Make it clearer: what is going on with reflexive graphs and loops

Chapter 1

You don't cite all papers we had discussed

Introduction

For Z&S there are more results we can describe from the Fitzpatrick paper.

There has been a robbery downtown and the robbers are escaping by car. Officers already on the streets are notified moments later. The robbers make a desperate dash for the highway but are spotted and soon tailed by police.

The robbers seem to be getting away – putting some distance between themselves and the sirens. Suddenly, the driver slams on the breaks. A squad car ahead has thrown out a strip of tire spikes! The left two tires are shredded, causing the driver to lose control. The vehicle veers off the road, flips upside down and eventually comes to a stop in the ditch. A media helicopter hovers overhead, capturing a chaotic scene flooded by the flashing lights of emergency vehicles.

Was there ever any hope of escape? Perhaps the robbers took the wrong route. They should have planned a vehicle swap. Or used a tunnel. Could it be that there were so many police officers that all routes were covered? That capture was inevitable? Perhaps the advantages of communication and central coordination allow the police to cut off likely escape routes, so that the probability of escape is low.

A (somewhat dispassionate) mind might watch these salacious stories on the news and wonder if you could apply math to these types of questions. To answer some of the above for sure. Vertex pursuit games are adversarial games played on graphs which model this sort of scenario. By having players take turns moving tokens on a graph (the game board, if you like) with the objective to capture (or evade) the other player, it is possible to simulate such chases.

Many variations of these graph pursuit games have been proposed. There are many rules and parameters to tweak to produce different games:

Can we cite 2 or 3 general books on the topic?

1. How much information do the players have?

2. Do they know each others positions? From how far away?

3. Do the players know the playing field, i.e., the graph?

4. Are the players restricted to vertices or edges?

5. Are players obligated to move?

6. Does the graph change over time?

it's not "new"

The combination of graph theory and game theory has led to the creation of a new field of inquiry about agents "chasing" or "following" each other. The Game of Cops and Robbers on Graphs [1] is perhaps the most well-known vertex pursuit game. It is a perfect information game with Cops trying to catch the Robber. In a perfect information game, all players know everything about the game. In this context, the players know each other's positions (they see each other) and they know the landscape (graph) around them [2].

very good

A variation called Zombies and Survivors (Z & S or Zombie Game) was recently proposed and studied [3, 4]. Z & S is the same as Cops and Robbers with the added twist that the zombies are required to move directly towards the Survivor. More precisely, the zombies has to move along an edge on a shortest path toward the Survivor.

This thesis has been an attempt to better understand this variant and, in particular, to see if the results obtained for Cops and Robbers still hold when the cops are constrained in their strategy. In particular, in Chapter 2 we give an example of a planar graph where 3 zombies always lose. Then in Chapter 3 we show how two zombies always win on a cycle with one chord.

Can we add something like: and in general, have a better understanding between "~~cop number~~ number of cops needed to win" and "constraints imposed to the cops"!

1.1 Notation

The following sections will use a few standard definitions from graph theory (and vertex-pursuit theory) which we include here for reference. Formally, a graph $G = (V, E)$ is composed of:

- A set V of vertices.
- A set E of edges $\{u, v\}$ where $u, v \in V$

no because we allow loops.

The graphs studied herein are finite, simple, connected, and undirected. This means that there is a finite number of vertices and that there exists a path connecting every pair of vertices.

Let $G = (V, E)$ be a graph with vertices $x, y \in V$. By undirected, we mean that an edge from x to y implies an edge from y to x. So we write $xy = yx$ for edge $\{x, y\}$ or $x, y \in E$. Playing on graphs with multiple connected components can be reduced to playing multiple games in parallel: the players are restricted to their starting connected component.

We say that vertices x and y are neighbours if $xy \in E$. That is, if there is an edge joining x to y. The set of all neighbours of x the *neighbourhood* of x which we denote $N(x) \subseteq V$. The *closed neighborhood* of x is the neighborhood of x along with x itself. The closed neighbourhood of x is denoted $N[x] = N(x) \cup \{x\} \subseteq V$

For example, in Figure 1.1 we have vertices $V = \{a, b, c, d, e, f, g, h\}$. Since a and b are connected by an edge, we have $ab \in E$. The neighbourhood of a is $N(a) = \{b, d, f\}$ and the closed neighbourhood of a is $N[a] = \{a, b, d, f\}$.

