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ESTIMATION IN NONLINEAR REGRESSION WITH HARRIS RECURRENT MARKOV CHAINS

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In this paper, we study parametric nonlinear regression under the Harris recurrent Markov chain framework. We first consider the non-linear least squares estimators of the parameters in the homoskedastic case, and establish asymptotic theory for the proposed estimators. Our results show that the convergence rates for the estimators rely not only on the properties of the nonlinear regression function, but also on the number of regenerations for the Harris recurrent Markov chain. Furthermore, we discuss the estimation of the parameter vector in a conditional volatility function, and apply our results to the nonlinear regression with $I(1)$ processes and derive an asymptotic distribution theory which is comparable to that obtained by Park and Phillips (2001). Some numerical studies including simulation and empirical application are provided to examine the finite sample performance of the proposed approaches and results.

1. Introduction. In this paper, we consider a parametric nonlinear regression model defined by

$$
\begin{aligned}
Y_t &= g(X_t, \theta_{01}, \theta_{02}, \cdots, \theta_{0d}) + e_t \\
(1.1) \qquad &=: g(X_t, \boldsymbol{\theta}_0) + e_t, \quad t = 1, 2, \cdots, n,
\end{aligned}
$$

where $\boldsymbol{\theta}_0$ is the true value of the d-dimensional parameter vector such that

$$
\boldsymbol{\theta}_0 = (\theta_{01}, \theta_{02}, \cdots, \theta_{0d})^\tau \in \Theta \subset \mathbb{R}^d
$$

and $g(\cdot, \cdot) : \ \mathbb{R}^{d+1} \to \mathbb{R}$ is assumed to be known. Throughout this paper, we assume that Θ is a compact set and $\boldsymbol{\theta}_0$ lies in the interior of Θ, which is a standard assumption in the literature. How to construct a consistent estimator for the parameter vector $\boldsymbol{\theta}_0$ and derive an asymptotic theory are important issues in modern statistics and econometrics. When the observations (Y_t, X_t) satisfy stationarity and weak dependence conditions, there is an extensive literature on the theoretical analysis and empirical application

1

of the above parametric nonlinear model and its extension, see, for example, Jennrich (1969), Malinvaud (1970) and Wu (1981) for some early references, and Severini and Wong (1992), Lai (1994), Skouras (2000) and Li and Nie (2008) for recent relevant works.

As pointed out in the literature, assuming stationarity is too restrictive and unrealistic in many practical applications. When tackling economic and financial issues from a time perspective, we often deal with nonstationary components. For instance, neither the consumer price index nor the share price index, nor the exchange rates constitute a stationary process. A traditional method to handle such data is to take the first order difference to eliminate possible stochastic or deterministic trends involved in the data, and then do the estimation for a stationary model. However, such differencing may lead to loss of useful information. Thus, the development of a modeling technique that takes both nonstationary and nonlinear phenomena into account in time series analysis is crucial. Without taking differences, Park and Phillips (2001) (hereafter PP) study the nonlinear regression (1.1) with the regressor $\{X_t\}$ satisfying a unit root (or I(1)) structure, and prove that the rates of convergence of the nonlinear least squares (NLS) estimator of $\boldsymbol{\theta}_0$ depend on the properties of $g(\cdot, \cdot)$. For an integrable $g(\cdot, \cdot)$, the rate of convergence is as slow as $n^{1/4}$, and for an asymptotically homogeneous $g(\cdot, \cdot)$, the rate of convergence can achieve the $\sqrt{n}$-rate and even n-rate of convergence. More recently, Chan and Wang (2012) consider the same model structure as proposed in the PP paper and then establish some corresponding results under certain technical conditions which are weaker than those used in the PP paper.

As also pointed out in a recent paper by Myklebust *et al* (2012), the null recurrent Markov process is a nonlinear generalization of the linear unit root process, and thus provides a more flexible framework in data analysis. For example, Gao *et al* (2013) show that the exchange rates between British pound and US dollar over the time period between January 1988 and February 2011 are nonstationary but do not necessarily follow a linear unit root process (see also Bec *et al* 2008 for a similar discussion of the exchange rates between French franc and German mark over the time period between December 1972 and April 1988). Hence, Gao *et al* (2013) suggest using the nonlinear threshold autoregressive (TAR) with stationary and unit root regimes, which can be proved as a 1/2-null recurrent Markov process, see, for example, Example 2.1 in Section 2.2 and Example 6.1 in the empirical application (Section 6).

Under the framework of null recurrent Markov chains, there has been an extensive literature on nonparametric and semiparametric estimation

(Karlsen and Tjøstheim 2001; Karlsen *et al* 2007, 2010; Lin *et al* 2009; Schienle 2011; Chen *et al* 2012; Gao *et al* 2014), by using the technique of the split chain (Nummelin 1984; Meyn and Tweedie 2009), and the generalised ergodic theorem and functional limit theorem developed in Karlsen and Tjøstheim (2001). As far as we know, however, there is virtually no work on the parametric estimation of the nonlinear regression model (1.1) when the regressor $\{X_t\}$ is generated by a class of Harris recurrent Markov processes that includes both stationary and nonstationary cases. This paper aims to fill this gap. If the function $g(\cdot, \cdot)$ is integrable, we can directly use some existing results for functions of Harris recurrent Markov processes to develop an asymptotic theory for the estimator of $\boldsymbol{\theta}_0$. The case that $g(\cdot, \cdot)$ belongs to a class of asymptotically homogeneous functions is much more challenging, as in this case the function $g(\cdot, \cdot)$ is no longer bounded. In nonparametric or semiparametric estimation theory, we do not have such problems because the kernel function is usually assumed to be bounded and have a compact support. Unfortunately, most of the existing results for the asymptotic theory of the null recurrent Markov process focus on the case where $g(\cdot, \cdot)$ is bounded and integrable (c.f., Chen 1999, 2000). Hence, in this paper, we first modify the conventional NLS estimator for the asymptotically homogeneous $g(\cdot, \cdot)$, and then use a novel method to establish asymptotic distribution as well as rates of convergence for the modified parametric estimator. Our results show that the rates of convergence for the parameter vector in nonlinear cointegrating models rely not only on the properties of the function $g(\cdot, \cdot)$, but also on the magnitude of the regeneration number for the null recurrent Markov chain.

In addition, we also study two important issues, which are closely related to nonlinear mean regression with Harris recurrent Markov chains. The first one is to study the estimation of the parameter vector in a conditional volatility function and its asymptotic theory. As the estimation method is based on the log-transformation, the rates of convergence for the proposed estimator would depend on the property of the log-transformed volatility function and its derivatives. Meanwhile, we also discuss the nonlinear regression with $I(1)$ processes when $g(\cdot, \cdot)$ is asymptotically homogeneous. By using Theorem 3.2 in Section 3, we obtain asymptotic normality for the parametric estimator with a stochastic normalized rate, which is comparable to Theorem 5.2 in PP. However, our derivation is done under Markov perspective, which carries with it the potential of extending the theory to nonlinear and nonstationary autoregressive processes, which seems to be hard to do with the approach of PP.

The rest of this paper is organised as follows. Some preliminary results

about Markov theory (especially Harris recurrent Markov chain) and function classes are introduced in Section 2. The main results of this paper and their extensions are given in Sections 3 and 4, respectively. Some simulation studies are carried out in Section 5 and the empirical application is given in Section 6. Section 7 concludes the paper. The outline of the proofs of the main results is given in an appendix. The supplemental document (Li *et al* 2015) includes some additional simulated examples, the detailed proofs of the main results and the proofs of some auxiliary results.

2. Preliminary results.   To make the paper self-contained, in this section, we first provide some basic definitions and preliminary results for a Harris recurrent Markov process $\{X_t\}$, and then define function classes in a way similar to those introduced in PP.

2.1. *Markov theory.*   Let $\{X_t, t \geq 0\}$ be a ϕ-irreducible Markov chain on the state space $(\mathbb{E}, \mathcal{E})$ with transition probability P. This means that for any set $A \in \mathcal{E}$ with $\phi(A) > 0$, we have $\sum_{t=1}^{\infty} \mathsf{P}^t(x, A) > 0$ for $x \in \mathbb{E}$. We further assume that the ϕ-irreducible Markov chain $\{X_t\}$ is Harris recurrent.

DEFINITION 2.1.   *A Markov chain $\{X_t\}$ is Harris recurrent if, given a neighborhood $\mathbb{B}_v$ of v with $\phi(\mathbb{B}_v) > 0$, $\{X_t\}$ returns to $\mathbb{B}_v$ with probability one, $v \in \mathbb{E}$.*

The Harris recurrence allows one to construct a split chain, which decomposes the partial sum of functions of $\{X_t\}$ into blocks of independent and identically distributed (i.i.d.) parts and two asymptotically negligible remaining parts. Let τ_k be the regeneration times, n the number of observations and $N(n)$ the number of regenerations as in Karlsen and Tjøstheim (2001), where they use the notation $T(n)$ instead of $N(n)$. For the process $\{G(X_t) : t \geq 0\}$, defining

$$
Z_k = \begin{cases}
\sum_{t=0}^{\tau_0} G(X_t), & k = 0, \\
\sum_{t=\tau_{k-1}+1}^{\tau_k} G(X_t), & 1 \leq k \leq N(n), \\
\sum_{t=\tau_{N(n)}+1}^{n} G(X_t), & k = N(n) + 1,
\end{cases}
$$

where $G(\cdot)$ is a real function defined on $\mathbb{R}$, then we have

$$
(2.1) \qquad S_n(G) = \sum_{t=0}^{n} G(X_t) = Z_0 + \sum_{k=1}^{N(n)} Z_k + Z_{N(n)+1}.
$$

From Nummelin (1984), we know that $\{Z_k, k \geq 1\}$ is a sequence of i.i.d. random variables, and Z_0 and $Z_{N(n)+1}$ converge to zero almost surely (a.s.)

when they are divided by the number of regenerations $N(n)$ (using Lemma 3.2 in Karlsen and Tjøstheim 2001).

The general Harris recurrence only yields stochastic rates of convergence in asymptotic theory of the parametric and nonparametric estimators (see, for example, Theorems 3.1 and 3.2 below), where distribution and size of the number of regenerations $N(n)$ have no a priori known structure but fully depend on the underlying process $\{X_t\}$. To obtain a specific rate of $N(n)$ in our asymptotic theory for the null recurrent process, we next impose some restrictions on the tail behavior of the distribution of the recurrence times of the Markov chain.

DEFINITION 2.2. *A Markov chain $\{X_t\}$ is β-null recurrent if there exist a small nonnegative function f, an initial measure λ, a constant $\beta \in (0, 1)$, and a slowly varying function $L_f(\cdot)$ such that*

$$(2.2) \qquad E_\lambda \left(\sum_{t=1}^n f(X_t) \right) \sim \frac{1}{\Gamma(1+\beta)} n^\beta L_f(n),$$

where E_λ stands for the expectation with initial distribution λ and $\Gamma(1+\beta)$ is the Gamma function with parameter $1 + \beta$.

