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# On dominating sets of maximal outerplanar graphs[☆]

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ABSTRACT

A dominating set of a graph is a set $S$ of vertices such that every vertex in the graph is either in $S$ or is adjacent to a vertex in $S$. The domination number of a graph $G$, denoted $\gamma(G)$, is the minimum cardinality of a dominating set of $G$. We show that if $G$ is an $n$-vertex maximal outerplanar graph, then $\gamma(G) \leq (n+t)/4$, where $t$ is the number of vertices of degree 2 in $G$. We show that this bound is tight for all $t \geq 2$. Upper-bounds for $\gamma(G)$ are known for a few classes of graphs.



## 1. Introduction

In this paper we only consider finite, undirected and simple graphs. A *dominating set* of a graph $G = (V, E)$ is a set $S \subseteq V$ such that every vertex in $G$ is either in $S$ or is adjacent to a vertex in $S$. The *domination number* of $G$, denoted $\gamma(G)$, is the minimum cardinality of a dominating set of $G$. Garey and Johnson [1] showed that deciding whether a given graph has domination number at most some given integer $k$ is an NP-complete problem; and remains so for planar graphs with maximum degree 3 and planar 4-regular graphs.

A graph $G$ is *outerplanar* if it has an embedding in the plane such that all vertices belong to the boundary of its outer face (the unbounded face). An outerplanar graph $G$ is *maximal* if $G + uv$ is not outerplanar for any two non-adjacent vertices $u$ and $v$ of $G$. In this paper we are concerned with upper bounds for $\gamma(G)$, when $G$ is a maximal outerplanar graph.

In 1996, Matheson and Tarjan [8] proved a tight upper bound for the dominating number on the class of *triangulated discs*: graphs that have an embedding in the plane such that all of their faces are triangles, except possibly one. They proved that $\gamma(G) \leq n/3$ for any $n$-vertex triangulated disc. They also showed that this bound is tight. Plummer and Zha [10] extended this bound to triangulations on the projective plane and proved that $\gamma(G) \leq \lceil n/3 \rceil$ for triangulations on the torus or Klein bottle. They also showed that this bound is tight. (A triangulated disc in which all faces are triangles and any two face boundaries intersect in a single edge, a single vertex, or not at all is called a triangulation.) Honjo et al. [5] extended the latter results showing the bound $n/3$ for triangulations on the torus and the Klein bottle and also for some other surfaces.

Matheson and Tarjan conjectured that $\gamma(G) \leq n/4$ for every $n$-vertex plane triangulation $G$ with $n$ sufficiently large. They noted that the octahedron, which has 6 vertices, has domination number 2. Recently, King and Pelsmajer [6] proved that this conjecture is true for graphs with maximum degree at most 6.

We observed that the graphs given by Matheson and Tarjan to show that the upper-bound $n/3$ is tight for triangulated discs are, in fact, outerplanar graphs. So, we came naturally to the question of whether this bound would also be the best

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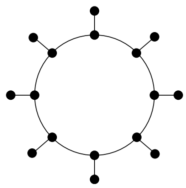


**Fig. 1.** An outerplanar graph with domination number $n/2$.

possible for *maximal outerplanar* graphs. For outerplanar graphs that are not maximal, the upper-bound $n/2$ is the best possible, as Fig. 1 shows. We prove a simple upper-bound for $\gamma(G)$ when $G$ is a maximal outerplanar graph, and show that this bound is tight.

## 2. Basic results

We first observe that if $G$ is a maximal outerplanar graph, then $G$ is 2-connected and Hamiltonian. It is also immediate that the following holds.

**Proposition 1.** *If $G$ is a maximal outerplanar graph, then there is an embedding of $G$ in the plane such that the boundary of the outer face is a Hamiltonian cycle and each inner face is a triangle.* □

A maximal outerplanar graph embedded in the plane as mentioned above will be called a *maximal outerplane graph*. For such a graph $G$, we denote by $H_G$ the (unique) Hamiltonian cycle which is the boundary of the outer face. We omit the subscript $G$, when $G$ is clear from the context.

