JOURNAL OF COMBINATORIAL THEORY, Series B 58, 22–33 (1993)

Graph Searching and a Min–Max Theorem for Tree-Width

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Received April 19, 1989

The tree-width of a graph G is the minimum k such that G may be decomposed into a "tree-structure" of pieces each with at most $k+1$ vertices. We prove that this equals the maximum k such that there is a collection of connected subgraphs, pairwise intersecting or adjacent, such that no set of $\leqslant k$ vertices meets all of them. A corollary is an analogue of LaPaugh's "monotone search" theorem for cops trapping a robber they can see (LaPaugh's robber was invisible). © 1993 Academic Press, Inc.

1. INTRODUCTION

Here is a cops-and-robber game, played on a finite, undirected graph G. (All graphs in this paper are undirected, and finite unless we say otherwise.) The robber stands on a vertex of the graph, and can at any time run at great speed to any other vertex along a path of the graph. He is not permitted to run through a cop, however. There are k cops, each of whom at any time either stands on a vertex or is in a helicopter (that is, is temporarily removed from the game). The objective of the player controlling the movement of the cops is to land a cop via helicopter on the vertex occupied by the robber, and the robber's objective is to elude capture. (The point of the helicopters is that cops are not constrained to move along paths of the graph—they move from vertex to vertex

* This research was performed under a consulting agreement with Bellcore and while visiting the DIMACS Center at Rutgers University.

0095-8956/93 $5.00

arbitrarily.) The robber can see the helicopter approaching its landing spot and may run to a new vertex before the helicopter actually lands.

There are two forms of this game. In the first, the robber is invisible, and so to capture him the cops must methodically search the whole graph. (An equivalent problem is that of clearing the vertices of some plague which infects along edges.) This and a similar search game were investigated by LaPaugh and others [2–5, 7–9]. In particular, LaPaugh [7] and Kirousis and Papadimitriou [4, 5] showed that if k cops can guarantee to catch the robber, then k cops can search the graph monotonely, that is, never returning to a vertex which a cop has previously vacated. See also [2, 3]. In this paper we are concerned with a second form of the game, where the cops can see the robber at all times—the difficulty is just to corner him somewhere. To see the difference, two cops suffice to catch a visible robber if G is a tree. (Put cop 1 on some vertex v, see which component of $G \setminus v$ contains the robber, and transport cop 2 to the neighbour of v in that component. Now repeat with the cops exchanged.) However, if the tree is not a caterpillar, two cops will not be able to catch a lucky invisible robber. One of our main results is, roughly, that if k cops can catch a visible robber then again there is a monotone search strategy; and on the other hand, if a visible robber can guarantee to elude k cops, then there is an escape strategy with a particularly simple form.

Let us make some definitions. We denote by $G \setminus X$ the graph obtained from G by deleting X (here X may be a vertex or an edge, or a set of vertices or edges). The vertex set of a component of $G \setminus X$ is called an *X-flap*. We denote by $[V]^{<k}$ the set of all subsets of V of cardinality $<k$.

Now we can state the game more precisely. A *position* is a pair (X, R), where $X \in [V(G)]^{<k}$ and R is an X-flap. (X is the set of vertices currently occupied by cops. R tells us where the robber is—since he can run arbitrarily fast, all that matters is which component of $G \setminus X$ contains him.) We set (X_0, R_0) to an initial position. In the normal game, $X_0 = \emptyset$ and the robber player chooses R_0 to be some component of G; however, in the analysis it will be useful to consider other initial positions. Now step 1 of the game begins. In general, at the start of the ith step we have a position (X_{i-1}, R_{i-1}). The cop player chooses a new set $X_i \in [V(G)]^{<k}$ such that either $X_{i-1} \subseteq X_i$ or $X_i \subseteq X_{i-1}$. (Thus, we are not constraining the cops to move one at a time; but it makes no difference.) Then the robber player chooses (if possible) an X_i-flap R_i satisfying $R_i \subseteq R_{i-1}$ or $R_{i-1} \subseteq R_i$, respectively. If this choice is impossible, that is, if $V(R_{i-1}) \subseteq X_i$, the cop player has won, and otherwise the game continues with step $i+1$. The robber player thus cannot win; his objective is to stop the cop player winning. (We could make the game finite, by declaring that the robber wins if he survives $|V(G)|^k$ steps, because it is easy to see that if the cops can win at all then they can win in at most this many steps. But it is more

convenient not to do so.) Note that, for convenience, we have changed k to $k-1$; now the cop player has only $k-1$ cops at his disposal.

