Zombies and Survivors

on Graphs

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Abstract

Lorem ipsum

Contents

Chapter 1

Zombies and Survivors

1.1 Introduction

There's been a robbery downtown and the robbers are escaping by car. Officers already on the streets are notified moments later. The robbers make a desperate dash for the highway but have a squad car in pursuit. A media helicopter captures the scene from above.

The robbers seem to be getting away, putting some distance between themselves and the sirens. Suddenly, the car slams into a strip of tire spikes. The left two tires are shredded, causing the driver to lose control. The vehicle veers off the road, flips upside down and eventually comes to a stop in the ditch. The crash is soon surrounded by flashing lights.

The advantage of communication and central control allows the police to employ an effective strategy: position officers to cut off likely escape routes. A mathematician (or computer scientist) might consider these types of scenarios and wonder: was there ever any hope of escape? Maybe the robbers took the wrong route. Or what if instead of cops, it's a bunch of mindless zombies?

Vertex pursuit games are adversarial games played on graphs and an attempt to model these types of problems. Graphs are used to formally describe structures such as networks and maps. Players take turns moving tokens on a graph (the game board, if you like) with the objective to capture (or evade) the other player.

Many variations of these graph pursuit games have been proposed. There are many rules and parameters to tweak to produce different games:

1. How much information do the players have?
2. Do they know each others positions? From how far away?
3. Do the players know the playing field, i.e., the graph?
4. Are the players restricted to vertices or edges?
5. Are players obligated to move?
6. Does the graph change over time somehow?

The Game of Cops and Robbers on Graphs [1] is perhaps the most well-known vertex pursuit game. It is a perfect information game with Cops trying to catch the Robber.

A variation called Zombies and Survivors (Z & S or Zombie Game) was recently proposed and studied in [2] and [3]. Z & S is the same as Cops and Robbers with the added twist that the zombies are required to move directly towards the survivor.

This thesis has been an attempt to better understand this variant and, in particular, to see if the results obtained for Cops and Robbers still hold when the cops are constrained in their strategy.

1.1.1 How to Play

To begin, the zombies choose starting vertices. Then, the survivor chooses a start position. Then on the next and each following round the zombies (must) move toward the survivor and, if uncaught, the survivor (may) move. Here a move is an instantaneous jump along an edge from one vertex to another.

The sophistication of the zombies' strategy gives them their name: you can imagine the zombies – arms outstretched – ambling directly towards the survivor. In this game, the players have complete information of the graph and the positions of the players. Indeed, the zombies need to know the position of the survivor to enact their strategy.

The zombies move, the survivor responds and these two turns make one round. It has been asked by new players if the order of play might be reversed but then zombies always win.

The game concludes when either:

- A zombie catches the survivor. That is, a zombie wins by moving onto the vertex occupied by the survivor.
- It becomes clear that the survivor will never be caught. In this case the survivor wins.

It is easy to determine that a zombie has won. It is perhaps not as obvious how to determine the latter. We discuss winning conditions in greater detail in a later section.

The Z & S games studied herein use a few zombies chasing a single survivor. The game can be adapted to multiple surivors by making the zombies move toward to the closest survivor but if a single survivor can't hope to escape then adding more "survivors" is a little gruesome.

1.1.2 Notation

The graphs are assumed to be finite and connected: that is, there is a finite number of vertices and there exists a path between every pair of vertices. Playing on graphs with multiple connected components does not make much sense in the context of these games.

The following sections will use a few definitions from graph theory. Formally, a graph $G = (V, E)$ is composed of:

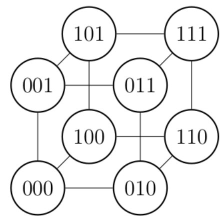


Figure 1.1: The Hypercube of Dimension 3

- A set V of vertices.
- A set $E \subset V \times V$ of edges described by a pair of endpoints.

If $G = (V, E)$ is a graph and $x, y \in V$ are vertices, we say that vertices x and y are neighbours if $(x, y) \in E$. That is, if there is an edge joining x to y. In these games, the edges are assumed to be undirected and so we may write $xy = yx$ and consider the two directions as a single edge. We call the set of all neighbours of x the neighbourhood of x which we denote $N(x) \subset V$.

For example, in Figure 1.1 we have vertices $V = \{000, 001, 010, 011, 100, 101, 110, 111\}$. Since 000 and 001 are connected, $(000, 001) \in E$. The neighbourhood of 000 is $N(000) = \{001, 010, 100\}$.

The degree of a vertex is the number of edges incident to that vertex. The minimum and maximum degrees of a graph are sometimes denoted as $\delta(G)$ and $\Delta(G)$, respectively.

1.1.3 Playing and Modeling the Game

The survivor is $s \in V(G)$ and $z_i \in V(G)$ are zombies with $i \in \{1, \ldots, k\}$. In the games studied there is a single survivor and a "small" number k of zombies.

