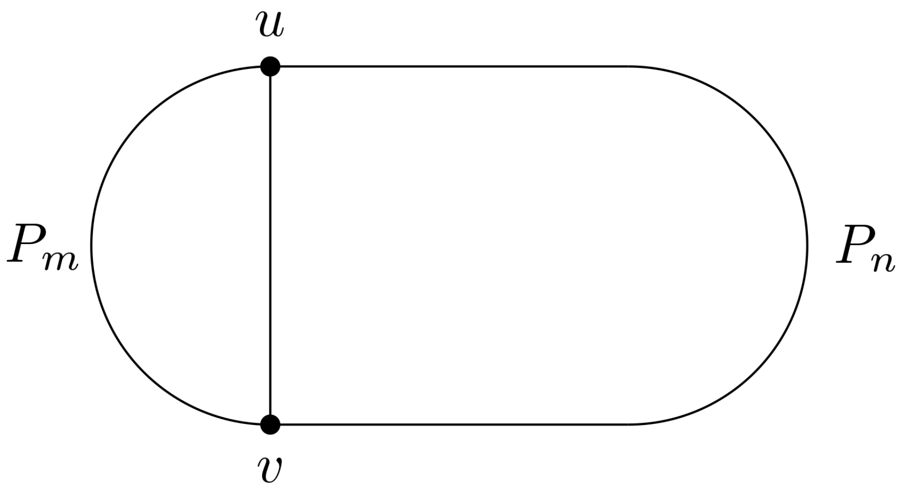
EDIT:

Definition 1. We define a family of graphs we call *bifurcated cycles* and denote as $Q_{m,n}$. As the name suggests, bifurcated cycles are cycles of length $m+n$ with a single chord which divides the cycle into paths P_m and P_n of lengths m and n, $m \leq n$.



Theorem 1. The Bifurcated cycle $Q_{m,n}$ is 2-zombie win.

Proof. First, we show that a certain game state is a losing position for the survivor. Second, we show how to position the zombies at the start of the game so that – no matter where the survivor starts – a losing position is inevitably reached.

We use the following notation. Denote as P_m and P_n the paths of lengths m and n respectively. Call the endpoints for the chord u and v. We think of $Q_{m,n}$ as embbeded in the plane with P_m – the shortest side – on the left. This does not limit the generality of the following and allows us to define (counter-)clockwise distance: the length of the path along the outer cycle with respect to this embedding.

Following the rules of the game, the zombies always move along a shortest path toward the survivor we call zs-paths. Let $Z_k = \{\exists \ell : z_k = u_{i,0}, u_{i,1}, u_{i,2}, \ldots, u_{i,\ell-1}, s = u_{i,\ell}\}$ be the set of i different zs-paths of length ℓ for zombie k.

There is at least one such path since our graph is presumed connected, thus $1 \leq i \leq p$.

If there is only one path, then z_k's next move is $u_{i,1}$. If all zs-paths include $u_{i,1}$, then again z_k's next move must be to that vertex.

If, however, there are multiple zs-paths which have different first moves, then the zombie could make multiple moves.

If P_1 and P_2 as two possible zs-paths with distinct next moves then

$$|P_1| \leq |P_2|$$

In our argument we will suppose that the zombie follows $|P_1|$ since that is a valid move.

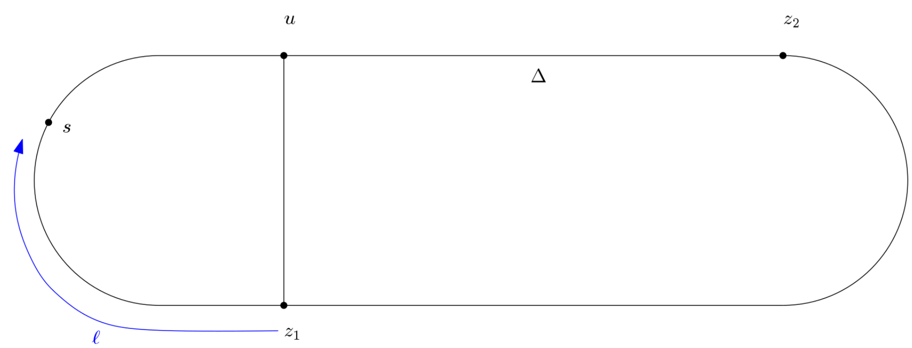
Part 1. Cornering the Survivor on the Smallest Cycle

Suppose that the game has reached the following state:

- the first zombie is on an endpoint of the chord, say v

- the second zombie is Δ vertices away from u, the other endpoint (counting clockwise from u to z_2).
- the survivor is somewhere on $P_m \setminus N(v)$.

