IE 360 HOMEWORK 2 2019402204 - Fatmanur Yaman

**1a)** Show that $Var(\varepsilon) = \frac{\alpha}{2-\alpha} \cdot \sigma^2$

$Y_t = \mu + \varepsilon_t$, $Var(\varepsilon_t) = \sigma^2$, $Var(Y_t) = \sigma^2$

$Var(Y_{t-1}) = \sigma^2$

$\vdots$

$Var(Y_0) = \sigma^2$

$S_t = \alpha Y_t + (1-\alpha) S_{t-1}$

$F_{t,t+\tau} = S_t$, $\tau \geq 1$ → recursive iteration

$$S_t = \alpha Y_t + (1-\alpha) Y_{t-1} + \alpha (1-\alpha)^2 Y_{t-2} + \dots + \alpha (1-\alpha)^t Y_0 = F_{t,t+\tau}$$

→ Take the variance:

$$Var(F_{t,t+\tau}) = Var(S_t) = Var(\alpha Y_t + \alpha(1-\alpha) Y_{t-1} + \alpha(1-\alpha)^2 Y_{t-2} + \dots + \alpha(1-\alpha)^t Y_0)$$

$$= Var(\alpha Y_t) + Var(\alpha(1-\alpha) Y_{t-1}) + Var(\alpha(1-\alpha)^2 Y_{t-2}) + \dots$$

$$= \alpha^2 Var(Y_t) + \alpha^2 (1-\alpha)^2 Var(Y_{t-1}) + \alpha^2 (1-\alpha)^4 Var(Y_{t-2}) + \dots + \alpha^2 (1-\alpha)^{2t} Var(Y_0)$$

$$= \alpha^2 \cdot \sigma^2 + \alpha^2 (1-\alpha)^2 \sigma^2 + \alpha^2 (1-\alpha)^4 \sigma^2 + \dots + \alpha^2 (1-\alpha)^{2t} \sigma^2$$

let's assume: $(1-\alpha)^2 = x$

$1 - \sqrt{x} = \alpha$

$$= \alpha^2 \sigma^2 (1 + (1-\alpha)^2 + (1-\alpha)^4 + \dots + (1-\alpha)^{2t})$$

$$= (1-\sqrt{x})^2 \sigma^2 (1 + x + x^2 + \dots + x^t)$$

$$= (1-\sqrt{x})^2 \sigma^2 \frac{1 - x^{t+1}}{1-x}$$

$$= (1-\sqrt{x}) \cancel{(1-\sqrt{x})} \sigma^2 \frac{1 - x^{t+1}}{\cancel{(1-\sqrt{x})}(1+\sqrt{x})}$$

$$= \alpha \cdot \sigma^2 \cdot \frac{1 - (1-\alpha)^{2t+2}}{1 + 1 - \alpha} = \alpha \sigma^2 \frac{1 - (1-\alpha)^{2t+2}}{2-\alpha} \quad //$$

For larger value of $t$ $(t \to \infty)$, $1 - (1-\alpha)^{2t+2} \to 1$ //

$$\boxed{= \frac{\alpha \cdot \sigma^2}{2-\alpha}} \quad //$$

**1b)** $Var(\varepsilon) = Var(Y_t - F_{t,t+\tau}) = Var(Y_t) + Var(F_{t,t+\tau}) - \underbrace{2Cov(Y_t, F_{t+t+\tau})}_{0\text{, I explained in the other side of the page.}}$

$$= \sigma^2 + \frac{\alpha \cdot \sigma^2}{2-\alpha} = \frac{2\sigma^2 - \cancel{\alpha\sigma^2} + \cancel{\alpha\sigma^2}}{2-\alpha} = \frac{2\sigma^2}{2-\alpha}$$

Having stable forecasting model means that having a small error variance. The error variance is the spread of the forecast around what should it be. We want smaller spread around the actual data to make better forecasts. Thus, the error variance should be small. $\frac{2\sigma^2}{2-\alpha}$ should be small. Since $\sigma^2$ is given, $\alpha$ will determine the variance.

Smaller $\alpha$ values make the variance smaller but it gives more importance to the last observations. There is a trade-off between error variance and the dependency to the recent data.

b) $F_{t,t+\tau} = F_{t-1,t} + \alpha\,(Y_t - F_{t-1,t})$

$$= \alpha Y_t + (1-\alpha)F_{t-1,t} = \alpha Y_t + \alpha(1-\alpha)Y_{t-1} + \alpha(1-\alpha)^2 Y_{t-2} + \dots + \alpha(1-\alpha)^t Y_0$$

$$Var(Y_{t+\tau} - F_{t,t+\tau}) = Var(Y_{t+\tau}) + Var(F_{t,t+\tau}) - 2\,Cov(Y_{t+\tau}, F_{t+\tau})$$

$$Cov(Y_{t+\tau}, F_{t+\tau}) = E[Y_{t+\tau} F_{t+\tau}] - \underbrace{E[Y_{t+\tau}]}_{\mu} E[F_{t+\tau}]$$

$$= E[Y_{t+\tau}\cdot(\alpha Y_t + \alpha(1-\alpha)Y_{t-1} + \dots + \alpha(1-\alpha)^t Y_0] - \mu\cdot E[\alpha Y_t + \alpha(1-\alpha)Y_{t-1} + \dots + \alpha(1-\alpha)^t Y_0]$$

$$= \alpha\cdot E[Y_{t+\tau} Y_t] + \alpha(1-\alpha)E[Y_{t+\tau}\cdot Y_{t-1}] + \dots + \alpha(1-\alpha)^t E[Y_t\cdot Y_0] - \mu\cdot(\alpha E[Y_t] + \alpha(1-\alpha)E[Y_{t-1}] \dots + \alpha(1-\alpha)^t E[Y_0])$$

