Zombies and Survivors on Graphs

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1 Introduction

This report presents and discusses the Game of Zombies and Survivors on Graphs (Z & S or the Zombie Game), a variation of the classic Game of Cops and Robbers recently studied in [3]. Z & S is a vertex-pursuit game in which players take turns moving tokens on a graph with the objective to capture (or evade) the other player. A graph is a mathematical model used to describe networks, maps and – appropriately enough – board games.

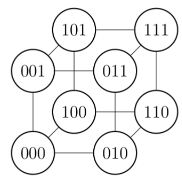


Figure 1: The Cube Graph

Many games can be modelled using graphs. Monopoly, for example, can be seen as a large cycle in which every vertex is joined to the twelve (?) following vertices in clockwise order. Rolling the die determines which edge to follow.

A graph theoretic perspective can be applied to video games, which often involve pathfinding, mazes and chases. In their introduction, the authors of [1] show how to convert the labyrinth in Pacman into a graph: the vertices are the intersections of the maze and vertices are connected by an edge whenever a corridor links the intersections.

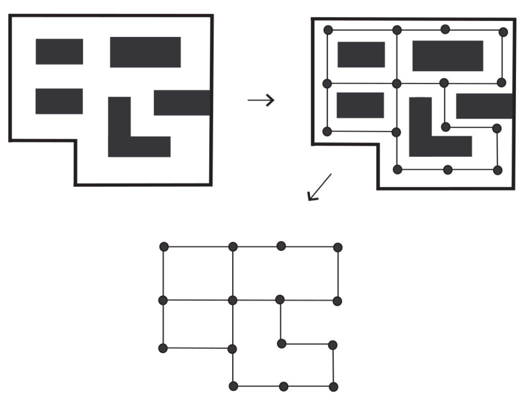


Figure 2: A Maze and its Corresponding Graph [1][p.2]

1.1 How to Play

The Game of Zombies and Survivors is a variant of Cops and Robbers, a well-studied game. See [1] for a comprehensive study of Cops and Robbers.

If familiar with the Game of Cops and Robbers, the reader may skip this section: the only difference between the two games is that the zombies must move along shortest paths towards the survivor. The cops are under no such restriction.

Zombies and Survivors is an adversarial game where one player controls the survivor(s), the other the zombie(s). Usually we keep it simple and have a few zombies chase a single survivor. After all, if a signel survivor can't hope to escape, then adding more is simply gruesome.

Each round, the players take turns moving from vertex to vertex along the edges of a graph. More precisely, the zombies all move, then the survivor moves. The zombies seek to capture the survivor (move onto the same vertex) while the survivor tries to escape indefinitely.

To begin, the zombies choose starting vertices to occupy. Next, the survivor chroooses a start position; ideally, one not too close to any of the zombies. On the next and each following round the zombies move toward the survivor along a shortest path.

The sophistication of the zombie's strategy gives them their name. You can almost imagine the zombies, arms outstretched, ambling directly towards the survivor. Once all the zombies have moved, the survivor can move (flee) to a neighbouring vertex or pass.

The game concludes when either:

- A zombie catches the survivor. That is, a zombie moves onto the vertex occupied by the survivor. This is a zombie win.
- It becomes clear that the survivor will never be caught. In this case we say it is survivor win.

We discuss winning conditions in greater detail in a later section.

The game can be uninteresting depending on the shape of the graph. For example, if there are more zombies than vertices, then the survivor cannot even start. If the board is a cycle, a path or a tree, then the game is decided by the opening round. The goal then becomes to create graphs to keep the game surprising. The outcome of a game in some way describes a feature of a graph: its complexity, its connectivity, or perhaps, its "survivability." Indeed, the number of zombies required to guarantee that the survivor is caught can be seen as a graph parameter.

Before the game begins, the players agree to play on a particular graph with a specific number of zombies. These could be considered parameters of the game.

The graphs are assumed to be finite and connected: that is, there is a finite number of vertices and there exists a path between every pair of vertices. Playing on graphs with multiple connected components does not make much sense in the context of these games. In our analysis, the players have complete information about the graph and the positions of the players.

2 Notation

The following sections will use a few basic ideas and definitions from graph theory. Formally, a graph $G = (V, E)$ is composed of:

- A set V of vertices.
- A set $E \subset V \times V$ of edges described by a pair of endpoints.

If $G = (V, E)$ is a graph and $x, y \in V$ are vertices, we say that vertices x and y are neighbours if $(x, y) \in E$. That is, if there is an edge joining x to y. In these games, the edges are assumed to be undirected and so we may write $xy = yx$ and consider the two directions as a single edge. We call the set of all neighbours of x the neighbourhood of x which we denote $N(x) \subset V$.

[Need to add degree, subgraph, paths, distance, diameter]

For example, in Figure 1 we have vertices $V = \{000, 001, 010, 011, 100, 101, 110, 111\}$. Since 000 and 001 are connected, $(000, 001) \in E$. The neighbourhood of 000 is $N(000) = \{001, 010, 100\}$.

2.1 Modeling the Game

We need a way to identify the players' positions over time so let $z_t^i \in V(G)$ be zombie i on round t. Similarly s_t is the survivor on round t. It might be tempting to group the zombies together into some tuple of the vertex set, but each zombie acts independently of the others and so this may not always be practical.

We typically use only one survivor, so we normally only use a few zombies, and say $i \in \{1, \ldots, k\}$ for "small" values of k.

Indeed, in some cases a single zombie may suffice to capture the survivor. (We examine this scenario in the next section.)

The zombies play, then the survivor plays and these two turns make one round. In particular, when we start the game we allow $z_0 \in V(G)$ to be any vertex of the graph.

2.2 Paths and Moves

The zombie strategy requires that we consider all shortests paths connecting the zombie to the survivor.

For zombie k, write $Z_k = \{\exists \ell : z_k = u_{i,0}, u_{i,1}, u_{i,2}, \ldots, u_{i,\ell-1}, s = u_{i,\ell}\}$ be the set of i different $z_k s$-paths of length ℓ.

There is at least one such path since our graph is presumed connected, so $i > 0$ and $Z_k \neq \emptyset$.

If there is only one path, then z_k's next move is $u_{i,1}$. If all zs-paths include $u_{i,1}$, then again z_k's next move must be to that vertex.

If, however, there are multiple zs-paths which have different first moves, then the zombie could make multiple moves.

We call all the set of all neighbours on a shortest path to the survivor the *zombie moves*, which formally are

$$Z[x] = \{y \in N(x) \mid d(y, s) = d(x, s) - 1$$

In these games we use graph distance (or, equivalently, assume the edges have unit cost). In other words, the distance between two vertices is the hop length of a (shortest possible) path connecting them. (EDIT: The existence and nature of shortest path may warrant further discussion, in lemmas & observations below.)

2.3 Rounds and Turns

We divide the game into rounds and turns. A round consists of two turns: a zombie turn and a survivor turn. This is necessary because we must consider distances between the players at different points of the game: at the beginning and middle of each round.

We track this by counting the turns. It is the zombie's turn on $t \equiv 0 \mod 2$ and the survivor's turn on $t \equiv 1 \mod 2$. Round r is given by $\lfloor \frac{t}{2} \rfloor$.

The game starts on round 0 with the zombies choosing initial vertices. The survivor follows. In a sense the game really begins in round 1 with the zombies finding, selecting and moving along a shortest path. The survivor responds. The game repeats in this way until the survivor is caught, or both players agree that the survivor will always escape.