If there is a winning strategy for the cop player, we say that "$<k$ cops can search the graph." If in addition the cop player can always win in such a way that the sequence $X_0, X_1, \ldots$ satisfies $X_i \cap X_{i''} \subseteq X_{i'}$ for $i \leqslant i' \leqslant i''$, we say that "$<k$ cops can monotonely search the graph." We show that if $<k$ cops can search G then they can monotonely search it.

Our results are also connected with the concept of "tree-width," introduced in [10]. A *tree-decomposition* of a graph G is a pair (T, W), where T is a tree and $W = (W_t : t \in V(T))$ is a family of subsets of $V(G)$, satisfying

(W1) $\bigcup_{t \in V(T)} W_t = V(G)$, and every edge of G has both ends in some W_t, and

(W2) if $t, t', t'' \in V(T)$ and t' lies on the path from t to t'' then $W_t \cap W_{t''} \subseteq W_{t'}$.

(In some papers a tree-decomposition has been defined as a family of subgraphs rather than as a family of subsets of $V(G)$, but for us the subset version is more convenient.) The *width* of a tree-decomposition is $\max(|W_t| - 1 : t \in V(T))$, and the *tree-width* of G is the minimum width of a tree-decomposition of G. We prove that if G has tree-width $k-1$, then it can be monotonely searched by $<k+1$ cops, and cannot be searched at all by $<k$ cops. Thus, if there is a winning strategy for the cops then there is a "nice" winning strategy. This is analogous to LaPaugh's theorem. However, it turns out in addition that if there is a non-losing strategy for the robber, then there is a nice non-losing strategy, in a sense we shall explain; and we find this rather unexpected, because determining tree-width (and hence whether k cops can win) is NP-hard [1].

First, how do we describe a strategy for the robber? That is answered by the following.

(1.1) *A graph G cannot be searched by $<k$ cops if and only if there is a function σ mapping each $X \in [V(G)]^{<k}$ to a non-empty union $\sigma(X)$ of X-flaps, such that if $X \subseteq Y \in [V(G)]^{<k}$ then $\sigma(X)$ is the union of all X-flaps which intersect $\sigma(Y)$.*

Proof. If such a function σ exists, the robber can remain uncaptured by choosing $R_i \in \sigma(X_i)$ at each step. Conversely, suppose that $<k$ cops cannot search the graph. For each $X \in [V(G)]^{<k}$, let $\sigma(X)$ be the union of all X-flaps R such that the cop player cannot guarantee to win when the initial position is set to be (X, R). Then σ satisfies the theorem. ∎

Two subsets $X, Y \subseteq V(G)$ *touch* if either $X \cap Y \neq \varnothing$ or some vertex in X has a neighbour in Y. Here is another, similar game, in which the cop player has slightly more power. We set (X_0, R_0) as before. At the start of the ith step we have a position (X_{i-1}, R_{i-1}). The cop player chooses a new set $X_i \in [V(G)]^{<k}$ with no restriction on X_i. Then the robber player chooses (if possible) an X_i-flap R_i which touches R_{i-1}. (In all other respects the game is unchanged.) We call this *jump-searching*; and define "$<k$ cops can jump-search G," etc., as before. Analogously to (1.1), and with a similar proof, we have the following.

(1.2) *A graph G cannot be jump-searched by $<k$ cops if and only if there is a function σ mapping each $X \in [V(G)]^{<k}$ to a non-empty union $\sigma(X)$ of X-flaps, such that every X-flap in $\sigma(X)$ touches $\sigma(Y)$ for all $X, Y \in [V(G)]^{<k}$.*

We show that $<k$ cops can search G if and only if they can jump-search it; and if they cannot, then there is a function σ as in (1.2) such $\sigma(X)$ is an X-flap for all X. For this reason, we define a *haven* in G of order k to be a function β which assigns an X-flap $\beta(X)$ to each $X \in [V(G)]^{<k}$, in such a way that $\beta(X)$ touches $\beta(Y)$ for all $X, Y \in [V(G)]^{<k}$. There is a related result of [11] that, for $k \geqslant 2$, G has "branch-width" $>k$ (a parameter similar to tree-width and within a constant factor of it) if and only if G has a "tangle" of order $k+1$.

