Definition 1. We define a family of graphs we call *bifurcated cycles* and denote as $Q_{m,n}$. As the name suggests, bifurcated cycles are cycles of length $m+n$ with a single chord which divides the cycle into paths P_1 and P_2 of lengths m and n.

$Q_{m,n}$

Theorem 1. The Bifurcated cycle $Q_{m,n}$ is 2-zombie win if m, n are even.

Proof. We place the two zombies on the longest half of the bifurcated cycle with z_1 at a distance of k from a chorded vertex and z_2 at a further distance of $k + \frac{n}{2}$.

Given this start configuration, we describe all winning strategies for s in terms of m, n, and k.

From that, we will show that for all m and n, there exists at least one value of k such that none of these winning strategies are viable and thus that the survivor will be captured.

Part 1. Notation

Formally, let $u, v \in V(Q_{m,n})$ denote the endpoints of the chord and P_1, P_2 denote the paths on either side of the chord.

By construction we have $|P_1| = m$ and $|P_2| = n$ and we can assume, without loss of generality, that $m \leq n$. We also assume $m, n \geq 2$, since otherwise the construction adds parallel edges or degenerates to K_2.

Let C_1 and C_2 be the subcycles of length $m+1$ and $n+1$ induced by P_1 and P_2 respectively.

Each round of the game is composed of two turns: first the zombies' turn, followed by the survivor's turn. We denote as $z_i^{(t)} \in V(Q_{m,n})$ the position of zombie i (and $s^{(t)}$ the position of the survivor) at round t.

Part 2. The Zombie Start Positions

Now, as mentioned above, we place the two zombies on vertices $z_1^{(0)}$ and $z_2^{(0)}$ on P_2 such that

1. The distance between the two zombies is $d(z_1^{(0)}, z_2^{(0)}) = n/2$, and
2. There is a path $P_5 = v, v_1, v_2, \ldots, v_k = z_1^{(0)}$ of length k between $z_1^{(0)}$ and the chorded vertex v. If $k = 0$, then P_5 is the trivial path v, and $z_1^{(0)} = v$.

diagram1.png

Without loss of generality, we can assume that $0 \leq k \leq n/4$, else we reflect the graph and rename the vertices.

These zombie positions divide P_2 into sub-paths $P_3 = u \ldots z_2^{(0)}$, $P_4 = z_2^{(0)} \ldots z_1^{(0)}$, and $P_5 = v \ldots z_1^{(0)}$.

Part 3. The First k Moves of the Game

Let us first assume that $k > 0$. We consider the case when $k = 0$ in the following part. Notice that if the survivor chooses to start on P_4, then the zombies are guaranteed to win since

$$2 \leq d(z_i^{(0)}, s^{(0)}) \leq n/2 - 2 \qquad \text{for } i = 1, 2$$

Play is effectively restricted to P_4. The zombies move in opposite directions towards the survivor and inevitably corner it.

So we can assume that the survivor does not start on P_4. The survivor must then be on P_1, P_3 or P_5 and, in all of these cases, the zombies' first k moves are clear: the zombies move towards the corded vertices u and v.

To see this, suppose first that the survivor is on P_3 or P_5. Then the zombies move in opposite directions because the survivor must be at distance at least two and so we have

$$2 \leq d(z_i^{(0)}, s^{(0)}) \leq |P_3| + |P_5| + 1 - 2 = n/2 - 1 \qquad \text{for } i = 1, 2$$

so neither zombie can choose to follow P_4.

