Assumption A
Assumption 3
Assumption I
etc.

You have not defined the notation C_{n+2}

Chapter 3

Cycle With One Chord

Games played on cycles are straightforward: if the zombies are too close, the survivor can lead the zombies in the same direction around the cycle. Otherwise, the zombies are too far apart and whichever side (sub-path of the cycle with the zombies at the end) the survivor may choose, the zombies will move in opposite directions and win. In this Chapter, we investigate the game on cycles augmented by a single chord.

be more precise. Can you give exact bounds on the distance?

Definition 1. Let $m, n \in \mathbb{Z}$ with $2 \leq m \leq n$. Consider a cycle of length $m + n$ and add a chord which divides the cycle into paths P_m and P_n of lengths m and n, respectively. We denote such a graph by $Q_{m,n}$.

See Figure 3.1 for an illustration of $Q_{7,8}$. The construction contains three sub-cycles which the survivor could use to fool the zombies. Let us first examine the construction for small values of m and n.

take the proofs out of the statements.

Lemma 3. Setting $m = n = 2$ yields a graph with two adjacent cliques K_3, which are dominated by a single vertex so it is zombie-win.

Lemma 4. For $m = 2$ and $n \geq 3$, two zombies win by starting on diametrically opposed vertices on the cycle C_{n+2}. The additional edge has no impact on a 2-zombie cycle strategy.

Lemma 5. For $3 \leq m \leq n \leq 5$, the zombie number is also two: placing the zombies on the endpoints of the chord divides the graph into two cycles of length at most 5 which can be guarded by two adjacent vertices.

For larger values of m and n the outcome is not as clear. Unfortunately for the survivor, we are able to show the existence of starting positions for the zombies (obtained as a function of m, n) which limits the survivor's options and prevents the zombies from being led in the same direction.

+ add a sentence saying that one zombie cannot win.

Theorem 6. Let $m, n \in Z$ with $3 \leq m \leq n$. The zombie number of $Q_{m,n}$ is 2.

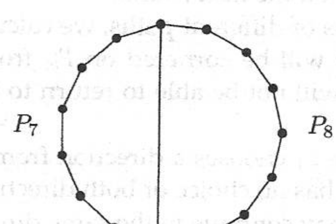


FIGURE 3.1: $Q_{7,8}$

Should we give labels to the most important equations in this chapter?

what is Δ?

We imagine $Q_{m,n}$ as embbeded in the plane with P_m – the shortest side – on the left. This does not limit the generality of the following and allows us to define (counter-)clockwise distance: the length of the path along a cycle with respect to the given direction on this embedding.

not part of the statement

Proof. First, observe that 1 zombie will not suffice for any graph containing an isometric subcycle of length at least 4. ~~Thus $Q_{2,2}$ is the only cycle with a chord which is zombie-win.~~ Second, the observations above show the statement to be true for ~~$m = 2$ and for~~ $3 \leq m \leq n \leq 5$, so for the remainder of the proof we assume that $m \geq 3$ and $n \geq 6$.

We seek a winning starting position for the zombies for $m \geq 3$ and $n \geq 6$. We describe a strategy in three separate parts, which we summarize here.

* First we ~~will~~ show how to position the zombies to guarantee a win ~~if the survivor is on P_m~~. We can find the intervals of Δ which guarantee the survivor will be sandwiched on P_m by considering all possible combinations of directions "chosen" by the zombies (refer to Part 3.1). The zombies' choice of direction is not really a choice, after all: the choice is forced by the position of the survivor and the length of the possible zombie-survivor paths.

assuming the survivor starts on P_m.

Next, we show how to position the zombies at the start of the game so that no matter where the survivor starts a losing position is guaranteed: we offset the zombies on the larger cycle with an additional parameter k, which ensures the zombies are not too close together and therefore guard C_{n+1} (refer to Part 3.2). After k rounds, the survivor will have no choice but to retreat to the smaller cycle and fall into the carefully orchestrated trap described in the first part of the proof.

In Part 3.3, we show that such a starting position is available to the zombies for any $m \geq 3$, $n \geq 6$. Finally in Part 3.4 we use these results to give winning Δ and k for a sample of m, n values.

3.1 Cornering the Survivor on C_{m+1}

You have not defined this notation

Part 1. Suppose that the game has reached the following state:

- the first zombie is on an endpoint of the chord, say v
- there are Δ vertices counting clockwise from u to z_2.
- the survivor is on P_m at a distance of ℓ vertices counting clockwise from v.

You have not defined vertices u and N. They should appear on Figure 3.1.

This configuration is illustrated in Figure 3.2. Note that we must have

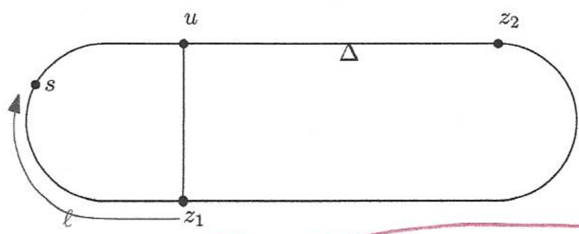
$$2 \leq \ell \leq m - 1$$

else z_1 captures the survivor on the next round.

By comparing the lengths of different paths, we calculate the values of Δ which guarantee that the survivor will be cornered on P_m from this start configuration. That is to say, the survivor will not be able to return to the endpoints of the chord before z_2.

