Zombies and Survivors on Graphs

Joël Faubert

University of Ottawa

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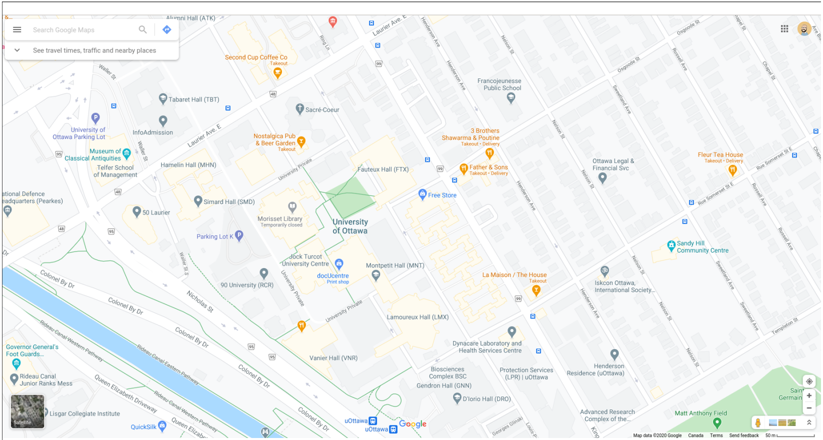
Outline

1. Introduction to Pursuit Games
2. A Planar Graph Where 3 Zombies Lose
3. Two Zombies Win on Cycle With One Chord
4. Conclusion
5. References

Games on Graphs



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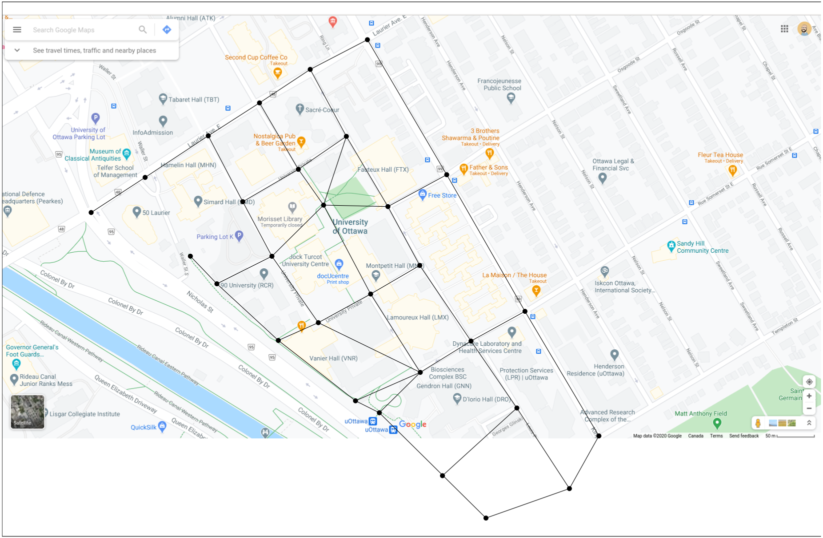

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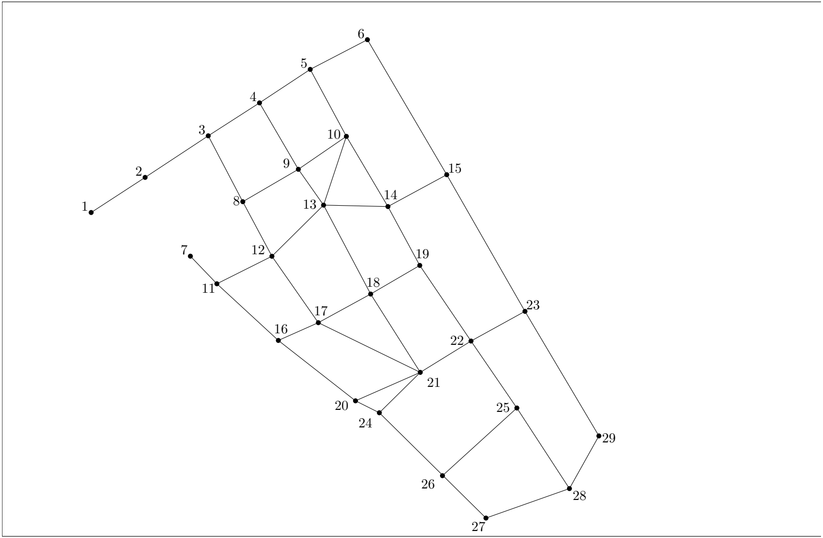

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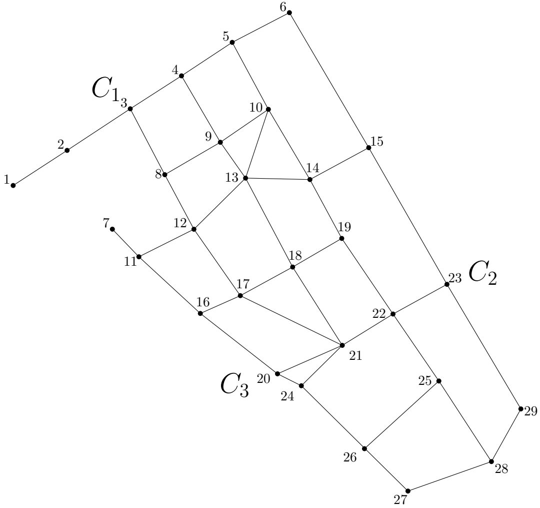

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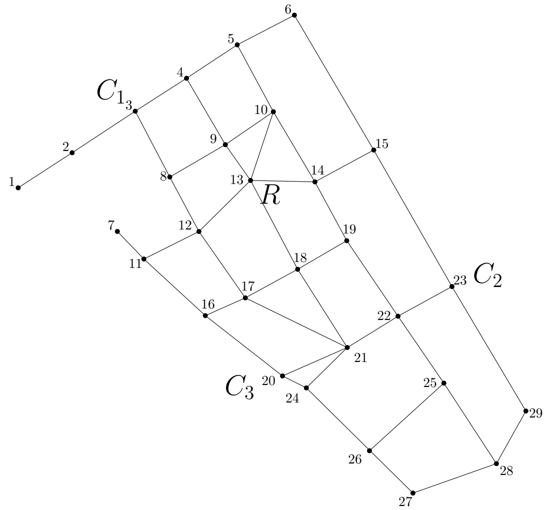

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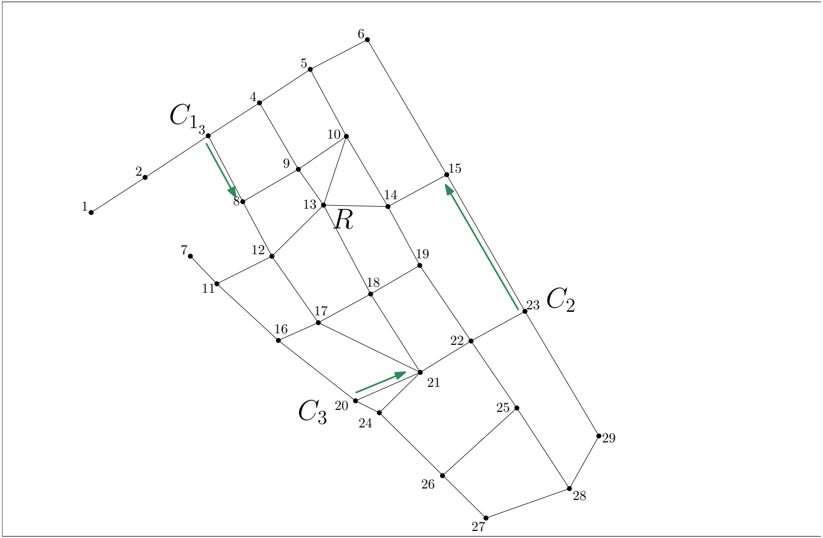