If the survivor is on P_1, then all shortest paths to the survivor must include u or v (since these are cut vertices) and the shortest paths to these vertices from $z_1^{(0)}$ and $z_2^{(0)}$ cannot include P_4 because

$$\begin{aligned} d(z_1^{(0)}, v) = |P_5| = k \leq \frac{n}{4} < \frac{n}{2} = |P_4| \qquad & \forall n > 0 \\ d(z_1^{(0)}, u) \leq |P_5| + 1 = k + 1 \leq \frac{n}{4} + 1 \leq \frac{n}{4} + \frac{n}{2} = \frac{3n}{4} < n - k = |P_4| + |P_3| \qquad & \forall n > 0, 0 \leq k \leq \frac{n}{4}, k \in \mathbb{Z} \\ d(z_2^{(0)}, u) = |P_3| = \frac{n}{2} - k < \frac{n}{2} = |P_4| \qquad & \forall n > 0 \\ d(z_2^{(0)}, v) \leq |P_3| + 1 = \frac{n}{2} - k + 1 < \frac{n}{2} + k = |P_4| + |P_5| \qquad & \forall n > 0, 0 < k \leq \frac{n}{4}, k \in \mathbb{Z} \end{aligned}$$

We inevitably reach the following scenario: $z_1^{(k)}$ is on the chorded vertex v and $z_2^{(k)}$ is approaching u at a distance of $n/2 - 2k$.

Part 4. Once z_1 Reaches the Chord

If the survivor lies on the path between u and $z_2^{(k)}$, then $z_1^{(k)}$ follows the chord across the cycle and the survivor is encircled. So we can assume now that, if the game is to continue, the survivor must be somewhere on P_1.

Note that if $k = 0$ and this is in fact the first turn of the game, then the survivor loses by starting anywhere on P_2. So we can still assume that the survivor is somewhere on P_1 at a distance $2 \leq \ell \leq m - 1$ from v.

Since s must be on P_1, we can consider all possible zombie decisions and their outcomes. First, z_2 can go clockwise or counterclockwise. Second, z_1 can go clockwise or take the chord to go counterclockwise.

Each zombie has two possible decisions (which depend on the position of the survivor) for a total of four possibilities. We systematically analyze each of these possibilities in the following way:

- Case I z_2 goes clockwise.
  - Case I (A) z_1 goes clockwise.
  - Case I (B) z_1 goes counterclockwise.
- Case II z_2 goes counterclockwise.
  - Case I (A) z_1 goes clockwise.
  - Case I (B) z_1 goes counterclockwise.

Case I: z_2 goes clockwise.

Let us first consider the possibility that z_2 goes clockwise as it is a little different: it is only possible if $k = 0$ since comparing lengths of available z_2v-paths shows

$$\frac{n}{2} + 2k \leq \frac{n}{2} - 2k + 1 \iff k = 0$$

So this outcome is possible only in the following situation: z_2 is exactly at the midpoint of P_2, with paths of length $n/2$ on either side.

Case I(A): z_1 goes clockwise.

Since we assume here that z_2 moves clockwise, we must have $k = 0$ and $\ell \leq m/2$. This eliminates the possibility of *Case I(B)*: if z_2 goes clockwise, z_1 cannot go counterclockwise.

Note that if $\ell = m/2$ then z_2 may go either way and we must include this possibility in both cases.

Since $\ell \leq m/2$, z_1 is forced to follow s clockwise around C_1.

The survivor wishes to maintain distance at least 2 and so is forced to move around C_1. We can assume the initial distance ℓ is preserved since the survivor passing (or even reversing) on its turn is equivalent to choosing smaller initial distance of ℓ.

We fast-forward the game and look at the next event: when the one of the players next attain the chord. Note that if s and z_2 reach u and v on the same round, then z_2 captures the survivor on the next turn.

So either

1. z_2 reaches v before s reaches u; or
2. s reaches u before z_2 reaches v.

Subcase I(A)1: z_2 reaches v before s reaches u.

Since z_2 was at a distance of $n/2$, this event must occur $n/2$ rounds later and z_1 will have pursued the survivor that length around P_1.

We have supposed here that s hasn't yet reached the chord, so there exists a path of length

$$m - \ell - n/2 \geq 1$$

between s and u.

