Discrete Structures Spring 2024 – Week 11

Discrete Probability

Lecture 1

Probability of an Event

Pierre-Simon Laplace
(1749-1827)

- Pierre-Simon Laplace introduced classical theory of probability in the 18th century when he analyzed games of chance

- An *experiment* is a procedure that yields one of a given set of possible outcomes.
- The *sample space* of the experiment is the set of possible outcomes.
- An *event* is a subset of the sample space.

Probability of an Event

- Laplace's definition of the probability of an event:
- **Definition**: If S is a finite sample space of equally likely outcomes, and E is an event, that is, a subset of S, then the *probability* of E is

$$p(E) = |E| / |S|$$

- For every event E, we have $0 \leq p(E) \leq 1$.

Applying Laplace's Definition

- **Example**: An urn contains four blue balls and five red balls. What is the probability that a ball chosen from the urn is blue?

- **Solution**: The probability that the ball is chosen is $4/9$ since there are nine possible outcomes, and four of these produce a blue ball.

Applying Laplace's Definition

- **Example**: What is the probability that when two dice are rolled, the sum of the numbers on the two dice is 7?

- **Solution**: By the product rule there are $6^2 = 36$ possible outcomes. Six of these sum to 7. Hence, the probability of obtaining a 7 is $6/36 =$

1/6.

Applying Laplace's Definition

- **Example**: In a lottery, a player wins a large prize when they pick four digits that match, in correct order, four digits selected by a random mechanical process. What is the probability that a player wins the prize?

- **Solution**: By the product rule there are $10^4 = 10{,}000$ ways to pick four digits. Since there is only 1 way to pick the correct digits, the probability of winning the large prize is $1/10{,}000 = 0.0001$.

Applying Laplace's Definition

- A smaller prize is won if only three digits are matched. What is the probability that a player wins the small prize?

- **Solution**: If exactly three digits are matched, one of the four digits must be incorrect and the other three digits must be correct. For the digit that is incorrect, there are 9 possible choices. Hence, by the sum rule, there a total of 36 possible ways to choose four digits that match exactly three of the winning four digits. The probability of winning the small price is $36/10{,}000 = 9/2500 = 0.0036$.

Applying Laplace's Definition

- **Example**: There are many lotteries that award prizes to people who correctly choose a set of six numbers out of the first n positive integers, where n is usually between 30 and 60. What is the probability that a person picks the correct six numbers out of 40?

- **Solution**: The number of ways to choose six numbers out of 40 is

$$C(40,6) = 40!/(34!6!) = 3,838,380.$$

Hence, the probability of picking a winning combination is $1/3,838,380 \approx 0.00000026$

Applying Laplace's Definition

- **Example**: What is the probability that the numbers $11, 4, 17, 39,$ and 23 are drawn in that order from a bin with 50 balls labeled with the numbers $1,2, \ldots, 50$ if

a) The ball selected is not returned to the bin.

b) The ball selected is returned to the bin before the next ball is selected.

Applying Laplace's Definition

- **Solution**: Use the product rule in each case.

a) *Sampling without replacement*: The probability is $1/254{,}251{,}200$ since there are $50 \cdot 49 \cdot 47 \cdot 46 \cdot 45 = 254{,}251{,}200$ ways to choose the five balls.

b) *Sampling with replacement*: The probability is $1/50^5 = 1/312{,}500{,}000$ since $50^5 = 312{,}500{,}000$.

The Probability of Complements

- **Theorem 1**: Let *E* be an event in sample space *S*. The probability of the event $\overline{E} = S - E$, the complementary event of E, is given by

$$p(\overline{E}) = 1 - p(E).$$

- **Proof**: Using the fact that $|\overline{E}| = |S| - |E|$,

$$p(\overline{E}) = \frac{|S| - |E|}{|S|} = 1 - \frac{|E|}{|S|} = 1 - p(E).$$

The Probability of Complements

- **Example**: A sequence of **10** bits is chosen randomly. What is the probability that at least one of these bits is 0?

- **Solution**: Let E be the event that at least one of the 10 bits is 0. Then $\overline{E}$ is the event that all of the bits are 1s. The size of the sample space S is 2^{10}. Hence,

$$p(E) = 1 - p(\overline{E}) = 1 - \frac{|\overline{E}|}{|S|} = 1 - \frac{1}{2^{10}} = 1 - \frac{1}{1024} = \frac{1023}{1024}.$$

The Probability of Unions of Events

- **Theorem 2**: Let E_1 and E_2 be events in the sample space S. Then

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$$

Proof: Given the inclusion-exclusion formula from Section 2.2, $|A \cup B| = |A| + |B| - |A \cap B|$, it follows that

$$p(E_1 \cup E_2) = \frac{|E_1 \cup E_2|}{|S|} = \frac{|E_1| + |E_2| - |E_1 \cap E_2|}{|S|}$$

$$= \frac{|E_1|}{|S|} + \frac{|E_2|}{|S|} - \frac{|E_1 \cap E_2|}{|S|}$$

$$= p(E_1) + p(E_2) - p(E_1 \cap E_2).$$

The Probability of Unions of Events

- **Example**: What is the probability that a positive integer selected at random from the set of positive integers not exceeding 100 is divisible by either 2 or 5?