Let $f$ be an inner face of a maximal outerplane graph $G$. If $f$ is adjacent to the outer face, then we say that $f$ is a *marginal triangle*; otherwise we say that $f$ is an *internal triangle*. A maximal outerplane graph $G$ without internal triangles is called *striped*. We may use the term *triangle* to refer to an inner face or to a subgraph that is isomorphic to $K_3$.

It is interesting to note the direct relation between the number of internal triangles and the number of vertices of degree 2 in a maximal outerplane graph. This is stated in the next proposition.

**Proposition 2.** *Let $G$ be a maximal outerplane graph of order $n \geq 4$. If $G$ has $k$ internal triangles, then $G$ has $k+2$ vertices of degree 2.*

**Proof.** The proof can be done by induction on $k$. For $k = 0$ the proof follows easily by induction on $n$ (one can contract an edge of $H_G$ that belongs to a marginal triangle which has two chords of $H_G$). □

The next result will be useful in what follows.

**Lemma 3.** *If $G$ is a maximal outerplane graph of order $n \geq 3$, then $G$ has $n-1$ faces and $H_G$ has $n-3$ chords.*

**Proof.** First note that if $G$ has $n_f$ faces and $H_G$ has $n_c$ chords, then $G$ has $n+n_c$ edges, and thus, by Euler's formula, $n_f = n_c+2$. The dual graph $G^*$ of $G$ has $n_f - 1$ vertices of degree 3 and one vertex of degree $n$. Adding the degrees of the vertices of $G^*$, we get that $3(n_f - 1) + n = 2(n + n_c)$. The two equations yield $n_f = n - 1$ and $n_c = n - 3$. □

## 3. Maximal outerplanar graphs without internal triangles

In this section we prove an upper-bound for the domination number of striped maximal outerplanar graphs. For that, we introduce first a terminology and notation that will be useful in this proof.

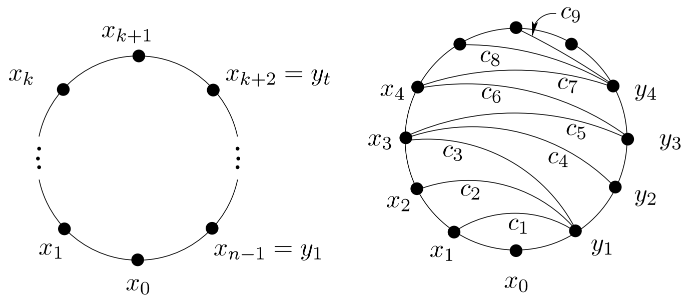
Let $G$ be a striped maximal outerplane graph of order $n \geq 4$, and let $H = (x_0, x_1, \ldots, x_{n-1})$ be the boundary of its outer face. From Proposition 2, we know that $G$ has exactly two vertices of degree 2. Suppose, without loss of generality, that $x_0$ and $x_{k+1}$ are the two vertices of degree 2 in $G$.

The removal of the vertices $x_0$ and $x_{k+1}$ breaks the cycle $H$ into two paths: the path $P = (x_1, x_2, \ldots, x_k)$ and the path $Q = (x_{k+2}, x_{k+3}, \ldots, x_{n-1})$. To refer to the reverse of $Q$, for ease of notation, we label its vertices in such a way that $Q^{-1} = (y_1, y_2, \ldots, y_t)$, as depicted in Fig. 2(a). We call this label assignment a *canonical vertex-labelling*.

We also consider that the chords of $H$ are ordered and named $c_1, c_2, \ldots, c_{n-3}$, according to the statement of the next lemma (see Fig. 2(b)).