We can assume that once z_1 chooses a direction from v, ~~that~~ any of it continues in that direction: either the zombie has no choice or both directions around the cycle are of the same length (and so z_1 may continue in the same direction). * *

We can also assume that on its turn the survivor will move away from z_1 and maintain a distance of ℓ (or $m - \ell + 1$, if they are moving counter-clockwise) since because you have not defined Case II yet!



add v to the figure

v

FIGURE 3.2: z_1 on v, s somewhere on P_m

a winning survivor strategy which involves waiting a turn or moving backwards is equivalent to a survivor strategy which always moves but starts with a smaller (or larger) value of ℓ.

These two assumptions allow us to "fast-forward" the game by Δ rounds (or $n - \Delta$ rounds, if we are in Case II) and determine when the survivor is captured. Since z_1 is already on the same sub-cycle as the survivor, there are two possibilites:

A. z_1 goes clockwise if $\ell \leq 1 + m - \ell$. Combined with the bounds on ℓ, this gives $4 \leq 2\ell \leq m + 1$

B. z_1 goes counter-clockwise if $1 + m - \ell \leq \ell$. Combined with the bounds on ℓ, we obtain $m + 1 \leq 2\ell \leq 2m - 2$

We must consider four possible paths from z_2 to the survivor:

- P_a of length $\Delta + (m - \ell)$
- P_b of length $\Delta + 1 + \ell$
- P_c of length $(n - \Delta) + 1 + (m - \ell)$
- P_d of length $(n - \Delta) + \ell$

These paths are illustrated in Figure 3.3.
Comparing path lengths we see that:

I. z_2 moves counter-clockwise if either

$$|P_a| \leq \min\{|P_c|, |P_d|\} \quad \text{or} \quad |P_b| \leq \min\{|P_c|, |P_d|\}$$

.

II. z_2 goes clockwise if either

$$|P_c| \leq \min\{|P_a|, |P_b|\} \quad \text{or} \quad |P_d| \leq \min\{|P_a|, |P_b|\}$$

We will examine all combinations of these possible "zombie-decisions" to show that there exist values of Δ which prevent the survivor's escape in any of the possible games (from this start configuration where the survivor is on P_m). We break it down as follows:

- I. z_2 goes counter-clockwise

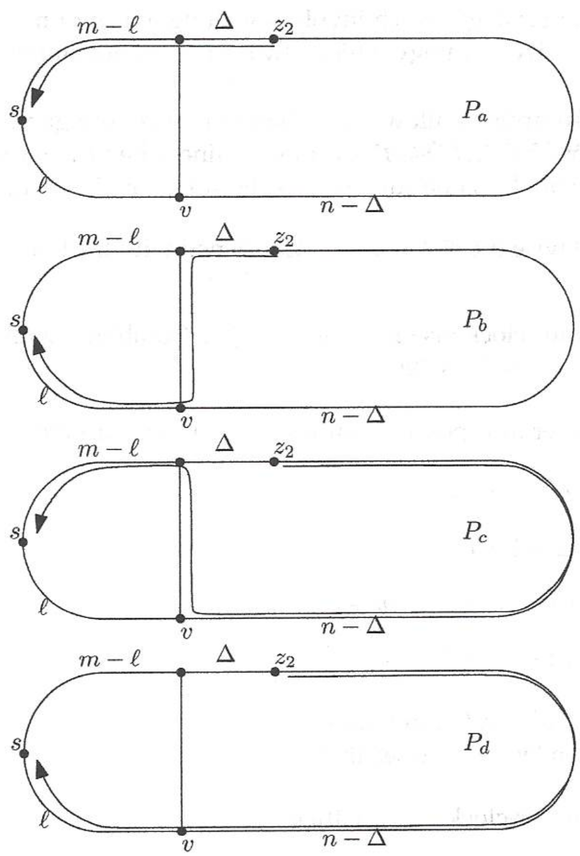


FIGURE 3.3: Possible paths from z_2 to s

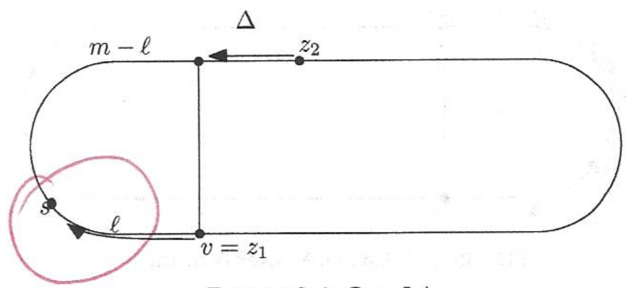


FIGURE 3.4: Case I.A.

- II. z_2 goes clockwise.
- A. z_1 goes clockwise
- B. z_1 goes counter-clockwise
- *Case I.A.* z_2 goes counter-clockwise and z_1 goes clockwise.
  Suppose the zombies will move as in Figure 3.4.
  We obtain the following constraints on ℓ from assumption A.

$$4 \leq 2\ell \leq m+1$$

and the following constraints on Δ from assumption I.

$$\Delta + (m-\ell) \leq n - \Delta + 1 + m - \ell \quad \text{and}$$
$$\Delta + (m-\ell) \leq n - \Delta + \ell$$

or

$$\Delta + 1 + \ell \leq n - \Delta + 1 + m - \ell \quad \text{and}$$
$$\Delta + 1 + \ell \leq n - \Delta + \ell$$

So that together with assumption A we can obtain:

$$2\Delta \leq n+1 \quad \text{and}$$
$$2\Delta \leq n - m + 2\ell \leq n+1$$

or

$$2\Delta \leq n + m - 2\ell \quad \text{and}$$
$$2\Delta \leq n - 1 \leq n + m - 2\ell$$

So for z_2 to follow either P_a or P_b and go counter-clockwise we must have

$$2\Delta \leq n - m + 2\ell \quad \text{or}$$
$$2\Delta \leq n - 1$$

We must determine which of s or z_2 reaches u first. Consider the game after Δ rounds, as illustrated in Figure 3.5.