- **Solution**: Let E_1 be the event that the integer is divisible by 2 and E_2 be the event that it is divisible 5? Then the event that the integer is divisible by 2 or 5 is $E_1 \cup E_2$ and $E_1 \cap E_2$ is the event that it is divisible by 2 and 5.

It follows that: $p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$

$$= 50/100 + 20/100 - 10/100 = 3/5.$$

Probability Theory

Lecture 2 and Lecture 3

Assigning Probabilities

- Laplace's definition from the previous section, assumes that all outcomes are equally likely. Now we introduce a more general definition of probabilities that avoids this restriction.
- Let S be a sample space of an experiment with a finite number of outcomes. We assign a probability $p(s)$ to each outcome s, so that:

i. $0 \leq p(s) \leq 1$ for each $s \in S$

ii. $\sum_{s \in S} p(s) = 1$

Assigning Probabilities

- The function *p* from the set of all outcomes of the sample space *S* is called a *probability distribution*.

Assigning Probabilities

- **Example:** What probabilities should we assign to the outcomes H(heads) and T (tails) when a fair coin is flipped? What probabilities should be assigned to these outcomes when the coin is biased so that heads comes up twice as often as tails?

- **Solution**: For a fair coin, we have $p(H) = p(T) = ½$.
- For a biased coin, we have $p(H) = 2p(T)$.
- Because $p(H) + p(T) = 1$, it follows that

$$2p(T) + p(T) = 3p(T) = 1.$$

Hence, $p(T) = 1/3$ and $p(H) = 2/3$.

Uniform Distribution

- **Definition**: Suppose that S is a set with n elements. The *uniform distribution* assigns the probability $1/n$ to each element of S. (Note that we could have used Laplace's definition here.)

- **Example**: Consider again the coin flipping example, but with a fair coin. Now $p(H) = p(T) = 1/2$.

Probability of an Event

- **Definition**: The probability of the event E is the sum of the probabilities of the outcomes in E.

$$p(E) = \sum_{s \in S} p(s)$$

- Note that now no assumption is being made about the distribution.

Example

- **Example**: Suppose that a die is biased so that 3 appears twice as often as each other number, but that the other five outcomes are equally likely. What is the probability that an odd number appears when we roll this die?

- **Solution**: We want the probability of the event $E = \{1,3,5\}$ • We have $p(3) = 2/7$ and

- $p(1) = p(2) = p(4) = p(5) = p(6) = 1/7$.

- Hence, $p(E) = p(1) + p(3) + p(5) = 1/7 + 2/7 + 1/7 = 4/7$.

Probabilities of Complements and Unions of

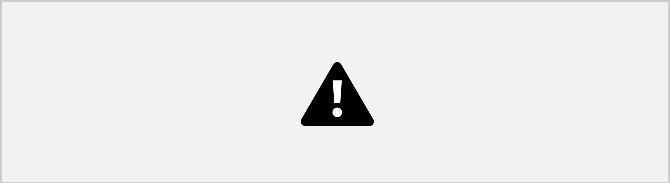
Events

- Complements: $p(\overline{E}) = 1 - p(E)$ still holds.
- Since each outcome is in either E or $\overline{E}$, but not both,

- Unions: also still holds under the new definition.

Conditional Probability

- Definition: Let E and F be events with $p(F) > 0$.
- The conditional probability of E given F, denoted by $P(E|F)$, is defined as:



Conditional Probability

- **Example**: A bit string of length four is generated at random so that each of the 16 bit strings of length 4 is equally likely. What is the probability that it contains at least two consecutive 0s, given that its first bit is a 0?
- **Solution**: Let E be the event that the bit string contains at least two consecutive 0s, and F be the event that the first bit is a 0.
- Since $E \cap F = \{0000, 0001, 0010, 0011, 0100\}$
- $p(E \cap F) = 5/16$.
- Because 8 bit strings of length 4 start with a 0, $p(F) = 8/16 = ½$. Hence,