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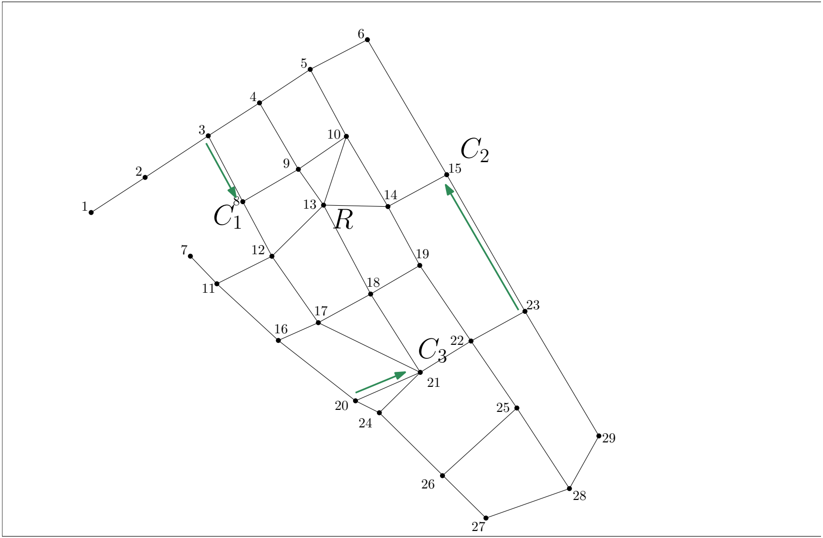

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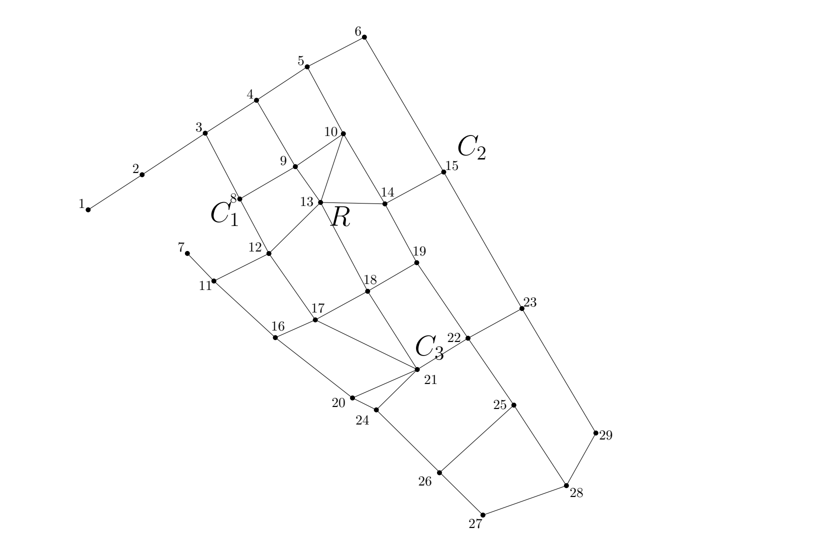

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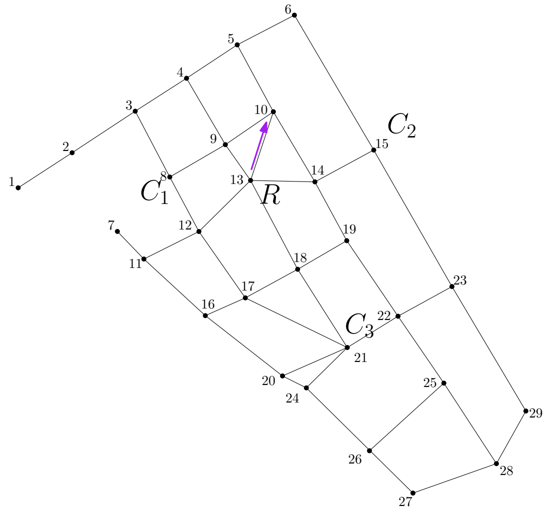

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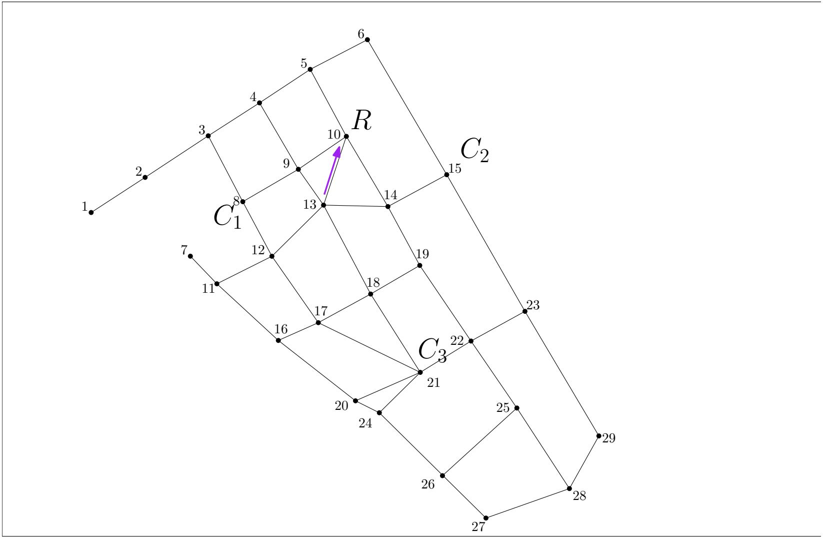

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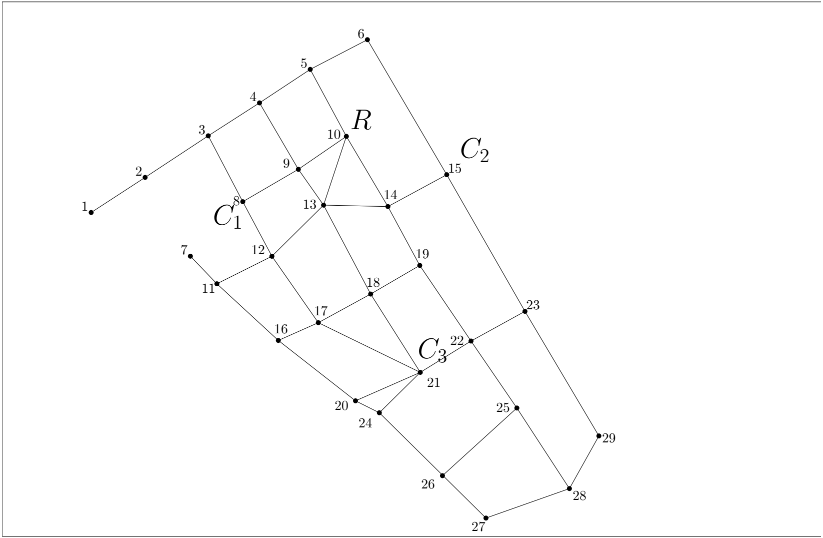
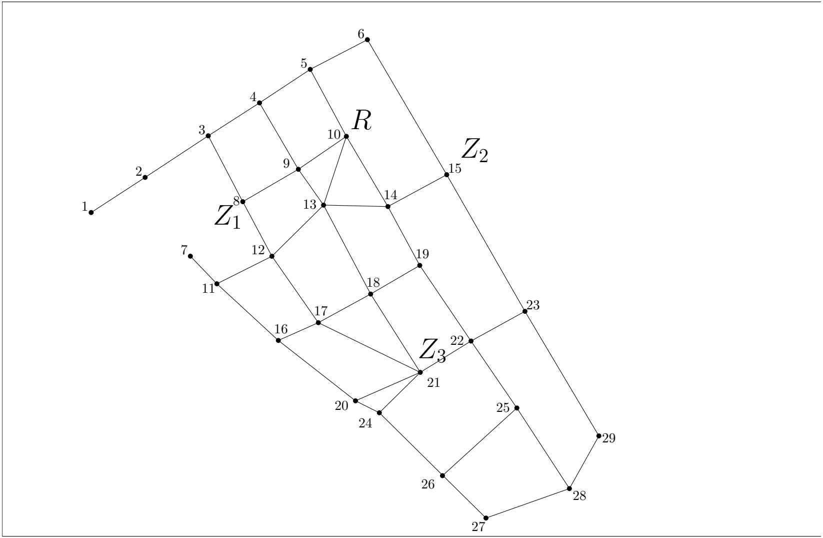


