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Zombies and Survivors

Author:
Joël FAUBERT

Supervisor:
Dr. Jean-Lou DE CARUFEL

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Abstract

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by Joël FAUBERT

Cops and Robbers on Graphs (C & R) is a vertex-to-vertex pursuit game played on graphs first introduced by Quilliot (in 1978) and Nowakowski (in 1983). The cop player starts the game by choosing a set of vertices which will be the cops' starting positions. The robber player responds by choosing its own start vertex. On each player's turn, the player may move its tokens to adjacent vertices. The cops win if the robber is captured (they occupy the same vertex). The robber wins if the robber can avoid capture indefinitely. A variation of C & R called Zombies and Survivors (Z & S) was recently proposed and studied by Fitzpatrick. Z & S is the same as C & R with the added twist that the zombies are required to move closer to the survivor. In the deterministic version of Z & S, the zombies are assumed to apply an optimal strategy (that is, they choose a winning start position and, whenever two shortest paths exist, make a winning move if such exists). Chapter 1 summarizes important results in vertex-pursuit games. In Chapter 2 we give an example of a planar graph where 3 zombies always lose, whereas Aigner and Fromme showed in 1984 that three robbers have a winning strategy on planar graphs. In Chapter 3 we show how two zombies can win on a cycle with one chord.

Acknowledgements

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C&R	**C**ops and **R**obbers
Z&S	**Z**ombies and **S**urvivors

For/Dedicated to/To my…

Chapter 1

Introduction

There has been a robbery downtown and the robbers are escaping by car. Officers already on the streets are notified only minutes later. The robbers seem to be ge5ting away – putting some distance between themselves and the chasing sirens. The driver slams on the breaks, too slow to react to the strip of tire spikes thrown by an ambushing officer! Two tires are shredded and the driver loses control. The robbers are quickly surrounded and apprehended. The media arrives; the crash is captured by a hovering helicopter.

Was there ever any hope of escape? Perhaps the robbers took the wrong route. They should have planned a vehicle swap. Or used a tunnel. Could it be that there were so many police officers that all routes were covered? That capture was inevitable? Perhaps the advantages of communication and central coordination allow the police to cut off likely escape routes, so that the probability of escape is low. A (somewhat dispassionate) mind might watch these salacious stories on the news and wonder how to apply math to these types of questions.

Vertex pursuit games are adversarial games played on graphs. Players take turns moving tokens on a graph (the game board) with the objective to capture (or evade) the other player, thereby simulating a chase or pursuit. Many variations of these graph pursuit games have been proposed [1], [2]. There are many rules and parameters to tweak to produce different games:

1. How much information do the players have?
2. Do they know each other's positions? From how far away?
3. Do the players know the playing field, i.e., the graph?
4. Are the players restricted to vertices or edges?
5. Are players obligated to move?
6. Does the graph change over time?

The Game of Cops and Robbers on Graphs (C & R) [3] is perhaps the most well-known vertex pursuit game. It is a perfect information game with cops trying to catch the robber. The cop player starts the game by choosing a set of vertices which will be the cops' starting positions. The robber player responds by choosing its own start vertex. On each player's turn, the player may move its tokens to adjacent vertices. The cops win if the robber is captured (they occupy the same vertex). The robber wins if the robber can avoid capture indefinitely. In a perfect information game, all players know everything about the game. In this context, the players know each other's positions (they see each other) and they know the landscape (graph) around them [4].

A variation called Zombies and Survivors (Z & S) was recently proposed and studied [5], [6]. Z & S is the same as C & R with the added twist that the zombies are required, on their turn, to move closer to the survivor.

This thesis has been an attempt to better understand this variant and to see if the results obtained for C & R still hold when the cops are constrained in their strategy. In general, we would like to know how different constraints imposed on the pursuers affects the number of pursuers required to win. We investigate "the cost of being undead", as Fitzpatrick [5] would call it. In particular, in Chapter 2 we give an example of a planar graph where 3 zombies always lose whereas 3 cops win in the classical version of the game (refer to Subsection 1.2.3). In Chapter 3 we show how two zombies can win on a cycle with one chord.

1.1 Notation

The following sections will use definitions from graph theory (and vertex-pursuit theory) which we include here for reference. Formally, a graph $G = (V, E)$ is composed of:

- A set V of vertices.
- A set E of edges $e = \{u, v\}$, with each edge e being a multiset of vertices $u, v \in V$.

We also write $V(G)$ for the set of vertices of G and $E(G)$ for the set of edges of G. Two different vertices $x, y \in V(G)$ are *neighbours* (also *adjacent*) if $xy = e \in E$. Vertices x and y are said to be *incident* to edge e. The graphs studied herein are *finite*, *connected*, *undirected* and *reflexive*. There is a *finite* number of vertices. A graph is *connected* if, for any two vertices x and y, there exists a sequence of consecutively adjacent vertices starting with x and ending with y. We limit ourselves to connected graphs because playing on graphs with multiple connected components can be reduced to playing multiple games in parallel: the players are restricted to their starting connected component. By undirected, we mean that an edge from x to y implies an edge from y to x so we treat the two directions as a single edge and write $\{x, y\}$ or simply $xy = yx \in E$. We do not allow parallel edges since they are redundant in modeling pursuit games. Lastly, in order to model a player's choice to pass on a turn, we suppose each vertex also has a loop (an edge $\{x, x\}$ to itself), making the graph *reflexive*. This way, players still choose an edge even though they do not move to a different vertex. These structures are also are more precisely known as *pseudographs* (graphs with loops), or as *multigraphs* (graphs which allow loops and multiple edges between two nodes).

We will have occasion to use a few more concepts of graph theory. The set $N(x) = \{y \in V | xy \in E \wedge x \neq y\} \subseteq V$ is the *neighbourhood* of x. The *closed neighborhood* of vertex x is the the neighborhood of x along with x itself and is denoted $N[x] = N(x) \cup \{x\} \subseteq V$. A set $S \subseteq V(G)$ is said to be *dominating* if $\cup_{x \in S} N(x) \supseteq V(G)$. The order of a dominating set is often denote $\gamma(G)$. The *degree* of a vertex is the cardinality of its neighbourhood $|N(x)|$. The *minimum* and *maximum degrees* of a graph are $\min\{|N(x)| : x \in V\}$ and $\max\{|N(x)| : x \in V\}$ and are denoted as $\delta(G)$ and $\Delta(G)$, respectively.

For example, in Figure 1.1 we have vertices $V = \{a, b, c, d, e, f, g, h\}$. Since a and b are connected by an edge, we have $ab \in E$. The neighbourhood of a is $N(a) = \{b, d, f\}$ and the closed neighbourhood of a is $N[a] = \{a, b, d, f\}$. In this example,

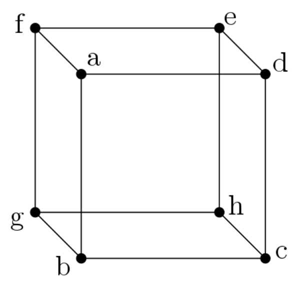

FIGURE 1.1: The Hypercube of Dimension 3

we also have that $\delta(G) = \Delta(G) = 3$ since all vertices have degree 3. Note that we do not draw loops in the figure (and, indeed, loops are omitted in all our figures for simplicity) and loops do not play a role in the degree.

Two basic classes of graphs are important in the study of these games: paths and cycles. A simple path $P = v_0, v_1, v_2, \ldots, v_n$ is a non-repeating sequence of adjacent vertices in a graph. A cycle C_n is a path of length $n \geq 3$ with an additional edge joining the last vertex back to the first (a so-called *closed* path). We say that a graph contains a path P if P is a *subgraph* of G, so $V(P) \subseteq V(G)$ and $E(P) \subseteq E(G)$. More generally, with $S \subseteq V(G)$ a set of vertices, we write $G[S]$ for the *induced subgraph*: the graph which contains S and the edges of G which join vertices of S. We write $G - u$ to mean $G[V(G) \setminus u]$.

Paths allow us to define a distance between vertices $d_G(x, y)$ as the length of the shortest path connecting x to y (or infinity if such a path does not exist which is never the case in our games). Computing such paths, also known as *geodesics*, is a classic problem in computer science. A geodesic has the additional property of being *isometric* [7], meaning that the distance between vertices of an isometric path is preserved in the subgraph induced by the path. The *diameter* $\mathrm{diam}(G)$ is the length of a longest possible shortest path in G. The *girth* of a graph is the length of the shortest cycle contained the graph.

Graphs can be combined in various ways to create new graphs. The *Cartesian product* of G and H is denoted $G \square H$ and defined as the graph with vertices

$$V(G \square H) = V(G) \times V(H)$$

and edges

$$E(G \square H) = \{(x_1, y_1)(x_2, y_2) | (x_1x_2 \in E(G) \wedge y_1 = y_2) \vee \\ (x_1 = x_2 \wedge y_1y_2 \in E(H))\} \quad .$$

The *strong product* of G and H, denoted $G \boxtimes H$, is similarly defined as the graph with vertices

$$V(G \boxtimes H) = V(G) \times V(H)$$

and edges

$$E(G \boxtimes H) = E(G \square H) \cup \{(x_1, y_1)(x_2, y_2) | (x_1x_2 \in E(G) \wedge y_1y_2 \in E(H))\} \quad .$$

These operations allow the construction of various families of graphs such as grids ($P_n \square P_m$), toroids ($C_n \square C_m$) and the hypercube (recursively by taking $Q_n = Q_{n-1} \square Q_{n-1}$, with $Q_1 = K_2$).

1.2 Classical Cops and Robbers

We start with an explanation of the Game of Cops and Robbers, then summarize some key results from C & R.

1.2.1 How to Play Cops and Robbers

C & R is a two player game: one player controls the cops (usually a handful), the other the robbers (usually just a single robber). The cops begin the game by choosing start vertices. Next, the robber chooses a start position. On each following round the each cop may move along an edge to a neighbouring vertex or remain in position. Here a move is an instantaneous jump between adjacent vertices. If the robber remains uncaught after all cops have had a chance to move, the robber then gets the opportunity to move along an edge.

In this game, the players have complete information of the graph and the positions of the players. The cops move, the robber responds and these two *turns* make one *round*.

The game is decided when either:

- A cop captures the robber. That is, the cop player wins if one of the cops move onto the vertex occupied by the robber.
- The robber wins if it can evade the cops indefinitely.

Consider a game of C & R played on the Hypercube of dimension 3 (refer to Figure 1.1). On this graph, a single cop will lose: the survivor may choose a vertex at distance 2 and preserve this distance indefinitely by running around a sub-cycle of length 4. However, two cops win handily wherever they may start. Suppose they choose to start on two adjacent vertices, say a and d. This start is not optimal for the cops – this graph is dominated by two vertices, so they could start directly in such a position. Nevertheless, in two turns the cops can move into a dominating position (like a, h) and capture the robber.

Study of vertex-pursuit games is first attributed to Quilliot [8], [9], as well as Nowakowski and Winkler [10]. These researchers independently consider games of C & R with a single cop and characterize by way of a relation those graphs where the cop always wins. These are now known as *cop-win* graphs and can be recognized by the existence of an ordering of the vertices called a *dismantling*; so-called because it is the successive deletion of *corners* resulting in a single vertex (see Subsection 1.2.2).

The *cop-number* of a graph, denoted $c(G)$, was introduced by Aigner and Fromme [11] and defined as the minimum number of cops required to guarantee cops win on graph G. In the example of the hypercube of dimension 3 (see Figure 1.1), a single cop loses but two cops always suffice, and so the cop-number of this graph is 2. The cop-number is well defined since a dominating set guarantees a win for the pursuers, so $c(g) \leq \gamma(G)$.

Later, Berarducci et al. and Hahn et al. generalized the characterization of cop-win graphs into *k-cop win* graphs [12], [13]. A graph is k-cop win if and only if there exists a function (defined on the strong product of the graph taken with itself k times, to represent the position of the cops) which satisfies certain properties; essentially it is a function which takes as input a position C of cops and returns the next position for the cops that guarantees a win (see [3][p. 119]). There exists a polynomial-time algorithm for deciding whether a graph is k-cop-win by iteratively solving for this function, so we can decide if $c(G) \leq k$ for any graph in polynomial time as long as k is fixed and not a function of $|V(G)|$ (in which case the problem is NP-hard).

Meyniel's conjecture posits that $\mathcal{O}(\sqrt{|V(G)|})$ is an upper bound on the cop-number [14]. Incremental progress toward this bound has been made on special classes of graphs as well as for graphs in general [15][p. 31] (see Subsection 1.2.3).

1.2.2 Dismantlings, Cop-win Trees

A vertex u is a *corner* if its closed neighbourhood is a subset of one of of its neighbours' closed neighbourhood. Formally, this is

$$u \in V(G) \qquad \text{and} \qquad \exists v \in V(G) : N[u] \subseteq N[v] \quad .$$

We say that corner u is *dominated* or *cornered* by v.

By supposing that a single cop wins on G, it can be shown that G must contain a corner. Consider the robber's last turn: if the robber loses on the next turn, it must be because the robber cannot stay in place nor can it move to a neighbour without being caught on the next turn. This is precisely the case when the cop is on a vertex which corners the robber. In a sense, a corner is a dead end for the robber, and thus would be avoided by a clever robber. It would be useful, then to study the game on the graph after removing or deleting the corner.

Let G be a reflexive graph with $x \in V(G)$ a fixed vertex. A (one-point) *retraction* is an edge-preserving function

$$f : G \to G \setminus v$$

(a homomorphism) such that $f(v) = x$ for some $x \neq v \in V(G)$ and f restricted on $H = G \setminus v$ is the identity:

$$f(v) = x \qquad f(u) = u \qquad \forall u \in V(G) \setminus \{v\}$$

and

$$xy \in E(G) \implies f(x)f(y) \in E(G \setminus \{v\}) \quad .$$

A retract maps a graph G to graph $H = G - u$ with one less vertex, and we say that H is a *retract* of G.

It is possible to define a retraction on a corner u: if u is a corner, then it is dominated by some $v \in V(G)$. So if $x \in V(G)$, $x \neq u$ and $xu \in E(G)$ then $xv \in E(G)$ since u is a corner. Therefore the map

$$f(x) = \begin{cases} v & \text{if } x = u \\ x & \text{otherwise} \end{cases}$$

is edge-preserving since $f(x)f(u) = xv$ and $xv \in E(G)$, so $xv \in E(H) = E(G - u)$. For other vertices $x, y \notin \{u, v\}$, $f(x)f(y) = xy \in E(G)$ so $f(x)f(y) \in E(G - u)$ also. This shows that f is a homomorphism and hence a retraction. Since the graphs studied herein are reflexive, a one-point retraction can be seen as the absorption of one vertex by another; the edge between the corner and its dominating vertex becomes another loop (which we discard since we do not need parallel edges).