2.4 Zombie Number

EDIT: Not quite. Redo

The minimum number zombies guaranteed to win.

If a single zombie is guaranteed to win on a graph, we say that it is zombie-win and that it has zombie number $z(G) = 1$. If it can be shown that k zombies win, then $z(G) = k$

3 Lemmas and Observations

3.1 The Zombie Strategy

The zombies are dumb but they see the survivor. The survivor's only hope is to flee forever using infinite stamina.

If a graph is connected, then there must exist some path connecting z, s. Indeed, there must be a shortest such path. But there can also be arbitrarily many shortest paths connecting two vertices.

[insert lattice type graph with many paths between two vertices]

Consider the set of all possible paths between the zombie and the survivor. We are primarily interested in the shortest possible paths since these paths dictate the zombie strategy at the beginning of each round. We sometimes refer to as zs-paths or "zombie-survivor paths".

Note that if there are multiple zs-paths for a given round then they must all have the same length. Otherwise something doesn't add up.

3.2 Keep Your Distance

Lemma 1. Consider an arbitrary zombie z_r at round r. Then

$$d(z_r, s_r) \leq \operatorname{diam}(G)$$

Moreover, for all $r \geq 1$ the sequence $d(z_r, s_r)$ of distances is non-inceasing, i.e.,

$$d(z_{r+1}, s_{r+1}) \leq d(z_r, s_r)$$

Proof. The first part follows from the definition of the diameter of the graph.

If $z_i = s_i$, then $d(z_i, s_i) = 0$ and the game is over. We may consider the game to be finished (and the sequence thus finite and non-increasing) or suppose that the zombie mirrors the survivor forever and thereby obtain a sequence of zeroes (which is non-increasing).

Otherwise, we have $z_i \neq s_i$ and, since G is connected, there exists a shortest $z_i s_i$-path. Say

EDIT: reverse order so that u_1 is next move?

$$P : z_i = u_0, u_1, \ldots, u_k = s_i$$

so that $d(z_i, s_i) = k$.

On round $i + 1$, the zombie must move to $z_{i+1} \in N(z_i)$ such that $d(z_{i+1}, s_i) < d(z_i, s_i)$. In fact, in the graph distance model we have precisely $d(z_{i+1}, s_i) = d(z_i, s_i) - 1$. We can suppose that $z_{i+1} = u_1$ is the next vertex along P.

In response, the survivor moves to $s_{i+1} \in N[s_i]$. Then

$$P' : z_{i+1} = u_1, u_2, \ldots, u_k, s_{i+1}$$

is a $z_{i+1}s_{i+1}$-path of length at most k. So the length of a shortest zombie-survivor path on round $i + 1$ is at most $k = d_i$. So $d_{i+1} \leq d_i$. □

EDIT: Remove next part? Useless. We can consider these distance sequences to be infinite.

Corollary 1. A zombie-win distance sequence has finite support. A survivor-win distance sequence must be eventually constant and $d_k > 1$ for every round $k \geq 1$.

3.3 There is No Hope

Which are the graphs for which a single zombie is guaranteed to capture the survivor?

It is pretty clear that the survivor has little chance if the graph is a a simple path P_n.

[insert graph of long path here haha]

Indeed, any finite acyclic graph is zombie win since an acyclic and connected graph has a unique path connecting any $z, s \in V(T)$. Every step the zombie takes towards the survivor limits the survivor's movement. And options.

Maximally outer-planar graph.

4 Survivor Strategy

Suppose we could agree on some algorithm to fully determine the zombies' behaviour. Or, perhaps, assume that all possible games will exhaustively be played by the computer. How then, should we program the survivor to maximize its chances of survival? On every round, the survivor may stay in place or move to one of its neighbours. However, if ever the survivor moves to a vertex adjacent to a zombie, then it loses immediately on the next round. So the *valid survivor moves* are the neighbours of the survivor (or its current position), minus those adjacent to one of the k zombies.

If the survivor is s and $Z = \{z_1, z_2, \ldots, z_k\}$ is the set of zombie positions, then

$$N[s] \setminus N[Z]$$

Where the neighbourhood of the set Z is the union of all of the zombies' neighbourhoods. These survivor moves can be computed by iterating through the neighbours of s and removing those that are neighbours of a zombie. Another approach would be to use the results of Floyd-Warshall, as with the zombies:

1. Scan row s of A for indexes x where $a_{s,x} = 1$. These are the neighbours of s. Add each neighbour a set S.
2. For each neighbour x and for each zombie z, $1 \leq z \leq k$, probe $a_{z,x}$. This is the distance from the neighbour to the zombie.

3. If $a_{z,x} = 1$, then x is adjacent to a zombie and so $S = S \setminus \{x\}$.
4. Return S

If the set of valid survivor moves is empty, then the survivor is cornered. The only remaining move is to pass, and be caught after another round. If the set returned is a singleton, then circumstances have forced the survivor's hand. If, however, there many possible moves, then how best do we choose among them?

Perhaps the simplest strategy is to invert the strategy used by the zombies: the survivor makes the move that maximizes its distance from all of the zombies. While running the algorithm described above, we could simultaneously compute $\sum_{i=1}^{k} d(x, z_i)$, the sum of all the distances from the neighbour to the zombies, and choose one the moves that maximizes this value.

This cowardly strategy is amusingly similar to that of the zombies. It is also a poor strategy. The only way to escape the zombies is to lead them into some sort of cycle, as we discuss next. So the survivor needs to act with more sophistication than just fleeing in the opposite direction. The game depicted below is an example where the survivor has an easy win, but the strategy above fails.

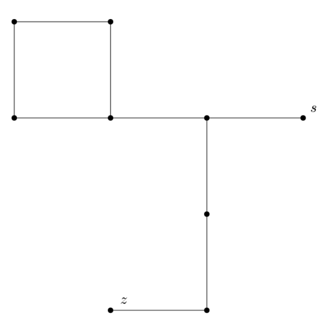


Figure 3: a game in which the cowardly strategy fails

5 Win Condition

The zombies win if they manage to move onto the vertex occupied by the survivor. That's fairly obvious. However, in the section How to Play ZAS, we also said that the game was won by the survivor "when it becomes obvious that the survivor will not be caught."

If played on a finite graph, this necessarily means that the survivor has managed to lead the zombies into a cycle.

6 Zombies on the Grid

EDITS: We must always clearly state when an action or consequence occurs: at the end of a turn or the end of a round. (A round being composed of two turns).

Add Theorem and proof environments.

Another theorem on worst-case capture time? Diameter of grid?

The Grid graph $G_{m,n}$ is a rectangular arrangement of mn vertices in m rows and n columns. Vertices are joined by an edge if they are on the same row or column. The goal of this section is to prove

Theorem 1. The zombie number of the Grid is 2 for any m, n.

