Table 5.2.1
INTEGRATION FORMULAS

DIFFERENTIATION FORMULA	INTEGRATION FORMULA	DIFFERENTIATION FORMULA	INTEGRATION FORMULA
1. $\frac{d}{dx}[x] = 1$	$\int dx = x + C$	8. $\frac{d}{dx}[-\csc x] = \csc x \cot x$	$\int \csc x \cot x\, dx = -\csc x + C$
2. $\frac{d}{dx}\left[\frac{x^{r+1}}{r+1}\right] = x^r \quad (r \neq -1)$	$\int x^r\, dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)$	9. $\frac{d}{dx}[e^x] = e^x$	$\int e^x\, dx = e^x + C$
3. $\frac{d}{dx}[\sin x] = \cos x$	$\int \cos x\, dx = \sin x + C$	10. $\frac{d}{dx}\left[\frac{b^x}{\ln b}\right] = b^x \quad (0 < b,\, b \neq 1)$	$\int b^x\, dx = \frac{b^x}{\ln b} + C \quad (0 < b,\, b \neq 1)$
4. $\frac{d}{dx}[-\cos x] = \sin x$	$\int \sin x\, dx = -\cos x + C$	11. $\frac{d}{dx}[\ln \lvert x\rvert] = \frac{1}{x}$	$\int \frac{1}{x}\, dx = \ln \lvert x\rvert + C$
5. $\frac{d}{dx}[\tan x] = \sec^2 x$	$\int \sec^2 x\, dx = \tan x + C$	12. $\frac{d}{dx}[\tan^{-1} x] = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}\, dx = \tan^{-1} x + C$
6. $\frac{d}{dx}[-\cot x] = \csc^2 x$	$\int \csc^2 x\, dx = -\cot x + C$	13. $\frac{d}{dx}[\sin^{-1} x] = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}\, dx = \sin^{-1} x + C$
7. $\frac{d}{dx}[\sec x] = \sec x \tan x$	$\int \sec x \tan x\, dx = \sec x + C$	14. $\frac{d}{dx}[\sec^{-1} \lvert x\rvert] = \frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}}\, dx = \sec^{-1} \lvert x\rvert + C$

▶ Example 3

$$\int (3x^6 - 2x^2 + 7x + 1)\, dx = 3\int x^6\, dx - 2\int x^2\, dx + 7\int x\, dx + \int 1\, dx$$
$$= \frac{3x^7}{7} - \frac{2x^3}{3} + \frac{7x^2}{2} + x + C \quad ◀$$

Sometimes it is useful to rewrite an integrand in a different form before performing the integration. This is illustrated in the following example.

▶ Example 4 Evaluate

$$\text{(a)} \int \frac{\cos x}{\sin^2 x}\, dx \qquad \text{(b)} \int \frac{t^2 - 2t^4}{t^4}\, dt \qquad \text{(c)} \int \frac{x^2}{x^2+1}\, dx$$

Solution (a).

$$\int \frac{\cos x}{\sin^2 x}\, dx = \int \frac{1}{\sin x}\frac{\cos x}{\sin x}\, dx = \int \csc x \cot x\, dx = -\csc x + C$$

Formula 8 in Table 5.2.1

Solution (b).

$$\int \frac{t^2 - 2t^4}{t^4}\, dt = \int \left(\frac{1}{t^2} - 2\right) dt = \int (t^{-2} - 2)\, dt$$
$$= \frac{t^{-1}}{-1} - 2t + C = -\frac{1}{t} - 2t + C$$

Solution (c). By adding and subtracting 1 from the numerator of the integrand, we can rewrite the integral in a form in which Formulas 1 and 12 of Table 5.2.1 can be applied:

$$\int \frac{x^2}{x^2+1}\, dx = \int \left(\frac{x^2+1}{x^2+1} - \frac{1}{x^2+1}\right) dx$$
$$= \int \left(1 - \frac{1}{x^2+1}\right) dx = x - \tan^{-1} x + C \quad ◀$$

