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Note

Dominating sets of maximal outerplanar graphs

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ABSTRACT

A dominating set $D \subset V$ of a graph G is a set such that each vertex $v \in V$ is either in the set or adjacent to a vertex in the set. We show that if G is an n-vertex maximal outerplanar graph with $n \geq 3$ having k vertices of degree 2, then G has a dominating set of size at most $\lfloor \frac{n+k}{4} \rfloor$, by a simple coloring method.

1. Introduction

For a graph $G = (V(G), E(G))$ and $S \subset V(G)$, let $N_G(S)$ denote the set of vertices which are adjacent to a vertex of S in G. For $S, T \subset V(G)$, we say that S *dominates* T if $T \subset S \cup N_G(S)$. If $D \subset V(G)$ dominates $V(G)$, then D is said to be a *dominating set* of G. The domination number of G, denoted as $\gamma(G)$, is defined as the minimum cardinality of a dominating set of G. For simplicity, we say that a graph G is a *triangulation* if it is a maximal plane graph. A plane graph G is said to be a *triangulated disc* if all of its faces except the infinite face are triangles.

In 1996, Matheson and Tarjan [4] proved that any triangulated disc G with n vertices satisfies $\gamma(G) \leq \lfloor \frac{n}{3} \rfloor$. Later Honjo, Kawarabayashi and Nakamoto [2] extended this result to triangulations of other surfaces. Matheson and Tarjan conjectured that $\gamma(G) \leq \frac{n}{4}$ for every n-vertex triangulation G with sufficiently large n. In 2010, King and Pelsmajer [3] proved this conjecture for graphs of maximum degree 6. Note that we need two vertices to dominate the six vertices of the octahedron graph, so we cannot omit the condition that n is sufficiently large.

In this paper, we consider this problem for maximal outerplanar graphs and prove the following theorem:

Theorem 1. *Suppose G is an n-vertex maximal outerplanar graph with $n \geq 3$ having k vertices of degree 2; then $\gamma(G) \leq \lfloor \frac{n+k}{4} \rfloor$.*

The same statement as in Theorem 1 is also shown by Campos and Wakabayashi [1], but we prove it independently and give a much shorter proof, by a coloring method.

We say that a graph G is *k-colored* if the vertices of G are colored by at most k-colors such that each vertex has a different color from any of its adjacent vertices. In Section 3, we consider related problems and present some conjectures.

2. Proof of the theorem

The main tool of the proof of Theorem 1 is the 4-coloring stated in the following lemma:

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Lemma 1. *A maximal outerplanar graph G can be 4-colored such that every cycle of length 4 in G has all four colors.*

Proof. We use induction on $|V(G)|$. It is well known that every maximal outerplanar graph has at least two vertices of degree 2. The conclusion of Lemma 1 is trivial if G is a triangle, so we assume $|V(G)| \geq 4$. Since any vertex $v \in V(G)$ of degree 2 belongs to only one cycle of length 4, say C, and $G - v$ is also maximal outerplanar, we can give a 4-coloring to $G - v$ by the induction hypothesis and color v with the remaining color which does not appear in $C - v$. □

In the rest of this section, let G be a maximal outerplanar graph with n vertices. It is easy to see that the following proposition holds:

Proposition 1. *Suppose G is 4-colored such that every cycle of length 4 in G has all four colors, and let $R \subset V(G)$ contain all vertices of some given color. Then R dominates all of the vertices of $V(G)$ except those of degree 2.*

Proof. Let v be an arbitrary vertex of G having degree more than 2, i.e., $|N_G(v)| \geq 3$. Let u, w and x be consecutive vertices of $N_G(v)$ in this order. Since $vuwx$ forms a cycle of length 4, one of $\{v, u, w, x\}$ belongs to R, i.e., v is dominated by R. □

Now let $S = \{v_1, v_2, \ldots, v_k\}$ be the set of vertices of G having degree 2 and u_i be one of the two vertices adjacent to v_i for each $i \in \{1, \ldots, k\}$. We then prepare a set of k additional vertices $S' = \{v'_1, v'_2, \ldots, v'_k\}$, and construct a graph G' such that

$$V(G') = V(G) \cup S' \quad \text{and}$$
$$E(G') = E(G) \cup \{v'_1 v_1, v'_2 v_2, \ldots, v'_k v_k\} \cup \{v'_1 u_1, v'_2 u_2, \ldots, v'_k u_k\}.$$

