Chapter 1

Introduction

There has been a robbery downtown and the robbers are escaping by car. Officers already on the streets are notified moments later. The robbers make a desperate dash for the highway but are spotted and soon tailed by police.

The robbers seem to be getting away – putting some distance between themselves and the sirens. Suddenly, the driver slams on the breaks. A squad car ahead has thrown out a strip of tire spikes! The left two tires are shredded, causing the driver to lose control. The vehicle veers off the road, flips upside down and eventually comes to a stop in the ditch. A media helicopter hovers overhead, capturing a chaotic scene flooded by the flashing lights of emergency vehicles.

Was there ever any hope of escape? Perhaps the robbers took the wrong route. They should have planned a vehicle swap. Or used a tunnel. Could it be that there were so many police officers that all routes were covered? That capture was inevitable? Perhaps the advantages of communication and central coordination allow the police to cut off likely escape routes, so that the probability of escape is low.

A (somewhat dispassionate) mind might watch these salacious stories on the news and wonder if you could apply math to these types of questions. To answer some of the above for sure. Vertex pursuit games are adversarial games played on graphs which model this sort of scenario. By having players take turns moving tokens on a graph (the game board, if you like) with the objective to capture (or evade) the other player, it is possible to simulate such chases.

Many variations of these graph pursuit games have been proposed [4], [5]. There are many rules and parameters to tweak to produce different games:

1. How much information do the players have?

2. Do they know each others positions? From how far away?

3. Do the players know the playing field, i.e., the graph?

4. Are the players restricted to vertices or edges?

5. Are players obligated to move?

6. Does the graph change over time?

The Game of Cops and Robbers on Graphs [6] is perhaps the most well-known vertex pursuit game. It is a perfect information game with Cops trying to catch the Robber. In a perfect information game, all players know everything about the game. In this context, the players know each other's positions (they see each other) and they know the landscape (graph) around them [7].

A variation called Zombies and Survivors (Z & S or Zombie Game) was recently proposed and studied [1], [2]. Z & S is the same as Cops and Robbers with the

added twist that the zombies are required to move directly towards the survivor. More precisely, the zombies have to move along an edge on a shortest path toward the survivor.

This thesis has been an attempt to better understand this variant and to see if the results obtained for Cops and Robbers still hold when the cops are constrained in their strategy. In general, we would like to know how different constraints imposed on the pursuers affects the number of pursuers required to win. We investigate "the cost of being undead", as Fitzpatrick [1] would call it. In particular, in Chapter 2 we give an example of a planar graph where 3 zombies always lose. Then in Chapter 3 we show how two zombies can win on a cycle with one chord.

[handwritten margin note: whereas 3 cops win in the classical version of the game (refer to Section 1.2.3)]

1.1   Notation

The following sections will use a few standard definitions from graph theory (and vertex-pursuit theory) which we include here for reference. Formally, a graph $G = (V, E)$ is composed of:

[handwritten margin note: \emph{.}]

- A set V of vertices.

- A set E of edges $\{u, v\}$ where $u, v \in V$.

We also write $V(G)$ for the set of vertices of G and $E(G)$ for the set of edges of G. Let $G = (V, E)$ be a graph with vertices $x, y \in V$. The graphs studied herein are finite, connected, undirected and reflexive. There is a *finite* number of vertices and *connected* means there exists a path connecting every pair of vertices. We limit ourselves to connected graphs because playing on graphs with multiple connected components can be reduced to playing multiple games in parallel: the players are restricted to their starting connected component. By undirected, we mean that an edge from x to y implies an edge from y to x so we treat the two directions as a single edge and write $\{x, y\}$ or simply $xy = yx \in E$. Lastly, in order to model a player's choice to pass on a turn, we suppose each vertex also has a loop (an edge to itself), making the graph reflexive. This way, players still choose an edge even though they do not move to a different vertex.