We divide the game into rounds and turns. A round consists of two turns: a zombie turn and a survivor turn.

The game starts on round 0 with the zombies choosing initial vertices. The survivor follows. In a sense the game really begins in round 1 with the zombies finding, selecting and moving along a shortest path. The survivor responds. The game repeats in this way until the survivor is caught, or both players agree that the survivor will always escape.

We can implement this aspect of the game fairly easily by couting the turns. It is the zombie's turn on $t \equiv 0 \mod 2$ and the survivor's turn on $t \equiv 1 \mod 2$. Round r is given by $\lfloor \frac{t}{2} \rfloor$.

It is occasionally useful to identify the players' positions over time, in which case let $z_t^i \in V(G)$ be zombie i on round t. Similarly s_t is the survivor on round t. This burdensome notation will be omitted when possible.

It might be tempting to group the zombies together into some tuple of the vertex set, but each zombie acts independently of the others and so this may not always be practical.

1.1.4 Paths and Moves

The zombie strategy requires that we consider all shortests paths connecting the zombie to the survivor. In the studied discrete graphs below, the distance between two vertices is the hop length of a (shortest possible) path connecting them.

A path $P = v_0, v_1, v_2, \ldots, v_n$ is a "strict" walk: a non-repeating sequence of adjacent vertices in a graph. A path is an example of a subgraph $G' = (V(P), E(P))$ since $V(P) \subseteq V$ and $E(P) \subseteq E$.

Paths allow us to define a distance $d(x, y)$ between vertices as the length of the shortest path connecting them (or infinity if such does not exist) and computing such paths is a classic problem in computer science.

The diameter and girth of a graph are useful properties which appear in some of the theorems herein. The diameter $\text{diam}(G)$ is the length of the longest possible path in G and the girth of a graph is the length of the minimum order subcycle.

Now consider zombie k. According to the rules of the game, on its turn the zombie "must move on a shortest path" towards the survivor. More precisely, this requires considering $Z_k = \{\exists \ell : z_k = u_{i,0}, u_{i,1}, u_{i,2}, \ldots, u_{i,\ell-1}, s = u_{i,\ell}\}$ the set of i different $z_k s$-paths of length ℓ.

There is at least one such path since our graph is presumed connected, so $i > 0$ and $Z_k \neq \emptyset$.

If there is only one path, then z_k's next move is $u_{i,1}$. If all zs-paths include $u_{i,1}$, then again z_k's next move must be to that vertex.

If, however, there are multiple zs-paths which have different first moves, then the zombie could make multiple moves.

We call all the set of all neighbours on a shortest path to the survivor the *zombie moves*, which could be denoted

$$Z[x; s] = \{y \in N(x) \mid d(y, s) = d(x, s) - 1$$

the zombies moves from x given survivor is on s.

1.1.5 Zombie Number

EDIT: Not quite. Redo

The minimum number zombies guaranteed to win.

If a single zombie is guaranteed to win on a graph, we say that it is zombie-win and that it has zombie number $z(G) = 1$. If it can be shown that k zombies win, then $z(G) = k$

1.2 Literature Review

The following is a summary of results relevant to the Game of Zombies and Survivors.

1.2.1 Cops and Robbers, Cop-Number

Study of vertex pursuit games is first attributed to Quilliot [4, 5], and Nowakowski and Winkler [6]. Both authors independently consider games of Cops and Robbers with a single Cop and characterize by way of a relation those graphs where the Cop always wins. These are now known as Cop-win graphs and can be recognized by the existence of an ordering of the vertices called a *dismantling*; so-called because it is the successive deletion of *corners* resulting in a single vertex (see the last section on dismantlings, cop-win trees and visiblity graphs).

The Cop-number of a graph (denoted $c(G)$) is introduced by Aigner and Fromme in [7]) and defined as the minimum number of Cops required to guarantee a Cop win on a graph G.

It is possible to recognize graphs for which k Cops are guaranteed to win, i.e. $c(G) = k$ [8] and [9]. A graph is k-cop win if and only if there exists a function (on a k-product of the graph to represent the position of the Cops) which satisfies certain properties; essentially it is a function which takes as input a position C of Cops and returns the next position for the Cops that guarantees a win (see [1][p. 119]). There exists a polynomial-time algorithm for deciding whether a graph is k-Cop-win by iteratively solving for this function.

Another important line of inquiry relating to the Cop-number is the investigation of Meyniel's conjecture, which posits that $\mathcal{O}(\sqrt{n})$ is an upper bound on the Cop-number [10]. Incremental progress has been made on special classes of graphs as well as for graphs in general. See also for a recent overview [11][p. 31].