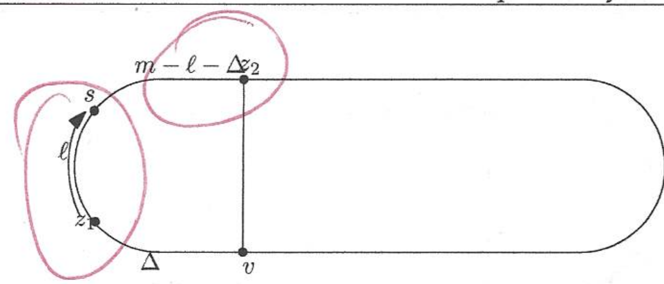


FIGURE 3.5: Case I.A. after Δ rounds

If $\Delta = m - \ell$ both z_2 and s reach u on the same round, with the survivor moving onto the zombie-occupied vertex (and losing). If we have $\Delta = m - \ell + 1$, then s reaches u first but is caught by z_2 on the following round. So, to guarantee the survivor has not escaped P_m we need

$$\Delta \leq m - \ell + 1$$

otherwise the survivor can reach the chord at least two rounds before z_2 can move to block. We wish to prevent this scenario since the survivor could then take the chord and possibly escape, pulling both zombies into a loop either on C_{m+1} or C_{n+1}.

That is not sufficient, however. We must also ensure that z_2 moves counter-clockwise (opposite to z_1) once it reaches u in order to trap the survivor. So we need

$$m - \ell - \Delta \leq 1 + \Delta + \ell$$

Or, in terms of Δ,

$$2\Delta \geq m - 2\ell - 1$$

When we combine all the restrictions we obtain the following characterization for Case I.A.:

z_1 goes clockwise:

$$4 \leq 2\ell \leq m + 1$$

and z_2 goes counter-clockwise:

$$2\Delta \leq n - m + 2\ell \qquad \text{or} \qquad 2\Delta \leq n - 1$$

the zombies win:

$$2\Delta \leq 2m - 2\ell + 2 \qquad \text{and} \qquad m - 2\ell - 1 \leq 2\Delta$$

- *Case I.B* z_2 and z_1 both go counter-clockwise.

  Suppose the zombies will move as in Figure 3.6.

  From assumption B and the constraint on ℓ, we must have

$$m + 1 \leq 2\ell \leq 2m - 2$$

  and the constraints on Δ from assumption I are again:

$$\Delta + (m - \ell) \leq n - \Delta + 1 + m - \ell \qquad \text{and}$$
$$\Delta + (m - \ell) \leq n - \Delta + \ell$$

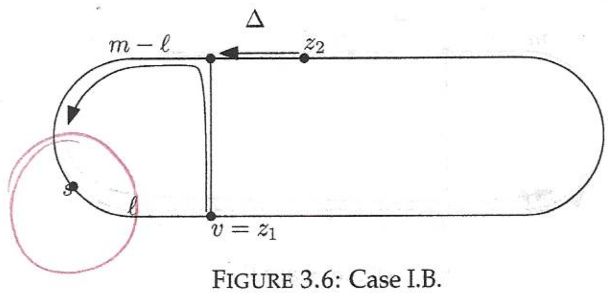


FIGURE 3.6: Case I.B.

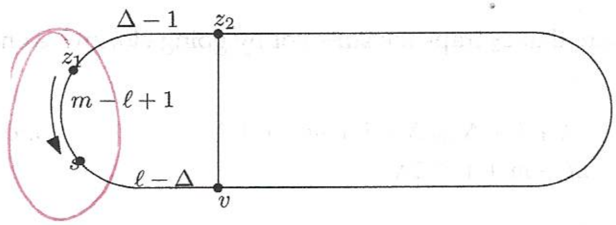


FIGURE 3.7: Case I.B. after Δ rounds

or

$$\begin{aligned} \Delta + 1 + \ell &\leq n - \Delta + 1 + m - \ell && \text{and} \\ \Delta + 1 + \ell &\leq n - \Delta + \ell \end{aligned}$$

Simplifying using assumption B yields:

$$\begin{aligned} 2\Delta &\leq n + 1 \leq n - m + 2\ell && \text{and} \\ 2\Delta &\leq n - m + 2\ell \end{aligned}$$

or

$$\begin{aligned} 2\Delta &\leq n + m - 2\ell \leq n - 1 && \text{and} \\ 2\Delta &\leq n - 1 \end{aligned}$$

So for z_2 to go counter-clockwise in this case we must have

$$\begin{aligned} 2\Delta &\leq n + 1 && \text{or} \\ 2\Delta &\leq n + m - 2\ell \end{aligned}$$

Again we must consider who reaches the chord first. Consider the game after Δ rounds, as illustrated in Figure 3.7.

