Zombies and Survivors

on Graphs

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Abstract

Lorem ipsum

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Chapter 1

Introduction

There has been a robbery downtown and the robbers are escaping by car. Officers already on the streets are notified moments later. The robbers make a desperate dash for the highway but are spotted and soon tailed by police.

The robbers seem to be getting away – putting some distance between themselves and the sirens. Suddenly, the driver slams on the breaks. A squad car ahead has thrown out a strip of tire spikes! The left two tires are shredded, causing the driver to lose control. The vehicle veers off the road, flips upside down and eventually comes to a stop in the ditch. A media helicopter hovers overhead, capturing a chaotic scene flooded by the flashing lights of emergency vehicles.

Was there ever any hope of escape? Perhaps the robbers took the wrong route. They should have planned a vehicle swap. Or used a tunnel. Could it be that there were so many police officers that all routes were covered? That capture was inevitable? Perhaps the advantages of communication and central coordination allow the police to cut off likely escape routes, so that the probability of escape is low.

A (somewhat dispassionate) mind might watch these salacious stories on the news and wonder if you could apply math to these types of questions. To answer some of the above for sure. Vertex pursuit games are adversarial games played on graphs which model this sort of scenario. By having players take turns moving tokens on a graph (the game board, if you like) with the objective to capture (or evade) the other player, it is possible to simulate such chases.

Many variations of these graph pursuit games have been proposed [TODO: Cite 2 or 3 general books on the topic]. There are many rules and parameters to tweak to produce different games:

1. How much information do the players have?
2. Do they know each others positions? From how far away?
3. Do the players know the playing field, i.e., the graph?
4. Are the players restricted to vertices or edges?

5. Are players obligated to move?

6. Does the graph change over time?

The Game of Cops and Robbers on Graphs [1] is perhaps the most well-known vertex pursuit game. It is a perfect information game with Cops trying to catch the Robber. In a perfect information game, all players know everything about the game. In this context, the players know each other's positions (they see each other) and they know the landscape (graph) around them [2].

A variation called Zombies and Survivors (Z & S or Zombie Game) was recently proposed and studied [3, 4]. Z & S is the same as Cops and Robbers with the added twist that the zombies are required to move directly towards the survivor. More precisely, the zombies have to move along an edge on a shortest path toward the survivor.

This thesis has been an attempt to better understand this variant and, in particular, to see if the results obtained for Cops and Robbers still hold when the cops are constrained in their strategy. In general, we would like to have a better understanding between the number of cops needed to win and different constraints imposed on the pursuers. In particular, in Chapter 2 we give an example of a planar graph where 3 zombies always lose. Then in Chapter 3 we show how two zombies always win on a cycle with one chord.

1.1 Notation

The following sections will use a few standard definitions from graph theory (and vertex-pursuit theory) which we include here for reference. Formally, a graph $G = (V, E)$ is composed of:

- A set V of vertices.
- A set E of edges $\{u, v\}$ where $u, v \in V$.

The graphs studied herein are finite, connected, and undirected. This means that there is a finite number of vertices and that there exists a path connecting every pair of vertices. We also write $V(G)$ for the set of vertices of G and $E(G)$ for the set of edges of G.

Let $G = (V, E)$ be a graph with vertices $x, y \in V$. By undirected, we mean that an edge from x to y implies an edge from y to x. So we treat the two directions as a single edge and write $\{x, y\}$ or simply $xy = yx \in E$. Playing on graphs with multiple connected components can be reduced to playing multiple games in parallel: the players are restricted to their starting connected component.

We say that vertices x and y are neighbours if $xy \in E$; that is, if there is an edge joining x to y. The set $N(x) = \{y \in V | xy \in E\} \subseteq V$ is the *neighbourhood* of x. The *closed neighborhood* of vertex x is the the neighborhood of x along with x itself and is denoted $N[x] = N(x) \cup \{x\} \subseteq V$.

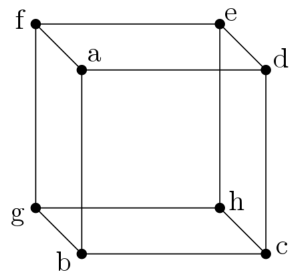


Figure 1.1: The Hypercube of Dimension 3

For example, in Figure 1.1 we have vertices $V = \{a, b, c, d, e, f, g, h\}$. Since a and b are connected by an edge, we have $ab \in E$. The neighbourhood of a is $N(a) = \{b, d, f\}$ and the closed neighbourhood of a is $N[a] = \{a, b, d, f\}$.

The *degree* of a vertex is the number of edges incident to that vertex (or equivalently the cardinality of its neighbourhood $|N(x)|$). The *minimum* and *maximum degrees* of a graph are $\min$ or $\max\{|N(x)||x \in V\}$ and are denoted as $\delta(G)$ and $\Delta(G)$, respectively.

Two basic classes of graphs are important in the study of these games: paths and cycles. A path $P = v_0, v_1, v_2, \ldots, v_n$ is a "strict" walk: a non-repeating sequence of adjacent vertices in a graph. A cycle C_n is a path of length $n \geq 3$ with an additional edge joining the last vertex back to the first (a so-called *closed* path).

We say that a graph contains a path P if P is a subgraph of G, so $V(P) \subseteq V(G)$ and $E(P) \subseteq E(G)$. More generally, with $S \subseteq V(G)$ a set of vertices, write $G[S]$ for the *induced subgraph*: the graph which contains S and the edges of G which join vertices of S.

Paths allow us to define a distance between vertices $d_G(x, y)$ as the length of the shortest path connecting x to y (or infinity if such a path does not exist; never the case in our games). Computing such paths, also known as *geodesics*, is a classic problem in computer science.

A geodesic has the additional property of being *isometric* [5], meaning that the distance between vertices of an isometric paths is preserved in the subgraph induced by the path. This property allows players to guard or patrol isometric paths [6], preventing the robber from entering (and thus crossing) the path without capture.

Finally, the *diameter* and *girth* of a graph are two useful graph properties which appear in some of the theorems herein. The diameter $\text{diam}(G)$ is the length of a longest possible shortest path in G. The girth of a graph is the length of the minimum order subcycle.

1.2 Classical Cops and Robbers

We start with an explanation of the game, then give some key results from C & R since much of our thesis is a comparison between the two games. We investigate "the cost of being undead", as Fitzpatrick [3] would call it.

1.2.1 How to Play Cops and Robbers

C & R is a two player game: one player controls the cops, the other the robber. The cops begin the game by choosing start vertices. Next, the robber chooses a start position. On each following round the cops may move along an edge to a neighbouring vertex or remain in position. Here a move is an instantaneous jump between adjacent vertices. If the robber remains uncaught after the cops have had a chance to move, the robber then gets the opportunity to move along an edge.

In this game, the players have complete information of the graph and the positions of the players. The cops move, the robber responds and these two *turns* make one *round*.

The game is decided when either:

- A cop captures the robber. That is, the cop player wins if one of the cops move onto the vertex occupied by the robber.
- The robber wins if it can evade the cops indefinitely.

1.2.2 Cops and robbers, Cop-Number

Study of vertex-pursuit games is first attributed to Quilliot [7, 8], and Nowakowski and Winkler [9]. These researchers independently consider games of C & R with a single cop and characterize by way of a relation those graphs where the cop always wins. These are now known as *cop-win* graphs and can be recognized by the existence of an ordering of the vertices called a *dismantling*; so-called because it is the successive deletion of *corners* resulting in a single vertex (see Subsection 1.2.5).

The *cop-number* of a graph, denoted $c(G)$, is introduced by Aigner and Fromme [6] and defined as the minimum number of cops required to guarantee they win on a graph G. Later, Berarducci et al. and Hahn et al. generalized the characterization of cop-win graphs into k*-cop win* graphs [10, 11].

A graph is k-cop win if and only if there exists a function (on a k-product of the graph to represent the position of the cops) which satisfies certain properties; essentially it is a function which takes as input a position C of cops and returns the next position for the cops that guarantees a win (see [1][p. 119]). There exists a polynomial-time algorithm for deciding whether a graph is k-cop-win by iteratively solving for this function, so we can decide if $c(G) \leq k$ for any graph in polynomial time as long as k is fixed and not a function of $|V(G)|$.

Another important line of inquiry relating to the cop-number is the investigation of Meyniel's conjecture, which posits that $\mathcal{O}(\sqrt{|V(G)|})$ is an upper bound on the cop-number [12]. Incremental progress has been made on special classes of graphs as well as for graphs in general [13][p. 31].

1.2.3 The Cop-Number and the Genus of the Graph

One of the most surprising results about the C & R is owed to Aigner and Fromme [6], who showed that the cop number of a planar graph is at most 3. Basically, a graph is planar if it can be drawn in the plane (say, on a piece of paper) without crossing any edges. Aigner and Fromme describe a 3-cop strategy which uses *isometric* paths of the graph to encircle and entrap the robber.

Outerplanar graphs are planar graphs which can be drawn such that all vertices belong to a common face (called the *outerface*). Clarke [14] showed that the cop number of outerplanar graphs is 2 by considering two possible cases: those with and without cut vertices. The 2 cops have a winning strategy on outerplanar graphs without cut vertices, and this strategy can be used to cordon off sections (blocks) of the outerplanar graph.

The game has also been studied for graphs embeddable in surfaces of higher order. In 2001, Schroeder conjectured [15] that for a graph of genus g, the cop-number is at most $g+3$. [TODO: Include here the best known bound].

1.2.4 Relation to the Girth and Minimum Degree of a Graph

Aigner and Fromme also show a relationship between the cop-number, the girth of a graph and its minimum degree [6]. More precisely, if G has girth at least 5, then $c(G) \geq \delta(G)$ where $\delta(G)$ is the minimum degree of G.

This result has since been refined [12]: if G has girth at least $8t-3$ and $\delta(G) = d$, then more than d^t cops are needed to win. In a recent seminar by B. Mohar (Graph Searching Online Seminar, held May 1, 2020) it was argued that a graph with girth g and $\delta(G) = d$ will require at least $\frac{1}{g}(d-1)^{\lfloor \frac{g-1}{4} \rfloor}$.

1.2.5 Dismantlings, Cop-win Trees

Quilliot and Nowakowski both indepedently characterized cop-win graphs as those which admit a *dismantling*.

Let G be a reflexive graph with $x \in V(G)$ a fixed vertex. A (one-point) *retract* is an edge preserving function $f : G \to H = G \setminus v$ (aka a homomorphism) such that $f(v) = x$ for some $x \neq v \in V(G)$ and f restricted on H is the identity.

Formally,

$$f(v) = x \qquad f(u) = u \qquad \forall u \in V(G) \setminus \{v\}$$

and

$$xy \in E(G) \implies f(x)f(y) \in E(G \setminus \{v\})$$

.

Since the graphs studied herein are reflexive, a one-point retract can be seen as the absorption of one vertex into another. The edge between two adjacent vertices becomes another loop. The retract maps a graph G to graph G' with one less vertex.

A vertex u is a *corner* if its closed neighbourhood is a subset of one of of its neighbours' closed neighbourhood, i.e.,

$$u \in V(G) \qquad \text{and} \qquad \exists v \in V(G) : N[u] \subseteq N[v]$$

.

It is possible to define a retract on corner u: if u is a corner, then it is dominated by some $v \in V(G)$. So if $x \in V(G)$, $x \neq u$ and $xu \in E(G)$ then $xv \in E(G)$ since u is a corner. Therefore the map

$$f(x) = \begin{cases} v & \text{if } x = u \\ x & \text{otherwise} \end{cases}$$

is edge-preserving since $f(x)f(u) = xv$ and $xv \in E(G)$, so $xv \in E(H) = E(G - u)$. For other vertices $x, y \notin \{u, v\}$, $f(x)f(y) = xy \in E(G)$ so $f(x)f(y) \in E(G - v)$ also. This shows that f is a homomorphism as required and hence a retract. This is a formal way of saying that a corner of a graph can be folded into a dominating vertex.

A *dismantling* is a sequence of retracts $f_1, f_2, \ldots, f_{n-1}$ such that the composition $F_{n-1} = f_{n-1} \circ f_{n-2} \circ \cdots f_2 \circ f_1$ gives a function for which $F_{n-1}(G) = K_1$. That is, there is a sequence of retracts which maps the graph to a single vertex.

Not all vertices of a graph need to be corners in order for there to exist a dismantling: it suffices to have an ordering where each v_i is a corner in $G[\{v_i, v_{i+1}, \ldots, v_n\}]$. Such a sequence of f_i's defines a copwin ordering

$$\mathcal{O} = (v_1, v_2, \ldots, v_n)$$

where v_1 is a corner in $G_1 = G$, v_2 is a corner in $G - v_1$, and so on. A fundamental result in C & R is that cop-win graphs – graphs for which a single cop is guaranteed to win – are characterized by the existence of such dismantlings. A graph is copwin if and only if it is dismantlable.