For a graph G with retract H, it can be shown [12] that

$$c(H) \leq c(G)$$

by using a *shadow strategy* argument, that is by playing parallel games on G and H: the winning cop strategy on G implies a winning cop strategy on H. Quilliot and Nowakowski both indepedently characterized cop-win graphs by combining these observations about corners and retractions into what is called a *dismantling*. Informally, a dismantling is the successive folding of a corner onto its dominating vertex until we are left with a single vertex.

Formally, a *dismantling* is a sequence of retracts $f_1, f_2, \ldots, f_{n-1}$ such that the composition $F_{n-1} = f_{n-1} \circ f_{n-2} \circ \cdots f_2 \circ f_1$ gives a function for which $F_{n-1}(G) = K_1$: a sequence of retracts which maps the graph to a single vertex.

Not all vertices of a graph need to be corners in order for there to exist a dismantling: it suffices to have an ordering where each v_i is a corner in $G[\{v_i, v_{i+1}, \ldots, v_n\}]$. Such a sequence of f_i's defines a *cop-win ordering*

$$\mathcal{O} = (v_1, v_2, \ldots, v_n)$$

where v_1 is a corner in $G_1 = G$, v_2 is a corner in $G - v_1$, and so on. A fundamental result in C & R is that cop-win graphs – graphs for which a single cop is guaranteed to win – are characterized by the existence of such dismantlings. A graph is copwin if and only if it is dismantlable; the dismantling provides a winning cop strategy as follows. Start the cop on v_n, the last vertex of the dismantling. Here we have $F_n(R) = v_n$, so that the cop has captured the image of the robber under F_n (which reduces the graph to a single vertex). With the the robber on u and the cop guards its "shadow" (or image) in $F_i(u)$, the cop onto vertex $F_{i-1}(u)$. That is to say, the cop re-captures the shadow of the robber at every turn.

A cop-win spanning tree combines the idea of a dismantling with a spanning tree and was first proposed in [16]. A cop-win spanning tree S is defined as a tree where $x, y \in V(G)$ and $xy \in E(S)$ if there exists a retract f_j in the dismantling $F_n = f_n \circ f_{n-1} \circ \cdots \circ f_2 \circ f_1$ such that $f_j(x) = y$ or $f_j(y) = x$ in $G[\{j, j+1, \ldots, n\}]$. Cop-win spanning trees give a strategy for the cops to follow: start at the root (the last vertex in the ordering) and descend the tree in the branch containing the robber. Lemmas 2.1.2 and 2.1.3 from [16] show that the cop can always stay in the same branch (and above) the robber in the tree. So the robber is eventually stuck in a leaf and caught.

1.2.3 The Cop-Number and the Genus of the Graph

One of the most surprising results about the C & R is owed to Aigner and Fromme [11], who showed that the cop number of a planar graph is at most 3. Basically, a graph is planar if it can be drawn in the plane (say, on a piece of paper) without crossing any edges. Aigner and Fromme describe a 3-cop strategy which uses isometric paths of the graph to encircle and entrap the robber.

Since geodesics are isometric paths, the cops can patrol (or guard) the path – preventing the robber from entering (and thus crossing) the path without being captured. The cops can "shadow the robber's projection on the path." More precisely, after a finite number of rounds, a cop can move onto a vertex of the path such that its distance to any other vertex of the path is the same as the robber's distance to that vertex. Or, more simply perhaps, both players reach any vertex of the path at the same time (with the cop moving into position before the robber can arrive). Consequently, the robber can never move onto or through the path.

Aigner and Fromme use three guards to enact an encircling strategy: two guards patrol two isometric paths with the same endpoint, thereby creating a boundary within which the robber cannot escape. The third cop moves into the robber's territory and guards another shortest path, thereby constricting its territory. Eventually, the robber's territory is empty and therefore must be caught.

Outerplanar graphs are planar graphs which can be drawn such that all vertices belong to a common face (called the *outerface*). Clarke [16] showed that the cop number of outerplanar graphs is 2 by considering two possible cases: those with and without cut vertices. The 2 cops have a winning strategy on outerplanar graphs without cut vertices, and this strategy can be used to cordon off sections (blocks) of the outerplanar graph.

The game has also been studied for graphs embeddable in surfaces of higher order. In 2001, Schroeder conjectured [17] that for a graph of genus g, the cop-number is at most $g+3$. Currently, the best known bound for graph G of genus g is $c(G) \leq \left\lfloor \frac{3}{2}g \right\rfloor + 3$ (refer to [18]).

1.2.4 Relation to the Girth and Minimum Degree of a Graph

Aigner and Fromme also showed a relationship between the cop-number, the girth of a graph and its minimum degree [11]. More precisely, if G has girth at least 5, then $c(G) \geq \delta(G)$ where $\delta(G)$ is the minimum degree of G.

This result has since been refined [14]: if G has girth at least $8t-3$ and $\delta(G) = d$, then more than d^t cops are needed to win. In a recent seminar by B. Mohar (Graph Searching Online Seminar, held May 1, 2020) it was argued that a graph with girth g and $\delta(G) = d$ will require at least $\frac{1}{g}(d-1)^{\lfloor \frac{g-1}{4} \rfloor}$ cops.

1.2.5 Cops and Computational Geometry

Intersection graphs are constructed by equating sets with vertices and adding an edge between vertices whenever the intersection of their respective sets are non-empty. It has been shown [19] that unit-disk intersection graphs (intersection graphs where the sets are disks of unit-length radii) have cop-number at most 9.

Gavenčiak et al. [20] also examined the game of C & R on similar constructions. First, the authors show that several classes of intersection graphs have unbounded cop-number. Second, they find that the cop-number of intersection graphs of arc-connected subsets is at most $10g+5$ for an orientable surface of genus g.

The *visibility graph* of a simple polygon P (a polgyon without holes or crossing edges) is a graph with the points of the polygon as vertices with edges connecting these vertices whenever two vertices "see" each other in P. That is, in a visibility graph, $x\ y$ whenever the segment xy is contained within P. Lubiw et al. [21] showed that the vertex-visibility graph of any simple polygon is dismantlable and hence cop-win.

1.3 Cops Turn Into Zombies

Zombies and Survivors (Z & S) is a variation of Cops and Robbers first proposed by Fitzpatrick [5] and is the game studied herein. In Z & S games, the cops are replaced by zombies which always move closer to the robber (who is now a survivor). The sophistication of the zombies' strategy gives them their name: arms outstretched, the zombies amble directly towards the survivor. However, there is some ambiguity without further precision: how exactly do the zombies move closer to the survivor? What if there are multiple options?

In the version first proposed by Fitzpatrick, the zombies choose their start positions. On their turn, the zombies each select a shortest path toward the survivor (a *geodesic*) and move along the edge to the next vertex of the path. If there are multiple such paths, the zombies are free to choose. These types of zombies are known as deterministic zombies (see 1.3.1). Alternately, the zombies could break ties randomly (which leads to a different game with so called probabilistic zombies; see 1.3.2). In this thesis, we study the deterministic version of the game which is concerned with the worst-case outcomes for the survivor.

The players have complete information of the graph and the positions of the players. Indeed, the zombies need both to enact their strategy. If uncaught, the survivor may move to one of its neighbouring vertices or stay in place.

The game is decided when:

- A zombie eats the survivor by moving to the survivor's vertex.
- The survivor can evade the zombies indefinitely.

We call $s \in V(G)$ the survivor and $z_i \in V(G)$ are zombies with $i \in \{1, \ldots, k\}$. This notation represents both a player and its position in the graph. In the games studied there is a single survivor and $k \geq 1$ zombies.

As in C & R, we divide the game into *rounds* and *turns*. A round consists of two turns: a zombie turn and a survivor turn. It is convenient to define the zombie's turn on $t \equiv 0(\mod 2)$ and the survivor's turn on $t \equiv 1(\mod 2)$. Round r is given by $\lfloor \frac{t}{2} \rfloor$.

It is occasionally useful to identify the players' positions over time, in which case let $z_r^i \in V(G)$ be zombie i's position on round r. Similarly s_r is the survivor's position on round r. This burdensome notation will be omitted when possible.

1.3.1 Deterministic Zombies

Since there can be multiple shortest paths linking a zombie z_k to a survivor s, the zombie may have to make a choice between neighbouring vertices on its turn. The possible *zombie moves* are those neighbours of z_k which lie on a shortest path toward the survivor, which we denote

$$Z[z_k, s] = \{y \in N(x) \mid d(y, s) = d(z_k, s) - 1\}$$

the zombies moves from z_k given survivor is on s.

There is at least one such move since the game graph is presumed connected, so $Z[z_k, s] \neq \emptyset$. If there is only one path, then z_k has no choice but to move to the next vertex of that path. If all possible shortest paths move through the same next vertex, then again z_k does not have any choice on its move. If, however, there are multiple shortest paths connecting the zombie to the survivor with different first moves, then the zombie could make multiple moves.

A *zombie strategy* or *zombie play* (respectively, *survivor strategy* or *survivor play*) is a sequence of vertices of the graph which represent a zombie's position over the course of a game of Z&S. A game of Z&S can be described with a collection of zombie plays (one for each zombie) together with a survivor play. We say that a zombie play is a winning play if it guarantees the survivor is caught.

In the deterministic version of the game we consider the worst possible outcomes for the survivor: when the zombies play optimally it is as though they coordinate before choosing their next move. A graph is survivor win only if the survivor escapes in every possible zombie-play. The *zombie number* of a graph $z(G)$ can now

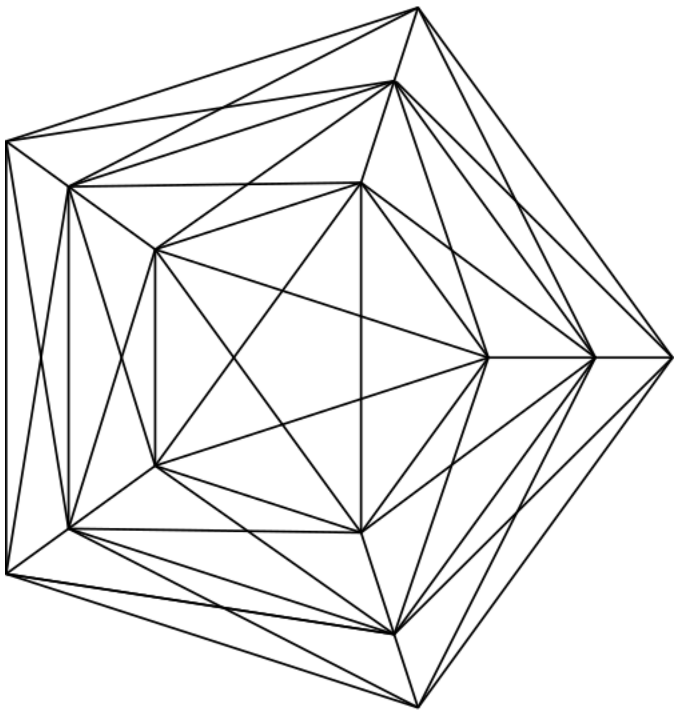

FIGURE 1.2: Cop-Win but not Zombie-Win

be defined analogously to the cop number: it is the number of zombies required to guarantee the survivor is captured given an optimal zombie-play. One of the first observations [5] about the zombie number is that:

Lemma 1. For any graph G, $c(G) \leq z(G)$.

Strategies are available to cops which are not available to the zombies, but the cops could apply a zombie strategy. The zombies are weaker versions of cops, similar in a way to the "fully active" cops from [22] wherein the cops are obligated to move on their turn. Nevertheless, on simpler types of graphs the zombies are just as effective as the cops [5]:

Observation 1.

1. $z(T) = c(T) = 1$ for any tree T.
2. $z(C) = c(C) = 2$ for any cycle of length $n \geq 4$.
3. $z(K_{m,n}) = c(K_{m,n}) = 2$ for any complete bipartite graph with $2 \leq m \leq n$.
4. $z(K_n) = c(K_n) = 1$ for $n \geq 1$.

It is also known [5] that cop-win graphs are not necessarily zombie-win. A counter-example is reproduced below (refer to Figure 1.2). This yields

Theorem 1. If a graph is zombie-win, then it is cop-win. However, a cop-win graph is not necessarily zombie-win.

Cop-win graphs are characterized by a dismantling (see Subsection 1.2.2). Does there exist a characterization for zombie-win graphs – for graphs on which a single zombie can always win? One has yet to be described. However, [5] observed that a graph is zombie-win if a specific cop-win spanning tree exists:

Theorem 2. If there exists a breadth-first search of a graph G such that the associated spanning tree is also a cop-win spanning tree, then G is zombie-win.

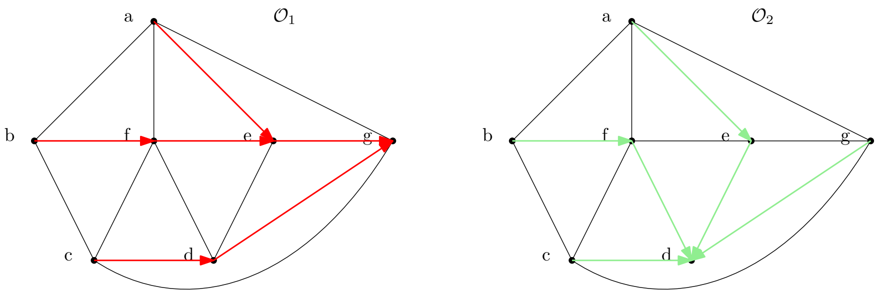

FIGURE 1.3: Two dismantlings and associated trees. The retracts are shown by directed edges. Only the tree associated to $\mathcal{O}_2$ results from breadth-first (shown in green).

Thus a sufficient condition for zombie-win graphs are those for which a specific cop-win tree exists: a cop-win tree obtainable as a breadth-first search of the graph from some root vertex. It is unknown if it is also a necessary condition. To illustrate these ideas, consider the following graph (refer to Figure 1.3), these two dismantlings, their orderings, and the resulting copwin spanning trees.

Let f_i be defined as

$$\begin{aligned} f_1(b) &= f \\ f_2(c) &= d \\ f_3(f) &= e \\ f_4(a) &= e \\ f_5(e) &= g \\ f_6(d) &= g \quad . \end{aligned}$$

The composition of the f_i is a dismantling with ordering $\mathcal{O}_1 = \{b, c, f, a, e, d, g\}$. Let also g_i be defined as

$$\begin{aligned} g_1(b) &= f \\ g_2(a) &= e \\ g_3(c) &= d \\ g_4(f) &= d \\ g_5(e) &= d \\ g_6(g) &= d \quad . \end{aligned}$$

These functions are also a dismantling with ordering $\mathcal{O}_2 = \{b, a, c, f, e, g, d\}$. However, only the second produces a copwin tree obtainable as a breadth-first search: they have different final nodes, and thus different roots. Observe that in the tree associated with $\mathcal{O}_1$, the tree-distance from g to c is 2 (and yet it is at distance 1 in the graph).