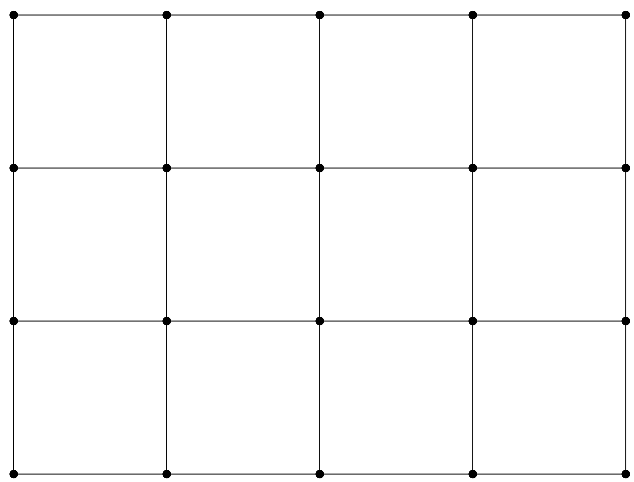


Figure 4: $G_{4,5}$

As a quick aside, note that the grid graph can be constructed by taking the Cartesian product of two paths $P_m \square P_n$. The Cartesian product of G_1 and G_2, denoted $G_1 \square G_2$ is a graph G whose vertices are all pairs of vertices of the two graphs. Two vertex pairs are connected by an edge if they are equal in one component and the other is joined in the original graph. In set notation, that is

$$V(G) = \{(u, v) \mid u \in V(G_1), v \in V(G_2)\}$$
$$E(G) = \{\{(u_1, v_1), (u_2, v_2)\} \mid (u_1 \sim_{G_1} u_2 \wedge v_1 = v_2) \vee (u_1 = u_2 \wedge v_1 \sim_{G_2} v_2)\}$$

We claim that two zombies suffice to win on this family of graphs since they can execute a guarding strategy. To demonstrate this, we need the following observations about shortest paths on the grid.

First, if the zombie and the survivor are on the same row (or column) then there is a single shortest path and hence only one valid zombie move: the zombie moves closer to the survivor along the row (or column).

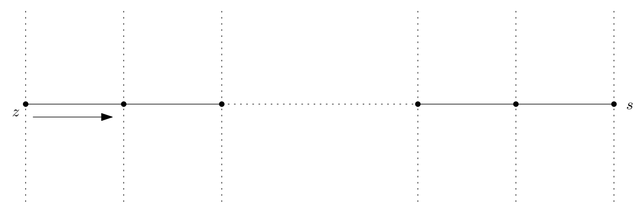


Figure 5: Zombie and Survivor on Same Row

Second, if the zombie and survivor are on different rows and columns then there are at least two shortest paths joining them and exactly two possible zombie moves: horizontal or vertical. The survivor and zombie are assumed on different rows and columns so the zombie can make progress in one or the other direction.

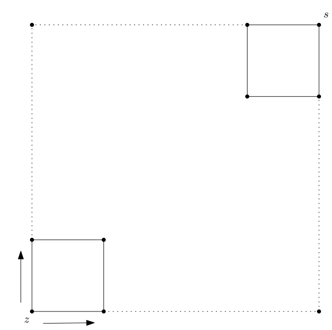


Figure 6: Zombie and Survivor on Different Row and Column

We now show that two zombies can play a shadowing strategy which is guaranteed to capture the survivor. We mimic a proof strategy from Cops and Robbers in which we show that the Robber Territory is shrinking at every round. We analoguously define the Survivor Territory $S_j \subset V(G)$ as the set of vertices to which the survivor may move on turn j without being eaten by a zombie.

To enact the shadowing strategy, the zombies may choose any starting position (and so the set of winning zombie starts $Z_W(G_{m,n}) = V(G_{m,n})$ in Fitzpatrick's notation – so the grid belongs to the family of graphs for which any zombie starting position will work – Fitzpatrick asks if there is a characterization of graphs for which any start will win. We do not have an answer but observe that the grid belongs to this family for 2 zombies). Each zombie will shadow the survivor's position along an axis: one horizontal, the other vertical.

6.1 The Shadowing Strategy

Proof. Let us consider one zombie at a time, say the zombie which will shadow the survivor's vertical shadow (i.e., its column), since the other zombie's behaviour is symmetric. We show that the zombie will eventually capture the vertical shadow – that the zombie will close the horizontal distance between the survivor and the zombie – and that, once it does so, the zombie can always recapture the survivor's shadow after the survivor moves.

Assuming that the zombie and survivor are on different columns, then the zombie may move one column closer since the vertex on the same row but one column closer lies on a shortest path to the survivor. If they are on the same row, then that is the only possible zombie move. In response, the survivor may:

- Remain in place in which case the zombie has closed the horizontal distance by 1.
- Move vertically (up or down) in which case again the zombie has closed the horizontal distance by 1.
- Move horizontally towards the zombie in which case the horizontal distance is reduced by 2.
- Move horizontally away from the zombie in which case the horizontal distance is preserved.

In the first three scenarios, the horizontal distance between the two adversaries has been reduced. The fourth scenario in which the survivor moves away cannot occur indefinitely since the grid is finite. Thus, in at most n rounds (the number of columns) the zombie will capture the survivor's vertical shadow.

Suppose now that the zombie has captured the survivor's vertical shadow; that they are now on the same column. It is clear that the zombie can recapture the survivor's shadow no matter how the survivor moves:

- If the survivor moves vertically or remains in place, then the zombie must move vertically and the survivor's vertical shadow remains captured.
- If the survivor moves horizontally, then the zombie may choose to mimic the move and thereby recapture the vertical shadow.

This argument shows that after a finite number of moves a zombie may capture the survivor's vertical shadow. Now observe that once the survivor's vertical shadow has been captured, the survivor can never enter the zombie-occupied row: any attempts to go around the zombie are immediately blocked. (Expand and clarify?)

6.2 Shrinking the Survivor Territory

By the previous discussion, after $\max\{m, n\}$ rounds, both the survivor's vertical and horizontal shadows have been captured. Suppose we have reached a point in the game where the zombies have moved and captured both horizontal and vertical shadows.

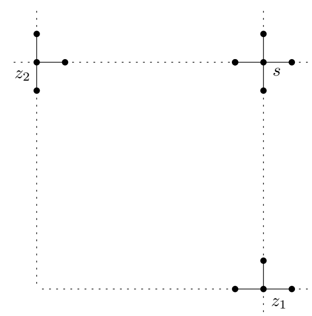


Figure 7: Once Both Shadows Are Captured

The survivor has five possible moves:

- Stay in place, in which case both zombies have a single shortest path and move closer along their current row/column. So the Survivor Territory has shrunk by a column and a row.
- Move vertically, in which case the zombie capturing the vertical shadow has no choice but to move closer, while the other zombie recaptures the horizontal shadow. Here the Survivor Territory has shrunk by one row.
- Move horizontally, in which case the zombie capturing the horizontal shadow has not choice but to move closer, while the other zombie recaptures the vertical shadow. Now the Survivor Territory has shrunk by one column.

In every scenario, at least one of the zombies is forced to move one step closer to the survivor. Since the survivor can never enter the rows and columns occupied by the shadowing zombies, this means that the Survivor Territory shrinks by at least one row or one column at every round. Since the grid is finite, the Survivor Territory is eventually empty and hence the survivor is captured.

□

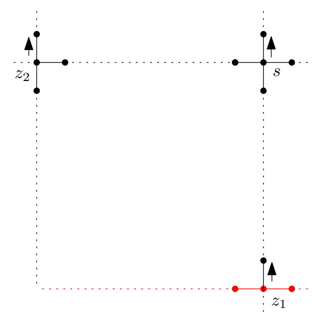


Figure 8: Shrinking Survivor Territory: The survivor moves up when shadowed; the zombie guarding the vertical shadow must move up, thereby eliminating the row below. The zombie guarding the horizontal shadow also moves up, so its column remains blocked.