A cop-win spanning tree combines the idea of a dismantling with a spanning tree and was first proposed in [14]. A cop-win spanning tree S is defined as a tree where $x, y \in V(G)$ and $xy \in E(S)$ if there exists a retract f_j in the dismantling $F_n = f_n \circ f_{n-1} \circ \cdots \circ f_2 \circ f_1$ such that $f_j(x) = y$ or $f_j(y) = x$ in $G[\{j, j+1, \ldots, n\}]$. Cop-win spanning trees give a strategy for the cops to follow: start at the root (the last vertex in the ordering) and descend the tree in the branch containing the robber. Lemmas 2.1.2 and 2.1.3 from [14] show that the cop can always stay in the same branch (and above) the robber in the tree. So the robber is eventually stuck in a leaf and caught.

1.3 Cops Turn Into Zombies

1.3.1 How to Play Zombies and Survivor

Zombies and Survivor is similar to C & R except that zombies move "directly" toward the survivor. More precisely, on the zombies' turn each zombie independently selects a shortest path toward the survivor (a *geodesic*) and moves along the edge to the next vertex of the path.

The sophistication of the zombies' strategy gives them their name: you can imagine the zombies – arms outstretched – ambling directly towards the survivor. As in C & R, the players have complete information of the graph and the positions of the players. Indeed, the zombies need to know the position of the survivor to enact their strategy.

If uncaught, the survivor may move to one of its neighbouring vertices or stay in place. Again, a move is an instantaneous jump along an edge from one vertex to another.

The game concludes when either:

- A zombie eats the survivor by moving to the survivor's vertex.
- The survivor evades the zombies indefinitely.

1.3.2 Modeling the Game

We call $s \in V(G)$ the survivor and $z_i \in V(G)$ are zombies with $i \in \{1, \ldots, k\}$. This notation represents both a player and its position in the graph. In the games studied there is a single survivor and $k \geq 1$ of zombies.

We divide the game into rounds and turns. A round consists of two turns: a zombie turn and a survivor turn. It is convenient to define the zombie's turn on $t \equiv 0 \mod 2$ and the survivor's turn on $t \equiv 1 \mod 2$. Round r is given by $\lfloor \frac{t}{2} \rfloor$.

It is occasionally useful to identify the players' positions over time, in which case let $z_r^i \in V(G)$ be zombie i on round r. Similarly s_r is the survivor on round r. This burdensome notation will be omitted when possible.

It might be tempting to group the zombies together into some tuple of the vertex set, but each zombie acts independently of the others and so this may not be practical.

1.3.3 Paths and Moves

The zombie strategy requires consideration of all geodesics connecting each zombie to the survivor. Let us be precisely define these terms.

Consider zombie k. According to the rules of the game, on its turn the zombie "must move on a shortest path" towards the survivor. The *zombie moves* are those neighbours which lie on a shortest path toward the survivor, and which could be denoted

$$Z[x, s] = \{y \in N(x) \mid d(y, s) = d(x, s) - 1\}$$

the zombies moves from x given survivor is on s.

[fix me here; simplify; clarify]

There is at least one such move since our graph is presumed connected, so $i > 0$ and $Z_k \neq \emptyset$.

If there is only one path, then z_k's next move is $u_{i,1}$. If all zs-paths include $u_{i,1}$, then again z_k's next move must be to that vertex.

If, however, there are multiple zs-paths which have different first moves, then the zombie could make multiple moves.

1.3.4 Deterministic Zombies

Zombies and Survivors (or more specifically, "deterministic zombies") are an interesting variation proposed in [3]. In these games, the Cops are replaced by Zombies which must follow a geodesic to the Survivor.

The Zombie Number is defined analogously to the Cop Number: it is the number of Zombies required to capture the Survivor. However, in Z & S there are two additional considerations: the zombie start and the zombie choices. In this type of game, the starting locations for the zombies is of utmost importance: consider how difficult it might be to evade adversaries which are clustered versus some that are well-dispersed. So we say $z(G) = k$ if k is the minimum number of zombies required to guaranteed a win given an appropriate (or optimal) start. Additionally, the rules of this game permit some agency to the zombies: when confronted with multiple geodesics, they may have a choice between neighbouring vertices. Zombie number also presumes that the zombies make the correct choices. Perhaps more precisely, the zombie number of a graph is k if k zombies, suitably positioned, can play a game which guarantees the survivor is caught.

Unlike Cops, these Zombies cannot apply a cornering strategy. Or any strategy. As a consequence, you need at least as many Zombies as you need Cops. This is one of the first observations in [3]: the Cop Number $c(G)$ is a lower bound of the Zombie Number. The Zombies are weaker versions of Cops, similar in a way to the "fully active" Cops from [16] where the Cops must move on their turn. Both active and "lazy" Cops have more freedom of choice than the zombies, and thus fewer are required to ensure victory.

Does a characterization exist for Zombie-win graphs? Those for which a single zombie can always win? One has yet to be described. Howeover, [3] showed that a graph is zombie-win if a specific spanning tree exists:

Theorem 1 (Fitzpatrick)**.** *If there exists a breadth-first search of a graph G such that the associated spanning tree is also a cop-win spanning tree, then G is zombie-win.*

Thus a sufficient condition for zombie-win graphs are those for which a specific copwin tree exists: one equivalent to a breadth-first search of the graph from some vertex. It is unknown if it is also a necessary condition.

A few questions: are cop-win graphs necessarily zombie-win? No. A counter example [3] is reproduced below 1.2.

Below 1.3 is an example of a graph and two dismantlings, one of which results in a BFS tree, and the other does not.

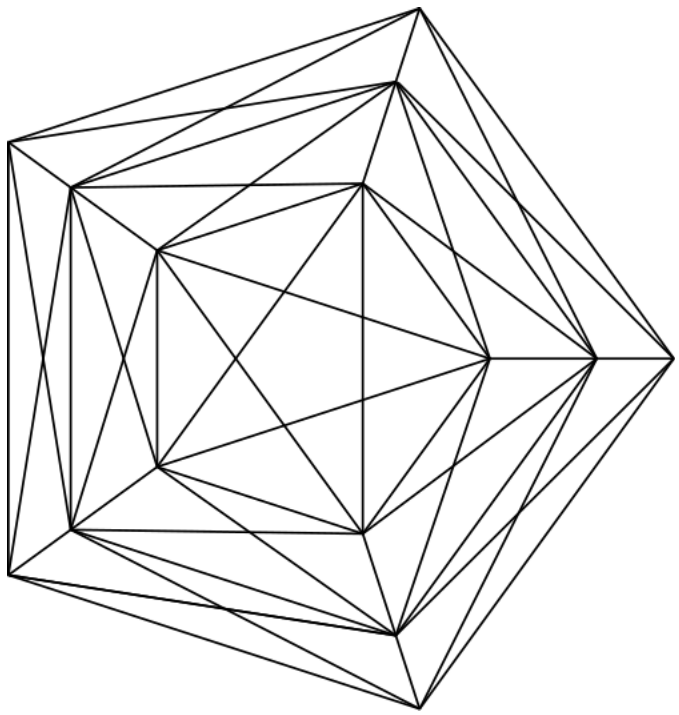


Figure 1.2: Cop-Win but not Zombie-Win

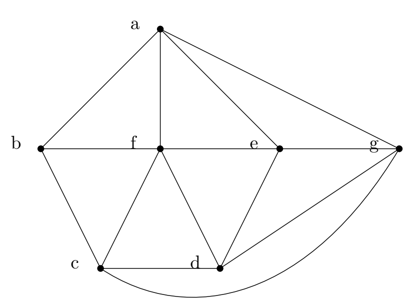


Figure 1.3: A Cop-win tree

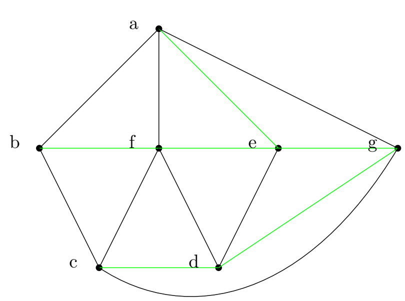
Here are two dismantlings, their orderings, and the resulting copwin spanning trees.

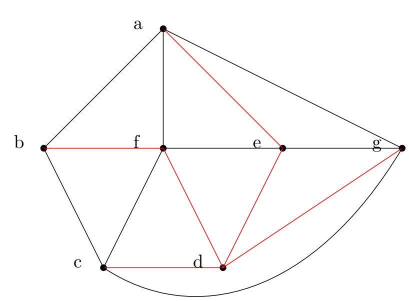
$$\begin{aligned} f_1(b) &= f \\ f_2(c) &= d \\ f_3(f) &= e \\ f_4(a) &= e \\ f_5(e) &= g \\ f_6(d) &= g \end{aligned}$$

Gives ordering $\mathcal{O}_1 = \{b, c, f, a, e, d, g\}$. Whereas

$$\begin{aligned} g_1(b) &= f \\ g_2(a) &= e \\ g_3(c) &= d \\ g_4(f) &= d \\ g_5(e) &= d \\ g_6(g) &= d \end{aligned}$$

Also gives a dismantling with ordering $\mathcal{O}_2 = \{b, a, c, f, e, g, d\}$. But only the second produces a copwin tree obtainable as a bread-first search.





Moreover, it would seem that a zombie loses if it starts on g, but not on d.

1.3.5 Probabilistic zombies

Zombies are often depicted as mindless or aimless. It is a common trope that zombies idle around, moving in random directions until they somehow (suddenly) distinguish the uninfected. It is only at this point that the zombies will charge.

Such behavior likely inspired another type of pursuit game [17] in which the zombies start randomly on the graph. Once the survivor chooses a start vertex, the zombies "notice" the survivor and start moving directly towards it (again by following a shortest path).

Without knowing where the zombies start, however, it is impossible to know the outcome with certainty. So study of these games becomes probabilistic; zombies win if they have at least a 50% chance of winning. The (probabilistic) zombie number of a graph is the minimum number of zombies required for a 50% chance of winning and this zombie number is obtained for several classes of graphs in [17] and for toroidal grids in [18].

The original paper on probabilistic zombies [17] also includes a lemma which is useful for our work in Chapters 2 and 3:

Lemma 1 (3.1, [17])**.** The survivor wins on C_n against $k \geq 2$ zombies if and only if all zombies are initially located on an induced subpath containing at most $\lceil \frac{n}{2} \rceil - 2$ vertices.

Chapter 2

Planar Zombies

Aigner and Fromme [6] showed that the cop number of a planar graph is at most three; three cops can guard isometric paths to constrict the robber territory over time. Unfortunately, Zombies aren't smart enough to apply this strategy. So could a survivor potentially evade an infinite number of Zombies, given the right planar graph? While we have not yet answered this question, we have found a planar graph for which the zombie number is greater than 3. This graph, which is illustrated in Figure 2.1, is an extension of the graph identified by Fitzpatrick [3][Fig. 2] which has $z(G) = 3 > 2 = c(G)$.

Our graph G is constructed by taking a 5-cycle and augmenting it by adding paths of length 5 which connect adjacent vertices of the cycle. We then connect each 5-path to neighbouring 5-paths by way of an edge from the 2nd (or 4th) vertices. Though arbitrary, we fix the embedding described in order to refer to the parts of the graph.

We will call vertices

$C = \{1, \ldots, 5\}$	the interior 5-cycle
$X = V(G) \setminus C$	those vertices not on the interior 5-cycle
$Y = \{7, 9, 12, 14, 17, 19, 22, 24, 27, 29\}$	the vertices of degree 3.
$S = \{7, 8, 9, 12, 13, 14, 17, 18, 19, 22, 23, 24, 27, 28, 29\}$	the outermost 15-cycle

We assume that the zombies choose distinct starting vertices to maximize their chances of winning since the game is easily won by the survivor if there are fewer than 3 zombies (for example, by adding another arbitrary zombie and following one of the strategies described below).

Our proof relies on a special characteristic of this graph: if the survivor and the three zombies are all on $G[S]$, the outermost 15-cycle, with the zombies on the same side of the survivor and within a distance of 2, 3, 4 or 5, then the survivor can win by fleeing away from the zombies around the outermost 15-cycle.