~~When we are thinking want to~~ We also write $V(G)$ for the set of vertices of G and $V(E)$ " " " edges " G.

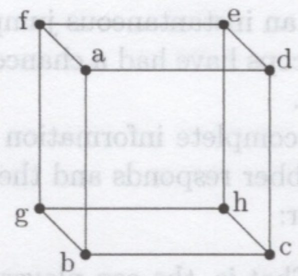

Figure 1.1: The Hypercube of Dimension 3

\emph{} degree \emph{} \emph{}

The degree of a vertex is the number of edges incident to that vertex (or equivalently the cardinality of its neighbourhood $|N(x)|$). The minimum and maximum degrees of a graph are sometimes denoted as $\delta(G)$ and $\Delta(G)$, respectively. → and how is it defined?

Two basic classes of graphs are important in the study of these games: paths and cycles. A path $P = v_0, v_1, v_2, \ldots, v_n$ is a "strict" walk: a non-repeating sequence of adjacent vertices in a graph. A cycle C_n is a path of length $n \geq 3$ with an additional edge joining the last vertex back to the first (a so-called *closed* path).

a

We say that a graph contains path P if P is a subgraph of G, so $V(P) \subseteq V(G)$ and $E(P) \subseteq E(G)$. More generally, With $S \subseteq V(G)$ a set of vertices, write $G[S]$ for the induced subgraph: the graph which contains S and the edges of G which join vertices of S.

\emph{}

Paths allow us to define a distance between vertices $d_G(x, y)$ as the length of the shortest path connecting x to y (or infinity if such does not exist; never the case in our games). Computing such paths, also known as *geodesics*, is a classic problem in computer science. a path

A geodesic has the additional property of being *isometric* [5], meaning that the distance between vertices of an isometric paths is preserved in the subgraph induced by the path. This property allows players to guard or patrol isometric paths [6], preventing the robber from entering (and thus crossing) the path without capture.

Finally, the diameter and girth of a graph are two useful graph properties which appear in some of the theorems herein. The diameter diam(G) is the length of a longest possible shortest path in G. The girth of a graph is the length of the minimum order subcycle.

\emph{} \emph{}

1.2 Classical Cops and Robbers

We start with an explanation of the game, then give some key results from C & R since much of our thesis is a comparison between the two games. We investigate *the cost of being undead*, as Fitzpatrick [3] would call it.

I would use quotation marks instead

1.2.1 How to Play Cops and Robbers

C & R is a two player game: one player controls the cops, the other the robber. The cops begin the game by choosing start vertices. Next, the robber choooses a start position. On each following round the cops may move along an edge to a neighbouring vertex or

Why not add ~~[illegible]~~ two short examples using Figure 1.1? One example with one cop. One example with two cops.

remain in position. Here a move is an instantaneous jump between adjacent vertices. If the robber remains uncaught after the cops have had a chance to move, the robber then gets the opportunity to move along an edge.

In this game, the players have complete information of the graph and the positions of the players. The cops move, the robber responds and these two turns make one round.

\emph{.} \emph{.}

The game concludes when either:

- A cop captures the robber. That is, the cop player wins if one of the cops move onto the vertex occupied by the robber.
- The robber wins if it can evade the cops indefinitely.

It's weird to say that the game "concludes" when... it lasts forever

there are 3 authors :)

1.2.2 Cops and robbers, Cop-Number

Study of vertex-pursuit games is first attributed to Quilliot [7, 8], and Nowakowski and Winkler [9]. Both authors independently consider games of C & R with a single cop and characterize by way of a relation those graphs where the cop always wins. These are now known as *cop-win* graphs and can be recognized by the existence of an ordering of the vertices called a *dismantling*; so-called because it is the successive deletion of *corners* resulting in a single vertex.

