- Punctuation with formulas
- ~~Cases (c) and d~~
- P_c vs P_d

Always add a sentence explaining why one zombie is not sufficient. Because otherwise, technically our argument only gives an upper bound.

Always declare m and n in the statements

Chapter 3

Cycle With One Chord

Let $m,n \in \mathbb{Z}$ with $2 \le m \le n$. Consider

Assumption A
Assumption I
etc.

Games played on cycles are straightforward: if the zombies are too close, the survivor can lead the zombies in the same direction around the cycle. Otherwise, the zombies are too far apart and whichever side (sub-path of the cycle with the zombies at the end) the survivor may choose, the zombies will move in opposite directions and win. In this Chapter, we investigate the game on cycles augmented by a single chord.

Definition 1. ~~Take~~ the a cycle of length $m+n$ and add a chord which divides the cycle into paths P_m and P_n of lengths m and n, respectively. ~~Without loss of generality $m \leq n$.~~ We denote such a ~~cycle as~~ graph by $Q_{m,n}$.

See Figure 3.1 for an illustration of $Q_{7,8}$. The construction contains three sub-cycles which the survivor could use to fool the zombies. Let us first examine the construction for small values of m and n.

- ~~Setting $m = n = 1$ gives K_2 with two added loops, which is zombie-win.~~ This graph contains parallel edges.
- With $m = n = 2$ we have two adjacent cliques K_3 which are dominated by a single vertex, so it is also zombie-win.
- For $m = 2$ and $n \geq 4$, 2 zombies win by starting on diametrically opposed vertices on the cycle C_{n+2}.

what about $Q_{3,3}$?

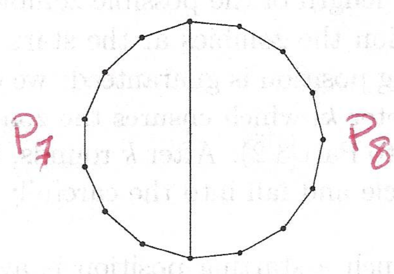


Figure 3.1: $Q_{7,8}$

Have a lemma for $m=n=2$.

Have a lemma for ~~$2 \le m \le \ldots$~~ ~~for~~ $m=2$ or $m=3$ and $n \le 5$ $((m,n) \neq (2,2))$

Then the theorem for $m \geq 3$, $n \geq 6$ and $n \geq m$.

not sure

- If $m = n = 3$ the zombie number is 2 since two zombies on the chord enpoints dominate the graph.

- For $m = 3$, $n = 4$, the zombie number is also 2: placing the zombies on the endpoints of the chord divides the graph into C_4 and C_5 and the zombies clearly win from this position.

prove

The zombie strategy that starts on the endpoints of the chord works for $Q_{3,6}$, $Q_{4,4}$, $Q_{4,5}$ and $Q_{5,5}$ but it does not work for $Q_{3,7}$, $Q_{4,6}$ nor indeed for any $Q_{m,n}$ for $m \geq 3$ and $n \geq 6$. Unfortunately for the survivor, we are able to show the existence of starting positions for the zombies (obtained as a function of m, n) which limits the survivor's options and prevents the zombies from being led in the same direction.

Theorem 3. The zombie number of $Q_{m,n}$ $(3 \leq m \leq n)$ is 2. Declare m and n.

In the proof below, we imagine $Q_{m,n}$ as embbeded in the plane with P_m – the shortest side – on the left. This does not limit the generality of the following and allows us to define (counter-)clockwise distance: the length of the path along a cycle with respect to the given direction on this embedding. Note also that if P_1 and P_2 are two possible zombie-survivor-paths with distinct next moves and

$$|P_1| \leq |P_2|$$

then in the following argument we suppose that the zombie follows $|P_1|$ since that is a valid move. no. we have to

Proof. We seek a winning ~~zombie start~~ for $m \geq 3$, and $n \geq 6$. We describe a strategy in three separate parts, which we summarize here.

The chord is the crux of the game, so we assume that one zombie is on an endpoint of the chord and that the other is at some distance Δ from the chord. We also assume that the survivor is somewhere on P_m, the shorter path (refer to Figure 3.2). In such a scenario, we know that the first zombie will chase the survivor around the cycle $C_{m+1} = P_m + uv$, which will force the survivor to flee (in the same direction) after at most $\lfloor \frac{m}{2} \rfloor - 2$ rounds.

could be improved

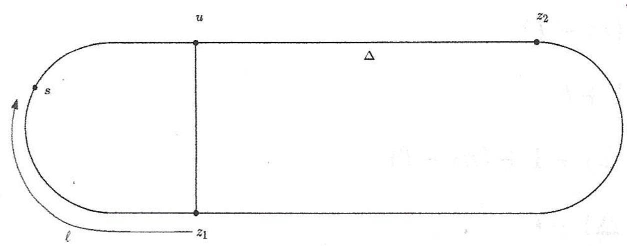
We can find the intervals of Δ which guarantee the survivor will be sandwiched on P_m by considering all possible combinations of directions "chosen" by the zombies (refer to Part 3.1). The zombies' choice of direction is not really a choice, after all: the choice is forced by the position of the survivor and the length of the possible zombie-survivor paths.

