PHYS 514 – COMP 514 - FINAL PROJECT

Fatma Cankara

1. a.

1.a.1.

In this part, we will show that we can derive hydrostatic equation as Lane-Emden equation. In the beginning, we have two differential equations for stellar structure. We know that mass is conserved and described by the continuity equation:

$$\frac{dm}{dr} = 4\pi r^2 \rho \tag{1}$$

where p is a function of r. The hydrostatic equation is

$$\frac{dp}{dr} = -\frac{Gm(r)\rho}{r^2} \tag{2}$$

We can write

$$\frac{r^2}{\rho}\frac{dp}{dr} = -Gm$$

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dp}{dr}\right) = -G\frac{dm}{dr} \tag{3}$$

Plugging Eq. 1 into Eq. 3, we get

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dp}{dr}\right) = -4\pi G r^2 \rho$$

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dp}{dr}\right) = -4\pi G \rho \tag{4}$$

Our two equations involve three unknowns: pressure P, density ρ, mass variable m, thus we need to define ρ as it is not explicitly given. We can use this relation between p and ρ in the following manner. In order to make this equation dimensionless we can use central density ρ_c.

Define θ such that

$$\rho(r) = \rho_c \theta^n(r) \tag{5}$$

For a polytrope, we assume that gas pressure is:

$$\begin{aligned} p(r) &= K\rho^{\gamma}(r) \\ &= K{\rho_c}^{\gamma}\theta^{\gamma n}(r) \end{aligned} \tag{6}$$

$$= K{\rho_c}^{1+1/n}\theta^{n+1}(r)$$

where $\gamma = \frac{n+1}{n} = 1 + \frac{1}{n}$

Eqn. 5 and Eqn. 6 define the distribution of pressure and density in the star give n, K and ρ_c. Now, using this information we derived, we can write hydrostatic equation as the following:

We have

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dp}{dr}\right) = -4\pi G\rho \tag{4}$$

and

$$\rho(r) = \rho_c\theta^n(r) \tag{5}$$

$$p = K\rho^\gamma = K{\rho_c}^\gamma\theta^{n+1} \tag{6}$$

So, we can write:

$$\frac{K\rho_c^\gamma}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho_c\theta^n}\frac{d\theta^{n+1}}{dr}\right) = -4\pi G\rho_c\theta^n$$

$$\frac{p_c}{r^2}\frac{d}{dr}\left(\frac{r^2}{\theta^n}\frac{d\theta^{n+1}}{dr}\right) = -4\pi G\rho_c\theta^n$$

$$\frac{p_c}{{\rho_c}^2}\frac{n+1}{r^2}\frac{d}{dr}\left(\frac{r^2\theta^n}{\theta^n}\frac{d\theta}{dr}\right) = -4\pi G\theta^n$$

$$\left(\frac{p_c(n+1)}{4\pi G{\rho_c}^2}\right)\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right) = -\theta^n \tag{7}$$

In equation 7, $\left(\frac{p_c(n+1)}{4\pi G{\rho_c}^2}\right)$ term has units length^2. If we define it to be $\alpha^2$for simplification, we get

$$\frac{\alpha^2}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right) = -\theta^n$$

where

$$\alpha = \left(\frac{p_c(n+1)}{4\pi G{\rho_c}^2}\right)^{1/2} \tag{8}$$

Now, we will substitute $r = \alpha\xi$ for ξ a dimensionless variable; we get:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n \tag{9}$$

which is the Lane-Emden equation. The Lane-Emden equation is a dimensionless form of the hydrostatic equilibrium equation. Here dimensionless variables θ and ξ relate to density ρ and radius r.