A subset $X \subseteq V(G)$ is *connected* if $X \neq \emptyset$ and the restriction (denoted by $G|X$) of G to X is connected. A *screen* in G is a set of connected subsets of $V(G)$, mutually touching. A screen $\mathscr{S}$ has *thickness* $\geqslant k$ (and we write $\tau(\mathscr{S}) \geqslant k$) if there is no $X \in [V(G)]^{<k}$ such that $X \cap H \neq \emptyset$ for all $H \in \mathscr{S}$. Screens are really just another way to describe havens, because for instance we have the following.

(1.3) *G has a screen of thickness $\geqslant k$ if and only if G has a haven of order $\geqslant k$.*

Proof. Let $\mathscr{S}$ be a screen in G with $\tau(\mathscr{S}) \geqslant k$. For each $X \in [V(G)]^{<k}$ there exists $H \in \mathscr{S}$ with $X \cap H = \emptyset$. Let $\beta(X)$ be the X-flap containing H (this exists, since H is connected). Then β is a haven of order k. Conversely, let β be a haven of order $\geqslant k$, and let $\mathscr{S} = \{\beta(X) : X \in [V(G)]^{<k}\}$. Then $\mathscr{S}$ is a screen with $\tau(\mathscr{S}) \geqslant k$. ∎

The following is our main result.

(1.4) *Let G be a graph, and $k \geqslant 1$ an integer. Then the following are equivalent:*

(i) *G has a screen of thickness $\geqslant k$,*

(ii) *G has a haven of order $\geqslant k$,*

(iii) $<k$ *cops cannot jump-search* G,

(iv) $<k$ *cops cannot search* G,

(v) $<k$ *cops cannot monotonely search* G,

(vi) G *has tree-width* $\geq k-1$.

Proof. We have seen in (1.3) that (i) $\Rightarrow$ (ii). That (ii) $\Rightarrow$ (iii) follows from (1.2), and it is easy to see that (iii) $\Rightarrow$ (iv). That (iv) $\Rightarrow$ (v) is trivial, and (v) $\Rightarrow$ (vi) is again easy to see. Thus, it remains to show that (vi) $\Rightarrow$ (i), and that is proved in the next section. ∎

2. Screens and Tree-Decompositions

The object of this section is to prove that (vi) $\Rightarrow$ (i) in (1.4). To do so we prove the following, from which (vi) $\Rightarrow$ (i) follows by setting $\mathcal{S}=\emptyset$.

(2.1) *Let* $\mathcal{S}$ *be a screen in* G, *and* $k \geq 1$ *an integer. There is no screen* $\mathcal{S}'$ *in* G *with* $\mathcal{S} \subseteq \mathcal{S}'$ *and* $\tau(\mathcal{S}') \geq k$, *if and only if there is a tree-decomposition* (T, W) *of* G *such that each* $t \in V(T)$ *with* $|W_t| \geq k$ *has valency* 1 *in* T *and satisfies* $W_t \cap H = \emptyset$ *for some* $H \in \mathcal{S}$.

The reason for proving the stronger statement (2.1) is that it permits the use of an inductive argument. For given G, k we assume that the result holds for all screens *larger* than $\mathcal{S}$, and prove that it also holds for $\mathcal{S}$. The proof requires several lemmas about tree-decompositions.

A pair (A, B) of subsets of $V(G)$ is a *separation* of G if $A \cup B = V(G)$ and no edge of G has one end in $A-B$ and the other in $B-A$. Thus, a separation is essentially a tree-decomposition of G using a two-vertex tree. Our first lemma is proved in [10], but we prove it again for completeness.

(2.2) *Let* (T, W) *be a tree-decomposition of a graph* G, *let* $f \in E(T)$ *with ends* t_1, t_2, *let* $T \backslash f$ *have components* T_1, T_2, *and let* $X_i = \bigcup (W_t : t \in V(T_i))$ $(i=1,2)$. *Then* (X_1, X_2) *is a separation of* G *and* $X_1 \cap X_2 = W_{t_1} \cap W_{t_2}$.

Proof. Since $\bigcup (W_t : t \in V(T)) = V(G)$ it follows that $X_1 \cup X_2 = V(G)$. For every edge $e \in E(G)$ there exists $t \in V(T)$ such that e has both ends in W_t, and so either e has both ends in X_1 or in X_2. Thus (X_1, X_2) is a separation. Certainly $W_{t_1} \cap W_{t_2} \subseteq X_1 \cap X_2$. For the converse, let $v \in X_1 \cap X_2$, and choose $s_i \in V(T_i)$ with $v \in W_{s_i}$ $(i=1,2)$. Then t_1, t_2 both lie on the path of T between s_1 and s_2, and so $v \in W_{t_1}, W_{t_2}$. Hence $X_1 \cap X_2 \subseteq W_{t_1} \cap W_{t_2}$, as required. ∎

That there is a connection between screens and tree-decompositions is indicated by the following.