Fitzpatrick also obtains upper bounds on the zombie number of various graph constructions.

Theorem 3. For any graph G and $n \geq 4$, $z(G \square C_n) \leq 3z(G)$.

Theorem 4. If T is a finite tree, then for any graph G, $z(G \boxtimes T) \leq 2z(G)$.

Theorem 5. Let H be a graph with m vertices and at least one vertex of degree $m-1$. For any graph G, $z(G \square H) \leq z(G) + 1$.

1.3.2 Probabilistic Zombies

Zombies are often depicted as mindless or aimless. It is a common trope that zombies idle around, moving in random directions until they somehow (suddenly) distinguish the uninfected. It is only at this point that the zombies will charge. Such behavior likely inspired another type of pursuit game [23] in which the zombies start randomly on the graph. Once the survivor chooses a start vertex, the zombies "notice" the survivor and start moving directly towards it (again by following a shortest path).

Without knowing where the zombies start, however, it is impossible to know the outcome with certainty. The *probabilistic zombie-number* of a graph is the minimum number of these random zombies required to give the zombies a 50% chance of winning. Accepting some uncertainty in the outcomes of games simplifies the problem of the zombie choice: if there is more than one possible zombie move then choose one randomly.

Firstly, the zombie-number is unbounded. A graph which requires arbitrarily many zombies can be constructed by taking a cycle of length at least 5 and attaching additional vertices to one of the vertices of the cycle. In this way, we can make the odds that the zombies start on vertices which "guard" the cycle arbitrarily small. Indeed, we have the following property [23]:

Lemma 2. The survivor wins on C_n against $k \geq 2$ zombies if and only if all zombies are initially located on an induced subpath containing at most $\lceil \frac{n}{2} \rceil - 2$ vertices.

The zombie-number of cycles, projective planes, hyper cubes, grids and toroidal grids has also been considered. Bonato et al. proved the following theorem [23] which gives the zombie-number of cycles.

Theorem 6. Let $n \in \mathbb{Z}$ with $n \geq 3$. The probabilistic zombie-number of cycle C_n is

$$z(C_n) = \begin{cases} 4 & \text{if } n \geq 27 \text{ or } n = 23, 25, \\ 3 & \text{if } 11 \leq n \leq 22 \text{ or } n = 9, 24, 26, \\ 2 & \text{if } 4 \leq n \leq 8 \text{ or } n = 10, \\ 1 & \text{if } n = 3 \end{cases} .$$

On the grid, two zombies are enough to get even odds [23] of capturing the survivor:

Theorem 7. Let G_n be the grid graph isomorphic to $P_n \square P_n$. The probabilistic zombie-number of G_n is 2.

We note that the proof strategy of this theorem in [23] also works for deterministic zombies. Two deterministic zombies thus are guaranteed to win by effecting a coordinate-shadowing strategy on the grid.

Theorem 8. The probabilistic zombie-number of a hyper cube Q_n of dimension n is $z(Q_n) = \frac{2n}{3} + \Theta(\sqrt{n})$, as $n \to \infty$.

1.4 Our Contributions

Fitzpatrick [5] gave an example of an outerplanar graph G for which $z(G) > 2$, showing that the upper bound of two for the cop-number on outerplanar graphs does not apply to zombies. In Chapter 2, we take inspiration from this counter-example and give a planar graph on which three zombies always lose. This shows that the upper bound of 3 for the cop-number of planar graphs does not apply either.

Cycles (and subcycles) are important in these games since the survivor cannot hope to win without one. Two zombies win on a cycle by choosing diametrically opposed vertices. Fitzpatrick showed that maximally outerplanar graphs are zombie-win. In Chapter 3 we show that two zombies win on a cycle with a single chord.

Chapter 2

Planar Zombies

Aigner and Fromme [11] showed that the cop number of a planar graph is at most three: three cops can guard isometric paths to constrict the robber territory over time. Unfortunately, zombies are not smart enough to apply this strategy. Could a survivor potentially evade an infinite number of zombies, given the right planar graph?

Consider the planar graph illustrated in Figure 2.1. This graph was inspired by the outerplanar graph identified by Fitzpatrick [5][Fig. 2] for which two zombies lose. This shows that the upper bound of two on the cop-number of outerplanar graphs does not hold for zombies. In this Chapter, we give a planar graph on which three zombies always lose. This shows that the upper bound of three for the cop-number of planar graphs does not apply either.

Our graph G is constructed by taking a 5-cycle and augmenting it by adding paths of length 5 which connect adjacent vertices of the cycle. We then connect each 5-path to neighbouring 5-paths by way of an edge from the 2nd (or 4th) vertices. Though arbitrary, we fix the embedding described in order to refer to the parts of the graph.

We will call vertices

$$\begin{aligned} C &= \{1, \ldots, 5\} \\ X &= V(G) \setminus C \\ Y &= \{7, 9, 12, 14, 17, 19, 22, 24, 27, 29\} && \text{the vertices of degree 3.} \\ S &= \{7, 8, 9, 12, 13, 14, 17, 18, 19, 22, 23, 24, 27, 28, 29\} && \text{the outermost 15-cycle} \end{aligned}$$

Our proof relies on a strategy available to the survivor on this graph which we call *Running Around the Outside*. Observe that if the three zombies and survivor are on $G[S]$ on an induced sub-path of length at most 6 (with an empty vertex between the survivor and the lead zombie), then the survivor wins by fleeing away from the zombies around the outermost 15-cycle.

To see this, let $E' = \{xy \in E(G) : x, y \in Y\}$ be the set of edges which connect the vertices of Y and let $G' = G - E'$ be the subgraph obtained by removing these edges. These edges are highlighted in red in Figure 2.1. The table below 2.1 compares the length of possible zombie-survivor paths of lengths at most 5 in G and G'.

This table shows that when the zombie and the survivor are both in S and within a distance of 4 or 5, then the shortest path from the zombie to the survivor is contained entirely in S and thus zombies never have the opportunity to leave the outermost 15-cycle. When the survivor and zombie are both on the outermost cycle at distances 2 or 3, then the zombies must stay on S since it requires at least 2 moves to exit S (so the shortest path must be the one contained in S). This shows that the survivor has won if the survivor and the zombies are all on an induced sub-path

z	s	shortest path in G	$d_G(z,s)$	shortest path in G'	$d_{G'}(z,s)$
7	13	7,8,9,12,13	4	7,6,1,2,11,12,13	6
7	14	7,8,9,12,13,14	5	7,6,1,2,3,15,14	6
8	14	8,9,12,13,14	4	8,9,10,2,3,15,14	6
8	17	8,9,12,13,14,17	5	8,9,10,2,3,16,17	6
9	17	9,12,13,14,17	4	9, 10, 2, 3, 16, 17	5
9	18	9,12,13,14,17	5	9,10,2,3,16,17,18	6

TABLE 2.1: Zombies cannot exit the outermost cycle if the survivor is also on the outermost cycle within distance 5.

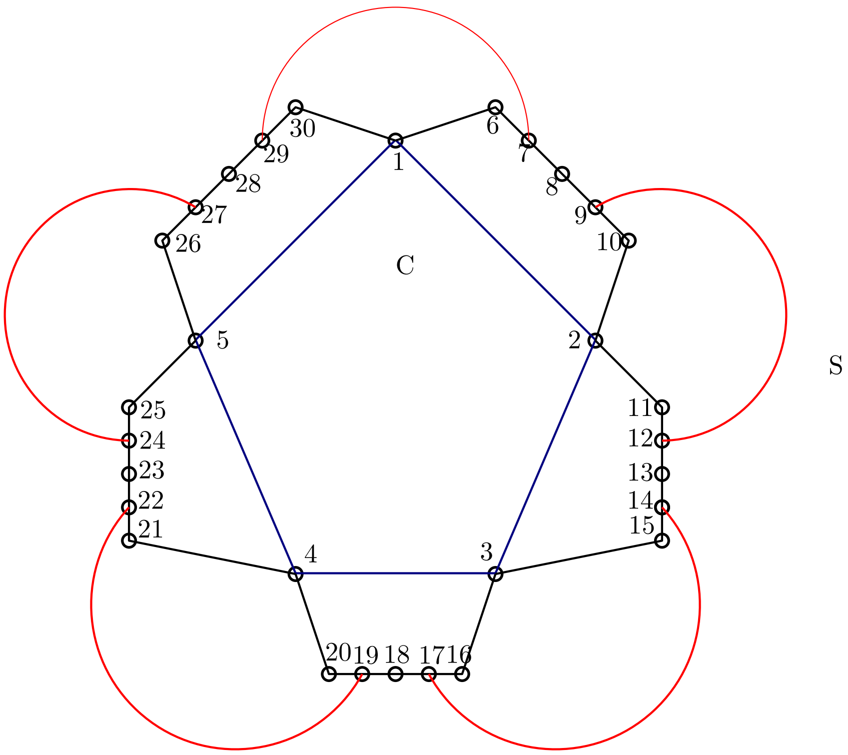

FIGURE 2.1: A graph with $z(G) > 3$. Edges of the 5-cycle C are blue and edges of E' are red. The vertices of S and edges of E' form a cycle of length 15.

of $G[S]$ of length at most 6 (and at least 3 since there needs to be an empty vertex between the leading zombie and survivor).

Theorem 9. The zombie-number of planar graphs is at least 4.

Proof. By counter-example: we give a planar graph G on which the survivor defeats 3 zombies. We must provide a winning survivor strategy (a safe starting vertex, and the opening for a winning sequence of survivor moves) for every possible 3 zombie start configuration on G. We divide all possible zombie-starts by the number of zombies which start on C.

- All the zombies start of C. That is, $z_i \in C$ for $1 \leq i \leq 3$ (refer to Case I).
- Two of the zombies are of C but one is not so $z_1, z_2 \in C$ and $z_3 \in X = V(G) \setminus C$ (refer to Case II).
- All three zombies start on vertices of X (refer to Case III).

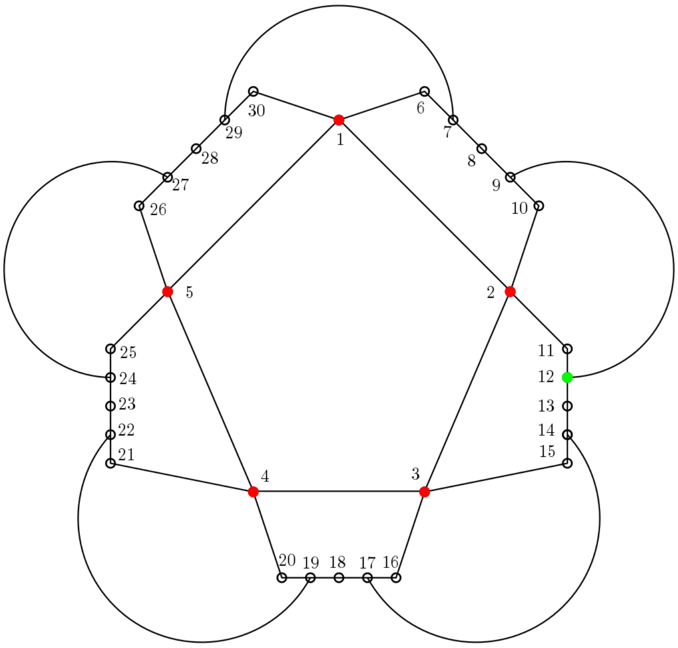

FIGURE 2.2: Case I, Round 0. The survivor is green; zombies are red.

- One zombie chooses a vertex of C, two others start on vertices of X, i.e. $z_1, z_2 \in X$ and $z_3 \in C$ (refer to Case IV).

Since these cases are exhaustive, the survivor can always respond to a zombie start with a winning strategy, and so $z(G) > 3$.

I. The three zombies choose vertices of C.

Instead of showing that the strategy works for all possible configurations of 3 zombies on C, we show that the survivor can win against 5 zombies occupying every vertex of C. Since the survivor defeats 5 such zombies, the same strategy will work on any subset of 3.

The zombies occupy the vertices 1–5 and the survivor chooses a vertex $y_1 \in Y$ of degree 3. Without loss of generality, say the survivor chooses 12.

If the survivor starts on $y_1 \in Y$, and moves to $y_2 \in Y$ using edge y_1y_2 and continues to flee in the same direction along the outermost 15-cycle, then the zombies will not be able to catch the survivor. Let us examine the first few rounds in detail. The game begins as illustrated in Figure 2.2.

On the first round, the zombies each have a single shortest path to the survivor on 12 and thus must move as follows:

- The zombie on 2 moves to 11.
- The zombies on 1 and 3 collide on 2.
- The zombies on 4 and 5 move to 3 and 1, respectively.

The survivor responds by moving to 9. Round 1 moves are illustrated in Figures 2.3:

Yet again the zombies have a single shortest path to the survivor on 9 and thus move as follows:

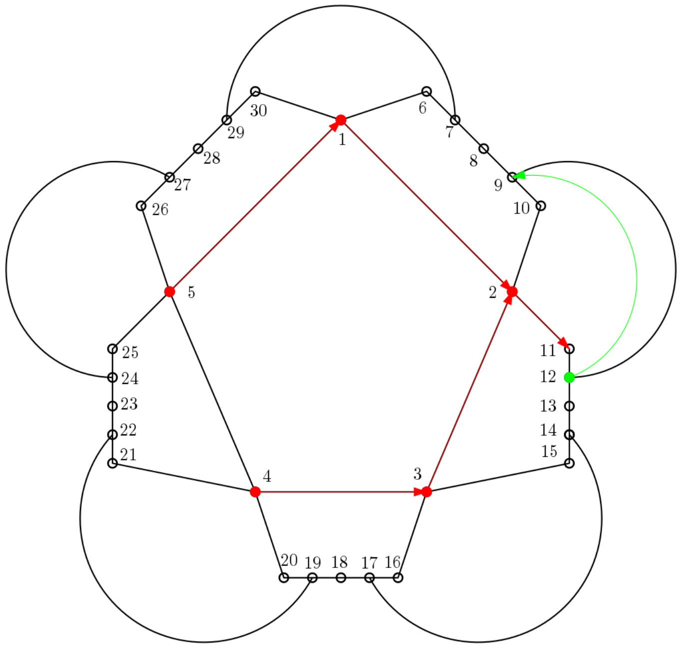

FIGURE 2.3: Case I, Round 1. Zombies are red; survivor is green. Arrows along edges indicate each players' next move.

- The zombie on 11 moves to 12.
- Zombies on 2 move to 10.
- Zombies on 1 and 3 collide on 2.

The survivor responds by moving to 8. These moves are illustrated in Figure 2.4:

After round 3 all zombies are within a distance of 3 of the survivor on the outermost 15-cycle. See Figure 2.5. The survivor wins by Running Around the Outside, i.e., by moving counter-clockwise on the cycle $G[S]$.