\emph{.} \emph{.}

The *cop-number* of a graph, denoted $c(G)$, is introduced by Aigner and Fromme [6] and defined as the minimum number of cops required to guarantee they win on a graph G. Later, [10] and [11] generalized the characterization of cop-win graphs into k-copwin graphs.

write the names of the authors and put this at the end of the sentence.

A graph is k-cop win if and only if there exists a function (on a k-product of the graph to represent the position of the cops) which satisfies certain properties; essentially it is a function which takes as input a position C of cops and returns the next position for the cops that guarantees a win (see [1][p. 119]). There exists a polynomial-time algorithm for deciding whether a graph is k-cop-win by iteratively solving for this function.

So we can compute the cop number of any graph in polynomial time?

Another important line of inquiry relating to the cop-number is the investigation of Meyniel's conjecture, which posits that $\mathcal{O}(\sqrt{|V(G)|})$ is an upper bound on the cop-number [12]. Incremental progress has been made on special classes of graphs as well as for graphs in general. See also for a recent overview [13][p. 31].

1.2.3 The Cop-Number and the Genus of the Graph

One of the most surprising results about the C & R is owed to Aigner and Fromme [6], who showed that the cop number of a planar graph is at most 3. Basically, a graph is planar if it can be drawn in the plane (say, on a piece of paper) without crossing any edges. Aigner and Fromme describe a 3-cop strategy which uses *isometric* paths of the graph to encircle and entrap the robber.

Outerplanar graphs are planar graphs which can be drawn such that all vertices belong to a common face (called the outer face). Clarke [14] showed that the cop number of outerplanar graphs is 2 by considering two possible cases: those with and without cut vertices. The 2

one word \emph{.}

cops have a winning strategy on outerplanar graphs without cut vertices, and this strategy can be used to cordon off sections (blocks) of the outerplanar graph.

The game has also been studied for graphs embeddable in surfaces of higher order. In 2001, Schroeder conjectured [15] that for a graph of genus g, the cop-number is at most $g+3$. [Include here the best known bound].

1.2.4 Relation to the Girth and Minimum Degree of a Graph

Aigner and Fromme also show a relationship between the cop-number, the girth of a graph and its minimum degree [6]. More precisely, if G has girth at least 5, then $c(G) \geq \delta(G)$ where $\delta(G)$ is the minimum degree of G.

This result has since been refined by [12] and again recently in a seminar by B. Mohar (Graph Searching Seminar, held online May 1, 2020).

write the result, at least for [12]. And why not for the seminar too?

1.2.5 Dismantlings, Cop-win Trees, Zombie-win Trees

There is no zombie in this section.

Quilliot and Nowakowski both indepedently characterized cop-win graphs as those which admit a *dismantling*.

Let x be a fixed vertex. A (one-point) *retract* is an edge preserving function $f : G \to H = G \setminus v$ (aka a homomorphism) such that $f(v) = x$ for some $x \neq v \in V(G)$ and f restricted on H is the identity.

Formally,

$$f(v) = x \quad f(u) = u \quad \forall u \in V(G) \setminus \{v\}$$

and

we need it otherwise this is not a homomorphism

$$xy \in E(G) \implies f(x)f(y) \in E(G \setminus \{v\})$$

If G is a reflexive graph, then a one-point retract can be seen as joining two vertices. The edge between two adjacent vertices becomes another loop. The retract maps a graph G to graph G' with one less vertex.

Recall that corners are vertices v whose closed neighbourhoods are a subset of a neighbours' closed neighbourhood, i.e.

$$u, v \in V(G) \qquad \text{and} \qquad N[v] \subseteq N[u]$$

We

You can define a retract on corner v: if v is a corner, then it is dominated by some $u \in V(G)$. So if $x \in V(G)$, $x \neq v$ and $xv \in E(G)$ then $xu \in E(G)$ by definition of a corner. Therefore the map

$$f(x) = \begin{cases} u & \text{if } x = v \\ x & \text{otherwise} \end{cases}$$

Let's discuss this.

it's the first time you mention them.

\emph{}

\emph{.} to?

is edge preserving since $f(x)f(v) = xu$ and $xu \in E(G)$, so $xu \in E(H) = E(G-v)$. For other vertices $x, y \notin \{u, v\}$, $f(x)f(y) = xy \in E(G)$ so $f(x)f(y) \in E(G-v)$ also. This shows that f is a homomorphism as required and hence a retract. This is a formal way of saying that a corner of a graph can be folded into a dominating vertex.