Next, we show how to position the zombies at the start of the game so that no matter where the survivor starts a losing position is guaranteed: we offset the zombies on the larger cycle with an additional parameter k, which ensures the zombies are not too close together and therefore guard C_{n+1} (refer to Part 3.2). After k rounds, the survivor will have no choice but to retreat to the smaller cycle and fall into the carefully orchestrated trap described in the first part of the proof.

great

In Part 3.3, we show that such a starting position is available to the zombies for any $m \geq 3$, $n \geq 6$. Finally in Part 3.4 we describe a simple algorithm to compute these winning start positions.

starting position for the zombies



labels are too small

Figure 3.2: z_1 on v, s somewhere on P_m

distances are expressed in two different ways.

3.1 Cornering the Survivor on C_{m+1}

Part 1. Suppose that the game has reached the following state:

- the first zombie is on an endpoint of the chord, say v
- there are Δ vertices counting clockwise from u to z_2.
- the survivor is on P_m at a distance of ℓ vertices counting clockwise from v.

This configuration is illustrated in Figure 3.2. Note that we must have $2 \leq \ell \leq m - 1$ else z_1 captures the survivor on the next round.

By comparing the lengths of different paths, we calculate the values of Δ which guarantee that the survivor will be cornered on P_m from this start configuration. That is to say, the survivor will not be able to return to the endpoints of the chord before z_2.

We can assume that once z_1 chooses a direction from v that it continues in that direction: either the zombie has no choice or both directions around the cycle are of the same length (and so z_1 may continue in the same direction).

We can also assume that on its turn the survivor will move away from z_1 and maintain a distance of ℓ (or $m - \ell + 1$, if they are moving counter-clockwise) since a winning survivor strategy which involves waiting a turn or moving backwards is equivalent to a survivor strategy which always moves but starts with a smaller (or larger) value of ℓ.

These two assumptions allow us to "fast-forward" the game by Δ rounds and determine when the survivor is captured. Since z_1 is already on the same sub-cycle as the survivor, there are two possibilites:

A. z_1 goes clockwise if $\ell \leq 1 + m - \ell$. Combined with the bounds on ℓ, this gives $4 \leq 2\ell \leq m + 1$

B. z_1 goes counter-clockwise if $1 + m - \ell \leq \ell$. Combined with the bounds on ℓ, we obtain $m + 1 \leq 2\ell \leq 2m - 2$

We must consider four possible ~~shortest~~ paths from z_2 to the survivor:

center this it is important

- P_a of length $\Delta + (m - \ell)$
- P_b of length $\Delta + 1 + \ell$
- P_c of length $(n - \Delta) + 1 + (m - \ell)$
- P_d of length $(n - \Delta) + \ell$

These paths are illustrated in Figure 3.3.
Comparing path lengths we see that:

I. z_2 moves counter-clockwise if either $|P_a| \leq \min\{|P_c|, |P_d|\}$ or $|P_b| \leq \min\{|P_c|, |P_d|\}$.

II. z_2 goes clockwise if either $|P_c| \leq \min\{|P_a|, |P_b|\}$ or $|P_d| \leq \min\{|P_a|, |P_b|\}$.

We will examine all combinations of these possible "zombie-decisions" to show that there exist values of Δ which prevent the survivor's escape in any of the possible games (from this start configuration where the survivor is on P_m). We break it down as follows:

- I. z_2 goes counter-clockwise
- II. z_2 goes clockwise.
- A. z_1 goes clockwise
- B. z_1 goes counter-clockwise

Case I.A. z_2 goes counter-clockwise and z_1 goes clockwise.
Suppose the zombies will move as in Figure 3.4.
We obtain the following constraints on ℓ from assumption A.

$$4 \leq 2\ell \leq m + 1$$

and the following constraints on Δ from assumption I.

$$\begin{aligned} \Delta + (m - \ell) &\leq n - \Delta + 1 + m - \ell && \text{and} \\ \Delta + (m - \ell) &\leq n - \Delta + \ell \end{aligned}$$