In order to show the behavior of regular solutions at the center, we will find a series solution to the differential equation (Eqn. 9) by an asymptotic approximation centered at x_0. For this purpose, we use Mathematica's AsymptoticDSolveValue function. Using this approximation, we can show that

$$\theta(\xi) = 1 - \frac{1}{6}\xi^2 + \frac{n}{120}\xi^4 \ldots$$

When we solve this for n = 1, we get;

$$-\frac{ie^{-i\xi}(-1+e^{2i\xi})}{2\xi}$$

When we simplify this term we get:

$$\frac{sin\,\xi}{\xi}$$

1.a.2.

In order to calculate the mass of the star, we can integrate density over the radius of the star. For this purpose, we can use Eqn. 1 as follows:

$$M = \int_0^R 4\pi r^2 \rho dr$$

We did change of variables such that:

$$r = \alpha\xi$$
$$\rho = \rho_c \theta^n$$
$$R = \alpha\xi_n$$

We get

$$M = 4\pi\alpha^3\rho_c \int_0^{\xi_n} \xi^2\theta^n d\xi \tag{10}$$

Furthermore, reordering the Lane-Emden equation, we know that:

$$\xi^2\theta^n = -\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)$$

Then, Eqn. 10 becomes:

$$M = -4\pi\alpha^3\rho_c \int_0^{\xi_n} \frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) d\xi$$

$$M = -4\pi\alpha^3\rho_c {\xi_n}^2 \left(\frac{d\theta}{d\xi}\right)_{\xi_n} \tag{11}$$

Using the equation R = $\alpha\xi_n$, we can rewrite as:

$$M = -4\pi R^3\rho_c \left(\frac{-\theta'(\xi_n)}{\xi_n}\right) \tag{12}$$

Finally, equation 12 gives the total mass.

1.a.3.

In order to find the relationship between mass and radius for a group of stars that share the same polytropic EOS, we can start by rewriting α using Eqn. 6:

$$p(r) = K\rho^{\gamma}(r) = K\rho^{\frac{n+1}{n}} \tag{6}$$

We have α from Eqn 8 as:

$$\alpha = \left(\frac{\mathrm{p_c}(n+1)}{4\pi\mathrm{G}\rho_c{}^2}\right)^{1/2} \tag{8}$$

Now, we reformulate as a function of ρ_c and K:

$$\alpha = \left(\frac{K\rho_c{}^{\frac{n+1}{n}}(n+1)}{4\pi\mathrm{G}\rho_c{}^2}\right)^{1/2}$$

$$\alpha = \left(\frac{K\rho_c{}^{\frac{1-n}{n}}(n+1)}{4\pi G}\right)^{1/2}$$

Then ρ_c can be written as:

$$\rho_c = \left(\frac{K(n+1)}{\alpha^2 4\pi\mathrm{G}}\right)^{n/n-1}$$

Now, we will replace ρ_c in Eqn. 11

$$M = -4\pi\alpha^3\rho_c\xi_n{}^2\left(\frac{d\theta}{d\xi}\right)_{\xi_n}$$

$$M = -4\pi\alpha^3\left(\frac{K(n+1)}{\alpha^2 4\pi\mathrm{G}}\right)^{n/n-1}\xi_n{}^2\left(\frac{d\theta}{d\xi}\right)_{\xi_n}$$

Reorganizing the terms, we get:

$$\left[-\frac{GM}{\xi_n{}^2\left(\frac{d\theta}{d\xi}\right)_{\xi_n}}\right]^{(n-1)}\alpha^{3-n} = \frac{[K(n+1)]^n}{4\pi G}$$

Previously we redefined *R* as :

$$R = \alpha\xi_n$$

To be able to get the relationship between R and M, we will replace α with R/ξ_n:

$$\left[-\frac{GM}{{\xi_n}^2\left(\frac{d\theta}{d\xi}\right)_{\xi_n}}\right]^{(n-1)}\left(\frac{R}{\xi_n}\right)^{3-n} = \frac{[K(n+1)]^n}{4\pi G}$$

