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VERTEX-TO-VERTEX PURSUIT IN A GRAPH

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A graph G is given and two players, a cop and a robber, play the following game: the cop chooses a vertex, then the robber chooses a vertex, then the players move alternately beginning with the cop. A move consists of staying at one's present vertex or moving to an adjacent vertex; each move is seen by both players. The cop wins if he manages to occupy the same vertex as the robber, and the robber wins if he avoids this forever.

We characterize the graphs on which the cop has a winning strategy, and connect the problem with the structure theory of graphs based on products and retracts.

1. Introduction

The following game was brought to our attention by G. Gabor. A Graph G is given and player I, henceforth known as the cop, chooses a vertex—the 'station'—on which to begin. Player II, the robber, then chooses his starting vertex, and the players move alternately thereafter beginning with the cop. A move consists of staying in one's place or moving along an edge of G to an adjacent vertex, and the cop wins if he 'catches' the robber after a finite number of moves. Since there is complete information in this game, either the cop or the robber must have a winning strategy; in the former case G will be called a *cop-win* graph, otherwise G is *robber-win.* Our objective is to characterize the cop-win graphs and to connect them with the structure theory of graphs developed in [2].

Note that this game is quite different from the one considered in [3], where the players move continuously and with no information. The motivation in that case was looking for a lost spelunker; here we envision a chase from intersection to intersection in a city. A knowledge of which graphs are cop-win might in theory help law-enforcement officers to decide where to put up roadblocks, although our model is certainly a huge oversimplification.

Since a player may stay at his present vertex it is convenient to regard all graphs as *reflexive*, i.e. equipped with loops at every vertex. An induced subgraph H of G is a *retract* of G is there is an edge-preserving map f from G onto H such that $f|_H$ is the identity map on H. (The loops allow two adjacent vertices to be mapped to the same vertex in H.) The following theorem gives a way to construct new cop-win graphs from old:

Theorem 1. *If G is a retract of a finite product of cop-win graphs, then G is a cop-win graph.*

Clearly any finite path is cop-win and any n-cycle, $n \geq 4$, is robber-win. Thus we have

Corollary. *If G has a retract which is an n-cycle, $n \geq 4$, then G is robber-win; if G is a retract of a finite product of paths, it is cop-win.*

Note that the finiteness of the product is important; the product of a collection of paths of unbounded length is not even connected and thus cannot be cop-win.

Every finite tree is a retract of a finite product of paths [2]. However, there are both robber-win and cop-win graphs which do not satisfy the corresponding conditions in the corollary; an example of each is given in Fig. 1 and Fig. 2 respectively.

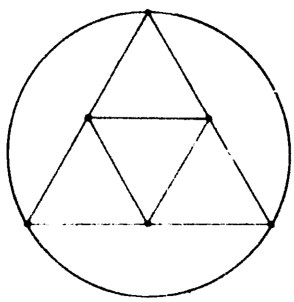

Fig. 1.

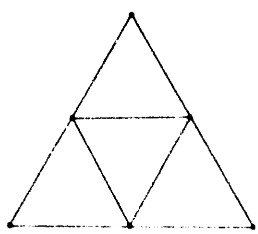

Fig. 2.

The *variety* generated by a collection of graphs is the closure of the collection under products and retracts. A graph G is *irreducible* if whenever G is a retract of a product of graphs, G is already a retract of one of the factors. The cop-win graphs could thus be characterized using Theorem 1 if all irreducible cop-win graphs could be found, but a simpler characterization is possible, with the help of a standard retrograde analysis.

Consider the position in a game just before the robber's last move. The robber is 'cornered', i.e. all vertices adjacent to his position are also adjacent to the cop's.

The *neighborhood* $N(v)$ of a vertex in G is the set of vertices adjacent to v, which in our reflexive graphs includes v itself. v is *irreducible* if for some $u \neq v$, $N(u) \supset N(v)$; in that case u is called a *cover* of v. G is *dismantlable* if there is a numbering $\{v_1, \ldots, v_n\}$ of the vertices of G such that for each $i < n$, v_i is irreducible in the subgraph induced by $\{v_i, \ldots, v_n\}$. (These notions correspond to those found in [1] for partially ordered sets.) From Theorem 1 we then have:

A *finite* graph is cop-win if and only if it is dismantlable.

Not all cop-win graphs are finite (viz. an infinite complete graph), but fortunately an extension of the above analysis yields a complete characterization and optimal strategies for the cop as well.

Let G be an arbitrary (reflexive) graph; we define for each ordinal α a binary relation $\leqslant_\alpha$ on the set $V(G)$ of vertices of G. Let $x \leqslant_0 y$ iff $x = y$, and for each $\alpha > 0$, set

$x \leqslant_\alpha y$ if and only if for all $u \in N(x)$, there exists a $v \in N(y)$ such that $u \leqslant_\rho v$ for some $\rho < \alpha$.

Now let α' be the least ordinal such that $\leqslant_{\alpha'} = \leqslant_{\alpha'+1}$ and define $\leqslant$ to be $\leqslant_{\alpha'}$.