(2.3) *Let (T, W) be a tree-decomposition of G, and let $\mathscr{S}$ be a screen in G. Then there exists $t \in V(T)$ such that $W_t \cap H \neq \emptyset$ for all $H \in \mathscr{S}$.*

Proof. Suppose that for all $t \in V(T)$ there exists $H_t \in \mathscr{S}$ such that $W_t \cap H_t = \emptyset$. Given t, choose $v \in H_t$, choose $t' \in V(T)$ with $v \in W_{t'}$, and let f_t be the first edge of the path of T from t to t'. (We see that $t' \neq t$ since $W_{t'} \cap H_t \neq \emptyset$.) Since $|E(T)| < |V(T)|$, there is an edge f of T with ends t_1, t_2 such that $f = f_{t_1} = f_{t_2}$. Let $T \backslash f$ have components T_1, T_2 where $t_i \in V(T_i)$ $(i = 1, 2)$, and let (X_1, X_2) be as in (2.2). Now some vertex of H_{t_1} is in X_2 by definition of f_{t_1}, and $H_{t_1} \cap (X_1 \cap X_2) = \emptyset$ by (2.2), since $H_{t_1} \cap W_{t_1} = \emptyset$. Since (X_1, X_2) is a separation and H_{t_1} is connected, it follows that $H_{t_1} \subseteq X_2 - X_1$. Similarly $H_{t_2} \subseteq X_1 - X_2$, and so H_{t_1}, H_{t_2} do not touch, a contradiction. The result follows. ▮

Our third lemma is the following.

(2.4) *Let (T, W) be a tree-decomposition of a graph G, and let $t_0 \in V(T)$. Let (A, B) be a separation of G such that $W_{t_0} \subseteq A$. Suppose that there is no $Y \subseteq V(G)$ with $|Y| < |A \cap B|$ meeting every path of G between W_{t_0} and $A \cap B$. Then there is a tree-decomposition (T, W') of $G | B$ with the following properties*:

(i) $W'_{t_0} = A \cap B$,

(ii) *for every* $t \in V(T)$, $|W'_t| \leqslant |W_t|$, *and*

(iii) *if* $t \in V(T)$ *has valency* 1 *and* $t \neq t_0$ *then* $W'_t \subseteq W_t$.

Proof. Let $|A \cap B| = k$. By Menger's theorem there are k paths of G from W_{t_0} to $A \cap B$, mutually vertex-disjoint, $P_1, ..., P_k$ say. Since $|A \cap B| = k$ it follows that each P_i has only one vertex in $A \cap B$, its last vertex, and consequently $V(P_i) \subseteq A$, since $W_{t_0} \subseteq A$. Let P_i have last vertex z_i $(1 \leqslant i \leqslant k)$; then $A \cap B = \{z_1, ..., z_k\}$.

For $t \in V(T)$, define $W'_t = W_t \cap B$ if $t \neq t_0$ and t has valency 1 in T; and otherwise define

$$W'_t = (W_t \cap B) \cup \{z_i : 1 \leqslant i \leqslant k,\ V(P_i) \cap W_t \neq \emptyset\}.$$

Let $W' = (W_t : t \in V(T))$. We show that (T, W') satisfies the theorem.

First, let us verify (W1) and (W2). Let x be a vertex or edge of $G | B$. If x is a vertex let $X = \{x\}$, and if x is an edge let X be the set of ends of x. Then $X \subseteq B$. Choose $t \in V(T)$ with $X \subseteq W_t$. Then $X \subseteq W_t \cap B \subseteq W'_t$. This verifies (W1).