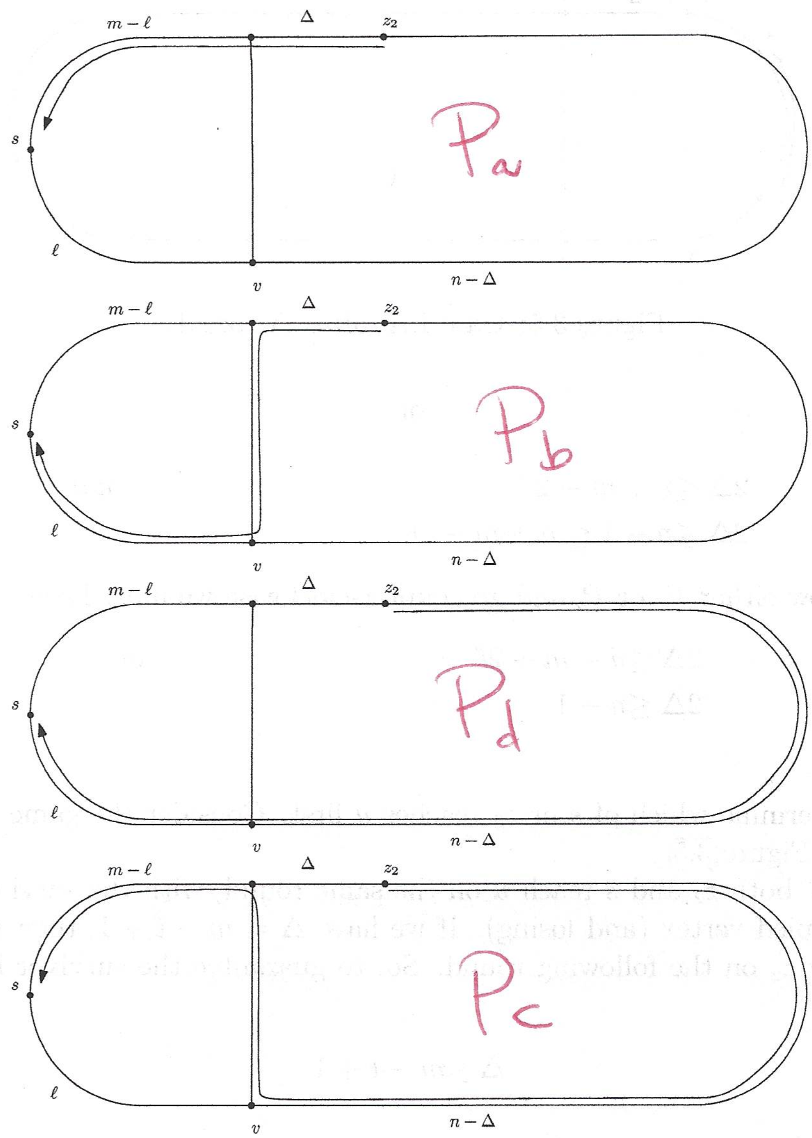
or

$$\begin{aligned} \Delta + 1 + \ell &\leq n - \Delta + 1 + m - \ell && \text{and} \\ \Delta + 1 + \ell &\leq n - \Delta + \ell \end{aligned}$$

So that together with assumption A we can obtain:

$$\begin{aligned} 2\Delta &\leq n + 1 && \text{and} \\ 2\Delta &\leq n - m + 2\ell \leq n + 1 \end{aligned}$$

labels are too small



swap the two in this figure

Figure 3.3: Possible paths from z_2 to s

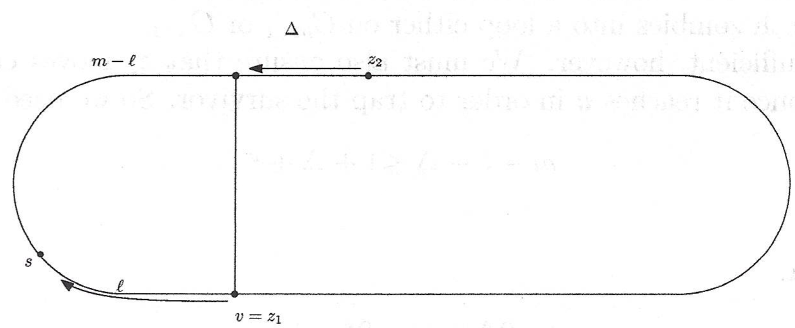


Figure 3.4: Case I.A.

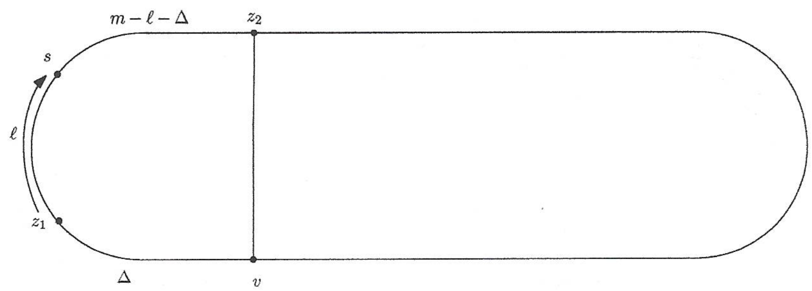


Figure 3.5: Case I.A. after Δ rounds

or

$$\begin{aligned} 2\Delta &\leq n+m-2\ell && \text{and} \\ 2\Delta &\leq n-1 \leq n+m-2\ell \end{aligned}$$

So for z_2 to follow either P_a or P_b and go counter-clockwise we must have

$$\begin{aligned} 2\Delta &\leq n-m+2\ell && \text{or} \\ 2\Delta &\leq n-1 \end{aligned}$$

We must determine which of s or z_2 reaches u first. Consider the game after Δ rounds, as illustrated in Figure 3.5.