Zombies and Survivors



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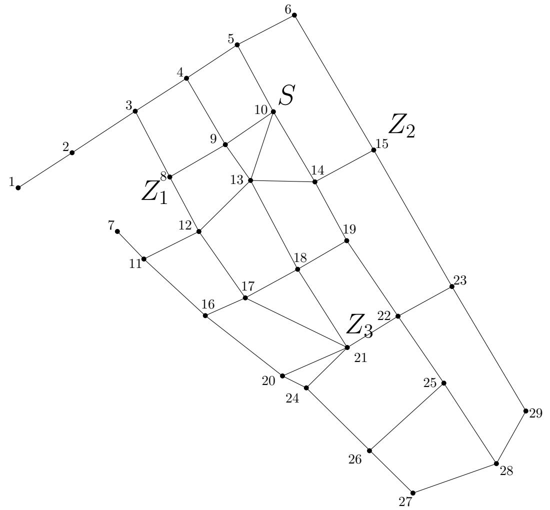

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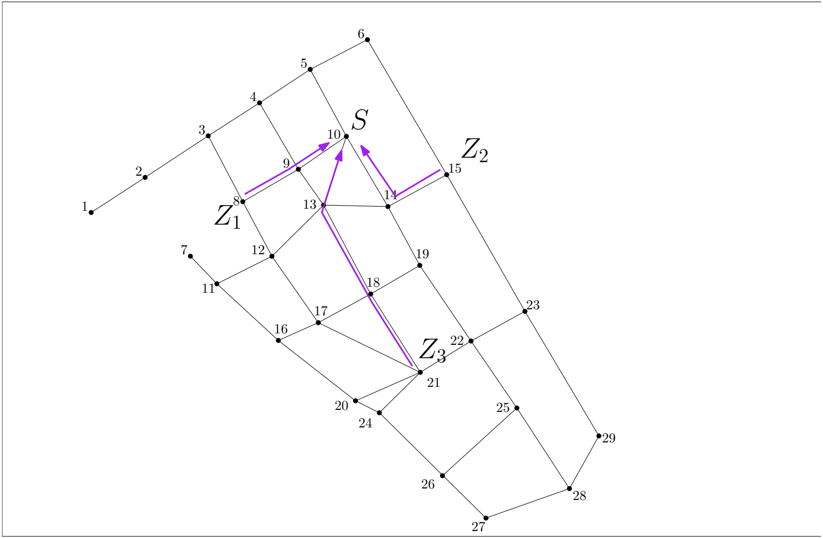

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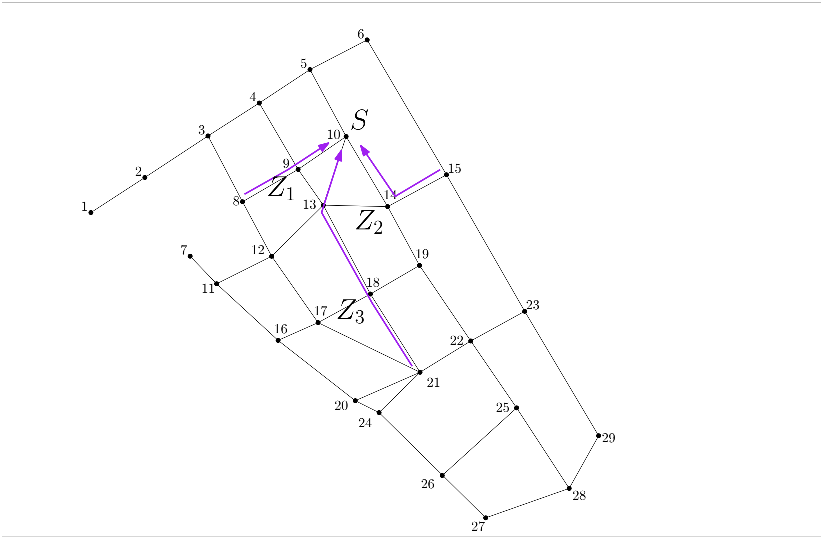

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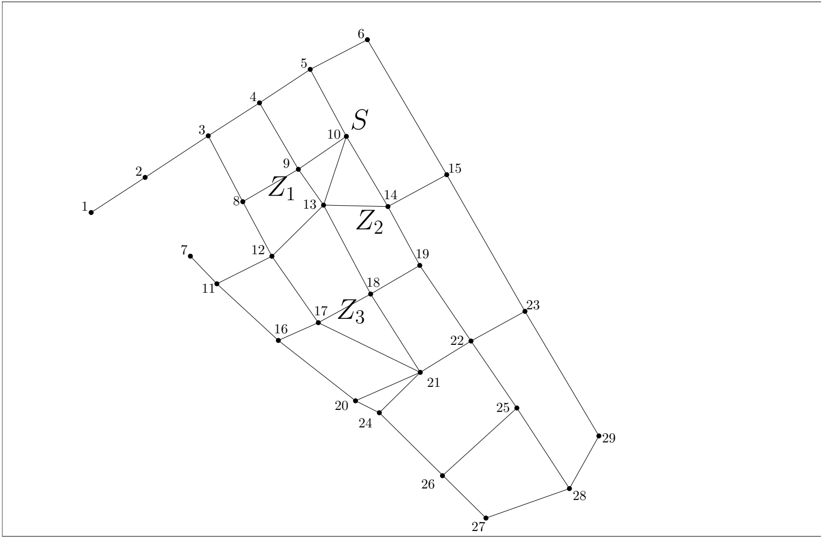

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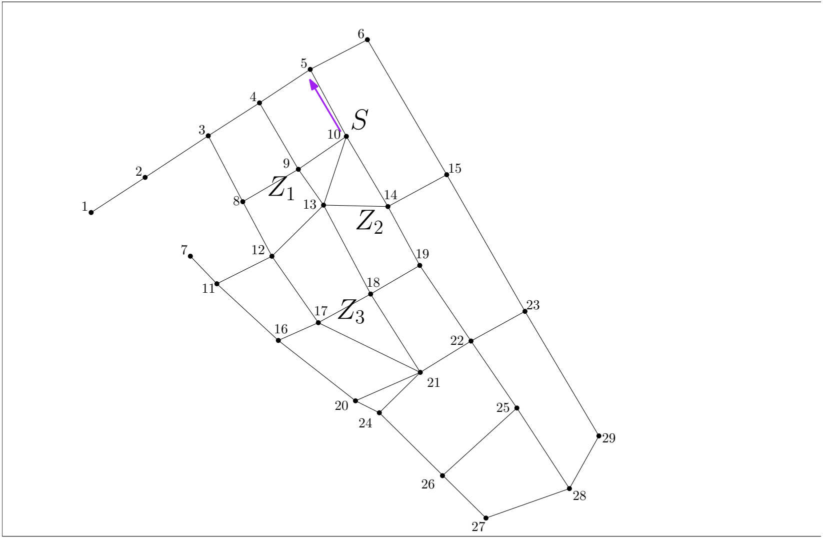

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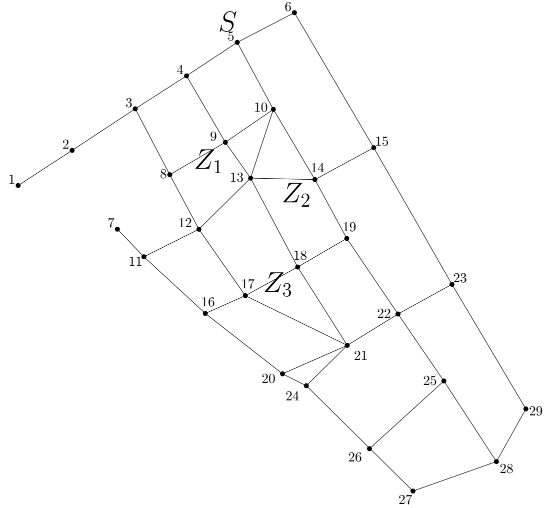