▶ **Example 4**

$$\int \frac{dx}{\left(\frac{1}{3}x-8\right)^5} = \int \frac{3\,du}{u^5} = 3\int u^{-5}\,du = -\frac{3}{4}u^{-4}+C = -\frac{3}{4}\left(\frac{1}{3}x-8\right)^{-4}+C \blacktriangleleft$$

$u = \frac{1}{3}x - 8$
$du = \frac{1}{3}\,dx$ or $dx = 3\,du$

▶ **Example 5** Evaluate $\displaystyle\int \frac{dx}{1+3x^2}$.

Solution. Substituting

$$u = \sqrt{3}x, \quad du = \sqrt{3}\,dx$$

yields

$$\int \frac{dx}{1+3x^2} = \frac{1}{\sqrt{3}}\int \frac{du}{1+u^2} = \frac{1}{\sqrt{3}}\tan^{-1} u + C = \frac{1}{\sqrt{3}}\tan^{-1}(\sqrt{3}x) + C \blacktriangleleft$$

With the help of Theorem 5.2.3, a complicated integral can sometimes be computed by expressing it as a sum of simpler integrals.

▶ **Example 6**

$$\begin{aligned}\int\left(\frac{1}{x}+\sec^2 \pi x\right)dx &= \int \frac{dx}{x} + \int \sec^2 \pi x\,dx \\ &= \ln|x| + \int \sec^2 \pi x\,dx \\ &= \ln|x| + \frac{1}{\pi}\int \sec^2 u\,du \\ &= \ln|x| + \frac{1}{\pi}\tan u + C = \ln|x| + \frac{1}{\pi}\tan \pi x + C \blacktriangleleft\end{aligned}$$

$u = \pi x$
$du = \pi\,dx$ or $dx = \frac{1}{\pi}\,du$

The next four examples illustrate a substitution $u = g(x)$ where $g(x)$ is a nonlinear function.

▶ **Example 7** Evaluate $\displaystyle\int \sin^2 x \cos x\,dx$.

Solution. If we let $u = \sin x$, then

$$\frac{du}{dx} = \cos x, \quad \text{so} \quad du = \cos x\,dx$$

Thus,

$$\int \sin^2 x \cos x\,dx = \int u^2\,du = \frac{u^3}{3} + C = \frac{\sin^3 x}{3} + C \blacktriangleleft$$

Example 1 Use the two methods above to evaluate $\int_0^2 x(x^2+1)^3\,dx$.

Solution by Method 1. If we let

$$u = x^2 + 1 \quad \text{so that} \quad du = 2x\,dx$$

then we obtain

$$\int x(x^2+1)^3\,dx = \frac{1}{2}\int u^3\,du = \frac{u^4}{8} + C = \frac{(x^2+1)^4}{8} + C$$

Thus,

$$\int_0^2 x(x^2+1)^3\,dx = \left[\int x(x^2+1)^3\,dx\right]_{x=0}^2$$

$$= \frac{(x^2+1)^4}{8}\bigg]_{x=0}^2 = \frac{625}{8} - \frac{1}{8} = 78$$

Solution by Method 2. If we make the substitution $u = x^2 + 1$ in (2), then

$$\text{if} \quad x = 0, \quad u = 1$$
$$\text{if} \quad x = 2, \quad u = 5$$

Thus,

$$\int_0^2 x(x^2+1)^3\,dx = \frac{1}{2}\int_1^5 u^3\,du$$

$$= \frac{u^4}{8}\bigg]_{u=1}^5 = \frac{625}{8} - \frac{1}{8} = 78$$

which agrees with the result obtained by Method 1. ◄

▸ Example 8 Evaluate $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$.