The definition of a small function f in the above definition can be found in some existing literature (c.f., pp.15 in Nummelin 1984). Assuming β-null recurrence restricts the tail behavior of the recurrence time of the process to be a regularly varying function. In fact, for all small functions f, by Lemma 3.1 in Karlsen and Tjøstheim (2001), we can find an $L_s(\cdot)$ such that (2.2) holds for the β-null recurrent Markov chain with $L_f(\cdot) = \pi_s(f)L_s(\cdot)$, where $\pi_s(\cdot)$ is an invariant measure of the Markov chain $\{X_t\}$, $\pi_s(f) = \int f(x)\pi_s(dx)$ and s is the small function in the minorization inequality (3.4) of Karlsen and Tjøstheim (2001). Letting $L_s(n) = L_f(n)/(\pi_s(f))$ and following the argument in Karlsen and Tjøstheim (2001), we may show that the regeneration number $N(n)$ of the β-null recurrent Markov chain $\{X_t\}$ has the following asymptotic distribution

$$(2.3) \qquad \frac{N(n)}{n^\beta L_s(n)} \xrightarrow{d} M_\beta(1),$$

where $M_\beta(t)$, $t \geq 0$ is the Mittag-Leffler process with parameter β (c.f., Kasahara 1984). Since $N(n) < n$ a.s. for the null recurrent case by (2.3), the rates of convergence for the nonparametric kernel estimators are slower than those for the stationary time series case (c.f., Karlsen *et al* 2007; Gao *et al* 2014). However, this is not necessarily the case for the parametric

estimator in our model (1.1). In Section 3 below, we will show that our rate of convergence in the null recurrent case is slower than that for the stationary time series for integrable $g(\cdot, \cdot)$ and may be faster than that for the stationary time series case for asymptotically homogeneous $g(\cdot, \cdot)$. In addition, our rates of convergence also depend on the magnitude of β, which measures the recurrence times of the Markov chain $\{X_t\}$.

2.2. *Examples of β-null recurrent Markov chains.* For a stationary or positive recurrent process, $\beta = 1$. We next give several examples of β-null recurrent Markov chains with $0 < \beta < 1$.

EXAMPLE 2.1. (1/2-null recurrent Markov chain).
(i) Let a random walk process be defined as

$$(2.4) \qquad X_t = X_{t-1} + x_t, \quad t = 1, 2, \cdots, \quad X_0 = 0,$$

where $\{x_t\}$ is a sequence of i.i.d. random variables with $\mathsf{E}[x_1] = 0$, $0 < \mathsf{E}[x_1^2] < \infty$ and $\mathsf{E}[|x_1|^4] < \infty$, and the distribution of x_t is absolutely continuous (with respect to the Lebesgue measure) with the density function $f_0(\cdot)$ satisfying $\inf_{x \in \mathbb{C}_0} f_0(x) > 0$ for all compact sets $\mathbb{C}_0$. Some existing papers including Kallianpur and Robbins (1954) have shown that $\{X_t\}$ defined by (2.4) is a 1/2-null recurrent Markov chain.
(ii) Consider a parametric TAR model of the form:

$$(2.5) \quad X_t = \alpha_1 X_{t-1} I\big(X_{t-1} \in \mathbb{S}\big) + \alpha_2 X_{t-1} I\big(X_{t-1} \in \mathbb{S}^c\big) + x_t, \quad X_0 = 0,$$

where $\mathbb{S}$ is a compact subset of $\mathbb{R}$, $\mathbb{S}^c$ is the complement of $\mathbb{S}$, $\alpha_2 = 1$, $-\infty < \alpha_1 < \infty$, $\{x_t\}$ satisfies the corresponding conditions in Example 2.1(i) above. Recently, Gao *et al* (2013) have shown that such a TAR process $\{X_t\}$ is a 1/2-null recurrent Markov chain. Furthermore, we may generalize the TAR model (2.5) to

$$X_t = H(X_{t-1}, \boldsymbol{\zeta}) I\big(X_{t-1} \in \mathbb{S}\big) + X_{t-1} I\big(X_{t-1} \in \mathbb{S}^c\big) + x_t,$$

where $X_0 = 0$, $\sup_{x \in \mathbb{S}} |H(x, \boldsymbol{\zeta})| < \infty$ and $\boldsymbol{\zeta}$ is a parameter vector. According to Teräsvirta *et al* (2010), the above autoregressive process is also a 1/2-null recurrent Markov chain.

EXAMPLE 2.2. (β-null recurrent Markov chain with $\beta \neq 1/2$).
Let $\{x_t\}$ be a sequence of i.i.d. random variables taking positive values, and $\{X_t\}$ be defined as

$$X_t = \begin{cases} X_{t-1} - 1, & X_{t-1} > 1, \\ x_t, & X_{t-1} \in [0, 1], \end{cases}$$

for $t \geq 1$, and $X_0 = C_0$ for some positive constant C_0. Myklebust *et al* (2012) prove that $\{X_t\}$ is β-null recurrent if and only if

$$\mathsf{P}([x_1] > n) \sim n^{-\beta} l^{-1}(n), \quad 0 < \beta < 1,$$

where $[\,\cdot\,]$ is the integer function and $l(\cdot)$ is a slowly varying positive function.

From the above examples, the β-null recurrent Markov chain framework is not restricted to linear processes (see Example 2.1(ii)). Furthermore, such a null recurrent class has the invariance property that if $\{X_t\}$ is β-null recurrent, then for a one-to-one transformation $\mathcal{T}(\cdot)$, $\{\mathcal{T}(X_t)\}$ is also β-null recurrent (c.f., Teräsvirta *et al* 2010). Such invariance property does not hold for the $I(1)$ processes. For other examples of the β-null recurrent Markov chain, we refer to Example 1 in Schienle (2011). For some general conditions on diffusion processes to ensure the Harris recurrence is satisfied, we refer to Höpfner and Löcherbach (2003) and Bandi and Phillips (2009).

2.3. *Function classes.* Similarly to Park and Phillips (1999, 2001), we consider two classes of parametric nonlinear functions: integrable functions and asymptotically homogeneous functions, which include many commonly-used functions in nonlinear regression. Let $\|\mathbf{A}\| = \sqrt{\sum_{i=1}^{q} \sum_{j=1}^{q} a_{ij}^2}$ for $\mathbf{A} = (a_{ij})_{q \times q}$, and $\|\mathbf{a}\|$ be the Euclidean norm of vector $\mathbf{a}$. A function $h(x) : \mathbb{R} \to \mathbb{R}^d$ is π_s-integrable if

$$\int_{\mathbb{R}} \|h(x)\| \pi_s(dx) < \infty,$$

where $\pi_s(\cdot)$ is the invariant measure of the Harris recurrent Markov chain $\{X_t\}$. When $\pi_s(\cdot)$ is differentiable such that $\pi_s(dx) = p_s(x)dx$, $h(x)$ is π_s-integrable if and only if $h(x)p_s(x)$ is integrable, where $p_s(\cdot)$ is the invariant density function for $\{X_t\}$. For the random walk case as in Example 2.1(i), the π_s-integrability reduces to the conventional integrability as $\pi_s(dx) = dx$.

DEFINITION 2.3. *A d-dimensional vector function $h(x, \boldsymbol{\theta})$ is said to be integrable on Θ if for each $\boldsymbol{\theta} \in \Theta$, $h(x, \boldsymbol{\theta})$ is π_s-integrable and there exist a neighborhood $\mathbb{B}_{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$ and $M : \mathbb{R} \to \mathbb{R}$ bounded and π_s-integrable such that $\|h(x, \boldsymbol{\theta}') - h(x, \boldsymbol{\theta})\| \leq \|\boldsymbol{\theta}' - \boldsymbol{\theta}\| M(x)$ for any $\boldsymbol{\theta}' \in \mathbb{B}_{\boldsymbol{\theta}}$.*

The above definition is comparable to Definition 3.3 in PP. However, in our definition, we do not need condition (b) in Definition 3.3 of their paper, which makes the integrable function family in this paper slightly more general. We next introduce a class of asymptotically homogeneous functions.

DEFINITION 2.4. *For a d-dimensional vector function $h(x, \boldsymbol{\theta})$, let $h(\lambda x, \boldsymbol{\theta}) = \kappa(\lambda, \boldsymbol{\theta})H(x, \boldsymbol{\theta}) + R(x, \lambda, \boldsymbol{\theta})$, where $\kappa(\cdot, \cdot)$ is nonzero. $h(\lambda x, \boldsymbol{\theta})$ is said to be asymptotically homogeneous on Θ if the following two conditions are satisfied: (i) $H(\cdot, \boldsymbol{\theta})$ is locally bounded uniformly for any $\boldsymbol{\theta} \in \Theta$ and continuous with respect to $\boldsymbol{\theta}$; (ii) the remainder term $R(x, \lambda, \boldsymbol{\theta})$ is of order smaller than $\kappa(\lambda, \boldsymbol{\theta})$ as $\lambda \to \infty$ for any $\boldsymbol{\theta} \in \Theta$. As in PP, $\kappa(\cdot, \cdot)$ is the asymptotic order of $h(\cdot, \cdot)$ and $H(\cdot, \cdot)$ is the limit homogeneous function.*

The above definition is quite similar to that of an H-regular function in PP except that the regularity condition (a) in Definition 3.5 of PP is replaced by the local boundness condition (i) in Definition 2.4. Following Definition 3.4 in PP, as $R(x, \lambda, \boldsymbol{\theta})$ is of order smaller than $\kappa(\cdot, \cdot)$, we have either

$$(2.6) \qquad\qquad R(x, \lambda, \boldsymbol{\theta}) = a(\lambda, \boldsymbol{\theta})A_R(x, \boldsymbol{\theta})$$

or

$$(2.7) \qquad\qquad R(x, \lambda, \boldsymbol{\theta}) = b(\lambda, \boldsymbol{\theta})A_R(x, \boldsymbol{\theta})B_R(\lambda x, \boldsymbol{\theta}),$$

where $a(\lambda, \boldsymbol{\theta}) = o(\kappa(\lambda, \boldsymbol{\theta}))$, $b(\lambda, \boldsymbol{\theta}) = O(\kappa(\lambda, \boldsymbol{\theta}))$ as $\lambda \to \infty$, $\sup_{\boldsymbol{\theta} \in \Theta} A_R(\cdot, \boldsymbol{\theta})$ is locally bounded, and $\sup_{\boldsymbol{\theta} \in \Theta} B_R(\cdot, \boldsymbol{\theta})$ is bounded and vanishes at infinity.

Note that the above two definitions can be similarly generalized to the case that $h(\cdot, \cdot)$ is a $d \times d$ matrix of functions. Details are omitted here to save space. Furthermore, when the process $\{X_t\}$ is positive recurrent, an asymptotically homogeneous function $h(x, \boldsymbol{\theta})$ might be also integrable on Θ as long as the density function of the process $p_s(x)$ is integrable and decreases to zero sufficiently fast when x diverges to infinity.

3. Main results. In this section, we establish some asymptotic results for the parametric estimators of $\boldsymbol{\theta}_0$ when $g(\cdot, \cdot)$ and its derivatives belong to the two classes of functions introduced in Section 2.3.

3.1. *Integrable function on Θ.* We first consider estimating model (1.1) by the NLS approach, which is also used by PP in the unit root framework. Define the loss function by

$$(3.1) \qquad\qquad L_{n,g}(\boldsymbol{\theta}) = \sum_{t=1}^{n} (Y_t - g(X_t, \boldsymbol{\theta}))^2.$$

We can obtain the resulting estimator $\widehat{\boldsymbol{\theta}}_n$ by minimizing $L_{n,g}(\boldsymbol{\theta})$ over $\boldsymbol{\theta} \in \Theta$, i.e.,

$$(3.2) \qquad\qquad \widehat{\boldsymbol{\theta}}_n = \arg\min_{\boldsymbol{\theta} \in \Theta} L_{n,g}(\boldsymbol{\theta}).$$

For $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_d)^\tau$, let

$$\dot{g}(x, \boldsymbol{\theta}) = \left(\frac{\partial g(x, \boldsymbol{\theta})}{\partial \theta_j} \right)_{d \times 1}, \quad \ddot{g}(x, \boldsymbol{\theta}) = \left(\frac{\partial^2 g(x, \boldsymbol{\theta})}{\partial \theta_i \partial \theta_j} \right)_{d \times d}.$$

Before deriving the asymptotic properties of $\widehat{\boldsymbol{\theta}}_n$ when $g(\cdot, \cdot)$ and its derivatives are integrable on Θ, we give some regularity conditions.

ASSUMPTION 3.1. (i) $\{X_t\}$ is a Harris recurrent Markov chain with invariant measure $\pi_s(\cdot)$.

(ii) $\{e_t\}$ is a sequence of i.i.d. random variables with mean zero and finite variance σ^2, and is independent of $\{X_t\}$.