1.2.2 The Cop-Number and the Genus of the Graph

One of the most surprising results about the Game of Cops and Robbers are owed to Aigner and Fromme [7], who showed that the cop number of a planar graph is at most 3. Basically, a graph is planar if it can be drawn in the plane (say, on a piece of paper) without crossing any edges. Aigner and Fromme describe a 3-Cop strategy which uses *isometric* paths of the graph to encircle and entrap the Robber.

Outerplanar graphs are planar graphs whose vertices all belong to the outer face. [12] shows that the cop number of outerplanar graphs is 2 by considering two possible cases: those with and without cut vertices.

Planar graphs of order n [13]

1.2.3 Relation to the Girth and Minimum Degree of a Graph

Aigner and Fromme also show a relationship between the Cop-number, the girth of a graph and its minimum degree [7]. More precisely, if G has girth at least 5, then $c(G) \geq \delta(G)$ where $\delta(G)$ is the minimum degree of G.

This result has since been refined by [] and again recently in [].

1.2.4 Dismantlings, Cop-win Trees, Zombie-win Trees

Quilliot and Nowakowski both indepedently characterized cop-win graphs as those which admit a *dismantling*.

A (one-point) retract is an edge preserving function $f : G \mapsto H = G \setminus v$ (aka a homomorphism) such that $f(v) = x$ for some $x \neq v \in V(G)$ and f restricted on H is the identity. Formally,

$$f(v) = x \qquad f(u) = u \qquad \forall u \in V(G) \setminus \{v\}$$

and

$$xy \in E(G) \implies f(x)f(y) \in E(G \setminus \{v\})$$

If G is a reflexive graph, then a one-point retract can be seen as joining two vertices. The edge between two adjacent vertices becomes another loop. The retract maps a graph G to graph G' with one less vertex.

Recall that corners are vertices v whose closed neighbourhoods are a subset of a neighbours' closed neighbourhood, i.e.

$$u, v \in V(G) \qquad \text{and} \qquad N[v] \subseteq N[u]$$

You can define a retract on corner v: if v is a corner, then it is dominated by some $u \in V(G)$. So if $x \in V(G)$, $x \neq v$ and $xv \in E(G)$ then $xu \in E(G)$ by definition of a corner. Therefore the map

$$f(x) = \begin{cases} u & \text{if } x = v \\ x & \text{otherwise} \end{cases}$$

is edge preserving since $f(x)f(v) = xu$ and $xu \in E(G)$, so $xu \in E(H) = E(G - v)$. For other vertices $x, y \notin \{u, v\}$, $f(x)f(y) = xy \in E(G)$ so $f(x)f(y) \in E(G - v)$ also. This shows that f is a homomorphism as required and hence a retract.

This is a formal way of saying that a corner of a graph can be folded into a dominating vertex: an astute Robber would never move into a corner.

A dismantling is a sequence of retracts $f_1, f_2, \ldots, f_{n-1}$ such that the composition $F_{n-1} = f_{n-1} \circ f_{n-2} \circ \cdots f_2 \circ f_1$ gives a function for which $F_{n-1}(G) = K_1$. That is, there is a sequence of retracts which maps the graph to a single vertex.

Not all vertices of a graph need be corners in order for there to exist a dismantling: it suffices to have an ordering where each v_i is a corner in $G[v_i, v_{i+1}, \ldots, v_n]$.

Such a sequence of f_j's defines a copwin ordering

$$\mathcal{O} = \{v_1, v_2, \ldots, v_n\}$$

where v_1 is a corner in $G_1 = G$, v_2 is a corner in $G - v_1$, and so on.

A fundamental result in C & R is that copwin graphs – graphs for which a single cop is guaranteed to win – are characterized by the existence of such dismantlings. A graph is copwin if and only if it is dismantlable.

A Cop-win spanning tree combines the idea of a dismantling with a spanning tree and was first proposed in [12].

A copwin spanning tree S is defined as a tree where $x, y \in V(G)$ $xy \in E(S)$ if there exists a retract f_j in the dismantling $F_n = f_n \circ f_{n-1} \circ \cdots \circ f_2 \circ f_1$ such that $f_j(x) = y$ or $f_j(y) = x$.

Copwin spanning trees give a strategy for the cops to follow: start at the root (the last vertex in the ordering) and descend the tree in the branch containing the robber. Lemmas 2.1.2 and 2.1.3 from [12] show that the cop can always stay in the same branch (and above) the robber in the tree. So the robber is eventually stuck in a leaf and caught.

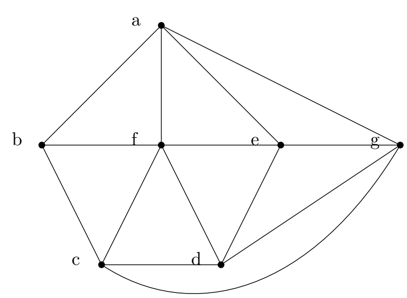
Zombies cannot apply a cornering strategy, however. Fitzpatrick showed that a graph is zombie-win if a specific spanning tree exists [2]:

Theorem 6. [Fitzpatrick] If there exists a breadth-first search of a graph G such that the associated spanning tree is also a cop-win spanning tree, then G is zombie-win.