Since all Y values are independent:

$$E[Y_m\cdot Y_n] = E[(\mu+\varepsilon_m)(\mu+\varepsilon_n)]$$
$$= E[\mu^2 + \mu\varepsilon_n + \varepsilon_m\mu + \varepsilon_m\varepsilon_n] \rightarrow \varepsilon_t \sim N(0,\sigma^2)$$
$$= \mu^2 + 0 + 0 + 0 = \mu^2$$

$$= \alpha\cdot\mu^2 + \alpha(1-\alpha)\mu^2 + \dots + \alpha(1-\alpha)^t\mu^2 - \mu(\alpha\cdot\mu + \alpha(1-\alpha)\mu + \dots + \alpha(1-\alpha)^t\mu)$$

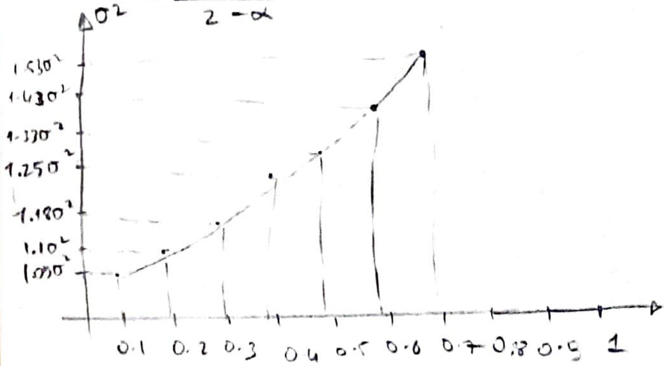
$$= \cancel{\alpha\mu^2} + \cancel{\alpha\mu^2(1-\alpha)} + \dots + \cancel{\alpha(1-\alpha)^t\mu^2} - \cancel{\alpha\mu^2} - \cancel{\alpha(1-\alpha)\mu^2} - \dots - \cancel{\alpha(1-\alpha)^t\mu}$$

$$= 0$$

$$\boxed{Var(Y_{t+\tau} - F_{t,t+\tau}) = Var(Y_t) + Var(F_{t,t+\tau})}$$

The Plot.

$$Var(\varepsilon) = \frac{2\sigma^2}{2-\alpha}$$




As can be seen from the plot, as $\alpha$ value increases, the error variance also increases. Thus, to have a stable forecasting model, $\alpha$ should be small.

2) $Y_t = \mu + C_t + \varepsilon_t \qquad \varepsilon_t \sim (0, \sigma^2)$

$\hat{L}(t) = \alpha(Y_t - \hat{C}_{t-s}) + (1-\alpha)\hat{L}(t-1)$

$\hat{C}_t = \gamma(Y_t - \hat{L}(t)) + (1-\gamma)\hat{C}_{t-s}$

$F_{t,t+\tau} = \hat{L}(t) + \hat{C}_{t-s+\tau}$

$e_t = Y_t - F_t$

2a) $\hat{L}_{\substack{june\\2022}} = 201 \qquad \hat{C}_{july2021} = 1.71 \qquad e_{july\,2022} = 5$

$\hat{L}_{\substack{july\\2022}} = 204 \qquad \hat{C}_{july2022} = 1.81$

$e_t = Y_t - F_t$

$5 = Y_t - [201 + 1.71]$

$\boxed{Y_t = 207.71}$

$204 = \alpha(Y_t - 1.71) + (1-\alpha) \cdot 201$

$= \alpha Y_t - 1.71\alpha + 201 - 201\alpha$

$3 = \alpha Y_t - 202.71\alpha$

$= \alpha(Y_t - 202.71)$

$\Rightarrow \alpha = \dfrac{3}{Y_t - 202.71}$

$= \dfrac{3}{207.71 - 202.71} = \dfrac{3}{5} = 0.6$

$1.81 = \gamma(Y_t - 204) + (1-\gamma)1.71$

$= \gamma Y_t - 204\gamma + 1.71 - 1.71\gamma$

$0.2 = \gamma(Y_t - 202.71)$

$\Rightarrow \gamma = \dfrac{0.2}{Y_t - 205.71} = \dfrac{0.2}{207.71 - 205.71} = \dfrac{0.2}{2} = 0.1$

2b) $SSE = \sum_{j=1}^{48} (Y_j - \hat{Y}_j)^2$ $\boxed{Y_t = \mu + c_t + \varepsilon_t}$ $\boxed{\hat{Y}_t = \hat{\mu} + \hat{c}_t}$ $\hat{c}_m = \hat{c}_{m+12k}$, $k=0,1,2,3$