Solution. If we let $u = \sqrt{x}$, then

$$\frac{du}{dx} = \frac{1}{2\sqrt{x}}, \quad \text{so} \quad du = \frac{1}{2\sqrt{x}}\,dx \quad \text{or} \quad 2\,du = \frac{1}{\sqrt{x}}\,dx$$

Thus,

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = \int 2e^u\,du = 2\int e^u\,du = 2e^u + C = 2e^{\sqrt{x}} + C \blacktriangleleft$$

▸ Example 9 Evaluate $\int t^4 \sqrt[3]{3 - 5t^5}\,dt$.

Solution.

$$\int t^4 \sqrt[3]{3-5t^5}\,dt = -\frac{1}{25}\int \sqrt[3]{u}\,du = -\frac{1}{25}\int u^{1/3}\,du$$

$$\boxed{\begin{array}{c} u = 3 - 5t^5 \\ du = -25t^4\,dt \text{ or } -\frac{1}{25}\,du = t^4\,dt \end{array}}$$

$$= -\frac{1}{25}\frac{u^{4/3}}{4/3} + C = -\frac{3}{100}\left(3 - 5t^5\right)^{4/3} + C \blacktriangleleft$$

▸ Example 10 Evaluate $\int \frac{e^x}{\sqrt{1 - e^{2x}}}\,dx$.

Solution. Substituting

$$u = e^x, \quad du = e^x\,dx$$

yields

$$\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx = \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C = \sin^{-1}(e^x) + C \blacktriangleleft$$

▸ Example 2 Evaluate

$$\text{(a)} \int_0^{\pi/8} \sin^5 2x \cos 2x\, dx \qquad \text{(b)} \int_2^5 (2x-5)(x-3)^9\, dx$$

Solution (a). Let

$$u = \sin 2x \quad \text{so that} \quad du = 2\cos 2x\, dx \quad \left(\text{or } \tfrac{1}{2}\, du = \cos 2x\, dx\right)$$

With this substitution,

$$\begin{aligned} &\text{if} \quad x = 0, \quad u = \sin(0) = 0 \\ &\text{if} \quad x = \pi/8, \quad u = \sin(\pi/4) = 1/\sqrt{2} \end{aligned}$$

so

$$\begin{aligned} \int_0^{\pi/8} \sin^5 2x \cos 2x\, dx &= \frac{1}{2}\int_0^{1/\sqrt{2}} u^5\, du \\ &= \frac{1}{2} \cdot \frac{u^6}{6}\bigg]_{u=0}^{1/\sqrt{2}} = \frac{1}{2}\left[\frac{1}{6(\sqrt{2})^6} - 0\right] = \frac{1}{96} \end{aligned}$$

Solution (b). Let

$$u = x - 3 \quad \text{so that} \quad du = dx$$

This leaves a factor of $2x - 5$ unresolved in the integrand. However,

$$x = u + 3, \quad \text{so} \quad 2x - 5 = 2(u+3) - 5 = 2u + 1$$

With this substitution,

$$\begin{aligned} &\text{if} \quad x = 2, \quad u = 2 - 3 = -1 \\ &\text{if} \quad x = 5, \quad u = 5 - 3 = 2 \end{aligned}$$

so

$$\begin{aligned} \int_2^5 (2x-5)(x-3)^9\, dx &= \int_{-1}^2 (2u+1)u^9\, du = \int_{-1}^2 (2u^{10} + u^9)\, du \\ &= \left[\frac{2u^{11}}{11} + \frac{u^{10}}{10}\right]_{u=-1}^2 = \left(\frac{2^{12}}{11} + \frac{2^{10}}{10}\right) - \left(-\frac{2}{11} + \frac{1}{10}\right) \\ &= \frac{52{,}233}{110} \approx 474.8 \end{aligned}$$

◂

INTEGRATION BY PARTS

▶ **Example 2** Evaluate $\int xe^x\,dx$.