Reorganizing this term, we get:

$$R = \left[\left(\frac{[K(n+1)]^n}{4\pi G}\right)\left[-\frac{G}{{\xi_n}^2\left(\frac{d\theta}{d\xi}\right)_{\xi_n}}\right]^{(1-n)} {\xi_n}^{(3-n)}\right]^{1/(3-n)} M^{\left(\frac{1-n}{3-n}\right)}$$

Or

$$M = \left[\left(\frac{[K(n+1)]^n}{4\pi G}\right)\left[-\frac{G}{{\xi_n}^2\left(\frac{d\theta}{d\xi}\right)_{\xi_n}}\right]^{(1-n)} {\xi_n}^{(3-n)}\right]^{1/(n-1)} R^{\left(\frac{3-n}{1-n}\right)} \tag{13}$$

Eqn. 13 shows us that there is a relationship between M and R such that:

$$M \propto R^{\frac{3-n}{1-n}}$$

For a given and K, the part in square brackets is constant. So, constant of proportionality C becomes:

$$C = \left[\left(\frac{[K(n+1)]^n}{4\pi G}\right)\left[-\frac{G}{{\xi_n}^2\left(\frac{d\theta}{d\xi}\right)_{\xi_n}}\right]^{(1-n)} {\xi_n}^{(3-n)}\right]^{1/(n-1)}$$

1.b.

A scatter and loglog plot for the M and R values can be seen in Figure 1. In the data, mass values are given in solar units, however, radius is given in CGS units. All values are evaluated in SI units that are obtained with relevant conversions. As question stated, surface gravity values are converted into radius values using the Newton's Law for universal gravitation, which is:

$$r = \frac{GM}{g^2}$$

where M is the mass of the object, r is its radius, and G is the gravitational constant.

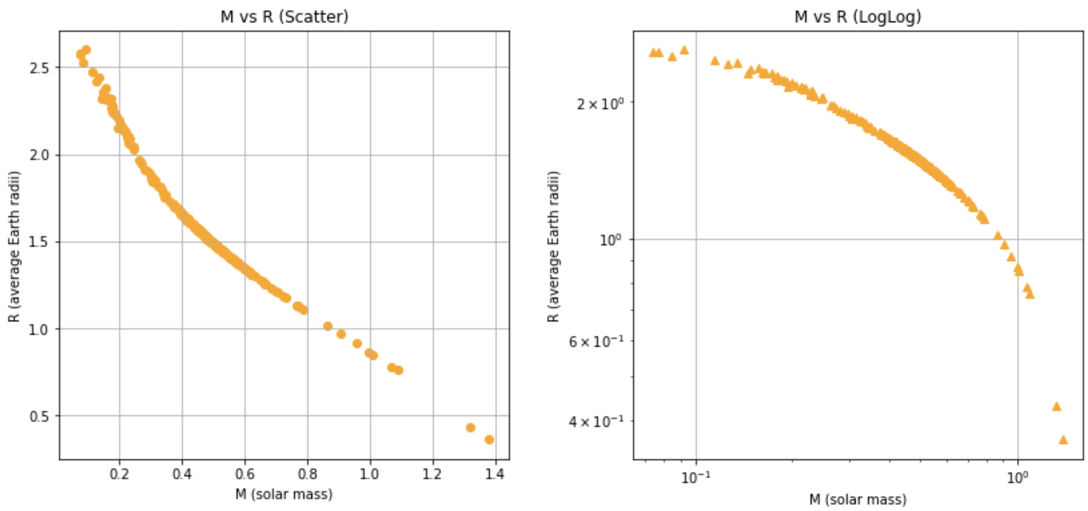

Figure 1. Distribution of mass (in solar mass) vs radius (in average Earth radii).

1.c.