[handwritten margin note: Say that graphs don't have parallel edges]

We will have occasion to use a few more concepts of graph theory. We say that vertices x and y are *neighbours* if $xy \in E$; that is, if there is an edge joining x to y. The set $N(x) = \{y \in V | xy \in E\} \subseteq V$ is the *neighbourhood* of x. The *closed neighborhood* of vertex x is the the neighborhood of x along with x itself and is denoted $N[x] = N(x) \cup \{x\} \subseteq V$. A set $S \subseteq V(G)$ is said to be *dominating* if $\cup_x N(x) \supseteq V(G)$. In the context of these games, a dominating set guarantees a win for the pursuers, since all possible start vertices are covered (and the evader loses in round 1).

[handwritten margin note: and x≠y. / something is wrong / and]

[handwritten margin note: and x≠y / be careful because the graph is reflexive]

The *degree* of a vertex is the number of edges incident to that vertex (or equivalently the cardinality of its neighbourhood $|N(x)|$). The minimum and maximum degrees of a graph are min or $\max\{|N(x)| : x \in V\}$ and are denoted as $\delta(G)$ and $\Delta(G)$, respectively.

[handwritten margin note: $\min\{|N(x)| : x \in V\}$]

For example, in Figure 1.1 we have vertices $V = \{a, b, c, d, e, f, g, h\}$. Since a and b are connected by an edge, we have $ab \in E$. The neighbourhood of a is $N(a) = \{b, d, f\}$ and the closed neighbourhood of a is $N[a] = \{a, b, d, f\}$. In this example, we also have that $\delta(G) = \Delta(G) = 3$ since all vertices have degree 3.

[handwritten margin note: we don't draw loops]

Two basic classes of graphs are important in the study of these games: paths and cycles. A path $P = v_0, v_1, v_2, \ldots, v_n$ is a "strict" walk: a non-repeating sequence of adjacent vertices in a graph. A cycle C_n is a path of length $n \geq 3$ with an additional
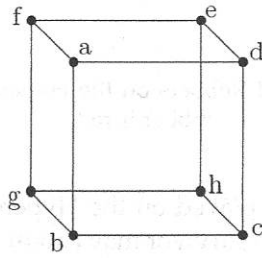
FIGURE 1.1: The Hypercube of Dimension 3

edge joining the last vertex back to the first (a so-called *closed* path). We say that a graph contains a path P if P is a *subgraph* of G, so $V(P) \subseteq V(G)$ and $E(P) \subseteq E(G)$. More generally, with $S \subseteq V(G)$ a set of vertices, write $G[S]$ for the *induced subgraph*: the graph which contains S and the edges of G which join vertices of S.

Paths allow us to define a distance between vertices $d_G(x, y)$ as the length of the shortest path connecting x to y (or infinity if such a path does not exist; never the case in our games). Computing such paths, also known as *geodesics*, is a classic problem in computer science. A geodesic has the additional property of being *isometric* [8], meaning that the distance between vertices of an isometric path is preserved in the subgraph induced by the path. This property allows players to guard or patrol isometric paths [3], preventing the robber from entering (and thus crossing) the path without capture.

Finally, the *diameter* and *girth* of a graph are two useful graph properties which appear in some of the theorems herein. The diameter diam(G) is the length of a longest possible shortest path in G. The girth of a graph is the length of the minimum order subcycle.

1.2 Classical Cops and Robbers

We start with an explanation of the game, then give some key results from C & R since much of our thesis is a comparison between the two games.

1.2.1 How to Play Cops and Robbers

C & R is a two player game: one player controls the cops, the other the robber. The cops begin the game by choosing start vertices. Next, the robber chooooses a start position. On each following round the cops may move along an edge to a neighbouring vertex or remain in position. Here a move is an instantaneous jump between adjacent vertices. If the robber remains uncaught after the cops have had a chance to move, the robber then gets the opportunity to move along an edge.

In this game, the players have complete information of the graph and the positions of the players. The cops move, the robber responds and these two *turns* make one *round*.

The game is decided when either:

- A cop captures the robber. That is, the cop player wins if one of the cops move onto the vertex occupied by the robber.