On the following round, z_2 can either follow z_1 clockwise along a hull edge or go counterclockwise using the chord edge. But since

$$m - \ell - \frac{n}{2} + 1 \leq m - 2 - \frac{n}{2} + 1 \leq n - 2 - \frac{n}{2} + 1 = \frac{n}{2} - 1 < \frac{n}{2} + \ell$$

We see that the shortest z_2s-path cannot follow the hull edge. So z_2 takes the chord and moves counterclockwise. After this zombie turn, we have

$$d(z_1, z_2) = \ell - 1 + m - \ell - \frac{n}{2} \leq \frac{m-1}{2}$$

So that the survivor is caught between two zombies on less than half the diameter of C_1. This allows us to conlude that if the zombies start with $k = 0$ and

$$m - \ell - \frac{n}{2} \geq 0$$

then the survivor will lose.

To avoid this scenario, the survivor must choose $\ell > m - \frac{n}{2}$ (i.e. $\ell \geq m - \frac{n}{2} + 1$) while still respecting the restriction that $\ell \leq \frac{m}{2}$.

In order to choose such ℓ we must have

$$m - \frac{n}{2} + 1 \leq \ell \leq \frac{m}{2}$$

or, simply,

$$m + 2 \leq n$$

Such choice for ℓ is impossible for the survivor whenever $m + 2 > n$, so we have a simple winning zombie-strategy for these configurations: choose $k = 0$.

Subcase I(A)2: s reaches v before z_2 reaches u.

It takes $m - \ell$ rounds for s to complete its circuit around C_1 and reach u. So we must have z_2 at distance now $n/2 - (m - \ell)$ from v. This means we require

$$\frac{n}{2} - (m - \ell) \geq 1$$

This inequality allows us to bound ℓ

$$m - \frac{n}{2} + 1 \leq \ell \leq \frac{m}{2}$$

which simplifies to

$$n \geq m + 2$$

Notice that the survivor has won in this scenario since

$$d(s, z_1) = \ell \leq \frac{m}{2} \leq \frac{n}{2}$$

and

$$d(s, z_2) = \frac{n}{2} - (m - \ell) + 1 \leq \frac{n}{2} - m + \left(\frac{m}{2} - 1\right) + 1 = \frac{n}{2} - \frac{m}{2} < \frac{n}{2}$$

That is to say, the two zombies are now on the same side of C_1 at distance at most $\frac{n}{2}$ from the survivor, so the survivor can win by circling clockwise around C_2.

Case II: z_2 goes counterclockwise.

Now, either

(A) $\ell \leq \frac{m}{2}$ which forces z_1 to follow a hull edge onto P_1, or
(B) $\ell \geq \frac{m}{2} + 1$, which forces z_1 take the chord edge to u.

Subcase II(A): We have $\ell \leq \frac{m}{2}$, so that z_1 follows a hull edge towards s.

Subcase II(A)1: z_2 reaches the chord before the survivor.

We have assumed that z_1 is following s in a clockwise direction. We must consider the distances at round $n/2 - k$, when z_2 attains the chord.

Here z_1 must continue in the same direction. In order for the survivor to win, we must have z_2 forced to take the chord on the next move and follow in clockwise direction. This implies that

$$\begin{aligned} 1 + n/2 - 2k + \ell &< m - \ell - (n/2 - 2k) \\ 2\ell &< m - n + 4k - 1 \\ 2\ell &\leq m - n + 4k - 2 \\ \ell &\leq \frac{m - n + 4k - 2}{2} \end{aligned}$$

Since we know $\ell \geq 2$, this allows us to bound ℓ:

$$2 \leq \ell \leq \frac{m - n + 4k - 2}{2}$$

In order to be able to choose ℓ, we must then have

$$2 \leq \frac{m - n + 4k - 2}{2}$$

or

$$k \geq \frac{n - m + 6}{4}$$

Subcase II(A)2: The survivor is able to reach the chord before z_2 closes in.