Pressure of white dwarfs are governed by quantum mechanical effect named electron degeneracy due to halting of their thermonuclear energy production because of the low mass. This governing equation is

$$P = C[x(2x^2 - 3)(x^2 + 1)^{0.5} + 3\, sinh^{-1} x] \qquad (14)$$

where

$$x = \left(\frac{\rho}{D}\right)^{\frac{1}{q}}$$

$$x \ll 1$$

When we apply a series expansion to Eqn. 14, we get:

$$1.6\left(\left(\frac{\rho}{D}\right)^{1/q}\right)^5 + 0\left(\left(\frac{\rho}{D}\right)^{1/q}\right)^6$$

$$= \ 1.6\left(\left(\frac{\rho}{D}\right)^{1/q}\right)^5$$

Which can also be written as the polytropic equation of Eqn 6 that is:

$$p(r) = \ K_* \rho^{\frac{n+1}{n}} \qquad (6)$$

Then the hidden values for K_* and n_*become:

$$K_* \ = \frac{8C}{5D^{5/q}}$$

$$n_* = q/(5 - q)$$

Now that we obtained electron degeneracy equation in the form of polytropic equation, we can use equation 13, to fit our data. From Eqn. 13, we know that M and R are related with some constant term relating them. Also, our dataset is composed of mass and radius values, thus using this equation can do the fit. However, we should note that Eqn. 6 is only valid when x << 1, thus not all datapoints are eligible for the fitting process. Let us remember Eqn. 13:

$$M = \left[\left(\frac{[K(n+1)]^n}{4\pi G} \right) \left[- \frac{G}{\xi_n{}^2 \left(\frac{d\theta}{d\xi} \right)_{\xi_n}} \right]^{(1-n)} \xi_n{}^{(3-n)} \right]^{1/(n-1)} R^{\left(\frac{3-n}{1-n}\right)}$$

In order to simplify this equation, we said:

$$M = CR^{\left(\frac{3-n}{1-n}\right)}$$

And represented the constant part with C only. Applying a non-linear fit to find C and n; then extrapolating on C to find K will give us the desired fit for the data. Please note that, at this stage

$$C = \left[\left(\frac{[K(n+1)]^n}{4\pi G} \right) \left[- \frac{G}{\xi_n{}^2 \left(\frac{d\theta}{d\xi} \right)_{\xi_n}} \right]^{(1-n)} \xi_n{}^{(3-n)} \right]^{1/(n-1)}$$

$$n_* = q/(5-q)$$

In order to achieve that, solution to the Lane-Emden equation is made possible by Python'a solve_ivp module. Termination conditions are evaluated as well. For initial value of the proportionality constant, I found an article named 'On the theory of white dwarf stars. The energy sources of white dwarfs' by L. Mestel. In this article, C is mentioned to be around 10^12 orders of magnitude. So, I used this value. For q, initial value is suggested to be 3. Also as mentioned above, equation X is only valid for low mass stars. For the threshold of low mass, I consulted Wikipedia and found low-mass helium white dwarfs has mass < 0.20 M⊙, so I took 0.2 as my threshold. Using these values, as a result of the analysis, we found that the fitted parameters are as follows:

q = 3.0, K = 4.1701771736549137e-07 and C = 3200000000000.0

From theory, we know that q is an integer. Since we cannot constrain a fitting parameter to be an integer, we will set q to be the nearest integer. Then, we will do a second fit for the other parameter, K∗. We had q to be around 3 as per the suggestion. When we do a second round of fitting by taking q as constant and only estimating K, we get the following parameters:

q = 3.0, K = 4.17017717e-07, C = 3.2e+12

Now, we will plot fitted values respect to M. I plotted both scatter and loglog scale as scatter plot did not reveal a good relationship.

