Discrete Structures Spring 2024 – Week 6

Mathematical Induction

Lecture 1

Background

- Many mathematical statements assert that a property is true for all positive integers.

1. For every positive integer n: $n! \leq n^n$
2. $n^3 - n$ is divisible by 3
3. The sum of the first n positive integers is $n(n + 1)/2$.

Background

- Mathematical Induction is used to prove such type of results
- Mathematical induction can be used to prove statements that assert that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function.

Background

- Proofs using mathematical induction have two parts. First, they show that the statement holds for the positive integer 1. Second, they show that if the statement holds for a positive integer then it must also hold for the next larger integer.

- Mathematical induction is based on the rule of inference that tells us that if $P(1)$ and $\forall k(P(k) \rightarrow P(k+1))$ are true for the domain of positive integers, then $\forall n\, P(n)$ is true.

Climbing an Infinite Ladder

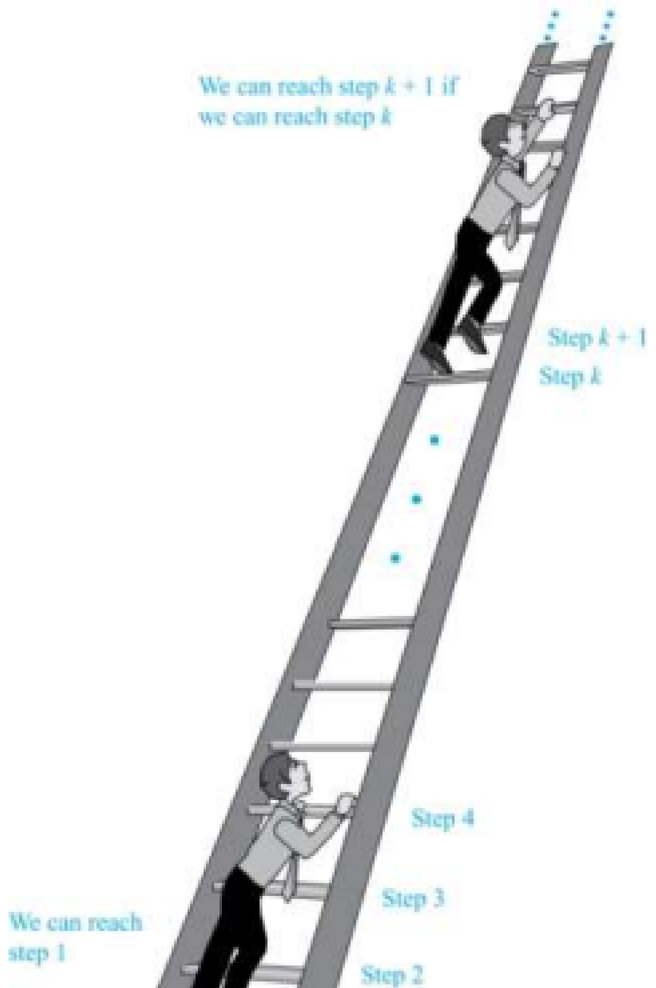
- Suppose we have an infinite ladder:

1. We can reach the first rung of the ladder.
2. If we can reach a particular rung of the ladder, then we can reach the next rung.

- From (1), we can reach the first rung. Then by applying (2), we can reach the second rung. Applying (2) again, the third rung. And so on. We can apply (2) any number of times to reach any particular rung, no matter how high up.

Climbing an Infinite Ladder

- This example motivates proof by mathematical induction.



Principle of Mathematical Induction

- To prove that P(n) is true for all positive integers n, we complete these steps:

- Basis Step: Show that P(1) is true.
- Inductive Step: Show that $P(k) \rightarrow P(k + 1)$ is true for all positive integers k.
  - To complete the inductive step, assuming the inductive hypothesis that P(k) holds for an arbitrary integer k, show that must P(k + 1) be true.