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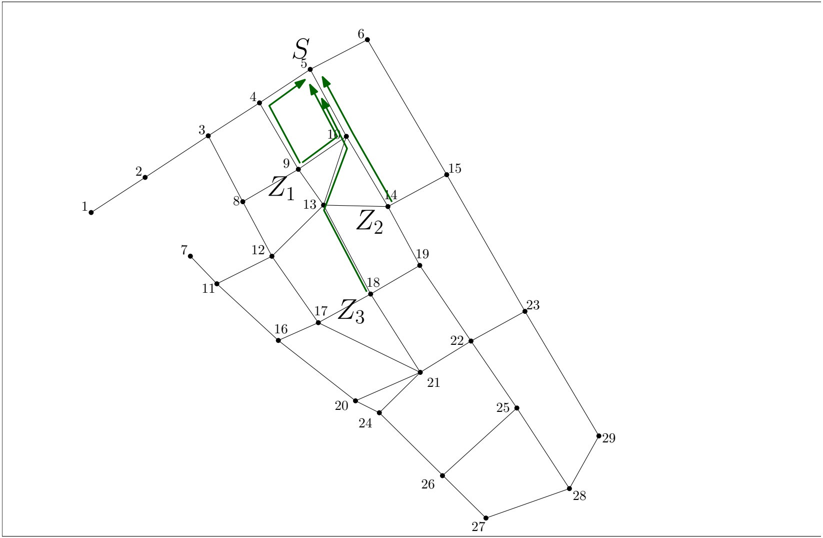

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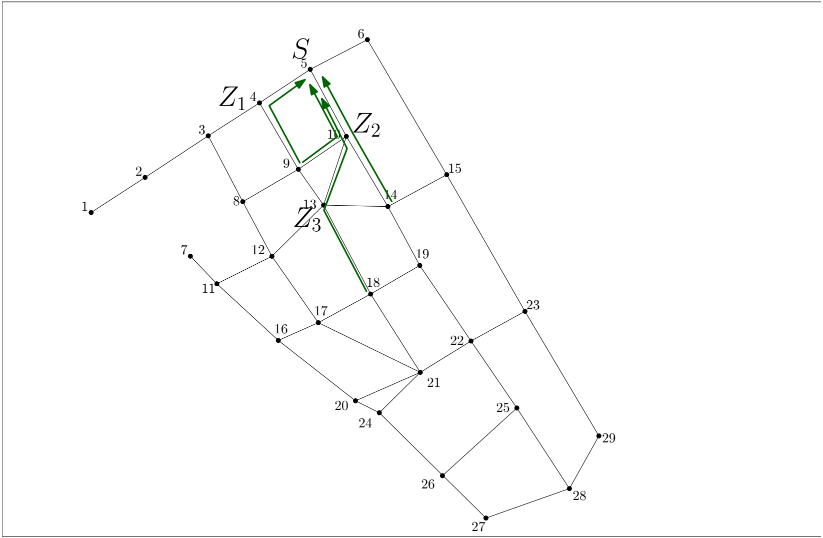

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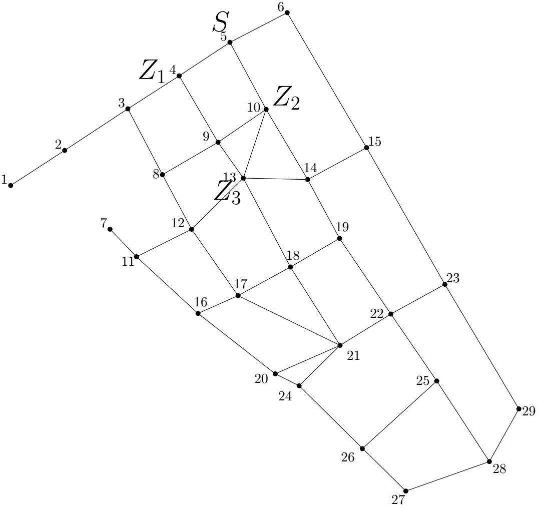

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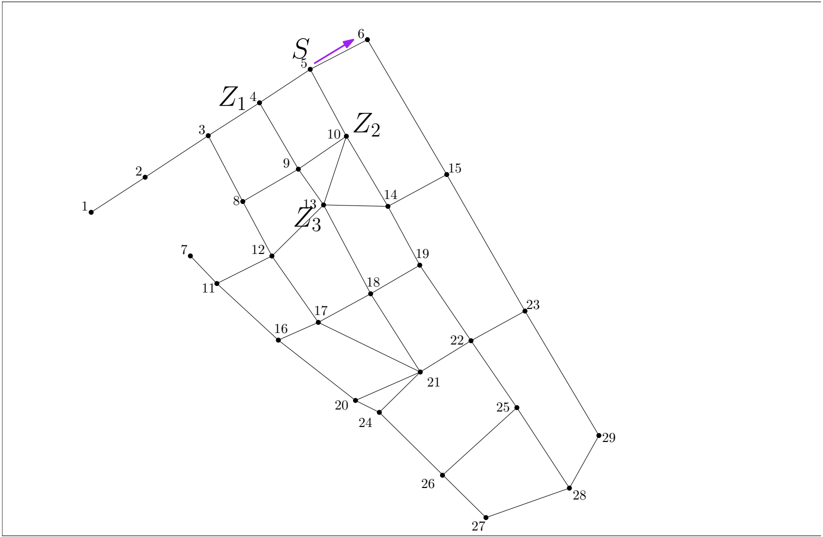


Cop-Number and Zombie-Number

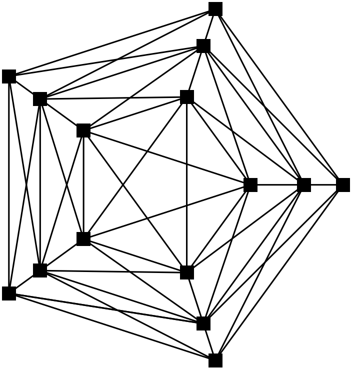


Figure: Cop-win but not zombie-win [FHMP16])

- Cop-Number $c(G)$ number of cops needed to guarantee a win on G.

Cop-Number and Zombie-Number

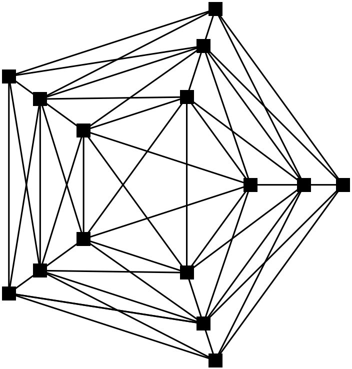


Figure: Cop-win but not zombie-win [FHMP16])

- Cop-Number $c(G)$ number of cops needed to guarantee a win on G.
- Zombie-Number $z(G)$ number of zombies needed to guarantee a win on G.

Cop-Number and Zombie-Number

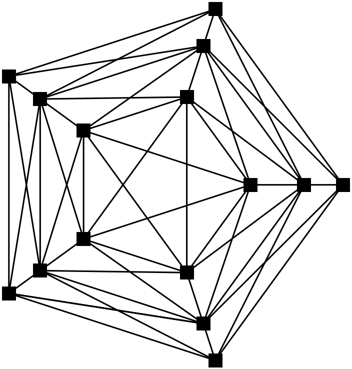


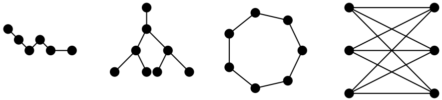
Figure: Cop-win but not zombie-win [FHMP16])

- Cop-Number $c(G)$ number of cops needed to guarantee a win on G.
- Zombie-Number $z(G)$ number of zombies needed to guarantee a win on G.

Lemma

For any graph G, $c(G) \leq z(G)$.

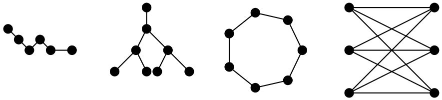
Cop-Number and Zombie-Number



G	$c(G)$	$z(G)$
Tree (acyclic)	1	1
C_3	1	1
C_n $(n \geq 4)$	2	2
K_n $(n \geq 1)$	1	1
$K_{n,m}$ $(n, m \geq 2)$	2	2
G planar	3	?
G outerplanar	2	?