ASSUMPTION 3.2. (i) $g(x, \boldsymbol{\theta})$ is integrable on Θ, and for all $\boldsymbol{\theta} \neq \boldsymbol{\theta}_0$, $\int \left[g(x, \boldsymbol{\theta}) - g(x, \boldsymbol{\theta}_0) \right]^2 \pi_s(dx) > 0$.

(ii) Both $\dot{g}(x, \boldsymbol{\theta})$ and $\ddot{g}(x, \boldsymbol{\theta})$ are integrable on Θ, and the matrix

$$\ddot{L}(\boldsymbol{\theta}) := \int \dot{g}(x, \boldsymbol{\theta}) \dot{g}^\tau(x, \boldsymbol{\theta}) \pi_s(dx)$$

is positive definite when $\boldsymbol{\theta}$ is in a neighborhood of $\boldsymbol{\theta}_0$.

REMARK 3.1. In Assumption 3.1(i), $\{X_t\}$ is assumed to be Harris recurrent, which includes both the positive and null recurrent Markov chains. The i.i.d. restriction on $\{e_t\}$ in Assumption 3.1(ii) may be replaced by the condition that $\{e_t\}$ is an irreducible, ergodic and strongly mixing process with mean zero and certain restriction on the mixing coefficient and moment conditions (c.f., Theorem 3.4 in Karlsen *et al* 2007). Hence, under some mild conditions, $\{e_t\}$ can include the well-known AR and ARCH processes as special examples. However, for this case, the techniques used in the proofs of Theorems 3.1 and 3.2 below need to be modified by noting that the compound process $\{X_t, e_t\}$ is Harris recurrent. Furthermore, the homoskedasticity on the error term can also be relaxed, and we may allow the existence of certain heteroskedasticity structure, i.e., $e_t = \sigma(X_t)\eta_t$, where $\sigma^2(\cdot)$ is the conditional variance function and $\{\eta_t\}$ satisfies Assumption 3.1(ii) with a unit variance. However, the property of the function $\sigma^2(\cdot)$ would affect the convergence rates given in the following asymptotic results. For example, to ensure the validity of Theorem 3.1, we need to further assume that $\sigma^2(\cdot)$ is π_s-integrable, which indicates that $\|\dot{g}(x, \boldsymbol{\theta})\|^2 \sigma^2(x)$ is integrable on Θ. As in the literature (c.f., Karlsen *et al* 2007), we need to assume the independence between $\{X_t\}$ and $\{\eta_t\}$.

Assumption 3.2 is quite standard and similar to the corresponding conditions in PP. In particular, Assumption 3.2(i) is a key condition to derive the global consistency of the NLS estimator $\widehat{\boldsymbol{\theta}}_n$.

We next give the asymptotic properties of $\widehat{\boldsymbol{\theta}}_n$. The following theorem is applicable for both stationary (positive recurrent) and nonstationary (null recurrent) time series.

THEOREM 3.1.   *Let Assumptions 3.1 and 3.2 hold.*
(a) The solution $\widehat{\boldsymbol{\theta}}_n$ which minimizes the loss function $L_{n,g}(\boldsymbol{\theta})$ over Θ is consistent, i.e.,

$$(3.3) \qquad\qquad \widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0 = o_P(1).$$

(b) The estimator $\widehat{\boldsymbol{\theta}}_n$ has an asymptotically normal distribution of the form:

$$(3.4) \qquad\qquad \sqrt{N(n)}\left(\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0\right) \xrightarrow{d} \mathsf{N}\left(\mathbf{0}_d, \sigma^2 \ddot{L}^{-1}(\boldsymbol{\theta}_0)\right),$$

where $\mathbf{0}_d$ is a d-dimensional null vector.

REMARK 3.2. Theorem 3.1 shows that $\widehat{\boldsymbol{\theta}}_n$ is asymptotically normal with a stochastic convergence rate $\sqrt{N(n)}$ for both the stationary and nonstationary cases. However, $N(n)$ is usually unobservable and its specific rate depends on β and $L_s(\cdot)$ if $\{X_t\}$ is β-null recurrent (see Corollary 3.2 below). We next discuss how to link $N(n)$ with a directly observable hitting time. Indeed, if $\mathbb{C} \in \mathcal{E}$ and $I_{\mathbb{C}}$ has a ϕ-positive support, the number of times that the process visits $\mathbb{C}$ up to the time n is defined by $N_{\mathbb{C}}(n) = \sum_{t=1}^n I_{\mathbb{C}}(X_t)$. By Lemma 3.2 in Karlsen and Tjøstheim (2001), we have

$$(3.5) \qquad\qquad \frac{N_{\mathbb{C}}(n)}{N(n)} \longrightarrow \pi_s(\mathbb{C}) \quad a.s.,$$

if $\pi_s(\mathbb{C}) = \pi_s I_{\mathbb{C}} = \int_{\mathbb{C}} \pi_s(dx) < \infty$. A possible estimator of β is

$$(3.6) \qquad\qquad \widehat{\beta} = \frac{\ln N_{\mathbb{C}}(n)}{\ln n},$$

which is strongly consistent as shown by Karlsen and Tjøstheim (2001). However, it is usually of somewhat limited practical use due to the slow convergence rate (c.f., Remark 3.7 of Karlsen and Tjøstheim 2001). A simulated example is given in Appendix B of the supplemental document to discuss the finite sample performance of the estimation method in (3.6).

By (3.5) and Theorem 3.1, we can obtain the following corollary directly.

COROLLARY 3.1. *Suppose that the conditions of Theorem 3.1 are satisfied, and let $\mathbb{C} \in \mathcal{E}$ such that $I_{\mathbb{C}}$ has a ϕ-positive support and $\pi_s(\mathbb{C}) < \infty$. Then the estimator $\widehat{\boldsymbol{\theta}}_n$ has an asymptotically normal distribution of the form:*

$$(3.7) \qquad \sqrt{N_{\mathbb{C}}(n)} \left(\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0 \right) \xrightarrow{d} N \left(\mathbf{0}_d, \sigma^2 \ddot{L}_{\mathbb{C}}^{-1}(\boldsymbol{\theta}_0) \right),$$

where $\ddot{L}_{\mathbb{C}}(\boldsymbol{\theta}_0) = \pi_s^{-1}(\mathbb{C})\ddot{L}(\boldsymbol{\theta}_0)$.

REMARK 3.3. In practice, we may choose $\mathbb{C}$ as a compact set such that $\phi(\mathbb{C}) > 0$ and $\pi_s(\mathbb{C}) < \infty$. In the additional simulation study (Example B.1) given in the supplemental document, for two types of 1/2-null recurrent Markov processes, we choose $\mathbb{C} = [-A, A]$ with the positive constant A carefully chosen, which works well in our setting. If $\pi_s(\cdot)$ has a continuous derivative function $p_s(\cdot)$, we can show that

$$\ddot{L}_{\mathbb{C}}(\boldsymbol{\theta}_0) = \int \dot{g}(x, \boldsymbol{\theta}_0)\dot{g}^\tau(x, \boldsymbol{\theta}_0)p_{\mathbb{C}}(x)dx \quad \text{with} \quad p_{\mathbb{C}}(x) = p_s(x)/\pi_s(\mathbb{C}).$$

The density function $p_{\mathbb{C}}(x)$ can be estimated by the kernel method. Then, replacing $\boldsymbol{\theta}_0$ by the NLS estimated value, we can obtain a consistent estimate for $\ddot{L}_{\mathbb{C}}(\boldsymbol{\theta}_0)$. Note that $N_{\mathbb{C}}(n)$ is observable and σ^2 can be estimated by calculating the variance of the residuals $\widehat{e}_t = Y_t - g(X_t, \widehat{\boldsymbol{\theta}}_n)$. Hence, for inference purposes, one may not need to estimate β and $L_s(\cdot)$ when $\{X_t\}$ is β-null recurrent, as $\ddot{L}_{\mathbb{C}}(\boldsymbol{\theta}_0)$, σ^2 and $N_{\mathbb{C}}(n)$ in (3.7) can be explicitly computed without knowing any information about β and $L_s(\cdot)$.

From (3.4) in Theorem 3.1 and (2.3) in Section 2.1 above, we have the following corollary.

COROLLARY 3.2. *Suppose that the conditions of Theorem 3.1 are satisfied. Furthermore, $\{X_t\}$ is a β-null recurrent Markov chain with $0 < \beta < 1$. Then we have*

$$(3.8) \qquad \widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0 = O_P\left(\frac{1}{\sqrt{n^\beta L_s(n)}} \right),$$

where $L_s(n)$ is defined in Section 2.1.

REMARK 3.4. As $\beta < 1$ and $L_s(n)$ is a slowly varying positive function, for the integrable case, the rate of convergence of $\widehat{\boldsymbol{\theta}}_n$ is slower than $\sqrt{n}$, the rate of convergence of the parametric NLS estimator in the stationary time series case. Combining (2.3) and Theorem 3.1, the result (3.8) can be strengthened to

$$(3.9) \qquad \sqrt{n^\beta L_s(n)}(\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0) \xrightarrow{d} \left[M_\beta^{-1}(1)\sigma^2 \ddot{L}^{-1}(\boldsymbol{\theta}_0) \right]^{1/2} \cdot N_d,$$

where N_d is a d-dimensional normal distribution with mean zero and covariance matrix being the identity matrix, which is independent of $M_\beta(1)$. A similar result is also obtained by Chan and Wang (2012). Corollary 3.2 and (3.9) complement the existing results on the rates of convergence of nonparametric estimators in β-null recurrent Markov processes (c.f., Karlsen *et al* 2007; Gao *et al* 2014). For the random walk case, which corresponds to $1/2$-null recurrent Markov chain, the rate of convergence is $n^{1/4}$, which is similar to a result obtained by PP for the processes that are of $I(1)$ type.

3.2. *Asymptotically homogeneous function on* Θ. We next establish an asymptotic theory for a parametric estimator of $\boldsymbol{\theta}_0$ when $g(\cdot,\cdot)$ and its derivatives belong to a class of asymptotically homogeneous functions. For a unit root process $\{X_t\}$, PP establish the consistency and limit distribution of the NLS estimator $\widehat{\boldsymbol{\theta}}_n$ by using the local time technique. Their method relies on the linear framework of the unit root process, the functional limit theorem of the partial sum process, and the continuous mapping theorem. The Harris recurrent Markov chain is a general process and allows for a possibly nonlinear framework, however. In particular, the null recurrent Markov chain can be seen as a nonlinear generalisation of the linear unit root process. Hence, the techniques used by PP for establishing the asymptotic theory is not applicable in such a possibly nonlinear Markov chain framework. Meanwhile, as mentioned in Section 1, the methods used to prove Theorem 3.1 cannot be applied here directly because the asymptotically homogeneous functions usually are not bounded and integrable. This leads to the violation of the conditions in the ergodic theorem when the process is null recurrent. In fact, most of the existing limit theorems for the null recurrent Markov process $h(X_t)$ (c.f., Chen, 1999, 2000) only consider the case where $h(\cdot)$ is bounded and integrable. Hence, it is quite challenging to extend Theorem 3.3 in PP to the case of general null recurrent Markov chains and establish an asymptotic theory for the NLS estimator for the case of asymptotically homogeneous functions.