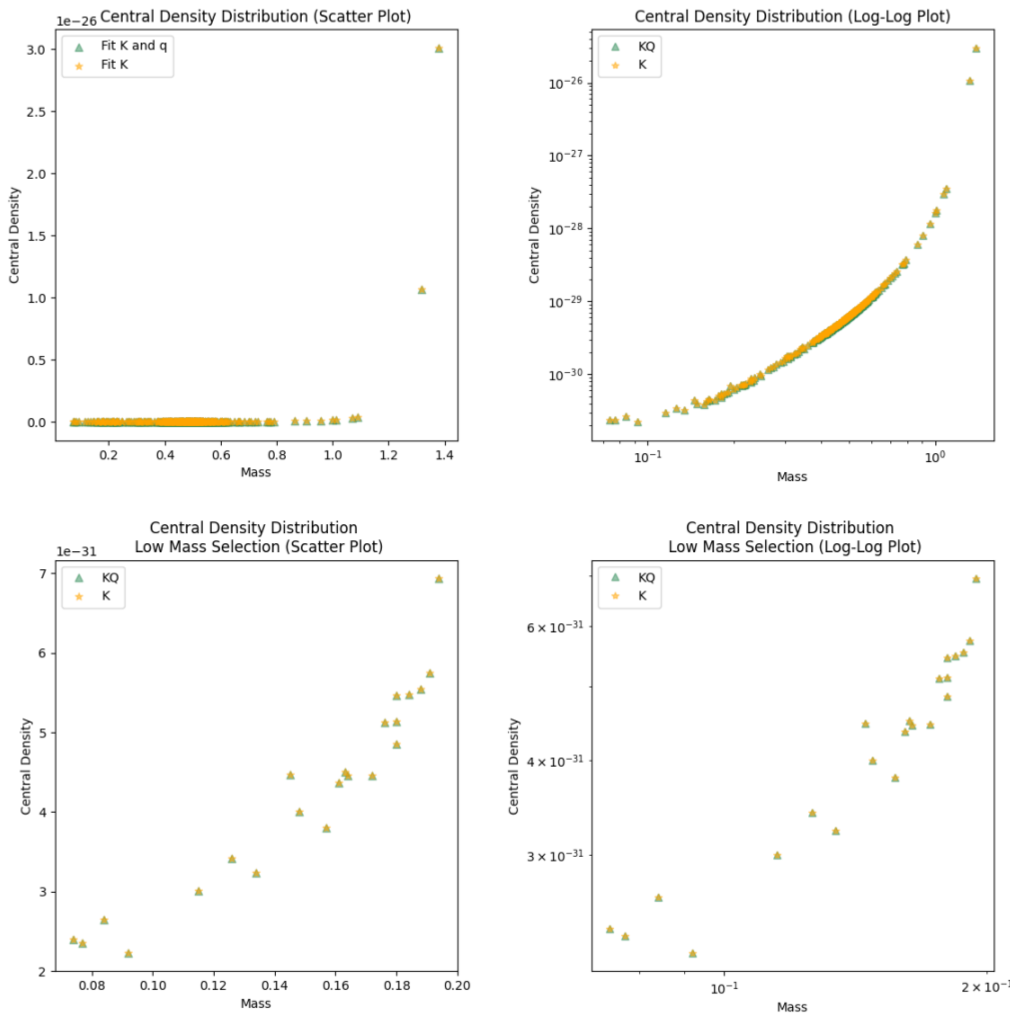

Figure 2. M vs Central Density . Plots were created for both full data and low mass selection. Loglog plots are created for better visualization.

As we can see from the plots, there does not seem a big difference between fitting K and q, or K only. This could be due to the random initial parameters that we selected; we might interpolate on this to see how changing the parameters really affect this. Also, for low mass stars, again the selection of mass threshold might make a difference.

1.d. I could not solve this part.

1.e. Theoretical values for q, D, C and K from the constants are given below, obtained using equation 10 and 11.

q: 3, D: 1947864333.345182, C: 6.002332185660436e+21, K: 3161128.6219916763

2.

In the case of neutron stars, the concept of gravity is understood by Einstein's general relativity approach as Newton's approach collapses.

2.a.