To see this, let $E' = \{xy \in E(G) : x, y \in Y\}$ be the set of edges which connect an exterior 5-path to another and let $G' = G - E'$ be the subgraph without these edges. These edges are highlighted in red in Figure 2.2. The table below 2.1 compares the length of possible

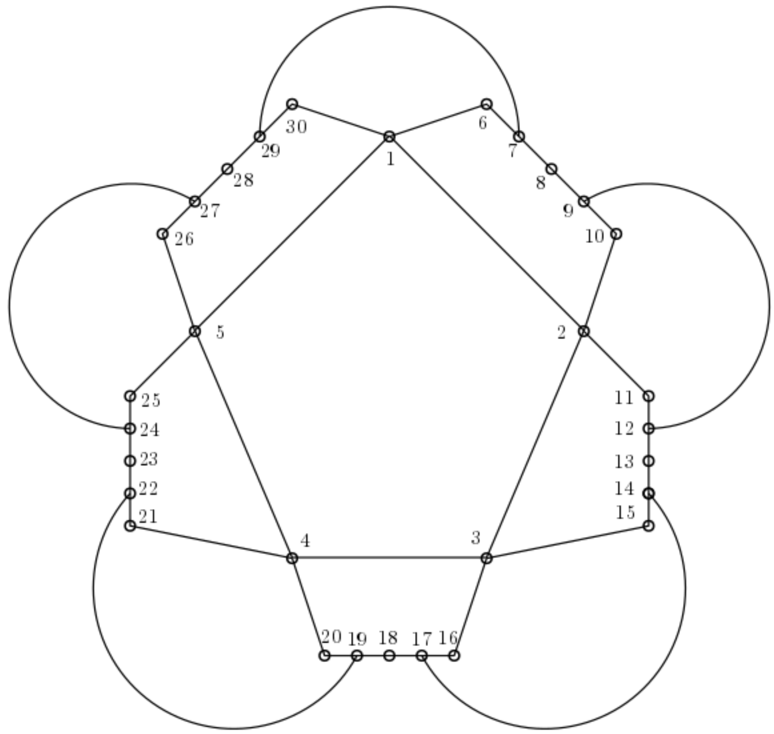


Figure 2.1: A graph with $z(G) > 3$

zombie-survivor paths of lengths at most 5 in both graphs (without loss of generality, the graph having symmetries).

If the survivor and zombie are both on the outermost cycle at distances 2 or 3, then the fact that the zombies must stay in S is clear. When the zombie and the survivor are both in S and within a distance of 4 or 5, then the shortest path from the zombie to the survivor is contained entirely in S and thus zombies never have the opportunity to leave the outermost 15-cycle.

Proof. We must provide a winning survivor strategy for every possible zombie-start arrangement. We divide all possible zombie-starts by the number of zombies which start on the interior 5-cycle.

z	s	shortest path in G	$d_G(z,s)$	shortest path in G'	$d_{G'}(z,s)$
7	14	7,8,9,12,13,14	5	7,6,1,2,3,15,14	6
8	17	8,9,12,13,14,17	5	8,9,10,2,3,16,17	6
9	18	9,12,13,14,17	5	9,10,2,3,16,17,18	6
8	14	8,9,12,13,14	4	8,9,10,2,3,15,14	6
9	17	9,12,13,14,17	4	9, 10, 2, 3, 16, 17	5
12	18	12, 13, 14, 17, 18	4	12, 11, 2, 3, 16, 17, 18	6

Table 2.1: Zombies Cannot Exit the Outermost Cycle

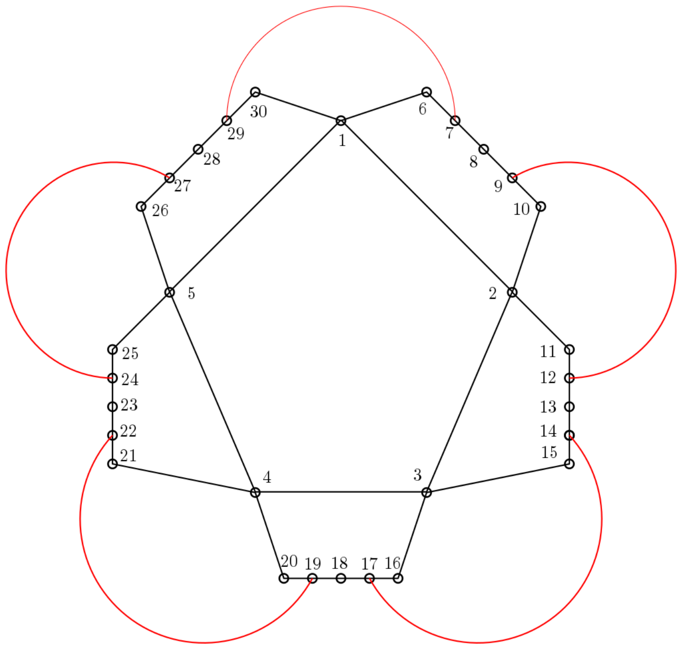


Figure 2.2: An Escape Strategy for the Survivor

- $z_i \in C$ for $1 \leq i \leq 3$: all the zombies start on the interior 5-cycle (refer to Case 2).
- $z_1, z_2 \in C$ and $z_3 \in X = V(G) \setminus C$: two of the zombies are on the interior 5-cycle but one is not (refer to Case 2).
- $z_1, z_2 \in X$: one zombie chooses a vertex on the interior 5-cycle, two others start on the exterior (refer to Case 2).
- All three zombies start on exterior vertices X (refer to Case 2).

Since these cases are exhaustive, the survivor can always respond to a zombie start with a winning strategy, and so $z(G) > 3$.

Case 1 : The three zombies choose vertices on the interior 5-cycle.

Instead of showing that the strategy works for all possible configurations of 3 zombies on the interior 5-cycle, we show that the survivor can win against 5 zombies occupying every vertex of the interior 5-cycle. Since the survivor defeats 5 such zombies, the same strategy will work on any subset of 3.

The zombies occupy the vertices 1–5 and the survivor chooses a vertex $y_1 \in Y$ of degree 3. Without loss of generality, say the survivor chooses 12.

If the survivor starts on $y_1 \in Y$, and moves to $y_2 \in Y$ using edge $y_1 y_2$ and continues to flee in the same direction along the outermost 15-cycle, then the zombies will not be able to catch the survivor. Let us examine the first few rounds in detail. The game begins as illustrated in Figure 2.3.

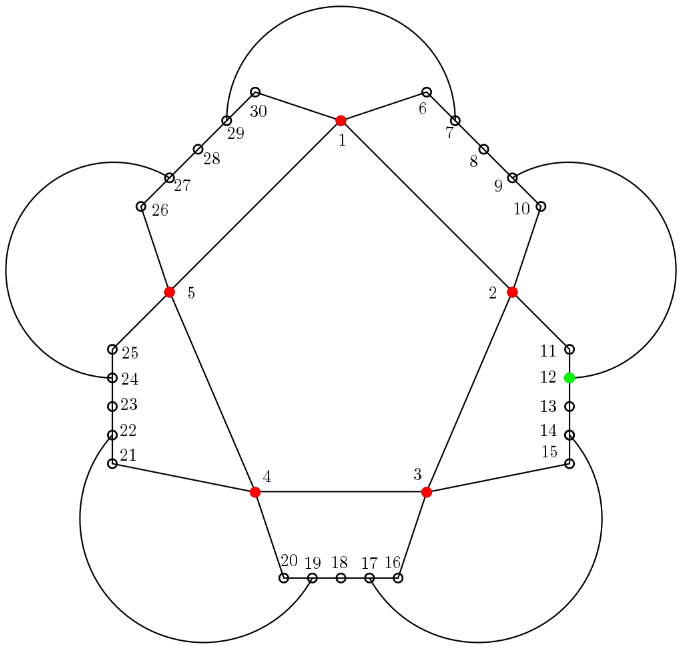


Figure 2.3: Case I, Round 0

On the first round, the zombies each have a single shortest path to the survivor on 12 and thus must move as follows:

- The zombie on 2 moves to 11.
- The zombies on 1 and 3 collide on 2.
- The zombies on 4 and 5 move to 3 and 1, respectively.

The survivor responds by moving to 9. Round 1 moves are illustrated in Figures 2.4:

Yet again the zombies have a single shortest path to the survivor on 9 and thus move as follows:

- The zombie on 11 moves to 12.
- Zombies on 2 move to 10.
- Zombies on 1 and 3 collide on 2.

The survivor responds by moving to 8. These moves are illustrated in Figure 2.5:

Finally, after round 3 all zombies are within a distance of 3 of the survivor on the outermost 15-cycle. See Figure 2.6. The survivor wins by running anti-clockwise on the cycle $G[S]$.

This shows that however the 3 zombies on the interior 5-cycle may be arranged in the initial round, they will not be able to corner the survivor following this strategy.

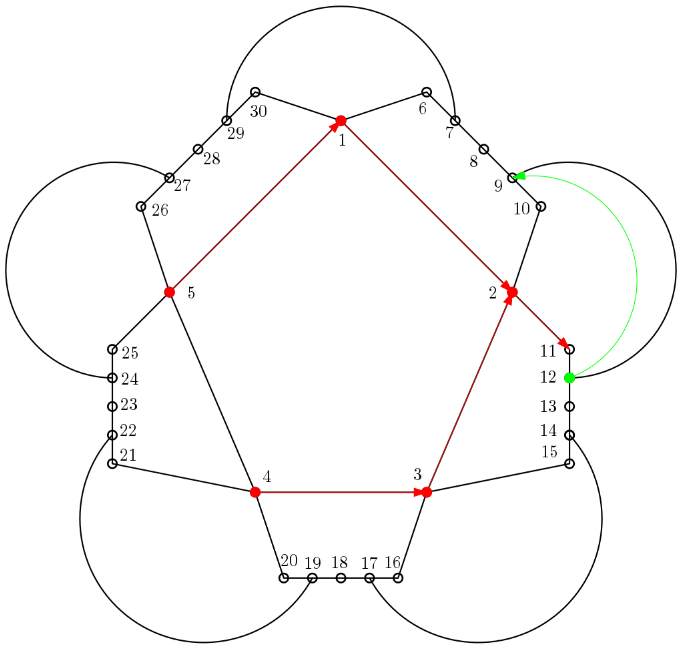


Figure 2.4: Case I, Round 1

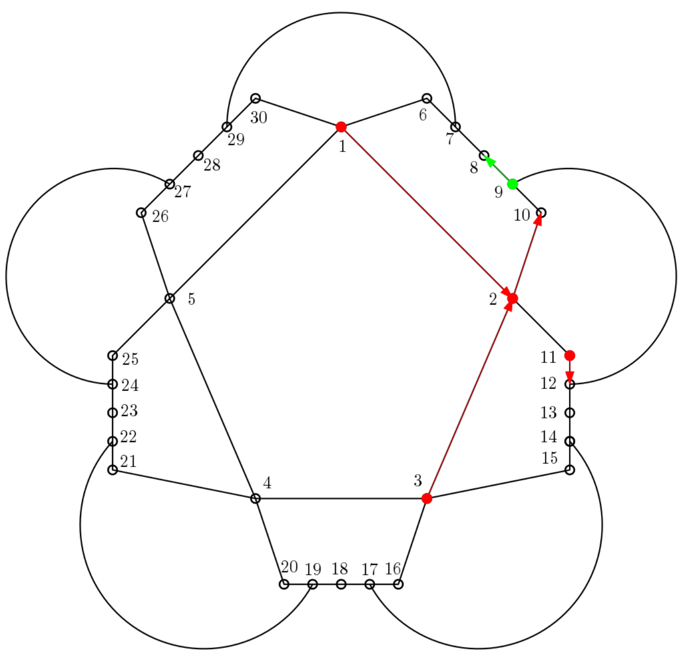


Figure 2.5: Case I, Round 2

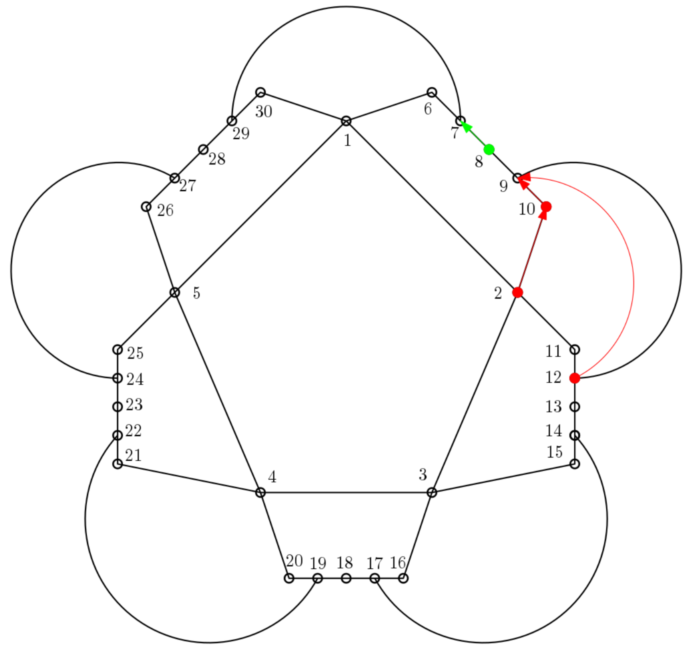


Figure 2.6: Case I, Round 3

Case II : Two zombies (z_1, z_2) choose vertices on the interior 5-cycle and one zombie (z_3) chooses a vertex in $X = \{6, \ldots, 30\}$, an exterior vertex.

We use the same strategy as in Case I with an additional restriction on the survivor's start vertex. The survivor starts on $s = y_1 \in Y$ (a vertex of degree 3) such that $3 \leq d_{G[X]}(s, z_3) \leq 4$ and so that the edge connecting y_1 to $y_2 \in Y$ is not on the shortest path between s and z_3. That is to say, the survivor can flee from z_3 along an edge connecting two exterior 5-paths.

This choice of start vertex is always available to the survivor. See Figure 2.7. Without loss of generality, assume that z_3 has chosen one of the vertices on the exterior 5-path 6-10.