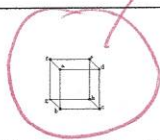- The robber wins if it can evade the cops indefinitely.

too small

FIGURE 1.2: Cops and Robbers on the Hypercube. Cops are blue; robber is red.

Consider a game of C & R played on the Hypercube of dimension 3. On this graph, a single cop will lose: the survivor may choose a vertex at distance 2 and preserve this distance indefinitely by running around a sub-cycle of length 4. However, two cops win handily wherever they may start. Suppose they choose to start on two adjacent vertices, say a and d, as in Figure 1.2. This start is not optimal for the cops – this graph is dominated by two vertices, so they could start directly in such a position. Nevertheless, in two turns the cops can move into a dominating position (like a, h) and capture the robber.

why not combine Fig. 1.1 and 1.2 into one single figure?

Study of vertex-pursuit games is first attributed to Quilliot [9], [10], as well as Nowakowski and Winkler [11]. These researchers independently consider games of C & R with a single cop and characterize by way of a relation those graphs where the cop always wins. These are now known as *cop-win* graphs and can be recognized by the existence of an ordering of the vertices called a *dismantling*; so-called because it is the successive deletion of *corners* resulting in a single vertex (see Subsection 1.2.2).

The *cop-number* of a graph, denoted $c(G)$, was introduced by Aigner and Fromme [3] and defined as the minimum number of cops required to guarantee cops win on graph G. In the example of the hypercube of dimension 2 (see Figure 1.2), a single cop loses but two cops always suffice, and so the cop-number of this graph is 2.

Later, Berarducci et al. and Hahn et al. generalized the characterization of cop-win graphs into *k-cop win* graphs [12], [13]. A graph is *k*-cop win if and only if there exists a function (defined on a *k*-product of the graph to represent the position of the cops) which satisfies certain properties; essentially it is a function which takes as input a position C of cops and returns the next position for the cops that guarantees a win (see [6][p. 119]). There exists a polynomial-time algorithm for deciding whether a graph is *k*-cop-win by iteratively solving for this function, so we can decide if $c(G) \leq k$ for any graph in polynomial time as long as k is fixed and not a function of $|V(G)|$.

Let's discuss this

Meyniel's conjecture posits that $\mathcal{O}(\sqrt{|V(G)|})$ is an upper bound on the cop-number [14]. Incremental progress has been made on special classes of graphs as well as for graphs in general [15][p. 31] (see Subsection 1.2.3).

1.2.2 Dismantlings, Cop-win Trees

A vertex u is a *corner* if its closed neighbourhood is a subset of one of of its neighbours' closed neighbourhood. Formally, this is

$$u \in V(G) \quad \text{and} \quad \exists v \in V(G) : N[u] \subseteq N[v]$$

We say that corner u is *dominated* or *cornered* by v.

By supposing that a single cop wins on G, it can be shown that G must contain a corner: consider the robber's last turn. If the robber loses on the next turn, it must be because the robber cannot stay in place nor can it move to a neighbour without being caught on the next turn. This is precisely the case when the cop is on a vertex which corners the survivor. Moreover, G is cop-win if and only if $G - u$ is cop-win. This

have you defined this notation?

second statement can be shown by playing two games (the *shadow* game strategy): one on G and one on $G - u$.

Quilliot and Nowakowski both indepedently characterized cop-win graphs by combining these observations into what is called a *dismantling*. Informally, a dismantling is the successive folding of a corner onto its dominating vertex until we are left with a single vertex.

Formally, let G be a reflexive graph with $x \in V(G)$ a fixed vertex. A (one-point) *retract* is an edge-preserving function $f : G \to H = G \setminus v$ (aka a homomorphism) such that $f(v) = x$ for some $x \neq v \in V(G)$ and f restricted on H is the identity.