This shows that however the 3 zombies may be arranged on C in the initial round, they will not be able to corner the survivor following this strategy.

II. Two zombies z_1 and z_2 choose vertices of C and one zombie z_3 chooses a vertex in $X = \{6, \ldots, 30\}$.

The survivor starts on $s = y_1 \in Y$ (a vertex of degree 3) such that

$$3 \leq d_{G[X]}(s, z_3) \leq 4$$

and so that the edge connecting y_1 to $y_2 \in Y$ is not on the shortest path between s and z_3.

This choice of start vertex is always available to the survivor (refer to Figure 2.6). Without loss of generality, assume that z_3 has chosen one of the vertices 6-10.

- if z_3 chooses to start at 7 or 6, then the survivor chooses 27, which is at a distance of 3 or 4 respectively.
- if z_3 chooses to start at 8, then the survivor can start at either 14 or 27, both of which are at a distance of 4.

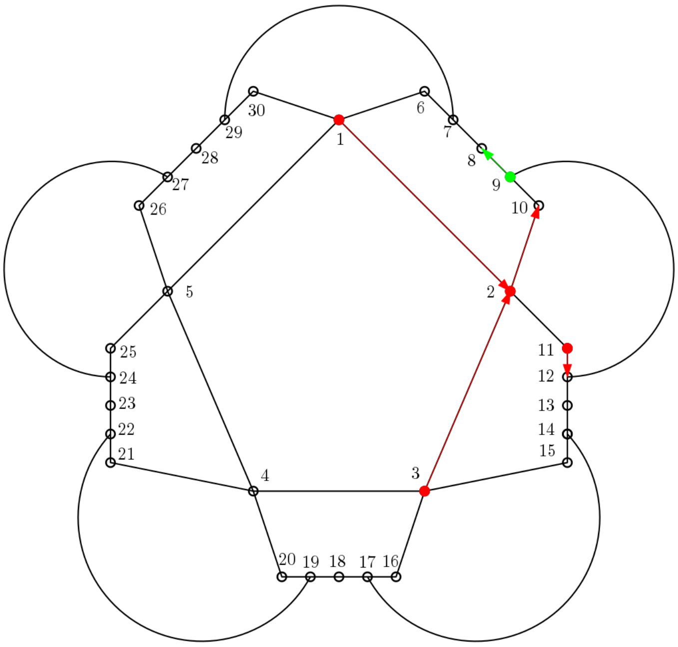

FIGURE 2.4: Case I, Round 2. Zombies are red; survivor is green. Arrows along edges indicate each players' next move.

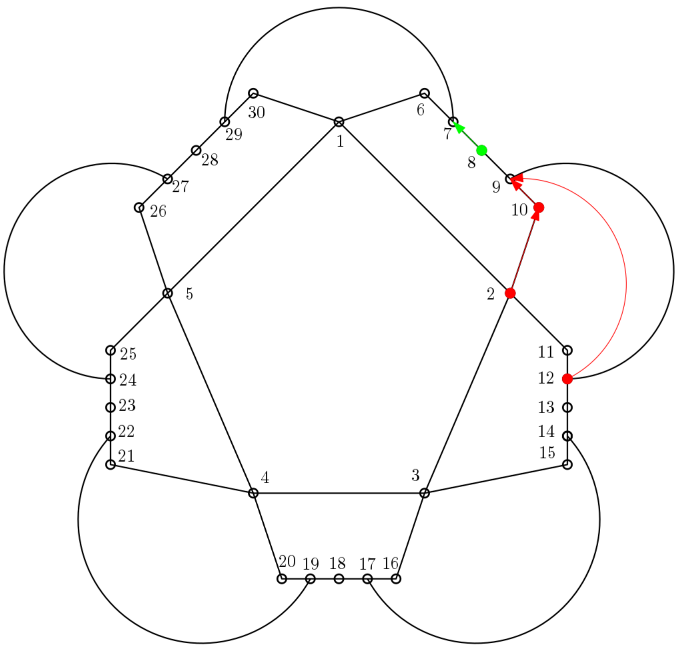

FIGURE 2.5: Case I, Round 3. Zombies are red; survivor is green. Arrows along edges indicate each players' next move.

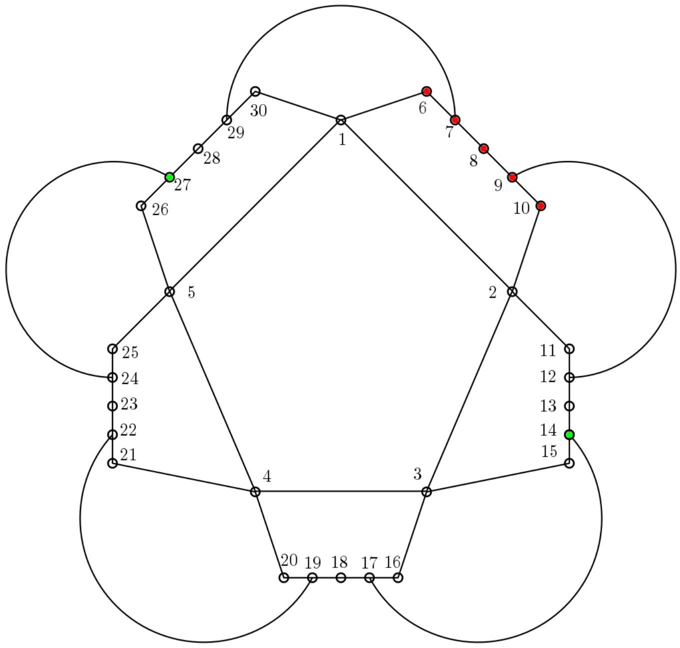

FIGURE 2.6: The survivor strategy in Case II. One of the red vertices has a zombie, and the two green vertices are survivor starts.

- if z_3 chooses to start at 9 or 10, then the survivor chooses 14, which is at a distance 3 or 4 respectively.

In round 1, if z_3 is not adjacent to C (either starting at 7, 8 or 9), then already the zombie has no choice but to pursue the survivor on the outermost 15-cycle.

If z_3 is adjacent to the C (either starting at 6 or 10), then z_3 may choose either to move onto the outermost 15-cycle or to cut through C since both are moves on a shortest s, z_3 paths.

However, as above, if z_3 chooses to move onto a vertex in S and follow along the outermost 15-cycle, then the game is already won for the survivor since $d(z_3, s) = 4$ and thus the third zombie can been forced to chase around the outermost 15-cycle forever.

If z_3 chooses to move to C then all three zombies are on C and we have reached a situation just as in Case I, Round 1 2.3: three zombies are on C, and the survivor is on a vertex $y \in Y$. The survivor wins using the strategy from Case I.

This shows that the survivor will always escape the third zombie following this strategy. Because this strategy is a restricted version of the strategy from Case I, we know that the zombies that start on C will not be able to corner the survivor. Therefore, this strategy defeats all possible start configurations where two zombies start on C and the third starts on the X.

III. All three zombies choose vertices in X.

We will show that the survivor can always start on C and wait (or flee) on C until the zombies collect in a subpath behind it, at which point the survivor can exit C and begin Running Around the Outside. We separate this case into sub-cases based on the number of moves required by the zombies to reach the interior cycle.

(a) All three zombies require the same number of rounds to reach C.

(b) Two zombies start adjacent to C, and the third is at distance 2 from C.

(c) Two zombies start at a distance of 2 from C and the third is at a distance of 3.

(d) Two zombies start adjacent to C, and the third is at distance 3 from C.

(e) One zombie starts adjacent to C, and the other two are at a distance of 2 from C.

Case III(a) : All three zombies require the same number of rounds to reach C.

Suppose all the zombies have chosen vertices in X which are adjacent to vertices in C. These are vertices $Q = \{6, 10, 11, 15, 16, 20, 21, 25, 26, 30\}$. Because there are 3 zombies and 5 interior vertices, there will always be at least two vertices in the interior cycle that are not threatened in round 0. The survivor starts on one of these safe vertices.

In round 1, the zombies have no choice but to enter C since the shortest path from a vertex $q \in Q$ to $s \in C$ necessarily includes the edge qc for some $c \in C$. Thus, after their first turn, the zombies all occupy vertices in C. The survivor responds by exiting C to $s' \in Q$.

In round 2, the zombies again have no choice but to approach the survivor using vertices on C. The survivor responds by moving to some $s'' \in Y$ and we have reached the scenario of Case I. The survivor wins by Running Around the Outside.

If all the zombies are at a distance of 2 from C (those vertices in Y) then the survivor can start on any vertex $s \in C$.

In round 1, the zombies approach the survivor by moving to vertices in Q. Let $q_0, q_1 \in Q \cap N(s)$ be the neighbours of the survivor which are not on C. Now, either:

(a) Both q_0 and q_1 are occupied by zombies. In this case, there is some $c \in N(s_0) \cap C$ which is not threatened by a zombie (since two of them are adjacent to s). Therefore the survivor can safely move onto another vertex on C and, on the following round, move to an occupied vertex in Q. After another round the survivor moves to a vertex in Y and we again have reached a situation as in Case I.

(b) At least one of q_0 and q_1 is not occupied by zombies. In this case, the survivor can exit C immediately by moving to a vertex in Q. After the next round, all three zombies are on C and the survivor moves to a vertex in Y and again we are in a situation like Case I.

If all the zombies are at a distance of 3 from C, then the survivor may start on any vertex of C and simply pass on the first round. The zombies, have no choice but to move to vertices in Y and so we find ourselves in the case described before.

Now we must deal with the cases where the zombies are at different distances from the center cycle.

Case III(b) : Two zombies start adjacent to C, and the third is at distance 2 from C.

Suppose that two of the zombies have chosen vertices in Q and the other has chosen a vertex in Y. That is, two zombies are adjacent to C while the third requires two rounds to reach C.

There are now at least three unthreatened vertices on C for the survivor to choose. The survivor can choose any unthreatened vertex on C.

In round 1, two zombies enter C and the third moves to a vertex $q \in Q$ adjacent to C. The survivor exits C to another vertex $q_0 \in Q$. This move is always available to the survivor since only one vertex in Q is occupied by a zombie and every vertex in C is adjacent to two vertices of Q.

After the next turn, all three zombies are on C: two are already on C; the other must follow a shortest path which uses interior vertices since the shortest path between any two vertices of Q goes through the interior. The survivor moves to a vertex $s_2 \in Y$ and wins using the strategy from Case I.

Case III(c) : Two zombies start at a distance of 2 from C and the third is at a distance of 3.

The survivor may start on any of the vertices on C since none are threatened by a zombie.

In round 1, two zombies move to vertices in Q and the third moves to a vertex in Y. If the survivor is unthreatened after the first round, it can simply pass. If the survivor is threatened by a zombies adjacent to C, then at least one of the survivor's neighbours on C is unthreatened since there are two zombies on Q.

In either case, after round 1 we find ourselves in the situation described in Case III(b).

Case III(d) : Two zombies start adjacent to C, and the third is at distance 3 from C.

Suppose without loss of generality that the zombie at distance 3 from C has chosen vertex 18. Since there are only two zombies adjacent to C, at least one of the vertices $\{1, 2, 5\}$ must be a safe start for the survivor. Now observe that the survivor can defeat the zombie which starts on 18 with following play:

Round	z	s
0	18	2
1	17	10
2	16	9
3	3	8
4	2	7
5	1	29
6	30	28

But this is the same strategy that defeats the two zombies at distance 1 described in Case III(a): the survivor starts on an interior vertex, then exits on its first turn. Thus after 7 rounds, the survivor has succesfully baited all three zombies onto a 5-path and so the game is won by Running Around the Outside.

Case III(e) : One zombie starts adjacent to C, and the other two are at a distance of 2 from C.

Again, the survivor's strategy in this case is to waste time on C in order to allow all the zombies to approach. Since only one of the zombies is adjacent to C, there are four potential start vertices for the survivor. Any of these will work.

In round 1, the zombie at distance 1 from C moves onto C and the other two move to vertices $q_0, q_1 \in Q$, which are adjacent to C.

Now, either:

(a) Both q_0 and q_1 are adjacent to s_0. In this case, the survivor moves to $s_1 \in N(s_0) \cap C$, the neighbour on C that is not occupied by the zombie that has already reached C. After the next turn, all three zombies have reached C and so the survivor can exit to some $s_2 \in Q$. Again, after another round we have returned to Case I.

(b) At least one of q_0 and q_1 is adjacent to s_0. In this case, the survivor can exit C by moving to a vertex $s_1 \in Q$. After the next round, all three zombies are on C and we are in a situation like Case I.

In either case, the survivor has a simple winning strategy.

Case III(f) : One zombie starts at a distance of 2 from C, and the other two are at a distance of 3.

The survivor starts in C. None of the vertices on C are threatened by the zombies, since they are at a distance at least 2.

In round 1, the zombies approach C. The zombie that started at distance 2 from C is now on a vertex in Q and the other two zombies are on vertices in Y. If unthreatened, the survivor simply passes. If the survivor is threatened by the zombie that is adjacent to C, then she moves to another vertex on C. The other two zombies pose no threat in this round.

There is now one zombie at distance of 1 from C and two zombies at a distance of 2, and so we have returned to the situation describe in Case III(e).

Case III(g) : One zombie starts at a distance of 1 from C, and the other two are at a distance of 3.

The survivor starts on one of the four safe vertices on C.

In round 1, one zombie steps onto C while the other two zombies move to vertices at distance 2 from C. Only the zombie on C can threaten the survivor at this point. If the survivor is safe, then she may pass. Otherwise, since there is only a single zombie on C, at most one of the survivor's neighbours on C is threatened. So the survivor has a safe move to a vertex on C.

In round 2, the zombie on C pursues the survivor ineffectually while the other two zombies move to vertices $q_0, q_1 \in Q$ which are adjacent to C. Now, as in Case III(e), either

(a) Both q_0 and q_1 are adjacent to s_0. In this case, the survivor moves to $s_1 \in N(s_0) \cap C$, the neighbour on C that is not occupied by the zombie that has already reached C. After the next turn, all three zombies have reached C and so the survivor can exit to some $s_2 \in Q$. Again, after another round we have returned to Case I.

(b) At least one of q_0 and q_1 is not adjacent to s_0. In this case, the survivor can exit C by moving to a vertex $s_1 \in Q$. After the next round, all three zombies are on C and we are in a situation like Case I.

Case III(h) : The three zombies are at different distances from C.

In particular, this means that the zombies are at distances 1, 2 and 3 from C.

Observe that there is always a vertex in C that is at distance at least 3 from all zombies. This is a start position for the survivor which will allow her to survive unthreatened for at least two rounds.

Round	z_1	z_2	z_3	s
0	1	8	15	28
1	30	7	3	27
2	29	29	4	24
3	28	28	5	23
4	27	27	25	22

TABLE 2.2: If the zombies start at 1, 8 and 15, then the survivor wins by choosing 28.