For (W2), let $t, t', t'' \in V(T)$, such that t' lies on the path of T from t to t''. We must show that $W'_t \cap W'_{t''} \subseteq W'_{t'}$. We may assume that $t, t'' \neq t'$, and so t' has valency $\geqslant 2$ in T. Let $v \in W'_t \cap W'_{t''}$. If $v \notin A \cap B$, then

$$v \in (W_t \cap B) \cap (W_{t''} \cap B) \subseteq W_{t'} \cap B \subseteq W'_{t'},$$

as required. We assume then that $v \in A \cap B$, and so $v = z_i$ for some i. Since $z_i \in W'_t$, it follows that $W_t \cap V(P_i) \neq \emptyset$, and similarly $W_{t''} \cap V(P_i) \neq \emptyset$. From (2.2) (applied to an edge incident with t' of the path from t to t'') we deduce that $W_{t'} \cap V(P_i) \neq \emptyset$, and so $v = z_i \in W'_{t'}$, as required. This verifies (W2).

Since $W_{t_0} \cap B \subseteq A \cap B$ and $W_{t_0} \cap V(P_i) \neq \emptyset$ for $1 \leqslant i \leqslant k$, we see that $W'_{t_0} = A \cap B$ and so (i) holds. Statement (iii) holds from the definition of W'_t. To see (ii), let $t \in V(T)$; by (iii) we may assume that $t = t_0$ or t has valency $\geqslant 2$ in T. Let

$$Z = \{z_i : 1 \leqslant i \leqslant k,\ W_t \cap V(P_i) \neq \emptyset\}.$$

Then

$$|Z| \leqslant \sum_{1 \leqslant i \leqslant k} |W_t \cap V(P_i)| \leqslant |W_t \cap A|$$

since each $V(P_i) \subseteq A$ and $P_1, ..., P_k$ are mutually disjoint. But $W'_t = (W_t \cap (B - A)) \cup Z$, and so

$$|W'_t| \leqslant |W_t \cap (B - A)| + |Z| \leqslant |W_t \cap (B - A)| + |W_t \cap A| = |W_t|.$$

Thus (ii) holds, as required. ∎

The last lemma we need is the following.

(2.5) *Let T be a tree, let $s \in V(T)$, and let $T_1, ..., T_r$ be subtrees of T with union T, each containing s and otherwise mutually disjoint. Let G be a graph, let $X \subseteq V(G)$ with r X-flaps $C_1, ..., C_r$, and for $1 \leqslant i \leqslant r$ let (T_i, W^i) be a tree-decomposition of $G|(X \cup C_i)$ with $W^i_s = X$. For $1 \leqslant i \leqslant r$ and each $t \in V(T_i)$, let $W_t = W^i_t$, and let $W = (W_t : t \in V(T))$. Then (T, W) is a tree-decomposition of G.*

Proof. Since $\bigcup_{1 \leqslant i \leqslant r} G|(X \cup C_i) = G$, it follows that (W1) holds. For (W2) let t, t', $t'' \in V(T)$ where t' lies on the path between t and t''. If $t, t'' \in V(T_i)$ for some i, then also $t' \in V(T_i)$ and

$$W_t \cap W_{t''} = W^i_t \cap W^i_{t''} \subseteq W^i_{t'} = W_{t'},$$

as required. We assume then that there is no such i. Without loss of generality we may therefore assume that $t, t' \in V(T_1)$ and $t'' \in V(T_2)$. Now t' lies on the path of T_1 between s and t, and so

$$X \cap W_t = W^i_s \cap W^i_t \subseteq W^i_{t'} = W_{t'}.$$

But $W_t \subseteq X \cup C_1$ and $W_{t''} \subseteq X \cup C_2$, and so

$$W_t \cap W_{t''} \subseteq (X \cup C_1) \cap (X \cup C_2) = X.$$

Hence $W_t \cap W_{t''} \subseteq X \cap W_t \subseteq W_{t'}$, as required. ∎

Proof of (2.1). First, let us show the easier "if" statement. Suppose that there is a screen $\mathscr{S}' \supseteq \mathscr{S}$ with $\tau(\mathscr{S}') \geqslant k$, and let (T, W) be a tree-decomposition of G. By (2.3) there exists $t \in V(T)$ such that $W_t \cap H \neq \varnothing$ for all $H \in \mathscr{S}'$. Hence $|W_t| \geqslant k$ (since $\tau(\mathscr{S}') \geqslant k$) and $W_t \cap H \neq \varnothing$ for all $H \in \mathscr{S}$ (since $\mathscr{S} \subseteq \mathscr{S}'$). Hence (T, W) does not satisfy the conditions of the theorem, as required.