If $\ell = \Delta$, then z_2 reaches u and s reaches v on the same round, and therefore s will be caught on the next. Therefore, to guarantee the survivor has not escaped P_m in this scenario we need

$$\Delta \leq \ell$$

Otherwise, the survivor reaches the chord before z_2 and could escape.

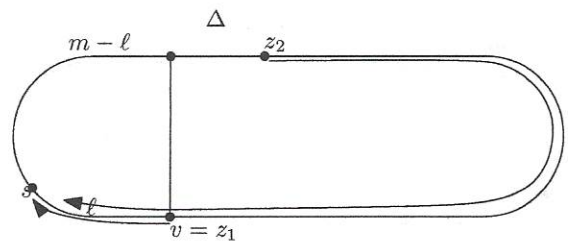


FIGURE 3.8: Case II.A.

Then, to ensure that z_2 traps the survivor by going clockwise once it reaches u we need

$$1+\ell-\Delta \leq \Delta-1+m-\ell+1 \quad \text{and}$$
$$2\ell-m+1 \leq 2\Delta$$

We obtain the following characterization for Case I.B.:

z_1 goes counter-clockwise:

$$m+1 \leq 2\ell \leq 2m-2$$

and z_2 goes counter-clockwise

$$2\Delta \leq n+1 \quad \text{or} \quad 2\Delta \leq n+m-2\ell$$

the zombies win:

$$2\Delta \leq 2\ell \quad \text{and} 2\ell-m+1 \leq 2\Delta$$

- *Case II.A* z_2 and z_1 both go clockwise.

  Suppose the zombies will move as in Figure 3.8.

  We have the following constraint on ℓ from assumption A.

$$4 \leq 2\ell \leq m+1$$

and the following constraints on Δ from assumption II.

$$n-\Delta+1+m-\ell \leq \Delta+m-\ell \quad \text{and}$$
$$n-\Delta+1+m-\ell \leq \Delta+1+\ell$$

or

$$n-\Delta+\ell \leq \Delta+m-\ell \quad \text{and}$$
$$n-\Delta+\ell \leq \Delta+1+\ell$$

Simplifying with a bit of algebra yields:

$$n-m+2\ell \leq 2\Delta \quad \text{and}$$
$$n-1 \leq 2\Delta$$

or

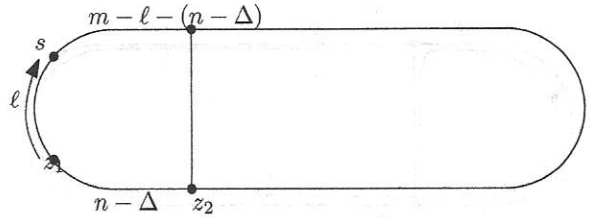


FIGURE 3.9: Case II.A. after $n-\Delta$ rounds

$$n+1 \leq 2\Delta \quad \text{and}$$
$$n+m-2\ell \leq 2\Delta$$

These inequalites are of the form

$$n-x \leq 2\Delta \quad \text{and}$$
$$n-1 \leq 2\Delta$$

or

$$n+x \leq 2\Delta \quad \text{and}$$
$$n+1 \leq 2\Delta$$

Where $x = m - 2\ell$.

Supposing $x \geq 0$, we have

$$n-x \leq n+x \leq 2\Delta \quad \text{and}$$
$$n-1 < n+1 \leq 2\Delta$$

and take the lowest bounds because of the disjunction, so that

$$2\Delta \geq n-x = n-m+2\ell \qquad \text{and} \qquad 2\Delta \geq n-1$$

suffices.

Since assumption A gives $m-2\ell \geq -1$, supposing $x < 0$ reduces the inequalites to

$$n+1 \leq 2\Delta \quad \text{and}$$
$$n-1 \leq 2\Delta$$

which is satisfied by $2\Delta \geq n-x = n-m+2\ell$ and $2\Delta \geq n-1$.

Thus z_2 will go clockwise under assumption A if

$$2\Delta \geq n-m+2\ell \quad \text{and}$$
$$2\Delta \geq n-1$$

Consider the game after $n-\Delta$ rounds, as illustrated in Figure 3.9.

We have assumed that z_1 is going clockwise. If $m-\ell = n-\Delta$, then z_2 reaches v and s reaches u on the same round and s will be caught on the next. Therefore,

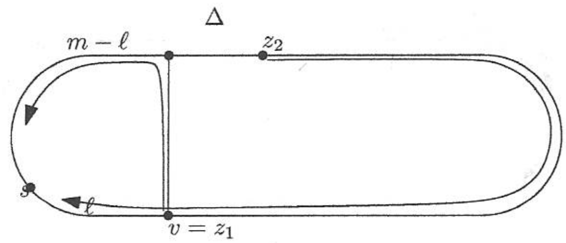


FIGURE 3.10: Case II.B.

to guarantee the survivor has not escaped P_m in this scenario we need

$$\begin{aligned} n - \Delta &\leq m - \ell \qquad \text{and} \\ \Delta &\geq n - m + \ell \end{aligned}$$

otherwise the survivor could reach the chord before z_2.