In round 1, the closest zombie (more precision here - give label) moves onto C, the second closest moves to a vertex adjacent to C and the third moves to a vertex at a distance of 2 from C. The survivor remains in place.

In round 2, the closest zombie threatens the survivor, the second closest zombie moves onto C, and the last one moves onto a vertex adjacent to a vertex in C. Now, at least one of the survivor's neighbours is an unoccupied vertex in Q, which she can take to escape C.

After the next round, all three zombies are on C or one step behind the survivor and the survivor has won the game by moving to a vertex in Y as in Case I.

IV. One zombie chooses a vertex on C, the two others choose vertices of X.

We were unable to develop an argument to show why the survivor wins in this case. Instead, Appendix A provides tables showing the first few moves of a winning survivor strategy for every possible zombie start in this case (without loss of generality). In each scenario, the survivor is able to position itself so as to eventually apply the Running Around the Outside strategy.

For example, suppose the zombies start on 1, 8 and 15, then the survivor can respond with 28 and win by Running Around the Outside after round 4. The first few moves of this game are detailed in Table 2.2.

□

Chapter 3

Cycle With One Chord

Games played on cycles are straightforward: by Lemma 2 if the zombies are too close to each other, the survivor can lead the zombies in the same direction around the cycle. Otherwise, the zombies are too far apart and whichever side (sub-path of the cycle with the zombies at the end) the survivor may choose, the zombies will move in opposite directions and win. In this Chapter, we investigate the game on cycles augmented by a single chord.

Definition 1. Let $m, n \in \mathbb{Z}$ with $2 \leq m \leq n$. Consider a cycle of length $m+n$ and add a chord which divides the cycle into paths P_m and P_n of lengths m and n, respectively. We denote such a graph as $Q_{m,n}$. Let u and v denote the endpoints of the chord. We refer to the subcycles of lengths $m+1$ and $n+1$ formed by paths and the chord as $C_{m+1} = Q_{m,n}[P_m]$ and $C_{n+1} = Q_{m,n}[P_n]$, respectively.

See Figure 3.1 for an illustration of $Q_{7,8}$. The construction contains three subcycles which the survivor could use to fool the zombies. Let us first examine the construction for small values of m and n.

Lemma 3. The zombie-number $z(Q_{2,2}) = 1$.

Proof. Setting $m = n = 2$ yields a graph with two adjacent cliques K_3, which are dominated by a single vertex so it is zombie-win. □

Lemma 4. Let $n \in \mathbb{Z}$ with $n \geq 3$. Then $z(Q_{2,n}) = 2$.

Proof. For $m = 2$ and $n \geq 3$, two zombies win by starting on diametrically opposed vertices on the cycle C_{n+1}. The additional edge has no impact on a two-zombie cycle strategy. A single zombie does not suffice because of the cycle of length at least 4. □

Lemma 5. Let $m, n \in \mathbb{Z}$ with $3 \leq m \leq n \leq 5$. Then $z(Q_{m,n}) = 2$.

Proof. For $Q_{m,n}$ with $3 \leq m \leq n \leq 5$, placing the zombies on the endpoints of the chord divides the graph into two cycles of length at most 5 which can be guarded by

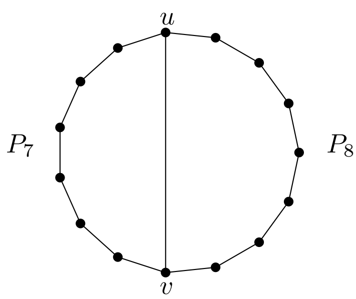

FIGURE 3.1: $Q_{7,8}$

two adjacent vertices. A single zombie does not suffice because of the cycle of length at least 4. □

For larger values of m and n the outcome is not as clear. Unfortunately for the survivor, we are able to show the existence of starting positions for the zombies (obtained as a function of m, n) which limits the survivor's options and prevents the zombies from being led in the same direction.

Theorem 10. Let $m, n \in Z$ with $3 \leq m \leq n$. The zombie number of $Q_{m,n}$ is 2.

We imagine $Q_{m,n}$ as embedded in the plane with P_m – the shortest side – on the left. This does not limit the generality of the following and allows us to define (counter-)clockwise distance: the length of the path along a cycle with respect to the given direction on this embedding.

Proof. First, observe that one zombie will not suffice for any graph containing an isometric subcycle of length at least four. Second, the lemmas above show the statement to be true for $3 \leq m \leq n \leq 5$, so for the remainder of the proof we assume that $m \geq 3$ and $n \geq 6$.

We seek a winning starting position for the zombies for $m \geq 3$ and $n \geq 6$. We describe a strategy in three separate parts, which we summarize here.

First we show how to position the zombies to guarantee a win assuming the survivor starts on P_m. We will position one of the zombies on an endpoint of the chord and another at some distance Δ from the other endpoint. We calculate the values of Δ which guarantee the survivor will be sandwiched on P_m by considering all possible combinations of directions "chosen" by the zombies (refer to Part 3.1). The zombies' choice of direction is not really a choice, after all: the choice is forced by the position of the survivor and the length of the possible zombie-survivor paths.

Next, we show how to position the zombies at the start of the game so that no matter where the survivor starts a losing position is guaranteed: we offset the zombies on the larger cycle with an additional parameter k, which ensures the zombies are not too close together and therefore guard C_{n+1} (refer to Part 3.2). After k rounds, the survivor will have no choice but to retreat to the smaller cycle and fall into the carefully orchestrated trap described in the first part of the proof.

In Part 3.3, we show that such a starting position is available to the zombies for any $m \geq 3$, $n \geq 6$. Finally in Part 3.4 we use these results to give winning Δ and k for any fixed m, n values.

3.1 Cornering the Survivor on C_{m+1}

As mentioned, let u and v denote the endpoints of the chord. Let z_1, z_2 and s denote the positions of the zombies and the survivor, respectively.

Part 1. Suppose that the game has reached the following state:

- the first zombie is on an endpoint of the chord, say v
- there are Δ vertices counting clockwise from u to z_2.
- the survivor is on P_m at a distance of ℓ vertices counting clockwise from v.

This configuration is illustrated in Figure 3.2. Note that we must have

$$2 \leq \ell \leq m - 1 \quad (1)$$

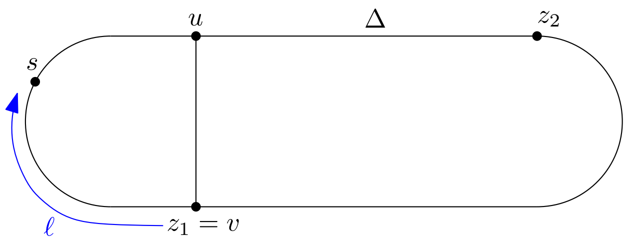

FIGURE 3.2: z_1 on v, s somewhere on P_m

else z_1 captures the survivor on the next round.

By comparing the lengths of different paths, we calculate the values of Δ which guarantees that the survivor will be cornered on P_m from this start configuration. That is to say, the survivor will not be able to return to any of the endpoints of the chord before z_2.

We can assume that once z_1 chooses a direction from v, it continues in that direction: either the zombie has no choice or both directions around the cycle are of the same length (and so z_1 may continue in the same direction).

We can also assume that on its turn the survivor will move away from z_1 and maintain a distance of ℓ (or $m - \ell + 1$, if they are moving counter-clockwise) since a winning survivor strategy which involves waiting a turn or moving backwards is equivalent to a survivor strategy which always moves but starts with a smaller (or larger) value of ℓ.

These two assumptions allow us to "fast-forward" the game by Δ rounds (or $n - \Delta$ rounds) and determine when the survivor is captured. Since z_1 is already on the same sub-cycle as the survivor, there are two possibilities:

A. z_1 goes clockwise if $\ell \leq 1 + m - \ell$. Combined with 1, we have

$$4 \leq 2\ell \leq m + 1 \quad . \tag{A}$$

B. z_1 goes counter-clockwise if $1 + m - \ell \leq \ell$. Combined 1, we obtain

$$m + 1 \leq 2\ell \leq 2m - 2 \quad . \tag{B}$$

We must consider four possible paths from z_2 to the survivor:

- P_a of length $\Delta + (m - \ell)$,
- P_b of length $\Delta + 1 + \ell$,
- P_c of length $(n - \Delta) + 1 + (m - \ell)$, and
- P_d of length $(n - \Delta) + \ell$.

These paths are illustrated in Figure 3.3.
Comparing path lengths we see that:
z_2 moves counter-clockwise if either

$$|P_a| \leq \min\{|P_c|, |P_d|\} \qquad \text{or} \qquad |P_b| \leq \min\{|P_c|, |P_d|\} \quad . \tag{I}$$

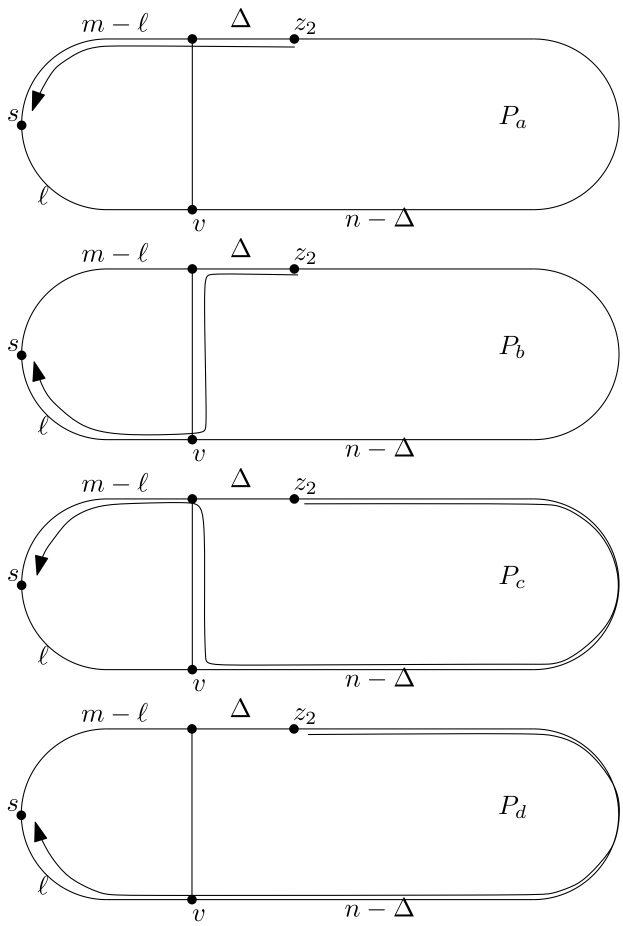

FIGURE 3.3: Possible paths from z_2 to s

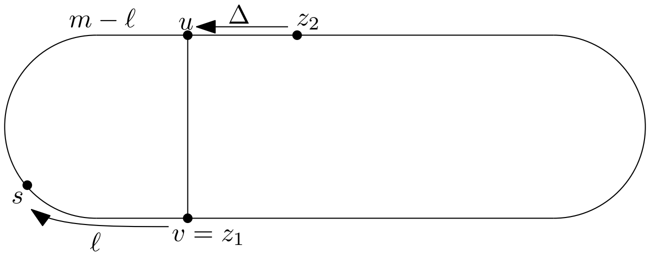

FIGURE 3.4: Case I.A.

z_2 goes clockwise if either

$$|P_c| \leq \min\{|P_a|, |P_b|\} \qquad \text{or} \qquad |P_d| \leq \min\{|P_a|, |P_b|\} \quad . \tag{II}$$

We will examine all combinations of these possible "zombie-decisions" to show that there exist values of Δ which prevent the survivor's escape in any of the possible games (from this start configuration where the survivor is on P_m). We break it down as follows:

- I. z_2 goes counter-clockwise
- II. z_2 goes clockwise.
- A. z_1 goes clockwise
- B. z_1 goes counter-clockwise
- *Case I.A.* z_2 goes counter-clockwise and z_1 goes clockwise.

  Suppose the zombies will move as in Figure 3.4.

  We obtain the following constraints on ℓ from A

  $$4 \leq 2\ell \leq m+1$$

  and the following constraints on Δ from I

  $$\begin{aligned} \Delta + (m-\ell) &\leq n - \Delta + 1 + m - \ell && \text{and} \\ \Delta + (m-\ell) &\leq n - \Delta + \ell \end{aligned}$$

  or

  $$\begin{aligned} \Delta + 1 + \ell &\leq n - \Delta + 1 + m - \ell && \text{and} \\ \Delta + 1 + \ell &\leq n - \Delta + \ell \quad . \end{aligned}$$

  Combining with A we can obtain:

  $$\begin{aligned} 2\Delta &\leq n+1 && \text{and} \\ 2\Delta &\leq n - m + 2\ell \leq n+1 \end{aligned}$$

  or

  $$\begin{aligned} 2\Delta &\leq n + m - 2\ell && \text{and} \\ 2\Delta &\leq n - 1 \leq n + m - 2\ell \end{aligned}$$

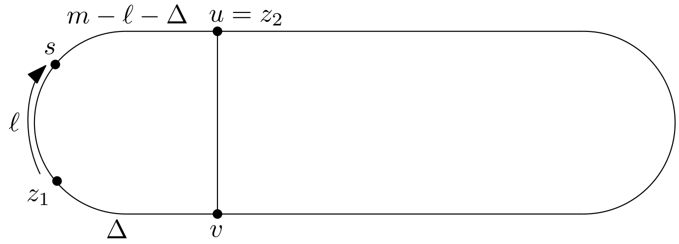

FIGURE 3.5: Case I.A. after Δ rounds

So for z_2 to follow either P_a or P_b and go counter-clockwise we must have

$$2\Delta \leq n-m+2\ell \quad \text{or}$$
$$2\Delta \leq n-1 \quad .$$

We must determine which of s or z_2 reaches u first. Consider the game after Δ rounds, as illustrated in Figure 3.5.

If $\Delta = m-\ell$ both z_2 and s reach u on the same round, with the survivor moving onto the zombie-occupied vertex (and losing). If we have $\Delta = m-\ell+1$, then s reaches u first but is caught by z_2 on the following round. So, to guarantee the survivor has not escaped P_m we need

$$\Delta \leq m-\ell+1$$

otherwise the survivor can reach the chord at least two rounds before z_2 can move to block. We wish to prevent this scenario since the survivor could then take the chord and possibly escape, pulling both zombies into a loop either on C_{m+1} or C_{n+1}.