G' is clearly also maximal outerplanar, so G' can be 4-colored such that every cycle of length 4 in G is 4-colored by Lemma 1. Let T be all of the vertices of some given color in the above mentioned coloring of G'. By choosing a suitable color, we can assume $|T| \leq \lfloor \frac{|V(G')|}{4} \rfloor = \lfloor \frac{n+k}{4} \rfloor$. Finally, let $T \cap S' = \{v'_{i_1}, v'_{i_2}, \ldots, v'_{i_{k'}}\}$ and $T' = (T - S') \cup \{v_{i_1}, v_{i_2}, \ldots, v_{i_{k'}}\}$. Note that each vertex of S has degree 3 in G'. Thus, applying Proposition 1 to G', we can see that T dominates $V(G)$ and so does T', i.e., T' is a dominating set of G satisfying $|T| \leq \lfloor \frac{n+k}{4} \rfloor$. □

3. Conjectures

To strengthen Matheson and Tarjan's theorem, we conjecture for 2-connected planar graphs as follows:

Conjecture 1. *Suppose G is a 2-connected planar graph with n vertices such that each of its vertices of degree 2 belongs to a triangle; then $\gamma(G) \leq \lfloor \frac{n}{3} \rfloor$.*

Note that if G is a cycle of length n, then $\gamma(G) \leq \lceil \frac{n}{3} \rceil$ clearly holds, and hence the same inequality holds if G is a 2-connected outerplanar graph with n vertices, since it has a Hamiltonian cycle. Further, it is easy to see that $\gamma(G) \leq \lfloor \frac{n}{3} \rfloor$ holds if G is outerplanar and satisfies the assumption of Conjecture 1. On the other hand, if we only assume that an n-vertex planar graph G is 2-connected, $\gamma(G) > \lceil \frac{n}{3} \rceil$ is possible. Let $C_1, C_2, \ldots, C_k$ be cycles of length 5, let u_i and v_i be non-adjacent vertices of C_i $(i = 1, 2, \ldots, k)$, and let $G = \bigcup_{i=1}^{k} C_i + \{u_1 v_2, u_2 v_3, \ldots, u_{k-1} v_k, u_k v_1\}$. Then G is 2-connected, and since each C_i needs two vertices to be dominated, $\gamma(G) = 2k = \frac{2}{5}n$. Moreover, it seems that there is no n-vertex connected graph with minimum degree 2 and $\gamma(G) > \frac{2}{5}n$ except the cycle of length 4, so we conjecture as follows:

Conjecture 2. *Suppose $n \geq 5$ and G is an n-vertex connected graph with minimum degree 2; then $\gamma(G) \leq \lfloor \frac{2}{5}n \rfloor$.*

Let us go back to the triangulated-disc case. In [4], Matheson and Tarjan also showed an infinite set of n-vertex triangulated discs with $\gamma(G) = n/3$, which are maximal outerplanar and also sharp examples of our theorem. Now we consider 3-connectivity. Outerplanar graphs are not 3-connected, and it seems that if we assume 3-connectivity, we can expect a better upper bound. Thus, for the 3-connected case, we conjecture as follows:

Conjecture 3. *Suppose G is a 3-connected triangulated disc with n vertices; then $\gamma(G) \leq \lfloor \frac{n+2}{4} \rfloor$.*

Note that the octahedron graph also satisfies the inequality in Conjecture 3, and there are many other examples attaining the equality.

If a triangulated disc G has a Hamiltonian cycle C, the subgraph H of G induced by the set of all of the edges on C or inside the region bounded by C is spanning and maximal outerplanar, so we can get an upper bound of $\gamma(G)$ by applying Theorem 1 to H. Considering Conjecture 3 and Tutte's famous theorem which states that every 4-connected planar graph has a Hamiltonian cycle, we also conjecture as follows for the 4-connected case:

Conjecture 4. *Every 4-connected triangulation has a maximal outerplanar spanning subgraph with two vertices of degree 2.*

Note that a maximal outerplanar subgraph of a triangulation G having k vertices of degree 2 corresponds to an induced tree of the dual graph of G having k vertices of degree 1. There are some results concerning induced trees or forests of planar graphs. Poh [5] proved that the vertices of a planar graph can be partitioned into three vertex sets, each of which induces a "linear forest", i.e., a forest of which every component is a path. Wu [6] shows that the vertices of a 4-connected planar graph can be partitioned into two vertex sets, each of which induces 'pseudoforests', i.e., graphs such that every block has two or three vertices. Using this result, Wu also shows that every triangle-free planar graph has a vertex partition into two induced forests. In view of these results and Conjecture 4, we propose the following conjecture, which includes the 'dual version' of Conjecture 4:

Conjecture 5. *Let G be a plane graph and suppose the dual graph of G is 4-connected. Then G has an induced path P such that $G - P$ is a tree.*

In Conjecture 5 we do not restrict the dual graphs to triangulations, so Conjecture 5 is stronger than Conjecture 4.

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