Conditional Probability

- **Example:** What is the conditional probability that a family with two children has two boys, given that they have at least one boy. Assume that each of the possibilities *BB*, *BG*, *GB*, and *GG* is equally likely where *B* represents a boy and *G* represents a girl.
- **Solution**: Let E be the event that the family has two boys and let F be the event that the family has at least one boy. Then $E = \{BB\}$, $F = \{BB, BG, GB\}$, and $E \cap F = \{BB\}$.
- It follows that p(F) = 3/4 and $p(E \cap F) = 1/4$.
- Hence,

Independence

- **Definition:** The events E and F are independent if and only if

$$p(E \cap F) = p(E)p(F).$$

Independence

- **Example**: Suppose E is the event that a randomly generated bit string of length four begins with a 1 and F is the event that this bit string contains an even number of 1s. Are E and F independent if the 16 bit strings of length four are equally likely?
- **Solution**: There are eight bit strings of length four that begin with a 1, and eight bit strings of length four that contain an even number of 1s. Since the number of bit strings of length 4 is 16,

$$p(E) = p(F) = 8/16 = ½.$$

- Since $E \cap F = \{1111, 1100, 1010, 1001\}$, $p(E \cap F) = 4/16 = 1/4$. • We conclude that E and F are independent, because $p(E \cap F) = 1/4 = (½)\,(½) = p(E)\,p(F)$

Independence

- **Example**: Assume that each of the four ways a family can have two children (*BB*, *GG*, *BG*,*GB*) is equally likely. Are the events E, that a family with two children has two boys, and F, that a family with two children has at least one boy, independent?
- **Solution**: Because $E = \{BB\}$, $p(E) = 1/4$.
- We saw previously that that $p(F) = 3/4$ and $p(E \cap F) = 1/4$.
- The events E and F are not independent since

$$p(E)\,p(F) = 3/16 \neq 1/4 = p(E \cap F)\ .$$

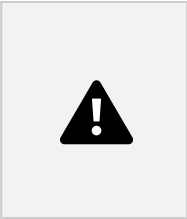
Pair-wise and Mutual Independence

- **Definition**: The events $E_1, E_2, \ldots, E_n$ are *pairwise independent* if and only if $p(E_i \cap E_j) = p(E_i)\,p(E_j)$ for all pairs i and j with $i \leq j \leq n$.
- The events are *mutually independent* if

whenever $i_j, j = 1,2,\ldots, m$, are integers with

$$1 \leq i_1 < i_2 < \cdots < i_m \leq n \text{ and } m \geq 2.$$

Bernoulli Trials James Bernoulli



(1854 – 1705)

- **Definition**: Suppose an experiment can have only two possible outcomes, *e.g.*, the flipping of a coin or the random generation of a bit. • Each performance of the experiment is called a *Bernoulli trial.*
  - One outcome is called a *success* and the other a *failure*.
  - If p is the probability of success and q the probability of failure, then $p + q = 1$. •

Many problems involve determining the probability of k successes when an experiment consists of n mutually independent Bernoulli trials.

Bernoulli Trials

- **Example:** A coin is biased so that the probability of heads is 2/3. What is the probability that exactly four heads occur when the coin is flipped seven times?
- **Solution:** There are $2^7 = 128$ possible outcomes. The number of ways

four of the seven flips can be heads is C(7,4). The probability of each of the outcomes is $(2/3)^4(1/3)^3$ since the seven flips are independent. Hence, the probability that exactly four heads occur is

- $C(7,4)\ (2/3)^4(1/3)^3 = (35\cdot 16)/\ 2^7 = 560/\ 2187.$

Practice Problems

Practice Problem

- What is the probability that a randomly selected integer chosen from the first 100 positive integers is odd?
- **Solution:** Determine the number of odd integers in the range and divide it by the total number of integers.

Practice Problem

- What is the probability that the sum of the numbers on two dice is even when they are rolled?
- **Solution:** Determine the number of favorable outcomes (even sums) and divide it by the total number of possible outcomes.

Practice Problem

- What is the probability that a fair die never comes up an even number when it is rolled six times?
- Solution: Since we want to find the probability of never rolling an even number in six rolls, we need to consider the favorable outcomes where only odd numbers appear on each roll.

Practice Problem

- What is the probability that a positive integer not exceeding 100 selected at random is divisible by 5 or 7?

Practice Problem

- What is the probability that Abby, Barry, and Sylvia win the first, second, and third prizes, respectively, in a drawing if 200 people enter a contest and
- **a)** no one can win more than one prize.
- **b)** winning more than one prize is allowed.

Practice Problem

- Find the probability of each outcome when a biased die is rolled, if rolling a 2 or rolling a 4 is three times as likely as rolling each of the other four numbers on the die and it is equally likely to roll a 2 or a 4.

Practice Problem

- What is the conditional probability that exactly four heads appear when a fair coin is flipped five times, given that the first flip came up heads?