To address the above concerns, we have to modify the NLS estimator $\widehat{\boldsymbol{\theta}}_n$. Let M_n be a positive and increasing sequence satisfying $M_n \to \infty$ as $n \to \infty$, but is dominated by a certain polynomial rate. We define the modified loss function by

$$(3.10) \qquad Q_{n,g}(\boldsymbol{\theta}) = \sum_{t=1}^{n} \left[Y_t - g(X_t, \boldsymbol{\theta})\right]^2 I\left(|X_t| \le M_n\right).$$

The modified NLS (MNLS) estimator $\overline{\boldsymbol{\theta}}_n$ can be obtained by minimizing

$Q_{n,g}(\boldsymbol{\theta})$ over $\boldsymbol{\theta} \in \Theta$,

$$(3.11) \qquad \overline{\boldsymbol{\theta}}_n = \arg\min_{\boldsymbol{\theta} \in \Theta} Q_{n,g}(\boldsymbol{\theta}).$$

The above truncation technique enables us to develop the limit theorems for the parametric estimate $\overline{\boldsymbol{\theta}}_n$ even when the function $g(\cdot, \cdot)$ or its derivatives are unbounded. A similar truncation idea is also used by Ling (2007) to estimate the ARMA-GARCH model when the second moment may not exist (it is called as the self-weighted method by Ling 2007). However, Assumption 2.1 in Ling (2007) indicates the stationarity for the model. The Harris recurrence considered in the paper is more general and includes both the stationary and nonstationary cases. As $M_n \to \infty$, for the integrable case discussed in Section 3.1, we can easily show that $\overline{\boldsymbol{\theta}}_n$ has the same asymptotic distribution as $\widehat{\boldsymbol{\theta}}_n$ under some regularity conditions. In Example 5.1 below, we compare the finite sample performance of these two estimators, and find that they are quite similar. Furthermore, when $\{X_t\}$ is positive recurrent, as mentioned in the last paragraph of Section 2.3, although the asymptotically homogeneous $g(x, \boldsymbol{\theta})$ and its derivatives are unbounded and not integrable on Θ, it may be reasonable to assume that $g(x, \boldsymbol{\theta})p_s(x)$ and its derivatives (with respect to $\boldsymbol{\theta}$) are integrable on Θ. In this case, Theorem 3.1 and Corollary 3.1 in Section 3.1 still hold for the estimation $\overline{\boldsymbol{\theta}}_n$ and the role of $N(n)$ (or $N_{\mathbb{C}}(n)$) is the same as that of the sample size, which implies that the root-n consistency in the stationary time series case can be derived. Hence, we only consider the null recurrent $\{X_t\}$ in the remaining subsection.

Let

$$B_i(1) = [i-1, i), \quad i = 1, 2, \cdots, [M_n], \quad B_{[M_n]+1}(1) = [[M_n], M_n],$$
$$B_i(2) = [-i, -i+1), \quad i = 1, 2, \cdots, [M_n], \quad B_{[M_n]+1}(2) = [-M_n, -[M_n]].$$

It is easy to check that $B_i(k)$, $i = 1, 2, \cdots, [M_n] + 1$, $k = 1, 2$, are disjoint, and $[-M_n, M_n] = \cup_{k=1}^{2} \cup_{i=1}^{[M_n]+1} B_i(k)$. Define

$$\zeta_0(M_n) = \sum_{i=0}^{[M_n]} \left[\pi_s(B_{i+1}(1)) + \pi_s(B_{i+1}(2)) \right],$$

and

$$
\begin{aligned}
\boldsymbol{\Lambda}_n(\boldsymbol{\theta}) &= \sum_{i=0}^{[M_n]} \dot{h}_g(\frac{i}{M_n},\boldsymbol{\theta})\dot{h}_g^\tau(\frac{i}{M_n},\boldsymbol{\theta})\pi_s(B_{i+1}(1)) + \\
&\quad \sum_{i=0}^{[M_n]} \dot{h}_g(\frac{-i}{M_n},\boldsymbol{\theta})\dot{h}_g^\tau(\frac{-i}{M_n},\boldsymbol{\theta})\pi_s(B_{i+1}(2)), \\
\widetilde{\Lambda}_n(\boldsymbol{\theta},\boldsymbol{\theta}_0) &= \sum_{i=0}^{[M_n]} \big[h_g(\frac{i}{M_n},\boldsymbol{\theta}) - h_g(\frac{i}{M_n},\boldsymbol{\theta}_0)\big]^2 \pi_s(B_{i+1}(1)) + \\
&\quad \sum_{i=0}^{[M_n]} \big[h_g(\frac{-i}{M_n},\boldsymbol{\theta}) - h_g(\frac{-i}{M_n},\boldsymbol{\theta}_0)\big]^2 \pi_s(B_{i+1}(2)),
\end{aligned}
$$

where $h_g(\cdot,\cdot)$ and $\dot{h}_g(\cdot,\cdot)$ will be defined in Assumption 3.3(i) below.

Some additional assumptions are introduced below to establish asymptotic properties for $\overline{\boldsymbol{\theta}}_n$.

ASSUMPTION 3.3. (i) $g(x,\boldsymbol{\theta})$, $\dot{g}(x,\boldsymbol{\theta})$ and $\ddot{g}(x,\boldsymbol{\theta})$ are asymptotically homogeneous on Θ with asymptotic orders $\kappa_g(\cdot)$, $\dot{\kappa}_g(\cdot)$ and $\ddot{\kappa}_g(\cdot)$, and limit homogeneous functions $h_g(\cdot,\cdot)$, $\dot{h}_g(\cdot,\cdot)$ and $\ddot{h}_g(\cdot,\cdot)$, respectively. Furthermore, the asymptotic orders $\kappa_g(\cdot)$, $\dot{\kappa}_g(\cdot)$ and $\ddot{\kappa}_g(\cdot)$ are independent of $\boldsymbol{\theta}$.

(ii) The function $h_g(\cdot,\boldsymbol{\theta})$ is continuous on the interval $[-1,1]$ for all $\boldsymbol{\theta} \in \Theta$. For all $\boldsymbol{\theta} \neq \boldsymbol{\theta}_0$, there exist a continuous $\widetilde{\Lambda}(\cdot,\boldsymbol{\theta}_0)$ which achieves unique minimum at $\boldsymbol{\theta} = \boldsymbol{\theta}_0$ and a sequence of positive numbers $\{\widetilde{\zeta}(M_n)\}$ such that

(3.12) $$\lim_{n\to\infty} \frac{1}{\widetilde{\zeta}(M_n)}\widetilde{\Lambda}_n(\boldsymbol{\theta},\boldsymbol{\theta}_0) = \widetilde{\Lambda}(\boldsymbol{\theta},\boldsymbol{\theta}_0).$$

For $\boldsymbol{\theta}$ in a neighborhood of $\boldsymbol{\theta}_0$, both $\dot{h}_g(\cdot,\boldsymbol{\theta})$ and $\ddot{h}_g(\cdot,\boldsymbol{\theta})$ are continuous on the interval $[-1,1]$ and there exist a continuous and positive definite matrix $\boldsymbol{\Lambda}(\boldsymbol{\theta})$ and a sequence of positive numbers $\{\zeta(M_n)\}$ such that

(3.13) $$\lim_{n\to\infty} \frac{1}{\zeta(M_n)}\boldsymbol{\Lambda}_n(\boldsymbol{\theta}) = \boldsymbol{\Lambda}(\boldsymbol{\theta}).$$

Furthermore, both $\zeta_0(M_n)/\widetilde{\zeta}(M_n)$ and $\zeta_0(M_n)/\zeta(M_n)$ are bounded, and $\zeta_0(l_n)/\zeta_0(M_n) = o(1)$ for $l_n \to \infty$ but $l_n = o(M_n)$.

(iii) The asymptotic orders $\kappa_g(\cdot)$, $\dot{\kappa}_g(\cdot)$ and $\ddot{\kappa}_g(\cdot)$ are positive and nondecreasing such that $\kappa_g(n) + \ddot{\kappa}_g(n) = O\big(\dot{\kappa}_g(n)\big)$ as $n \to \infty$.

(iv) For each $x \in [-M_n, M_n]$, $\mathcal{N}_x(1) := \{y : x - 1 < y < x + 1\}$ is a small set and the invariant density function $p_s(x)$ is bounded away from zero and infinity.

REMARK 3.5. Assumption 3.3(i) is quite standard, see, for example, condition (b) in Theorem 5.2 in PP. The restriction that the asymptotic orders are independent of $\boldsymbol{\theta}$ can be relaxed at the cost of more complicated assumptions and more lengthy proofs. For example, to ensure the global consistency of $\overline{\boldsymbol{\theta}}_n$, we need to assume that there exist $\epsilon_* > 0$ and a neighborhood $\mathbb{B}_{\boldsymbol{\theta}_1}$ of $\boldsymbol{\theta}_1$ for any $\boldsymbol{\theta}_1 \neq \boldsymbol{\theta}_0$ such that

$$\inf_{|p - \bar{p}| < \epsilon_*, \; |q - \bar{q}| < \epsilon_*} \; \inf_{\boldsymbol{\theta} \in \mathbb{B}_{\boldsymbol{\theta}_1}} \; \left| p\kappa_g(n, \boldsymbol{\theta}) - q\kappa_g(n, \boldsymbol{\theta}_0) \right| \to \infty$$

for $\bar{p}, \bar{q} > 0$. And to establish the asymptotic normality of $\overline{\boldsymbol{\theta}}_n$, we need to impose additional technical conditions on the asymptotic orders and limit homogeneous functions, similar to condition (b) in Theorem 5.3 of PP. The explicit forms of $\boldsymbol{\Lambda}(\boldsymbol{\theta})$, $\zeta(M_n)$, $\widetilde{\Lambda}(\boldsymbol{\theta}, \boldsymbol{\theta}_0)$ and $\widetilde{\zeta}(M_n)$ in Assumption 3.3(ii) can be derived for some special cases. For example, when $\{X_t\}$ is generated by a random walk process, we have $\pi_s(dx) = dx$ and

$$\begin{aligned}
\boldsymbol{\Lambda}_n(\boldsymbol{\theta}) &= (1 + o(1)) \sum_{i=-[M_n]}^{[M_n]} \dot{h}_g\left(\frac{i}{M_n}, \boldsymbol{\theta}\right) \dot{h}_g^\tau\left(\frac{i}{M_n}, \boldsymbol{\theta}\right) \\
&= (1 + o(1)) M_n \int_{-1}^1 \dot{h}_g(x, \boldsymbol{\theta}) \dot{h}_g^\tau(x, \boldsymbol{\theta}) dx,
\end{aligned}$$

which implies that $\zeta(M_n) = M_n$ and $\boldsymbol{\Lambda}(\boldsymbol{\theta}) = \int_{-1}^1 \dot{h}_g(x, \boldsymbol{\theta}) \dot{h}_g^\tau(x, \boldsymbol{\theta}) dx$ in (3.13). The explicit forms of $\widetilde{\Lambda}(\boldsymbol{\theta}, \boldsymbol{\theta}_0)$ and $\widetilde{\zeta}(M_n)$ can be derived similarly for the above two cases and details are thus omitted.

Define $\mathbf{J}_g(n, \boldsymbol{\theta}_0) = \dot{\kappa}_g^2(M_n)\zeta(M_n)\boldsymbol{\Lambda}(\boldsymbol{\theta}_0)$. We next establish an asymptotic theory for $\overline{\boldsymbol{\theta}}_n$ when $\{X_t\}$ is null recurrent.

THEOREM 3.2. *Let $\{X_t\}$ be a null recurrent Markov process, Assumptions 3.1(ii) and 3.3 hold.*
(a) The solution $\overline{\boldsymbol{\theta}}_n$ which minimizes the loss function $Q_{n,g}(\boldsymbol{\theta})$ over Θ is consistent, i.e.,

$$\tag{3.14} \overline{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0 = o_P(1).$$

(b) The estimator $\overline{\boldsymbol{\theta}}_n$ has the asymptotically normal distribution,

$$\tag{3.15} N^{1/2}(n)\mathbf{J}_g^{1/2}(n, \boldsymbol{\theta}_0)\left(\overline{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0\right) \overset{d}{\longrightarrow} N\left(\mathbf{0}_d, \sigma^2 I_d\right),$$

where I_d is a $d \times d$ identity matrix.

REMARK 3.6. From Theorem 3.2, the asymptotic distribution of $\overline{\boldsymbol{\theta}}_n$ for the asymptotically homogeneous regression function is quite different from that of $\widehat{\boldsymbol{\theta}}_n$ for the integrable regression function when the process is null recurrent. Such finding is comparable to those in PP. The choice of M_n in the estimation method and asymptotic theory will be discussed in Corollaries 3.3 and 3.4 below.