Principle of Mathematical Induction

- Climbing an Infinite Ladder Example:

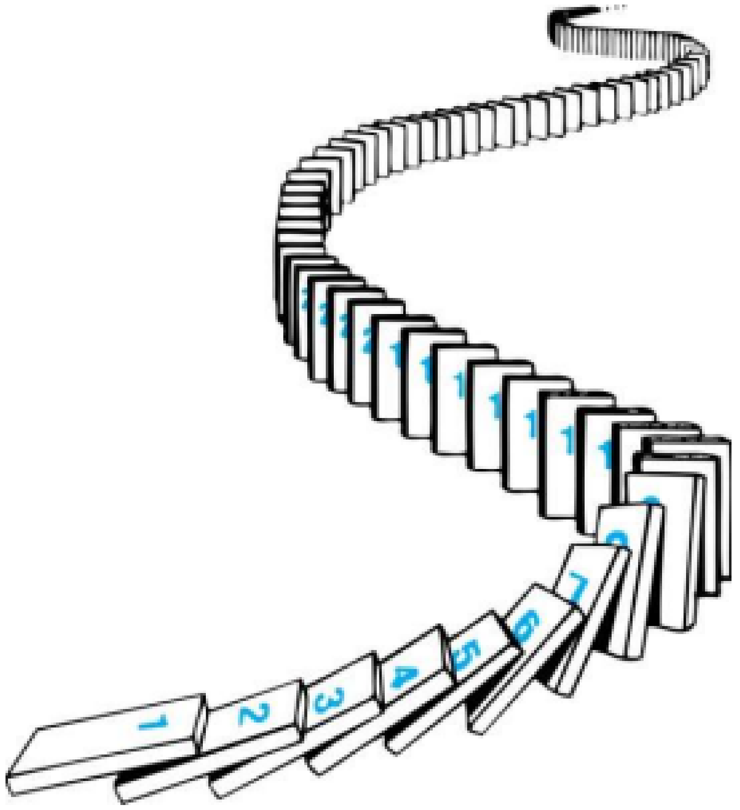
- Basis Step: By (1), we can reach rung 1.
- Inductive Step: Assume the inductive hypothesis that we can reach rung k. Then by (2), we can reach rung k + 1.
- Hence, $P(k) \rightarrow P(k + 1)$ is true for all positive integers k. We can reach every rung on the ladder.

Important Points About Using Mathematical Induction

- Mathematical induction can be expressed as the rule of inference
- $$(P(1) \land \forall k\,(P(k) \to P(k+1))) \to \forall n\, P(n),$$
- where the domain is the set of positive integers.
- In a proof by mathematical induction, we don’t assume that P(k) is true for all positive integers! We show that if we assume that P(k) is true, then P(k + 1) must also be true.
- Proofs by mathematical induction do not always start at the integer 1. In such a case, the basis step begins at a starting point b where b is an integer

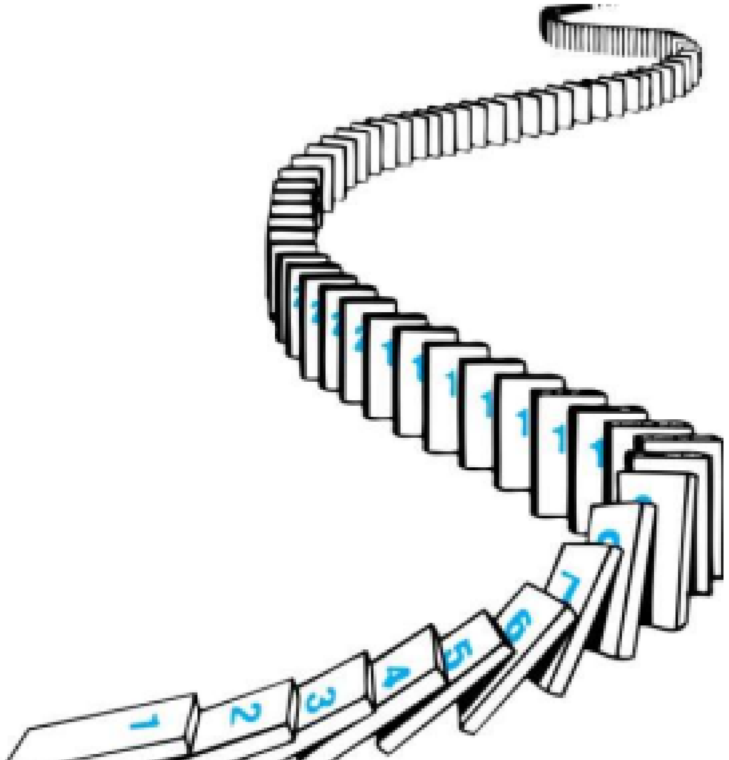
How Mathematical Induction Works – Another Example

- Consider an infinite sequence of dominoes, labeled 1,2,3, …, where each domino is standing
- Let P(n) be the proposition that the nth domino is knocked over
- We know that the first domino is knocked down, i.e., P(1) is true .



