Discrete Structures

Spring 2024 – Week 13

Boolean Algebra

Introduction to Boolean Algebra

- Boolean algebra has rules for working with elements from the set $\{0, 1\}$ together with the operators $+$ (Boolean sum), $\cdot$ (Boolean product), and $\bar{}$ (complement).
- These operators are defined by:
- *Boolean sum*: $1 + 1 = 1, 1 + 0 = 1, 0 + 1 = 1, 0 + 0 = 0$
- *Boolean product*: $1 \cdot 1 = 1, 1 \cdot 0 = 0, 0 \cdot 1 = 0, 0 \cdot 0 = 0$
- *Complement*: $\overline{0} = 1, \overline{1} = 0$

Introduction to Boolean Algebra

Example: Find the value of $1 \cdot 0 + \overline{(0+1)}$

Solution : $1 \cdot 0 + \overline{(0+1)} = 0 + \bar{1}$

$= 0 + 0$

$= 0$

Boolean Expressions and Boolean Functions

Definition:

Let $B = \{0, 1\}$

Then $B^n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in B \text{ for } 1 \leq i \leq n\}$ is the set of all possible n-tuples of 0s and 1s.

The variable x is called a *Boolean variable* if it assumes values only from B, that is, if its only possible values are 0 and 1.

A function from B^n to B is called a *Boolean function of degree n*.

Boolean Expressions and Boolean Functions

Example: The function $F(x, y) = x\bar{y}$ from the set of ordered pairs of Boolean variables to the set $\{0, 1\}$ is a Boolean function of degree 2.

TABLE 1

x	y	$F(x, y)$
1	1	0
1	0	1
0	1	0
0	0	0

Boolean Expressions and Boolean Functions (*continued*)

Example: Find the values of the Boolean function represented by $F(x, y, z) = xy + \bar{z}$.

Solution: We use a table with a row for each combination of values of x, y, and z to compute the values of $F(x,y,z)$.

TABLE 2

x	y	z	xy	$\bar{z}$	$F(x, y, z) = xy + \bar{z}$
1	1	1	1	0	1
1	1	0	1	1	1
1	0	1	0	0	0
1	0	0	0	1	1
0	1	1	0	0	0
0	1	0	0	1	1
0	0	1	0	0	0
0	0	0	0	1	1

Boolean Expressions and Boolean Functions (*continued*)

- **Definition**: Boolean functions F and G of n variables are equal if and only if $F(b_1, b_2, \ldots, b_n) = G(b_1, b_2, \ldots, b_n)$ whenever $b_1, b_2, \ldots, b_n$ belong to B. Two different Boolean expressions that represent the same function are *equivalent*.

- **Definition**: The complement of the Boolean function F is the function $\bar{F}$, where $\bar{F}(x_1, x_2, \ldots, x_n) = \overline{F(x_1, x_2, \ldots, x_n)}$.

Boolean Expressions and Boolean Functions (*continued*)

Definition: Let F and G be Boolean functions of degree n. The Boolean sum $F + G$ and the Boolean product FG are defined by

$$(F + G)(x_1, x_2, \ldots, x_n) = F(x_1, x_2, \ldots, x_n) + G(x_1, x_2, \ldots, x_n)$$

$$(FG)(x_1, x_2, \ldots, x_n) = F(x_1, x_2, \ldots, x_n)G(x_1, x_2, \ldots, x_n)$$

Identities of Boolean Algebra

TABLE 5 **Boolean Identities.**

Identity	*Name*
$\overline{\overline{x}} = x$	Law of the double complement
$x + x = x$ $x \cdot x = x$	Idempotent laws
$x + 0 = x$ $x \cdot 1 = x$	Identity laws
$x + 1 = 1$ $x \cdot 0 = 0$	Domination laws
$x + y = y + x$ $xy = yx$	Commutative laws

Each identity can be proved using a table.

All identities in Table 5, except for the first and the last two come in pairs. Each element of the pair is the *dual* of the other (obtained by switching Boolean sums and Boolean products and 0's and 1's.

The Boolean identities correspond to the identities of propositional logic (Section 1.3) and the set identities (Section 2.2).

Identities of Boolean Algebra

$x + (y + z) = (x + y) + z$ $x(yz) = (xy)z$	Associative laws
$x + yz = (x + y)(x + z)$ $x(y + z) = xy + xz$	Distributive laws
$\overline{(xy)} = \overline{x} + \overline{y}$ $\overline{(x + y)} = \overline{x}\,\overline{y}$	De Morgan's laws
$x + xy = x$ $x(x + y) = x$	Absorption laws
$x + \overline{x} = 1$	Unit property
$x\overline{x} = 0$	Zero property

Each identity can be proved using a table.

All identities in Table 5, except for the first and the last two come in pairs. Each element of the pair is the *dual* of the other (obtained by switching Boolean sums and Boolean products and 0's and 1's.