A dismantling is a sequence of retracts $f_1, f_2, \ldots, f_{n-1}$ such that the composition $F_{n-1} = f_{n-1} \circ f_{n-2} \circ \cdots f_2 \circ f_1$ gives a function for which $F_{n-1}(G) = K_1$. That is, there is a sequence of retracts which maps the graph to a single vertex.

Not all vertices of a graph need be corners in order for there to exist a dismantling: it suffices to have an ordering where each v_i is a corner in $G[v_i, v_{i+1}, \ldots, v_n]$.

same paragraph

Such a sequence of f_j's defines a copwin ordering

why not using index -?

$$\mathcal{O} = (v_1, v_2, \ldots, v_n)$$

and?

where v_1 is a corner in $G_1 = G$, v_2 is a corner in $G - v_1$, and so on.

A fundamental result in C & R is that cop-win graphs – graphs for which a single cop is guaranteed to win – are characterized by the existence of such dismantlings. A graph is copwin if and only if it is dismantlable.

same paragraph

A cop-win spanning tree combines the idea of a dismantling with a spanning tree and was first proposed in [14].

A cop-win spanning tree S is defined as a tree where $x, y \in V(G)$ $xy \in E(S)$ if there exists a retract f_j in the dismantling $F_n = f_n \circ f_{n-1} \circ \cdots \circ f_2 \circ f_1$ such that $f_j(x) = y$ or $f_j(y) = x$ in $G[j]$.

Cop-win spanning trees give a strategy for the cops to follow: start at the root (the last vertex in the ordering) and descend the tree in the branch containing the robber. Lemmas 2.1.2 and 2.1.3 from [14] show that the cop can always stay in the same branch (and above) the robber in the tree. So the robber is eventually stuck in a leaf and caught.

here j is an index we have not defined this notation.

1.3 Cops Turn Into Zombies

1.3.1 How to Play Zombies and Survivor

Zombies and Survivor is similar to C & R except that zombies move "directly" toward the survivor. More precisely, on the zombies' turn each zombie independently selects a shortest path toward the survivor (a *geodesic*) and moves along the edge to the next vertex of the path.

The sophistication of the zombies' strategy gives them their name: you can imagine the zombies – arms outstretched – ambling directly towards the survivor. As in C & R, the players have complete information of the graph and the positions of the players. Indeed, the zombies need to know the position of the survivor to enact their strategy.

If uncaught, the survivor may move to one of its neighbouring vertices or stay in place. Again, a move is an instantaneous jump along an edge from one vertex to another.

The game concludes when either:

- A zombie eats the survivor by moving to the survivor's vertex.

We should say here what happens when there is more than one shortest path.

it depends... when there is more than one option...

As in C&R,

\emph{.}

It's weird to say that the game "concludes" when... it lasts forever...

- The survivor evades the zombies indefinitely.

1.3.2 Modeling the Game

We call $s \in V(G)$ the survivor and $z_i \in V(G)$ are zombies with $i \in \{1, \ldots, k\}$. This notation represents both a player and its position in the graph. In the games studied there is a single survivor and $k \geq 1$ of zombies.

We divide the game into rounds and turns. A round consists of two turns: a zombie turn and a survivor turn. It is convenient to define the zombie's turn on $t \equiv 0 \mod 2$ and the survivor's turn on $t \equiv 1 \mod 2$. Round r is given by $\lfloor \frac{t}{2} \rfloor$.

It is occasionally useful to identify the players' positions over time, in which case let $z_r^i \in V(G)$ be zombie i on round r. Similarly s_r is the survivor on round r. This burdensome notation will be omitted when possible.

~~It might be tempting to group the zombies together into some tuple of the vertex set, but each zombie acts independently of the others and so this may not be practical.~~

1.3.3 Paths and Moves

The zombie strategy requires consideration of all geodesics connecting each zombie to the survivor. Let us be precisely define these terms.