Solution. In this case the integrand is the product of the algebraic function x with the exponential function e^x. According to LIATE we should let

$$u = x \quad \text{and} \quad dv = e^x\,dx$$

so that

$$du = dx \quad \text{and} \quad v = \int e^x\,dx = e^x$$

Thus, from (3)

$$\int xe^x\,dx = \int u\,dv = uv - \int v\,du = xe^x - \int e^x\,dx = xe^x - e^x + C \blacktriangleleft$$

▶ **Example 3** Evaluate $\int \ln x\,dx$.

Solution. One choice is to let $u = 1$ and $dv = \ln x\,dx$. But with this choice finding v is equivalent to evaluating $\int \ln x\,dx$ and we have gained nothing. Therefore, the only reasonable choice is to let

$$u = \ln x \qquad dv = dx$$

$$du = \frac{1}{x}\,dx \qquad v = \int dx = x$$

With this choice it follows from (3) that

$$\int \ln x\,dx = \int u\,dv = uv - \int v\,du = x\ln x - \int dx = x\ln x - x + C \blacktriangleleft$$

▶ **Example 4** Evaluate $\int x^2e^{-x}\,dx$.

Solution. Let

$$u = x^2, \quad dv = e^{-x}\,dx, \quad du = 2x\,dx, \quad v = \int e^{-x}\,dx = -e^{-x}$$

so that from (3)

$$\begin{aligned}\int x^2e^{-x}\,dx &= \int u\,dv = uv - \int v\,du \\ &= x^2(-e^{-x}) - \int -e^{-x}(2x)\,dx \\ &= -x^2e^{-x} + 2\int xe^{-x}\,dx \qquad (4)\end{aligned}$$

The last integral is similar to the original except that we have replaced x^2 by x. Another integration by parts applied to $\int xe^{-x}\,dx$ will complete the problem. We let

$$u = x, \quad dv = e^{-x}\,dx, \quad du = dx, \quad v = \int e^{-x}\,dx = -e^{-x}$$

so that

$$\int xe^{-x}\,dx = x(-e^{-x}) - \int -e^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x} + C$$

Finally, substituting this into the last line of (4) yields

$$\begin{aligned}\int x^2e^{-x}\,dx &= -x^2e^{-x} + 2\int xe^{-x}\,dx = -x^2e^{-x} + 2(-xe^{-x} - e^{-x}) + C \\ &= -(x^2 + 2x + 2)e^{-x} + C\end{aligned}$$ ◀

The LIATE method suggests that integrals of the form

$$\int e^{ax}\sin bx\,dx \quad \text{and} \quad \int e^{ax}\cos bx\,dx$$

can be evaluated by letting $u = \sin bx$ or $u = \cos bx$ and $dv = e^{ax}\,dx$. However, this will require a technique that deserves special attention.

Example 5 Evaluate $\int e^x \cos x \, dx$.

Solution. Let

$$u = \cos x, \quad dv = e^x \, dx, \quad du = -\sin x \, dx, \quad v = \int e^x \, dx = e^x$$

Thus,

$$\int e^x \cos x \, dx = \int u \, dv = uv - \int v \, du = e^x \cos x + \int e^x \sin x \, dx \tag{5}$$

Since the integral $\int e^x \sin x \, dx$ is similar in form to the original integral $\int e^x \cos x \, dx$, it seems that nothing has been accomplished. However, let us integrate this new integral by parts. We let

$$u = \sin x, \quad dv = e^x \, dx, \quad du = \cos x \, dx, \quad v = \int e^x \, dx = e^x$$

Thus,

$$\int e^x \sin x \, dx = \int u \, dv = uv - \int v \, du = e^x \sin x - \int e^x \cos x \, dx$$

Together with Equation (5) this yields

$$\int e^x \cos x \, dx = e^x \cos x + e^x \sin x - \int e^x \cos x \, dx \tag{6}$$

which is an equation we can solve for the unknown integral. We obtain

$$2 \int e^x \cos x \, dx = e^x \cos x + e^x \sin x$$

and hence

$$\int e^x \cos x \, dx = \tfrac{1}{2} e^x \cos x + \tfrac{1}{2} e^x \sin x + C$$