That is not sufficient, however. We must also ensure that z_2 moves counter-clockwise (opposite to z_1) once it reaches u in order to trap the survivor. So we need

$$m-\ell-\Delta \leq 1+\Delta+\ell$$

Or, in terms of Δ,

$$2\Delta \geq m-2\ell-1 \quad .$$

When we combine all the restrictions we obtain the following characterization for Case I.A.:

z_1 goes clockwise:

$$4 \leq 2\ell \leq m+1$$

and z_2 goes counter-clockwise:

$$2\Delta \leq n-m+2\ell \quad \text{or} \quad 2\Delta \leq n-1 \quad .$$

The zombies win:

$$2\Delta \leq 2m-2\ell+2 \quad \text{and} \quad m-2\ell-1 \leq 2\Delta \quad .$$

- *Case I.B.* z_2 and z_1 both go counter-clockwise.

  Suppose the zombies will move as in Figure 3.6.

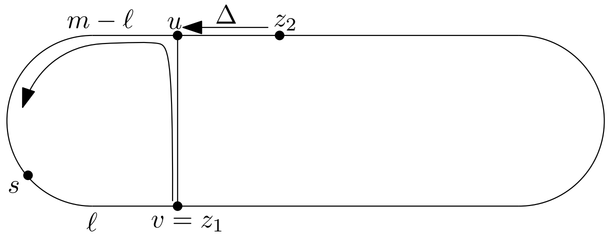

FIGURE 3.6: Case I.B.

From B and the constraint on ℓ, we must have

$$m + 1 \leq 2\ell \leq 2m - 2$$

and the constraints on Δ from I are again:

$$\begin{aligned} &\Delta + (m - \ell) \leq n - \Delta + 1 + m - \ell && \text{and} \\ &\Delta + (m - \ell) \leq n - \Delta + \ell \end{aligned}$$

or

$$\begin{aligned} &\Delta + 1 + \ell \leq n - \Delta + 1 + m - \ell && \text{and} \\ &\Delta + 1 + \ell \leq n - \Delta + \ell \quad . \end{aligned}$$

Simplifying using B yields:

$$\begin{aligned} &2\Delta \leq n + 1 \leq n - m + 2\ell && \text{and} \\ &2\Delta \leq n - m + 2\ell \end{aligned}$$

or

$$\begin{aligned} &2\Delta \leq n + m - 2\ell \leq n - 1 && \text{and} \\ &2\Delta \leq n - 1 \quad . \end{aligned}$$

So for z_2 to go counter-clockwise in this case we must have

$$\begin{aligned} &2\Delta \leq n + 1 && \text{or} \\ &2\Delta \leq n + m - 2\ell \quad . \end{aligned}$$

Again we must consider who reaches the chord first. Consider the game after Δ rounds, as illustrated in Figure 3.7.

If $\ell = \Delta$, then z_2 reaches u and s reaches v on the same round, and therefore s will be caught on the next. Therefore, to guarantee the survivor has not escaped P_m in this scenario we need

$$\Delta \leq \ell$$

Otherwise, the survivor reaches the chord before z_2 and could escape.

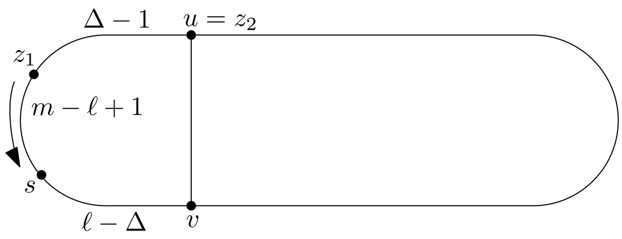

FIGURE 3.7: Case I.B. after Δ rounds

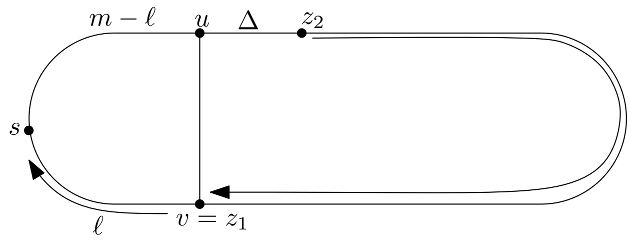

FIGURE 3.8: Case II.A.

Then, to ensure that z_2 traps the survivor by going clockwise once it reaches u we need

$$1 + \ell - \Delta \le \Delta - 1 + m - \ell + 1 \qquad \text{and}$$
$$2\ell - m + 1 \le 2\Delta \quad .$$

We obtain the following characterization for Case I.B.:

z_1 goes counter-clockwise:

$$m + 1 \le 2\ell \le 2m - 2$$

and z_2 goes counter-clockwise

$$2\Delta \le n + 1 \qquad \text{or} \qquad 2\Delta \le n + m - 2\ell \quad .$$

The zombies win:

$$2\Delta \le 2\ell \qquad \text{and} \qquad 2\ell - m + 1 \le 2\Delta \quad .$$

- *Case II.A.* z_2 and z_1 both go clockwise.
  Suppose the zombies will move as in Figure 3.8.
  We have the following constraint on ℓ from B

  $$4 \le 2\ell \le m + 1$$

  and the following constraints on Δ from II

  $$n - \Delta + 1 + m - \ell \le \Delta + m - \ell \qquad \text{and}$$
  $$n - \Delta + 1 + m - \ell \le \Delta + 1 + \ell$$

  or

$$\begin{aligned} n - \Delta + \ell &\leq \Delta + m - \ell && \text{and} \\ n - \Delta + \ell &\leq \Delta + 1 + \ell && . \end{aligned}$$

Simplifying with a bit of algebra yields:

$$\begin{aligned} n - m + 2\ell &\leq 2\Delta && \text{and} \\ n - 1 &\leq 2\Delta && \end{aligned}$$

$$\text{or}$$

$$\begin{aligned} n + 1 &\leq 2\Delta && \text{and} \\ n + m - 2\ell &\leq 2\Delta && . \end{aligned}$$

These inequalites are of the form

$$\begin{aligned} n - x &\leq 2\Delta && \text{and} \\ n - 1 &\leq 2\Delta && \end{aligned}$$

$$\text{or}$$

$$\begin{aligned} n + x &\leq 2\Delta && \text{and} \\ n + 1 &\leq 2\Delta && , \end{aligned}$$

where $x = m - 2\ell$.

Supposing $x \geq 0$, we have

$$\begin{aligned} n - x \leq n + x &\leq 2\Delta && \text{and} \\ n - 1 < n + 1 &\leq 2\Delta && \end{aligned}$$

and take the lowest bounds because of the disjunction, so that

$$2\Delta \geq n - x = n - m + 2\ell \qquad \text{and} \qquad 2\Delta \geq n - 1$$

suffices.

Since B gives $m - 2\ell \geq -1$, supposing $x < 0$ reduces the inequalites to

$$\begin{aligned} n + 1 &\leq 2\Delta && \text{and} \\ n - 1 &\leq 2\Delta && \end{aligned}$$

which is satisfied by $2\Delta \geq n - x = n - m + 2\ell$ and $2\Delta \geq n - 1$.

Thus z_2 will go clockwise under B if

$$\begin{aligned} 2\Delta &\geq n - m + 2\ell && \text{and} \\ 2\Delta &\geq n - 1 && \end{aligned}$$

Consider the game after $n - \Delta$ rounds, as illustrated in Figure 3.9.

We have assumed that z_1 is going clockwise. If $m - \ell = n - \Delta$, then z_2 reaches v and s reaches u on the same round and s will be caught on the next. Therefore,

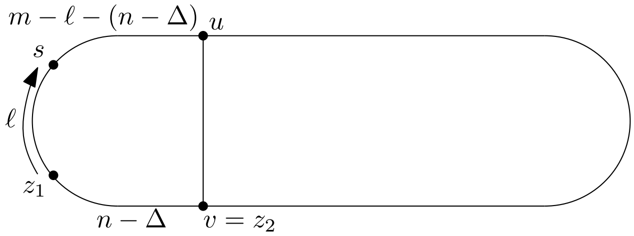

FIGURE 3.9: Case II.A. after $n-\Delta$ rounds

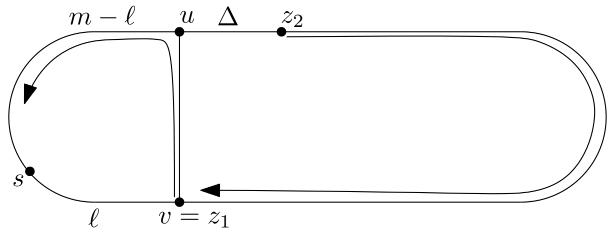

FIGURE 3.10: Case II.B.

to guarantee the survivor has not escaped P_m in this scenario we need

$$n-\Delta \leq m-\ell \quad \text{and} \quad \Delta \geq n-m+\ell \quad ,$$

otherwise the survivor could reach the chord before z_2.
To ensure that z_2 goes counter-clockwise once it reaches v, we need

$$1+m-\ell-(n-\Delta) \leq n-\Delta+\ell$$
$$2\Delta \leq 2n+2\ell-m-1 \quad .$$

We obtain the following characterization for Case II.B.:
z_1 goes clockwise:

$$4 \leq 2\ell \leq m+1 \quad ,$$

and z_2 goes clockwise

$$n-m+2\ell \leq 2\Delta \quad \text{and} \quad n-1 \leq 2\Delta \quad .$$

The zombies win:

$$2\Delta \geq 2n-2m+2\ell \quad \text{and} \quad 2\Delta \leq 2n+2\ell-m-1 \quad .$$

- *Case II.B.* z_2 goes clockwise and z_1 goes counter-clockwise.
  Suppose the zombies will move as in Figure 3.10.
  We have the following constraint on ℓ from B

$$m+1 \leq 2\ell \leq 2m-2 \quad ,$$

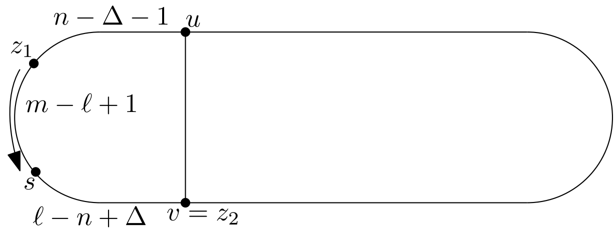

FIGURE 3.11: Case II.B. after $n-\Delta$ rounds

and the following constraints on Δ from II

$$n-\Delta+1+m-\ell \leq \Delta+m-\ell \quad \text{and}$$
$$n-\Delta+1+m-\ell \leq \Delta+1+\ell$$

or

$$n-\Delta+\ell \leq \Delta+m-\ell \quad \text{and}$$
$$n-\Delta+\ell \leq \Delta+1+\ell \quad .$$

These can be simplified with a bit of algebra:

$$n-m+2\ell \leq 2\Delta \quad \text{and}$$
$$n-1 \leq 2\Delta$$

or

$$n+1 \leq 2\Delta \quad \text{and}$$
$$n+m-2\ell \leq 2\Delta \quad .$$

These inequalities are of the form

$$n-x \leq 2\Delta \quad \text{and}$$
$$n-1 \leq 2\Delta$$

or

$$n+1 \leq 2\Delta \quad \text{and}$$
$$n+x \leq 2\Delta \quad ,$$

where $x = m-2\ell$. Since B gives $m-2\ell \leq -1$, we see that

$$n-1 \leq n+1 \leq n-x \leq 2\Delta$$

or

$$n+x \leq n-1 \leq n+1 \leq 2\Delta$$

Consider the game after $n-\Delta$ rounds, as illustrated in Figure 3.11.

If $n - \Delta = \ell$, then they both reach u at the same time, with the survivor moving onto the z_2-occupied vertex (and losing). If we have $n - \Delta = \ell + 1$, then s reaches u first but is caught by z_2 on the following round. So, to guarantee the survivor has not escaped P_m we need

$$n - \Delta \leq \ell + 1 \quad ,$$

otherwise the survivor reaches the chord before z_2 can move to block. If the survivor reaches the chord first, then it could take the chord and possibly escape.

To ensure that z_2 goes clockwise once it reaches v, we need

$$\begin{aligned} \ell - (n - \Delta) &\leq 1 + (n - \Delta - 1) + (m - \ell + 1) \\ 2\Delta &\leq 2n + m - 2\ell + 1 \quad . \end{aligned}$$

We obtain the following characterization for Case II.B.:

z_1 goes counter-clockwise:

$$m + 1 \leq 2\ell \leq 2m - 2 \quad ,$$

and z_2 goes clockwise:

$$n + 1 \leq 2\Delta \quad .$$

The zombies win:

$$n - \Delta \leq \ell + 1 \qquad \text{and} \qquad 2\Delta \leq 2n + m - 2\ell + 1 \quad .$$

We will show (in Section 3.3) that with $\Delta = \left\lfloor \frac{m}{2} \right\rfloor$, the zombies can always (successfully) execute this cornering strategy. Of course, this is not sufficient to show the zombies win: there is no guarantee that the survivor will choose to start along P_m as is assumed here, so we cannot simply start with this zombie configuration. Instead, we must force the survivor's hand.

3.2 Guarding the Largest Cycle C_{n+1}

Part 2. We consider the game on $Q_{m,n}$ in general and show how we can position the zombies on C_{n+1} to limit the survivor's options and thereby guarantee it will be caught.

Choose k such that positioning

1. z_2 at $\Delta + k$ clockwise from u
2. z_1 at k counter-clockwise from v

forces the survivor into a losing position: it is either immediately sandwiched on C_{n+1}, or falls into the trap described above on C_{m+1}.

The survivor cannot start next to the zombies else it loses right away. So we choose k such that, even if the survivor is as far away from one of the zombies as possible on C_{n+1}, the zombies still move in opposite directions. This is when the survivor is at distance two from one of the zombies (refer to Figure 3.12) and leads to the following inequalities:

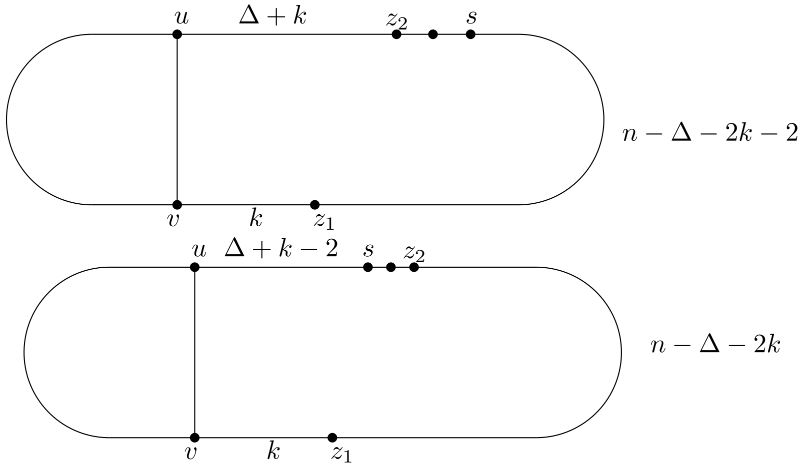

FIGURE 3.12: Preventing the zombies from turning in same direction on C_{m+1}

$$n - \Delta - 2k - 2 \leq \Delta + k + 1 + k + 2 \qquad \text{and}$$
$$\Delta + 2k - 1 \leq n - \Delta - 2k + 2 \quad .$$

Solving for k gives

$$n - 2\Delta - 5 \leq 4k \leq n - 2\Delta + 3 \quad .$$

A choice of k which satisfies these constraints guarantees that the zombies move in opposite directions if the survivor starts on C_n.