Table: Cop and zombie number of a few graph families

Cop-Number and Zombie-Number



G	$c(G)$	$z(G)$
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K_n $(n \geq 1)$	1	1
$K_{n,m}$ $(n, m \geq 2)$	2	2
G planar	3	at least 4
G outerplanar	2	at least 3

Table: Cop and zombie number of a few graph families

Zombies vs. Cops on Planar Graphs

Aigner and Fromme [AF84] described a winning 3 cop strategy: enclose the robber into a shrinking *territory* using *isometric paths*.

On Outerplanar Graphs

A graph is *outerplanar* if it has a planar drawing where all vertices are on the *outer face*.

In [Cla02], Clarke showed that 2 cops suffice to win on outerplanar graphs.

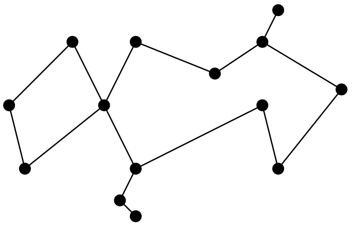

Figure: An outerplanar graph

Zombies vs Cops on Outerplanar Graphs

But 2 zombies lose on this outer-planar graph.

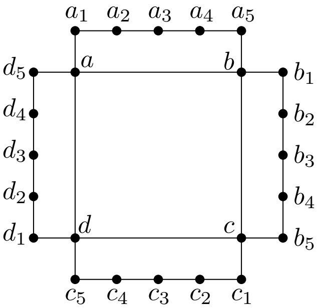

Figure: Two cops win but two zombies lose (Fig 2 of [FHMP16])

First Result

Observation 1

The zombie-number of planar graphs is at least 4.

Foiling 3 Zombies on the Plane

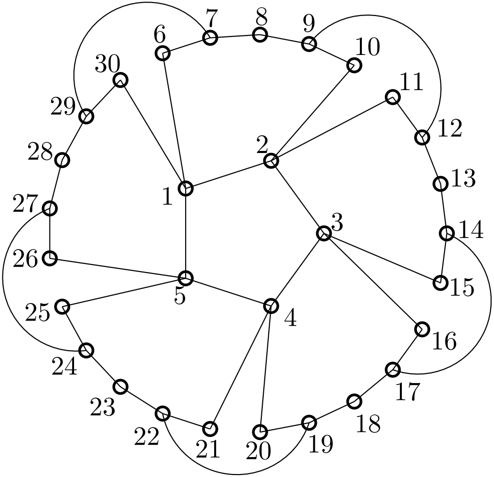

Figure: A planar graph where 3 zombies lose

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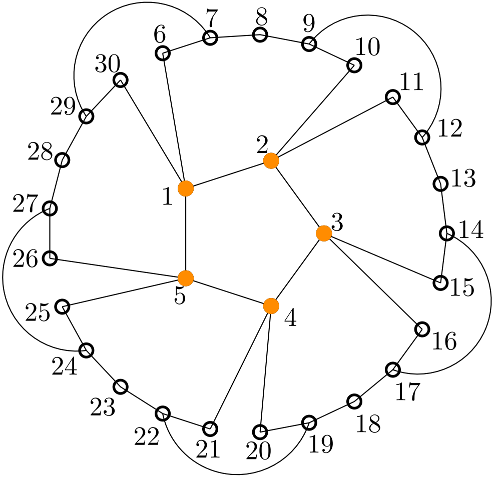

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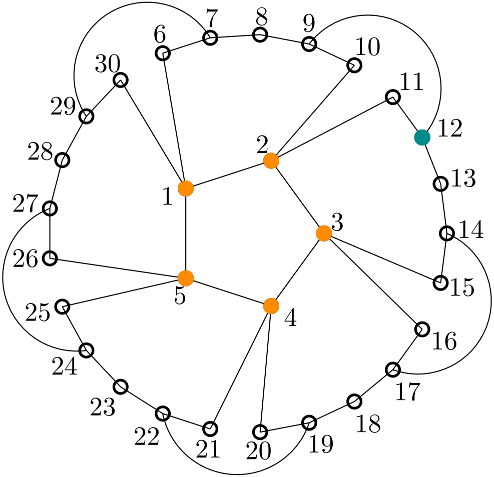

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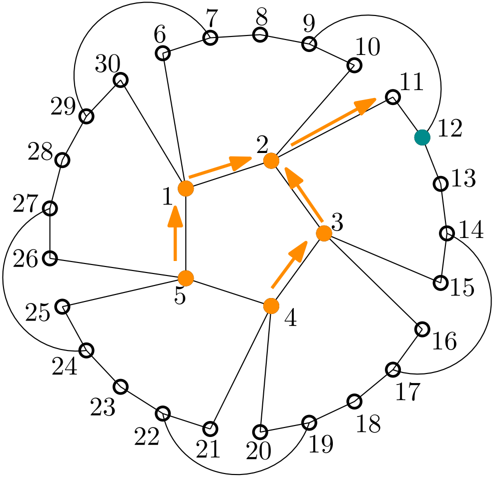

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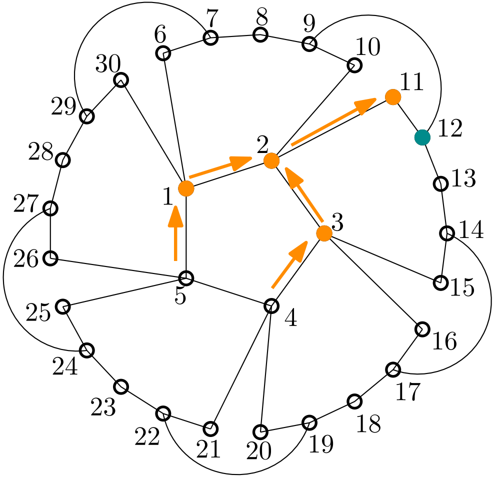

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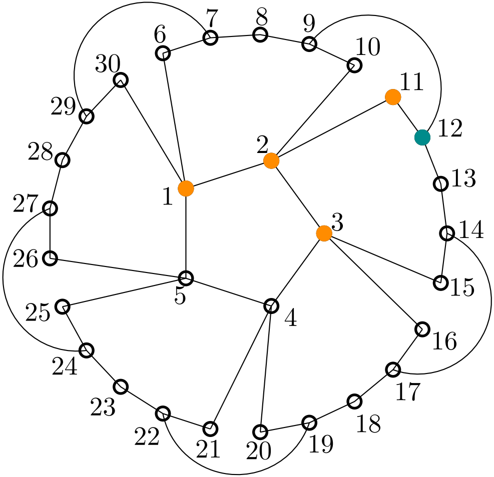

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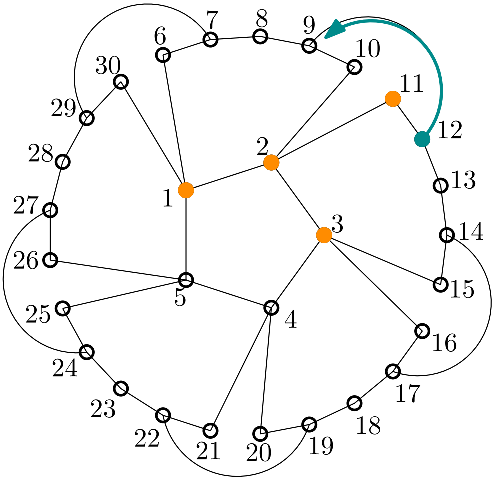

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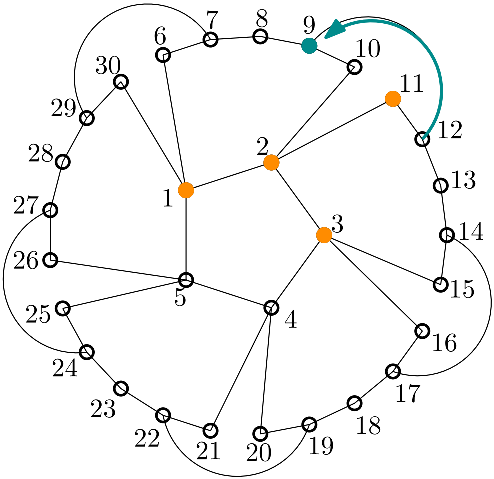

