Definition 1. *A graph G is* planar *if it can be embedded in the plane such that its edges intersect only at vertices [1].*

Definition 2. *A graph G is said to be* outer-planar *if it is planar and has and embedding in which all of the vertices of G lie on the outer face [2].*

Definition 3. *A graph G is* maximally outer planar *if the addition of any edge violates outer-planarity.*

Given an outer planar graph G and an embedding, we define:

Definition 4. *The* outer face *is the Hamilton cycle formed by the vertice and the edges connecting them on the outer face.*

Definition 5. *A* chord *is an edge which is not on the outer face.*

Definition 6. *An* ear *is vertex of degree 2. A maximally outer-planar graph has at least 2 ears.*

A maximally outer-planar graph is composed of triangles. In [2], they distinguish two types of triangles:

Definition 7. *A* marginal *triangle shares at least one edge with the outer face while an* internal *triangle has no edges on the outer face.*

There is an interesting relationship between internal triangles and the number of ears [2]. If k is the number of internal triangles, then there are $k+2$ ears. This is because it takes 3 marginal triangles to create an internal triangle, and adding another ear (without violating outer-planarity) creates a new internal triangle.

Definition 8. *Finally, a maximally outer-planar graph is called* striped *if it does not contain any internal triangles.*

Definition 9. *A* round *of the game consists of two turns: the zombie's turn followed by the survivor's turn.*

With these definitions in mind, we seek to prove the following:

Theorem 1. *Maximally outer-planar graphs are zombie-win, regardless of the zombie's initial position. That is to say, a single zombie playing on a maximally outer-planar graph will always catch the survivor by following the greedy strategy.*

Proof. Our approach is to show that the *survivor zone*, the set of vertices $S \subseteq V$ which the survivor can occupy without losing on the next round shrinks at every round of the game.

Observe that K_3 is zombie-win, so we can assume that $|V| = n > 3$.

The zombie and survivor choose initial positions $z^{(1)}, s^{(1)} \in V$, the zombie player going first.

First assume that $z^{(1)}$ is an ear. If $s^{(1)} \in N(z^{(1)})$, then the game is won by the zombie in round 1. Otherwise, $z^{(2)} \in N(z^{(1)})$ is not an ear (since $n > 3$) and so has degree at least 3. The survivor either passes or moves to some $s^{(2)}$ and, by the end of this round, the zombie is not on an ear.

If the survivor is on an ear adjacent to the zombie, then the zombie captures the survivor on its turn. Otherwise, we claim that the zombie does not move to an ear. Indeed, if the zombie is on a vertex without any adjacent ears, then the claim is clear. Assume then, that the zombie is on a vertex adjacent to an ear. The survivor is not on this ear vertex by assumption. Now supposing that the shortest path includes this ear leads to a contradiction by the triangle inequality.

So the zombie always moves to non-ear vertices unless it is to capture the survivor, and thus we can now assume that $z^{(r)}$ (for $r > 1$) is a vertex of degree at least 3.

Given an embedding of the graph, label the vertices $v_0, v_1, \ldots, v_{n-1}$ clockwise such that

$$H_G = v_0v_1, v_1v_2 \ldots, v_{n-2}v_{n-1}, v_{n-1}v_0$$

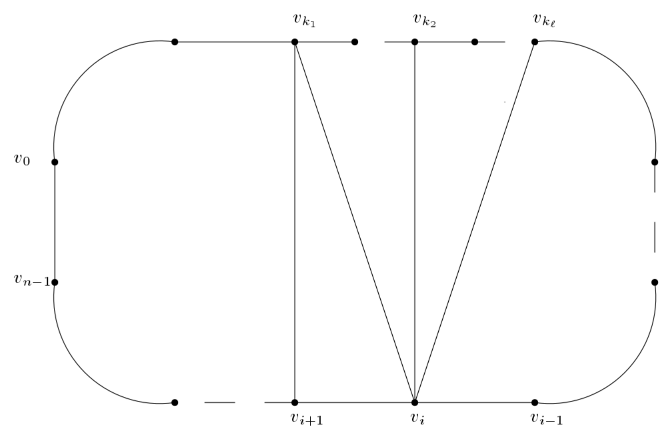
is the Hamilton cycle formed by the outer face (subscripts modulo n).

Now say that $z^{(r)} = v_i$ for some $1 \leq i \leq n$. The vertices v_{i-1} and v_{i+1} are the predecessor and successor of v_i on the outer face.

Since v_i is not an ear and $n > 3$, we also have that the set of vertices attached to v_i by a chord:

$$U = \{u \in V \setminus \{v_{i-1}, v_{i+1}\} | \, v_i u \in E\} = \{v_{k_1}, v_{k_2}, \ldots, v_{k_\ell}\}$$

is not empty.



Notice that because the graph is a triangulation and $v_iv_{k_1}$ is assumed to be the first clockwise chord from v_i, there must also be an edge $v_{i+1}v_{k_1}$.