A TABULAR METHOD FOR REPEATED INTEGRATION BY PARTS

Integrals of the form

$$\int p(x) f(x)\, dx$$

where $p(x)$ is a polynomial, can sometimes be evaluated using repeated integration by parts in which u is taken to be $p(x)$ or one of its derivatives at each stage. Since du is computed by differentiating u, the repeated differentiation of $p(x)$ will eventually produce 0, at which point you may be left with a simplified integration problem. A convenient method for organizing the computations into two columns is called ***tabular integration by parts***.

Tabular Integration by Parts

Step 1. Differentiate $p(x)$ repeatedly until you obtain 0, and list the results in the first column.

Step 2. Integrate $f(x)$ repeatedly and list the results in the second column.

Step 3. Draw an arrow from each entry in the first column to the entry that is one row down in the second column.

Step 4. Label the arrows with alternating $+$ and $-$ signs, starting with a $+$.

Step 5. For each arrow, form the product of the expressions at its tip and tail and then multiply that product by $+1$ or -1 in accordance with the sign on the arrow. Add the results to obtain the value of the integral.

This process is illustrated in Figure 7.2.1 for the integral $\int (x^2 - x) \cos x\, dx$.

REPEATED DIFFERENTIATION		REPEATED INTEGRATION
$x^2 - x$	$+$	$\cos x$
$2x - 1$	$-$	$\sin x$
2	$+$	$-\cos x$
0		$-\sin x$

$$\boxed{\begin{aligned}\int (x^2 - x) \cos x\, dx &= (x^2 - x) \sin x + (2x - 1) \cos x - 2 \sin x + C \\ &= (x^2 - x - 2) \sin x + (2x - 1) \cos x + C\end{aligned}}$$

▶ **Figure 7.2.1**

S.No.	Form of the rational function	Form of the partial fraction
1.	$\frac{px+q}{(x-a)(x-b)}$, $a \neq b$	$\frac{A}{x-a}+\frac{B}{x-b}$
2.	$\frac{px+q}{(x-a)^2}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}$
3.	$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$
4.	$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{x-b}$
5.	$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\frac{A}{x-a}+\frac{Bx+C}{x^2+bx+c}$

Express $\dfrac{x+7}{(x-2)(x+1)}$ in partial fractions.

Solution

Note that the denominator contains **distinct linear factors**.

$$\frac{x+7}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$$
$$= \frac{A(x+1)+B(x-2)}{(x-2)(x+1)}$$

Hence $x+7 = A(x+1)+B(x-2)$.

To eliminate the first bracket and find the value of B, put $x=-1$:

$$6 = A(0)+B(-3)$$
$$\Rightarrow \quad -3B = 6$$
$$\Rightarrow \quad B = -2$$

To eliminate the second bracket and find the value of A, put $x=2$:

$$9 = A(3)+B(0)$$
$$\Rightarrow \quad 3A = 9$$
$$\Rightarrow \quad A = 3$$

Hence $\dfrac{x+7}{(x-2)(x+1)} = \dfrac{3}{x-2} - \dfrac{2}{x+1}$.

[*Check*:
$$\frac{3}{x-2} - \frac{2}{x+1} = \frac{3(x+1)-2(x-2)}{(x-2)(x+1)}$$
$$= \frac{3x+3-2x+4}{(x-2)(x+1)}$$
$$= \frac{x+7}{(x-2)(x+1)}$$
]

Express $\frac{x^2-7x+9}{(x+2)(x-1)^2}$ in partial fractions.