- if z_3 chooses to start at 7 or 6, then the survivor chooses 27, which is at a distance of 3 or 4 respectively.
- if z_3 chooses to start at 8, then the survivor can start at either 14 or 27, both of which are at a distance of 4.
- if z_3 chooses to start at 9 or 10, then the survivor chooses 14, which is at a distance 3 or 4 respectively.

In round 1, if z_3 is not adjacent to the interior 5-cycle (either starting at 7, 8 or 9), then already the zombie has no choice but to pursue the survivor on the outermost 15-cycle.

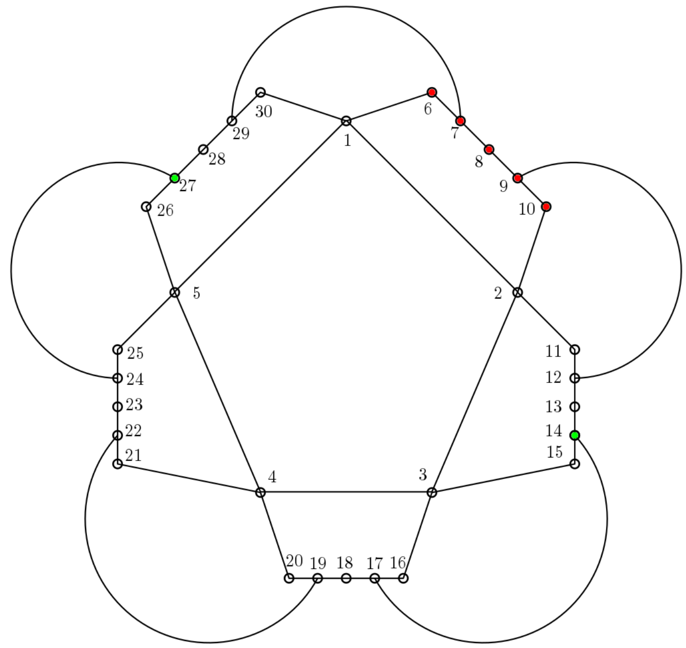


Figure 2.7: Case 2, Round 0

If z_3 is adjacent to the interior 5-cycle (either starting at 6 or 10), then z_3 may choose either to move onto the outermost 15-cycle or to cut through the interior 5-cycle since both are moves on a shortest s, z_3 paths.

However, as above, if z_3 chooses to move onto a vertex in S and follow along the outermost 15-cycle, then the game is already won for the survivor since $d(z_3, s) = 4$ and thus the third zombie can been forced to chase around the outermost 15-cycle forever.

If z_3 chooses to move to the interior cycle then all three zombies are on the interior 5-cycle and we have reached a situation just as in Case I, Round 1 2.4: three zombies are on the interior 5-cycle, and the survivor is on a vertex $y \in Y$. The survivor wins using the strategy from Case I.

This shows that the survivor will always escape the third zombie following this strategy. Now because this strategy is a restricted version of the strategy from Case 1, we know that the zombies that start on the interior 5-cycle will not be able to corner the survivor. Therefore, this strategy defeats all possible start configurations where two zombies start on the interior 5-cycle and the third starts on the exterior.

Case III : One zombie chooses a vertex on the interior 5-cycle, the two others choose vertices on the exterior.

We were unable to develop an argument to show why the survivor wins in this case. Instead, Appendix A.1 provides tables showing the first few moves of a winning survivor strategy for every possible zombie start (without loss of generality).

Case IV : All three zombies choose exterior vertices in X.

We separate this case again into sub-cases based on the number of moves required by the zombies to reach the interior cycle.

Case IV(a) : All three zombies require the same number of rounds to reach the interior 5-cycle.

Suppose all the zombies have chosen vertices in X which are adjacent to vertices in C. These are vertices $Q = \{6, 10, 11, 15, 16, 20, 21, 25, 26, 30\}$. Because there are 3 zombies and 5 interior vertices, there will always be at least two vertices in the interior cycle that are not threatened in round 0. The survivor starts on one of these safe vertices.

In round 1, the zombies have no choice but to enter the interior 5-cycle since the shortest path from a vertex $q \in Q$ to $s \in C$ necessarily includes the edge qc for some $c \in C$. Thus, after their first turn, the zombies all occupy vertices in the interior 5-cycle. The survivor responds by exiting the interior 5-cycle to $s' \in Q$.

In round 2, the zombies again have no choice but to approach the survivor using vertices on the interior 5-cycle. The survivor responds by moving to some $s'' \in Y$ and we have reached a scenario just like in Case I and so the survivor has a winning strategy.

If all the zombies are at a distance of 2 from the interior 5-cycle (those vertices in Y) then the survivor can start on any vertex $s \in C$.

In round 1, the zombies approach the survivor by moving to vertices in Q. Let $q_0, q_1 \in Q \cap N(s)$ be the neighbours of the survivor which are not on the interior 5-cycle. Now, either:

1. q_0 and q_1 are occupied by zombies. In this case, there is some $c \in N(s^0) \cap C$ which is not threatened by a zombie (since two of them are adjacent to s). Therefore the survivor can safely move onto another vertex on the interior 5-cycle and, on the following round, move to an occupied vertex in Q. After another round the survivor moves to a vertex in Y and we again have reached a situation as in Case I.
2. q_0 and q_1 are not both occupied by zombies. In this case, the survivor can exit the interior 5-cycle immediately by moving to a vertex in Q. After the next round, all three zombies are on the interior 5-cycle and the survivor moves to a vertex in Y and again we are in a situation like Case I.

If all the zombies are at a distance of 3 from the interior 5-cycle, then the survivor may start on any vertex of C and simply pass on the first round. The zombies, have no choice but to move to vertices in Y and so we find ourselves in the case described before.

Now we must deal with the cases where the zombies are at different distances from the center cycle.

Case IV(b) : Two zombies start adjacent to the interior 5-cycle, and the third is at distance 2 from the interior 5-cycle.

Suppose that two of the zombies have chosen vertices in Q and the other has chosen a vertex in Y. That is, two zombies are adjacent to the interior 5-cycle while the third requires two rounds to reach the interior 5-cycle.

There are now at least three unthreatened vertices on the interior 5-cycle for the survivor to choose. The survivor can choose any unthreatened vertex on the interior 5-cycle.

In round 1, two zombies enter the interior 5-cycle and the third moves to a vertex $q \in Q$ adjacent to the interior 5-cycle. The survivor exits the interior 5-cycle to another vertex $q_0 \in Q$. This move is always available to the survivor since only one vertex in Q is occupied by a zombie and every vertex in C is adjacent to two verties in Q.

After the next turn, all three zombies are on the interior 5-cycle and the survivor is on a vertex $s^2 \in Y$ and so the survivor has a winning strategy.

Case IV(c) : Two zombies start at a distance of 2 from the interior 5-cycle and the third is at a distance of 3.

The survivor may start on any of the vertices on the interior 5-cycle since none are threatened by a zombie.

In round 1, two zombies move to vertices in Q and the third moves to a vertex in Y. If the survivor is unthreatened after the first round, she may simply pass. If the survivor is threatened by one of the zombies adjacent to the interior 5-cycle, then at least one of her neighbours on the interior 5-cycle is unthreatened.

In either case, after round 1 we find ourselves in the situation described in Case IV(b).

Case IV(d) : Two zombies start adjacent to the interior 5-cycle, and the third is at distance 3 from the interior 5-cycle.

This scenario is slightly more complicated as the survivor must avoid being trapped by the third zombie. Consider, for example, the start configuration $\bar{z} = (10, 26, 18)$. If the survivor chooses to start at 4, then the game plays out as follows:

Round	z_1	z_2	z_3	s
0	10	26	18	4
1	2	5	19	21
2	3	4	22	21

The survivor is cornered by the zombies approaching from the interior 5-cycle and by the third zombie which uses the edge 19-22. However, the survivor could have started at 1, in which case the game is won by the survivor as follows:

Round	z_1	z_2	z_3	s
0	10	26	18	1
1	2	5	17 or 19	6
2	1	1	16 or 20	7
3	6	6	3 or 4	29

And we see that the survivor has a winning strategy for this start configuration.

Suppose without loss of generality that the zombie at distance 3 from the interior 5-cycle has chosen vertex 18. Since there are two zombies adjacent to the interior 5-cycle, at least one of the vertices $\{1, 2, 5\}$ must be a safe start for the survivor.

We may disregard the zombies that started at a distance of 1 from the interior 5-cycle in this next analysis since the survivor's strategy will be the same as in Case IV(a) and so these zombies will not be able to capture the survivor. Having shown above that if 1 is a safe start for the survivor, it remains to show that the strategy works if only 2 or 5 are safe starts. Since they are symmetric, we show that the strategy works if 2 is a safe start for the survivor.

Round	z	s
0	18	2
1	17	10
2	16	9
3	3	8
4	2	7
5	1	29
6	30	28

Thus after 7 rounds, the survivor has succesfully baited all three zombies onto an exterior 5-path and so the game is won.

Case IV(e) : One zombie starts adjacent to the interior 5-cycle, and the other two are at a distance of 2 from the interior 5-cycle.

Again, the survivor's strategy in this case is to waste time on the interior 5-cycle in order to allow all the zombies to approach. Since only one of the zombies is adjacent to the interior 5-cycle, there are four potential start vertices for the survivor. Any of these will work.

In round 1, the zombie at distance 1 from the interior 5-cycle moves onto the interior 5-cycle and the other two move to vertices $q_0, q_1 \in Q$, which are adjacent to the interior 5-cycle.

Now, either:

1. q_0 and q_1 are adjacent to s^0. In this case, the survivor moves to $s^1 \in N(s^0) \cap C$, the neighbour on the interior 5-cycle that is not occupied by the zombie that has already reached the interior 5-cycle. After the next turn, all three zombies have

reached the interior 5-cycle and so the survivor can exit to some $s^2 \in Q$. Again, after another round we have returned to Case I.

2. q_0 and q_1 are not both adjacent to s^0. In this case, the survivor can exit the interior 5-cycle by moving to a vertex $s^1 \in Q$. After the next round, all three zombies are on the interior 5-cycle and we are in a situation like Case I.

In either case, the survivor has a simple winning strategy.

Case IV(f) : One zombie starts at a distance of 2 from the interior 5-cycle, and the other two are at a distance of 3.

The survivor starts in the interior 5-cycle. None of the vertices on the interior 5-cycle are threatened by the zombies, since they are at a distance at least 2.

In round 1, the zombies approach the interior 5-cycle. The zombie that started at distance 2 from the interior 5-cycle is now on a vertex in Q and the other two zombies are on vertices in Y. If unthreatened, the survivor simply passes. If the survivor is threatened by the zombie that is adjacent to the interior 5-cycle, then she moves to another vertex on the interior 5-cycle. The other two zombies pose no threat in this round.

There is now one zombie at distance of 1 from the interior 5-cycle and two zombies at a distance of 2, and so we have returned to the situation describe in Case IV(e).

Case IV(g) : One zombie starts at a distance of 1 from the interior 5-cycle, and the other two are at a distance of 3.

The survivor starts on one of the four safe vertices on the interior 5-cycle.

In round 1, one zombie steps onto the interior 5-cycle while the other two zombies move to vertices at distance 2 from the interior 5-cycle. Only the zombie on the interior 5-cycle can threaten the survivor at this point. If the survivor is safe, then she may pass. Otherwise, since there is only a single zombie on the interior 5-cycle, at most one of the survivor's neighbours on the interior 5-cycle is threatened. So the survivor has a safe move to a vertex on interior 5-cycle.

In round 2, the zombie on the interior 5-cycle pursues the survivor ineffectually while the other two zombies move to vertices $q_0, q_1 \in Q$ which are adjacent to the interior 5-cycle. Now, as in Case IV(e), either

1. q_0 and q_1 are adjacent to s^0. In this case, the survivor moves to $s^1 \in N(s^0) \cap C$, the neighbour on the interior 5-cycle that is not occupied by the zombie that has already reached the interior 5-cycle. After the next turn, all three zombies have reached the interior 5-cycle and so the survivor can exit to some $s^2 \in Q$. Again, after another round we have returned to Case I.

2. q_0 and q_1 are not both adjacent to s^0. In this case, the survivor can exit the interior 5-cycle by moving to a vertex $s^1 \in Q$. After the next round, all three zombies are on the interior 5-cycle and we are in a situation like Case I.

Case IV(h) : The three zombies are at different distances from the interior 5-cycle.

In particular, this means that the zombies are at distances 1, 2 and 3 from the interior 5-cycle.

Observe that there is always a vertex in the interior 5-cycle that is at distance at least 3 from all zombies. This is a start position for the survivor which will allow her to survive unthreatened for at least two rounds.

In round 1, the closest zombie (more precision here - give label) moves onto the interior 5-cycle, the second closest moves to a vertex adjacent to the interior 5-cycle and the third moves to a vertex at a distance of 2 from the interior 5-cycle. The survivor remains in place.

In round 2, the closest zombie threatens the survivor, the second closest zombie moves onto the interior 5-cycle, and the last one moves onto a vertex adjacent to the interior 5-cycle. Now, at least one of the survivor's neighbours is an unoccupied vertex in Q, which she can take to escape the interior 5-cycle.