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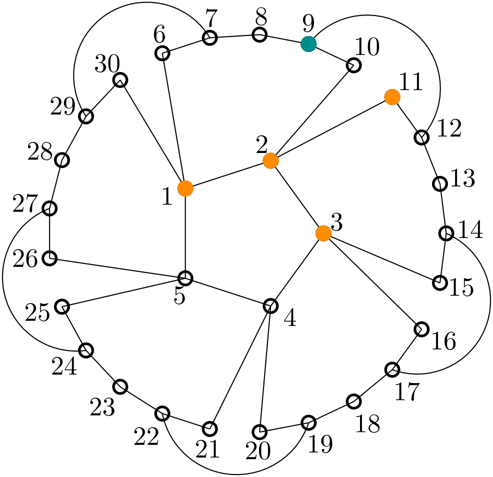

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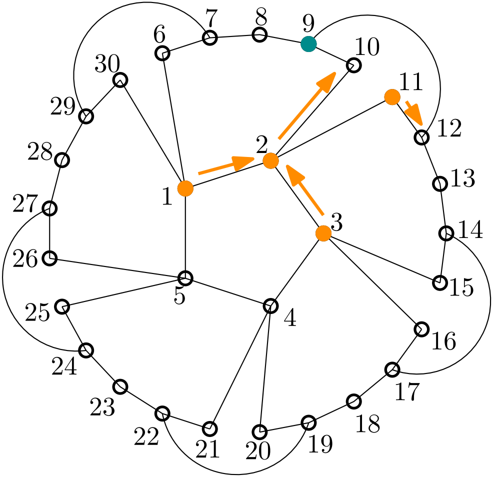

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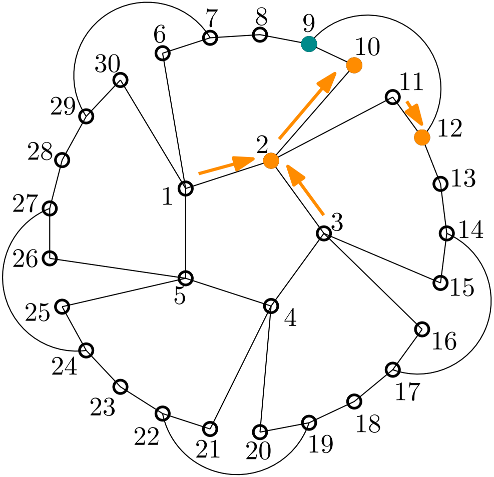

Figure: A planar graph where 3 zombies lose

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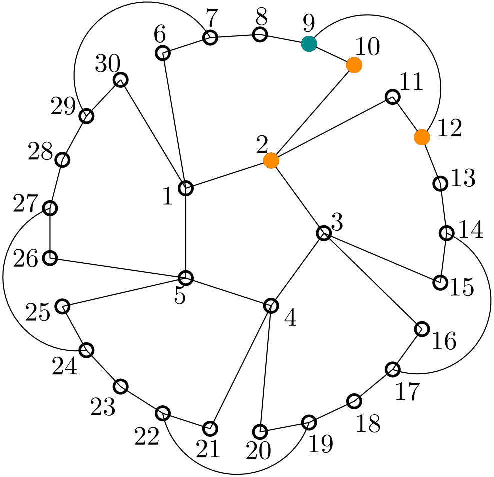

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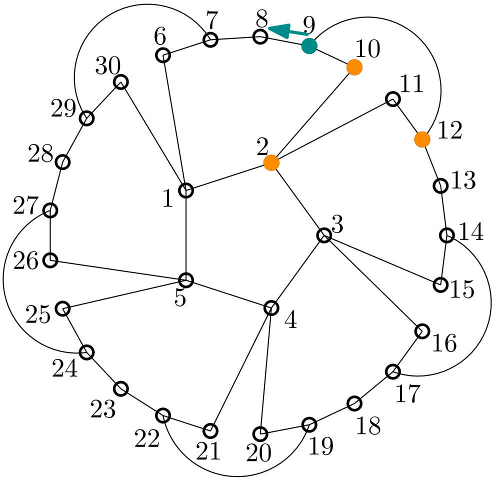

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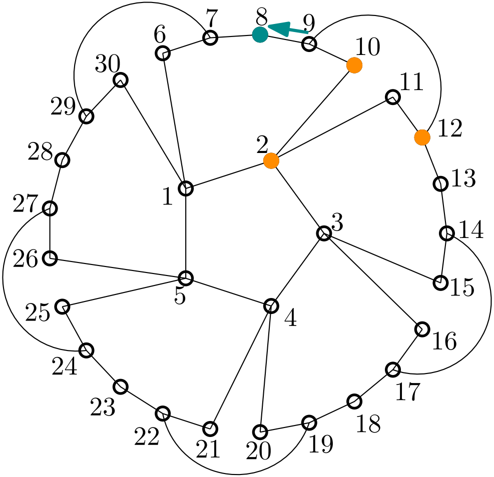

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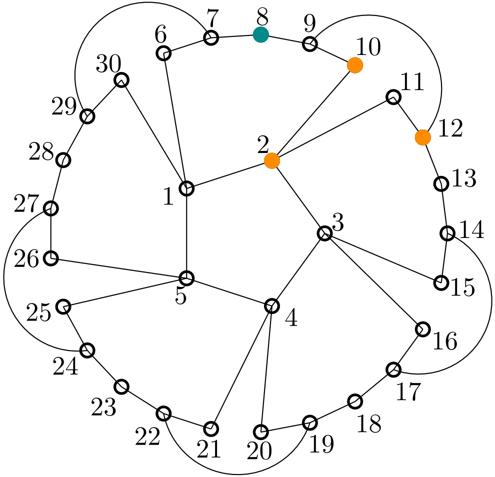

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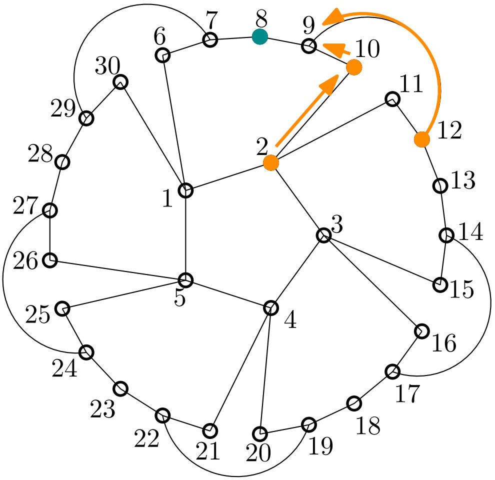

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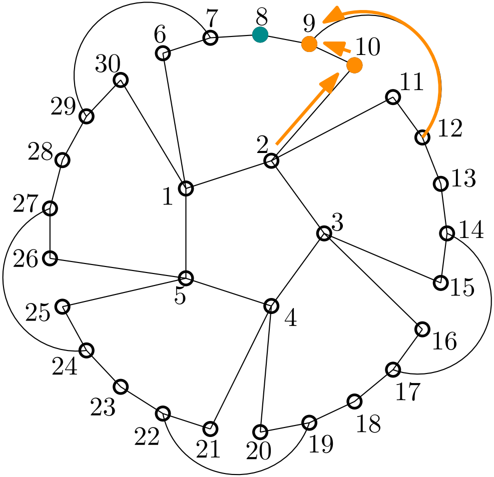

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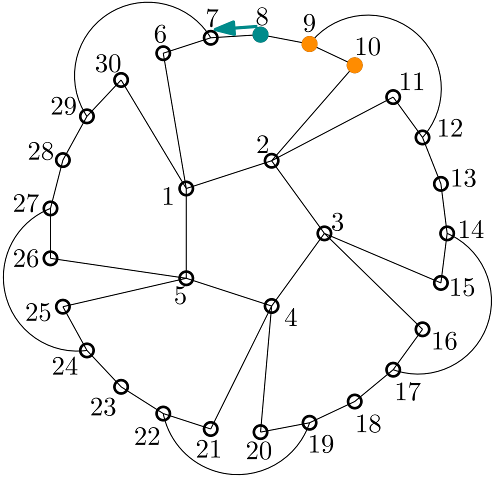