To ensure that z_2 goes counter-clockwise once it reaches v, we need

$$\begin{aligned} 1 + m - \ell - (n - \Delta) &\leq n - \Delta + \ell \\ 2\Delta &\leq 2n + 2\ell - m - 1 \end{aligned}$$

We obtain the following characterization for Case II.B.:

z_1 goes clockwise:

$$4 \leq 2\ell \leq m + 1$$

and z_2 goes clockwise

$$n - m + 2\ell \leq 2\Delta \qquad \text{and} \qquad n - 1 \leq 2\Delta$$

the zombies win:

$$2\Delta \geq 2n - 2m + 2\ell \qquad \text{and} \qquad 2\Delta \leq 2n + 2\ell - m - 1$$

- *Case II.B.* z_2 goes clockwise and z_1 goes counter-clockwise.
  Suppose the zombies will move as in Figure 3.10.
  We have the following constraint on ℓ from assumption B.

$$m + 1 \leq 2\ell \leq 2m - 2$$

and the following constraints on Δ from assumption II.

$$\begin{aligned} n - \Delta + 1 + m - \ell &\leq \Delta + m - \ell \qquad \text{and} \\ n - \Delta + 1 + m - \ell &\leq \Delta + 1 + \ell \end{aligned}$$

or

$$\begin{aligned} n - \Delta + \ell &\leq \Delta + m - \ell \qquad \text{and} \\ n - \Delta + \ell &\leq \Delta + 1 + \ell \end{aligned}$$

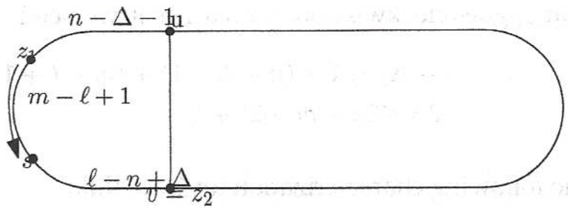


FIGURE 3.11: Case II.B. after $n - \Delta$ rounds

These can be simplified with a bit of algebra:

$$\begin{aligned} n - m + 2\ell &\leq 2\Delta \qquad \text{and} \\ n - 1 &\leq 2\Delta \end{aligned}$$

or

$$\begin{aligned} n + 1 &\leq 2\Delta \qquad \text{and} \\ n + m - 2\ell &\leq 2\Delta \end{aligned}$$

These inequalities are of the form

$$\begin{aligned} n - x &\leq 2\Delta \qquad \text{and} \\ n - 1 &\leq 2\Delta \end{aligned}$$

or

$$\begin{aligned} n + 1 &\leq 2\Delta \qquad \text{and} \\ n + x &\leq 2\Delta, \end{aligned}$$

where $x = m - 2\ell$. Since assumption B gives $m - 2\ell \leq -1$, we see that

$$n - 1 \leq n + 1 \leq n - x \leq 2\Delta$$

or

$$n + x \leq n - 1 \leq n + 1 \leq 2\Delta$$

Consider the game after $n - \Delta$ rounds, as illustrated in Figure 3.11.

If $n - \Delta = \ell$, then they both reach u at the same time, with the survivor moving onto the z_2-occupied vertex (and losing). If we have $n - \Delta = \ell + 1$, then s reaches u first but is caught by z_2 on the following round. So, to guarantee the survivor has not escaped P_m we need

$$n - \Delta \leq \ell + 1$$

otherwise the survivor reaches the chord before z_2 can move to block. If the survivor reaches the chord first, then it could take the chord and possibly escape.

To ensure that z_2 goes clockwise once it reaches v, we need

$$\begin{aligned}\ell - (n-\Delta) &\leq 1 + (n-\Delta-1) + (m-\ell+1)\\ 2\Delta &\leq 2n + m - 2\ell + 1\end{aligned}$$

We obtain the following characterization for Case II.B.:

z_1 goes counter-clockwise:

$$m+1 \leq 2\ell \leq 2m-2$$

and z_2 goes clockwise:

$$n+1 \leq 2\Delta$$

the zombies win:

$$n-\Delta \leq \ell+1 \qquad \text{and} \qquad 2\Delta \leq 2n+m-2\ell+1$$

We will show (in Section 3.4) that with $\Delta = \lfloor \frac{m}{2} \rfloor$, the zombies can always (successfully) execute this cornering strategy. Of course, this is not sufficient to show the zombies win: there is no guarantee that the survivor will choose to start along P_m as is assumed here, so we cannot simply start with this zombie configuration. Instead, we must force the survivor's hand.

3.2 Guarding the large cycle C_{n+1}

Part 2. We consider the game on this type of graph in general and show how we can position the zombies on C_{n+1} to limit the survivor's options and thereby guarantee it will be caught.

Choose k such that positioning

1. z_2 at $\Delta + k$ clockwise from u
2. z_1 at k counter-clockwise from v

forces the survivor into a losing position: it is either immediately sandwiched on C_{n+1}, or falls into the trap described above on C_{m+1}.

The survivor cannot start next to the zombies else it loses right away. So we choose k such that, even if the survivor is as far away from one of the zombies as possible on C_n, then the zombies still move in opposite directions (refer to Figure 3.12). This leads to the following inequalities

$$\begin{aligned}n-\Delta-2k-2 &\leq \Delta+k+1+k+2 \qquad \text{and}\\ \Delta+2k-1 &\leq n-\Delta-2k+2.\end{aligned}$$

Solving for k gives

$$n-2\Delta-5 \leq 4k \leq n-2\Delta+3$$

A choice of k which satisfies these constraints guarantees that the zombies move in opposite directions if the survivor starts on C_n.