Thus a sufficient condition for zombie-win graphs are those for which a specific copwin tree exists: one equivalent to a breadth-first search of the graph from some vertex. It remains unclear if it is also a necessary condition.

A few questions: are copwin graphs necessarily zombie win? (No. Smallest counter example?) What is the dismantling of this copwin but not-zombie win graph. Since a dismantling exists, a copwin spanning tree exists.

Here is an example of a graph and two dismantlings, one of which results in a BFS tree, and the other does not.



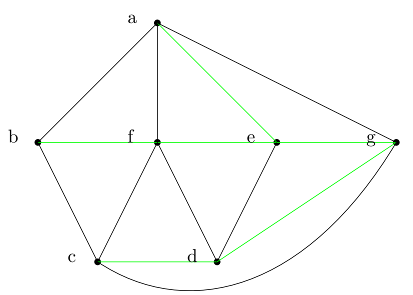
Here are two dismantlings, their orderings, and the resulting copwin spanning trees.

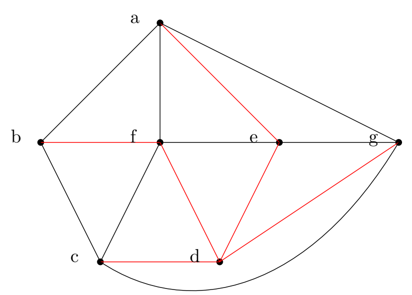
$$f_1(b) = f$$
$$f_2(c) = d$$
$$f_3(f) = e$$
$$f_4(a) = e$$
$$f_5(e) = g$$
$$f_6(d) = g$$

Gives ordering $\mathcal{O}_1 = \{b, c, f, a, e, d, g\}$. Whereas

$$g_1(b) = f$$
$$g_2(a) = e$$
$$g_3(c) = d$$
$$g_4(f) = d$$
$$g_5(e) = d$$
$$g_6(g) = d$$

Also gives a dismantling with ordering $\mathcal{O}_2 = \{b, a, c, f, e, g, d\}$. But only the second produces a copwin tree obtainable as a bread-first search.





Moreover, it would seem that a zombie loses if it starts on g, but not on d.

1.2.5 Probabilistic zombies

Zombies are often depicted as mindless or aimless. It is a common trope that zombies idle around, moving in random directions until they somehow (suddenly) distinguish the uninfected. It is only at this point that the zombies will charge.

Such behavior likely inspired another type of pursuit game in which the zombies start randomly on the graph. Once the survivor chooses a start vertex, the zombies "notice" the survivor and start moving directly towards it.

Without knowing where the zombies start, however, it is impossible to know the outcome with certainty. So study of these games becomes probabilistic; zombies win if they have at least a 50% chance of winning. The (probabilistic) zombie number of a graph is the number of zombies required for a 50% chance of winning and this zombie number is obtained for several classes of graphs in [14] and for toroidal grids in [15].

1.2.6 Deterministic zombies

This is the version of the game we study below.

1.3 Survivor Strategy

Suppose we could agree on some algorithm to fully determine the zombies' behaviour. Or, perhaps, assume that all possible games will exhaustively be played by the computer. How then, should we program the survivor to maximize its chances of survival? On every round, the survivor may stay in place or move to one of its neighbours. However, if ever the survivor moves to a vertex adjacent to a zombie, then it loses immediately on the next round. So the *valid survivor moves* are the neighbours of the survivor (or its current position), minus those adjacent to one of the k zombies.

If the survivor is s and $Z = \{z_1, z_2, \ldots, z_k\}$ is the set of zombie positions, then

$$N[s] \setminus N[Z]$$

Where the neighbourhood of the set Z is the union of all of the zombies' neighbourhoods. These survivor moves can be computed by iterating through the neighbours of s and removing those that are neighbours of a zombie. Another approach would be to use the results of Floyd-Warshall, as with the zombies:

1. Scan row s of A for indexes x where $a_{s,x} = 1$. These are the neighbours of s. Add each neighbour a set S.

2. For each neighbour x and for each zombie z, $1 \leq z \leq k$, probe $a_{z,x}$. This is the distance from the neighbour to the zombie.

3. If $a_{z,x} = 1$, then x is adjacent to a zombie and so $S = S \setminus \{x\}$.

4. Return S

If the set of valid survivor moves is empty, then the survivor is cornered. The only remaining move is to pass, and be caught after another round. If the set returned is a singleton, then circumstances have forced the survivor's hand. If, however, there many possible moves, then how best do we choose among them?