Denote as ℓ the length of the clockwise path from $z_1 = v$ to s. Note that we must have $2 \leq \ell \leq m-1$ else the survivor loses on the next round.



Because z_1 and s are on the sub-cycle C_{m+1} formed by $P_m + \{uv\}$, z_1's move on the next turn depends on the value of ℓ. If $2\ell < m+1$ then z_1 goes clockwise on the subcycle. If $2\ell > m+1$ then z_1 takes the chord and goes counter-clockwise. If we have equality, then z_1 may choose either direction since both are paths of equal lengths.

We assume that once z_1 chooses a direction on its move from v, that it will continue in that direction: either the zombie has no choice, having been captured in the survivor's orbit, or both directions around the cycle are of the same length (and so may continue in the same direction).

We can also assume that on its turn the survivor will move away from z_1 and keep a distance of ℓ (or $m - \ell + 1$). Any strategy involving waiting a turn or moving backwards is equivalent to a survivor strategy which always moves starting with a smaller (or larger) value of ℓ.

We will examine all combinations of the possible decisions made by the zombies from this configuration:

- I. z_2 goes counter-clockwise
- II. z_2 goes clockwise.
- A. z_1 goes clockwise
- B. z_1 goes counter-clockwise

Since z_1 is already on the same cycle as the survivor, it only has two options:

A. z_1 goes clockwise if $\ell \leq 1 + m - \ell$. Combined with the bounds on ℓ, this gives $4 \leq 2\ell \leq m+1$
B. z_1 goes counter-clockwise if $1+m-\ell \leq \ell$. Combined with the bounds on ℓ, we obtain $m+1 \leq 2\ell \leq 2m-2$

For z_2, there are four possible shortest paths to the survivor:

- P_a of length $\Delta + (m - \ell)$

- P_b of length $\Delta + 1 + \ell$
- P_c of length $(n - \Delta) + 1 + (m - \ell)$
- P_d of length $(n - \Delta) + \ell$

Comparing path lengths we see that:

I. z_2 moves counter-clockwise if either $|P_a| \leq \min\{|P_c|, |P_d|\}$ or $|P_b| \leq \min\{|P_c|, |P_d|\}$.
II. z_2 goes clockwise if either $|P_c| \leq \min\{|P_a|, |P_b|\}$ or $|P_d| \leq \min\{|P_a|, |P_b|\}$.

Case I.A We have the following constraint on ℓ from assumption A.

$$4 \leq 2\ell \leq m + 1$$

and the following constraints on Δ from assumption I.

$$\begin{aligned} \Delta + (m - \ell) \leq& n - \Delta + 1 + m - \ell && \text{and} \\ \Delta + (m - \ell) \leq& n - \Delta + \ell \end{aligned}$$

or

$$\begin{aligned} \Delta + 1 + \ell \leq& n - \Delta + 1 + m - \ell && \text{and} \\ \Delta + 1 + \ell \leq& n - \Delta + \ell \end{aligned}$$

These can be simplified further with a bit of algebra and assumption A:

$$\begin{aligned} 2\Delta \leq& n + 1 && \text{and} \\ 2\Delta \leq& n - m + 2\ell \leq n + 1 \end{aligned}$$

or

$$\begin{aligned} 2\Delta \leq& n + m - 2\ell && \text{and} \\ 2\Delta \leq& n - 1 \leq n + m - 2\ell \end{aligned}$$

From which we obtain

$$\begin{aligned} 2\Delta \leq& n - m + 2\ell && \text{or} \\ 2\Delta \leq& n - 1 \end{aligned}$$

So for z_2 to follow either P_a or P_b we must have

$$2\Delta \leq \max\{n - m + 2\ell, n - 1\}$$

Next we consider: which of s or z_2 reaches u first? If $\Delta = m - \ell$, then they both reach u at the same time, with the survivor moving onto the z_2-occupied vertex (and losing). If we have $\Delta = m - \ell + 1$, then s reaches u first but is caught by z_2 on the following round. So, to guarantee the survivor has not escaped P_m we need

$$\Delta \leq m - \ell + 1$$

otherwise the survivor reaches the chord before z_2 can move to block. If the survivor reaches the chord first, then it could take the chord and possibly escape. (more detail??)