Now we prove the "only if" statement. Let us fix G and k, and let $\mathscr{S}$ be a screen in G such that

(1) *There is no screen $\mathscr{S}'$ in G with $\mathscr{S} \subseteq \mathscr{S}'$ and $\tau(\mathscr{S}') \geqslant k$.*

We show that there is a tree-decomposition as in the theorem. We may assume (by induction on, say $2^{|V(G)|} - |\mathscr{S}|$) that the result holds for all screens in G larger than $\mathscr{S}$.

Choose $X \subseteq V(G)$ with $|X|$ minimum such that

(2) $X \cap H \neq \varnothing$ *for all* $H \in \mathscr{S}$.

By (1), $|X| < k$. If $X = V(G)$ then $|V(G)| < k$ and there is a tree-decomposition of G satisfying the theorem where the tree has only one vertex. We assume then that $X \neq V(G)$. Let the X-flaps be $C_1, ..., C_r$, where $r \geqslant 1$.

We claim that we may assume the following.

(3) *For* $1 \leqslant i \leqslant r$ *there is a tree-decomposition* (T_i, W^i) *of* $G|(X \cup C_i)$ *and a vertex* $t_i \in V(T_i)$ *such that*

- (i) $W^i_{t_i} = X$, *and*
- (ii) *for each* $t \in V(T_i)$ *with* $|W^i_t| \geqslant k$, t *has valency* 1 *in* T_i *and there exists* $H \in \mathscr{S}$ *with* $W^i_t \cap H = \varnothing$.

For let $1 \leqslant i \leqslant r$. Suppose first that $\mathscr{S} \cup \{C_i\}$ is not a screen. Since C_i is connected it follows that there exists $H \in \mathscr{S}$ such that C_i, H do not touch. Let

$$D = \{x \in X : x \text{ has a neighbour in } C_i\}.$$

Then $H \cap (C_i \cup D) = \varnothing$ and $(X, C_i \cup D)$ is a separation of $G|(X \cup C_i)$. Thus in this case (3) holds with T_i a 2-vertex tree.

We may assume then that $\mathscr{S} \cup \{C_i\}$ is a screen. By (1), there is no screen $\mathscr{S}'$ with $\mathscr{S} \cup \{C_i\} \subseteq \mathscr{S}'$ and with $\tau(\mathscr{S}') \geqslant k$. Since $C_i \notin \mathscr{S}$ by (2), it follows from the inductive hypothesis that there is a tree-decomposition (T_i, W) of G such that each $t \in V(T_i)$ with $|W_t| \geqslant k$ has valency 1 in T_i and there exists $H \in \mathscr{S} \cup \{C_i\}$ with $W_t \cap H = \varnothing$. We may assume that (T_i, W) does not satisfy the theorem for $\mathscr{S}$, and so there exists $t_i \in V(T_i)$ such that

(4) $|W_{t_i}| \geqslant k$, $W_{t_i} \cap C_i = \varnothing$, *and* $W_{t_i} \cap H \neq \varnothing$ *for all* $H \in \mathscr{S}$.

We claim also that

(5) *For all $Y \subseteq V(G)$ with $|Y| < |X|$, there is a path between W_{t_i} and X with no vertex in Y.*

For from the minimality of $|X|$, there exists $H \in \mathscr{S}$ with $Y \cap H = \emptyset$. But $W_{t_i} \cap H \neq \emptyset$ from (4), and $X \cap H \neq \emptyset$ from (2), and H is connected; and the claim follows.

Let $A = V(G) - C_i$, $B = X \cup C_i$; then (A, B) is a separation of G and $A \cap B = X$ and $W_{t_i} \subseteq A$. By (2.4), there is a tree-decomposition (T_i, W^i) of $G|(X \cup C_i)$ such that

(i) $W^i_{t_i} = X$,

(ii) for every $t \in V(T_i)$, $|W^i_t| \leqslant |W_t|$, and

(iii) if $t \in V(T_i)$ has valency 1 and $t \neq t_i$, then $W^i_t \subseteq W_t$.

We claim that (T_i, W^i) satisfies (3). For let $t \in V(T_i)$ with $|W^i_t| \geqslant k$. Then $|W_t| \geqslant |W^i_t| \geqslant k$ by (ii); and so t has valency 1 in T_i and there exists $H \in \mathscr{S} \cup \{C_i\}$ with $W_t \cap H = \emptyset$. Since $|X| < k$ and so $t \neq t_i$ by (i), it follows from (iii) that $W^i_t \subseteq W_t$ and so $W^i_t \cap H = \emptyset$. Now $W^i_t \subseteq X \cup C_i$ and $|W^i_t| \geqslant k > |X|$, and so $W^i_t \cap C_i \neq \emptyset$. Hence $H \neq C_i$, and so $H \in \mathscr{S}$. This proves that (T_i, W^i) satisfies (3).