Formally,

$$f(v) = x \qquad f(u) = u \qquad \forall u \in V(G) \setminus \{v\}$$

and

$$xy \in E(G) \implies f(x)f(y) \in E(G \setminus \{v\})$$

Since the graphs studied herein are reflexive, a one-point retract can be seen as the absorption of one vertex by another. The edge between two adjacent vertices becomes another loop. The retract maps a graph G to graph G' with one less vertex.

It is possible to define a retract on corner u: if u is a corner, then it is dominated by some $v \in V(G)$. So if $x \in V(G)$, $x \neq u$ and $xu \in E(G)$ then $xv \in E(G)$ since u is a corner. Therefore the map

$$f(x) = \begin{cases} v & \text{if } x = u \\ x & \text{otherwise} \end{cases}$$

is edge-preserving since $f(x)f(u) = xv$ and $xv \in E(G)$, so $xv \in E(H) = E(G - u)$. For other vertices $x, y \notin \{u, v\}$, $f(x)f(y) = xy \in E(G)$ so $f(x)f(y) \in E(G - v)$ also. This shows that f is a homomorphism as required and hence a retract.

A *dismantling*, then, is a sequence of retracts $f_1, f_2, \ldots, f_{n-1}$ such that the composition $F_{n-1} = f_{n-1} \circ f_{n-2} \circ \cdots f_2 \circ f_1$ gives a function for which $F_{n-1}(G) = K_1$: a sequence of retracts which maps the graph to a single vertex.

Not all vertices of a graph need to be corners in order for there to exist a dismantling: it suffices to have an ordering where each v_i is a corner in $G[\{v_i, v_{i+1}, \ldots, v_n\}]$. Such a sequence of f_i's defines a *cop-win ordering*

$$\mathcal{O} = (v_1, v_2, \ldots, v_n)$$

where v_1 is a corner in $G_1 = G$, v_2 is a corner in $G - v_1$, and so on. A fundamental result in C & R is that cop-win graphs – graphs for which a single cop is guaranteed to win – are characterized by the existence of such dismantlings. A graph is copwin if and only if it is dismantlable; the dismantling provides a winning cop strategy as follows. Start the cop on v_n, the last vertex of the dismantling. On round i with robber on u, move the cop onto vertex $f_{n-i}(u)$. That is to say, the cop captures the "shadow" of the robber at every turn.

A cop-win spanning tree combines the idea of a dismantling with a spanning tree and was first proposed in [16]. A cop-win spanning tree S is defined as a tree where $x, y \in V(G)$ and $xy \in E(S)$ if there exists a retract f_j in the dismantling $F_n = f_n \circ f_{n-1} \circ \cdots \circ f_2 \circ f_1$ such that $f_j(x) = y$ or $f_j(y) = x$ in $G[\{j, j+1, \ldots, n\}]$. Cop-win spanning trees give a strategy for the cops to follow: start at the root (the

last vertex in the ordering) and descend the tree in the b ranch containing the robber. Lemmas 2.1.2 and 2.1.3 from [16] show that the cop can always stay in the same branch (and above) the robber in the tree. So the robber is eventually stuck in a leaf and caught.

1.2.3   The Cop-Number and the Genus of the Graph

One of the most surprising results about the C & R is owed to Aigner and Fromme [3], who showed that the cop number of a planar graph is at most 3. Basically, a graph is planar if it can be drawn in the plane (say, on a piece of paper) without crossing any edges. Aigner and Fromme describe a 3-cop strategy which uses *iso-metric* paths of the graph to encircle and entrap the robber.

Outerplanar graphs are planar graphs which can be drawn such that all vertices belong to a common face (called the *outerface*). Clarke [16] showed that the cop number of outerplanar graphs is 2 by considering two possible cases: those with and without cut vertices. The 2 cops have a winning strategy on outerplanar graphs without cut vertices, and this strategy can be used to cordon off sections (blocks) of the outerplanar graph.

The game has also been studied for graphs embeddable in surfaces of higher order. In 2001, Schroeder conjectured [17] that for a graph of genus g, the cop-number is at most $g + 3$. Currently, the best known bound for graph G of genus g is $c(G) \leq \lfloor \frac{3}{2}g \rfloor + 3$ [18].