If $\Delta = m-\ell$ both z_2 and s reach u on the same round, with the survivor moving onto the zombie-occupied vertex (and losing). If we have $\Delta = m-\ell+1$, then s reaches u first but is caught by z_2 on the following round. So, to guarantee the survivor has not escaped P_m we need

$$\Delta \leq m-\ell+1$$

otherwise the survivor can reach the chord at least two rounds before z_2 can move to block. We wish to prevent this scenario since the survivor could then take the chord and possibly escape, pulling both zombies into a loop either on C_{m+1} or C_{n+1}.

That is not sufficient, however. We must also ensure that z_2 moves counter-clockwise (opposite to z_1) once it reaches u in order to trap the survivor. So we need

$$m-\ell-\Delta \leq 1+\Delta+\ell$$

Or, in terms of Δ,

$$2\Delta \geq m-2\ell-1$$

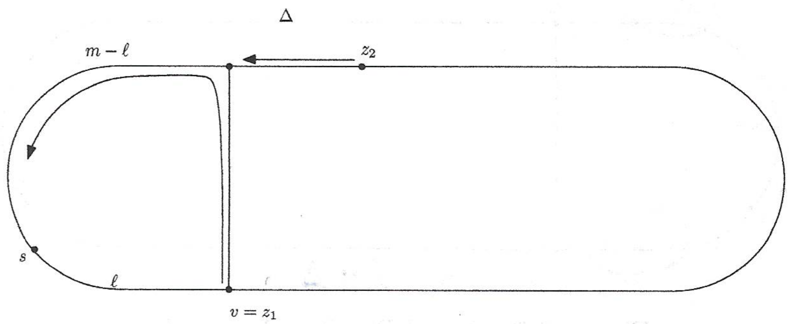


Figure 3.6: Case I.B.

When we combine all the restrictions we obtain the following carachterization:
~~*Case I.A. Summary*~~
z_1 goes clockwise:

$$4 \leq 2\ell \leq m+1$$

and z_2 goes counter-clockwise

$$2\Delta \leq n - m + 2\ell \qquad \text{or}$$
$$2\Delta \leq n - 1$$

on one line

the zombies win:

$$2\Delta \leq 2m - 2\ell + 2 \qquad \text{and}$$
$$m - 2\ell - 1 \leq 2\Delta$$

on one line

Case I.B z_2 and z_1 both go counter-clockwise.
Suppose the zombies will move as in Figure 3.6.
From assumption B and the constraint on ℓ, we must have

$$m+1 \leq 2\ell \leq 2m-2$$

and the constraints on Δ from assumption I are again:

$$\Delta + (m-\ell) \leq n - \Delta + 1 + m - \ell \qquad \text{and}$$
$$\Delta + (m-\ell) \leq n - \Delta + \ell$$

or

$$\Delta + 1 + \ell \leq n - \Delta + 1 + m - \ell \qquad \text{and}$$
$$\Delta + 1 + \ell \leq n - \Delta + \ell$$

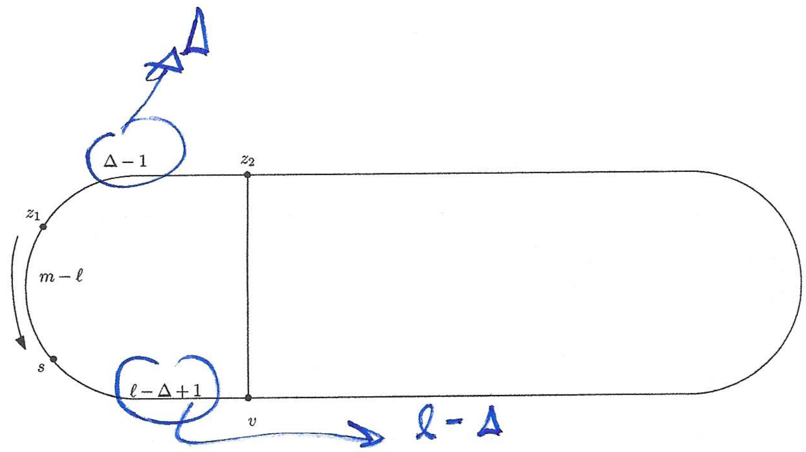


Figure 3.7: Case I.B. after Δ rounds

~~Simplified using~~ Simplifying using assumption B:

$$\begin{aligned} 2\Delta &\le n+1 \le n-m+2\ell && \text{and} \\ 2\Delta &\le n-m+2\ell \end{aligned}$$

or

$$\begin{aligned} 2\Delta &\le n+m-2\ell \le n-1 && \text{and} \\ 2\Delta &\le n-1 \end{aligned}$$

So for z_2 to go counter-clockwise in this case we must have

$$\begin{aligned} 2\Delta &\le n+1 && \text{or} \\ 2\Delta &\le n+m-2\ell \end{aligned}$$

Again we must consider who reaches the chord first. Consider the game after Δ rounds, as illustrated in Figure 3.7.

If $\ell = \Delta$, then z_2 reaches u and s reaches v on the same round, and therefore s will be caught on the next. Therefore, to guarantee the survivor has not escaped P_m in this scenario we need

$$\Delta \le \ell$$

Otherwise, the survivor reaches the chord before z_2 and could escape.