6.3 Defining Shadowing Rigorously

Label the vertices of the grid using the integer points of the first quadrant of the plane and consider $z_1, z_2, s \in V(G_{m,n})$ as points in $[0, m-1] \times [0, n-1]$. Rows and columns are then first and second coordinates of points in a finite lattice. Say $s^j = (x_0^j, y_0^j), z_1^j = (x_1^j, y_1^j), z_2^j = (x_2^j, y_2^j)$ are the positions of the players s, z_1 and z_2 on round $j \geq 0$. The players cannot escape the bounds of the grid and so $0 \leq x_i^j \leq n-1$ and $0 \leq y_i^j \leq m-1$ for $i \in \{0, 1, 2\}$ and for $j \geq 0$.

Let's show that – after a finite number of turns – a zombie can mirror the survivor's x-coordinate. Formally, there exists a round $k \geq 0$ such that $x_0^k = x_1^{k+1}$ and that for all $j > k$, $x_0^j = x_1^{j+1}$. That is to say, from turn k onwards, the zombie can always move onto the x-projection of the survivor on its turn.

Note that have $N[z_i^j] \subseteq \{(x_i^j, y_i^j), (x_i^j \pm 1, y_i^j), (x_i^j, y_i^j \pm 1)\}$ and the inclusion is strict when z_i^j is on the boundary of the grid.

Suppose we already have $x_0^k = x_1^k$. As mentioned above, in this case the zombie has a single shortest path to $(x_1^k, y_1^k \pm 1)$ where for simplicity we will assume that the move is “upwards” to $(x_1^k, y_1^k + 1)$. The zombie has moved onto the survivor's x-coordinate. The survivor has five possible responses.

Now, we can assume that $x_0^j > x_1^j$ (the opposite being symmetric). The zombie may follow two shortest paths, one of which is to $(x_1^j + 1, y_1^j)$. ...

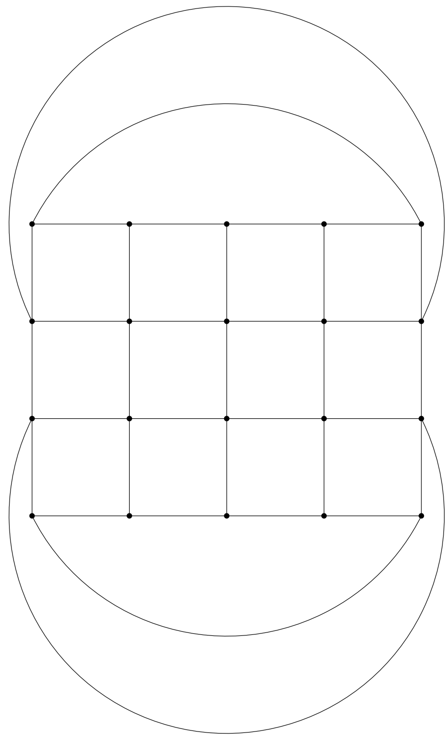


Figure 9: $C_{4,5}$

7 Zombies on a Cylinder

The Cylinder graph $C_{m,n}$ is a rectangular arrangement of mn vertices in m rows and n columns much like the Grid, except that vertices on one boundary edge are joined to vertices on the opposite side.

Again, the Cylinder graph can be considered as the Cartesian product $C_{m,n} = P_m \square C_n$ of a cycle and a path. Note that this a planar graph for any m and n.

We claim now that three zombies suffice to win on this family of graphs since they can execute a guarding strategy similar to the one detailed in the previous section.

Place two zombies on a row such that $d(z_1, z_2) = \lfloor \frac{n}{2} \rfloor$. Now observe that if the survivor finishes its turn on a different row, the zombies may move to a vertex of the same column but closer row. If the survivor finishes its turn on the same row, then the zombies have a single zombie move on the same row but to a closer column.

Thus, after a finite number of rounds, the zombies can shadow the survivor's horizontal shadow. Indeed, if we place these two zombies in the middle of the cylinder, then the survivor's horizontal shadow is captured in at most $\lceil \frac{m}{2} \rceil$ rounds.

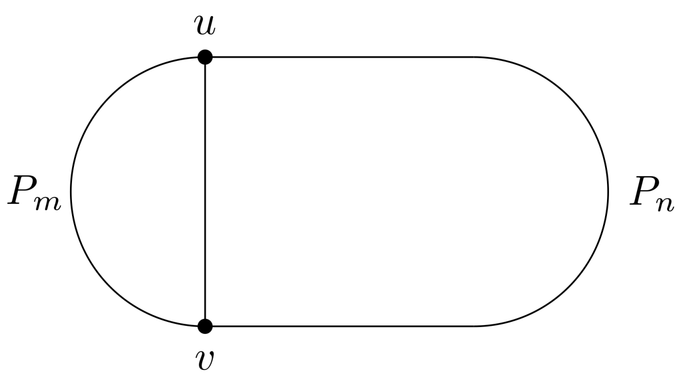
The survivor is now trapped between these two zombies since they can always move to recapture its horizontal shadow. However, it could alternate between rows and thereby defeat the shadowing zombies.

We add another zombie to capture the survivor's vertical shadow. Once the zombie's horizontal and vertical shadows are captured, the survivor cannot remain on the same row and is unable to change row indefinitely and thus will be cornered.

8 Cycle With One Chord

We analyze the Game of Zombies & Survivors on a cycle with a single chord.

Definition 1. Take a cycle of length $m+n$ and add a chord which divides the cycle into paths P_m and P_n of lengths m and n. Without loss of generality $m \leq n$. We denote such a cycle as $Q_{m,n}$.



Theorem 2. The zombie number of a cycle $Q_{m,n}$ $(3 \leq m \leq n)$ with a chord dividing the cycle into paths of lengths m and n is 2.

Proof. Denote as P_m and P_n the paths of lengths m and n respectively. We think of $Q_{m,n}$ as embbeded in the plane with P_m – the shortest side – on the left. This does not limit the generality of the following and allows us to define (counter-)clockwise distance: the length of the path along a cycle with respect to this embedding.

Setting $m=n=1$ gives K_2 with two added loops, which is zombie-win.

With $m=n=2$ we have two adjacent cliques K_3 which are dominated by a single vertex, so it is also zombie-win.

For $m = 2$ and $n \geq 4$, 2 zombies win by starting on diametrically opposed vertices on the cycle C_{n+2}.

If $m = n = 3$ the zombie number is 2 since two zombies on the chord enpoints dominate the graph.

For $m = 3$, $n = 4$, the zombie number is also 2: placing the zombies on the endpoints of the chord divides the graph into C_4 and C_5 and the zombies clearly win from this position.

The same strategy works for $Q_{3,6}$, $Q_{4.4}$, $Q_{4,5}$ and $Q_{5,5}$ but it does not work for $Q_{3,7}$, $Q_{4,6}$ nor indeed for any $Q_{m,n}$ for $m \geq 3$ and $n \geq 6$.

We seek a winning zombie strategy (that is, a zombie start) for $m \geq 3$, $n \geq 6$. The chord is the crux of the game, so first we assume that one zombie is on the chord and another at some distance Δ while the survivor is somewhere on P_m. We know the first zombie chases the survivor around the cycle, so we need to control the arrival of the second zombie so that the survivor cannot escape, nor can it trick the second zombie into spinning the same direction as the first.

Second, we show how to position the zombies at the start of the game so that – no matter where the survivor starts – a losing position is guaranteed. Either the survivor is stuck on a path between the two zombies (so that capture is obviously inevitable) or the survivor will be pushed into the carefully orchestrated scenario described in the first part of the proof.

Lastly, we show that such a starting position is always available to the zombies for any $m \geq 3$, $n \geq 6$.