REMARK 3.7. As in Corollary 3.1, we can modify (3.15) for inference purposes. Define $\mathbf{J}_{g,\mathbb{C}}(n, \boldsymbol{\theta}_0) = \dot{\kappa}_g^2(M_n)\boldsymbol{\Lambda}_{n,\mathbb{C}}(\boldsymbol{\theta}_0)$, where

$$
\boldsymbol{\Lambda}_{n,\mathbb{C}}(\boldsymbol{\theta}) = \sum_{i=0}^{[M_n]} \dot{h}_g\left(\frac{i}{M_n}, \boldsymbol{\theta}\right)\dot{h}_g^{\tau}\left(\frac{i}{M_n}, \boldsymbol{\theta}\right)\frac{\pi_s(B_{i+1}(1))}{\pi_s(\mathbb{C})} +
$$
$$
\sum_{i=0}^{[M_n]} \dot{h}_g\left(\frac{-i}{M_n}, \boldsymbol{\theta}\right)\dot{h}_g^{\tau}\left(\frac{-i}{M_n}, \boldsymbol{\theta}\right)\frac{\pi_s(B_{i+1}(2))}{\pi_s(\mathbb{C})},
$$

where $\mathbb{C}$ satisfies the conditions in Corollary 3.1. Then, by (3.5) and (3.15), we can show that

$$
(3.16) \qquad N_{\mathbb{C}}^{1/2}(n)\mathbf{J}_{g,\mathbb{C}}^{1/2}(n, \boldsymbol{\theta}_0)\left(\overline{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0\right) \xrightarrow{d} \mathsf{N}\left(\mathbf{0}_d, \sigma^2 I_d\right).
$$

When $\{X_t\}$ is β-null recurrent, we can use the asymptotically normal distribution theory (3.16) to conduct statistical inference without knowing any information of β as $N_{\mathbb{C}}(n)$ is observable and $\mathbf{J}_{g,\mathbb{C}}(n, \boldsymbol{\theta}_0)$ can be explicitly computed through replacing $\boldsymbol{\Lambda}_{n,\mathbb{C}}(\boldsymbol{\theta}_0)$ by the plug-in estimated value.

From (3.15) in Theorem 3.2 and (2.3) in Section 2 above, we have the following two corollaries. The rate of convergence in (3.17) below is quite general for β-null recurrent Markov processes. When $\beta = 1/2$, it is the same as the convergence rate in Theorem 5.2 of PP.

COROLLARY 3.3. *Suppose that the conditions of Theorem 3.2 are satisfied. Furthermore, let $\{X_t\}$ be a β-null recurrent Markov chain with $0 < \beta < 1$. Taking $M_n = M_0 n^{1-\beta} L_s^{-1}(n)$ for some positive constant M_0, we have*

$$
(3.17) \qquad \overline{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0 = O_P\left((n\dot{\kappa}_g^2(M_n))^{-1/2}\right).
$$

COROLLARY 3.4. *Suppose that the conditions of Theorem 3.2 are satisfied. Let $g(x, \boldsymbol{\theta}_0) = x\theta_0$, $\{X_t\}$ be a random walk process and $M_n = M_0 n^{1/2}$ for*

some positive constant M_0. Then we have

(3.18) $$\overline{\theta}_n - \theta_0 = O_P\left(n^{-1}\right),$$

where $\overline{\theta}_n$ is the MNLS estimator of θ_0. Furthermore,

(3.19) $$\sqrt{M_0^3 N(n) n^{3/2}}\left(\overline{\theta}_n - \theta_0\right) \overset{d}{\longrightarrow} \mathsf{N}\left(0, 3\sigma^2/2\right).$$

REMARK 3.8. For the simple linear regression model with regressors generated by a random walk process, (3.18) and (3.19) imply the existence of super consistency. Corollaries 3.3 and 3.4 show that the rates of convergence for the parametric estimator in nonlinear cointegrating models rely not only on the properties of the function $g(\cdot,\cdot)$, but also on the magnitude of β.

In the above two corollaries, we give the choice of M_n for some special cases. In fact, for the random walk process $\{X_t\}$ defined as in Example 2.1(i) with $\mathsf{E}[x_1^2] = 1$, we have

$$\frac{1}{\sqrt{n}}X_{[nr]} = \frac{1}{\sqrt{n}}\sum_{i=1}^{[nr]} x_i \Rightarrow B(r),$$

where $B(r)$ is a standard Brownian motion and " $\Rightarrow$ " denotes the weak convergence. Furthermore, by the continuous mapping theorem (c.f., Billingsley 1968),

$$\sup_{0\leq r\leq 1} \frac{1}{\sqrt{n}}X_{[nr]} \Rightarrow \sup_{0\leq r\leq 1} B(r),$$

which implies that it is reasonable to let $M_n = C_\alpha n^{1/2}$, where C_α may be chosen such that

(3.20) $$\alpha = \mathsf{P}\left(\sup_{0\leq r\leq 1} B(r) \geq C_\alpha\right) = \mathsf{P}\left(|B(1)| \geq C_\alpha\right) = 2\left(1 - \Phi\left(C_\alpha\right)\right),$$

where the second equality is due to the reflection principle and $\Phi(x) = \int_{-\infty}^{x}\left(e^{-\frac{u^2}{2}}/\sqrt{2\pi}\right)du$. This implies that C_α can be obtained when α is given, such as $\alpha = 0.05$. For the general β-null recurrent Markov process, the choice of the optimal M_n remains as an open problem. We conjecture that it may be an option to take $M_n = \widetilde{M}n^{1-\widehat{\beta}}$ with $\widehat{\beta}$ defined in (3.6) and $\widetilde{M}$ chosen by a data-driven method, and will further study this issue in future research.

4. Discussions and extensions.   In this section, we discuss the applications of our asymptotic results in estimating the nonlinear heteroskedastic regression and nonlinear regression with $I(1)$ processes. Furthermore, we also discuss possible extensions of our model to the cases of multivariate regressors and nonlinear autoregression.

4.1. *Nonlinear heteroskedastic regression.*   We introduce an estimation method for a parameter vector involved in the conditional variance function. For simplicity, we consider the model defined by

$$(4.1) \qquad\qquad Y_t = \sigma(X_t, \boldsymbol{\gamma}_0) e_{t*} \quad \text{for} \quad \boldsymbol{\gamma}_0 \in \Upsilon \subset \mathbb{R}^p,$$

where $\{e_{t*}\}$ satisfies Assumption 3.1(ii) with a unit variance, $\sigma^2(\cdot, \cdot) : \mathbb{R}^{p+1} \to \mathbb{R}$ is positive, and $\boldsymbol{\gamma}_0$ is the true value of the p-dimensional parameter vector involved in the conditional variance function. Estimation of the parametric nonlinear variance function defined in (4.1) is important in empirical applications as many scientific studies depend on understanding the variability of the data. When the covariates are integrated, Han and Park (2012) study the maximum likelihood estimation of the parameters in the ARCH and GARCH models. A recent paper by Han and Kristensen (2014) further considers the quasi maximum likelihood estimation in the GARCH-X models with stationary and nonstationary covariates. We next consider the general Harris recurrent Markov process $\{X_t\}$ and use a robust estimation method for model (4.1).

Letting ϖ_0 be a positive number such that $\mathsf{E}\left[\log(e_{t*}^2)\right] = \log(\varpi_0)$, we have

$$
\begin{aligned}
\log(Y_t^2) &= \log\left(\sigma^2(X_t, \boldsymbol{\gamma}_0)\right) + \log(e_{t*}^2) \\
&= \log\left(\sigma^2(X_t, \boldsymbol{\gamma}_0)\right) + \log(\varpi_0) + \log(e_{t*}^2) - \log(\varpi_0) \\
(4.2) \qquad &=: \log\left(\varpi_0 \sigma^2(X_t, \boldsymbol{\gamma}_0)\right) + \zeta_t,
\end{aligned}
$$

where $\mathsf{E}(\zeta_t) = 0$. Since our main interest lies in the discussion of the asymptotic theory for the estimator of $\boldsymbol{\gamma}_0$, we first assume that ϖ_0 is known to simplify our discussion. Model (4.2) can be seen as another nonlinear mean regression model with parameter vector $\boldsymbol{\gamma}_0$ to be estimated. The log-transformation would make data less skewed, and thus the resulting volatility estimator may be more robust in terms of dealing with heavy-tailed $\{e_{t*}\}$. Such transformation has been commonly used to estimate the variability of the data in the stationary time series case (c.f., Peng and Yao 2003; Gao 2007; Chen *et al* 2009). However, any extension to Harris recurrent Markov chains which may be nonstationary has not been done in the literature.

Our estimation method will be constructed based on (4.2). Noting that ϖ_0 is assumed to be known, define

$$(4.3) \qquad \sigma_*^2(X_t, \boldsymbol{\gamma}_0) = \varpi_0 \sigma^2(X_t, \boldsymbol{\gamma}_0) \quad \text{and} \quad g_*(X_t, \boldsymbol{\gamma}_0) = \log(\sigma_*^2(X_t, \boldsymbol{\gamma}_0)).$$

CASE (I). If $g_*(X_t, \boldsymbol{\gamma})$ and its derivatives are integrable on Υ, the log-transformed nonlinear least squares (LNLS) estimator $\widehat{\boldsymbol{\gamma}}_n$ can be obtained by minimizing $L_{n,\sigma}(\boldsymbol{\gamma})$ over $\boldsymbol{\gamma} \in \Upsilon$, where

$$(4.4) \qquad L_{n,\sigma}(\boldsymbol{\gamma}) = \sum_{t=1}^{n} \left[\log(Y_t^2) - g_*(X_t, \boldsymbol{\gamma}) \right]^2.$$

Letting Assumptions 3.1 and 3.2 be satisfied with e_t and $g(\cdot, \cdot)$ replaced by ζ_t and $g_*(\cdot, \cdot)$, respectively, then the asymptotic results developed in Section 3.1 still hold for $\widehat{\boldsymbol{\gamma}}_n$.

CASE (II). If $g_*(X_t, \boldsymbol{\gamma})$ and its derivatives are asymptotically homogeneous on Υ, the log-transformed modified nonlinear least squares (LMNLS) estimator $\overline{\boldsymbol{\gamma}}_n$ can be obtained by minimizing $Q_{n,\sigma}(\boldsymbol{\gamma})$ over $\boldsymbol{\gamma} \in \Upsilon$, where

$$(4.5) \qquad Q_{n,\sigma}(\boldsymbol{\gamma}) = \sum_{t=1}^{n} \left[\log(Y_t^2) - g_*(X_t, \boldsymbol{\gamma}) \right]^2 I\left(|X_t| \leq M_n \right),$$

where M_n is defined as in Section 3.2. Then, the asymptotic results developed in Section 3.2 still hold for $\overline{\boldsymbol{\gamma}}_n$ under some regularity conditions such as a slightly modified version of Assumptions 3.1 and 3.3. Hence, *it is possible to achieve the super-consistency result for $\overline{\boldsymbol{\gamma}}_n$ when $\{X_t\}$ is null recurrent.*

In practice, however, ϖ_0 is usually unknown and needs to be estimated. We next briefly discuss this issue for Case (ii). We may define the loss function by

$$Q_n(\boldsymbol{\gamma}, \varpi) = \sum_{t=1}^{n} \left[\log(Y_t^2) - \log\left(\varpi \sigma^2(X_t, \boldsymbol{\gamma}) \right) \right]^2 I\left(|X_t| \leq M_n \right).$$

Then, the estimators $\overline{\boldsymbol{\gamma}}_n$ and $\overline{\varpi}_n$ can be obtained by minimizing $Q_n(\boldsymbol{\gamma}, \varpi)$ over $\boldsymbol{\gamma} \in \Upsilon$ and $\varpi \in \mathbb{R}^+$. A simulated example (Example B.2) is given in Appendix B of the supplemental document to examine the finite sample performance of the LNLS and LMNLS estimations considered in Cases (i) and (ii), respectively.