Then, to ensure that z_2 traps the survivor by going clockwise once it reaches u we need

$$\begin{aligned} 1+\ell-\Delta &\le \Delta + m - \ell \\ 2\ell - m + 1 &\le 2\Delta \end{aligned}$$

Case I.B. Summary
z_1 goes counter-clockwise: Same format as for Case I.A

$$m+1 \le 2\ell \le 2m-2$$

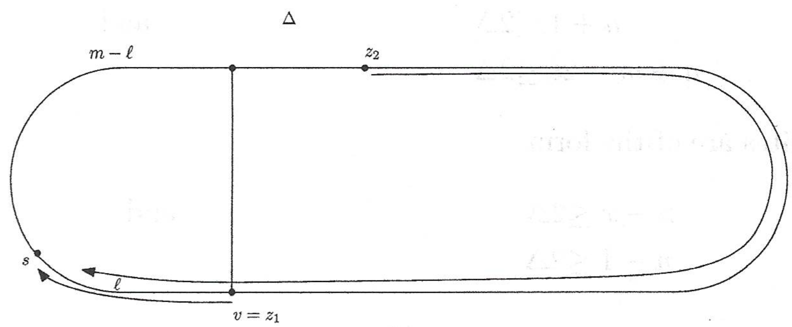


Figure 3.8: Case II.A.

and z_2 goes counter-clockwise

$$2\Delta \leq n+1 \qquad \text{or}$$
$$2\Delta \leq n+m-2\ell$$

the zombies win:

$$2\Delta \leq 2\ell$$
$$2\ell - m + 1 \leq 2\Delta$$

Case II.A z_2 and z_1 both go clockwise.
Suppose the zombies will move as in Figure $\boxed{3.8}$.
We have the following constraint on ℓ from assumption A.

$$4 \leq 2\ell \leq m+1$$

no parentheses

and the following constraints on Δ from assumption II.

Swap

$$n - \Delta + \ell \leq \Delta + (m-\ell) \qquad \text{and}$$
$$n - \Delta + \ell \leq \Delta + 1 + \ell$$

or

$$n - \Delta + 1 + m - \ell \leq \Delta + (m-\ell) \qquad \text{and}$$
$$n - \Delta + 1 + m - \ell \leq \Delta + 1 + \ell$$

plified with a bit of algebra:

$$n - m + 2\ell \leq 2\Delta \qquad \text{and}$$
$$n - 1 \leq 2\Delta$$

or

swap

$$
\begin{aligned}
n + 1 &\leq 2\Delta \qquad \text{and} \\
n + m - 2\ell &\leq 2\Delta
\end{aligned}
$$

These inequalites are of the form

$$
\begin{aligned}
n - x &\leq 2\Delta \qquad \text{and} \\
n - 1 &\leq 2\Delta
\end{aligned}
$$

or

$$
\begin{aligned}
n + x &\leq 2\Delta \qquad \text{and} \\
n + 1 &\leq 2\Delta
\end{aligned}
$$

Where $x = m - 2\ell$.

Supposing $x \geq 0$, we have

$$
\begin{aligned}
n - x &\leq n + x \leq 2\Delta \qquad \text{and} \\
n - 1 &< n + 1 \leq 2\Delta
\end{aligned}
$$

and take the lowest bounds because of the disjunction, so that $2\Delta \geq n - x = n - m + 2\ell$ and $2\Delta \geq n - 1$ suffices.

Since assumption A gives $m - 2\ell \geq -1$, supposing $x < 0$ reduces the inequalites to

$$
\begin{aligned}
n + 1 &\leq 2\Delta \qquad \text{and} \\
n - 1 &\leq 2\Delta
\end{aligned}
$$

which is satisfied by $2\Delta \geq n - x = n - m + 2\ell$ and $2\Delta \geq n - 1$.

Thus z_2 will go clockwise under assumption A if

$$
\begin{aligned}
2\Delta &\geq n - m + 2\ell \qquad \text{and} \\
2\Delta &\geq n - 1
\end{aligned}
$$

Consider the game after $n - \Delta$ rounds, as illustrated in Figure 3.9.

We have assumed that z_1 is going clockwise. If $m - \ell = n - \Delta$, then z_2 reaches v and s reaches u on the same round and s will be caught on the next. Therefore, to guarantee the survivor has not escaped P_m in this scenario we need

$$
\begin{aligned}
n - \Delta &\leq m - \ell \\
\Delta &\geq n - m + \ell
\end{aligned}
$$

otherwise the survivor could reach the chord before z_2.