After the next round, all three zombies are on the interior 5-cycle or one step behind the survivor and the survivor has won the game by moving to a vertex in Y as in Case I.

$\square$

Chapter 3

Cycle With One Chord

Games played on cycles are straightforward: if the zombies are too close, the survivor can lead the zombies in the same direction around the cycle. Otherwise, the zombies are too far apart and whichever side (sub-path of the cycle) the survivor may choose, the zombies will move in opposite directions and win. In this Chapter, we investigate the game on cycles augmented by a single chord.

Definition 1. Take a cycle of length $m+n-2$ and add a chord which divides the cycle into paths P_m and P_n of lengths m and n. Without loss of generality $m \leq n$. We denote such a cycle as $Q_{m,n}$.

Figure 3.1 is an example of such a graph.

The construction contains three sub-cycles which could potentially trip up the zombies as the survivor could trick the zombies into circling on one of these sub-cycles. Let us first examine the construction for small values of m and n.

- Setting $m = n = 1$ gives K_2 with two added loops, which is zombie-win.
- With $m = n = 2$ we have two adjacent cliques K_3 which are dominated by a single vertex, so it is also zombie-win.
- For $m = 2$ and $n \geq 4$, 2 zombies win by starting on diametrically opposed vertices on the cycle C_{n+2}.

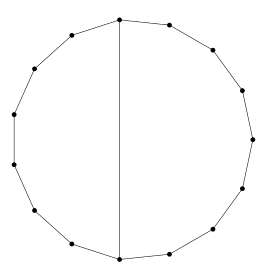


Figure 3.1: $Q_{7,8}$

- If $m = n = 3$ the zombie number is 2 since two zombies on the chord enpoints dominate the graph.
- For $m = 3$, $n = 4$, the zombie number is also 2: placing the zombies on the endpoints of the chord divides the graph into C_4 and C_5 and the zombies clearly win from this position.

The zombie strategy that starts on the endpoints of the chord works for $Q_{3,6}$, $Q_{4.4}$, $Q_{4,5}$ and $Q_{5,5}$ but it does not work for $Q_{3,7}$, $Q_{4,6}$ nor indeed for any $Q_{m,n}$ for $m \geq 3$ and $n \geq 6$.

Unfortunately for the survivor, we are able to show the existence of starting positions for the zombies (obtained as a function of m, n) which limits the survivor's options and prevents the zombies from being led in the same direction.

Theorem 2. The zombie number of $Q_{m,n}$ $(3 \leq m \leq n)$ is 2.

In the proof below, we imagine $Q_{m,n}$ as embbeded in the plane with P_m – the shortest side – on the left. This does not limit the generality of the following and allows us to define (counter-)clockwise distance: the length of the path along a cycle with respect to the given direction on this embedding. Note also that if P_1 and P_2 are two possible zombie-survivor-paths with distinct next moves and

$$|P_1| \leq |P_2|$$

then in the following argument we suppose that the zombie follows $|P_1|$ since that is a valid move.

Proof. We seek a winning zombie start for $m \geq 3$, $n \geq 6$. We describe a strategy in three separate parts, which we summarize here.

The chord is the crux of the game, so we assume that one zombie is on an endpoint of the chord and that the other is at some distance Δ from the chord. We also assume that the survivor is somewhere on P_m, the smaller path. In such a scenario, we know that the first zombie will chase the survivor around the cycle $C_{m+1} = P_m + uv$, which will force the survivor to flee (in the same direction) after at most $\lfloor \frac{m}{2} \rfloor - 2$ rounds.

We can find the intervals of Δ which guarantee the survivor will be cornered by considering all possible combinations of directions chosen by the zombies (refer to Part 3). The zombies' choice of direction is not really a choice, after all: the choice of a direction implies something about the the position of the survivor and the length of the possible zombie-survivor paths.

Next, we show how to position the zombies at the start of the game so that no matter where the survivor starts a losing position is guaranteed: we offset the zombies on the larger cycle with an additional parameter k, which ensures the zombies are not too close together and therefore guard C_{n+1} (refer to Part 3). After k rounds, the survivor will have no choice but to retreat to the smaller cycle and fall into the carefully orchestrated trap described in the first part of the proof.

In Part 3, we show that such a starting position is available to the zombies for any $m \geq 3$, $n \geq 6$. Finally in Part 3 we describe a simple algorithm to compute these winning start positions.

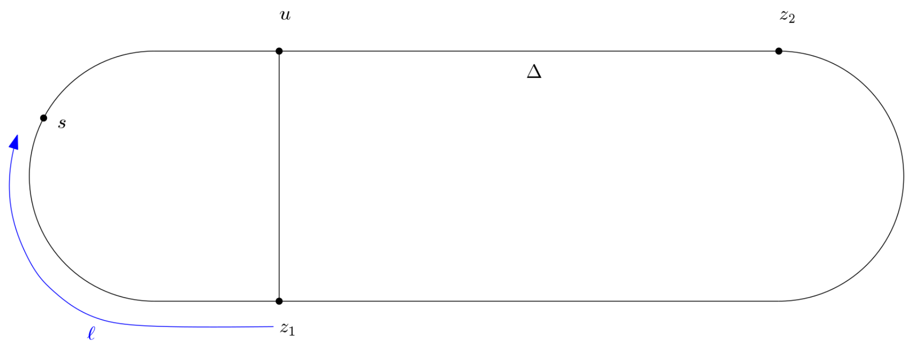


Figure 3.2: One zombie on the chord

Cornering the Survivor on C_{m+1}. Suppose that the game has reached the following state:

- the first zombie is on an endpoint of the chord, say v
- there are Δ vertices counting clockwise from u to z_2.
- the survivor is on P_m at a distance of ℓ vertices counting clockwise from v.

By comparing the lengths of different paths, we calculate the values of Δ which guarantee that the survivor will be cornered on P_m. That is to say, the survivor will be intercepted by z_2 before it can reach any vertex in $Q_{m,n} \setminus P_m$.

Denote as ℓ the length of the clockwise path from v to s. Note that we must have $2 \leq \ell \leq m - 1$ else z_1 captures the survivor on the next round.

We can assume that once z_1 chooses a direction from v that it will continue in that direction: either the zombie has no choice or both directions around the cycle are of the same length (and so may continue in the same direction).

We can also assume that on its turn the survivor will move away from z_1 and maintain a distance of ℓ (or $m - \ell + 1$, if they are moving counter-clockwise) since a winning survivor strategy which involves waiting a turn or moving backwards is equivalent to a survivor strategy which always moves but starts with a smaller (or larger) value of ℓ.

These two assumptions allow us to "fast-forward" the game by Δ rounds and determine when the survivor is captured.

Since z_1 is already on the same cycle as the survivor, it has two options:

A. z_1 goes clockwise if $\ell \leq 1 + m - \ell$. Combined with the bounds on ℓ, this gives $4 \leq 2\ell \leq m + 1$

B. z_1 goes counter-clockwise if $1 + m - \ell \leq \ell$. Combined with the bounds on ℓ, we obtain $m + 1 \leq 2\ell \leq 2m - 2$

There are four possible shortest paths for z_2 to the survivor:

- P_a of length $\Delta + (m - \ell)$
- P_b of length $\Delta + 1 + \ell$
- P_c of length $(n - \Delta) + 1 + (m - \ell)$
- P_d of length $(n - \Delta) + \ell$

Comparing path lengths we see that:

I. z_2 moves counter-clockwise if either $|P_a| \leq \min\{|P_c|, |P_d|\}$ or $|P_b| \leq \min\{|P_c|, |P_d|\}$.

II. z_2 goes clockwise if either $|P_c| \leq \min\{|P_a|, |P_b|\}$ or $|P_d| \leq \min\{|P_a|, |P_b|\}$.

We will examine all combinations of the possible decisions made by the zombies from this configuration:

- I. z_2 goes counter-clockwise
- II. z_2 goes clockwise.
- A. z_1 goes clockwise
- B. z_1 goes counter-clockwise

Case I.A We have the following constraint on ℓ from assumption A.

$$4 \leq 2\ell \leq m + 1$$

and the following constraints on Δ from assumption I.

$$\begin{aligned} \Delta + (m - \ell) &\leq n - \Delta + 1 + m - \ell && \text{and} \\ \Delta + (m - \ell) &\leq n - \Delta + \ell \end{aligned}$$

or

$$\begin{aligned} \Delta + 1 + \ell &\leq n - \Delta + 1 + m - \ell && \text{and} \\ \Delta + 1 + \ell &\leq n - \Delta + \ell \end{aligned}$$

These can be simplified with a bit of algebra and assumption A:

$$\begin{aligned} 2\Delta &\leq n + 1 && \text{and} \\ 2\Delta &\leq n - m + 2\ell \leq n + 1 \end{aligned}$$

or

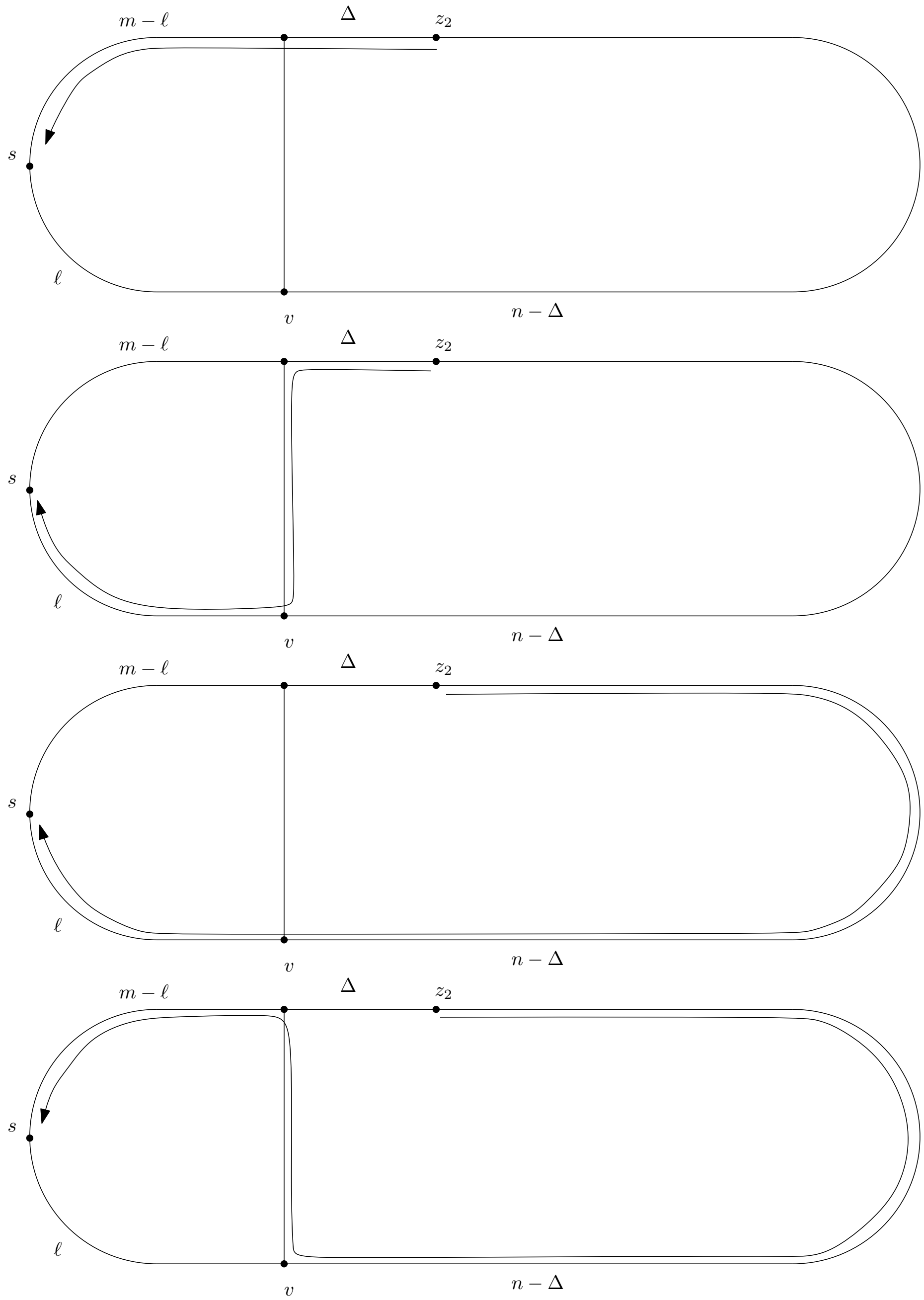


Figure 3.3: Four possible outcomes

$$\begin{aligned} 2\Delta &\leq n+m-2\ell && \text{and} \\ 2\Delta &\leq n-1 \leq n+m-2\ell \end{aligned}$$

So for z_2 to follow either P_a or P_b and go counter-clockwise we must have

$$\begin{aligned} 2\Delta &\leq n-m+2\ell && \text{or} \\ 2\Delta &\leq n-1 \end{aligned}$$

Next we consider: which of s or z_2 reaches u first? If $\Delta = m - \ell$ both z_2 and s reach u on the same round, with the survivor moving onto the zombie-occupied vertex (and losing). If we have $\Delta = m - \ell + 1$, then s reaches u first but is caught by z_2 on the following round. So, to guarantee the survivor has not escaped P_m we need

$$\Delta \leq m - \ell + 1$$

otherwise the survivor can reach the chord at least two rounds before z_2 can move to block. We wish to prevent this scenario since the survivor could then take the chord and possibly escape, pulling both zombies into a loop either on C_m or C_n. This constraint on Δ guarantees that the survivor cannot escape C_m before z_2's arrival in Case I.A.