When we consider dense objects such as neutron stars, as Newton's approach does not apply well, we need to use Einstein's general relativity s mentioned above. Due to these relativistic effects, hydrostatic equations of part 1 need to be modified into Tolman-Oppenheimer-Volkoff (TOV) equations as below:

$$m' = 4\pi r^2 \rho$$

$$v' = 2\frac{m + 4\pi r^3 p}{r(r - 2m)}$$

$$p' = -\frac{m + 4\pi r^3 p}{r(r - 2m)}(\rho + p) = -\frac{1}{2}(\rho + p)v'$$

where ' represents derivative with respect to r. As in the previous part, *m* and *p* represent the mass and pressure. Here the term v represents time dilation. For regular stars the effect of gravitational time dilation on their matter field dynamics is negligible. However, for the compact stars, such as the neutron stars, the effect of gravitational time dilation is significant. Time dilation is a function of gravity and can be thought as escape velocity from a gravitational field.

Now we are asked to obtain a plot for M vs R by integrating TOV equations from the center out. Initial conditions are

$$m(0) = 0$$

$$p(0) = p_c$$

$$v(0) = 0$$

Like in the part 1 case, equations involve three unknowns: pressure P, density ρ, mass variable m, thus we need to define ρ as it is not explicitly given. Since the interactions are complicated and we do not know the NS equation of state for the time being, we will use a polytope with n =

1 as neutron stars are approximately polytropes with index 0.5 < n < 1. So, we will relate p and ρ as :

$$p = K_{NS}\rho^2$$

The relationship between mass and radius can be observed from Figure 3 below. This graph is obtained by integrating TOV equation over a range of values of central density

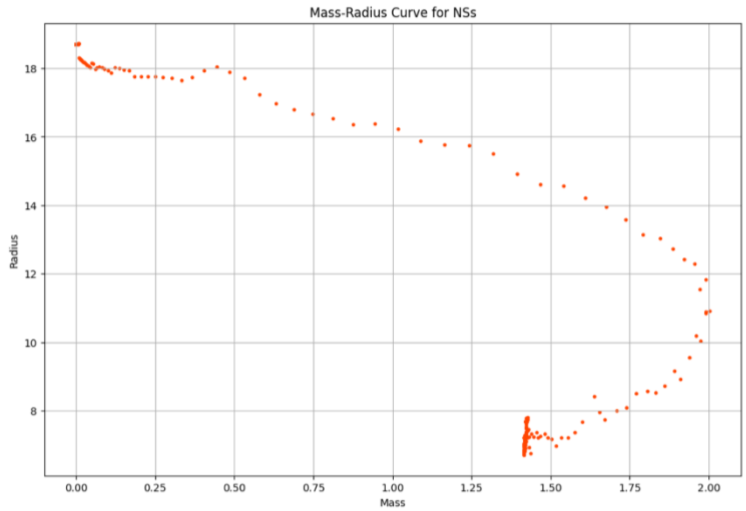

Figure 3. Mass-radius relationship.

2.b.

The m(r) function that we calculated above does not give the sum of the rest masses of all the particles in the neutron star, rather it contains the rest mass of the neutrons and the negative contribution from the binding energy. The rest mass or the baryonic mass, is given by the equation :

$$m_p' = 4\pi\left(1 - \frac{2m}{r}\right)^{0.5} r^2\rho$$

where m_p is the baryonic mass. From here, we can calculate fractional binding energy as:

$$\Delta = \frac{M_p - M}{M}$$

Using mentioned formula, we can plot the relation between fractional binding energy and radius, as well. This. Data is also obtained in the integration part, using Equation 16 and 17.