Perhaps the simplest strategy is to invert the strategy used by the zombies: the survivor makes the move that maximizes its distance from all of the zombies. While running the algorithm described above, we could simultaneously compute $\sum_{i=1}^{k} d(x, z_i)$, the sum of all the distances from the neighbour to the zombies, and choose one the moves that maximizes this value.

This cowardly strategy is amusingly similar to that of the zombies. It is also a poor strategy. The only way to escape the zombies is to lead them into some sort of cycle, as we discuss next. So the survivor needs to act with more sophistication than just fleeing in the opposite direction. The game depicted below is an example where the survivor has an easy win, but the strategy above fails.

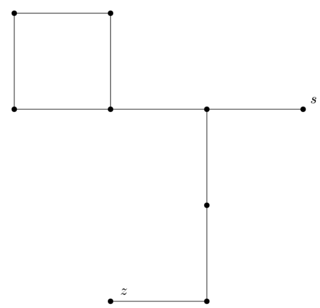


Figure 1.2: The Cowardly Strategy Fails

1.4 Planar Zombies

In [7], Aigner and Fromme showed that the cop number for a planar graph is at most three. A natural question, then, is to ask whether there exists an upper bound on the zombie number for planar graphs. While we have not yet answered this question, we have found a planar graph for which the zombie number is greater than 3.

Such a graph G consists of an interior 5-cycle with 5 outer paths connecting two adjacent vertices of the interior cycle. There are also edges connecting the second and before-last outer paths to allow the survivor to escape in certain situations. This graph, which is illustrated in Figure 1.3, is an extension of the graph in [2][Fig. 2] which has $z(G) = 3 > 2 = c(G)$.

We refer to the beginning of the game where the zombies and the survivor choose their starting positions as round zero or the starting round. We assume that the zombies choose distinct starting vertices to maximize their chances of winning since the game is easily won by the survivor if there are fewer than 3 zombies (for example, by adding another arbitrary zombie and following one of the strategies described below).

We call vertices

$C = \{1, \ldots, 5\}$	the interior 5-cycle
$X = V(G) \setminus C$	those vertices not on the interior 5-cycle
$Y = \{7, 9, 12, 14, 17, 19, 22, 24, 27, 29\}$	the vertices of degree 3.
$S = \{7, 8, 9, 12, 13, 14, 17, 18, 19, 22, 23, 24, 27, 28, 29\}$	the outermost 15-cycle

With this notation, we describe how the survivor can escape 3 zombies by providing a strategy for the three possible zombie start configurations:

- $z_i \in C$ for $1 \leq i \leq 3$: all the zombies start on the interior 5-cycle.
- $z_1, z_2 \in C$ and $z_3 \in X = V(G) \setminus C$: two of the zombies are on the interior 5-cycle but one is not.
- $z_1, z_2 \in X$: at least two of the zombies are not on the interior 5-cycle.

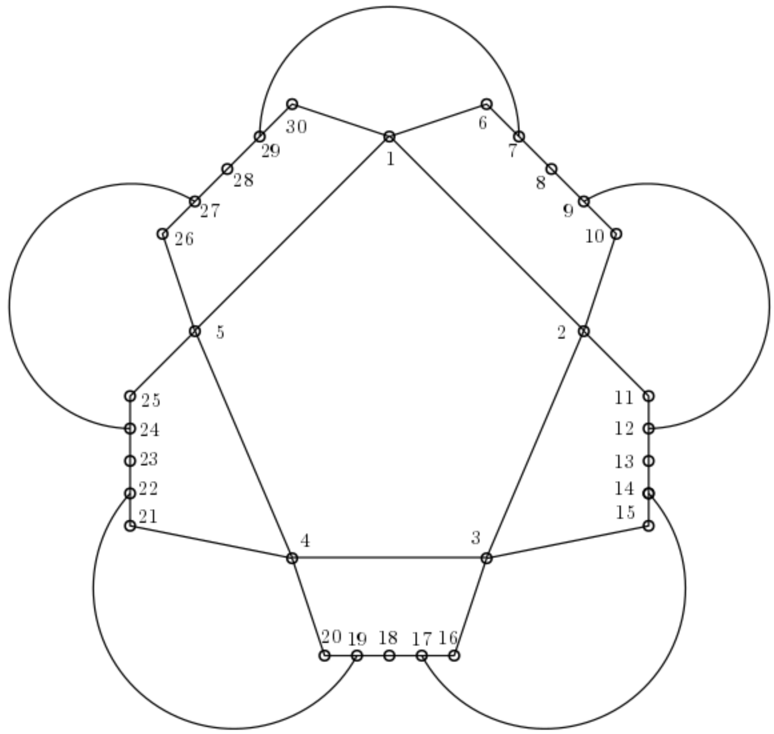


Figure 1.3: A graph with $z(G) > 3$

Our proof relies on a special characteristic of this graph: if the survivor and the three zombies are all on $G[S]$, the outermost 15-cycle, with the zombies on the same side of the survivor and within a distance of 2, 3, 4 or 5, then the survivor can win by fleeing away from the zombies around the outermost 15-cycle.