4.2. *Nonlinear regression with $I(1)$ processes.* As mentioned before, PP consider the nonlinear regression (1.1) with the regressors $\{X_t\}$ generated by

$$(4.6) \qquad X_t = X_{t-1} + x_t, \quad x_t = \sum_{j=0}^{\infty} \phi_j \varepsilon_{t-j},$$

where $\{\varepsilon_j\}$ is a sequence of i.i.d. random variables and $\{\phi_j\}$ satisfies some summability conditions. For simplicity, we assume that $X_0 = 0$ throughout this subsection. PP establish a suite of asymptotic results for the NLS estimator of the parameter $\boldsymbol{\theta}_0$ involved in (1.1) when $\{X_t\}$ is defined by (4.6). An open problem is how to establish such results by using the β-null recurrent Markov chain framework. This is quite challenging as $\{X_t\}$ defined by (4.6) is no longer a Markov process except for some special cases (for example, $\phi_j = 0$ for $j \geq 1$).

We next consider solving this open problem for the case where $g(\cdot, \cdot)$ is asymptotically homogeneous on Θ and derive an asymptotic theory for $\overline{\boldsymbol{\theta}}_n$ by using Theorem 3.2 (the discussion for the integrable case is more complicated, and will be considered in a future study). Our main idea is to approximate X_t by X_t^* which is defined by

$$X_t^* = \phi \sum_{s=1}^{t} \varepsilon_s, \quad \phi := \sum_{j=0}^{\infty} \phi_j \neq 0,$$

and then show that the asymptotically homogeneous function of X_t is asymptotically equivalent to the same function of X_t^*. As $\{X_t^*\}$ is a random walk process under the Assumption E.1 (see Appendix E of the supplemental document), we can then make use of Theorem 3.2. Define

$$(4.7) \qquad \mathbf{J}_{g*}(n, \boldsymbol{\theta}_0) = \dot{\kappa}_g^2(M_n) M_n \left(\int_{-1}^{1} \dot{h}_g(x, \boldsymbol{\theta}_0) \dot{h}_g^{\tau}(x, \boldsymbol{\theta}_0) dx \right).$$

We next give some asymptotic results for $\overline{\boldsymbol{\theta}}_n$ for the case where $\{X_t\}$ is a unit root process (4.6), and the proof is provided in Appendix E of the supplemental document.

THEOREM 4.1. *Let Assumptions E.1 and E.2 in Appendix E of the supplemental document hold, and $n^{-\frac{1}{2(2+\delta)}} M_n \to \infty$, where $\delta > 0$ is defined in Assumption E.1(i).*

(a) *The solution $\overline{\boldsymbol{\theta}}_n$ which minimizes the loss function $Q_{n,g}(\boldsymbol{\theta})$ over Θ is consistent, i.e.,*

$$(4.8) \qquad\qquad \overline{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0 = o_P(1).$$

(b) *The estimator $\overline{\boldsymbol{\theta}}_n$ has the asymptotically normal distribution,*

$$(4.9) \qquad N_\varepsilon^{1/2}(n)\mathbf{J}_{g*}^{1/2}(n,\boldsymbol{\theta}_0)\left(\overline{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0\right) \xrightarrow{d} N\left(\mathbf{0}_d, \sigma^2 I_d\right),$$

where $N_\varepsilon(n)$ is the number of regenerations for the random walk $\{X_t^*\}$.

REMARK 4.1. Theorem 4.1 establishes an asymptotic theory for $\overline{\boldsymbol{\theta}}_n$ when $\{X_t\}$ is a unit root process (4.6). Our results are comparable with Theorems 5.2 and 5.3 in PP. However, we establish asymptotic normality in (4.9) with stochastic rate $N_\varepsilon^{1/2}(n)\mathbf{J}_{g*}^{1/2}(n,\boldsymbol{\theta}_0)$, and PP establish their asymptotic mixed normal distribution theory with a deterministic rate. As $N_\varepsilon^{1/2}(n)\mathbf{J}_{g*}^{1/2}(n,\boldsymbol{\theta}_0) \propto n^{1/4}\mathbf{J}_{g*}^{1/2}(n,\boldsymbol{\theta}_0)$ in probability, if we take $M_n = M_0\sqrt{n}$ as in Corollary 3.4, we will find that our rate of convergence of $\overline{\boldsymbol{\theta}}_n$ is the same as that derived by PP.

4.3. *Extensions to multivariate regression and nonlinear autoregression.* The theoretical results developed in Section 3 are limited to nonlinear regression with a univariate Markov process. A natural question is whether it is possible to extend them to the more general case with multivariate covariates. In the unit root framework, it is well known that it is difficult to derive the limit theory for the case of multivariate unit root processes, as the vector Brownian motion is transient when the dimension is larger than (or equal to) 3. In contrast, under the framework of the Harris recurrent Markov chains, it is possible for us to generalize the theoretical theory to the multivariate case (with certain restrictions). For example, it is possible to extend the theoretical results to the case with one nonstationary regressor and several other stationary regressors. We next give an example of vector autoregressive (VAR) process which may be included in our framework under certain conditions.

EXAMPLE 4.1. Consider a q-dimensional VAR (1) process $\{\mathbf{X}_t\}$ which is defined by

$$(4.10) \qquad\qquad \mathbf{X}_t = \mathbf{A}\mathbf{X}_{t-1} + \mathbf{b} + \mathbf{x}_t, \quad t = 1, 2, \cdots,$$

where $\mathbf{X}_0 = \mathbf{0}_q$, $\mathbf{A}$ is a $q \times q$ matrix, $\mathbf{b}$ is a q-dimensional vector and $\{\mathbf{x}_t\}$ is a sequence of i.i.d. q-dimensional random vectors with mean zero. If all

the eigenvalues of the matrix $\mathbf{A}$ are inside the unit circle, under some mild conditions on $\{\mathbf{x}_t\}$, Theorem 3 in Myklebust *et al* (2012) shows that the VAR (1) process $\{\mathbf{X}_t\}$ in (4.10) is geometric ergodic, which belongs to the category of positive recurrence. On the other hand, if the matrix $\mathbf{A}$ has exactly one eigenvalue on the unit circle, under some mild conditions on $\{\mathbf{x}_t\}$ and $\mathbf{b}$, Theorem 4 in Myklebust *et al* (2012) shows that the VAR (1) process $\{\mathbf{X}_t\}$ in (4.10) is β-null recurrent with $\beta = 1/2$. For this case, the asymptotic theory developed in Section 3 is applicable. However, when $\mathbf{A}$ has two eigenvalues on the unit circle, under different restrictions, $\{\mathbf{X}_t\}$ might be null recurrent (but not β-null recurrent) or transient. If $\mathbf{A}$ has three or more eigenvalues on the unit circle, the VAR (1) process $\{\mathbf{X}_t\}$ would be transient, which indicates that the limit theory developed in this paper would be not applicable.

We next briefly discuss a nonlinear autoregressive model of the form:

$$(4.11) \qquad X_{t+1} = g(X_t, \boldsymbol{\theta}_0) + e_{t+1}, \quad t = 1, 2, \cdots, n.$$

For this autoregression case, $\{e_t\}$ is not independent of $\{X_t\}$, and thus the proof strategy developed in this paper needs to be modified. Following the argument in Karlsen and Tjøstheim (2001), in order to develop an asymptotic theory for the parameter estimation in the nonlinear autoregression (4.11), we may need that the process $\{X_t\}$ is Harris recurrent but not that the compound process $\{(X_t, e_{t+1})\}$ is also Harris recurrent. This is because we essentially have to consider sums of products like $\dot{g}(X_t, \boldsymbol{\theta}_0)e_{t+1} = \dot{g}(X_t, \boldsymbol{\theta}_0)(X_{t+1} - g(X_t, \boldsymbol{\theta}_0))$, which are of the general form treated in Karlsen and Tjøstheim (2001). The verification of the Harris recurrence of $\{X_t\}$ has been discussed by Lu (1998) and Example 2.1 given in Section 2.2 above. How to establish an asymptotic theory for the parameter estimation of $\boldsymbol{\theta}_0$ in model (4.11) will be studied in our future research.

5. Simulated examples. In this section, we provide some simulation studies to compare the finite sample performance of the proposed parametric estimation methods and to illustrate the developed asymptotic theory.

EXAMPLE 5.1. Consider the generalised linear model defined by

$$(5.1) \qquad Y_t = \exp\{-\theta_0 X_t^2\} + e_t, \quad \theta_0 = 1, \quad t = 1, 2, \cdots, n,$$

where $\{X_t\}$ is generated by one of the three Markov processes:
   (i) AR(1) process: $X_t = 0.5X_{t-1} + x_t$,

(ii) Random walk process: $X_t = X_{t-1} + x_t$,

(iii) TAR(1) process: $X_t = 0.5X_{t-1}I(|X_{t-1}| \leq 1) + X_{t-1}I(|X_{t-1}| > 1) + x_t$, $X_0 = 0$ and $\{x_t\}$ is a sequence of i.i.d. standard normal random variables for the above three processes. The error process $\{e_t\}$ is a sequence of i.i.d. $N(0, 0.5^2)$ random variables and independent of $\{x_t\}$. In this simulation study, we compare the finite sample behavior of the NLS estimator $\widehat{\theta}_n$ with that of the MNLS estimator $\overline{\theta}_n$, and the sample size n is chosen to be 500, 1000 and 2000. The aim of this example is to illustrate the asymptotic theory developed in Section 3.1 as the regression function in (5.1) is integrable when $\theta_0 \neq 0$. Following the discussion in Section 2.2, the AR(1) process defined in (i) is positive recurrent, and the random process defined in (ii) and the TAR(1) process defined in (iii) are 1/2-null recurrent.

We generate 500 replicated samples for this simulation study, and calculate the means and standard errors for both of the parametric estimators in 500 simulations. In the MNLS estimation procedure, we choose $M_n = C_\alpha n^{1-\beta}$ with $\alpha = 0.01$, where C_α is defined in (3.20), $\beta = 1$ for case (i), and $\beta = 1/2$ for cases (ii) and (iii). It is easy to find that $C_{0.01} = 2.58$.

Table 5.1. Means and standard errors for the estimators in Example 5.1

sample size	500	1000	2000
The regressor X_t is generated in case (i)			
NLS	1.0036 (0.0481)	1.0002 (0.0339)	1.0000 (0.0245)
MNLS	1.0036 (0.0481)	1.0002 (0.0339)	1.0000 (0.0245)
The regressor X_t is generated in case (ii)			
NLS	0.9881 (0.1783)	0.9987 (0.1495)	0.9926 (0.1393)
MNLS	0.9881 (0.1783)	0.9987 (0.1495)	0.9926 (0.1393)
The regressor X_t is generated in case (iii)			
NLS	0.9975 (0.1692)	1.0028 (0.1463)	0.9940 (0.1301)
MNLS	0.9975 (0.1692)	1.0028 (0.1463)	0.9940 (0.1301)

The simulation results are reported in Table 5.1, where the numbers in the parentheses are the standard errors of the NLS (or MNLS) estimator in the 500 replications. From Table 5.1, we have the following interesting findings. (a) The parametric estimators perform better in the stationary case (i) than in the nonstationary cases (ii) and (iii). This is consistent with the asymptotic results obtained in Section 3.1 such as Theorem 3.1 and Corollaries 3.1 and 3.2, which indicate that the convergence rates of the parametric estimators can achieve $O_P(n^{-1/2})$ in the stationary case, but only $O_P(n^{-1/4})$ in the 1/2-null recurrent case. (b) The finite sample behavior of the MNLS estimator is the same as that of NLS estimator since $\alpha =$

0.01 means little sample information is lost. (c) Both of the two parametric estimators improve as the sample size increases. (d) In addition, for case (i), the ratio of the standard errors between 500 and 2000 is 1.9633 (close to the theoretical ratio $\sqrt{4} = 2$); for case (iii), the ratio of the standard errors between 500 and 2000 is 1.3005 (close to the theoretical ratio $4^{1/4} = 1.4142$). Hence, this again confirms that our asymptotic theory is valid.