The Boolean identities correspond to the identities of propositional logic (Section 1.3) and the set identities (Section 2.2).

Identities of Boolean Algebra

Example: Show that the distributive law $x(y + z) = xy + xz$ is valid.

Solution: We show that both sides of this identity always take the same value by constructing this table.

TABLE 6 **Verifying One of the Distributive Laws.**

x	y	z	$y+z$	xy	xz	$x(y+z)$	$xy+xz$
1	1	1	1	1	1	1	1
1	1	0	1	1	0	1	1
1	0	1	1	0	1	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

Formal Definition of a Boolean Algebra

Definition: A *Boolean algebra* is a set B with two binary operations $\vee$ and $\wedge$, elements 0 and 1, and a unary operation $\bar{\ }$ such that for all x, y, and z in B:

$x \vee 0 = x$
$x \wedge 1 = x$

identity laws

$x \vee \bar{x} = 1$
$x \wedge \bar{x} = 0$

complement laws

$(x \vee y) \vee z = x \vee (y \vee z)$
$(x \wedge y) \wedge z = x \wedge (y \wedge z)$

associative laws

$x \vee y = y \vee x$
$x \wedge y = y \wedge x$

commutative laws

$x \vee (y \wedge z) = (x \vee y) \wedge (y \vee z)$
$x \wedge (y \vee z) = (x \wedge y) \vee (y \wedge z)$

distributive laws

The set of propositional variables with the operators $\wedge$ and $\vee$, elements **T** and **F**, and the negation operator $\neg$ is a Boolean algebra.

The set of subsets of a universal set with the operators $\cup$ and $\cap$, the empty set ($\varnothing$), universal set (U), and the set complementation operator $\overline{(\ )}$ is a Boolean algebra.

Representing Boolean Functions

Sum-of-Products Expansion

Example: Find Boolean expressions that represent the functions (i) $F(x, y, z)$ and (ii) $G(x, y, z)$ in Table 1.

Solution:

(i) To represent F we need the one term $x\bar{y}z$ because this expression has the value 1 when $x = z = 1$ and $y = 0$.

(ii) To represent the function G, we use the sum $xy\bar{z} + \bar{x}y\bar{z}$ because this expression has the value 1 when $x = y = 1$ and $z = 0$, or $x = z = 0$ and $y = 1$.

TABLE 1

x	y	z	F	G
1	1	1	0	0
1	1	0	0	1
1	0	1	1	0
1	0	0	0	0
0	1	1	0	0
0	1	0	0	1
0	0	1	0	0
0	0	0	0	0

The general principle is that each combination of values of the variables for which the function has the value 1 requires a term in the Boolean sum that is the Boolean product of the variables or their complements.

Sum-of-Products Expansion (*cont*)

Definition: A *literal* is a Boolean variable or its complement. A *minterm* of the Boolean variables $x_1, x_2, \ldots, x_n$ is a Boolean product $y_1 y_2 \cdots y_n$, where $y_i = x_i$ or $y_i = \overline{x}_i$. Hence, a minterm is a product of n literals, with one literal for each variable.

The minterm $y_1, y_2, \ldots, y_n$ has value has value 1 if and only if each x_i is 1.This occurs if and only if $x_i = 1$ when $y_i = x_i$ and $x_i = 0$ when $y_i = \overline{x}_i$.

Definition: The sum of minterms that represents the function is called the *sum-of-products expansion* or the *disjunctive normal form* of the Boolean function.

Sum-of-Products Expansion (*cont*)

Example: Find the sum-of-products expansion for the function $F(x,y,z) = (x + y)\bar{z}$.

Solution: We use two methods, first using a table and second using Boolean identities.

(i) Form the sum of the minterms corresponding to each row of the table that has the value 1.

Including a tem for each row of the table for which $F(x,y,z) = 1$ gives us $F(x, y, z) = xy\bar{z} + x\bar{y}\bar{z} + \bar{x}y\bar{z}$.

TABLE 2

x	y	z	$x + y$	$\bar{z}$	$(x + y)\bar{z}$
1	1	1	1	0	0
1	1	0	1	1	1
1	0	1	1	0	0
1	0	0	1	1	1
0	1	1	1	0	0
0	1	0	1	1	1
0	0	1	0	0	0
0	0	0	0	1	0

Sum-of-Products Expansion (*cont*)