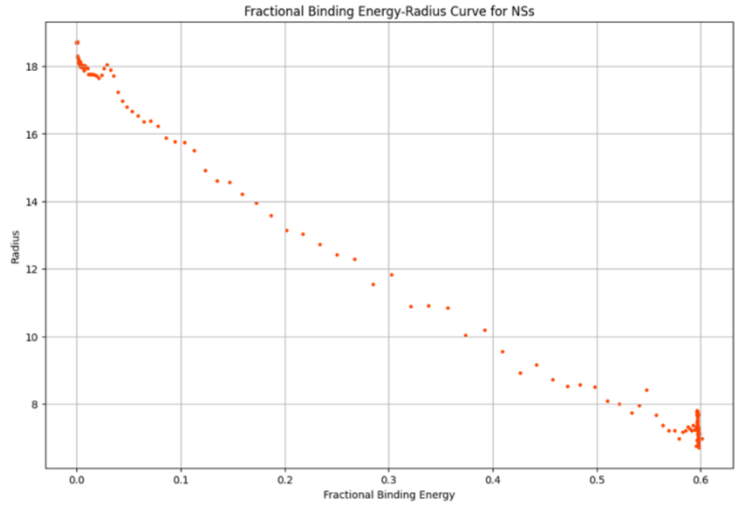

Figure 4. Resting binding energy vs radius.

The baryonic mass and above estimated mass gets closer to each other as radius increases. However, as we go to lower radius regions, we observe. That even though baryonic mass follows a simpler trend, relative mass tends to decrease after an increase.

2.c.

In this question, we plotted M vs ρ_C curve to understand NS stability. The criterion for stability is given in Equation 18 and 19. Some stars in Figure 2 are not stable and might be collapsing. In Figure 5, stable and unstable stars are shown.

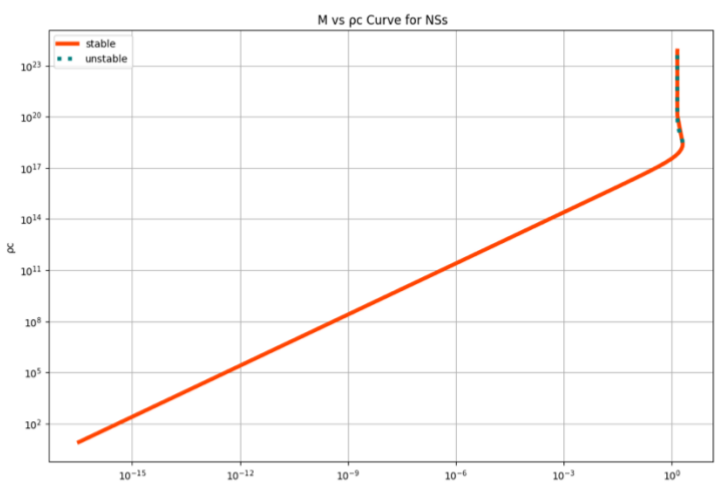

Figure 5. M vs ρ_C curve

2.d.

In this question, we calculated the maximally allowed masses for EOS with different values of K and plotted $M_{max}(K)$ vs K graph. Figure 6 shows the relationship between these two quantities. In our analysis so far, we set polytropic constant to be 100 as it was suggested in the assignment. We can also investigate the effect of changing K over M_{max}. We know that the most massive neutron star to be observed so far has a mass of $2.14M_\odot$. Using this value, we can also show the effect of the changes of on the maximum mass that can a neutron star have.