EXAMPLE 5.2. Consider the quadratic regression model defined by

$$(5.2) \qquad Y_t = \theta_0 X_t^2 + e_t, \quad \theta_0 = 0.5, \quad t = 1, 2, \cdots, n,$$

where $\{X_t\}$ is generated either by one of the three Markov processes introduced in Example 5.1, or by (iv) the unit root process:

$$X_t = X_{t-1} + x_t, \quad x_t = 0.2x_{t-1} + v_t,$$

in which $X_0 = x_0 = 0$, $\{v_t\}$ is a sequence of i.i.d. $\mathsf{N}(0, 0.75)$ random variables, and the error process $\{e_t\}$ is defined as in Example 5.1. In this simulation study, we are interested in the finite sample behavior of the MNLS estimator to illustrate the asymptotic theory developed in Section 3.2 as the regression function in (5.2) is asymptotically homogeneous. For the comparison purpose, we also investigate the finite sample behavior of the NLS estimation, although we do not establish the related asymptotic theory under the framework of null recurrent Markov chains. The sample size n is chosen to be 500, 1000 and 2000 as in Example 5.1 and the replication number is $R = 500$. In the MNLS estimation procedure, as in the previous example, we choose $M_n = 2.58n^{1-\beta}$, where $\beta = 1$ for case (i), and $\beta = 1/2$ for cases (ii)–(iv).

The simulation results are reported in Table 5.2, from which, we have the following conclusions. (a) For the regression model with asymptotically homogeneous regression function, the parametric estimators perform better in the nonstationary cases (ii)–(iv) than in the stationary case (i). This finding is consistent with the asymptotic results obtained in Sections 3.2 and 4.2. (b) The MNLS estimator performs as well as the NLS estimator (in particular for the nonstationary cases). Both the NLS and MNLS estimations improve as the sample size increases.

Table 5.2. Means and standard errors for the estimators in Example 5.2

sample size	500	1000	2000
The regressor X_t is generated in case (i)			
NLS	0.5002 (0.0095)	0.4997 (0.0068)	0.4998 (0.0050)
MNLS	0.5003 (0.0126)	0.4998 (0.0092)	0.4997 (0.0064)
The regressor X_t is generated in case (ii)			
NLS	0.5000 (2.4523×10^{-4})	0.5000 (6.7110×10^{-5})	0.5000 (2.7250×10^{-5})
MNLS	0.5000 (2.4523×10^{-4})	0.5000 (6.7112×10^{-5})	0.5000 (2.7251×10^{-5})
The regressor X_t is generated in case (iii)			
NLS	0.5000 (2.6095×10^{-4})	0.5000 (8.4571×10^{-5})	0.5000 (3.1268×10^{-5})
MNLS	0.5000 (2.6095×10^{-4})	0.5000 (8.4572×10^{-5})	0.5000 (3.1268×10^{-5})
The regressor X_t is generated in case (iv)			
NLS	0.5000 (2.1698×10^{-4})	0.5000 (7.1500×10^{-5})	0.5000 (2.6017×10^{-5})
MNLS	0.5000 (2.1699×10^{-4})	0.5000 (7.1504×10^{-5})	0.5000 (2.6017×10^{-5})

6. Empirical application. In this section we give an empirical application of the proposed parametric model and estimation methodology.

EXAMPLE 6.1. Consider the logarithm of the UK to US export and import data (in £). These data come from the website: https://www.uktradeinfo.com/, spanning from January 1996 to August 2013 monthly and with the sample size $n = 212$. Let X_t be defined as $\log(E_t) + \log(p_t^{UK}) - \log(p_t^{US})$, where $\{E_t\}$ is the monthly average of the nominal exchange rate, and $\{p_t^i\}$ denotes the consumption price index of country i. In this example, we let $\{Y_t\}$ denote the logarithm of either the export or the import value.

The data X_t and Y_t are plotted in Figures 6.1 and 6.2, respectively. Meanwhile, the real data application considered by Gao *et al* (2013) suggests that $\{X_t\}$ may follow the threshold autoregressive model proposed in that paper, which is shown to be a 1/2-null recurrent Markov process. Furthermore, an application of the estimation method by (3.6) gives $\beta_0 = 0.5044$. This further supports that $\{X_t\}$ roughly follows a β-null recurrent Markov chain with $\beta = 1/2$.

To avoid possible confusion, let $Y_{ex,t}$ and $Y_{im,t}$ be the export and import data, respectively. We are interested in estimating the parametric relationship between $Y_{ex,t}$ (or $Y_{im,t}$) and X_t. In order to find a suitable parametric relationship, we first estimate the relationship nonparametrically based on $Y_{ex,t} = m_{ex}(X_t) + e_{t1}$ and $Y_{im,t} = m_{im}(X_t) + e_{t2}$ (c.f., Karlsen *et al* 2007), where $m_{ex}(\cdot)$ and $m_{im}(\cdot)$ are estimated by
(6.1)
$$\widehat{m}_{ex}(x) = \frac{\sum_{t=1}^{n} K\left(\frac{X_t - x}{h}\right) Y_{ex,t}}{\sum_{t=1}^{n} K\left(\frac{X_t - x}{h}\right)} \quad \text{and} \quad \widehat{m}_{im}(x) = \frac{\sum_{t=1}^{n} K\left(\frac{X_t - x}{h}\right) Y_{im,t}}{\sum_{t=1}^{n} K\left(\frac{X_t - x}{h}\right)},$$
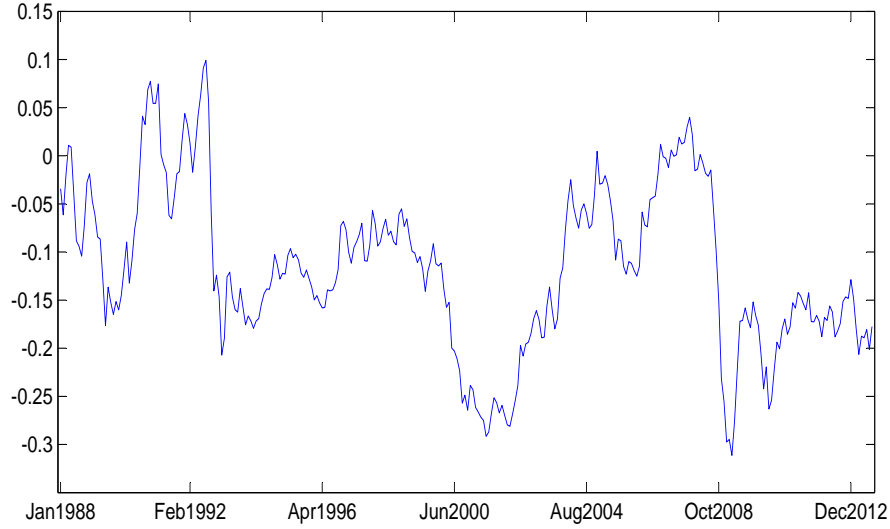
Figure 6.1: Plot of the real exchange rate X_t.

where $K(\cdot)$ is the probability density function of the standard normal distribution and the bandwidth h is chosen by the conventional leave-one-out cross-validation method. Then, a parametric calibration procedure (based on the preliminary nonparametric estimation) suggests using a third-order polynomial relationship of the form

$$(6.2) \qquad Y_{ex,t} = \theta_{ex,0} + \theta_{ex,1} X_t + \theta_{ex,2} X_t^2 + \theta_{ex,3} X_t^3 + e_{ex,t}$$

for the export data, where the estimated values (by using the method in Section 3.2) of $\theta_{ex,0}, \theta_{ex,1}, \theta_{ex,2}$ and $\theta_{ex,3}$ are 21.666, 5.9788, 60.231 and 139.36, respectively, and

$$(6.3) \qquad Y_{im,t} = \theta_{im,0} + \theta_{im,1} X_t + \theta_{im,2} X_t^2 + \theta_{im,3} X_t^3 + e_{im,t}$$

for the import data, where the estimated values of $\theta_{im,0}, \theta_{im,1}, \theta_{im,2}$ and $\theta_{im,3}$ are 21.614, 3.5304, 37.789 and 87.172, respectively. Their plots are given in Figures 6.3 and 6.4, respectively.

While Figures 6.3 and 6.4 suggest some relationship between the exchange rate and either the export or the import variable, the true relationship may also depend on some other macroeconomic variables, such as, the real interest rate in the UK during the period. As discussed in Section 4.3, we would like to extend the proposed models from the univariate case to the multivariate case. As a future application, we should be able to find a more accurate
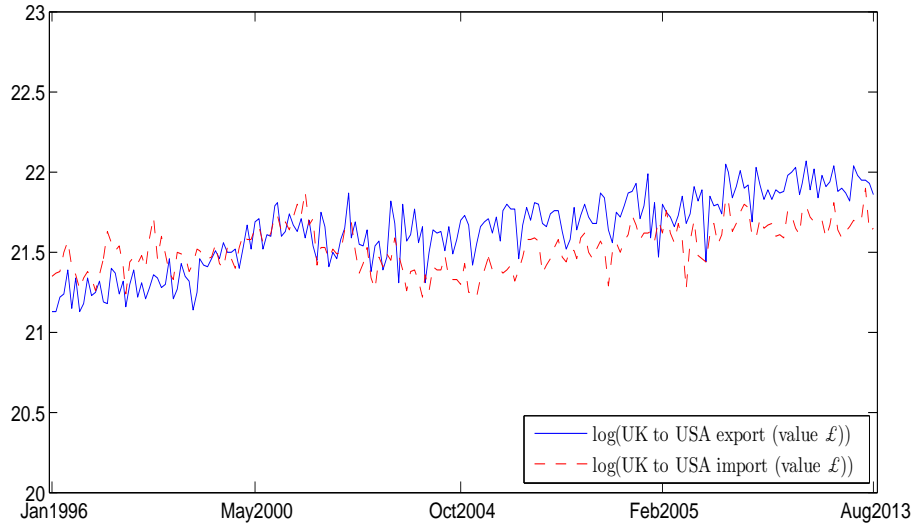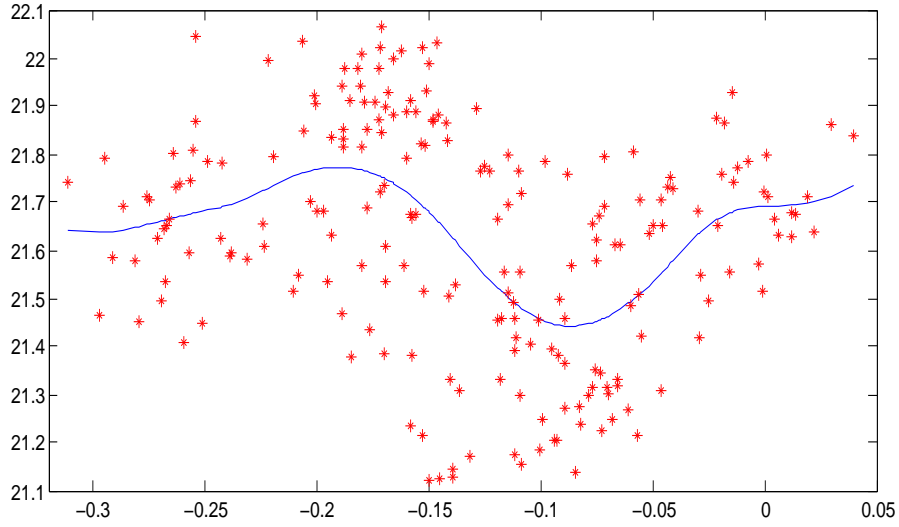
Figure 6.2: Plot of the logarithm of the export and import data Y_t.

Figure 6.3: Plot of the polynomial model fitting (6.2).

relationship among the export or the import variable with the exchange rate and some other macroeconomic variables.