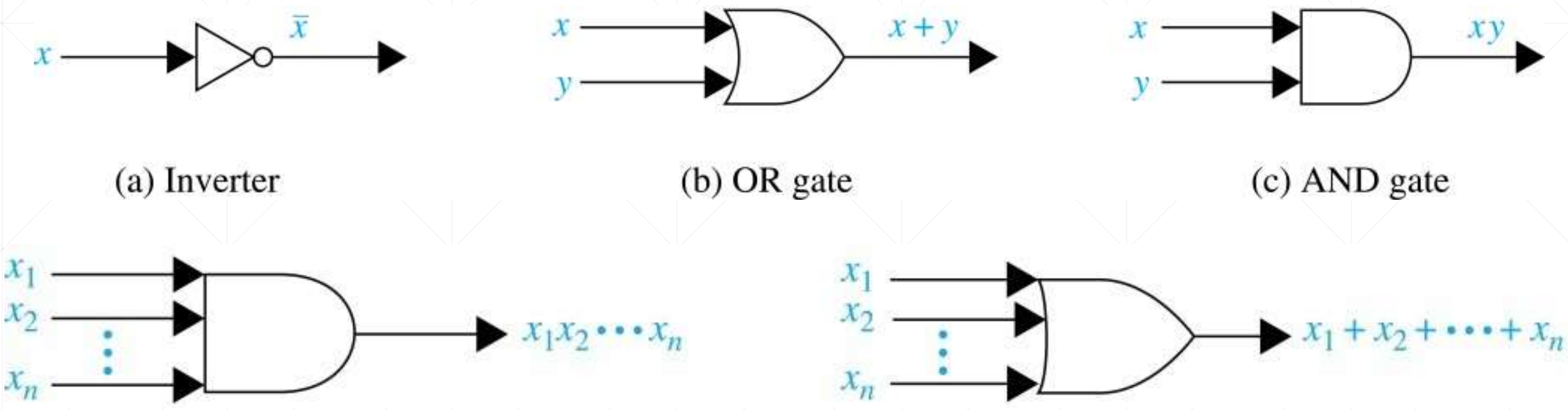
(ii) We now use Boolean identities to find the disjunctive normal form of $F(x,y,z)$:

$$
\begin{aligned}
F(x,y,z) &= (x + y)\,\bar{z} \\
&= x\bar{z} + y\bar{z} \quad \textit{distributive law} \\
&= x1\bar{z} + 1y\bar{z} \quad \textit{identity law} \\
&= x(y + \bar{y})\bar{z} + (x + \bar{x})y\bar{z} \quad \textit{unit property} \\
&= xy\bar{z} + x\bar{y}\bar{z} + xy\bar{z} + \bar{x}y\bar{z} \quad \textit{distributive law} \\
&= xy\bar{z} + x\bar{y}\bar{z} + \bar{x}y\bar{z} \quad \textit{idempotent law}
\end{aligned}
$$

Logic Gates

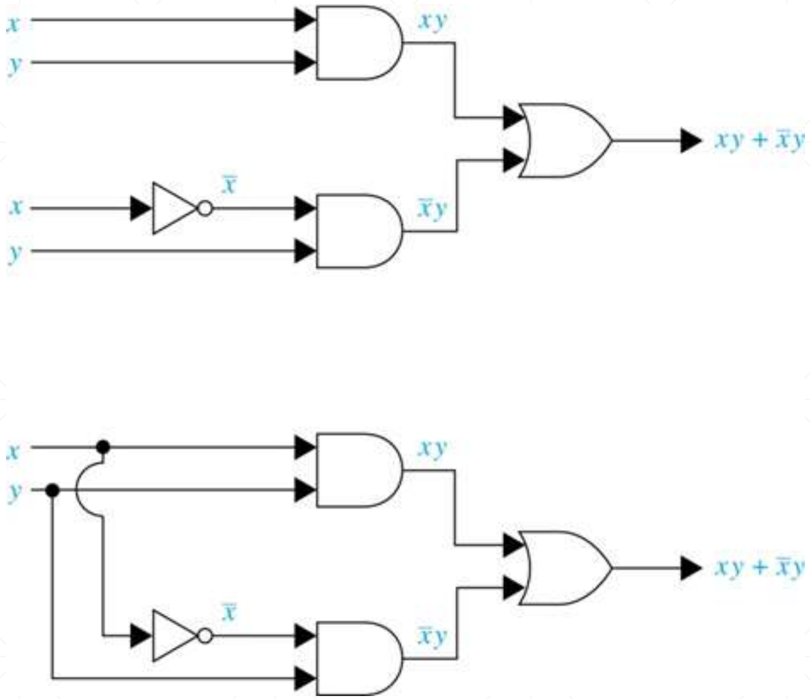
Logic Gates

- We construct circuits using *gates*, which take as input the values of two or more Boolean variables and produce one or more bits as output, and *inverters*, which take the value of a Boolean variable as input and produce the complement of this value as output.



Combinations of Gates

- Combinatorial circuits can be constructed using a combination of inverters, OR gates, and AND gates. Gates may share input and the output of one or more gates may be input to another.
- We show two ways of constructing a circuit that produces the output $xy + \bar{x}y$.



Combinations of Gates

Example: Construct circuits that produce these outputs

(a) $(x+y)\bar{x}$

(b) $\bar{x}\,\overline{(y+\bar{z})}$

(c) $(x+y+z)(\bar{x}\,\bar{y}\,\bar{z})$