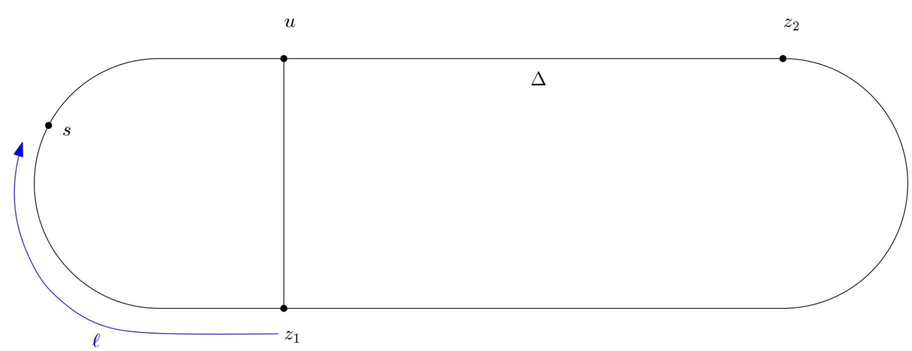
Note that if P_1 and P_2 are two possible zs-paths with distinct next moves and

$$|P_1| \leq |P_2|$$

then in the following argument we suppose that the zombie follows $|P_1|$ since that is a valid move.

Part 1. Cornering the Survivor on the Smallest Cycle

Suppose that the game has reached the following state:



- the first zombie is on an endpoint of the chord, say v
- there are Δ vertices counting clockwise from u to z_2.
- the survivor is on P_m at a distance of ℓ vertices counting clockwise from v.

By comparing the lengths of different paths, we calculate the values of Δ which guarantee that the survivor will be cornered on P_m. That is to say, the survivor will be intercepted by z_2 before it can reach any vertex in $Q_{m,n} \setminus P_m$.

Denote as ℓ the length of the clockwise path from v to s. Note that we must have $2 \le \ell \le m-1$ else z_1 captures the survivor on the next round.

We can assume that once z_1 chooses a direction from v that it will continue in that direction: either the zombie has no choice or both directions around the cycle are of the same length (and so may continue in the same direction).

We can also assume that on its turn the survivor will move away from z_1 and maintain a distance of ℓ (or $m-\ell+1$, if they are moving counter-clockwise) since a winning survivor strategy which involves waiting a turn or moving backwards is equivalent to a survivor strategy which always moves but starts with a smaller (or larger) value of ℓ.

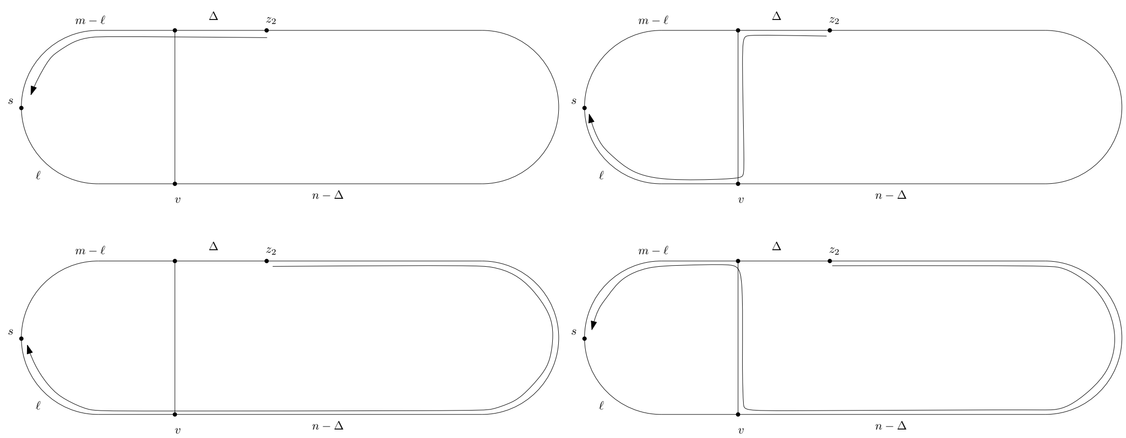
These two assumptions allow us to "fast-forward" the game by Δ rounds and determine when the survivor is captured.

Since z_1 is already on the same cycle as the survivor, it has two options:

A. z_1 goes clockwise if $\ell \leq 1 + m - \ell$. Combined with the bounds on ℓ, this gives $4 \leq 2\ell \leq m + 1$

B. z_1 goes counter-clockwise if $1 + m - \ell \leq \ell$. Combined with the bounds on ℓ, we obtain $m + 1 \leq 2\ell \leq 2m - 2$

There are four possible shortest paths for z_2 to the survivor:



- P_a of length $\Delta + (m - \ell)$
- P_b of length $\Delta + 1 + \ell$
- P_c of length $(n - \Delta) + 1 + (m - \ell)$
- P_d of length $(n - \Delta) + \ell$

Comparing path lengths we see that:

I. z_2 moves counter-clockwise if either $|P_a| \leq \min\{|P_c|, |P_d|\}$ or $|P_b| \leq \min\{|P_c|, |P_d|\}$.

II. z_2 goes clockwise if either $|P_c| \leq \min\{|P_a|, |P_b|\}$ or $|P_d| \leq \min\{|P_a|, |P_b|\}$.

We will examine all combinations of the possible decisions made by the zombies from this configuration:

- I. z_2 goes counter-clockwise
- II. z_2 goes clockwise.

- A. z_1 goes clockwise
- B. z_1 goes counter-clockwise

Case I.A We have the following constraint on ℓ from assumption A.

$$4 \leq 2\ell \leq m+1$$

and the following constraints on Δ from assumption I.

$$\begin{aligned} \Delta + (m-\ell) \leq& n - \Delta + 1 + m - \ell & \text{and} \\ \Delta + (m-\ell) \leq& n - \Delta + \ell \end{aligned}$$

or

$$\begin{aligned} \Delta + 1 + \ell \leq& n - \Delta + 1 + m - \ell & \text{and} \\ \Delta + 1 + \ell \leq& n - \Delta + \ell \end{aligned}$$

These can be simplified with a bit of algebra and assumption A:

$$\begin{aligned} 2\Delta \leq& n+1 & \text{and} \\ 2\Delta \leq& n - m + 2\ell \leq n+1 \end{aligned}$$

or

$$\begin{aligned} 2\Delta \leq& n + m - 2\ell & \text{and} \\ 2\Delta \leq& n - 1 \leq n + m - 2\ell \end{aligned}$$

So for z_2 to follow either P_a or P_b and go counter-clockwise we must have

$$\begin{aligned} 2\Delta \leq& n - m + 2\ell & \text{or} \\ 2\Delta \leq& n - 1 \end{aligned}$$

Next we consider: which of s or z_2 reaches u first? If $\Delta = m - \ell$ both z_2 and s reach u on the same round, with the survivor moving onto the zombie-occupied vertex (and losing).

If we have $\Delta = m - \ell + 1$, then s reaches u first but is caught by z_2 on the following round. So, to guarantee the survivor has not escaped P_m we need

$$\Delta \leq m - \ell + 1$$

otherwise the survivor can reach the chord at least two rounds before z_2 can move to block. We wish to prevent this scenario since the survivor could then take the chord and possibly escape, pulling both zombies into a loop either on C_m or C_n. This constraint on Δ guarantees that the survivor cannot escape C_m before z_2's arrival in Case I.A.