3.3 Existence of Δ and k for any m, n

Part 3. We wish to show that, for any m, n, there exist Δ and k which guarantee the survivor is caught. First, we show that $\Delta = \lfloor \frac{m}{2} \rfloor$ always works for the Cornering Strategy.

Note that

$$2\Delta = 2\left\lfloor \frac{m}{2} \right\rfloor = \begin{cases} m & \text{if } m \text{ is even} \\ m - 1 & \text{if } m \text{ is odd} \end{cases} \quad ,$$

and so $m - 1 \leq 2\lfloor \frac{m}{2} \rfloor \leq m$.

Suppose that we are in Case I.A. and $\Delta = \lfloor \frac{m}{2} \rfloor$. Case I.A. is characterized by the following constraints:

$$4 \leq 2\ell \leq m + 1$$
$$\text{and}$$
$$2\Delta \leq n - m + 2\ell \qquad \text{or} \qquad 2\Delta \leq n - 1 \quad .$$

The zombies win if

$$2\Delta \leq 2m - 2\ell + 2 \qquad \text{and} \qquad m - 2\ell - 1 \leq 2\Delta \quad .$$

In this case, the zombies win since

$$2\Delta = 2\lfloor\frac{m}{2}\rfloor \leq m < 2m-(m+1)+2 \leq 2m-2\ell+2$$

and

$$m-2\ell-1 \leq m-5 < 2\lfloor\frac{m}{2}\rfloor = 2\Delta$$

shows that the zombie-win requirements of Case I.A. are met.

Suppose that we are not in Case I.A. Recall that in all cases we must have $2 \leq \ell \leq m-1$. Therefore, negating the constraints of Case I.A. gives

$$2\ell \leq 3 \qquad \text{or} \qquad m+2 \leq 2\ell$$

or

$$2\Delta \geq n-m+2\ell+1 \qquad \text{and} \qquad 2\Delta \geq n-1+1$$

But $2\ell \leq 3$ is only possible if $\ell = 1$, and this is not a valid value for ℓ (it puts the survivor too close to z_1). With the upper bound on ℓ, the game is not in Case I.A. if

$$m+2 \leq 2\ell \leq 2m-2$$

or

$$2\Delta \geq n-m+2\ell+1 \qquad \text{and} \qquad 2\Delta \geq n-1+1 = n \tag{*}$$

.

Let us examine the consequences of assuming this second line * to be true.

If we assume that m is odd and $2\Delta \geq n$ then we obtain a contradiction since

$$2\Delta = 2\lfloor\frac{m}{2}\rfloor = m-1 \geq n$$

and we have assumed that $m \leq n$.

If m is even and $2\Delta \geq n$, then we must have $m = n$. If also $2\Delta \geq n-m+2\ell+1$ then we must have

$$\begin{aligned} 2\Delta &\geq n-m+2\ell+1 \\ m &\geq m-m+2\ell+1 \\ m &\geq 2\ell+1 \quad . \end{aligned}$$

Thus, if we set $\Delta = \lfloor\frac{m}{2}\rfloor$, we are in Case 1.A. unless

1. $m+2 \leq 2\ell \leq 2m-2$, or
2. $m = n$ are even and $m \geq 2\ell+1$.

In the first case, with $m+2 \leq 2\ell \leq 2m-2$, the zombies can apply Case I.B. since it is characterized by the following constraints:

$$m+1 \leq 2\ell \leq 2m-2$$

and

$$2\Delta \leq n+1 \qquad \text{or} \qquad 2\Delta \leq n+m-2\ell \quad .$$

Because $\Delta = \lfloor \frac{m}{2} \rfloor$ and $m+2 \leq 2\ell \leq 2m-2$, satisfies these constraints, the zombies can enact the strategy of Case I.B. They will win since this choice of Δ also satisfies the win conditions of Case I.B.:

$$2\Delta \leq 2\ell \qquad \text{and} \qquad 2\ell - m + 1 \leq 2\Delta \quad .$$

The first win condition is satisfied since $2\Delta \leq m < m+2 \leq 2\ell$, the second satisfied because $2\ell - m + 1 \leq (2m-2) - m + 1 = m - 2 < m - 1 \leq 2\Delta$.

In the second case, we have $m = n$ are even and $m \geq 2\ell + 1$. In this case, the zombies can play as in Case II.A. since it is characterized by

$$4 \leq 2\ell \leq m+1$$

and

$$n - m + 2\ell \leq 2\Delta \qquad \text{and} \qquad n - 1 \leq 2\Delta \quad .$$

Because $\Delta = \lfloor \frac{m}{2} \rfloor$ and $2\ell \leq m-1$ satisfies these constraints, the zombies can enact the strategy of Case II.A. They will win since this choice of Δ also satisfies the win conditions Case II.A.:

$$2\Delta \geq 2n - 2m + 2\ell \qquad \text{and} \qquad 2\Delta \leq 2n + 2\ell - m - 1 \quad .$$

The first win condition is satisfied since $2\Delta = m \geq 2\ell + 1 = 2n - 2m + 2\ell + 1 > 2n - 2m + 2\ell$, the second satisfied because $2\Delta = m \leq m + 1 = 2n + 2 - m - 1 \leq 2n + 2\ell - m - 1$.

It remains to show there exists a suitable value for k. Since k is constrained by the following inequalities

$$n - 2\Delta - 5 \leq 4k \leq n - 2\Delta + 3$$

it suffices to show that the interval

$$[\frac{n - 2\Delta - 5}{4}, \frac{n - 2\Delta + 3}{4}]$$

contains an integer, which must be so since

$$\left| \frac{n - 2\Delta + 3}{4} - \frac{n - 2\Delta - 5}{4} \right| = 2 \quad .$$

To show that there exists $k \geq 0$, suppose we have

$$n - 2\Delta + 3 < 0 \quad ,$$

which means

$$n < 2\Delta - 3 \quad .$$

With $\Delta = \lfloor \frac{m}{2} \rfloor$ we obtain a contradiction since we have presumed that $m \leq n$.

□

3.4 Putting It All Together

From the proof we obtain the following strategy:

Corollary 1. Two zombies win on $Q_{m,n}$ by placing z_1 at a counter-clockwise distance of

$$k = \left\lfloor \frac{n - 2\left\lfloor \frac{m}{2} \right\rfloor + 3}{4} \right\rfloor$$

from u and placing z_2 at a clockwise distance of

$$\left\lfloor \frac{m}{2} \right\rfloor + k$$

from v.

Proof. Place the zombies as described. If the survivor starts on P_n between the two zombies, then it loses because the large cycle is guarded as per Section 3.2. Otherwise, the zombies will push the survivor towards P_m. If, after k rounds, the survivor is still on P_m, then it is on a path of length at most $\Delta = \lfloor \frac{m}{2} \rfloor - 2$ (it cannot be adjacent to either z_1 or z_2). The two zombies and the survivor are on an induced subpath smaller than half the cycle C_{n+1} and so the survivor has lost. The only remaining possibility is that the survivor is somewhere on P_m, with z_1 on v and z_2 at a distance Δ from u, so the survivor loses by Section 3.1. $\square$

Chapter 4

Conclusion

In Chapter 2, we showed the existence of a graph for which 3 zombies always lose, thereby showing that the upper bound on the cop-number for planar graphs does not apply to zombies. This is hardly surprising, since the 3 Cops must effect a sophisticated strategy in order to capture the Robber, and the Zombies cannot coordinate in this way.

It remains to be shown if there is in fact an upper bound on the zombie-number for planar graphs. The example obtained in this thesis was a sort of extrapolation from the example given [5], which showed that the cop-number need not always equal the zombie-number, specifically in the case of outerplanar graphs. Is it possible to construct increasingly elaborate graphs (while still being planar) which would always provide the survivor with a winning strategy?

We wanted to investigate a simpler class of graphs: outerplanar ones. In this case, as we have noted, it has been shown [16] that 2 cops suffice to guarantee a win. It is also known that maximally-outerplanar graphs are zombie-win [5] and it is clear that 2 zombies suffice for a cycle, but what can be said about those outerplanar graphs in between the two extremes?

It has been our experience that 2 zombies often suffice on outerplanar graphs; but not always. The choice of zombie start is critical. This is the motivation for our work on $Q_{m,n}$ – the cycle with a single chord. Perhaps if we could segment or decompose an outerplanar graph into simpler components, then we could at least give an upper bound: perhaps 1 or 2 zombies per block. It is not clear how we can generalize our findings however. Adding a single extra chord changes the entire game.

We conclude with a few open questions which have yet to be addressed.

4.1 Open Questions

We have shown that zombies are not as effective as cops on planar graphs. What is the effectiveness of a zombie strategy on planar and outerplanar graphs. An upper bound for the zombie number of these classes of graphs has yet to be found.

In Chapter 3 we found a sort of interval for calculating which vertices lead to a two-zombie win. It would be possible to count these intervals and study the game from the probabilistic point of view. That is to say, finding how many zombies one would need to have even odds of winning on $Q_{m,n}$.

Graphs of the form $Q_{m,n}$ are basically two cycles which share an edge. Perhaps our findings could be generalized for any two overlapping cycles, though these graphs are no longer outerplanar. It is not clear if two zombies always win on these constructions.

Fitzpatrick showed that a graph is zombie-win if it admits a particular spanning tree (a zombie-win tree, see Subsection 1.3.1). Is the existence of a zombie-win tree also a necessary condition?

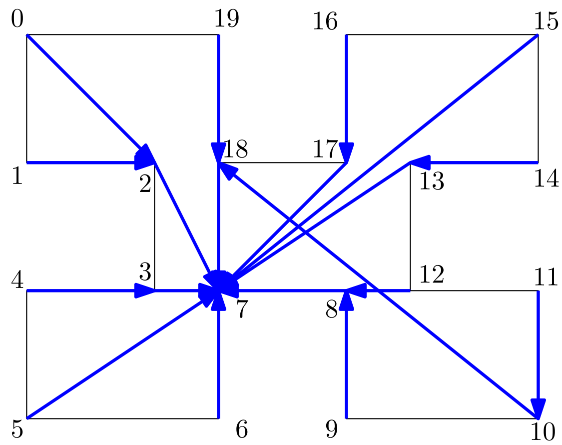

FIGURE 4.1: A Polygon Inscribed with a BFS Cop-win Tree

Finally, a recent result about visibility-augmenting edges [24] was used to conclude that visibility graphs of simple polygons are cop-win [21]. Are visibility graphs also zombie-win? Zombies seem to win handily on these graphs, but it is not clear how they could be proven zombie-win.

We have implemented tools which allow us to search – brute force – for zombie-win trees. So far, every polygon tested produces a visibility graph which admits such a tree (see 4.1 for an example).

Appendix A

Proof of Theorem 5, Case IV

In this Appendix, we provide full details for Case IV of the proof of Theorem 9. Here are all the possible start configurations (without loss of generality) of Case IV with the first few moves demonstrating that the survivor wins. By the end of each scenario, the survivor is able to apply the Running Around the Outside strategy.

Round	z_1	z_2	z_3	s
0	1	6	11	3
1	2	1	2	4
2	3	5	3	20
3	4	4	4	19

Round	z_1	z_2	z_3	s
0	1	6	12	3
1	2	1	11	4
2	3	5	2	20
3	4	4	3	19

Round	z_1	z_2	z_3	s
0	1	6	13	3
1	2	1	14	4
2	3	5	15	20
3	4	4	3	19

Round	z_1	z_2	z_3	s
0	1	6	14	4
1	5	1	15	21
2	4	5	3	22

Round	z_1	z_2	z_3	s
0	1	6	15	4
1	5	1	3	21
2	4	5	4	22

Round	z_1	z_2	z_3	s
0	1	7	11	3
1	2	6	2	4
2	3	1	3	21
3	4	5	4	22

Round	z_1	z_2	z_3	s
0	1	7	12	3
1	2	6	11	4
2	3	1	2	21
4	4	5	3	22

Round	z_1	z_2	z_3	s
0	1	7	13	3
1	2	6	14	4
2	3	1	15	21
3	4	5	3	22

Round	z_1	z_2	z_3	s
0	1	7	14	3
1	2	6	15	4
2	3	1	3	21
3	4	5	3	22

Round	z_1	z_2	z_3	s
0	1	7	15	4
1	5	6	3	21
2	4	1	4	22

Round	z_1	z_2	z_3	s
0	1	8	11	3
1	2	9	2	4
2	3	10	3	5
3	4	2	4	26
4	5	1	5	27
5	26	5	26	28

Round	z_1	z_2	z_3	s
0	1	8	12	3
1	2	9	11	4
2	3	10	2	5
3	4	2	1	26
4	5	1	5	27

Round	z_1	z_2	z_3	s
0	1	8	13	3
1	2	9	14	4
2	3	10	15	5
3	4	2	3	26
4	5	1	4	27
5	26	5	5	28
6	27	26	26	29

Round	z_1	z_2	z_3	s
0	1	8	14	3
1	2	9	15	4
2	3	10	3	5
3	4	2	4	26
4	5	1	5	27

Round	z_1	z_2	z_3	s
0	1	8	15	28
1	30	7	3	27
2	29	29	4	24
3	28	28	5	23
4	27	27	25	22

Round	z_1	z_2	z_3	s
0	1	8	15	17
1	2	9	14	18
2	3	12	17	19
3	16	13	18	22

Round	z_1	z_2	z_3	s
0	1	9	11	3
1	2	10	2	4
2	3	2	3	5
3	4	1	4	26
4	5	5	5	27

Round	z_1	z_2	z_3	s
0	1	9	12	3
1	2	10	11	4
2	3	2	2	5
3	4	1	1	26
4	5	5	5	27

Round	z_1	z_2	z_3	s
0	1	9	13	3
1	2	10	14	4
2	3	2	15	5
3	4	1	3	26
4	5	5	4	27
5	26	26	5	28
6	27	27	26	29

Round	z_1	z_2	z_3	s
0	1	9	14	3
1	2	10	15	4
2	3	2	3	5
3	4	1	4	26
4	5	5	5	27

Round	z_1	z_2	z_3	s
0	1	9	15	4
1	5	10	3	21
2	4	2	4	22
3	21	3	21	23
4	22	4	22	24
5	23	5	23	27
6	24	26	24	28