**Lemma 4.** *Let $G$ be a striped maximal outerplane graph of order $n \geq 4$ endowed with a canonical vertex-labelling. Then, the chords of $H$ can be ordered from $c_1$ to $c_{n-3}$ in such a way that:*

(i) *$c_1 := x_1y_1$ and*
(ii) *if $c_p = x_iy_j$, then $c_{p+1} = x_iy_{j+1}$ or $c_{p+1} = x_{i+1}y_j$.* □




(a) Canonical vertex-labelling. (b) Ordering of the chords.

**Fig. 2.** Striped maximal outerplane graph.

An easy way to understand the ordering of the chords of $H$ mentioned in the previous lemma is to build the weak dual graph of $G$, say $G^*$, and then consider in $G^*$ the path of length $n-3$ that goes from the vertex corresponding to the marginal triangle containing the vertex $x_0$ to the vertex corresponding to the marginal triangle containing the other vertex of degree 2. Traversing this path, which we call $P(G^*)$, the order of the edges visited defines the ordering of the chords of $H$.

We are now ready to show a tight upper-bound for the domination number of striped maximal outerplanar graphs.

**Theorem 5.** *Let $G$ be a striped maximal outerplanar graph with $n \geq 3$ vertices. Then,*

$$\gamma(G) \leq \begin{cases} \lfloor n/4 \rfloor, & \text{if } n \equiv 0, 1 \pmod 4; \\ \lceil n/4 \rceil, & \text{otherwise}. \end{cases}$$

**Proof.** Let $G$ be a striped maximal outerplane graph and $H$ be the Hamiltonian cycle which is the boundary of the outer face. Suppose $G$ is endowed with a canonical vertex-labelling, and the chords of $H$ are ordered as stated in Lemma 4.

The proof is by induction on $n$. Suppose that $n \geq 7$, as otherwise the proof is simple. The idea is to delete from $G$ a set $S$ of four vertices which includes the vertex $x_0$. More precisely, we delete from $G$ the four faces corresponding to the first four vertices of the path $P(G^*)$ of the weak dual graph $G^*$. (In other words, we delete from $G$ the four vertices "below" the chord $c_4$.) See Fig. 2(b). We apply induction on the resulting graph, say $G'$, and obtain a minimum dominating set $D'$. Then we show that $D'$ with an additional vertex from $S$ is a dominating set of $G$, concluding this way that $\gamma(G)$ has the desired properties.

Without loss of generality suppose $c_2 = x_2y_1$ (the case $c_2 = x_1y_2$ is symmetric). Then, $c_3 = x_3y_1$ or $c_3 = x_2y_2$.

*Case* 1: $c_3 = x_3y_1$. Take $S := \{x_0, x_1, x_2, z\}$, where $z = y_1$ if $c_4 = x_3y_2$, or $z = x_3$ if $c_4 = x_4y_1$. Clearly, $y_1$ is adjacent to all vertices in $S \setminus \{y_1\}$. Let $G' = G - S$ and let $D'$ be a minimum dominating set of $G'$. Thus, $D := D' \cup \{y_1\}$ is a dominating set of $G$.

By the induction hypothesis, $\gamma(G') \leq \lfloor (n-4)/4 \rfloor$ if $n \equiv 0, 1 \pmod 4$ and $\gamma(G') \leq \lceil (n-4)/4 \rceil$ if $n \equiv 2, 3 \pmod 4$. Thus, $\gamma(G)$ is bounded by the values stated in the theorem.

*Case* 2: $c_3 = x_2y_2$. Take $S := \{x_0, x_1, x_2, z\}$, where $z = x_2$ if $c_4 = x_3y_2$, or $z = y_1$ if $c_4 = x_2y_3$. In this case, analogously to the previous case, we consider $G' = G - S$, take a minimum dominating set $D'$ of $G'$ and set $D := D' \cup \{y_1\}$. The proof follows as in the previous case. □

In Fig. 3 we exhibit a family of striped maximal outerplanar graphs that shows that the bound given by Theorem 5 is tight. The family is recursively constructed from the basic graphs depicted in Fig. 3(a) and the gadget shown in Fig. 3(b). The pair of dashed edges in the basic graphs indicate the positions where a gadget can be inserted so as to build a graph of higher order. (Note that the top part and bottom part of the gadget have to coincide with the two edges of the pair.) Graphs in the figure are just sketches; non-triangular faces can be triangulated in any way. Fig. 3(c) shows an example of a graph of order 15 that was obtained from the fourth basic graph by adding 2 gadgets (and some extra edges).