[handwritten: does not read well.]

[handwritten margin: no need for \emph{.} because you already defined it.]

1.2.4   Relation to the Girth and Minimum Degree of a Graph

Aigner and Fromme also showed a relationship between the cop-number, the girth of a graph and its minimum degree [3]. More precisely, if G has girth at least 5, then $c(G) \geq \delta(G)$ where $\delta(G)$ is the minimum degree of G.

This result has since been refined [14]: if G has girth at least $8t - 3$ and $\delta(G) = d$, then more than d^t cops are needed to win. In a recent seminar by B. Mohar (Graph Searching Online Seminar, held May 1, 2020) it was argued that a graph with girth g and $\delta(G) = d$ will require at least $\frac{1}{8}(d - 1)^{\lfloor \frac{g-1}{4} \rfloor}$.

[handwritten: cops?]

[handwritten margin: looks very good. it shows you're up to date.]

1.2.5   Cops and Computational Geometry

Intersection graphs are constructed by equating sets with vertices and adding an edge between vertices whenever the intersection of their respective sets are non-empty. Gavenčiak et al. [19] examined the game of C & R on such constructions. First, the authors show that several classes of intersection graphs have unbounded cop-number. Second, they find that the cop-number of intersection graphs of arc-connected subsets is at most $10g + 5$ for an orientable surface of genus g.

[handwritten: Add the specific result for unit disk graphs (Jit really likes them)]

1.2.6   Playing on a Changing Landscape

The game has also been studied on edge-periodic graphs [20]. These are graphs whose edges appear and disappear according to a discrete time function.

1.3   Cops Turn Into Zombies

Zombies and Survivors is a variation of Cops and Robbers first proposed by Fitz-patrick [1] and is the game studied herein. In Z & S games, the cops are replaced

by zombies which always move closer to the robber (who is now a survivor). The sophistication of the zombies' strategy gives them their name: you can imagine the zombies – arms outstretched – ambling directly towards the survivor. However, there is some ambiguity without further precision: how exactly do the zombies move closer to the survivor? What if there are multiple options?

In the version first proposed by Fitzpatrick, the zombies choose their start positions. On their turn, they select a shortest path toward the survivor (a *geodesic*) and moves along the edge to the next vertex of the path. If there are multiple such paths, we assume that a zombie makes an optimal choice (these are deterministic zombies; see 1.3.1) or we choose randomly (so called probabilistic zombies; see 1.3.2). In this thesis, we study the later type of game which is concerned with the worst-case outcomes for the survivor.

The players have complete information of the graph and the positions of the players. Indeed, the zombies need both to enact their strategy. If uncaught, the survivor may move to one of its neighbouring vertices or stay in place.

The game is decided when:

- A zombie eats the survivor by moving to the survivor's vertex.

- The survivor can evade the zombies indefinitely.

We call $s \in V(G)$ the survivor and $z_i \in V(G)$ are zombies with $i \in \{1, \ldots, k\}$. This notation represents both a player and its position in the graph. In the games studied there is a single survivor and $k \geq 1$ of zombies.

As in C & R, we divide the game into *rounds* and *turns*. A round consists of two turns: a zombie turn and a survivor turn. It is convenient to define the zombie's turn on $t \equiv 0 \pmod 2$ and the survivor's turn on $t \equiv 1 \pmod 2$. Round r is given by $\lfloor \frac{t}{2} \rfloor$.

It is occasionally useful to identify the players' positions over time, in which case let $z_r^i \in V(G)$ be zombie i on round r. Similarly s_r is the survivor on round r. This burdensome notation will be omitted when possible.

1.3.1 Deterministic Zombies

Since there can be multiple shortest paths linking a zombie z_k to a survivor s, the zombie may have some limited agency on its turn. The possible *zombie moves* are those neighbours of z_k which lie on a shortest path toward the survivor, which we denote

$$Z[z_k, s] = \{y \in N(x) \mid d(y, s) = d(z_k, s) - 1\}$$

the zombies moves from z_k given survivor is on s.