After $n - \Delta$ rounds, we have (insert diagram)

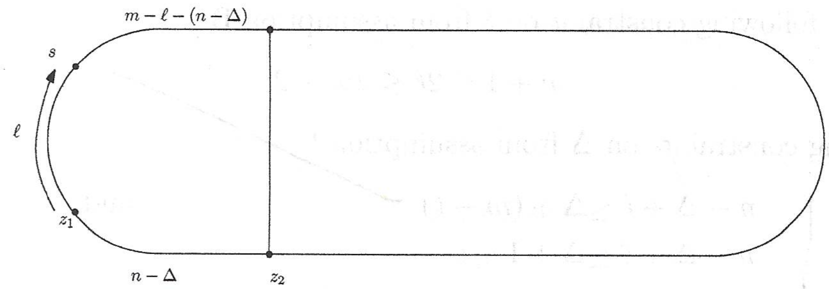


Figure 3.9: Case II.A. after $n-\Delta$ rounds

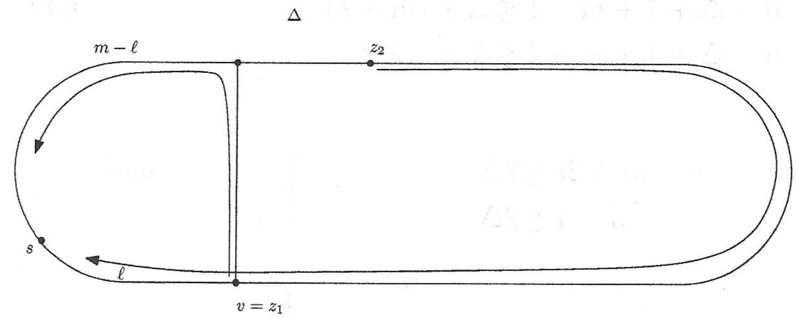


Figure 3.10: Case II.B.

Then, to ensure that z_2 goes counter-clockwise once it reaches v, we need

$$\begin{aligned} 1+m-\ell-(n-\Delta) &\leq n-\Delta+\ell \\ 2\Delta &\leq 2n+2\ell-m-1 \end{aligned}$$

All together this gives *Case II.A. Summary* Same format as previous cases

z_1 goes clockwise:

$$4 \leq 2\ell \leq m+1$$

and z_2 goes clockwise

$$\begin{aligned} n-m+2\ell &\leq 2\Delta \\ n-1 &\leq 2\Delta \end{aligned} \qquad \text{and}$$

the zombies win:

$$\begin{aligned} 2\Delta &\geq 2n-2m+2\ell \\ 2\Delta &\leq 2n+2\ell-m-1 \end{aligned}$$

Case II.B. z_2 goes clockwise and z_1 goes counter-clockwise.
Suppose the zombies will move as in Figure 3.10.

We have the following constraint on ℓ from assumption B.

no parentheses

$$m + 1 \leq 2\ell \leq 2m - 2$$

and the following constraints on Δ from assumption II.

Swap

$$n - \Delta + \ell \leq \Delta + (m - \ell) \quad \text{and}$$
$$n - \Delta + \ell \leq \Delta + 1 + \ell$$

or

$$n - \Delta + 1 + m - \ell \leq \Delta + (m - \ell) \quad \text{and}$$
$$n - \Delta + 1 + m - \ell \leq \Delta + 1 + \ell$$

We obtain:

$$n - m + 2\ell \leq 2\Delta \quad \text{and}$$
$$n - 1 \leq 2\Delta$$

or

$$n + 1 \leq 2\Delta \quad \text{and}$$
$$n + m - 2\ell \leq 2\Delta$$

Same thing

We have

$$n - \Delta + \ell \leq \Delta + (m - \ell) \quad \text{and}$$
$$n - \Delta + \ell \leq \Delta + 1 + \ell$$

or

same thing

$$n - \Delta + 1 + m - \ell \leq \Delta + (m - \ell) \quad \text{and}$$
$$n - \Delta + 1 + m - \ell \leq \Delta + 1 + \ell$$

These can be simplified further with a bit of algebra:

$$n - m + 2\ell \leq 2\Delta \quad \text{and}$$
$$n - 1 \leq 2\Delta$$

or

$$n + 1 \leq 2\Delta \quad \text{and}$$
$$n + m - 2\ell \leq 2\Delta$$

These inequalites are of the form

$$n - x \leq 2\Delta \quad \text{and}$$
$$n - 1 \leq 2\Delta$$

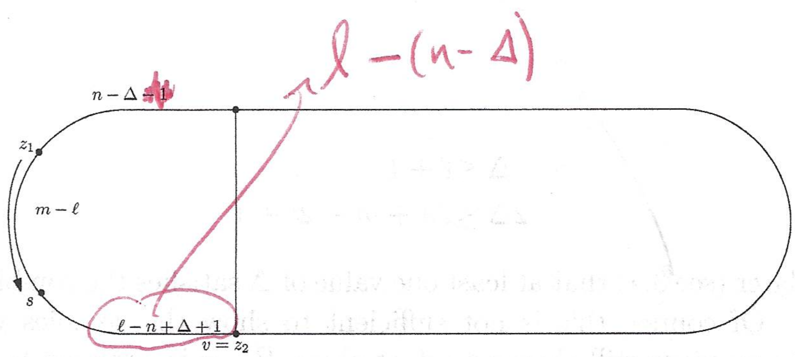


Figure 3.11: Case II.B. after $n - \Delta$ rounds

or

$$n + 1 \leq 2\Delta \qquad \text{and}$$
$$n + x \leq 2\Delta$$

where ~~Where~~ $x = m - 2\ell$. Since assumption B gives $m - 2\ell \leq -1$, we see that

$$n - 1 \leq n + 1 \quad \cancel{n+1} \leq n - x \leq 2\Delta$$

or

$$n + x \leq \cancel{n - 1 \leq 2\Delta} \quad n - 1 \leq n + 1 \leq 2\Delta$$

Consider the game after $n - \Delta$ rounds, as illustrated in Figure $\boxed{3.11}$.