That is not sufficient, however. We must also ensure that z_2 moves counter-clockwise (opposite to z_1) once it reaches u in order to trap the survivor. So we need

$$m - \ell - \Delta \leq 1 + \Delta + \ell$$

Or, in terms of Δ,

$$2\Delta \geq m - 2\ell - 1$$

When we combine all the restrictions we obtain

Case I.A. Summary

z_1 goes clockwise:

$$4 \leq 2\ell \leq m + 1$$

and z_2 goes counter-clockwise

$$\begin{aligned} 2\Delta &\leq n-m+2\ell && \text{or} \\ 2\Delta &\leq n-1 \end{aligned}$$

the zombies win:

$$\begin{aligned} 2\Delta \leq & 2m - 2\ell + 2 && \text{and} \\ m - 2\ell - 1 \leq & 2\Delta \end{aligned}$$

Case I.B From assumption B and the constraint on ℓ, we must have

$$m + 1 \leq 2\ell \leq 2m - 2$$

and the constraints on Δ from assumption I are again:

$$\begin{aligned} \Delta + (m - \ell) \leq & n - \Delta + 1 + m - \ell && \text{and} \\ \Delta + (m - \ell) \leq & n - \Delta + \ell \end{aligned}$$

or

$$\begin{aligned} \Delta + 1 + \ell \leq & n - \Delta + 1 + m - \ell && \text{and} \\ \Delta + 1 + \ell \leq & n - \Delta + \ell \end{aligned}$$

These can be simplified using assumption B:

$$\begin{aligned} 2\Delta \leq & n + 1 \leq n - m + 2\ell && \text{and} \\ 2\Delta \leq & n - m + 2\ell \end{aligned}$$

or

$$\begin{aligned} 2\Delta \leq & n + m - 2\ell \leq n - 1 && \text{and} \\ 2\Delta \leq & n - 1 \end{aligned}$$

So for z_2 to go counter-clockwise in this case we must have

$$\begin{aligned} 2\Delta \leq & n + 1 && \text{or} \\ 2\Delta \leq & n + m - 2\ell \end{aligned}$$

Again we must consider who reaches the chord first. We have assumed that z_1 is going counter-clockwise. If $\ell = \Delta$, then z_2 reaches u and s reaches v on the same round, and therefore s will be caught on the next. Therefore, to guarantee the survivor has not escaped P_m in this scenario we need

$$\Delta \leq \ell$$

otherwise the survivor reaches the chord before z_2 and could escape.

Then, to ensure that z_2 traps the survivor by going clockwise once it reaches u we need

$$\begin{aligned} 1+\ell-\Delta \leq& \Delta-1+m-\ell+1 \\ 2\ell-m+1 \leq& 2\Delta \end{aligned}$$

Case I.B. Summary

z_1 goes counter-clockwise:

$$m+1 \leq 2\ell \leq 2m-2$$

and z_2 goes counter-clockwise

$$\begin{aligned} 2\Delta \leq& n+1 && \text{or} \\ 2\Delta \leq& n+m-2\ell \end{aligned}$$

the zombies win:

$$\begin{aligned} 2\Delta \leq& 2\ell \\ 2\ell-m+1 \leq& 2\Delta \end{aligned}$$

Case II.A We have the following constraint on ℓ from assumption A.

$$4 \leq 2\ell \leq m+1$$

and the following constraints on Δ from assumption II.

$$\begin{aligned} n-\Delta+\ell \leq& \Delta+(m-\ell) && \text{and} \\ n-\Delta+\ell \leq& \Delta+1+\ell \end{aligned}$$

or

$$\begin{aligned} n-\Delta+1+m-\ell \leq& \Delta+(m-\ell) && \text{and} \\ n-\Delta+1+m-\ell \leq& \Delta+1+\ell \end{aligned}$$

These can be simplified with a bit of algebra:

$$\begin{aligned} n-m+2\ell \leq& 2\Delta && \text{and} \\ n-1 \leq& 2\Delta \end{aligned}$$

or

$$\begin{aligned} n+1 \leq& 2\Delta && \text{and} \\ n+m-2\ell \leq& 2\Delta \end{aligned}$$

These inequalites are of the form

$$\begin{aligned} n-x \leq& 2\Delta && \text{and} \\ n-1 \leq& 2\Delta \end{aligned}$$

or

$$\begin{aligned} n+x \leq& 2\Delta && \text{and} \\ n+1 \leq& 2\Delta \end{aligned}$$

Where $x = m - 2\ell$.

Supposing $x \geq 0$, we have

$$\begin{aligned} n-x \leq& n+x \leq 2\Delta && \text{and} \\ n-1 <& n+1 \leq 2\Delta \end{aligned}$$

and take the lowest bounds because of the disjunction, so that $2\Delta \geq n-x = n-m+2\ell$ and $2\Delta \geq n-1$ suffices.

Since assumption A gives $m - 2\ell \geq -1$, supposing $x < 0$ reduces the inequalites to

$$\begin{aligned} n+1 \leq& 2\Delta && \text{and} \\ n-1 \leq& 2\Delta \end{aligned}$$

which is satisfied by $2\Delta \geq n - x = n - m + 2\ell$ and $2\Delta \geq n - 1$.

Thus z_2 will go clockwise under assumption A if

$$\begin{aligned} 2\Delta \geq& n-m+2\ell && \text{and} \\ 2\Delta \geq& n-1 \end{aligned}$$

We have assumed that z_1 is going clockwise. If $m - \ell = n - \Delta$, then z_2 reaches v and s reaches u on the same round and s will be caught on the next. Therefore, to guarantee the survivor has not escaped P_m in this scenario we need

$$\begin{aligned} n-\Delta \leq& m-\ell \\ \Delta \geq& n-m+\ell \end{aligned}$$

otherwise the survivor could reach the chord before z_2.

After $n - \Delta$ rounds, we have (insert diagram)

Then, to ensure that z_2 goes counter-clockwise once it reaches v, we need

$$\begin{aligned} 1 + m - \ell - (n - \Delta) \leq & n - \Delta + \ell \\ 2\Delta \leq & 2n + 2\ell - m - 1 \end{aligned}$$

All together this gives *Case II.A. Summary*

z_1 goes clockwise:

$$4 \leq 2\ell \leq m + 1$$

and z_2 goes clockwise

$$\begin{aligned} n - m + 2\ell \leq & 2\Delta && \text{and} \\ n - 1 \leq & 2\Delta \end{aligned}$$

the zombies win:

$$\begin{aligned} 2\Delta \geq & 2n - 2m + 2\ell \\ 2\Delta \leq & 2n + 2\ell - m - 1 \end{aligned}$$

Case II.B We have the following constraint on ℓ from assumption B.

$$m + 1 \leq 2\ell \leq 2m - 2$$

and the following constraints on Δ from assumption II.

$$\begin{aligned} n - \Delta + \ell \leq & \Delta + (m - \ell) && \text{and} \\ n - \Delta + \ell \leq & \Delta + 1 + \ell \end{aligned}$$

or

$$\begin{aligned} n - \Delta + 1 + m - \ell \leq & \Delta + (m - \ell) && \text{and} \\ n - \Delta + 1 + m - \ell \leq & \Delta + 1 + \ell \end{aligned}$$

These can be simplified further with a bit of algebra:

$$\begin{aligned} n - m + 2\ell \leq & 2\Delta && \text{and} \\ n - 1 \leq & 2\Delta \end{aligned}$$

or

$$\begin{aligned} n + 1 \leq & 2\Delta && \text{and} \\ n + m - 2\ell \leq & 2\Delta \end{aligned}$$

We have

$$\begin{aligned} n - \Delta + \ell \leq & \Delta + (m - \ell) && \text{and} \\ n - \Delta + \ell \leq & \Delta + 1 + \ell \end{aligned}$$

or

$$\begin{aligned} n-\Delta+1+m-\ell \leq& \Delta+(m-\ell) && \text{and} \\ n-\Delta+1+m-\ell \leq& \Delta+1+\ell \end{aligned}$$

These can be simplified further with a bit of algebra:

$$\begin{aligned} n-m+2\ell \leq& 2\Delta && \text{and} \\ n-1 \leq& 2\Delta \end{aligned}$$

or

$$\begin{aligned} n+1 \leq& 2\Delta && \text{and} \\ n+m-2\ell \leq& 2\Delta \end{aligned}$$

These inequalites are of the form

$$\begin{aligned} n-x \leq& 2\Delta && \text{and} \\ n-1 \leq& 2\Delta \end{aligned}$$

or

$$\begin{aligned} n+1 \leq& 2\Delta && \text{and} \\ n+x \leq& 2\Delta \end{aligned}$$

Where $x=m-2\ell$. Now since assumption B gives $m-2\ell \leq -1$, we see that

$$\begin{gathered} n-1 \leq n-x \leq 2\Delta \\ \text{or} \\ n+x \leq n+1 \leq 2\Delta \end{gathered}$$

Now we consider: which of s or z_2 reaches v first? If $n-\Delta=\ell$, then they both reach u at the same time, with the survivor moving onto the z_2-occupied vertex (and losing). If we have $n-\Delta=\ell+1$, then s reaches u first but is caught by z_2 on the following round. So, to guarantee the survivor has not escaped P_m we need

$$n-\Delta \leq \ell+1$$

otherwise the survivor reaches the chord before z_2 can move to block. If the survivor reaches the chord first, then it could take the chord and possibly escape. (more detail??)

Then, to ensure that z_2 takesgoes clockwise once it reaches v, we need

$$\begin{aligned} \ell - (n - \Delta) \leq& 1 + (n - \Delta - 1) + (m - \ell + 1) \\ 2\Delta \leq& 2n + m - 2\ell + 1 \end{aligned}$$

Case II.B. Summary

z_1 goes counter-clockwise:

$$m + 1 \leq 2\ell \leq 2m - 2$$

and z_2 goes clockwise

$$n + 1 \leq 2\Delta$$

the zombies win:

$$\begin{aligned} n - \Delta \leq& \ell + 1 \\ 2\Delta \leq& 2n + m - 2\ell + 1 \end{aligned}$$

Guarding the large cycle C_{m+1}.

Choose k such that positioning

1. z_2 at $\Delta + k$ clockwise from u
2. z_1 at k counter-clockwise from v

forces the survivor into a losing position: it is either immediately sandwiched on C_{n+1}, or falls into the trap described above on C_{m+1}.

The survivor cannot start next to the zombies else it loses right away. So we choose k such that, even if the survivor is as far away from one of the zombies as possible on C_n, then the zombies still move in opposite directions. This leads to the following inequalities

$$\begin{aligned} n - \Delta - 2k - 2 \leq& \Delta + k + 1 + k + 2 \qquad \text{and} \\ \Delta + 2k - 1 \leq& n - \Delta - 2k + 2x` \end{aligned}$$

Solving for k gives

$$n - 2\Delta - 5 \leq 4k \leq n - 2\Delta + 3$$

Such k guarantees that the zombies start on vertices such that they must move in opposite directions if the survivor plays on C_n.

If the survivor starts between the zombies such that access to the chord is blocked, then clearly it has lost. Otherwise, the zombies must move towards the chord and in k rounds we reach the scenario described in Part 1 when z_1 reaches the chord and z_2 is Δ away. With suitable Δ, then, the survivor cannot win.

Existence of Δ and k for any m, n

We wish to show that, for any m, n, there exist Δ and k which guarantee the survivor is caught. First we show that $\Delta = \lfloor \frac{m}{2} \rfloor$ always works for the cornering strategy.

Note that

$$2\Delta = 2\left\lfloor \frac{m}{2} \right\rfloor = \begin{cases} m & \text{if } m \text{ is even} \\ m-1 & \text{if } m \text{ is odd} \end{cases}$$

and so $m - 1 \leq 2\lfloor \frac{m}{2} \rfloor \leq m$.

Suppose that we are in Case I. A. and $\Delta = \lfloor \frac{m}{2} \rfloor$. Case I. A is characterized by the following constraints:

$$4 \leq 2\ell \leq m + 1$$

and

$$2\Delta \leq n - m + 2\ell$$

or

$$2\Delta \leq n - 1$$

The zombies win if

$$\begin{aligned} 2\Delta \leq& 2m - 2\ell + 2 && \text{and} \\ m - 2\ell - 1 \leq& 2\Delta \end{aligned}$$

So if we are in Case I. A. and $\Delta = \left\lfloor \frac{m}{2} \right\rfloor$ the zombies win since

$$\begin{aligned} 2\Delta = 2\lfloor \frac{m}{2} \rfloor \leq m < 2m - (m+1) + 2 \leq 2m - 2\ell + 2 && \text{and} \\ m - 2\ell - 1 \leq m - 5 < 2\lfloor \frac{m}{2} \rfloor = 2\Delta \end{aligned}$$

Which shows that the zombie-win requirements are met.