Now consider the survivor's options when the zombies start as described here. If the survivor starts between the zombies such that access to the chord is blocked, then clearly the survivor has lost. Otherwise, in k rounds z_1 reaches the endpoint of

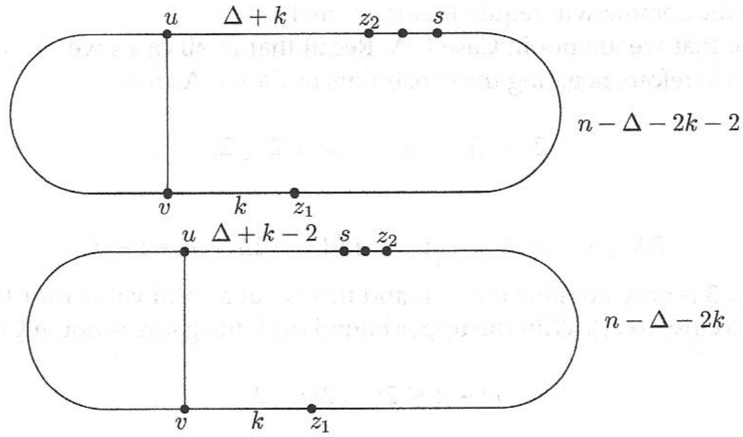


FIGURE 3.12: Preventing the zombies from turning in same direction on C_{m+1}

the chord. If the survivor is on the subpath of C_{n+1} of length Δ (connecting u to z_2), then on the next move z_1 will take the chord and trap the survivor. Therefore, after k rounds the survivor must be on $P_m \setminus N[u]$ or it has already lost. But this is precisely the scenario described in Part 3.1 when z_1 reaches the chord and z_2 is Δ away. With suitable Δ, then, the survivor is cornered on the smaller cycle.

good

3.3 Existence of Δ and k for any m, n

Every time you refer to Case I.A. Case II.A. etc. and winning conditions, I have not checked if there are typos.

Part 3. We wish to show that, for any m, n, there exist Δ and k which guarantee the survivor is caught. First, we show that $\Delta = \lfloor \frac{m}{2} \rfloor$ always works for the cornering strategy.

Note that

$$2\Delta = 2\left\lfloor \frac{m}{2} \right\rfloor = \begin{cases} m & \text{if } m \text{ is even} \\ m-1 & \text{if } m \text{ is odd} \end{cases}$$

and so $m - 1 \leq 2\lfloor \frac{m}{2} \rfloor \leq m$.

Suppose that we are in Case I. A. and $\Delta = \lfloor \frac{m}{2} \rfloor$. Case I. A is characterized by the following constraints:

$$4 \leq 2\ell \leq m+1$$

and

$$2\Delta \leq n - m + 2\ell \quad \text{or} \quad 2\Delta \leq n - 1$$

The zombies win if

$$2\Delta \leq 2m - 2\ell + 2 \quad \text{and} \quad m - 2\ell - 1 \leq 2\Delta$$

So if we are in Case I. A. and $\Delta = \lfloor \frac{m}{2} \rfloor$. In this case, the zombies win since

$$2\Delta = 2\lfloor \frac{m}{2} \rfloor \leq m < 2m - (m+1) + 2 \leq 2m - 2\ell + 2$$

and

$$m - 2\ell - 1 \leq m - 5 < 2\lfloor \frac{m}{2} \rfloor = 2\Delta$$

shows that the zombie-win requirements are met.

Suppose that we are not in Case I. A. Recall that in all cases we must have $2 \leq \ell \leq m-1$. Therefore, negating the constraints of Case I. A. gives

$$2\ell \leq 3 \qquad \text{or} \qquad m+2 \leq 2\ell$$

$$\text{or}$$

$$2\Delta \geq n-m+2\ell+1 \qquad \text{and} \qquad 2\Delta \geq n-1+1$$

But $2\ell \leq 3$ is only possible if $\ell = 1$, and this is not a valid value for ℓ (it puts the survivor too close to z_1). With the upper bound on ℓ, the game is not in Case I.A. if

$$m+2 \leq 2\ell \leq 2m-2$$

$$\text{or}$$

$$2\Delta \geq n-m+2\ell+1 \qquad \text{and} \qquad 2\Delta \geq n-1+1 = n$$

I would say "line". Better: give a number to the line.

Let us examine the consequences of assuming this second equation to be true.

If we assume that m is odd and $2\Delta \geq n$ then we obtain a contradiction since

$$2\Delta = 2\lfloor\frac{m}{2}\rfloor = m-1 \geq n$$

and we have assumed that $m \leq n$.

If m is even and $2\Delta \geq n$, then we must have $m = n$. If also $2\Delta \geq n-m+2\ell+1$ then we must have

$$\begin{aligned} 2\Delta &\geq n-m+2\ell+1 \\ m &\geq m-m+2\ell+1 \\ m &\geq 2\ell+1 \\ \cancel{2\ell} &\cancel{\leq m-1} \end{aligned}$$

~~So, if $m = n$ and they are even, then we are in Case I. A unless $2\ell \leq m-1$.~~ ?

To recap: If we set $\Delta = \lfloor\frac{m}{2}\rfloor$, we are in Case 1.A unless

1. $m+2 \leq 2\ell \leq 2m-2$, or
2. $m = n$ are even and $m \geq 2\ell+1$.