There is at least one such move since the game graph is presumed connected, so $Z[z_k, s] \neq \emptyset$. If there is only one path, then z_k's has no choice but to move to the next vertex of that path. If all possible shortest paths move through the same next vertex, then again z_k does not have any choice on its move. If, however, there are multiple shortest paths connecting the zombie to the survivor with different first moves, then the zombie could make multiple moves.

A *zombie strategy* or *zombie play* (respectively, *survivor strategy* or *survivor play*) is a sequence of vertices of the graph which represent a zombie's position over the course of a game of Z&S. A game of Z&S can be described with a collection of zombie plays (one for each zombie) together with a survivor play. We say that a zombie play is a winning play if it guarantees the survivor is caught.
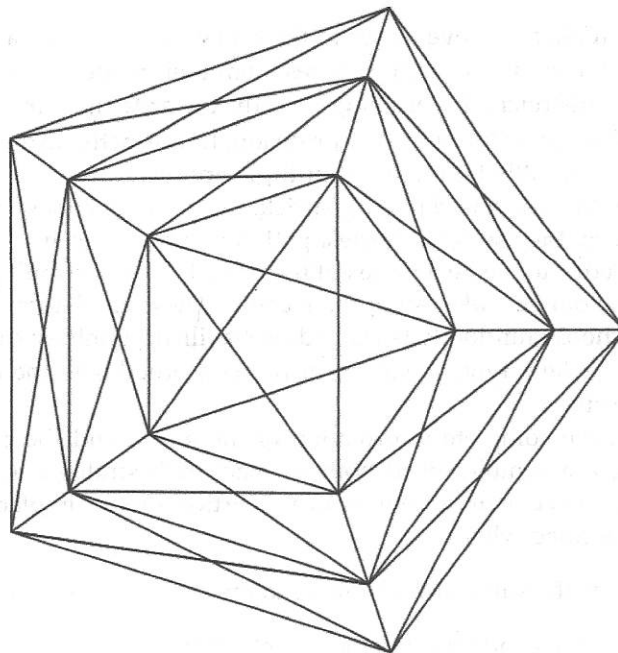
FIGURE 1.3: Cop-Win but not Zombie-Win

In the deterministic version of the game we consider the worst possible outcomes for the survivor: when the zombies play optimally it is as though they coordinate before choosing their next move. A graph is survivor win only if the survivor escapes in every possible zombie-play. The *zombie number* of a graph $z(G)$ can now be defined analogously to the cop number: it is the number of zombies required to guarantee the survivor is captured given an optimal zombie-play. One of the first observations [1] about the zombie number is that:

Lemma 1. For any graph G, $c(G) \leq z(G)$.

Strategies are available to cops which are not available to the zombies, but the cops could apply a zombie strategy. The zombies are weaker versions of cops, similar in a way to the "fully active" cops from [21] wherein the cops are obligated to move on their turn. Nevertheless, on simpler types of graphs the zombies are just as effective as the cops [1]:

Observation 1.   1. $z(T) = c(T) = 1$ for any tree T.

    2. $z(C) = c(C) = 2$ for any cycle of length $n \geq 4$.

    3. $z(K_{m,n}) = c(K_{m,n}) = 2$ for any complete bipartite graph with $2 \leq m \leq n$.

    4. $z(K_n) = c(K_n) = 1$ for $n \geq$.

It is also known that cop-win graphs are not necessarily zombie-win. A counterexample [1] is reproduced below (refer to Figure 1.3). This yields

Theorem 1. If a graph is zombie-win, then it is cop-win. However, a cop-win graph is not necessarily zombie-win.

Cop-win graphs are characterized by a dismantling (see 1.2.2). Does there exist a characterization for zombie-win graphs – for graphs on which a single zombie can

always win? One has yet to be described. Howeover, [1] observed that a graph is zombie-win if a specific cop-win spanning tree exists:

Theorem 2. If there exists a breadth-first search of a graph G such that the associated spanning tree is also a cop-win spanning tree, then G is zombie-win.