How Mathematical Induction Works – Another Example

- We also know that if whenever the k^{th} domino is knocked over, it knocks over the $(k + 1)^{st}$ domino, i.e, $P(k) \rightarrow P(k + 1)$ is true for all positive integers k.
- Hence, all dominos are knocked



over.

- P(n) is true for all positive integers n.

Using Mathematical Induction

- Use mathematical induction to show that
- $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$, for all nonnegative integers n.
- *Solution:* Let $P(n)$ be the proposition that $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$ for the integer n
- *Basis Step:* $P(0)$ is true because $2^0 = 2^{0+1} - 1$

- *Inductive Hypothesis:* For the inductive hypothesis, we assume that $P(k)$ is true for an arbitrary nonnegative integer k. That is, we assume that $1 + 2 + 2^2 + \cdots + 2^k = 2^{k+1} - 1$

Using Mathematical Induction

- To carry out the inductive step using this assumption, we must show that when we assume that $P(k)$ is true, then $P(k + 1)$ is also true.
- That is, we must show that $1 + 2 + 2^2 + \cdots + 2^k + 2^{k+1} = 2^{k+2} - 1$, assuming the inductive hypothesis $P(k)$.

- $1 + 2 + 2^2 + \cdots + 2^k + 2^{k+1} = (1 + 2 + 2^2 + \cdots + 2^k) + 2^{k+1}$
- $= (2^{k+1} - 1) + 2^{k+1}$ (inductive hypothesis)
- $= 2 \cdot 2^{k+1} - 1$
- $= 2^{k+2} - 1$

Proving a Summation Formula by Mathematical Induction

- Example: Show that: $\sum_{i=1}^{n} = \frac{n(n+1)}{2}$
- Basis Step: P(1) is true since $1(1 + 1)/2 = 1$.

- Inductive Step: Assume true for P(k).
- The inductive hypothesis is $\sum_{i=1}^{k} = \frac{k(k+1)}{2}$
- Under this assumption, $1+2+\ldots+k+(k+1) = \frac{k(k+1)}{2} + (k+1)$

$$= \frac{k(k+1)+2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

Recursively Defined Functions Lecture 2

Recursively Defined Functions

- Definition: A recursive or inductive definition of a function consists of two steps.
- Basis Step: Specify the value of the function at zero.
- Recursive Step: Give a rule for finding its value at an integer from its values at smaller integers.
- A function f(n) is the same as a sequence $a_0, a_1, \ldots$, where a_i, where $f(i) = a_i$

Recursively Defined Functions

- Example: Suppose f is defined by:
- $f(0) = 3$,
- $f(n + 1) = 2f(n) + 3$, Find $f(1)$, $f(2)$, $f(3)$, $f(4)$
- Solution: $f(1) = 2f(0) + 3 = 2 \cdot 3 + 3 = 9$
- $f(2) = 2f(1) + 3 = 2 \cdot 9 + 3 = 21$
- $f(3) = 2f(2) + 3 = 2 \cdot 21 + 3 = 45$

- $f(4) = 2f(3) + 3 = 2 \cdot 45 + 3 = 93$

Recursively Defined Functions

- Example: Give a recursive definition of the factorial function n!
- Solution:
- $f(0) = 1$
- $f(n + 1) = (n + 1) \cdot f(n)$

Recursively Defined Functions

$$\sum_{k=0}^{n} a_k$$

- Example: Give a recursive definition of: • Solution:
- The first part of the definition is

- The second part is

$$\sum_{k=0}^{0} a_k = a_0.$$

$$\sum_{k=0}^{n+1} a_k = \left(\sum_{k=0}^{n} a_k\right) + a_{n+1}.$$

Fibonacci Numbers

- Example : The Fibonacci numbers are defined as follows: • $f_0 = 0$, $f_1 = 1$, $f_n = f_{n-1} + f_{n-2}$
- Find f_2, f_3, f_4, f_5
- $f_2 = f_1 + f_0 = 1 + 0 = 1$

- $f_3 = f_2 + f_1 = 1 + 1 = 2$
- $f_4 = f_3 + f_2 = 2 + 1 = 3$
- $f_5 = f_4 + f = 3 + 2 = 5$