Consider zombie k. According to the rules of the game, on its turn the zombie "must move on a shortest path" towards the survivor. The *zombie moves* are those neighbours which lie on a shortest path toward the survivor, and which could be denoted

$$Z[x, s] = \{y \in N(x) \mid d(y, s) = d(x, s) - 1\}$$

the zombies moves from x given survivor is on s.

[fix me here; simplify; clarify]

There is at least one such move since our graph is presumed connected, so $i > 0$ and $Z_k \neq \emptyset$.

What is i?

what is Z_k?

If there is only one path, then z_k's next move is $u_{i,1}$. If all zs-paths include $u_{i,1}$, then again z_k's next move must be to that vertex.

If, however, there are multiple zs-paths which have different first moves, then the zombie could make multiple moves. And zombies are allowed to discuss and agree on a decision.

what is $U_{i,1}$?

1.3.4 Deterministic Zombies

Zombies and Survivors (or more specifically, "deterministic zombies") are an interesting variation proposed in [3]. In these games, the Cops are replaced by Zombies which must follow a geodesic to the Survivor.

The Zombie Number is defined analogously to the Cop Number: it is the number of Zombies required to capture the Survivor. However, in Z & S there are two additional considerations: the zombie start and the zombie choices. In this type of game, the starting

This is a repetition of §1.3.1.

We

locations for the zombies is of utmost importance: consider how difficult it might be to evade adversaries which are clustered versus some that are well-dispersed. So we say $z(G) = k$ if k is the minimum number of zombies required to guaranteed a win given an appropriate (or optimal) start. Additionally, the rules of this game permit some agency to the zombies: when confronted with multiple geodesics, they may have a choice between neighbouring vertices. Zombie number also presumes that the zombies make the correct choices. Perhaps more precisely, the zombie number of a graph is k if k zombies, suitably positioned, can play a game which guarantees the survivor is caught.

Let's discuss this.

Unlike Cops, these Zombies cannot apply a cornering strategy. Or any strategy. As a consequence, you need at least as many Zombies as you need Cops. This is one of the first observations in [3]: the Cop Number $c(G)$ is a lower bound of the Zombie Number. The Zombies are weaker versions of Cops, similar in a way to the "fully active" Cops from [16] where the Cops must move on their turn. Both active and "lazy" Cops have more freedom of choice than the zombies, and thus fewer are required to ensure victory.

Does a characterization exist for Zombie-win graphs? Those for which a single zombie can always win? One has yet to be described. Howoever, [3] showed that a graph is zombie-win if a specific spanning tree exists:

Theorem 1 (Fitzpatrick). If there exists a breadth-first search of a graph G such that the associated spanning tree is also a cop-win spanning tree, then G is zombie-win.

Thus a sufficient condition for zombie-win graphs are those for which a specific copwin tree exists: one equivalent to a breadth-first search of the graph from some vertex. It is unknown if it is also a necessary condition.

A few questions: are cop-win graphs necessarily zombie-win? No. A counter example [3] is reproduced below 1.2.

Below 1.3 is an example of a graph and two dismantlings, one of which results in a BFS tree, and the other does not.

Here are two dismantlings, their orderings, and the resulting copwin spanning trees.

(refer to Figure 1.2)

$$f_1(b) = f$$
$$f_2(c) = d$$
$$f_3(f) = e$$
$$f_4(a) = e$$
$$f_5(e) = g$$
$$f_6(d) = g$$

Gives ordering $\mathcal{O}_1 = \{b, c, f, a, e, d, g\}$. Whereas

(refer to Figure 1.3)