Thus a sufficient condition for zombie-win graphs are those for which a specific cop-win tree exists: a cop-win tree obtainable as a breadth-first search of the graph from some root vertex. It is unknown if it is also a necessary condition. To illustrate these ideas, consider the following graph (refer to Figure 1.4), these two dismantlings, their orderings, and the resulting copwin spanning trees.

Let f_i be defined as

$$f_1(b) = f$$
$$f_2(c) = d$$
$$f_3(f) = e$$
$$f_4(a) = e$$
$$f_5(e) = g$$
$$f_6(d) = g.$$

The composition of the f_i is a dismantling with ordering $\mathcal{O}_1 = \{b, c, f, a, e, d, g\}$. Let also g_i be defined as

$$g_1(b) = f$$
$$g_2(a) = e$$
$$g_3(c) = d$$
$$g_4(f) = d$$
$$g_5(e) = d$$
$$g_6(g) = d$$

. These functions are also a dismantling with ordering $\mathcal{O}_2 = \{b, a, c, f, e, g, d\}$. However, only the second produces a copwin tree obtainable as a breadth-first search: they have different final nodes, and thus different roots. Observe that in the tree associated with $\mathcal{O}_1$, the tree-distance from g to c is 2 (and yet it is at distance 1 in the graph).
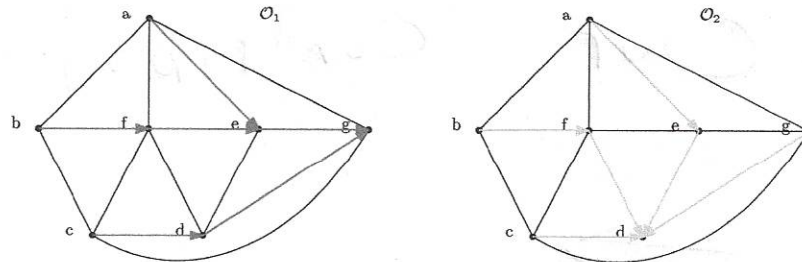

FIGURE 1.4: Two dismantlings and associated trees. The retracts are shown by directed edges. Only the tree associated to $\mathcal{O}_2$ results from breadth-first (shown in green).

Interestingly, a zombie loses if it starts on g (the survivor can use the isometric subcycle $c - e - f - g$ to escape), but the zombie wins if it starts on d (in accordance with 2). The following upper bounds are also known for the strong product of graphs with cycles and trees:

Theorem 3. For any graph G and $n \geq 4$, $z(G \square C_n) \leq 3z(G)$.

Theorem 4. If T is a finite tree, then for any graph G, $z(G \square T) \leq 2z(G)$.

1.3.2   Probabilistic zombies

Zombies are often depicted as mindless or aimless. It is a common trope that zombies idle around, moving in random directions until they somehow (suddenly) distinguish the uninfected. It is only at this point that the zombies will charge.

Such behavior likely inspired another type of pursuit game [22] in which the zombies start randomly on the graph. Once the survivor chooses a start vertex, the zombies "notice" the survivor and start moving directly towards it (again by following a shortest path).

Without knowing where the zombies start, however, it is impossible to know the outcome with certainty. So study of these games becomes probabilistic; zombies win if they have at least a 50% chance of winning. The (probabilistic) zombie number of a graph is the minimum number of zombies required for a 50% chance of winning and this zombie number is obtained for several classes of graphs in [22] and for toroidal grids in [23]. Accepting some uncertainty in the outcomes of games simplifies the problem of the zombie choice: if there is more than one possible zombie move then choose one randomly.

The original paper on probabilistic zombies [22] also includes a lemma which is useful for our work in Chapters 2 and 3:

Lemma 2 (3.1, [22]). The survivor wins on C_n against $k \geq 2$ zombies if and only if all zombies are initially located on an induced subpath containing at most $\lceil \frac{n}{2} \rceil - 2$ vertices.