For $1 \leqslant i \leqslant r$, let (T_i, W^i), t_i be as in (3). We may assume that $t_1 = \cdots = t_r = s$ say, and $T_1, ..., T_r$ are otherwise disjoint. Let T be the tree $T_1 \cup \cdots \cup T_r$, and define W as in (2.5). By (2.5), (T, W) is a tree-decomposition of G, and it satisfies the theorem by (3)(ii), as required. ∎

3. An Extension to Infinite Graphs

So far we have only been concerned with finite graphs, but havens and screens in infinite graphs are also of interest (see [12]). Let us extend (1.4) to the infinite case. We define havens, screens, and tree-decompositions for infinite graphs just as in the finite case, except that the tree T in a tree-decomposition (T, W) now may be infinite. However, in contrast with the finite case, a graph may have both a haven of order k and a tree-decomposition of width $< k - 1$. For instance, if G is a 1-way infinite path then it has a haven of every finite order (let $\beta(X)$ be the infinite component of $G \backslash X$) and yet it has a tree-decomposition of width 1. What we need to do is restrict the trees T in our tree-decompositions to be *rayless*, that is, to have no 1-way infinite path. We prove the following. (The cops-and-robber part of (1.4) may also be extended, in the natural way.)

(3.1) *Let G be a (possibly infinite) graph, and $k \geqslant 1$ an integer. Then the following are equivalent*:

(i) *G has no tree-decomposition* (T, W) *of width* $<k-1$ *such that* T *is rayless,*

(ii) *G has a haven of order* $\geqslant k$,

(iii) *G has a screen of thickness* $\geqslant k$.

For the proof we need the following lemma. If T is a tree and $t_1, t_2 \in V(T)$ we denote the vertex set of the path of T between t_1 and t_2 by $T(t_1, t_2)$. A tree-decomposition (T, W) of G is *linked* if for all $n \geqslant 0$ and all $t_1, t_2 \in V(T)$, either there are n mutually vertex-disjoint paths of G between W_{t_1} and W_{t_2}, or $|W_t| < n$ for some $t \in T(t_1, t_2)$.

(3.2) *Let G be a graph, possibly infinite but rayless, and let* $k \geqslant 0$ *be an integer. If every finite subgraph of G has tree-width* $<k$, *then G has a linked tree-decomposition* (T, W) *of width* $<k$ *such that T is rayless.*

Proof.

(1) *G has a linked tree-decomposition* (T, W) *of width* $<k$.

This follows from the theorem of [6].

Choose (T, W) as in (1), and choose $t_0 \in V(T)$. For each $t \in V(T) - \{t_0\}$, let $f(t)$ be the neighbour of t in $T(t_0, t)$. Take a well-order $\leqslant$ of $V(T)$ such that $f(t) \leqslant t$ for all $t \in V(T) - \{t_0\}$. For each $t \in V(T) - \{t_0\}$, let $f'(t)$ be the vertex $t' \in V(T) - \{t\}$ with $W_{t'} = W_{f(t)}$, chosen first in the well-order. Let T' be the tree with $V(T') = V(T)$ in which t and $f'(t)$ are adjacent for each $t \in V(T) - \{t_0\}$. (This is a tree since each $t \in V(T') - \{t_0\}$ has exactly one earlier neighbour.)

(2) (T', W) *is a tree-decomposition of width* $<k$.

It suffices to show that if $t_1, t_2 \in V(T)$ are distinct and not adjacent in T', and $v \in W_{t_1} \cap W_{t_2}$, then $v \in W_t$ for some $t \in T'(t_1, t_2) - \{t_1, t_2\}$. From the symmetry we may assume that $t_1 \leqslant t_2$. Let $t_3 = f(t_2)$ and $t'_3 = f'(t_2)$. Then $t_3 \in T(t_1, t_2)$ and so $v \in W_{t_3} = W_{t'_3}$. Moreover, $t'_3 \in T'(t_1, t_2) - \{t_1, t_2\}$, since $t_1 \leqslant t_2$, and the claim follows.