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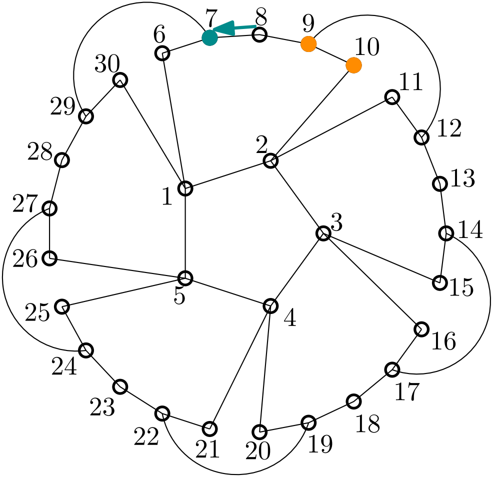


Figure: A planar graph where 3 zombies lose

Second Result

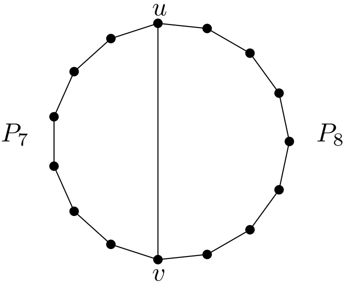


Figure: $Q_{7,8}$

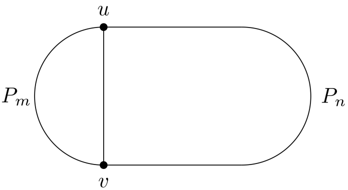


Figure: $Q_{m,n}$

Theorem

Let $m, n \in Z$ with $3 \leq m \leq n$. The zombie number of $Q_{m,n}$ is 2.

Two Zombie Strategy

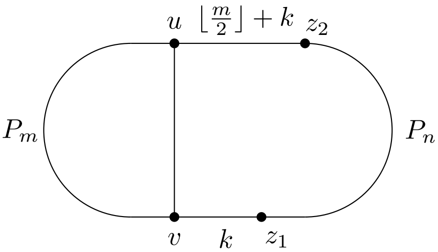


Figure: A winning 2-zombie start for a cycle with one chord.

where

$$k = \left\lfloor \frac{n - 2\left\lfloor \frac{m}{2} \right\rfloor + 3}{4} \right\rfloor .$$

Proof: Cornering the Survivor on C_{m+1}

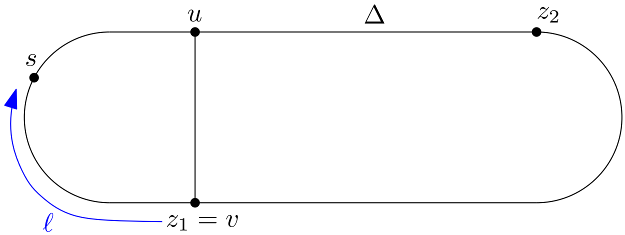


Figure: z_1 on v, z_2 is Δ away from u, and s on P_m

Which Way Do the Zombies Turn?

Depends on value of $2 \leq \ell \leq m-1$.

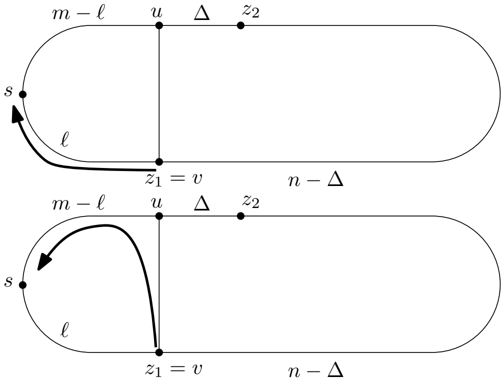
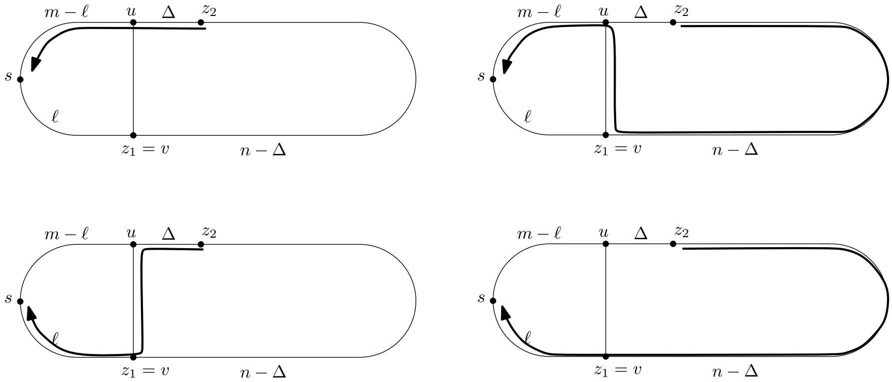


Figure: z_1 Goes Clockwise or Counter-clockwise

Possible z_2s-paths

Depends on value of ℓ and Δ.



Path Lengths

Four possible paths for z_2:

- $|P_a| = \Delta + (m - \ell)$,
- $|P_b| = \Delta + 1 + \ell$,
- $|P_c| = (n - \Delta) + 1 + (m - \ell)$,
- $|P_d| = (n - \Delta) + \ell$,

but only two possible choices: CW or CCW.

Four Possible Cases

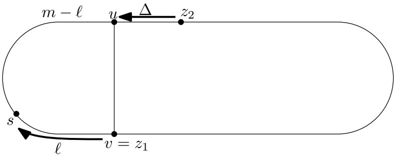


Figure: Case I.A.

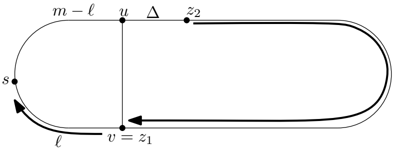


Figure: Case II.A.

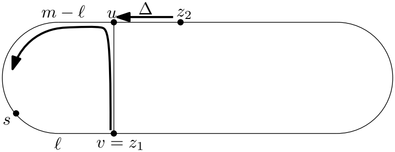


Figure: Case I.B.

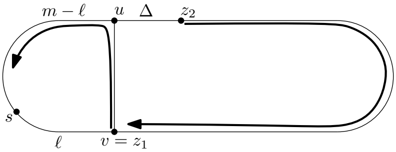


Figure: Case II.B.

Assumptions I or II, A or B

$$2 \leq \ell \leq m - 1 \qquad (*)$$

z_1 goes clockwise if $\ell \leq 1 + m - \ell$. Combined with *, we have

$$4 \leq 2\ell \leq m + 1. \qquad \text{(A)}$$

z_1 goes counter-clockwise if $1 + m - \ell \leq \ell$. Combined *, we obtain

$$m + 1 \leq 2\ell \leq 2m - 2. \qquad \text{(B)}$$

z_2 moves counter-clockwise if either

$$|P_a| \leq \min\{|P_c|, |P_d|\} \quad \text{or} \quad |P_b| \leq \min\{|P_c|, |P_d|\}. \qquad \text{(I)}$$

z_2 goes clockwise if either

$$|P_c| \leq \min\{|P_a|, |P_b|\} \quad \text{or} \quad |P_d| \leq \min\{|P_a|, |P_b|\}. \qquad \text{(II)}$$

Case I.A.

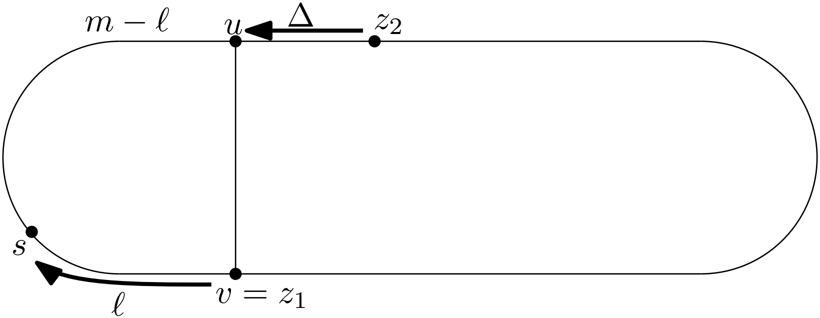


Figure: Case I.A.