7. Conclusions. In this paper, we have systematically studied the nonlinear regression under the general Harris recurrent Markov chain framework, which includes both the stationary and nonstationary cases. Note
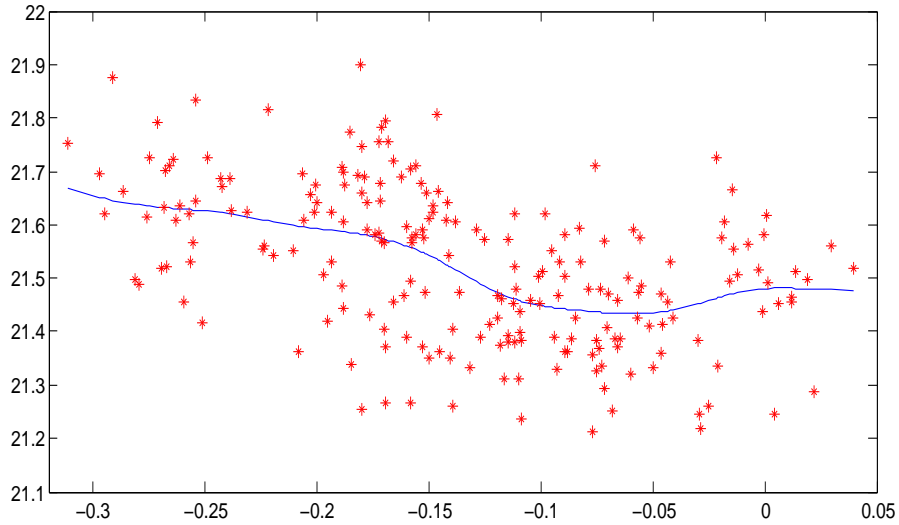
Figure 6.4: Plot of the polynomial model fitting (6.3).

that the nonstationary null recurrent process considered in this paper is under Markov perspective, which, unlike PP, indicates that our methodology has the potential of being extended to the nonlinear autoregressive case. In this paper, we not only develop an asymptotic theory for the NLS estimator of $\boldsymbol{\theta}_0$ when $g(\cdot, \cdot)$ is integrable, but also propose using a modified version of the conventional NLS estimator for the asymptotically homogeneous $g(\cdot, \cdot)$ and adopt a novel method to establish an asymptotic theory for the proposed modified parametric estimator. Furthermore, by using the log-transformation, we discuss the estimation of the parameter vector in a conditional volatility function. We also apply our results to the nonlinear regression with $I(1)$ processes which may be non-Markovian, and establish an asymptotic distribution theory, which is comparable to that obtained by PP. The simulation studies and empirical applications have been provided to illustrate our approaches and results.

Appendix A: Outline of the main proofs

In this appendix, we outline the proofs of the main results in Section 3. The detailed proofs of these results are given in Appendix C of the supplemental document. The major difference between our proof strategy and that based on the unit root framework (c.f., PP and Kristensen and Rahbek, 2013) is that our proofs rely on the limit theorems for functions of the Harris recurrent Markov process (c.f., Lemmas A.1 and A.2 below) whereas PP and Kristensen and Rahbek (2013)'s proofs use the limit theorems for integrated time series. We start with two technical lemmas which are crucial for the the proofs of Theorems 3.1 and 3.2. The proofs for these two lemmas are given in Appendix D of the supplemental document by Li, Tjøstheim and Gao (2015).

LEMMA A.1. *Let $h_I(x, \boldsymbol{\theta})$ be a d-dimensional integrable function on Θ and suppose that Assumption 3.1(i) is satisfied for $\{X_t\}$.*
(a) Uniformly for $\boldsymbol{\theta} \in \Theta$, we have

$$(A.1) \qquad \frac{1}{N(n)} \sum_{t=1}^{n} h_I(X_t, \boldsymbol{\theta}) = \int h_I(x, \boldsymbol{\theta}) \pi_s(dx) + o_P(1).$$

(b) If $\{e_t\}$ satisfies Assumption 3.1(ii), we have, uniformly for $\boldsymbol{\theta} \in \Theta$,

$$(A.2) \qquad \sum_{t=1}^{n} h_I(X_t, \boldsymbol{\theta}) e_t = O_P\big(\sqrt{N(n)}\big).$$

Furthermore, if $\int h_I(x, \boldsymbol{\theta}_0) h_I^{\tau}(x, \boldsymbol{\theta}_0) \pi_s(dx)$ is positive definite, we have

$$(A.3) \qquad \frac{1}{\sqrt{N(n)}} \sum_{t=1}^{n} h_I(X_t, \boldsymbol{\theta}_0) e_t \xrightarrow{d} \mathsf{N}\big(\mathbf{0}_d, \ \sigma^2 \int h_I(x, \boldsymbol{\theta}_0) h_I^{\tau}(x, \boldsymbol{\theta}_0) \pi_s(dx)\big).$$

LEMMA A.2. *Let $h_{AH}(x, \boldsymbol{\theta})$ be a d-dimensional asymptotically homogeneous function on Θ with asymptotic order $\kappa(\cdot)$ (independent of $\boldsymbol{\theta}$) and limit homogeneous function $\overline{h}_{AH}(\cdot, \cdot)$. Suppose that $\{X_t\}$ is a null recurrent Markov process with the invariant measure $\pi_s(\cdot)$ and Assumption 3.3(iv) are satisfied, and $\overline{h}_{AH}(\cdot, \boldsymbol{\theta})$ is continuous on the interval $[-1, 1]$ for all $\boldsymbol{\theta} \in \Theta$. Furthermore, letting*

$$\begin{aligned}
\boldsymbol{\Delta}_{AH}(n, \boldsymbol{\theta}) &= \sum_{i=0}^{[M_n]} \overline{h}_{AH}(\frac{i}{M_n}, \boldsymbol{\theta}) \overline{h}_{AH}^{\tau}(\frac{i}{M_n}, \boldsymbol{\theta}) \pi_s(B_{i+1}(1)) + \\
&\qquad \sum_{i=0}^{[M_n]} \overline{h}_{AH}(\frac{-i}{M_n}, \boldsymbol{\theta}) \overline{h}_{AH}^{\tau}(\frac{-i}{M_n}, \boldsymbol{\theta}) \pi_s(B_{i+1}(2))
\end{aligned}$$

with $B_i(1)$ and $B_i(2)$ defined in Section 3.2, there exist a continuous and positive definite matrix $\boldsymbol{\Delta}_{AH}(\boldsymbol{\theta})$ and a sequence of positive numbers $\{\zeta_{AH}(M_n)\}$ such that $\zeta_0(M_n)/\zeta_{AH}(M_n)$ is bounded, $\zeta_0(l_n)/\zeta_0(M_n) = o(1)$ for $l_n \to \infty$ but $l_n = o(M_n)$, and

$$\lim_{n\to\infty} \frac{1}{\zeta_{AH}(M_n)} \boldsymbol{\Delta}_{AH}(n, \boldsymbol{\theta}) = \boldsymbol{\Delta}_{AH}(\boldsymbol{\theta}),$$

where $\zeta_0(\cdot)$ is defined in Section 3.2.

(a) Uniformly for $\boldsymbol{\theta} \in \Theta$, we have

$$(A.4) \quad [N(n)\mathbf{J}_{AH}(n, \boldsymbol{\theta})]^{-1} \sum_{t=1}^{n} h_{AH}(X_t, \boldsymbol{\theta}) h_{AH}^{\tau}(X_t, \boldsymbol{\theta}) I\left(|X_t| \le M_n\right) = I_d + o_P(1),$$

where $\mathbf{J}_{AH}(n, \boldsymbol{\theta}) = \kappa^2(M_n)\zeta_{AH}(M_n)\boldsymbol{\Delta}_{AH}(\boldsymbol{\theta})$.

(b) If $\{e_t\}$ satisfies Assumption 3.1(ii), we have, uniformly for $\boldsymbol{\theta} \in \Theta$,

$$(A.5) \qquad \mathbf{J}_{AH}^{-1/2}(n, \boldsymbol{\theta}) \sum_{t=1}^{n} h_{AH}(X_t, \boldsymbol{\theta}) I\left(|X_t| \le M_n\right) e_t = O_P\left(\sqrt{N(n)}\right),$$

and furthermore

$$(A.6) \quad N^{-1/2}(n)\mathbf{J}_{AH}^{-1/2}(n, \boldsymbol{\theta}_0) \sum_{t=1}^{n} h_{AH}(X_t, \boldsymbol{\theta}_0) I\left(|X_t| \le M_n\right) e_t \xrightarrow{d} \mathsf{N}\left(\mathbf{0}_d, \ \sigma^2 I_d\right).$$

PROOF OF THEOREM 3.1. For Theorem 3.1(a), we only need to verify the following sufficient condition for the weak consistency (Jennrich, 1969): for a sequence of positive numbers $\{\lambda_n\}$,

$$(A.7) \qquad\qquad \frac{1}{\lambda_n}\left[L_{n,g}(\boldsymbol{\theta}) - L_{n,g}(\boldsymbol{\theta}_0)\right] = L^*(\boldsymbol{\theta}, \boldsymbol{\theta}_0) + o_P(1)$$

uniformly for $\boldsymbol{\theta} \in \Theta$, where $L^*(\cdot, \boldsymbol{\theta}_0)$ is continuous and achieves a unique minimum at $\boldsymbol{\theta}_0$. This sufficient condition can be proved by using (A.1) and (A.2) in Lemma A.1, and (3.3) in Theorem 3.1(a) is thus proved. Combining the so-called Cramér-Wold device (Billingsley, 1968) and (A.3) in Lemma A.1(b), we can complete the proof of the asymptotically normal distribution in (3.4). Details can be found in Appendix C of the supplemental document. □

PROOF OF COROLLARY 3.1. The asymptotic distribution (3.7) can be proved by using (3.5) and Theorem 3.1(b). □

PROOF OF COROLLARY 3.2. The convergence result (3.8) can be proved by using (2.3) and (3.4), and following the proof of Lemma A.2 in Gao *et al* (2014). A detailed proof is given in Appendix C of the supplemental document. □

PROOF OF THEOREM 3.2. The proof is similar to the proof of Theorem 3.1 above. To prove the weak consistency, similar to (A.7), we need to verify the sufficient condition: for a sequence of positive numbers $\{\lambda_n^*\}$,

$$(A.8) \qquad \frac{1}{\lambda_n^*}\big[Q_{n,g}(\boldsymbol{\theta}) - Q_{n,g}(\boldsymbol{\theta}_0)\big] = Q^*(\boldsymbol{\theta}, \boldsymbol{\theta}_0) + o_P(1)$$

uniformly for $\boldsymbol{\theta} \in \Theta$, where $Q^*(\cdot, \boldsymbol{\theta}_0)$ is continuous and achieves a unique minimum at $\boldsymbol{\theta}_0$. Using Assumption 3.3(ii) and following the proofs of (A.4) and (A.5) in Lemma A.2 (see Appendix D in the supplemental document), we may prove (A.8) and thus the weak consistency result (3.14). Combining the Cramér-Wold device and (A.6) in Lemma A.2(b), we can complete the proof of the asymptotically normal distribution for $\overline{\boldsymbol{\theta}}_n$ in (3.15). More details are given in Appendix C of the supplemental document. $\qquad\square$

PROOF OF COROLLARY 3.3. By using Theorem 3.2(b) and (2.3), and following the proof of Lemma A.2 in Gao *et al* (2014), we can directly prove (3.17). $\qquad\square$

PROOF OF COROLLARY 3.4. The convergence result (3.18) follows from (3.17) in Corollary 3.3 and (3.19) can be proved by using (3.15) in Theorem 3.2(b). $\qquad\square$

SUPPLEMENTARY MATERIAL

Supplement to "Estimation in Nonlinear Regression with Harris Recurrent Markov Chains". We provide some additional simulation studies, the detailed proofs of the main results in Section 3, the proofs of Lemmas A.1 and A.2 and Theorem 4.1.

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