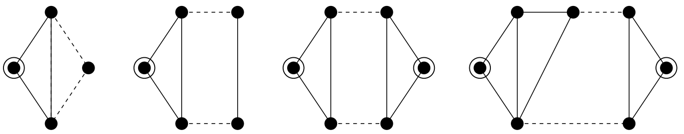
## 4. Maximal outerplanar graphs with internal triangles

In the previous section we considered maximal outerplanar graphs that do not have internal triangles. In this section we generalise the result shown, considering now maximal outerplanar graphs $G$ that have $k$ internal triangles, and prove the following result.

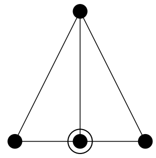
**Theorem 6.** *Let $G$ be a maximal outerplanar graph with $k$ internal triangles and $n \geq 3$ vertices. Then,*

$$\gamma(G) \leq \frac{n+k+2}{4}.$$

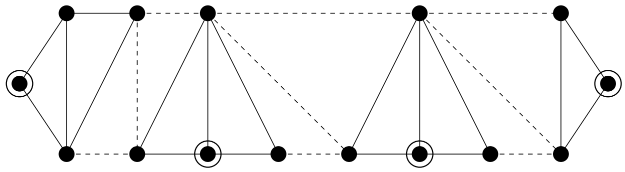
**Proof.** The proof is by induction on $n+k$. If $n \leq 6$, then it is easy to check that the result follows. If $k = 0$, then the result follows by Theorem 5. So, let us assume that $k \geq 1$.



(a) Basic graphs.



(b) Gadget.



(c) Example.

**Fig. 3.** Family of graphs showing that the bounds of Theorem 5 are tight. Note that the subgraphs induced by the set consisting of a marked vertex and its neighbours are mutually disjoint. Thus, a dominating set of the graph has to contain at least one vertex of each such a set.

Let $H$ be the Hamiltonian cycle of the outer face of an embedding of $G$ and $v_0, \ldots, v_{n-1}$ be a cyclic clockwise order of its vertices. If $v_iv_j$ is a chord of $H$, then we denote by $G[i, j]$ the subgraph of $G$ induced by the vertices in the path obtained when we traverse $H$ from $v_i$ to $v_j$ in the clockwise direction. We say that a chord $v_iv_j$ is *simple* if the subgraph $G[i, j]$ does not contain an internal triangle of $G$ (and therefore is striped).

We also say that $v_iv_j$ is a $|j - i|$-*chord* (of $H$). Relabel the vertices of $H$, if necessary, so that $T = (v_0v_iv_j)$, $1 < i < j$, is an internal triangle of $H$ such that $v_0v_i$ is a simple chord with the property that the subgraph $G[v_0, v_i]$ is of maximum order. Let us analyse the possible values for the index $i$.

*Case* 1: $i = 2$.

Let $\mathcal{T}$ be the set of internal triangles of $G$ that have at least one edge which is a 2-chord. Note that $T \in \mathcal{T}$.

Suppose, without loss of generality that, among all possible choices of a triangle in $\mathcal{T}$, we have that $T = (v_0, v_2, v_j)$ is a choice with $j$ minimum. In this case, $v_2v_j$ is a simple chord, otherwise an internal triangle in $G[v_2, v_j]$ would contradict the choice of $T$. Also, by the choice of $T$, we can conclude that $j = 4$. In fact, if we had $j > 4$, then $v_2v_j$ would be an $\ell$-chord with $\ell > 2$, contradicting the choice of $v_0v_2$.