Solution

Note that the denominator contains a **repeated linear factor**.

$$\frac{x^2-7x+9}{(x+2)(x-1)^2} = \frac{A}{x+2} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$
$$= \frac{A(x-1)^2 + B(x+2)(x-1) + C(x+2)}{(x+2)(x-1)^2}$$

Hence $x^2-7x+9 = A(x-1)^2 + B(x+2)(x-1) + C(x+2)$.

Put $x=1$ $\Rightarrow$ $3 = A(0)^2 + B(3)(0) + C(3)$
$\Rightarrow$ $3C = 3$
$\Rightarrow$ $C = 1$

Put $x=-2$ $\Rightarrow$ $27 = A(-3)^2 + B(0)(-3) + C(0)$
$\Rightarrow$ $9A = 27$
$\Rightarrow$ $A = 3$

To find the value of B, we must substitute a third number for x. It is convenient to use a simple number, such as $x=0$.

Put $x=0$ $\Rightarrow$ $9 = A(-1)^2 + B(2)(-1) + C(2)$
$\Rightarrow$ $A - 2B + 2C = 9$
$\Rightarrow$ $3 - 2B + 2 = 9$
$\Rightarrow$ $5 - 2B = 9$
$\Rightarrow$ $-2B = 4$
$\Rightarrow$ $B = -2$

Hence $\frac{x^2-7x+9}{(x+2)(x-1)^2} = \frac{3}{x+2} - \frac{2}{x-1} + \frac{1}{(x-1)^2}$.

$iii.\ \dfrac{x^2-3}{(x+2)(x^2+1)}$

Solution:

$$\frac{x^2-3}{(x+2)(x^2+1)}$$

$$Let\ \frac{x^2-3}{(x+2)(x^2+1)}=\frac{A}{x+2}+\frac{Bx+C}{x^2+1}$$

$Multiply\ both\ sides\ with\ (x+2)(x^2+1)$

$$x^2-3=A(x^2+1)+(Bx+C)(x+2)$$

$If\ x=-2,\ then$

$$(-2)^2-3=A([-2]^2+1)+[(B[-2]+C)([-2]+2)]$$

$$1=A(5)+0\Rightarrow A(5)=1\Rightarrow A=\frac{1}{5}$$

$Equating\ the\ coefficients\ of\ x^2$

$$1=A+B\Rightarrow B=1-A=1-\frac{1}{5}=\frac{4}{5}$$

$$\Rightarrow B=\frac{4}{5}$$

$Equating\ the\ constants$

$$-3=A+2C\Rightarrow 2C=-3-\frac{1}{5}=\frac{-16}{5}$$

$$\Rightarrow C=\frac{-16}{5\times 2}=\frac{-8}{5}\Rightarrow C=\frac{-8}{5}$$

$$\therefore\frac{x^2-3}{(x+2)(x^2+1)}=\frac{1}{5(x+2)}+\frac{4x-8}{5(x^2+1)}$$

Partial Fractions: Repeated Factors: Solutions

Find the following integrals by first writing each integral as a series of partial fractions

Exercise 1

$$\int \frac{4x^2+3x+6}{x^2(x+2)}\,dx$$

$$\frac{4x^2+3x+6}{x^2(x+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2}$$

$$= \frac{Ax(x+2)+B(x+2)+Cx^2}{x^2(x+2)}$$

Equating numerators:

$$4x^2+3x+6 = Ax(x+2) + B(x+2) + Cx^2$$

When $x=0$:
$6 = 2B$, so $B=3$

When $x = -2$:
$16 = 4C$, so $C = 4$

Now to find A, either choose another value of x, or equate one of the coefficients e.g. x