$\hookrightarrow$ seasonality with length of s

$= (Y_1 - \hat{\mu} - \hat{c}_1)^2 + (Y_2 - \hat{\mu} - \hat{c}_2)^2 + \dots + (Y_{48} - \hat{\mu} - \hat{c}_{12})^2$

To find the estimates of $\mu$ and $c$ values, the partial derivative should be taken:

$$\frac{\partial SSE}{\partial \hat{\mu}} = -2(Y_1 - \hat{\mu} - \hat{c}_1) - 2(Y_2 - \hat{\mu} - \hat{c}_2) - \dots - 2(Y_{48} - \hat{\mu} - \hat{c}_{12}) = 0$$

$$\frac{\partial SSE}{\partial \hat{c}_1} = -2(Y_1 - \hat{\mu} - \hat{c}_1) - 2(Y_{13} - \hat{\mu} - \hat{c}_1) - 2(Y_{25} - \hat{\mu} - \hat{c}_1) - 2(Y_{37} - \hat{\mu} - \hat{c}_1) = 0$$

$$\frac{\partial SSE}{\partial \hat{c}_2} = -2(Y_2 - \hat{\mu} - \hat{c}_2) - 2(Y_{14} - \hat{\mu} - \hat{c}_2) - 2(Y_{26} - \hat{\mu} - \hat{c}_2) - 2(Y_{38} - \hat{\mu} - \hat{c}_2) = 0$$

$\vdots$

$$\frac{\partial SSE}{\partial \hat{c}_{12}} = -2(Y_{12} - \hat{\mu} - \hat{c}_{12}) - 2(Y_{24} - \hat{\mu} - \hat{c}_{12}) - 2(Y_{36} - \hat{\mu} - \hat{c}_{12}) - 2(Y_{48} - \hat{\mu} - \hat{c}_{12}) = 0$$

$\hookrightarrow + \hat{c}_1 \cdot 4 - (Y_1 - \hat{\mu}) - (Y_{13} - \hat{\mu}) - (Y_{25} - \hat{\mu}) - (Y_{37} - \hat{\mu}) = 0$

$$\hat{c}_1 = \frac{(Y_1 - \hat{\mu}) + (Y_{13} - \hat{\mu}) + (Y_{25} - \hat{\mu}) + (Y_{37} - \hat{\mu})}{4}$$

Similarly:

$\hat{c}_2 = [(Y_2 - \hat{\mu}) + (Y_{14} - \hat{\mu}) + (Y_{26} - \hat{\mu}) + (Y_{38} - \hat{\mu})] / 4$

$\hat{c}_3 = [(Y_3 - \hat{\mu}) + (Y_{15} - \hat{\mu}) + (Y_{27} - \hat{\mu}) + (Y_{39} - \hat{\mu})] / 4$

$\hat{c}_4 = [(Y_4 - \hat{\mu}) + (Y_{16} - \hat{\mu}) + (Y_{28} - \hat{\mu}) + (Y_{40} - \hat{\mu})] / 4$

$\hat{c}_5 = [(Y_5 - \hat{\mu}) + (Y_{17} - \hat{\mu}) + (Y_{29} - \hat{\mu}) + (Y_{41} - \hat{\mu})] / 4$

$\hat{c}_6 = [(Y_6 - \hat{\mu}) + (Y_{18} - \hat{\mu}) + (Y_{30} - \hat{\mu}) + (Y_{42} - \hat{\mu})] / 4$

$\hat{c}_7 = [(Y_7 - \hat{\mu}) + (Y_{19} - \hat{\mu}) + (Y_{31} - \hat{\mu}) + (Y_{43} - \hat{\mu})] / 4$

$\hat{c}_8 = [(Y_8 - \hat{\mu}) + (Y_{20} - \hat{\mu}) + (Y_{32} - \hat{\mu}) + (Y_{44} - \hat{\mu})] / 4$

$\hat{c}_9 = [(Y_9 - \hat{\mu}) + (Y_{21} - \hat{\mu}) + (Y_{33} - \hat{\mu}) + (Y_{45} - \hat{\mu})] / 4$

$\hat{c}_{10} = [(Y_{10} - \hat{\mu}) + (Y_{22} - \hat{\mu}) + (Y_{34} - \hat{\mu}) + (Y_{46} - \hat{\mu})] / 4$

$\hat{c}_{11} = [(Y_{11} - \hat{\mu}) + (Y_{23} - \hat{\mu}) + (Y_{35} - \hat{\mu}) + (Y_{47} - \hat{\mu})] / 4$

$\hat{c}_{12} = [(Y_{12} - \hat{\mu}) + (Y_{24} - \hat{\mu}) + (Y_{36} - \hat{\mu}) + (Y_{48} - \hat{\mu})] / 4$

In addition to this, to estimate $\hat{\mu}$, the average of the values should be taken:

$$\hat{\mu} = \frac{1}{T} \sum_{t \in T} Y_t = \frac{1}{48} \sum_{t=1}^{48} Y_t$$

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