If $n - \Delta = \ell$, then they both reach u at the same time, with the survivor moving onto the z_2-occupied vertex (and losing). If we have $n - \Delta = \ell + 1$, then s reaches u first but is caught by z_2 on the following round. So, to guarantee the survivor has not escaped P_m we need

$$n - \Delta \leq \ell + 1$$

otherwise the survivor reaches the chord before z_2 can move to block. If the survivor reaches the chord first, then it could take the chord and possibly escape. (more detail??)

Then, to ensure that z_2 goes clockwise once it reaches v, we need X

$$\ell - (n - \Delta) \leq 1 + (n - \Delta - 1) + (m - \ell + 1)$$
$$2\Delta \leq 2n + m - 2\ell + 1$$

~~*Case II.B. Summary*~~ Same structure

z_1 goes counter-clockwise:

$$m + 1 \leq 2\ell \leq 2m - 2$$

and z_2 goes clockwise

$$n + 1 \leq 2\Delta$$

Section

there is no

the zombies win:

$$\begin{aligned} n - \Delta &\leq \ell + 1 \\ 2\Delta &\leq 2n + m - 2\ell + 1 \end{aligned}$$

We will show later (see $\boxed{3.4}$) that at least one value of Δ satisfies the zombie-win conditions of all four cases. Of course, this is not sufficient to show the zombies win: there's not guarantee that the survivor will choose to start along P_m as is assumed here, so we cannot simply start with this zombie configuration. Instead, we must force the survivor's hand.

3.2 Guarding the large cycle C_{m+1}

Do you mean C_{n+1}?

Part 2. We consider the game on this type of graph in general and show how we can position the zombies on C_{n+1} to limit the survivor's options and thereby guarantee it will be caught.

Choose k such that positioning

explain how you got this.

1. z_2 at $\Delta + k$ clockwise from u
2. z_1 at k counter-clockwise from v

forces the survivor into a losing position: it is either immediately sandwiched on C_{n+1}, or falls into the trap described above on C_{m+1}.

The survivor cannot start next to the zombies else it loses right away. So we choose k such that, even if the survivor is as far away from one of the zombies as possible on C_n, then the zombies still move in opposite directions. This leads to the following inequalities

one more figure?

$$\begin{aligned} n - \Delta - 2k - 2 &\leq \Delta + k + 1 + k + 2 \\ \Delta + 2k - 1 &\leq n - \Delta - 2k + 2 \end{aligned} \quad \text{and}$$

typo

C_n or C_{n+1}?

Solving for k gives

$$n - 2\Delta - 5 \leq 4k \leq n - 2\Delta + 3$$

Such k guarantees that the zombies start on vertices such that they must move in opposite directions if the survivor plays on C_n. C_n or C_{n+1}?

If the survivor starts between the zombies such that access to the chord is blocked, then clearly it has lost. Otherwise, the zombies must move towards the chord and in k rounds we reach the scenario described in Part 1 when z_1 reaches the chord and z_2 is Δ away. With suitable Δ, then, the survivor cannot win.

3.3 Existence of Δ and k for any m, n

Part 3. We wish to show that, for any m, n, there exist Δ and k which guarantee the survivor is caught. First we show that $\Delta = \lfloor \frac{m}{2} \rfloor$ always works for the cornering strategy.

Note that

$$2\Delta = 2\left\lfloor \frac{m}{2} \right\rfloor = \begin{cases} m & \text{if } m \text{ is even} \\ m-1 & \text{if } m \text{ is odd} \end{cases}$$

and so $m - 1 \leq 2\lfloor \frac{m}{2} \rfloor \leq m$.

Suppose that we are in Case I. A. and $\Delta = \lfloor \frac{m}{2} \rfloor$. Case I. A is characterized by the following constraints:

$$4 \leq 2\ell \leq m + 1$$

and

$$2\Delta \leq n - m + 2\ell$$

or

$$2\Delta \leq n - 1$$

one line

The zombies win if

$$2\Delta \leq 2m - 2\ell + 2 \qquad \text{and}$$
$$m - 2\ell - 1 \leq 2\Delta$$

one line

So if we are in Case I. A. and $\Delta = \lfloor \frac{m}{2} \rfloor$ the zombies win since

$$2\Delta = 2\lfloor \frac{m}{2} \rfloor \leq m < 2m - (m+1) + 2 \leq 2m - 2\ell + 2 \qquad \text{and}$$
$$\cancel{m - 2\ell - 1 \leq} \; m - 5 < \cancel{2\lfloor \frac{m}{2} \rfloor = 2\Delta}$$

$m - 2\ell - 1 \leq m - 2\ell - 4 - 1 \leq$ $m - 5$ $< m - 1 \leq 2\lfloor \frac{m}{2} \rfloor = 2\Delta$

Which shows that the zombie-win requirements are met.
Suppose that we are not in Case 1. A. Negating the constraints of Case I. A. gives

$$2\Delta \geq n - m + 2\ell + 1$$

and

$$2\Delta \geq n - 1 + 1 = n$$

or

$$m + 1 \leq 2\ell \leq 2m - 2$$

how did you get this? I'm a little confused

If we assume that m is odd and $2\Delta \geq n$ then we obtain a contradiction since

$$2\Delta = 2\lfloor \frac{m}{2} \rfloor = m - 1 \geq n$$

is

and we have assumed that $m \leq n$.