$3 = 2A + B$
$3 = 2A + 3$ (B=3)
so $A = 0$

Thus

$$\frac{4x^2+3x+6}{x^2(x+2)} = \frac{3}{x^2} + \frac{4}{x+2}$$

and

$$\int \frac{4x^2+3x+6}{x^2(x+2)}\,dx$$

$$= \int \frac{3}{x^2} + \frac{4}{x+2}\,dx$$

$$= \frac{-3}{x} + 4\ln|x+2| + C$$

Exercise 2

$$\int \frac{4}{(x-1)(x+1)^2}\,dx$$

$$\frac{4}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2}$$

$$= \frac{A(x+1)^2+B(x-1)(x+1)+C(x-1)}{(x-1)(x+1)^2}$$

Equating numerators:

$$4 = A(x+1)^2+B(x-1)(x+1)+C(x-1)$$

When $x=1$:
$4 = 4A$, so $A = 1$

When $x = -1$:
$4 = -2C$, so $C = -2$

Now to find B, either choose another value of x, or equate one of the coefficients e.g. x^2:

$0 = A + B$
$0 = 1 + B$, so $B = -1$

Thus

$$\frac{4}{(x-1)(x+1)^2} = \frac{1}{x-1} + \frac{-1}{(x+1)} + \frac{-2}{(x+1)^2}$$

and

$$\int \frac{4}{(x-1)(x+1)^2}\,dx$$

$$= \int \frac{1}{x-1} + \frac{-1}{(x+1)} + \frac{-2}{(x+1)^2}\,dx$$

$$= \ln|x-1| - \ln|x+1| + \frac{2}{(x+1)} + C$$

Exercise 3

$$\int \frac{3x^3-2x-2}{(1+x)x^3}\,dx$$

$$\frac{3x^3-2x-2}{(1+x)x^3} = \frac{A}{(1+x)} + \frac{B}{x} + \frac{C}{x^2} + \frac{D}{x^3}$$

$$= \frac{Ax^3+Bx^2(1+x)+Cx(1+x)+D(1+x)}{(1+x)x^3}$$

Equating numerators:

$$3x^3-2x-2 = Ax^3+Bx^2(1+x)+Cx(1+x)+D(1+x)$$

When $x = -1$:
$-3 = -A$, so $A = 3$

Equating coefficients

x^3: $3 = A + B$, so $B=0$

x^2: $0 = C$

x: $-2 = D$

Thus

$$\frac{3x^3-2x-2}{(1+x)x^3} = \frac{3}{(1+x)} + \frac{-2}{x^3}$$

and

$$\int \frac{3x^3-2x-2}{(1+x)x^3}\,dx$$

$$= \int \frac{3}{(1+x)} + \frac{-2}{x^3}\,dx$$

$$= 3\ln|1+x| + \frac{1}{x^2} + C$$

Example 11.29

Evaluate : (i) $\int \frac{3x+7}{x^2-3x+2}\,dx$ (ii) $\int \frac{x+3}{(x+2)^2(x+1)}\,dx.$

Solution

(i)
$$\int \frac{3x+7}{x^2-3x+2}\,dx = \int \frac{13}{x-2}\,dx - \int \frac{10}{x-1}\,dx \quad \left\{\begin{array}{l}\text{Resolving into}\\ \text{partial fractions}\end{array}\right.$$
$$= 13\log|x-2| - 10\log|x-1| + c$$

(ii)
$$\int \frac{x+3}{(x+2)^2(x+1)}\,dx = \int \frac{-2}{x+2}\,dx - \int \frac{1}{(x+2)^2}\,dx + \int \frac{2}{x+1}\,dx \quad \left\{\begin{array}{l}\text{Resolving into}\\ \text{partial fractions}\end{array}\right.$$
$$= -2\int \frac{1}{x+2}\,dx - \int \frac{1}{(x+2)^2}\,dx + 2\int \frac{1}{x+1}\,dx$$
$$= -2\log|x+2| - \int (x+2)^{-2}\,dx + 2\log|x+1| + c$$
$$= -2\log|x+2| + \frac{1}{x+2} + 2\log|x+1| + c.$$