Round	z_1	z_2	z_3	s
0	1	10	11	3
1	2	2	2	16
2	3	3	3	17

Round	z_1	z_2	z_3	s
0	1	10	12	3
1	2	2	11	4
2	3	3	2	5

Round	z_1	z_2	z_3	s
0	1	10	13	3
1	2	2	14	4
2	3	3	15	5
3	4	4	3	1

Round	z_1	z_2	z_3	s
0	1	10	14	3
1	2	2	15	4
2	3	3	3	5

Round	z_1	z_2	z_3	s
0	1	10	15	4
1	5	2	3	20
2	4	3	4	19

Round	z_1	z_2	z_3	s
0	1	6	16	4
1	5	1	3	21
2	4	5	4	22
3	21	4	21	23
4	22	21	22	24

Round	z_1	z_2	z_3	s
0	1	6	17	4
1	5	1	16	21
2	4	5	3	22
3	21	4	4	23
4	22	21	21	24

Round	z_1	z_2	z_3	s
0	1	6	18	4
1	5	1	19	3
2	4	2	20	16
3	3	3	4	17
4	16	16	3	18
5	17	17	16	19

Round	z_1	z_2	z_3	s
0	1	6	19	4
1	5	1	20	3
2	4	2	4	16
3	3	3	3	17

Round	z_1	z_2	z_3	s
0	1	6	20	3
1	2	1	4	16
2	3	2	3	17
3	16	3	16	18
4	17	16	17	19

Round	z_1	z_2	z_3	s
0	1	7	16	4
1	5	6	3	21
2	4	1	4	22
3	21	5	21	19
4	22	4	22	18
5	19	20	19	17

Round	z_1	z_2	z_3	s
0	1	7	17	3
1	2	6	16	4
2	3	1	3	21
3	4	5	4	22
4	21	4	21	23
5	22	21	22	24

Round	z_1	z_2	z_3	s
0	1	7	18	3
1	2	6	17	4
2	3	1	16	21
3	4	5	3	22

Round	z_1	z_2	z_3	s
0	1	7	19	4
1	5	6	20	3
2	4	1	4	15
3	3	2	3	14

Round	z_1	z_2	z_3	s
0	1	7	20	3
1	2	6	4	15
2	3	1	3	14

Round	z_1	z_2	z_3	s
0	1	8	16	4
1	5	7 or 9	3	21
2	4	6 or 10	4	22
3	21	1 or 2	21	23
4	22	5 or 1	22	24
5	23	25 or 5	23	27
6	24	24 or 26	24	28
7	27	27	27	29

Round	z_1	z_2	z_3	s
0	1	8	17	3
1	2	9	16	4
2	3	10	3	5
3	4	2	4	26
4	5	1	5	27

Round	z_1	z_2	z_3	s
0	1	8	18	3
1	2	9	17	4
2	3	10	16	5
3	4	2	3	26
4	5	1	4	27

Round	z_1	z_2	z_3	s
0	1	8	19	4
1	5	7 or 8	20	3
2	4	6 or 10	4	15
3	3	1 or 2	3	14

Round	z_1	z_2	z_3	s
0	1	8	19	4
1	5	7 or 8	20	3
2	4	6 or 10	4	15
3	3	1 or 2	3	14

Round	z_1	z_2	z_3	s
0	1	8	20	14
1	2	9	4	14
2	3	12	3	14
3	15	13	3	17
4	14	14	16	18
5	17	17	17	19

Round	z_1	z_2	z_3	s
0	1	9	16	4
1	5	10	3	21
2	4	2	4	22

Round	z_1	z_2	z_3	s
0	1	9	17	3
1	2	10	16	4
2	3	2	3	5
3	4	1	4	26
4	5	5	5	27

Round	z_1	z_2	z_3	s
0	1	9	18	3
1	2	10	17	4
2	3	2	16	5
3	4	1	3	26
4	5	5	4	27

Round	z_1	z_2	z_3	s
0	1	9	19	3
1	2	10	20	16
2	3	2	4	17

Round	z_1	z_2	z_3	s
0	1	9	20	3
1	2	10	4	16
2	3	2	3	17

Round	z_1	z_2	z_3	s
0	1	10	16	4
1	5	2	3	21
2	4	3	4	22

Round	z_1	z_2	z_3	s
0	1	10	17	3
1	2	2	16	4
2	3	3	3	5

Round	z_1	z_2	z_3	s
0	1	10	18	3
1	2	2	17	4
2	3	3	16	5
3	4	4	3	26
4	5	5	4	27

Round	z_1	z_2	z_3	s
0	1	10	19	3
1	2	2	20	15
2	3	3	4	14

Round	z_1	z_2	z_3	s
0	1	10	20	3
1	2	2	4	15
2	3	3	3	14

Round	z_1	z_2	z_3	s
0	1	6	21	3
1	2	1	4	16
2	3	2	3	17

Round	z_1	z_2	z_3	s
0	1	6	22	4
1	5	1	21	3
2	4	2	4	16
3	3	3	3	17

Round	z_1	z_2	z_3	s
0	1	6	23	4
1	5	1	22	3
2	4	2	21	16
3	3	3	4	17

Round	z_1	z_2	z_3	s
0	1	6	24	4
1	5	1	25	3
2	4	2	5	16
3	3	3	4	17

Round	z_1	z_2	z_3	s
0	1	6	25	4
1	5	1	5	3
2	4	2	4	16
3	3	3	3	17

Round	z_1	z_2	z_3	s
0	1	7	21	3
1	2	6	4	16
2	3	1	3	17

Round	z_1	z_2	z_3	s
0	1	7	22	4
1	5	6	21	3
2	4	1	4	16
3	3	2	3	17

Round	z_1	z_2	z_3	s
0	1	7	23	4
1	5	6	22	3
2	4	1	21	16
3	3	2	4	17

Round	z_1	z_2	z_3	s
0	1	7	24	4
1	5	6	25	3
2	4	1	5	16
3	3	2	4	17

Round	z_1	z_2	z_3	s
0	1	7	25	4
1	5	6	5	3
2	4	1	4	16
3	3	2	3	16

Round	z_1	z_2	z_3	s
0	1	8	21	3
1	2	9	4	16
2	3	10	3	17
3	16	2	16	14
4	17	3	17	13
5	14	15	14	12

Round	z_1	z_2	z_3	s
0	1	8	22	3
1	2	9	21	16
2	3	10	4	17
3	16	2	3	14
4	17	3	15	13
5	14	15	14	12

Round	z_1	z_2	z_3	s
0	1	8	23	3
1	2	9	22	4
2	3	10	21	5
3	4	2	4	26
4	5	1	5	27

Round	z_1	z_2	z_3	s
0	1	8	24	3
1	2	9	25	4
2	3	10	5	21
3	4	2	4	22

Round	z_1	z_2	z_3	s
0	1	8	25	3
1	2	9	5	16
2	3	10	4	17
3	16	2	3	14
4	17	3	15	13
5	14	15	14	12

Round	z_1	z_2	z_3	s
0	1	9	21	3
1	2	10	4	16
2	3	2	3	17

Round	z_1	z_2	z_3	s
0	1	9	22	4
1	5	10	21	3
2	4	2	4	16
3	3	3	3	17

Round	z_1	z_2	z_3	s
0	1	9	23	4
1	5	10	22	3
2	4	2	21	16
3	3	3	4	17

Round	z_1	z_2	z_3	s
0	1	9	24	4
1	5	10	25	3
2	4	2	5	16
3	3	3	4	17

Round	z_1	z_2	z_3	s
0	1	9	25	4
1	5	10	5	3
2	4	2	4	16
3	3	3	3	17

Round	z_1	z_2	z_3	s
0	1	10	21	3
1	2	2	4	16
2	3	3	3	17

Round	z_1	z_2	z_3	s
0	1	10	22	3
1	2	2	21	16
2	3	3	4	17

Round	z_1	z_2	z_3	s
0	1	10	23	3
1	2	2	22	15
2	3	3	21	14
3	15	15	4	17
4	14	14	3	18
5	17	17	16	19

Round	z_1	z_2	z_3	s
0	1	10	24	3
1	2	2	25	4
2	3	3	5	20
3	4	4	4	19

Round	z_1	z_2	z_3	s
0	1	10	25	3
1	2	2	5	16
2	3	3	4	17

Round	z_1	z_2	z_3	s
0	1	6	26	4
1	5	1	5	3
2	4	2	4	16
3	3	3	3	17

Round	z_1	z_2	z_3	s
0	1	6	27	4
1	5	1	26	3
2	4	2	5	16
3	3	3	4	17

Round	z_1	z_2	z_3	s
0	1	6	28	4
1	5	1	27	3
2	4	2	26	16
3	3	3	5	17

Round	z_1	z_2	z_3	s
0	1	6	29	4
1	5	1	30	3
2	4	2	1	16
3	3	3	2	17

Round	z_1	z_2	z_3	s
0	1	6	30	4
1	5	1	1	3
2	4	2	2	16
3	3	3	3	17

Round	z_1	z_2	z_3	s
0	1	7	26	4
1	5	6	5	3
2	4	1	4	16
3	3	2	3	17

Round	z_1	z_2	z_3	s
0	1	7	27	4
1	5	6	26	3
2	4	1	5	16
3	3	2	4	17

Round	z_1	z_2	z_3	s
0	1	7	28	4
1	5	6	27	3
2	4	1	26	16
3	3	2	5	17

Round	z_1	z_2	z_3	s
0	1	7	29	4
1	5	6	30	3
2	4	1	1	16
3	3	2	2	17

Round	z_1	z_2	z_3	s
0	1	7	30	4
1	5	6	1	3
2	4	1	2	16
3	3	2	3	17

Round	z_1	z_2	z_3	s
0	1	8	26	3
1	2	9	5	16
2	3	10	4	17
3	16	2	3	18
4	17	3	16	19
5	18	4	17	22
6	19	21	18	23

Round	z_1	z_2	z_3	s
0	1	8	27	3
1	2	9	26	4
2	3	10	5	21
3	4	2	4	22

Round	z_1	z_2	z_3	s
0	1	8	28	3
1	2	9	27 or 29	4
2	3	10	26 or 30	20
3	4	2	5 or 1	19

Round	z_1	z_2	z_3	s
0	1	8	29	3
1	2	9	30	4
2	3	10	1	20
3	4	2	5	19

Round	z_1	z_2	z_3	s
0	1	8	30	3
1	2	9	1	4
2	3	10	5	20
3	4	2	4	19

Round	z_1	z_2	z_3	s
0	1	9	26	3
1	2	10	5	16
2	3	2	4	17

Round	z_1	z_2	z_3	s
0	1	9	27	3
1	2	10	26	4
2	3	2	5	20
3	4	3	4	19

Round	z_1	z_2	z_3	s
0	1	9	28	4
1	5	10	27	3
2	4	2	26	16
3	3	3	5	17

Round	z_1	z_2	z_3	s
0	1	9	29	4
1	5	10	30	3
2	4	2	1	16
3	3	3	2	17

Round	z_1	z_2	z_3	s
0	1	9	30	4
1	5	10	1	3
2	4	2	2	16
3	3	3	3	17

Round	z_1	z_2	z_3	s
0	1	10	26	4
1	5	2	5	20
2	4	3	4	19

Round	z_1	z_2	z_3	s
0	1	10	27	4
1	5	2	26	20
2	4	3	5	19

Round	z_1	z_2	z_3	s
0	1	10	28	4
1	5	2	27	20
2	4	3	26	19
3	20	4	5	18
4	19	20	4	17
5	18	19	3	14
6	17	18	15	13
7	14	17	14	12

Round	z_1	z_2	z_3	s
0	1	10	29	4
1	2	2	30	3
2	3	3	1	20
3	4	4	5	19

Round	z_1	z_2	z_3	s
0	1	10	30	4
1	5	2	1	20
2	4	3	5	19

Round	z_1	z_2	z_3	s
0	1	11	16	4
1	5	2	3	20
2	4	3	4	21

Round	z_1	z_2	z_3	s
0	1	11	17	3
1	2	2	16	4
2	3	3	3	20

Round	z_1	z_2	z_3	s
0	1	11	18	3
1	2	2	17	4
2	3	3	16	20

Round	z_1	z_2	z_3	s
0	1	11	19	4
1	5	2	20	21
2	4	3	4	22

Round	z_1	z_2	z_3	s
0	1	11	20	3
1	2	2	4	16
2	3	3	3	17

Round	z_1	z_2	z_3	s
0	1	12	16	4
1	5	11	3	20
2	4	2	4	19

Round	z_1	z_2	z_3	s
0	1	12	17	3
1	2	11	16	4
2	3	2	3	20
3	4	3	4	19

Round	z_1	z_2	z_3	s
0	1	12	18	3
1	2	11	17	4
2	3	2	16	20
3	4	3	3	19

Round	z_1	z_2	z_3	s
0	1	12	19	4
1	5	11	20	3
2	4	2	4	16
3	3	3	3	17

Round	z_1	z_2	z_3	s
0	1	12	20	3
1	2	11	4	16
2	3	2	3	17

Round	z_1	z_2	z_3	s
0	1	13	16	4
1	5	14	3	21
2	4	15	4	22
3	21	3	21	23
4	22	4	22	24
5	23	5	23	27
6	24	26	24	28

Round	z_1	z_2	z_3	s
0	1	13	17	3
1	2	14	16	4
2	3	15	3	5
3	4	3	4	26
4	5	2	5	27

Round	z_1	z_2	z_3	s
0	1	13	18	3
1	2	14	17	4
2	3	15	16	5
3	4	3	3	26
4	5	4	4	27

Round	z_1	z_2	z_3	s
0	1	13	19	4
1	5	14	20	21
2	4	15	4	22
3	21	3	21	23
4	22	4	22	24
5	23	5	23	27
6	24	26	24	28

Round	z_1	z_2	z_3	s
0	1	13	20	9
1	2	12	4	8
2	10	9	3 or 5	7
3	9	8	2 or 1	29
4	8	7	1 or 30	28
5	7	29	30 or 29	27
6	29	28	29 or 28	24

Round	z_1	z_2	z_3	s
0	1	14	16	4
1	5	15	3	21
2	4	3	4	22

Round	z_1	z_2	z_3	s
0	1	14	17	4
1	5	15	16	21
2	4	3	3	22

Round	z_1	z_2	z_3	s
0	1	14	18	3
1	2	15	17	4
2	3	3	16	5
3	4	4	3	26
4	5	5	4	27

Round	z_1	z_2	z_3	s
0	1	14	19	4
1	5	15	20	21
2	4	3	4	22

Round	z_1	z_2	z_3	s
0	1	14	20	3
1	2	15	4	16
2	3	3	3	17

Round	z_1	z_2	z_3	s
0	1	15	16	4
1	5	3	3	21
2	4	4	4	22

Round	z_1	z_2	z_3	s
0	1	15	17	4
1	5	3	16	21
2	4	4	3	22

Round	z_1	z_2	z_3	s
0	1	15	19	4
1	5	3	20	21
2	4	4	4	22

Round	z_1	z_2	z_3	s
0	1	15	18	12
1	2	14	17	9
2	10	13	14	8
3	9	12	13	7

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