In the first case, with $m+2 \leq 2\ell \leq 2m-2$, the zombies can apply Case I.B since it is characterized by the following constraints:

$$m+1 \leq 2\ell \leq 2m-2$$

and

$$2\Delta \leq n+1 \qquad \text{or} \qquad 2\Delta \leq n+m-2\ell$$

Because $\Delta = \lfloor\frac{m}{2}\rfloor$ and $m+2 \leq 2\ell \leq 2m-2$, satisfies these constraints, the zombies can enact the strategy of Case I.B. They will win since this choice of Δ also satisfies the win conditions: of Case I.B.:

$$2\Delta \leq 2\ell \qquad \text{and} \qquad 2\ell-m+1 \leq 2\Delta$$

The first win condition is satisfied since $2\Delta \leq m < m+2 \leq 2\ell$, the second satisfied because $2\ell - m + 1 \leq (2m-2) - m + 1 = m - 2 < m - 1 \leq 2\Delta$.

In the second case, we have $m = n$ are even and ~~$2\ell \leq m-1$~~. In this case, the zombies can play as in Case II.A. since it is characterized by

$m \geq 2\ell + 1$.

$$4 \leq 2\ell \leq m+1$$

and

$$n - m + 2\ell \leq 2\Delta \qquad \text{and} \qquad n - 1 \leq 2\Delta$$

Because $\Delta = \lfloor \frac{m}{2} \rfloor$ and $2\ell \leq m-1$ satisfies these constraints, the zombies can enact the strategy of Case II.A. They will win since this choice of Δ also satisfies the win conditions: of Case II.A. :

$$2\Delta \geq 2n - 2m + 2\ell \qquad \text{and} \qquad 2\Delta \leq 2n + 2\ell - m - 1$$

The first win condition is satisfied since $2\Delta \geq m - 1 \leq 2n - 2 = 2n - 2m + 2(m-1) \geq 2n - 2m + 2\ell$, the second satisfied because $2\Delta = m \leq m + 1 = 2n + 2 - m - 1 \leq 2n + 2\ell - m - 1$.

what you want is
$2\Delta = m$
$\geq 2\ell + 1$
$= 2n - 2m + 2\ell + 1$
$> 2n - 2m + 2\ell$

It remains to show there exists a suitable value for k. Since k is constrained by the following inequalities

$$n - 2\Delta - 5 \leq 4k \leq n - 2\Delta + 3$$

it suffices to show that the interval

$$\left[\frac{n - 2\Delta - 5}{4}, \frac{n - 2\Delta + 3}{4}\right]$$

contains at least one non-negative integer.

We know there is at least one integer since

$$\left| \frac{n - 2\Delta + 3}{4} - \frac{n - 2\Delta - 5}{4} \right| = 2$$

.

To show that there exists $k \geq 0$, suppose we have

$$n - 2\Delta + 3 < 0$$

which means

$$n < 2\Delta - 3$$

With $\Delta = \lfloor \frac{m}{2} \rfloor$ we obtain a contradiction since we have presumed that $m \leq n$.

□

3.4 Computing Δ and *k*

An algorithm to calculate possible values of Δ loops over $0 \leq \Delta \leq n$ and over $2 \leq \ell \leq m-1$ and tests, for each Δ and each ℓ, which of the four cases is applicable and, if in one of the cases, whether the zombie-win constraints are satisfied. A value of Δ is accepted if, for every value of ℓ, the zombies win.

Too long and I think this is not a sentence.

Once we have obtained possible Δ, we can then determine k by calculating the bounds

$$n - 2\Delta - 5 \leq 4k \leq n - 2\Delta + 3$$

. An implementation of this algorithm generated winning zombie-start values for Δ and k which are listed in Table 3.1

this is not a sentence

Observe that $\Delta = \lfloor \frac{n}{2} \rfloor$ is always found, as predicted by our proof.

m	n	Δ	k
4	4	[0 2 3]	[1]
4	5	[0 2 3 4]	[1]
4	6	[0 2 3 4]	[1 1]
4	7	[0 2 3 5]	[1 1]
4	8	[0 2 3]	[1 2 1 1]
4	9	[0 2 3]	[1 2 1 1]
4	10	[0 2 3]	[1 2 1 2 1]
4	11	[0 2 3]	[1 2 1 2 1]
5	5	[0 2 3 4]	[1]
5	6	[0 2 3 4]	[1 1]
5	7	[0 2 3 4 5]	[1 1]
5	8	[0 2 3 5]	[1 2 1 1]
5	9	[0 2 3 6]	[1 2 1 1]
5	10	[0 2 3]	[1 2 1 2 1]
5	11	[0 2 3]	[1 2 1 2 1]
6	6	[3 4]	[1 1]
6	7	[3 4 5]	[1 1]
6	8	[3 4 5]	[1 2 1 1]
6	9	[3 4 6]	[1 2 1 1]
6	10	[3 4]	[1 2 1 2]
6	11	[3 4]	[1 2 1 2]
7	7	[3 4 5]	[1 1]
7	8	[3 4 5]	[1 2 1 1]
7	9	[3 4 5 6]	[1 2 1 1]
7	10	[3 4 6]	[1 2 1 2 1]
7	11	[3 4 7]	[1 2 1 2 1]
8	8	[4 5]	[1 2 1]
8	9	[4 5 6]	[1 2 1 1]
8	10	[4 5 6]	[1 2 1 2 1]
8	11	[4 5 7]	[1 2 1 2 1]
9	9	[4 5 6]	[1 2 1 1]
9	10	[4 5 6]	[1 2 1 2 1]
9	11	[4 5 6 7]	[1 2 1 2 1 1]
10	10	[5 6]	[1 2 1 2]
10	11	[5 6 7]	[1 2 1 2 1]
11	11	[5 6 7]	[1 2 1 2 1]