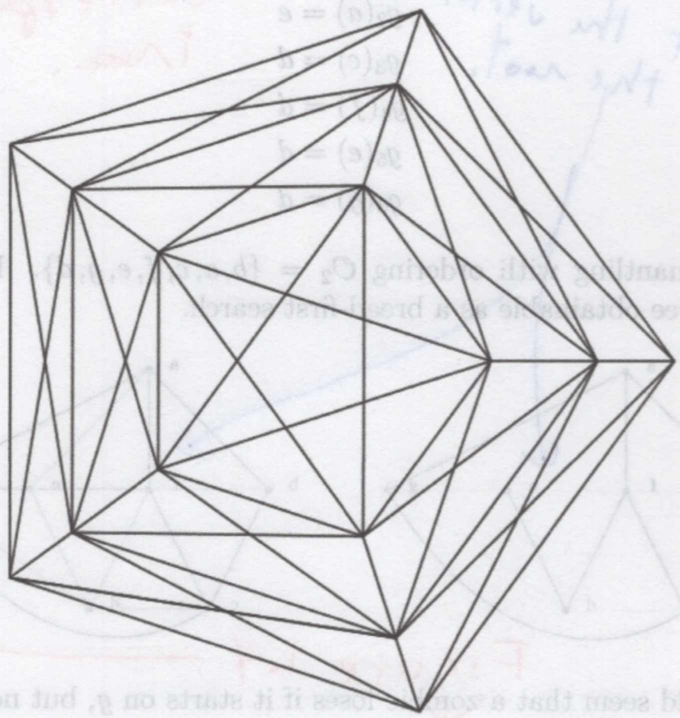

Figure 1.2: Cop-Win but not Zombie-Win

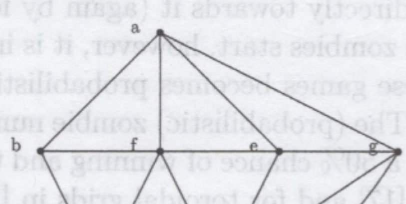

Figure 1.3: A Cop-win tree

In the caption of the figure, indicate what edges correspond to the tree. What are the roots? Why it does or does not correspond to a BFS-tree.

use a special symbol to represent the vertex that is the root.

$$g_1(b) = f$$
$$g_2(a) = e$$
$$g_3(c) = d$$
$$g_4(f) = d$$
$$g_5(e) = d$$
$$g_6(g) = d$$

Also gives a dismantling with ordering $\mathcal{O}_2 = \{b, a, c, f, e, g, d\}$. But only the second produces a copwin tree obtainable as a bread-first search.

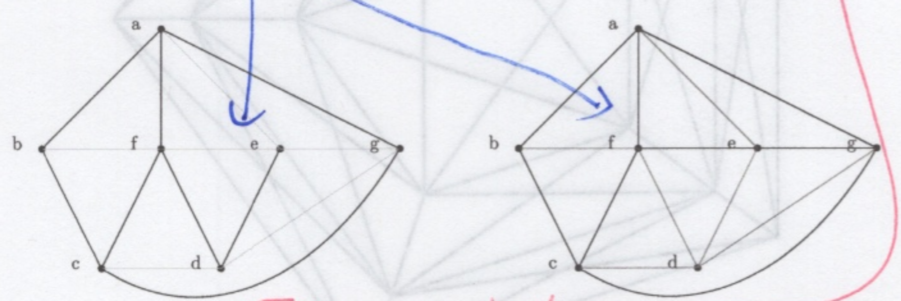

Figure 1.4

Moreover, it would seem that a zombie loses if it starts on g, but not on d.

1.3.5 Probabilistic zombies

Zombies are often depicted as mindless or aimless. It is a common trope that zombies idle around, moving in random directions until they somehow (suddenly) distinguish the uninfected. It is only at this point that the zombies will charge.

Such behavior likely inspired another type of pursuit game [17] in which the zombies start randomly on the graph. Once the survivor chooses a start vertex, the zombies "notice" the survivor and start moving directly towards it (again by following a shortest path).

Without knowing where the zombies start, however, it is impossible to know the outcome with certainty. So study of these games becomes probabilistic; zombies win if they have at least a 50% chance of winning. The (probabilistic) zombie number of a graph is the minimum number of zombies required for a 50% chance of winning and this zombie number is obtained for several classes of graphs in [17] and for toroidal grids in [18].

So what are the known results?

The original paper on probabilistic zombies [17] also includes a lemma which is useful for our work in Chapters 2 and 3:

Lemma 1 (3.1, [17]). The survivor wins on C_n against $k \geq 2$ zombies if and only if all zombies are initially located on an induced subpath containing at most $\lceil \frac{n}{2} \rceil - 2$ vertices.

Lemma 3.1?

What is the impact on the probability of winning?