That is not sufficient, however. We must also ensure that z_2 moves counter-clockwise (opposite to z_1) once it reaches u in order to trap the survivor. So we need

$$m - \ell - \Delta \leq 1 + \Delta + \ell$$

Or, in terms of Δ,

$$2\Delta \geq m - 2\ell - 1$$

When we combine all the restrictions we obtain

Case I.A. Summary

z_1 goes clockwise:

$$4 \leq 2\ell \leq m + 1$$

and z_2 goes counter-clockwise

$$\begin{aligned} 2\Delta &\leq n - m + 2\ell && \text{or} \\ 2\Delta &\leq n - 1 \end{aligned}$$

the zombies win:

$$\begin{aligned} 2\Delta &\leq 2m - 2\ell + 2 && \text{and} \\ m - 2\ell - 1 &\leq 2\Delta \end{aligned}$$

Case I.B From assumption B and the constraint on ℓ, we must have

$$m + 1 \leq 2\ell \leq 2m - 2$$

and the constraints on Δ from assumption I are again:

$$\begin{aligned} \Delta + (m - \ell) &\leq n - \Delta + 1 + m - \ell && \text{and} \\ \Delta + (m - \ell) &\leq n - \Delta + \ell \end{aligned}$$

or

$$\begin{aligned} \Delta + 1 + \ell \leq & n - \Delta + 1 + m - \ell & \text{and} \\ \Delta + 1 + \ell \leq & n - \Delta + \ell \end{aligned}$$

These can be simplified using assumption B:

$$\begin{aligned} 2\Delta \leq & n + 1 \leq n - m + 2\ell & \text{and} \\ 2\Delta \leq & n - m + 2\ell \end{aligned}$$

or

$$\begin{aligned} 2\Delta \leq & n + m - 2\ell \leq n - 1 & \text{and} \\ 2\Delta \leq & n - 1 \end{aligned}$$

So for z_2 to go counter-clockwise in this case we must have

$$\begin{aligned} 2\Delta \leq & n + 1 & \text{or} \\ 2\Delta \leq & n + m - 2\ell \end{aligned}$$

Again we must consider who reaches the chord first. We have assumed that z_1 is going counter-clockwise. If $\ell = \Delta$, then z_2 reaches u and s reaches v on the same round, and therefore s will be caught on the next. Therefore, to guarantee the survivor has not escaped P_m in this scenario we need

$$\Delta \leq \ell$$

otherwise the survivor reaches the chord before z_2 and could escape.

Then, to ensure that z_2 traps the survivor by going clockwise once it reaches u we need

$$\begin{aligned} 1 + \ell - \Delta \leq & \Delta - 1 + m - \ell + 1 \\ 2\ell - m + 1 \leq & 2\Delta \end{aligned}$$

Case I.B. Summary

z_1 goes counter-clockwise:

$$m + 1 \leq 2\ell \leq 2m - 2$$

and z_2 goes counter-clockwise

$$\begin{aligned} 2\Delta &\leq n+1 && \text{or} \\ 2\Delta &\leq n+m-2\ell \end{aligned}$$

the zombies win:

$$\begin{aligned} 2\Delta &\leq 2\ell \\ 2\ell - m + 1 &\leq 2\Delta \end{aligned}$$

Case II.A We have the following constraint on ℓ from assumption A.

$$4 \leq 2\ell \leq m+1$$

and the following constraints on Δ from assumption II.

$$\begin{aligned} n - \Delta + \ell &\leq \Delta + (m-\ell) && \text{and} \\ n - \Delta + \ell &\leq \Delta + 1 + \ell \end{aligned}$$

or

$$\begin{aligned} n - \Delta + 1 + m - \ell &\leq \Delta + (m-\ell) && \text{and} \\ n - \Delta + 1 + m - \ell &\leq \Delta + 1 + \ell \end{aligned}$$

These can be simplified with a bit of algebra:

$$\begin{aligned} n - m + 2\ell &\leq 2\Delta && \text{and} \\ n - 1 &\leq 2\Delta \end{aligned}$$

or

$$\begin{aligned} n + 1 &\leq 2\Delta && \text{and} \\ n + m - 2\ell &\leq 2\Delta \end{aligned}$$

These inequalites are of the form

$$\begin{aligned} n - x &\leq 2\Delta && \text{and} \\ n - 1 &\leq 2\Delta \end{aligned}$$

or

$$
\begin{aligned}
n+x &\leq 2\Delta \qquad\qquad \text{and} \\
n+1 &\leq 2\Delta
\end{aligned}
$$

Where $x = m - 2\ell$.

Supposing $x \geq 0$, we have

$$
\begin{aligned}
n-x &\leq n+x \leq 2\Delta \qquad\qquad \text{and} \\
n-1 &< n+1 \leq 2\Delta
\end{aligned}
$$

and take the lowest bounds because of the disjunction, so that $2\Delta \geq n - x = n - m + 2\ell$ and $2\Delta \geq n - 1$ suffices.

Since assumption A gives $m - 2\ell \geq -1$, supposing $x < 0$ reduces the inequalites to

$$
\begin{aligned}
n+1 &\leq 2\Delta \qquad\qquad \text{and} \\
n-1 &\leq 2\Delta
\end{aligned}
$$

which is satisfied by $2\Delta \geq n - x = n - m + 2\ell$ and $2\Delta \geq n - 1$.

Thus z_2 will go clockwise under assumption A if

$$
\begin{aligned}
2\Delta &\geq n - m + 2\ell \qquad\qquad \text{and} \\
2\Delta &\geq n - 1
\end{aligned}
$$

We have assumed that z_1 is going clockwise. If $m - \ell = n - \Delta$, then z_2 reaches v and s reaches u on the same round and s will be caught on the next. Therefore, to guarantee the survivor has not escaped P_m in this scenario we need

$$
\begin{aligned}
n - \Delta &\leq m - \ell \\
\Delta &\geq n - m + \ell
\end{aligned}
$$

otherwise the survivor could reach the chord before z_2.

After $n - \Delta$ rounds, we have (insert diagram)

Then, to ensure that z_2 goes counter-clockwise once it reaches v, we need

$$
\begin{aligned}
1 + m - \ell - (n - \Delta) &\leq n - \Delta + \ell \\
2\Delta &\leq 2n + 2\ell - m - 1
\end{aligned}
$$

All together this gives *Case II.A. Summary*

z_1 goes clockwise:

$$4 \leq 2\ell \leq m+1$$

and z_2 goes clockwise

$$\begin{aligned} n-m+2\ell \leq& 2\Delta \qquad \text{and} \\ n-1 \leq& 2\Delta \end{aligned}$$

the zombies win:

$$\begin{aligned} 2\Delta \geq& 2n-2m+2\ell \\ 2\Delta \leq& 2n+2\ell-m-1 \end{aligned}$$

Case II.B We have the following constraint on ℓ from assumption B.

$$m+1 \leq 2\ell \leq 2m-2$$

and the following constraints on Δ from assumption II.

$$\begin{aligned} n-\Delta+\ell \leq& \Delta+(m-\ell) \qquad \text{and} \\ n-\Delta+\ell \leq& \Delta+1+\ell \end{aligned}$$

or

$$\begin{aligned} n-\Delta+1+m-\ell \leq& \Delta+(m-\ell) \qquad \text{and} \\ n-\Delta+1+m-\ell \leq& \Delta+1+\ell \end{aligned}$$

These can be simplified further with a bit of algebra:

$$\begin{aligned} n-m+2\ell \leq& 2\Delta \qquad \text{and} \\ n-1 \leq& 2\Delta \end{aligned}$$

or

$$\begin{aligned} n+1 \leq& 2\Delta \qquad \text{and} \\ n+m-2\ell \leq& 2\Delta \end{aligned}$$