Now, let $G' := G - \{v_1, v_2, v_3\}$. Note that $G'$ is a maximal outerplanar graph with $n' = n - 3 \geq 4$ vertices and $k' \leq k - 1$ internal triangles. Thus, by the induction hypothesis, $\gamma(G') \leq (n' + k' + 2)/4$. Take a minimum dominating set of $G'$, say $D$. Then, $D \cup \{v_2\}$ is a dominating set of $G$. Hence,

$$\gamma(G) \leq |D| + 1 = \gamma(G') + 1 \leq \frac{n' + k' + 2}{4} + 1 \leq \frac{n + k + 2}{4}.$$

*Case* 2: $i = 3$.

In this case, we construct a simple graph $G'$ by removing the vertices $v_1$ and $v_2$, and contracting the edge $v_0v_3$. Clearly, $G'$ is a maximal outerplanar graph with $n' = n - 3 \geq 4$ vertices and $k' \leq k - 1$ internal triangles. By the induction hypothesis, $\gamma(G') \leq (n' + k' + 2)/4 = (n + k - 2)/4$.

Let $D$ be a minimum dominating set of $G'$. Then, either $v_0v_2 \in E(G)$ and $D \cup \{v_0\}$ is a dominating set of $G$, or $v_1v_3 \in E(G)$ and $D \cup \{v_3\}$ is a dominating set of $G$. Thus,

$$\gamma(G) \leq |D| + 1 = \gamma(G') + 1 \leq \frac{n + k - 2}{4} + 1 = \frac{n + k + 2}{4}.$$

*Case* 3: $i = 4$.

Let $G_1 := G[v_0, v_4]$ and $G_2 := G[v_4, v_0]$. As $G_1$ is a maximal outerplanar striped graph with 5 vertices, by Theorem 5, $\gamma(G_1) = 1$. The graph $G_2$ is maximal outerplanar with $n - 3$ vertices and at most $k - 1$ internal triangles. Thus, by the induction hypothesis, $\gamma(G_2) \leq (n + k - 2)/4$.

Since $G = G_1 \cup G_2$, it follows that $\gamma(G) \leq \gamma(G_1) + \gamma(G_2)$, and therefore,

$$\gamma(G) \leq 1 + \frac{n + k - 2}{4} \leq \frac{n + k + 2}{4}.$$

*Case* 4: $i \geq 5$.

Let $G_1 := G[v_0, v_i]$. As there is no internal triangle $T'$ of $G$, such that $T'$ is a subgraph of $G_1$, the triangle in $G_1$ that contains $v_0v_1$ also contains an edge of the outer face of $G$ (defined by $H$). Thus, either $v_0v_{i-1} \in E(G)$ or $v_1v_i \in E(G)$, and therefore, either $v_i$ or $v_0$ has degree 2 in $G_1$.

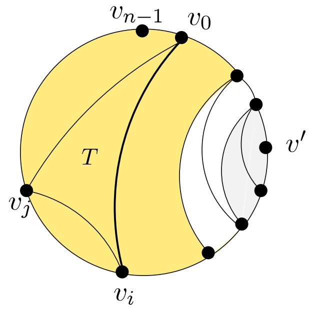
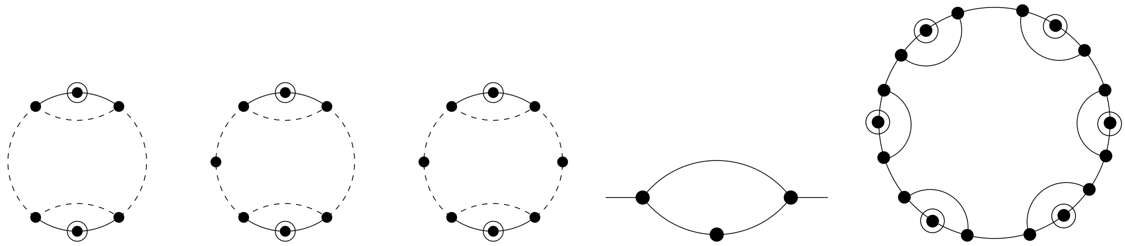



**Fig. 4.** Sketch of the graph $G$ with the subgraphs $G'$ and $G[S]$.



(a) Smaller graphs. (b) Gadget. (c) Example where $t = 6$ and $r = 0$.