Suppose now that we are not in Case 1. A. Negating the constraints of Case I. A. gives

$$2\Delta \geq n - m + 2\ell + 1$$

and

$$2\Delta \geq n - 1 + 1$$

or

$$m + 1 \leq 2\ell \leq 2m - 2$$

If we assume that m is odd and $2\Delta \geq n$ then we obtain a contradiction since

$$2\Delta = 2\lfloor\frac{m}{2}\rfloor = m - 1 \geq n$$

and we have assumed that $m \leq n$.

If m even, $m = n$ and $2\Delta \geq n - m + 2\ell + 1$ then

$$\begin{aligned} 2\Delta &\geq n - m + 2\ell + 1 \\ m &\geq m - m + 2\ell + 1 \\ m &\geq 2\ell + 1 \\ 2\ell &\leq m - 1 \end{aligned}$$

So, if $m = n$ and they are even, then we are in Case 1. A unless $2\ell \leq m - 1$.

To recap: If we set $\Delta = \lfloor\frac{m}{2}\rfloor$, we are in Case 1.A unless

$$m = n \qquad \text{and they are even}$$

$$\Delta = \lfloor\frac{m}{2}\rfloor = \frac{m}{2}$$

$$4 \leq 2\ell \leq m - 1$$

Now, can we be in Case 1. B? Case 1. B is described by the following constraints:

$$m + 1 \leq 2\ell \leq 2m - 2$$

and

$$2\Delta \leq n + 1$$

or

$$2\Delta \leq n + m - 2\ell$$

The negation of which is:

$$2\Delta \geq n + 1 + 1$$

and

$$2\Delta \geq n + m - 2\ell + 1$$

or

$$4 \leq 2\ell \leq m + 1$$

But this leads to the contradiction:

$$n \geq m \geq 2\Delta \geq n + 2$$

In remains to check if we win in Case 2. A.

Assuming still that

$$m = n \qquad \text{they are even}$$
$$\Delta = \frac{m}{2}$$
$$4 \leq 2\ell \leq m - 1$$

The win conditions require

$$\begin{aligned} 2n - 2m + 2\ell \leq 2\Delta &\leq 2n + 2\ell - m - 1 \\ 2m - 2m + m - 1 \leq 2\Delta &\leq 2m + 4 - m - 1 \\ m - 1 \leq 2\Delta &\leq m + 3 \end{aligned}$$

Which holds for $\Delta = \frac{m}{2}$.

Computing Δ

Given m and n, we choose Δ so that whenever we reach the scenario described in the first part, the survivor will be cornered. Such Δ must satisfy the following constraints for any possible value of ℓ.

Case I.A. Summary

z_1 goes clockwise:

$$4 \leq 2\ell \leq m + 1$$

and z_2 goes counter-clockwise

$$\begin{aligned} 2\Delta \leq& n - m + 2\ell && \text{or} \\ 2\Delta \leq& n - 1 \end{aligned}$$

the zombies win:

$$\begin{aligned} 2\Delta \leq& 2m - 2\ell + 2 && \text{and} \\ m - 2\ell - 1 \leq& 2\Delta \end{aligned}$$

Case I.B. Summary

z_1 goes counter-clockwise:

$$m + 1 \leq 2\ell \leq 2m - 2$$

and z_2 goes counter-clockwise

$$\begin{aligned} 2\Delta \leq & n + 1 \\ 2\Delta \leq & n + m - 2\ell \end{aligned} \qquad \text{or}$$

the zombies win:

$$\begin{aligned} 2\Delta \leq & 2\ell \\ 2\ell - m + 1 \leq & 2\Delta \end{aligned}$$

Case II.A. Summary

z_1 goes clockwise:

$$4 \leq 2\ell \leq m + 1$$

and z_2 goes clockwise

$$\begin{aligned} n - m + 2\ell \leq & 2\Delta \\ n - 1 \leq & 2\Delta \end{aligned} \qquad \text{and}$$

the zombies win:

$$\begin{aligned} 2\Delta \geq & 2n - 2m + 2\ell \\ 2\Delta \leq & 2n + 2\ell - m - 1 \end{aligned}$$

Case II.B. Summary

z_1 goes counter-clockwise:

$$m + 1 \leq 2\ell \leq 2m - 2$$

and z_2 goes clockwise

$$n + 1 \leq 2\Delta$$

the zombies win:

$$\begin{aligned} n - \Delta \leq & \ell + 1 \\ 2\Delta \leq & 2n + m - 2\ell + 1 \end{aligned}$$

A simple algorithm to calculate possible values of Δ loops over $0 \leq \Delta \leq n$ and over $2 \leq \ell \leq m - 1$ and tests, for each Δ and each ℓ, to determine which of the four cases is applicable and, if in one of the cases, whether the zombie-win constrains are satisfied. A value of Δ is accepted if, for every value of ℓ, the zombies win.

Once we have obtained possible Δ, we can then determine k by calculating the bounds

$$n - 2\Delta - 5 \leq 4k \leq n - 2\Delta + 3$$

□

Chapter 4

Conclusion, Future Works

In Chapter 2, we showed the existence of a graph for which 3 zombies always lose, thereby showing that the upper bound on the cop-number for planar graphs does not apply to zombies. This is hardly surprising, since the 3 Cops must effect a sophisticated strategy in order to capture the Robber, and the Zombies cannot coordinate in this way.

It remains to be shown if there is in fact an upper bound on the zombie-number for planar graphs. The example obtained in this thesis was a sort of extrapolation from the example given [3], which showed that the cop-number need not always equal the zombie-number. Is it possible to construct increasingly elaborate graphs (while still being planar) which would always provide the survivor with a winning strategy?

Having made no further progress in this direction, we decided to investigate a simpler class of graphs: outerplanar ones. In this case, as we have noted, it has been shown [14] that 2 Cops suffice to guarantee a win.

It is also known that maximally-outerplanar graphs are zombie-win [3] and it is clear that 2 Zombies suffice for a cycle, but what can be said about those outerplanar graphs in between the two extremes?

It has been our experience that 2 Zombies often suffice on outerplanar graphs. But not always. The choice of zombie start is critical. This is the motivation for our work on $Q_{m,n}$ – the cycle with a single chord. Perhaps if we could segment or decompose an outerplanar graph into simpler components, then we could at least give an upper bound: perhaps 1 or 2 Zombies per block. It is not clear how we can generalize our findings however. Adding a single extra chord changes the entire game.

Finally, we spent some considerable time pondering games of Z & S on visibility graphs. Recently, [19] applied a result about visibility-augmenting edges from [20] to conclude that visibility graphs of simple polygons are cop-win. A natural question then is to wonder if they are also zombie-win.

We have implemented tools which allow us to search, brute force, for Breadth-First Search dismantling trees (i.e., zombie-win trees). So far, every polygon tested produces a visibility graph which admits such a tree. See 4.1 for an example.

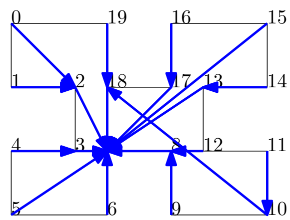


Figure 4.1: A Polygon Inscribed with a BFS Cop-win Tree

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Appendix A

End Matter

A.1 Planar Zombies Counter-Example Case III

Here are all the possible start configurations (without loss of generality) of Case III with the first few moves demonstrating that the survivor wins.

Round	z_1	z_2	z_3	s
0	1	6	11	3
1	2	1	2	4
2	3	5	3	20
3	4	4	4	19

Round	z_1	z_2	z_3	s
0	1	6	12	3
1	2	1	11	4
2	3	5	2	20
3	4	4	3	19

Round	z_1	z_2	z_3	s
0	1	6	13	3
1	2	1	14	4
2	3	5	15	20
3	4	4	3	19

Round	z_1	z_2	z_3	s
0	1	6	14	4
1	5	1	15	21
2	4	5	3	22

Round	z_1	z_2	z_3	s
0	1	6	15	4
1	5	1	3	21
2	4	5	4	22

Round	z_1	z_2	z_3	s
0	1	7	11	3
1	2	6	2	4
2	3	1	3	21
3	4	5	4	22

Round	z_1	z_2	z_3	s
0	1	7	12	3
1	2	6	11	4
2	3	1	2	21
4	4	5	3	22

Round	z_1	z_2	z_3	s
0	1	7	13	3
1	2	6	14	4
2	3	1	15	21
3	4	5	3	22

Round	z_1	z_2	z_3	s
0	1	7	14	3
1	2	6	15	4
2	3	1	3	21
3	4	5	3	22

Round	z_1	z_2	z_3	s
0	1	7	15	4
1	5	6	3	21
2	4	1	4	22

Round	z_1	z_2	z_3	s
0	1	8	11	3
1	2	9	2	4
2	3	10	3	5
3	4	2	4	26
4	5	1	5	27
5	26	5	26	28

Round	z_1	z_2	z_3	s
0	1	8	12	3
1	2	9	11	4
2	3	10	2	5
3	4	2	1	26
4	5	1	5	27

Round	z_1	z_2	z_3	s
0	1	8	13	3
1	2	9	14	4
2	3	10	15	5
3	4	2	3	26
4	5	1	4	27
5	26	5	5	28
6	27	26	26	29

Round	z_1	z_2	z_3	s
0	1	8	14	3
1	2	9	15	4
2	3	10	3	5
3	4	2	4	26
4	5	1	5	27

Round	z_1	z_2	z_3	s
0	1	8	15	4
1	5	7; 9	3	21
2	4	6; 10	4	22
3	21	1; 2	21	23
4	22	5; 1 or 3	22	24
5	23	25; 5; 2; 4	23	27
6	24	24; 26; 1; 5	24	28
7	27	27; 27; 30; 26	29	

Round	z_1	z_2	z_3	s
0	1	8	15	17
1	2	9	14	18
2	3	12	17	19
3	16	13	18	22

Round	z_1	z_2	z_3	s
0	1	9	11	3
1	2	10	2	4
2	3	2	3	5
3	4	1	4	26
4	5	5	5	27

Round	z_1	z_2	z_3	s
0	1	9	12	3
1	2	10	11	4
2	3	2	2	5
3	4	1	1	26
4	5	5	5	27

Round	z_1	z_2	z_3	s
0	1	9	13	3
1	2	10	14	4
2	3	2	15	5
3	4	1	3	26
4	5	5	4	27
5	26	26	5	28
6	27	27	26	29

Round	z_1	z_2	z_3	s
0	1	9	14	3
1	2	10	15	4
2	3	2	3	5
3	4	1	4	26
4	5	5	5	27

Round	z_1	z_2	z_3	s
0	1	9	15	4
1	5	10	3	21
2	4	2	4	22
3	21	3	21	23
4	22	4	22	24
5	23	5	23	27
6	24	26	24	28

Round	z_1	z_2	z_3	s
0	1	10	11	3
1	2	2	2	16
2	3	3	3	17

Round	z_1	z_2	z_3	s
0	1	10	12	3
1	2	2	11	4
2	3	3	2	5

Round	z_1	z_2	z_3	s
0	1	10	13	3
1	2	2	14	4
2	3	3	15	5
3	4	4	3	1

Round	z_1	z_2	z_3	s
0	1	10	14	3
1	2	2	15	4
2	3	3	3	5

Round	z_1	z_2	z_3	s
0	1	10	15	4
1	5	2	3	20
2	4	3	4	19

Round	z_1	z_2	z_3	s
0	1	6	16	4
1	5	1	3	21
2	4	5	4	22
3	21	4	21	23
4	22	21	22	24

Round	z_1	z_2	z_3	s
0	1	6	17	4
1	5	1	16	21
2	4	5	3	22
3	21	4	4	23
4	22	21	21	24

Round	z_1	z_2	z_3	s
0	1	6	18	4
1	5	1	19	3
2	4	2	20	16
3	3	3	4	17
4	16	16	3	18
5	17	17	16	19

Round	z_1	z_2	z_3	s
0	1	6	19	4
1	5	1	20	3
2	4	2	4	16
3	3	3	3	17

Round	z_1	z_2	z_3	s
0	1	6	20	3
1	2	1	4	16
2	3	2	3	17
3	16	3	16	18
4	17	16	17	19

Round	z_1	z_2	z_3	s
0	1	7	16	4
1	5	6	3	21
2	4	1	4	22
3	21	5	21	19
4	22	4	22	18
5	19	20	19	17

Round	z_1	z_2	z_3	s
0	1	7	17	3
1	2	6	16	4
2	3	1	3	21
3	4	5	4	22
4	21	4	21	23
5	22	21	22	24