We have

$$\begin{aligned} n-\Delta+\ell \leq& \Delta+(m-\ell) \qquad \text{and} \\ n-\Delta+\ell \leq& \Delta+1+\ell \end{aligned}$$

or

$$
\begin{aligned}
n - \Delta + 1 + m - \ell \leq & \Delta + (m - \ell) && \text{and} \\
n - \Delta + 1 + m - \ell \leq & \Delta + 1 + \ell
\end{aligned}
$$

These can be simplified further with a bit of algebra:

$$
\begin{aligned}
n - m + 2\ell \leq & 2\Delta && \text{and} \\
n - 1 \leq & 2\Delta
\end{aligned}
$$

or

$$
\begin{aligned}
n + 1 \leq & 2\Delta && \text{and} \\
n + m - 2\ell \leq & 2\Delta
\end{aligned}
$$

These inequalites are of the form

$$
\begin{aligned}
n - x \leq & 2\Delta && \text{and} \\
n - 1 \leq & 2\Delta
\end{aligned}
$$

or

$$
\begin{aligned}
n + 1 \leq & 2\Delta && \text{and} \\
n + x \leq & 2\Delta
\end{aligned}
$$

Where $x = m - 2\ell$. Now since assumption B gives $m - 2\ell \leq -1$, we see that

$$
\begin{aligned}
n - 1 \leq & n - x \leq 2\Delta \\
& \text{or} \\
n + x \leq & n + 1 \leq 2\Delta
\end{aligned}
$$

Now we consider: which of s or z_2 reaches v first? If $n - \Delta = \ell$, then they both reach u at the same time, with the survivor moving onto the z_2-occupied vertex (and losing). If we have $n - \Delta = \ell + 1$, then s reaches u first but is caught by z_2 on the following round. So, to guarantee the survivor has not escaped P_m we need

$$n - \Delta \leq \ell + 1$$

otherwise the survivor reaches the chord before z_2 can move to block. If the survivor reaches the chord first, then it could take the chord and possibly escape. (more detail??)

Then, to ensure that z_2 takesgoes clockwise once it reaches v, we need

$$\begin{aligned}\ell-(n-\Delta) \leq& 1+(n-\Delta-1)+(m-\ell+1)\\ 2\Delta \leq& 2n+m-2\ell+1\end{aligned}$$

Case II.B. Summary

z_1 goes counter-clockwise:

$$m+1 \leq 2\ell \leq 2m-2$$

and z_2 goes clockwise

$$n+1 \leq 2\Delta$$

the zombies win:

$$\begin{aligned}n-\Delta \leq& \ell+1\\ 2\Delta \leq& 2n+m-2\ell+1\end{aligned}$$

Part 2. Forcing the Survivor into a Losing Position. We now consider the game on this graph in general and show how we can guarantee the survivor will be caught.

Given m, n and Δ as computed below, we place the zombies on C_{n+1} so that the zombies move in opposite direction wherever the survivor may start. We need only consider the cycle C_{n+1} since, if the survivor starts on $C_{m+1} \setminus \{u, v\}$, then the zombies play as though the survivor is on u or v.

We choose k such that positioning

1. z_2 at $\Delta+k$ clockwise from u
2. z_1 at k counter-clockwise from v

forces the survivor into a losing position: it is either immediately sandwiched on C_{n+1}, or falls into the trap described above on C_{m+1}.

The survivor cannot start next to the zombies else it loses right away. So we choose k such that, even if the survivor is as far away from one of the zombies as possible on C_n, then the zombies still move in opposite directions. This leads to the following inequalities

$$
\begin{aligned}
n - \Delta - 2k - 2 \leq& \Delta + k + 1 + k + 2 && \text{and} \\
\Delta + 2k - 1 \leq& n - \Delta - 2k + 2
\end{aligned}
$$

Solving for k gives

$$n - 2\Delta - 5 \leq 4k \leq n - 2\Delta + 3$$

Such k guarantees that the zombies start on vertices such that they must move in opposite directions if the survivor plays on C_n.

If the survivor starts between the zombies such that access to the chord is blocked, then clearly it has lost. Otherwise, the zombies must move towards the chord and in k rounds we reach the scenario described in Part 1 when z_1 reaches the chord and z_2 is Δ away. With suitable Δ, then, the survivor cannot win.

Part 3. Computing the Winning Zombie Start

Given m and n, we choose Δ so that whenever we reach the scenario described in the first part, the survivor will be cornered. Such Δ must satisfy the following constraints for any possible value of ℓ.

Case I.A. Summary

z_1 goes clockwise:

$$4 \leq 2\ell \leq m + 1$$

and z_2 goes counter-clockwise

$$
\begin{aligned}
2\Delta \leq& n - m + 2\ell && \text{or} \\
2\Delta \leq& n - 1
\end{aligned}
$$

the zombies win:

$$
\begin{aligned}
2\Delta \leq& 2m - 2\ell + 2 && \text{and} \\
m - 2\ell - 1 \leq& 2\Delta
\end{aligned}
$$

Case I.B. Summary

z_1 goes counter-clockwise:

$$m + 1 \leq 2\ell \leq 2m - 2$$

and z_2 goes counter-clockwise

$$\begin{aligned} 2\Delta \leq& n+1 \qquad\qquad\qquad\qquad \text{or} \\ 2\Delta \leq& n+m-2\ell \end{aligned}$$

the zombies win:

$$\begin{aligned} 2\Delta \leq& 2\ell \\ 2\ell - m + 1 \leq& 2\Delta \end{aligned}$$

Case II.A. Summary

z_1 goes clockwise:

$$4 \leq 2\ell \leq m+1$$

and z_2 goes clockwise

$$\begin{aligned} n-m+2\ell \leq& 2\Delta \qquad\qquad\qquad\qquad \text{and} \\ n-1 \leq& 2\Delta \end{aligned}$$

the zombies win:

$$\begin{aligned} 2\Delta \geq& 2n-2m+2\ell \\ 2\Delta \leq& 2n+2\ell-m-1 \end{aligned}$$

Case II.B. Summary

z_1 goes counter-clockwise:

$$m+1 \leq 2\ell \leq 2m-2$$

and z_2 goes clockwise

$$n+1 \leq 2\Delta$$

the zombies win:

$$\begin{aligned} n-\Delta \leq& \ell+1 \\ 2\Delta \leq& 2n+m-2\ell+1 \end{aligned}$$

A simple algorithm to calculate possible values of Δ loops over $0 \leq \Delta \leq n$ and over $2 \leq \ell \leq m-1$ and tests, for each Δ and each ℓ, to determine which of the four cases is applicable and, if in one of the cases, whether the zombie-win constrains are satisfied. A value of Δ is accepted if, for every value of ℓ, the zombies win.

Once we have obtained possible Δ, we can then determine k by calculating the bounds

$$n-2\Delta-5 \leq 4k \leq n-2\Delta+3$$

□

9 Existence of Winning Start

We wish to show that, for any m, n, there exist Δ and k which guarantee the survivor is caught. First we show that $\Delta = \lfloor \frac{m}{2} \rfloor$ always works for the cornering strategy.

Note that

$$2\Delta = 2\left\lfloor \frac{m}{2} \right\rfloor = \begin{cases} m & \text{if } m \text{ is even} \\ m-1 & \text{if } m \text{ is odd} \end{cases}$$

and so $m - 1 \leq 2\lfloor \frac{m}{2} \rfloor \leq m$.