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$\int x\sin(x)dx$

Solution

$-x\cos(x)+\sin(x)+C$

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Solution steps

$\int x\sin(x)dx$

Apply Integration By Parts: $-x\cos(x)-\int-\cos(x)dx$

$=-x\cos(x)-\int-\cos(x)dx$

$\int-\cos(x)dx=-\sin(x)$

$=-x\cos(x)-(-\sin(x))$

Simplify

$=-x\cos(x)+\sin(x)$

Add a constant to the solution

$=-x\cos(x)+\sin(x)+C$

$\int x^2 \sin(x)dx$

Solution

$-x^2\cos(x) + 2(x\sin(x) + \cos(x)) + C$

Hide Step

Solution steps

$\int x^2 \sin(x)dx$

Apply Integration By Parts: $-x^2\cos(x) - \int -2x\cos(x)dx$

$= -x^2\cos(x) - \int -2x\cos(x)dx$

$\int -2x\cos(x)dx = -2(x\sin(x) + \cos(x))$

$= -x^2\cos(x) - (-2(x\sin(x) + \cos(x)))$

Simplify

$= -x^2\cos(x) + 2(x\sin(x) + \cos(x))$

Add a constant to the solution

$= -x^2\cos(x) + 2(x\sin(x) + \cos(x)) + C$

$\int x^2 \, cos \, x \, dx$

Steps Graph Related Examples

$\int x^2 \cos(x) dx$

Solution

$x^2 \sin(x) - 2(-x\cos(x) + \sin(x)) + C$

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Solution steps

$\int x^2 \cos(x) dx$

Apply Integration By Parts: $x^2 \sin(x) - \int 2x \sin(x) dx$

$= x^2 \sin(x) - \int 2x \sin(x) dx$

$\int 2x \sin(x) dx = 2(-x\cos(x) + \sin(x))$

$= x^2 \sin(x) - 2(-x\cos(x) + \sin(x))$

Add a constant to the solution

$= x^2 \sin(x) - 2(-x\cos(x) + \sin(x)) + C$

$$\int_{-\pi}^{\pi} x\sin(x)\,dx$$

Apply Integration By Parts: $\left[-x\cos(x) - \int -\cos(x)dx\right]_{-\pi}^{\pi}$

$$= \left[-x\cos(x) - \int -\cos(x)dx\right]_{-\pi}^{\pi}$$

$\int -\cos(x)dx = -\sin(x)$

$$= \left[-x\cos(x) - \left(-\sin(x)\right)\right]_{-\pi}^{\pi}$$

Simplify

$$= \left[-x\cos(x) + \sin(x)\right]_{-\pi}^{\pi}$$

Compute the boundaries: 2π

$$= 2\pi$$

$\int_0^{2\pi} x\sin(x)dx$

Apply Integration By Parts: $\left[-x\cos(x) - \int -\cos(x)dx\right]_0^{2\pi}$

$= \left[-x\cos(x) - \int -\cos(x)dx\right]_0^{2\pi}$

$\int -\cos(x)dx = -\sin(x)$

$= \left[-x\cos(x) - (-\sin(x))\right]_0^{2\pi}$

Simplify

$= \left[-x\cos(x) + \sin(x)\right]_0^{2\pi}$

Compute the boundaries: -2π

$= -2\pi$

$\int_0^{2\pi} x^2 \sin(x)\,dx$

Apply Integration By Parts: $\left[-x^2\cos(x) - \int -2x\cos(x)\,dx\right]_0^{2\pi}$

$= \left[-x^2\cos(x) - \int -2x\cos(x)\,dx\right]_0^{2\pi}$

$\int -2x\cos(x)\,dx = -2\left(x\sin(x) + \cos(x)\right)$

$= \left[-x^2\cos(x) - \left(-2\left(x\sin(x) + \cos(x)\right)\right)\right]_0^{2\pi}$

Simplify

$= \left[-x^2\cos(x) + 2\left(x\sin(x) + \cos(x)\right)\right]_0^{2\pi}$

Compute the boundaries: $-4\pi^2$

$= -4\pi^2$