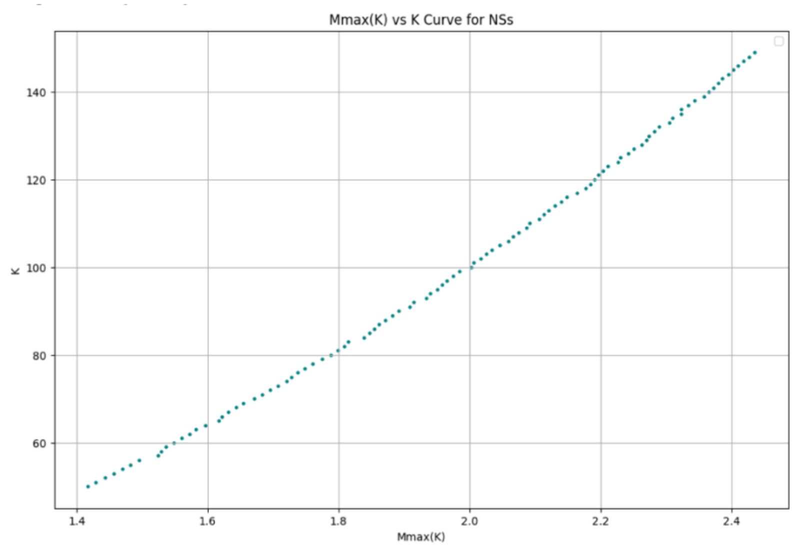

Figure 6. M_{max} vs K.

When we plot the first and last values of M_{max} and K to define the boundaries, we obtain the lines in Figure 7. Heaviest neutron star with $M = 2.14M_\odot$ corresponds to the intersection point in Figure 7.

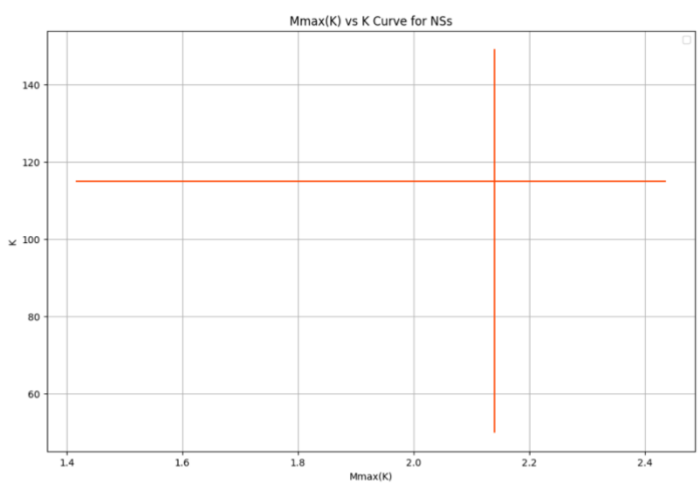

Figure 7. Boundaries for M_{max} and K

Then, we we put the M_{max} vs K graph on top of it, from the lines we can see that the maximum allowable polytropic constant is just a few points above 115.

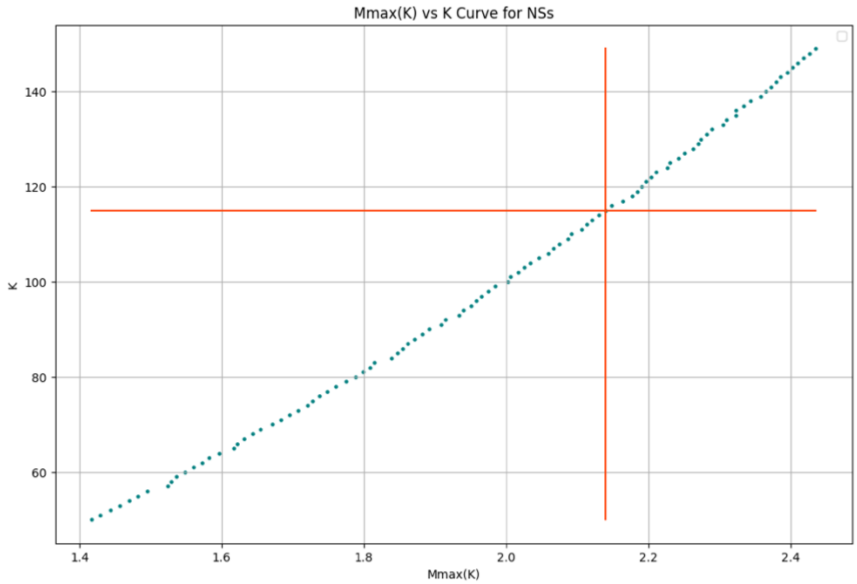

Figure 7. M_{max} vs K, showing maximum allowable polytropic constant.