If m even, $m = n$ and $2\Delta \geq n - m + 2\ell + 1$ then

$$\begin{aligned} 2\Delta &\geq n - m + 2\ell + 1 \\ m &\geq m - m + 2\ell + 1 \\ m &\geq 2\ell + 1 \\ 2\ell &\leq m - 1 \end{aligned}$$

So, if $m = n$ and they are even, then we are in Case 1. A unless $2\ell \leq m - 1$.

To recap: If we set $\Delta = \lfloor \frac{m}{2} \rfloor$, we are in Case 1.A unless

$$m = n \qquad \text{and they are even}$$

$$\Delta = \lfloor \frac{m}{2} \rfloor = \frac{m}{2}$$

$$4 \leq 2\ell \leq m - 1$$

Is it possible to have Case 1. B? Case 1. B is described by the following constraints:

$$m + 1 \leq 2\ell \leq 2m - 2$$

and

$$2\Delta \leq n + 1$$

or

$$2\Delta \leq n + m - 2\ell$$

The negation of which is:

$$2\Delta \geq n + 1 + 1$$

and

$$2\Delta \geq n + m - 2\ell + 1$$

or

$$4 \leq 2\ell \leq m + 1$$

But this leads to the contradiction:

$$n \geq m \geq 2\Delta \geq n + 2$$

In remains to check if we win in Case 2. A.

Assuming still that

$$m = n \qquad \text{they are even}$$

$$\Delta = \frac{m}{2}$$

$$4 \leq 2\ell \leq m - 1$$

The win conditions require

$$
\begin{aligned}
2n - 2m + 2\ell \leq 2\Delta &\leq 2n + 2\ell - m - 1 \\
2m - 2m + m - 1 \leq 2\Delta &\leq 2m + 4 - m - 1 \\
m - 1 \leq 2\Delta &\leq m + 3
\end{aligned}
$$

Which holds for $\Delta = \frac{m}{2}$.

□

3.4 Computing Δ and k

Given m and n, we choose Δ so that whenever we reach the scenario described in the first part, the survivor will be cornered. Such Δ must satisfy the following constraints for any possible value of ℓ.

Case I.A. Summary

z_1 goes clockwise:

$$4 \leq 2\ell \leq m + 1$$

and z_2 goes counter-clockwise

$$
\begin{aligned}
2\Delta &\leq n - m + 2\ell \qquad \text{or} \\
2\Delta &\leq n - 1
\end{aligned}
$$

the zombies win:

$$
\begin{aligned}
2\Delta &\leq 2m - 2\ell + 2 \qquad \text{and} \\
m - 2\ell - 1 &\leq 2\Delta
\end{aligned}
$$

Case I.B. Summary

z_1 goes counter-clockwise:

$$m + 1 \leq 2\ell \leq 2m - 2$$

and z_2 goes counter-clockwise

$$
\begin{aligned}
2\Delta &\leq n + 1 \qquad \text{or} \\
2\Delta &\leq n + m - 2\ell
\end{aligned}
$$

the zombies win:

$$
\begin{aligned}
2\Delta &\leq 2\ell \\
2\ell - m + 1 &\leq 2\Delta
\end{aligned}
$$

Case II.A. Summary

z_1 goes clockwise:

$$4 \leq 2\ell \leq m+1$$

and z_2 goes clockwise

$$\begin{aligned} n-m+2\ell \leq& 2\Delta \\ n-1 \leq& 2\Delta \end{aligned} \qquad \text{and}$$

the zombies win:

$$\begin{aligned} 2\Delta \geq& 2n-2m+2\ell \\ 2\Delta \leq& 2n+2\ell-m-1 \end{aligned}$$

Case II.B. Summary

z_1 goes counter-clockwise:

$$m+1 \leq 2\ell \leq 2m-2$$

and z_2 goes clockwise

$$n+1 \leq 2\Delta$$

the zombies win:

$$\begin{aligned} n-\Delta \leq& \ell+1 \\ 2\Delta \leq& 2n+m-2\ell+1 \end{aligned}$$

A simple algorithm to calculate possible values of Δ loops over $0 \leq \Delta \leq n$ and over $2 \leq \ell \leq m-1$ and tests, for each Δ and each ℓ, to determine which of the four cases is applicable and, if in one of the cases, whether the zombie-win constrains are satisfied. A value of Δ is accepted if, for every value of ℓ, the zombies win.

Once we have obtained possible Δ, we can then determine k by calculating the bounds

$$n-2\Delta-5 \leq 4k \leq n-2\Delta+3$$