(3) (T', W) *is linked.*

Let $t_1, t_2 \in V(T)$, such that there do not exist n mutually vertex-disjoint paths of G between W_{t_1} and W_{t_2}. We prove that $|W_t| < n$ for some $t \in T'(t_1, t_2)$ by induction on $|T'(t_1, t_2)|$. From the symmetry, we may assume that $t_1 \leqslant t_2$. Let $t_3 = f(t_2)$ and $t'_3 = f'(t_2)$. Since (T, W) is linked there exists $t \in T(t_1, t_2)$ with $|W_t| < n$. If $t = t_2$ we are done, since $t_2 \in T'(t_1, t_2)$. If not, then $t \in T(t_1, t_3)$, and so there do not exist n mutually vertex-disjoint paths between W_{t_1} and $W_{t_3} = W_{t'_3}$ by (2.2) applied to (T, W) and an edge incident with t of the path from t_1 to t_3 (we remark that the proof of (2.2) works equally well for infinite graphs G and tree-decompositions (T, W) with T infinite). Since $T'(t_1, t'_3) \subsetneq T'(t_1, t_2)$, it follows from our inductive hypothesis that $|W_{t'}| < n$ for some $t' \in T'(t_1, t'_3) \subseteq T'(t_1, t_2)$, as required.

(4) *G has a linked tree-decomposition* (T, W) *of width* $<k$ *such that* $W_{t_1} \neq W_{t_2}$ *for all distinct* $t_1, t_2 \in V(T)$.

Let (T', W) be as in (2) and (3). Let T'' be the graph obtained from T' by deleting each vertex t with $W_t = W_{t'}$ for some $t' < t$. Since each such $t \in V(T')$ has valency 1 in T' from the construction of T', it follows that T'' is a tree; and if $W'' = (W_t : t \in V(T''))$, then (T'', W'') satisfies (4).

(5) *If* (T, W) *is as in* (4) *then T is rayless.*

Suppose that P is a 1-way infinite path of T. Choose n minimum such that $|W_t| = n$ for infinitely many $t \in V(P)$; then, by replacing P by an infinite subpath, we may assume that $|W_t| \geq n$ for all $t \in V(P)$. Choose $t_1, t_2, \ldots \in V(P)$, in order, such that $|W_{t_i}| = n$ for all $i \geq 1$. Since (T, W) is linked, there are n mutually vertex-disjoint paths $P_i^1, \ldots, P_i^n$ of G between W_{t_i} and $W_{t_{i+1}}$ for each $i \geq 1$. It follows easily that

$$\bigcup_{i \geq 1} P_i^1 \cup \cdots \cup P_i^n$$

is the union of n paths $P_1, \ldots, P_n$ say, with $W_{t_i} \subseteq V(P_1 \cup \cdots \cup P_n)$ for each $i \geq 1$. Since $W_{t_1}, W_{t_2}, \ldots$ are all distinct we deduce that $V(P_1 \cup \cdots \cup P_n)$ is infinite and so some P_i is infinite, contrary to the hypothesis. This proves (5).

From (4) and (5) the result follows. ∎

Proof of (3.1). The equivalence of (ii) and (iii) is proved just as (1.3) is proved in the finite case. To see that (iii)$\Rightarrow$(i), let $\mathscr{S}$ be a screen of thickness $\geq k$, and let (T, W) be a tree-decomposition of G such that T is rayless. Now (2.3) can be extended to infinite graphs G and rayless trees T. (In the proof, we replace the statement that $|E(T)| < |V(T)|$ by the statement that since T is rayless there is no injective map of $V(T)$ into $E(T)$ mapping each vertex to an incident edge.) Hence, by this extended form of (2.3), there exists $t \in V(T)$ such that $W_t \cap H \neq \emptyset$ for all $H \in \mathscr{S}$. Since $\tau(\mathscr{S}) \geq k$ it follows that $|W_t| \geq k$; and so (i) holds.

It remains to prove that (i)$\Rightarrow$(iii). Suppose that (iii) is false; we prove that (i) is false.

(1) *G is rayless.*

If P is a 1-way infinite path of G, let $\mathscr{S}$ be the set of all infinite subpaths of P; then $\mathscr{S}$ is a screen in G of thickness $\geq k$, a contradiction.

(2) *Every finite subgraph of G has tree-width* $<k-1$.

Suppose that H is a finite subgraph of G with tree-width $\geq k-1$. By (1.4), H has a screen of thickness $\geq k$, and hence so does G, a contradiction.

Now the result follows from (1), (2), and (3.2). ∎

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