Theorem 2. *G is a cop-win graph if and only if the relation $\leqslant$ described above is trivial, i.e. $x \leqslant y$ for every $x, y \in V(G)$.*

2. Proofs of Theorems 1 and 2

The proof of Theorem 1 is in two parts. First we show the product H of a finite collection $\{G_1, G_2, \ldots, G_n\}$ of cop-win graphs is cop-win. There is a projection-induced one-to-one correspondence which associates a move in the game on H with simultaneous moves in the games on $G_1, \ldots, G_n$. The cop plays his winning strategy on each G_i; once he catches the robber on one graph, he can stay with his quarry until he has won all n games and is now on the same vertex as the robber on the product graph. (H is called in [4] the *selection compound* of the graphs G_i.)

To show that a retract H of a cop-win graph G is cop-win, let f be the retraction map of G onto H. The cop plays on H the f-image of his winning strategy on G, regarding the robber's moves as moves in the graph G which happen to be restricted to H. The final capture will be made in H, where f is the identity. □

For the proof of Theorem 2 a few facts about the relation $\leqslant_\alpha$ are noted.

First, if $p < \alpha$, then $\leqslant_p \subseteq \leqslant_\alpha$. This follows by induction from the definition, and guarantees the existence of the ordinal α'. In fact $\alpha' \leqslant n(n-1)$ when G is finite and has n vertices; otherwise $\alpha' \leqslant |V(G)|$.

Secondly, each $\leqslant_\alpha$ is a quasiorder, i.e. is reflexive and transitive. Reflexivity is immediate since each $\leqslant_\alpha$ contains $\leqslant_0$; assume transivity holds for all $p<\alpha$, and suppose that $x\leqslant_\alpha y$ and $y\leqslant_\alpha z$. Let $u\in N(x)$ and find $v\in N(y)$ and $\rho<\alpha$ with $u\leqslant_\rho v$; then find $w\in N(z)$ and $\rho'<\alpha$ with $v\leqslant_{\rho'} w$. Then $u\leqslant_{\max\{\rho,\rho'\}} w$ and so $x\leqslant_\alpha z$ as desired.

Now let α' be the least ordinal such that $\leqslant_{\alpha'}=\leqslant_{\alpha'+1}$. Suppose that $\leqslant$ is trivial, and let the cop choose an arbitrary station x_0. Let y_1 be the starting vertex for the robber; since $y_1\leqslant_{\alpha'}x_0$, there is some $x_1\in N(x_0)$ and some $\alpha(1)<\alpha'$ with $y_1\leqslant_{\alpha(1)}x_1$. In general, after the cop's ith move he is at x_i and the robber is at y_i with $y_i\leqslant_{\alpha(i)}x_i$. When the robber moves to $y_{i+1}\in N(y_i)$ there is an $x_{i\in 1}\in N(x_i)$ such that $y_{i+1}\leqslant_{\alpha(i+1)}x_{i+1}$ with $\alpha(i+1)<\alpha(i)$; the cop moves to x_{i+1}.

Since the $\alpha(i)$'s constitute a strictly decreasing sequence of ordinals, the sequence must stop after a finite number k of moves; thus $\alpha(k)=0$, $x_k=y_k$ and the cop has won.

Conversely, let us suppose that $y_0\nleqslant x_0$ for some x_0, y_0 in $V(G)$. Assume for the moment that the cop is obliged to begin at x_0; the robber begins at a point $y_1\in N(y_0)$ such that for any $x\in N(x_0)$, $y_1\nleqslant x$. (If there were no such y_1 then by definition $y_0\leqslant_{\alpha'+1}x_0$, a contradiction.) Proceeding in this manner, the robber guarantees that for any i there is a vertex $y_{i+1}\in N(y_i)$ such that for any $x\in N(x_i)$, $y_{i+1}\nleqslant x$; here x_i and y_i are the respective positions of the cop and robber after the cop's ith move. Since in particular there is always a $y_{i+1}\notin N(x_i)$, the robber is never caught.

It remains only to note that if the cop has a winning strategy entailing an initial station at vertex v, then the graph G is certainly connected. Hence if forced to begin at x_0 the cop could simply migrate to v and win from there. □

3. Some remarks

It is not difficult to verify that for a finite number k, $y\leqslant_k x$ but not $y\leqslant_{k-1}x$ if and only if when the cop is at x and the robber at y, with the robber to move, best strategy by both players leads to a win by the cop in exactly k moves of each player. Let G be a fixed cop-win graph and let ρ' be the least ordinal such that the quasiorder $\leqslant_{\rho'}$ has a *maximum* element, i.e. a vertex v such that $u\leqslant_{\rho'}v$ for every vertex u. If ρ' is a finite number k, then the cop can assure himself a win in at most k moves by beginning at v, and at every turn making sure that his position dominates the robber's in the least-numbered quasiorder possible. It is straightforward to check that this is an optimal strategy for the cop.

If ρ' is infinite, then for any finite number k, the robber has a strategy that keeps him alive for at least k moves (by which time the statute of limitations on his theft may have run out). This situation exists, for example, in the graph G composed of a collection of finite paths of unbounded length with a common endpoint.

It is easily seen that for a *regular* graph, α' can never be greater than 1; it follows that every incomplete regular graph is robber-win. Similar considerations might lead one to conclude that the forces of law would have an advantage in cities (such as Boston, MA) which have street-intersections of varied degree; we suspect that this might be difficult to verify.

Note added in proof

It has come to the attention of the authors that a finite-case characterization of cop-win graphs was obtained by Alain Quilliot, in his Thesis 3rd cycle, Université de Paris VI, 1978.

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