Then, to ensure that z_2 continues counter-clockwise once it reaches u, we need

$$\begin{aligned} m - \ell - \Delta \leq & 1 + \Delta + \ell \\ 2\Delta \geq & m - 2\ell - 1 \end{aligned}$$

When we combine all the restrictions we obtain

$$\begin{aligned} m - 2\ell - 1 \leq & 2\Delta \leq \min\{2m - 2\ell + 2, \max\{n-1, n-m+2\ell\}\} && \text{and} \\ 4 \leq & 2\ell \leq m+1 \end{aligned}$$

Case I.B From assumption B and the constraint on ℓ, we must have

$$m + 1 \leq 2\ell \leq 2m - 2$$

and the constraints on Δ from assumption I are again:

$$\begin{aligned} \Delta + (m - \ell) \leq & n - \Delta + 1 + m - \ell && \text{and} \\ \Delta + (m - \ell) \leq & n - \Delta + \ell \end{aligned}$$

or

$$\begin{aligned} \Delta + 1 + \ell \leq & n - \Delta + 1 + m - \ell && \text{and} \\ \Delta + 1 + \ell \leq & n - \Delta + \ell \end{aligned}$$

These can be simplified using assumption B:

$$\begin{aligned} 2\Delta \leq & n + 1 \leq n - m + 2\ell && \text{and} \\ 2\Delta \leq & n - m + 2\ell \end{aligned}$$

or

$$\begin{aligned} 2\Delta \leq & n + m - 2\ell \leq n - 1 && \text{and} \\ 2\Delta \leq & n - 1 \end{aligned}$$

This gives

$$\begin{aligned} 2\Delta \leq & n + 1 && \text{or} \\ 2\Delta \leq & n + m - 2\ell \end{aligned}$$

So

$$2\Delta \leq \max\{n+1, n+m-2\ell\}$$

Again we must consider who reaches the chord first. We have assumed that z_1 is going counter-clockwise. If $\ell = \Delta$, then z_2 reaches u and s reaches v on the same round, and therefore s will be caught on the next. Therefore, to guarantee the survivor has not escaped P_m in this scenario we need

$$\Delta \leq \ell$$

otherwise the survivor reaches the chord before z_2 and could escape.

Then, to ensure that z_2 takes the chord and goes clockwise once it reaches u, we need

$$\begin{aligned} 1+\ell-\Delta \leq& \Delta-1+m-\ell+1 \\ 2\ell-m+1 \leq& 2\Delta \end{aligned}$$

We now combine all the restrictions. This case is characterized by

$$\begin{aligned} 2\ell-m+1 \leq& 2\Delta \leq \min\{2\ell, \max\{n-1, n-m+2\ell\} && \text{and} \\ m+1 \leq& 2\ell \leq 2m-2 \end{aligned}$$

Case II.A We have the following constraint on ℓ from assumption A.

$$4 \leq 2\ell \leq m+1$$

and the following constraints on Δ from assumption II.

$$\begin{aligned} n-\Delta+\ell \leq& \Delta+(m-\ell) && \text{and} \\ n-\Delta+\ell \leq& \Delta+1+\ell \end{aligned}$$

or

$$\begin{aligned} n-\Delta+1+m-\ell \leq& \Delta+(m-\ell) && \text{and} \\ n-\Delta+1+m-\ell \leq& \Delta+1+\ell \end{aligned}$$

These can be simplified further with a bit of algebra:

$$\begin{aligned} n-m+2\ell \leq& 2\Delta && \text{and} \\ n-1 \leq& 2\Delta \end{aligned}$$

or

$$\begin{aligned} n+1 \leq& 2\Delta && \text{and} \\ n+m-2\ell \leq& 2\Delta \end{aligned}$$

These inequalites are of the form

$$\begin{aligned} n-x \leq& 2\Delta && \text{and} \\ n-1 \leq& 2\Delta \end{aligned}$$

or

$$\begin{aligned} n+x \leq& 2\Delta && \text{and} \\ n+1 \leq& 2\Delta \end{aligned}$$

Where $x = m-2\ell$.