In order for the survivor to win in this scenario, we must have s able to reach the chord before z_2 gets to u's neighbour on P_2. This implies that

$$\begin{aligned}\frac{n}{2} - 2k - (m - \ell) \geq 2 \\ \ell \geq m + 2k - \frac{n}{2} + 2\end{aligned}$$

Now since $\ell \leq \frac{m}{2}$ we have

$$m + 2k - \frac{n}{2} + 2 \leq \ell \leq \frac{m}{2}$$

So to be able to choose ℓ to make this strategy viable we require

$$m + 2k - \frac{n}{2} + 2 \leq \frac{m}{2}$$

And solving for k gives

$$k \leq \frac{n - m - 4}{4}$$

Subcase II(B): We have $\ell \geq \frac{m}{2} + 1$, so that z_1 follows the chord edge towards s.

Subcase II(B)1: z_2 reaches the chord before s.

We again assume that the survivor preserves its distances of $m - \ell + 1$ from z_1, since moving back or staying still is equivalent to choosing a larger initial value of ℓ. In order for the survivor to win, we must have z_2 forced to follow in the same direction. This implies that

$$\begin{aligned}\frac{n}{2} - 2k - 1 + (m - \ell + 1) < 1 + 2k + \ell - \frac{n}{2} \\ n + m < 1 + 4k + 2\ell \\ 2\ell > n + m - 4k - 1 \\ 2\ell \geq n + m - 4k \\ \ell \geq \frac{n + m - 4k}{2}\end{aligned}$$

Since $\ell \leq m - 1$, we see that

$$\frac{n + m - 4k}{2} \leq \ell \leq m - 1$$

So in order to choose ℓ to enact this strategy we need

$$\frac{n+m-4k}{2} \leq m-1$$

Which allows us to conclude that

$$k \geq \frac{n-m+2}{4}$$

Subcase II(B)2: z_1 follows the chord edge and s reaches the chord before z_2

We start with the same scenario as in (B)1; z_1 is forced to take the chord edge since $\ell \geq \frac{m}{2}+1$.

z_2 was at a distance of $n/2-2k$ from the chorded vertex u and s requires ℓ turns in order to reach v. Thus, in order for the survivor to escape we must have

$$\frac{n}{2} - 2k - \ell \geq 1$$

Solving for ℓ gives

$$\ell \leq \frac{n}{2} - 2k - 1$$

Combined with our lower bound for ℓ this gives

$$\frac{m+2}{2} \leq \ell \leq \frac{n}{2} - 2k - 1$$

So to be able to choose ℓ to make this strategy viable we need

$$\frac{m+2}{2} \leq \frac{n}{2} - 2k - 1$$

Solving for k gives

$$k \leq \frac{n-m-4}{4}$$

Part 5. Conclusion

All together now, we have the following constraints for the different survivor-win scenarios:

II(A)1. $k \geq \frac{n-m+6}{4}$
II(A)2. $k \leq \frac{n-m-4}{4}$
II(B)1. $k \geq \frac{n-m+2}{4}$
II(B)2. $k \leq \frac{n-m-4}{4}$

If any of these conditions on k are true, then the surivor has a winning strategy. So, to guarantee that none of these strategies will work, we must choose k such that

II(A)1. $k < \frac{n-m+6}{4}$
II(A)2. $k > \frac{n-m-4}{4}$
II(B)1. $k < \frac{n-m+2}{4}$
II(B)2. $k > \frac{n-m-4}{4}$

Are all satisfied. Or, equivalently,

II(A)1. $k \leq \frac{n-m+5}{4}$
II(A)2. $k \geq \frac{n-m-3}{4}$
II(B)1. $k \leq \frac{n-m+1}{4}$
II(B)2. $k \geq \frac{n-m-3}{4}$

Now because

$$\frac{n-m-3}{4} < \frac{n-m+1}{4} < \frac{n-m+5}{4}$$

We must choose $k \in [\frac{n-m-3}{4}, \frac{n-m+1}{4}]$. We know there exists such an integer k since:

$$\frac{n-m+1}{4} - \frac{n-m-3}{4} = 1$$

$\square$