**Fig. 5.** Sketch of a family of graphs that shows that the bounds of Theorem 6 are tight. Note that the neighbourhoods of each of the $t$ vertices of degree 2 are disjoint: so any dominating set must have at least $t$ vertices.

As $G_1$ is a maximal outerplanar striped graph, it has only one more vertex, say $v'$, of degree 2. Consider that $G_1$ is embedded in the plane, and let $H'$ be the Hamiltonian cycle starting at $v'$ that defines the outer face of $G_1$. Suppose $G_1$ is endowed with a canonical vertex-labelling, and the chords of $H'$ are ordered as stated in Lemma 4. Since $G_1$ has at least six vertices, it has at least three chords. Applying now the same idea used in the proof of Theorem 5, we can delete from $G$ a set $S$ of four vertices, including the vertex $v'$, which corresponds to the deletion of the four faces corresponding to the first four vertices of the path $P(G_1^*)$ of the weak dual graph $G_1^*$. See Fig. 4. In this case, the resulting graph, say $G' := G - S$, has $n - 4$ vertices and is maximal outerplanar. Thus, by the induction hypothesis, $\gamma(G') \leq ((n-4)+k+2)/4$. As in the proof of Theorem 5, we can also conclude that $\gamma(G[S]) = 1$. Thus,

$$\gamma(G) \leq \gamma(G[S]) + \gamma(G') \leq 1 + \frac{n+k-2}{4} \leq \frac{n+k+2}{4}. \quad \square$$

Now, we construct a family of maximal outerplanar graphs that shows that the bound given by Theorem 6 is tight. Each graph in the family has $3t + r$ vertices, where $t$ is the number of vertices of degree 2, and $r \in \{0, 1, 2\}$. In Fig. 5(a) we sketch the smaller graphs in the family. The dashed boundary can be triangulated in any way so as to obtain a maximal outerplanar graph. Since the neighbourhoods of each of the $t$ vertices of degree 2 are disjoint, any dominating set must have at least $t$ vertices.

Let $G$ be a graph in the family. From $G$, we can construct a new graph in the family by substituting one edge of the Hamiltonian cycle, non-incident with a degree-two vertex, with the gadget shown in Fig. 5(b). Fig. 5(c) sketches a case where $t = 6$ and $r = 0$.

Using Proposition 2, the bounds given by Theorem 6 can be written in terms of the number of vertices of degree 2 in the graph. We obtain this way the following result.

**Corollary 7.** *Let $G$ be a maximal outerplanar graph with $n \geq 4$ vertices and $t$ vertices of degree 2. Then,*

$$\gamma(G) \leq \frac{n+t}{4}. \quad \square$$

## 5. Final remarks

The problem of determining the domination number of a graph is NP-hard. Due to the importance of theoretical and practical aspects of this problem, much investigation on this topic has been carried out. These works have approached the problem in a variety of ways: establishing bounds for the domination number of some classes of graphs [8,10,5–7,9], characterising classes of graphs for which the domination number has a fixed value, or even imposing some more properties on the dominating sets [2–4].

In this work we have established upper-bounds for the domination number of maximal outerplanar graphs and showed that these bounds are tight for an infinite family of graphs. This study was motivated by the conjecture of Matheson and Tarjan [8], which claims that every $n$-vertex plane triangulation $G$, with $n$ sufficiently large, has $\gamma(G) \leq n/4$. We believe that trying to establish the status of this conjecture would be a very fascinating endeavour. Also, it would be interesting to determine good bounds for dominating numbers of other classes of planar graphs, especially when these results give rise to polynomial algorithms to construct dominating sets within these bounds. The proofs we have shown here give rise to such type of algorithms. Although these proofs are not complicated, the challenge was to find the correct bound.

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