Supposing $x \geq 0$, we have

$$\begin{aligned} n - x \leq& n + x \leq 2\Delta \\ n - 1 \leq& n + 1 \leq 2\Delta \end{aligned} \qquad \text{and}$$

Whereas if $x < 0$, then from assumption A we must have $m - 2\ell = -1$, so that our constraints reduce to

$$\begin{aligned} n + 1 \leq& 2\Delta \\ n - 1 \leq& 2\Delta \end{aligned} \qquad \text{and}$$

Thus, in any case we must have

$$\begin{aligned} \max\{n - x, n - 1\} \leq& 2\Delta \\ \max\{n - m + 2\ell, n - 1\} \leq& 2\Delta \end{aligned}$$

Again we must consider who reaches the chord first. We have assumed that z_1 is going clockwise. If $m - \ell = n - \Delta$, then z_2 reaches v and s reaches u on the same round, and therefore s will be caught on the next. Therefore, to guarantee the survivor has not escaped P_m in this scenario we need

$$\begin{aligned} m - \ell \leq& n - \Delta \\ \Delta \leq& n - m + \ell \end{aligned}$$

otherwise the survivor reaches the chord before z_2 and could escape. (EDIT: More details)

Then, to ensure that z_2 takes the chord and goes counter-clockwise once it reaches v, we need

$$1 + m - \ell - (n - \Delta) \leq n - \Delta + \ell$$

Solving for 2Δ gives

$$2\Delta \leq 2n + 2\ell - m - 1$$

We now combine all the restrictions. This case is characterized by

$$\max\{n - m + 2\ell, n - 1\} \leq 2\Delta \leq \min\{2n + 2\ell - m - 1, 2n - 2m + 2\ell\}$$

Case II.B We have the following constraint on ℓ from assumption B.

$$m + 1 \leq 2\ell \leq 2m - 2$$

and the following constraints on Δ from assumption II.

$$\begin{aligned} n - \Delta + \ell \leq& \Delta + (m - \ell) \\ n - \Delta + \ell \leq& \Delta + 1 + \ell \end{aligned} \qquad \text{and}$$

or

$$\begin{aligned} n - \Delta + 1 + m - \ell \leq& \Delta + (m - \ell) \\ n - \Delta + 1 + m - \ell \leq& \Delta + 1 + \ell \end{aligned} \qquad \text{and}$$

These can be simplified further with a bit of algebra:

$$\begin{aligned} n - m + 2\ell \leq& 2\Delta && \text{and} \\ n - 1 \leq& 2\Delta \end{aligned}$$

or

$$\begin{aligned} n + 1 \leq& 2\Delta && \text{and} \\ n + m - 2\ell \leq& 2\Delta \end{aligned}$$

These inequalites are of the form

$$\begin{aligned} n - x \leq& 2\Delta && \text{and} \\ n - 1 \leq& 2\Delta \end{aligned}$$

$$\text{or}$$

$$\begin{aligned} n + 1 \leq& 2\Delta && \text{and} \\ n + x \leq& 2\Delta \end{aligned}$$

Where $x = m - 2\ell$. Now since assumption B gives $m - 2\ell \leq -1$, we see that

$$\begin{aligned} n - 1 \leq& n - x \leq 2\Delta \\ &\text{or} \\ n + x \leq& n + 1 \leq 2\Delta \end{aligned}$$

From which we conclude that $n + 1 \leq 2\Delta$.

Now we consider: which of s or z_2 reaches v first? If $n - \Delta = \ell$, then they both reach u at the same time, with the survivor moving onto the z_2-occupied vertex (and losing). If we have $n - \Delta = \ell + 1$, then s reaches u first but is caught by z_2 on the following round. So, to guarantee the survivor has not escaped P_m we need

$$n - \Delta \leq \ell + 1$$

otherwise the survivor reaches the chord before z_2 can move to block. If the survivor reaches the chord first, then it could take the chord and possibly escape. (more detail??)

Then, to ensure that z_2 takesgoes clockwise once it reaches v, we need

$$\ell - (n - \Delta) \leq 1 + (n - \Delta - 1) + (m - \ell + 1)$$

Solving for 2Δ gives

$$2\Delta \leq 2n + m - 2\ell + 1$$

We now combine all the restrictions. This case is characterized by

$$\begin{aligned} n + 1 \leq& 2\Delta \\ 2\Delta \leq& 2n - 2\ell - 2 \\ 2\Delta \leq& 2n + m - 2\ell + 1 \end{aligned}$$

So

$$n + 1 \leq 2\Delta \leq 2n - 2\ell - 2$$

Now: group up all inequalities, color coding whether they represent which case we are in, or whether survivor could win.

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