TABLE 3.1: Winning zombie-starts with Δ and k for a few values of m and n.

specifically in the case of outerplanar graph

Chapter 4

Conclusion, Future Works

In Chapter 2, we showed the existence of a graph for which 3 zombies always lose, thereby showing that the upper bound on the cop-number for planar graphs does not apply to zombies. This is hardly surprising, since the 3 Cops must effect a sophisticated strategy in order to capture the Robber, and the Zombies cannot coordinate in this way.

It remains to be shown if there is in fact an upper bound on the zombie-number for planar graphs. The example obtained in this thesis was a sort of extrapolation from the example given [1], which showed that the cop-number need not always equal the zombie-number. Is it possible to construct increasingly elaborate graphs (while still being planar) which would always provide the survivor with a winning strategy?

too negative

Having made no further progress in this direction, we decided to investigate a simpler class of graphs: outerplanar ones. In this case, as we have noted, it has been shown [16] that 2 Cops suffice to guarantee a win.

It is also known that maximally-outerplanar graphs are zombie-win [1] and it is clear that 2 Zombies suffice for a cycle, but what can be said about those outerplanar graphs in between the two extremes?

It has been our experience that 2 Zombies often suffice on outerplanar graphs. But not always. The choice of zombie start is critical. This is the motivation for our work on $Q_{m,n}$ – the cycle with a single chord. Perhaps if we could segment or decompose an outerplanar graph into simpler components, then we could at least give an upper bound: perhaps 1 or 2 Zombies per block. It is not clear how we can generalize our findings however. Adding a single extra chord changes the entire game.

Finally, we spent some considerable time pondering games of Z & S on visibility graphs. Recently, [24] applied a result about visibility-augmenting edges from [25] to conclude that visibility graphs of simple polygons are cop-win. A natural question then is to wonder if they are also zombie-win.

We have implemented tools which allow us to search, brute force, for Breadth-First Search dismantling trees (i.e., zombie-win trees). So far, every polygon tested produces a visibility graph which admits such a tree. See 4.1 for an example.

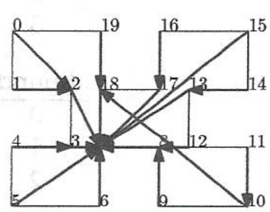


FIGURE 4.1: A Polygon Inscribed with a BFS Cop-win Tree

Appendix

Here are all the possible start configurations (without loss of generality) of Case III with the first few moves demonstrating that the survivor wins.

Round	z_1	z_2	z_3	s
0	1	6	11	3
1	2	1	2	4
2	3	5	3	20
3	4	4	4	19

Round	z_1	z_2	z_3	s
0	1	6	12	3
1	2	1	11	4
2	3	5	2	20
3	4	4	3	19

Round	z_1	z_2	z_3	s
0	1	6	13	3
1	2	1	14	4
2	3	5	15	20
3	4	4	3	19

Round	z_1	z_2	z_3	s
0	1	6	14	4
1	5	1	15	21
2	4	5	3	22

Round	z_1	z_2	z_3	s
0	1	6	15	4
1	5	1	3	21
2	4	5	4	22

Round	z_1	z_2	z_3	s
0	1	7	11	3
1	2	6	2	4
2	3	1	3	21
3	4	5	4	22

Round	z_1	z_2	z_3	s
0	1	7	12	3
1	2	6	11	4
2	3	1	2	21
4	4	5	3	22

Round	z_1	z_2	z_3	s
0	1	7	13	3
1	2	6	14	4
2	3	1	15	21
3	4	5	3	22

Round	z_1	z_2	z_3	s
0	1	7	14	3
1	2	6	15	4
2	3	1	3	21
3	4	5	3	22

Round	z_1	z_2	z_3	s
0	1	7	15	4
1	5	6	3	21
2	4	1	4	22

Round	z_1	z_2	z_3	s
0	1	8	11	3
1	2	9	2	4
2	3	10	3	5
3	4	2	4	26
4	5	1	5	27
5	26	5	26	28

Round	z_1	z_2	z_3	s
0	1	8	12	3
1	2	9	11	4
2	3	10	2	5
3	4	2	1	26
4	5	1	5	27

Round	z_1	z_2	z_3	s
0	1	8	13	3
1	2	9	14	4
2	3	10	15	5
3	4	2	3	26
4	5	1	4	27
5	26	5	5	28
6	27	26	26	29

Round	z_1	z_2	z_3	s
0	1	8	14	3
1	2	9	15	4
2	3	10	3	5
3	4	2	4	26
4	5	1	5	27

Round	z_1	z_2	z_3	s
0	1	8	15	28
1	30	7	3	27
2	29	29	4	24
3	28	28	5	23
4	27	27	25	22

Round	z_1	z_2	z_3	s
0	1	8	15	17
1	2	9	14	18
2	3	12	17	19
3	16	13	18	22

Round	z_1	z_2	z_3	s
0	1	9	11	3
1	2	10	2	4
2	3	2	3	5
3	4	1	4	26
4	5	5	5	27