Case I.A. Inequalities

Constraints on ℓ from A

$$4 \leq 2\ell \leq m + 1$$

Constraints on Δ from I

$$\Delta + (m - \ell) \leq n - \Delta + 1 + m - \ell \qquad \text{and}$$
$$\Delta + (m - \ell) \leq n - \Delta + \ell$$

or

$$\Delta + 1 + \ell \leq n - \Delta + 1 + m - \ell \qquad \text{and}$$
$$\Delta + 1 + \ell \leq n - \Delta + \ell.$$

After Algebraic Simplification

For z_2 to go counter-clockwise (following either P_a or P_b) we must have

$$2\Delta \leq n - m + 2\ell$$

or

$$2\Delta \leq n - 1.$$

Scenario Outcome

What happens Δ turns later?

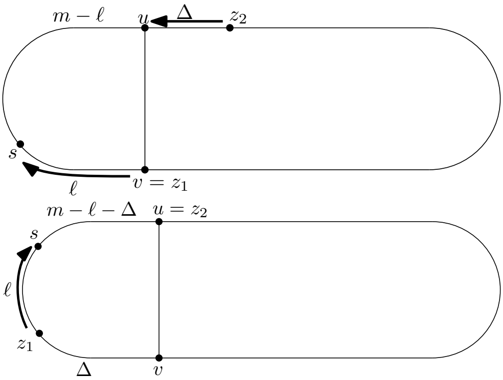


Figure: Case I.A. before/after Δ rounds

Winning Condition

$$\Delta \leq m - \ell + 1.$$

If $\Delta = m - \ell$ both z_2 and s reach u on the same round, with the survivor moving onto the zombie-occupied vertex (and losing).
If we have $\Delta = m - \ell + 1$, then s reaches u first but is caught by z_2 on the following round.

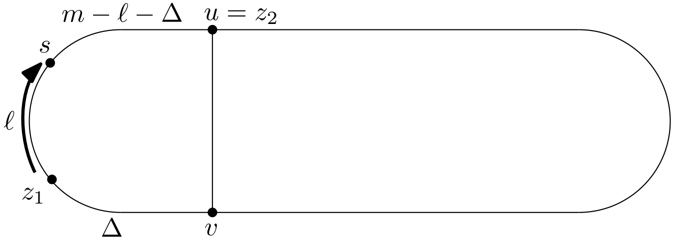


Figure: Case I.A. after Δ rounds

Win Condition II

Must also ensure that z_2 moves counter-clockwise (opposite to z_1) once it reaches u in order to trap the survivor. So we also need

$$m - \ell - \Delta \leq 1 + \Delta + \ell$$

Or, in terms of Δ,

$$2\Delta \geq m - 2\ell - 1.$$

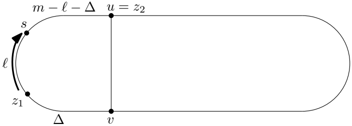


Figure: Case I.A. after Δ rounds

Case I.A.

z_1 clockwise: $4 \leq 2\ell \leq m+1$
z_2 counter-clockwise:
$2\Delta \leq n-m+2\ell$ or $2\Delta \leq n-1$
zombies win:
$2\Delta \leq 2m-2\ell+2$ and
$m-2\ell-1 \leq 2\Delta$

Case II.A.

z_1 clockwise: $4 \leq 2\ell \leq m+1$
z_2 clockwise:
$n-m+2\ell \leq 2\Delta$ and $n-1 \leq 2\Delta$
zombies win:
$2\Delta \geq 2n-2m+2\ell$ and
$2\Delta \leq 2n+2\ell-m-1$

Case I.B.

z_1 counter-clockwise:
$m+1 \leq 2\ell \leq 2m-2$
z_2 counter-clockwise:
$2\Delta \leq n+1$ or $2\Delta \leq n+m-2\ell$
zombies win:
$\Delta \leq 2\ell$ and
$2\ell-m+1 \leq 2\Delta$

Case II.B.

z_1 counter-clockwise:
$m+1 \leq 2\ell \leq 2m-2$
z_2 clockwise:
$n+1 \leq 2\Delta$
zombies win:
$n-\Delta \leq \ell+1$ and
$2\Delta \leq 2n+m-2\ell+1$

Guarding C_{n+1}

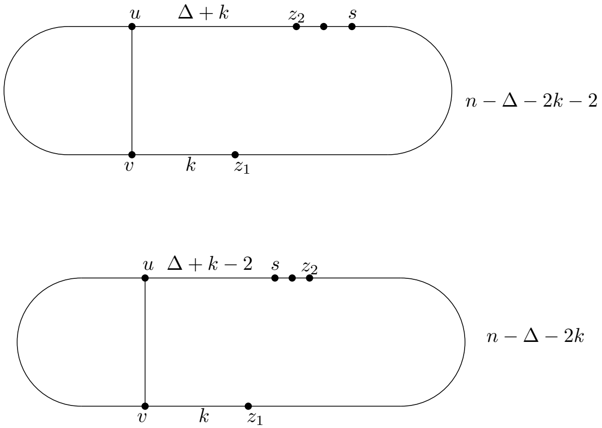


Figure: Preventing the zombies from turning in same direction on C_{m+1}

Guarding C_{n+1} II

To ensure zombies turn in opposite directions, we need:

$$n - \Delta - 2k - 2 \leq \Delta + k + 1 + k + 2 \qquad \text{and}$$
$$\Delta + 2k - 1 \leq n - \Delta - 2k + 2.$$

Solving for k gives

$$n - 2\Delta - 5 \leq 4k \leq n - 2\Delta + 3$$

How the Strategy Works

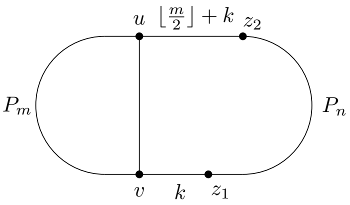


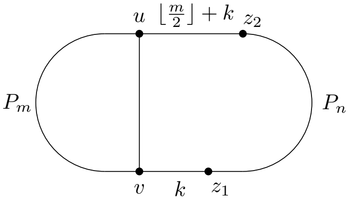
Figure: A winning 2-zombie start for a cycle with one chord.

where

$$k = \left\lfloor \frac{n - 2\left\lfloor \frac{m}{2} \right\rfloor + 3}{4} \right\rfloor.$$

- If s starts between the two zombies on C_{n+1} – on subpath that does not include the chord – then survivor loses.

How the Strategy Works



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- Otherwise, s must be on other half of C_{n+1} or on C_{m+1}.

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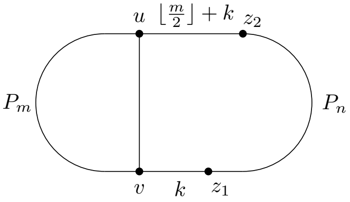

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- Pushed by zombies onto P_m.

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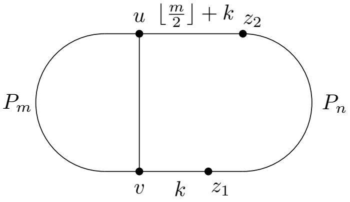

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- If s starts between the two zombies on C_{n+1} – on subpath that does not include the chord – then survivor loses.
- Otherwise, s must be on other half of C_{n+1} or on C_{m+1}.
- Pushed by zombies onto P_m.
- Falls into Δ trap after k turns.

Research Directions

- Is there an upper bound on the zombie-number for planar graphs?

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- Is there an upper bound on the zombie-number for planar graphs?
- Is there an upper bound on the zombie-number of outerplanar graphs?
- What about cycles with more than one chord?
- Can the proof strategy employed for $Q_{m,n}$ be generalized to any two cycles with shared path? Or bowtie graphs?

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