Suppose that we are in Case I. A. and $\Delta = \lfloor \frac{m}{2} \rfloor$. Case I. A is characterized by the following constraints:

$$4 \leq 2\ell \leq m + 1$$

and

$$2\Delta \leq n - m + 2\ell$$

or

$$2\Delta \leq n - 1$$

The zombies win if

$$\begin{aligned} 2\Delta \leq& 2m - 2\ell + 2 & \text{and} \\ m - 2\ell - 1 \leq& 2\Delta \end{aligned}$$

So if we are in Case I. A. and $\Delta = \lfloor \frac{m}{2} \rfloor$ the zombies win since

$$\begin{aligned} 2\Delta = 2\lfloor \frac{m}{2} \rfloor \leq m < 2m - (m+1) + 2 \leq 2m - 2\ell + 2 & \qquad \text{and} \\ m - 2\ell - 1 \leq m - 5 < 2\lfloor \frac{m}{2} \rfloor = 2\Delta \end{aligned}$$

Which shows that the zombie-win requirements are met.

Suppose now that we are not in Case 1. A. Negating the constraints of Case I. A. gives

$$2\Delta \geq n - m + 2\ell + 1$$

and

$$2\Delta \geq n - 1 + 1$$

or

$$m + 1 \leq 2\ell \leq 2m - 2$$

If we assume that m is odd and $2\Delta \geq n$ then we obtain a contradiction since

$$2\Delta = 2\lfloor \frac{m}{2} \rfloor = m - 1 \geq n$$

and we have assumed that $m \leq n$.

If m even, $m = n$ and $2\Delta \geq n - m + 2\ell + 1$ then

$$\begin{aligned} 2\Delta &\geq n - m + 2\ell + 1 \\ m &\geq m - m + 2\ell + 1 \\ m &\geq 2\ell + 1 \\ 2\ell &\leq m - 1 \end{aligned}$$

So, if $m = n$ and they are even, then we are in Case 1. A unless $2\ell \leq m - 1$.

To recap: If we set $\Delta = \lfloor \frac{m}{2} \rfloor$, we are in Case 1.A unless

$$m = n \qquad \text{and they are even}$$
$$\Delta = \lfloor \frac{m}{2} \rfloor = \frac{m}{2}$$
$$4 \leq 2\ell \leq m - 1$$

Now, can we be in Case 1. B? Case 1. B is described by the following constraints:

$$m + 1 \leq 2\ell \leq 2m - 2$$

and

$$2\Delta \leq n + 1$$

or

$$2\Delta \leq n + m - 2\ell$$

The negation of which is:

$$2\Delta \geq n + 1 + 1$$

and

$$2\Delta \geq n + m - 2\ell + 1$$

or

$$4 \leq 2\ell \leq m + 1$$

But this leads to the contradiction:

$$n \geq m \geq 2\Delta \geq n + 2$$

In remains to check if we win in Case 2. A.

Assuming still that

$$m = n \qquad \text{they are even}$$
$$\Delta = \frac{m}{2}$$
$$4 \leq 2\ell \leq m - 1$$

The win conditions require

$$\begin{aligned} 2n - 2m + 2\ell \leq 2\Delta &\leq 2n + 2\ell - m - 1 \\ 2m - 2m + m - 1 \leq 2\Delta &\leq 2m + 4 - m - 1 \\ m - 1 \leq 2\Delta &\leq m + 3 \end{aligned}$$

Which holds for $\Delta = \frac{m}{2}$.

10 Literature Review

- Cops and Robbers results?
- Deterministic zombies
- Probabilistic zombies

References

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[2] M. Aigner and M. Fromme, "A game of cops and robbers," *Discrete Applied Mathematics*, vol. 8, no. 1, pp. 1–12, 1984.

[3] S. Fitzpatrick, J. Howell, M. Messinger, and D. Pike, "A deterministic version of the game of zombies and survivors on graphs," *Discrete Applied Mathematics*, vol. 213, pp. 1–12, 2016.

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[5] C. Birch, "Understanding pac-man ghost behavior," http://gameinternals.com/post/2072558330/understanding-pac-man-ghost-behavior, December 2010, (Accessed on 14/08/2017).

[6] K. H. Rosen, "Discrete mathematics and its applications," *AMC*, vol. 10, p. 12, 2007.

1 $Q_{m,n}$ Appendices

1 Simplifying z_2's inequalities for Case II.A.

We have

$$\begin{aligned} n - \Delta + \ell \leq& \Delta + (m - \ell) && \text{and} \\ n - \Delta + \ell \leq& \Delta + 1 + \ell \end{aligned}$$

or

$$\begin{aligned} n - \Delta + 1 + m - \ell \leq& \Delta + (m - \ell) && \text{and} \\ n - \Delta + 1 + m - \ell \leq& \Delta + 1 + \ell \end{aligned}$$

These can be simplified further with a bit of algebra:

$$\begin{aligned} n - m + 2\ell \leq& 2\Delta && \text{and} \\ n - 1 \leq& 2\Delta \end{aligned}$$

or

$$\begin{aligned} n + 1 \leq& 2\Delta && \text{and} \\ n + m - 2\ell \leq& 2\Delta \end{aligned}$$

These inequalites are of the form

$$\begin{aligned} n - x \leq& 2\Delta && \text{and} \\ n - 1 \leq& 2\Delta \end{aligned}$$

or

$$\begin{aligned} n + x \leq& 2\Delta && \text{and} \\ n + 1 \leq& 2\Delta \end{aligned}$$

Where $x = m - 2\ell$.

Supposing $x \geq 0$, we have

$$\begin{aligned} n - x \leq& n + x \leq 2\Delta && \text{and} \\ n - 1 \leq& n + 1 \leq 2\Delta \end{aligned}$$

Whereas if $x < 0$, then from assumption A we must have $m - 2\ell = -1$, so that our constraints reduce to

$$\begin{aligned} n + 1 \leq& 2\Delta && \text{and} \\ n - 1 \leq& 2\Delta \end{aligned}$$

2 Simplifying z_2's inequalities for Case II.B.

We have

$$\begin{aligned} n - \Delta + \ell \leq& \Delta + (m - \ell) && \text{and} \\ n - \Delta + \ell \leq& \Delta + 1 + \ell \end{aligned}$$

or

$$\begin{aligned} n - \Delta + 1 + m - \ell \leq& \Delta + (m - \ell) && \text{and} \\ n - \Delta + 1 + m - \ell \leq& \Delta + 1 + \ell \end{aligned}$$

These can be simplified further with a bit of algebra:

$$\begin{aligned} n - m + 2\ell \leq& 2\Delta && \text{and} \\ n - 1 \leq& 2\Delta \end{aligned}$$

or

$$\begin{aligned} n + 1 \leq& 2\Delta && \text{and} \\ n + m - 2\ell \leq& 2\Delta \end{aligned}$$

These inequalites are of the form

$$\begin{aligned} n - x \leq& 2\Delta && \text{and} \\ n - 1 \leq& 2\Delta \end{aligned}$$

or

$$\begin{aligned} n + 1 \leq& 2\Delta && \text{and} \\ n + x \leq& 2\Delta \end{aligned}$$

Where $x = m - 2\ell$. Now since assumption B gives $m - 2\ell \leq -1$, we see that

$$\begin{aligned} n - 1 \leq& n - x \leq 2\Delta \\ &\text{or} \\ n + x \leq& n + 1 \leq 2\Delta \end{aligned}$$