If the survivor is adjacent to v_i then the zombie captures the survivor. Otherwise, observe that $s^{(r)}$ is in $\{v_{i+1}, \ldots, v_{k_\ell}\}$ or $\{v_{k_1}, \ldots, v_{i-1}\}$ (or both). Assume without loss of generality that $s^{(r)}$ is in $\{v_{i+1}, \ldots, v_{k_\ell}\}$. Therefore, the survivor zone at round r is $S^{(r)} = \{v_{i+1}, \ldots, v_{k_\ell}\} \setminus (\{v_{i+1}\} \cup \{v_{k_1}, \ldots, v_{k_\ell}\})$.

Now, the shortest zs-path either starts with (1) a hull edge or (2) a chord edge.

1. If the shortest zs-path starts with a hull edge then it is v_iv_{i+1}. In this case, the survivor must be in $\{v_{i+2}, \ldots, v_{k_1-1}\}$, else we obtain a contradiction using the triangle inequality. Since the graph is outer-planar, the survivor cannot leave this set without passing through either v_{i+1} or v_{k_1}.

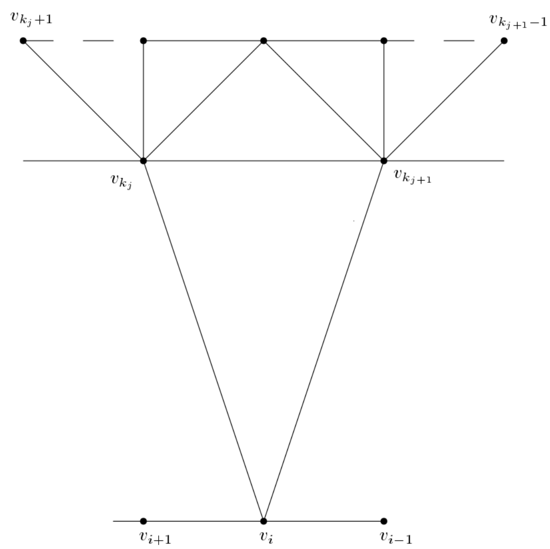
   Therefore, the survivor zone at round $r+1$ is $S^{(r+1)} \subseteq \{v_{i+2}, \ldots, v_{k_1-1}\} \setminus \{v_{i+2}\} \subsetneq S^{(r)}$. This shows that the survivor zone shrank by at least one vertex.

2. Assume that the shortest zs-path starts with a chord edge. We consider two subcases: (a) $s^{(r)} \in \{v_{i+2}, \ldots, v_{k_1-1}\}$, or (b) $s^{(r)} \in \{v_{k_1}, v_{k_1+1}, \ldots, v_{k_\ell-1}, v_{k_\ell}\}$

   (a) Subcase $s^{(r)} \in \{v_{i+2}, \ldots, v_{k_1-1}\}$. In this case, the chord edge of the shortest zs-path must $v_iv_{k_1}$, since it is the first in clockwise order. As in case 1, the survivor cannot escape $\{v_{i+2}, \ldots, v_{k_1-1}\}$ without passing through either v_{i+1} or v_{k_1}.

   So we see that $S^{(r+1)} \subseteq \{v_{i+2}, \ldots, v_{k_1-1}\} \setminus \{v_{k_1-1}\} \subsetneq S^{(r)}$ has again shrunk by at least one vertex.

(b) Subcase $s^{(r)} \in \{v_{k_1}, v_{k_1+1}, \ldots, v_{k_\ell -1}, v_{k_\ell}\}$. In this case, there exists $v_{k_j} \in U$ such that the survivor is somewhere in the arc defined by v_{k_j} and $v_{k_{j+1}}$. That is, the survivor is in $v_{k_j+1}, v_{k_j+2}, \ldots, v_{k_{j+1}-1}, v_{k_{j+1}}$ the vertices of the outer face within the span of said arc. Observe that the first edge on the shortest zs-path must be $v_i v_{k_j}$ or $v_i v_{k_{j+1}}$.



As before, the survivor now cannot escape $\{v_{k_j}, v_{k_j+1}, \ldots, v_{k_{j+1}}\}$ without passing through either v_{k_j} or $v_{k_{j+1}}$. Since $v_i v_{k_j}$ and $v_i v_{k_{j+1}}$ are two consecutive clockwise chords attached to v_i and the graph is maximally outer planar, we must have edge $v_{k_j} v_{k_{j+1}}$. Therefore, $S^{(r+1)} \subseteq \{v_{k_j}, v_{k_j+1}, \ldots, v_{k_{j+1}}\} \setminus \{v_{k_j}, v_{k_j+1}\} \subsetneq S^{(r)}$.

In these three exhaustive cases, we have shown that the survivor zone shrinks by at least one vertex. Since the graph is finite, the survivor is inevitably restricted to a single vertex adjacent to the zombie and the survivor is caught on the next round.

□

References

[1] J. A. Bondy and U. S. R. Murty, "Graph theory, volume 244 of graduate texts in mathematics," 2008.

[2] C. Campos and Y. Wakabayashi, "On dominating sets of maximal outerplanar graphs," *Discrete Applied Mathematics*, vol. 161, no. 3, pp. 330–335, 2013.