion of $(x + y)^n$ where n is a positive integer.
- To illustrate this idea, we first look at the process of expanding $(x + y)^3$.

- $(x + y)\,(x + y)\,(x + y)$ expands into a sum of terms that are the product of a term from each of the three sums.
- Terms of the form x^3, x^2y, xy^2, y^3 arise. The question is what are the coefficients?

Powers of Binomial Expressions

- To obtain x^3, an x must be chosen from each of the sums. There is only one way to do this. So, the coefficient of x^3 is 1.
- To obtain x^2y, an x must be chosen from two of the sums and a y from the other. There $\binom{3}{2}$ are ways to do this and so the coefficient of x^2y is 3 • To obtain xy^2, an x must $\binom{3}{1}$ be chosen from any of the sums and a y from the

other two . There are ways to do this and so the coefficient of xy^2 is 3 • To obtain y^3, a y must be chosen from each of the sums. There is only one way to do this. So, the coefficient of y3 is 1.

- We have used a counting argument to show that $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$
- Next we present the binomial theorem gives the coefficients of the terms in the expansion of $(x + y)^n$

Binomial Theorem

- **Binomial Theorem**: Let x and y be variables, and n a nonnegative integer. Then:

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

- **Proof**: We use combinatorial reasoning . The terms in the expansion of $(x + y)^n$ are of the form $x^{n-j}y^j$ for $j = 0,1,2,\ldots,n$.
- To form the term $x^{n-j}y^j$, it is necessary to choose $n-j$ xs from the n sums. Therefore, the coefficient of $x^{n-j}y^j$ is which equals .

Using the Binomial Theorem

- **Example**: What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$?
- **Solution**: We view the expression as $(2x + (-3y))^{25}$. By the binomial theorem

Consequently, the coefficient of $x^{12}y^{13}$ in the expansion is obtained when j = 13.

Practice Problems

Practice Problems

- In how many different orders can five runners finish a race if no ties are allowed?
- Permutations/Combination?

- N = ?
- R = ?

Practice Problems

- How many ways are there for four men and five women to stand in a line so that

a. All men stand together?

b. All women stand together?

Permutation/Combination?

N=?, R=?

Any special case?

Practice Problems

- A club has 25 members.

a. How many ways are there to choose four members of the club to serve on an executive committee?

b. How many ways are there to choose a president, vice president, secretary, and treasurer of the club, where no person can hold more than one office?

Permutation/Combination?

N=? R=?

Practice Problems

- Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with six members if it must have the same number of men and women?