Round	z_1	z_2	z_3	s
0	1	7	18	3
1	2	6	17	4
2	3	1	16	21
3	4	5	3	22

Round	z_1	z_2	z_3	s
0	1	7	19	4
1	5	6	20	3
2	4	1	4	15
3	3	2	3	14

Round	z_1	z_2	z_3	s
0	1	7	20	3
1	2	6	4	15
2	3	1	3	14

Round	z_1	z_2	z_3	s
0	1	8	16	4
1	5	7 or 9	3	21
2	4	6 or 10	4	22
3	21	1 or 2	21	23
4	22	5 or 1	22	24
5	23	25 or 5	23	27
6	24	24 or 26	24	28
7	27	27	27	29

Round	z_1	z_2	z_3	s
0	1	8	17	3
1	2	9	16	4
2	3	10	3	5
3	4	2	4	26
4	5	1	5	27

Round	z_1	z_2	z_3	s
0	1	8	18	3
1	2	9	17	4
2	3	10	16	5
3	4	2	3	26
4	5	1	4	27

Round	z_1	z_2	z_3	s
0	1	8	19	4
1	5	7 or 8	20	3
2	4	6 or 10	4	15
3	3	1 or 2	3	14

Round	z_1	z_2	z_3	s
0	1	8	19	4
1	5	7 or 8	20	3
2	4	6 or 10	4	15
3	3	1 or 2	3	14

Round	z_1	z_2	z_3	s
0	1	8	20	14
1	2	9	4	14
2	3	12	3	14
3	15	13	3	17
4	14	14	16	18
5	17	17	17	19

Round	z_1	z_2	z_3	s
0	1	9	16	4
1	5	10	3	21
2	4	2	4	22

Round	z_1	z_2	z_3	s
0	1	9	17	3
1	2	10	16	4
2	3	2	3	5
3	4	1	4	26
4	5	5	5	27

Round	z_1	z_2	z_3	s
0	1	9	18	3
1	2	10	17	4
2	3	2	16	5
3	4	1	3	26
4	5	5	4	27

Round	z_1	z_2	z_3	s
0	1	9	19	3
1	2	10	20	16
2	3	2	4	17

Round	z_1	z_2	z_3	s
0	1	9	20	3
1	2	10	4	16
2	3	2	3	17

Round	z_1	z_2	z_3	s
0	1	10	16	4
1	5	2	3	21
2	4	3	4	22

Round	z_1	z_2	z_3	s
0	1	10	17	3
1	2	2	16	4
2	3	3	3	5

Round	z_1	z_2	z_3	s
0	1	10	18	3
1	2	2	17	4
2	3	3	16	5
3	4	4	3	26
4	5	5	4	27

Round	z_1	z_2	z_3	s
0	1	10	19	3
1	2	2	20	15
2	3	3	4	14

Round	z_1	z_2	z_3	s
0	1	10	20	3
1	2	2	4	15
2	3	3	3	14

Round	z_1	z_2	z_3	s
0	1	6	21	3
1	2	1	4	16
2	3	2	3	17

Round	z_1	z_2	z_3	s
0	1	6	22	4
1	5	1	21	3
2	4	2	4	16
3	3	3	3	17

Round	z_1	z_2	z_3	s
0	1	6	23	4
1	5	1	22	3
2	4	2	21	16
3	3	3	4	17

Round	z_1	z_2	z_3	s
0	1	6	24	4
1	5	1	25	3
2	4	2	5	16
3	3	3	4	17

Round	z_1	z_2	z_3	s
0	1	6	25	4
1	5	1	5	3
2	4	2	4	16
3	3	3	3	17

Round	z_1	z_2	z_3	s
0	1	7	21	3
1	2	6	4	16
2	3	1	3	17

Round	z_1	z_2	z_3	s
0	1	7	22	4
1	5	6	21	3
2	4	1	4	16
3	3	2	3	17

Round	z_1	z_2	z_3	s
0	1	7	23	4
1	5	6	22	3
2	4	1	21	16
3	3	2	4	17

Round	z_1	z_2	z_3	s
0	1	7	24	4
1	5	6	25	3
2	4	1	5	16
3	3	2	4	17

Round	z_1	z_2	z_3	s
0	1	7	25	4
1	5	6	5	3
2	4	1	4	16
3	3	2	3	16

Round	z_1	z_2	z_3	s
0	1	8	21	3
1	2	9	4	16
2	3	10	3	17
3	16	2	16	14
4	17	3	17	13
5	14	15	14	12

Round	z_1	z_2	z_3	s
0	1	8	22	3
1	2	9	21	16
2	3	10	4	17
3	16	2	3	14
4	17	3	15	13
5	14	15	14	12

Round	z_1	z_2	z_3	s
0	1	8	23	3
1	2	9	22	4
2	3	10	21	5
3	4	2	4	26
4	5	1	5	27

Round	z_1	z_2	z_3	s
0	1	8	24	3
1	2	9	25	4
2	3	10	5	21
3	4	2	4	22

Round	z_1	z_2	z_3	s
0	1	8	25	3
1	2	9	5	16
2	3	10	4	17
3	16	2	3	14
4	17	3	15	13
5	14	15	14	12

Round	z_1	z_2	z_3	s
0	1	9	21	3
1	2	10	4	16
2	3	2	3	17

Round	z_1	z_2	z_3	s
0	1	9	22	4
1	5	10	21	3
2	4	2	4	16
3	3	3	3	17

Round	z_1	z_2	z_3	s
0	1	9	23	4
1	5	10	22	3
2	4	2	21	16
3	3	3	4	17

Round	z_1	z_2	z_3	s
0	1	9	24	4
1	5	10	25	3
2	4	2	5	16
3	3	3	4	17

Round	z_1	z_2	z_3	s
0	1	9	25	4
1	5	10	5	3
2	4	2	4	16
3	3	3	3	17

Round	z_1	z_2	z_3	s
0	1	10	21	3
1	2	2	4	16
2	3	3	3	17

Round	z_1	z_2	z_3	s
0	1	10	22	3
1	2	2	21	16
2	3	3	4	17

Round	z_1	z_2	z_3	s
0	1	10	23	3
1	2	2	22	15
2	3	3	21	14
3	15	15	4	17
4	14	14	3	18
5	17	17	16	19

Round	z_1	z_2	z_3	s
0	1	10	24	3
1	2	2	25	4
2	3	3	5	20
3	4	4	4	19

Round	z_1	z_2	z_3	s
0	1	10	25	3
1	2	2	5	16
2	3	3	4	17

Round	z_1	z_2	z_3	s
0	1	6	26	4
1	5	1	5	3
2	4	2	4	16
3	3	3	3	17

Round	z_1	z_2	z_3	s
0	1	6	27	4
1	5	1	26	3
2	4	2	5	16
3	3	3	4	17

Round	z_1	z_2	z_3	s
0	1	6	28	4
1	5	1	27	3
2	4	2	26	16
3	3	3	5	17

Round	z_1	z_2	z_3	s
0	1	6	29	4
1	5	1	30	3
2	4	2	1	16
3	3	3	2	17

Round	z_1	z_2	z_3	s
0	1	6	30	4
1	5	1	1	3
2	4	2	2	16
3	3	3	3	17

Round	z_1	z_2	z_3	s
0	1	7	26	4
1	5	6	5	3
2	4	1	4	16
3	3	2	3	17

Round	z_1	z_2	z_3	s
0	1	7	27	4
1	5	6	26	3
2	4	1	5	16
3	3	2	4	17

Round	z_1	z_2	z_3	s
0	1	7	28	4
1	5	6	27	3
2	4	1	26	16
3	3	2	5	17

Round	z_1	z_2	z_3	s
0	1	7	29	4
1	5	6	30	3
2	4	1	1	16
3	3	2	2	17

Round	z_1	z_2	z_3	s
0	1	7	30	4
1	5	6	1	3
2	4	1	2	16
3	3	2	3	17

Round	z_1	z_2	z_3	s
0	1	8	26	3
1	2	9	5	16
2	3	10	4	17
3	16	2	3	18
4	17	3	16	19
5	18	4	17	22
6	19	21	18	23

Round	z_1	z_2	z_3	s
0	1	8	27	3
1	2	9	26	4
2	3	10	5	21
3	4	2	4	22

Round	z_1	z_2	z_3	s
0	1	8	28	3
1	2	9	27 or 29	4
2	3	10	26 or 30	20
3	4	2	5 or 1	19

Round	z_1	z_2	z_3	s
0	1	8	29	3
1	2	9	30	4
2	3	10	1	20
3	4	2	5	19

Round	z_1	z_2	z_3	s
0	1	8	30	3
1	2	9	1	4
2	3	10	5	20
3	4	2	4	19

Round	z_1	z_2	z_3	s
0	1	9	26	3
1	2	10	5	16
2	3	2	4	17

Round	z_1	z_2	z_3	s
0	1	9	27	3
1	2	10	26	4
2	3	2	5	20
3	4	3	4	19

Round	z_1	z_2	z_3	s
0	1	9	28	4
1	5	10	27	3
2	4	2	26	16
3	3	3	5	17

Round	z_1	z_2	z_3	s
0	1	9	29	4
1	5	10	30	3
2	4	2	1	16
3	3	3	2	17

Round	z_1	z_2	z_3	s
0	1	9	30	4
1	5	10	1	3
2	4	2	2	16
3	3	3	3	17

Round	z_1	z_2	z_3	s
0	1	10	26	4
1	5	2	5	20
2	4	3	4	19

Round	z_1	z_2	z_3	s
0	1	10	27	4
1	5	2	26	20
2	4	3	5	19

Round	z_1	z_2	z_3	s
0	1	10	28	4
1	5	2	27	20
2	4	3	26	19
3	20	4	5	18
4	19	20	4	17
5	18	19	3	14
6	17	18	15	13
7	14	17	14	12

Round	z_1	z_2	z_3	s
0	1	10	29	4
1	2	2	30	3
2	3	3	1	20
3	4	4	5	19

Round	z_1	z_2	z_3	s
0	1	10	30	4
1	5	2	1	20
2	4	3	5	19

Round	z_1	z_2	z_3	s
0	1	11	16	4
1	5	2	3	20
2	4	3	4	21

Round	z_1	z_2	z_3	s
0	1	11	17	3
1	2	2	16	4
2	3	3	3	20

Round	z_1	z_2	z_3	s
0	1	11	18	3
1	2	2	17	4
2	3	3	16	20

Round	z_1	z_2	z_3	s
0	1	11	19	4
1	5	2	20	21
2	4	3	4	22

Round	z_1	z_2	z_3	s
0	1	11	20	3
1	2	2	4	16
2	3	3	3	17

Round	z_1	z_2	z_3	s
0	1	12	16	4
1	5	11	3	20
2	4	2	4	19

Round	z_1	z_2	z_3	s
0	1	12	17	3
1	2	11	16	4
2	3	2	3	20
3	4	3	4	19

Round	z_1	z_2	z_3	s
0	1	12	18	3
1	2	11	17	4
2	3	2	16	20
3	4	3	3	19

Round	z_1	z_2	z_3	s
0	1	12	19	4
1	5	11	20	3
2	4	2	4	16
3	3	3	3	17

Round	z_1	z_2	z_3	s
0	1	12	20	3
1	2	11	4	16
2	3	2	3	17

Round	z_1	z_2	z_3	s
0	1	13	16	4
1	5	14	3	21
2	4	15	4	22
3	21	3	21	23
4	22	4	22	24
5	23	5	23	27
6	24	26	24	28

Round	z_1	z_2	z_3	s
0	1	13	17	3
1	2	14	16	4
2	3	15	3	5
3	4	3	4	26
4	5	2	5	27

Round	z_1	z_2	z_3	s
0	1	13	18	3
1	2	14	17	4
2	3	15	16	5
3	4	3	3	26
4	5	4	4	27

Round	z_1	z_2	z_3	s
0	1	13	19	4
1	5	14	20	21
2	4	15	4	22
3	21	3	21	23
4	22	4	22	24
5	23	5	23	27
6	24	26	24	28

Round	z_1	z_2	z_3	s
0	1	13	20	9
1	2	12	4	8
2	10	9	3 or 5	7
3	9	8	2 or 1	29
4	8	7	1 or 30	28
5	7	29	30 or 29	27
6	29	28	29 or 28	24

Round	z_1	z_2	z_3	s
0	1	14	16	4
1	5	15	3	21
2	4	3	4	22

Round	z_1	z_2	z_3	s
0	1	14	17	4
1	5	15	16	21
2	4	3	3	22

Round	z_1	z_2	z_3	s
0	1	14	18	3
1	2	15	17	4
2	3	3	16	5
3	4	4	3	26
4	5	5	4	27

Round	z_1	z_2	z_3	s
0	1	14	19	4
1	5	15	20	21
2	4	3	4	22

Round	z_1	z_2	z_3	s
0	1	14	20	3
1	2	15	4	16
2	3	3	3	17

Round	z_1	z_2	z_3	s
0	1	15	16	4
1	5	3	3	21
2	4	4	4	22

Round	z_1	z_2	z_3	s
0	1	15	17	4
1	5	3	16	21
2	4	4	3	22

Round	z_1	z_2	z_3	s
0	1	15	18	12
1	2	14	17	9
2	10	13	14	8
3	9	12	13	7

Round	z_1	z_2	z_3	s
0	1	15	19	4
1	5	3	20	21
2	4	4	4	22