To see this, let $E' = \{xy \in E(G) : x, y \in Y\}$ be the set of edges which connect an exterior 5-path to another and let $G' = G - E'$ be the subgraph without these edges. These edges are highlighted in red in Figure 1.4:

If the survivor and zombie are both on the outermost cycle at distances 2 or 3 then the fact that the zombies must stay in S is obvious. The following table shows that when the zombie and the survivor are both in S and within a distance of 4 or 5, then the shortest path from the zombie to the survivor is contained entirely in S and thus zombies never have the opportunity to leave the outermost 15-cycle.

z	s	shortest path in G	$d_G(z,s)$	shortest path in G'	$d_{G'}(z,s)$
7	14	7,8,9,12,13,14	5	7,6,1,2,3,15,14	6
8	17	8,9,12,13,14,17	5	8,9,10,2,3,16,17	6
9	18	9,12,13,14,17	5	9,10,2,3,16,17,18	6
8	14	8,9,12,13,14	4	8,9,10,2,3,15,14	6
9	17	9,12,13,14,17	4	9, 10, 2, 3, 16, 17	5
12	18	12, 13, 14, 17, 18	4	12, 11, 2, 3, 16, 17, 18	6

We now give winning survivor strategies for each of the possible zombie-start configura-

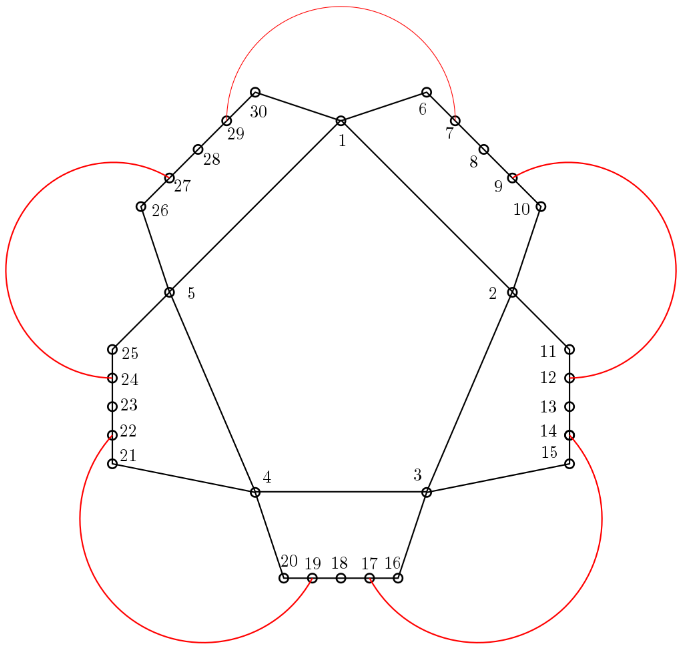


Figure 1.4: An escape strategy for the survivor

tions.

Proof. Case I: The three zombies choose vertices on the interior 5-cycle.

Instead of showing that the strategy works for all possible start configurations of 3 zombies on the interior 5-cycle, we show that the survivor can escape 5 zombies if they all start on the interior 5-cycle. The zombies occupy the vertices 1–5 and the survivor chooses a vertex of degree 3. Without loss of generality, say the survivor chooses 12.

If the survivor starts on $y_1 \in Y$ (one of the vertices of degree 3), and moves to $y_2 \in Y$ using edge y_1y_2 and continues to flee in the same direction along the outermost 15-cycle, then the zombies will not be able to catch the survivor.

Notice that as soon as all three zombies are within a distance of 5 of the survivor on the outermost 15-path, then by the discussion above the game is won by the survivor.

On the first round, the zombies each have a single shortest path to the survivor on 12 and thus must move as follows:

- The zombie on 2 moves to 11.
- The zombies on 1 and 3 collide on 2.
- The zombies on 4 and 5 move to 3 and 1, respectively.

The survivor responds by moving to 9. These moves are illustrated in the following figures:

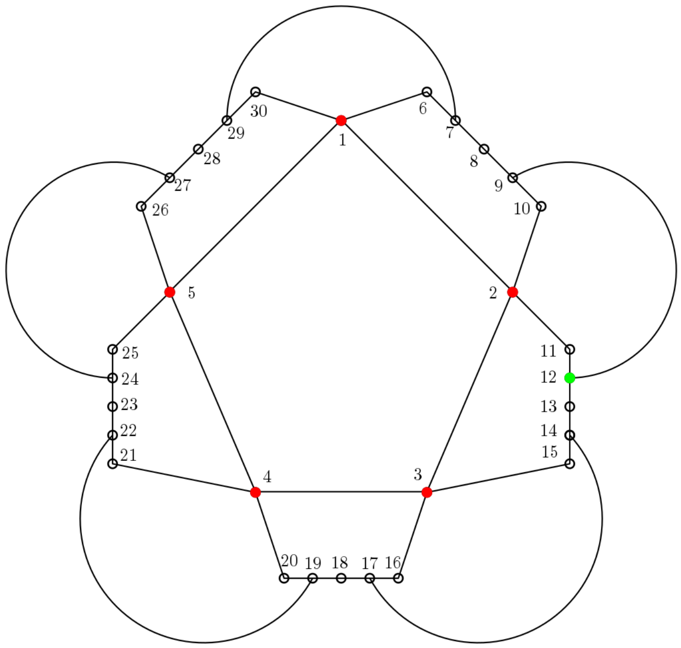


Figure 1.5: Case I, Round 0

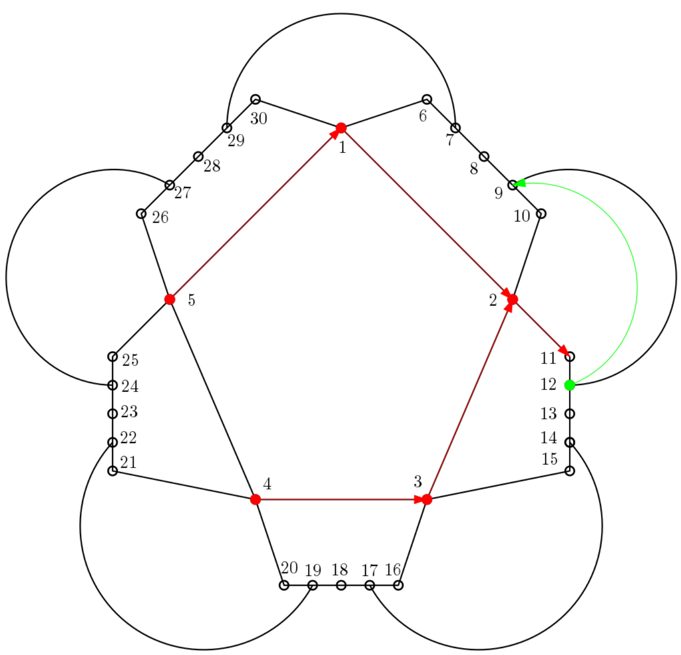


Figure 1.6: Case I, Round 1

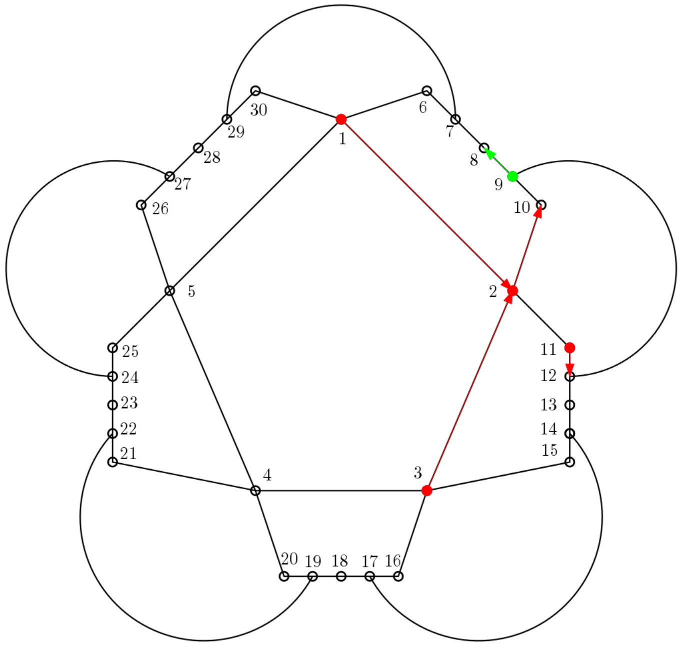


Figure 1.7: Case I, Round 2

Yet again the zombies have a single shortest path to the survivor on 9 and thus move as follows:

- The zombie on 11 moves to 12.
- Zombies on 2 move to 10.
- Zombies on 1 and 3 collide on 2.

The survivor responds by moving to 8. These moves are illustrated in the following figure:

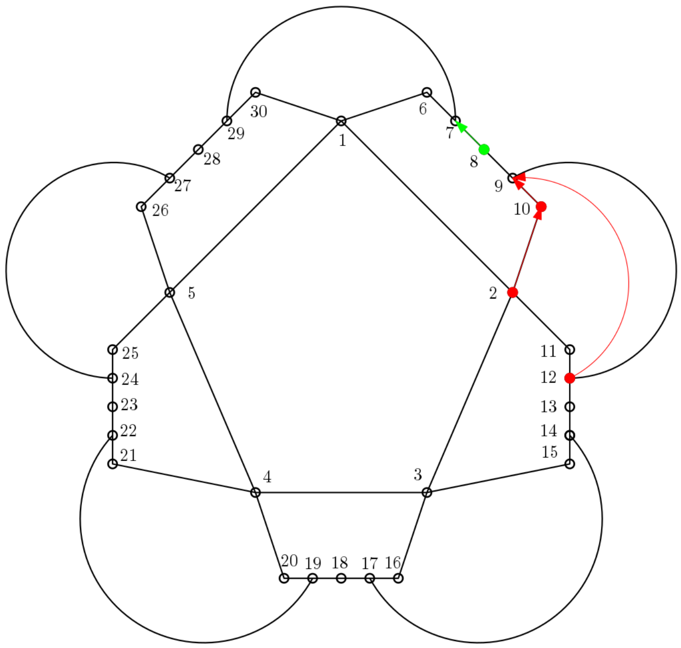


Figure 1.8: Case I, Round 3

Finally, after round 3 all zombies are within a distance of 3 of the survivor on the outermost 15-cycle, and so the survivor wins by running anti-clockwise on the cycle $G[S]$.

This shows that however the 3 zombies on the interior 5-cycle may be arranged in the initial round, they will not be able to corner the survivor following this strategy.

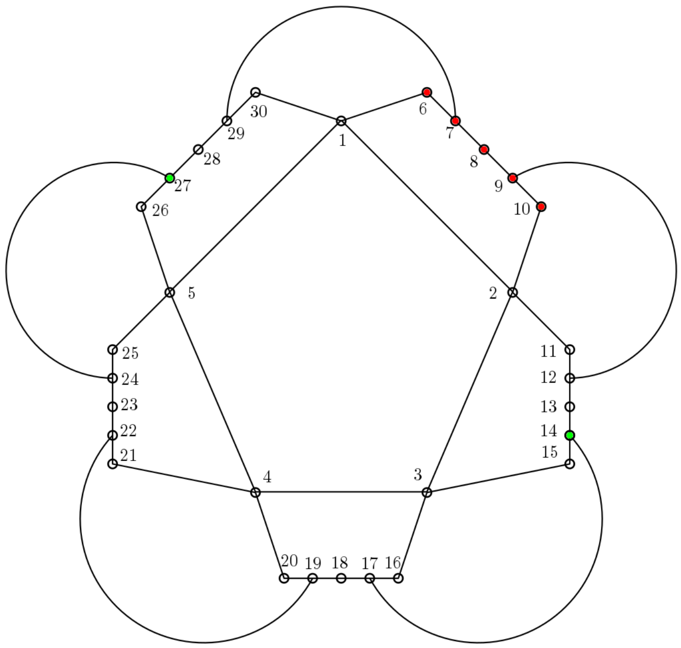


Figure 1.9: Case 2, Round 0

Case II: Two zombies (z_1, z_2) choose vertices on the interior 5-cycle and one zombie (z_3) chooses a vertex in $X = \{6, \dots, 30\}$, an exterior vertex.

We use the same strategy as in Case I with an additional restriction on the survivor's start vertex. The survivor starts on $s = y_1 \in Y$ (a vertex of degree 3) such that $3 \leq d_{G[X]}(s, z_3) \leq 4$ and so that the edge connecting y_1 to $y_2 \in Y$ is not on the shortest path between s and z_3. That is to say, the survivor can flee from z_3 along an edge connecting two exterior 5-paths.

This choice of start vertex is always available to the survivor. Without loss of generality, assume that z_3 has chosen one of the vertices on the exterior 5-path 6-10.

- if z_3 chooses to start at 7 or 6, then the survivor chooses 27, which is at a distance of 3 or 4 respectively.

- if z_3 chooses to start at 8, then the survivor can start at either 14 or 27, both of which are at a distance of 4.

- if z_3 chooses to start at 9 or 10, then the survivor chooses 14, which is at a distance 3 or 4 respectively.

In round 1, if z_3 is not adjacent to the interior 5-cycle (either starting at 7, 8 or 9), then already the zombie has no choice but to pursue the survivor on the outermost 15-cycle.

If z_3 is adjacent to the interior 5-cycle (either starting at 6 or 10), then z_3 may choose either to move onto the outermost 15-cycle or to cut through the interior 5-cycle since both are moves on a shortest s, z_3 paths.

However, as above, if z_3 chooses to move onto a vertex in S and follow along the outermost 15-cycle, then the game is already won for the survivor since $d(z_3, s) = 4$ and thus the third zombie can been forced to chase around the outermost 15-cycle forever.

If z_3 chooses to move to the interior cycle then all three zombies are on the interior 5-cycle and we have reached a situation just as in Case I, Round 1.

This shows that the survivor will always escape the third zombie following this strategy. Now because this strategy is a restricted version of the strategy from Case 1, we know that the zombies that start on the interior 5-cycle will not be able to corner the survivor. Therefore, this strategy defeats all possible start configurations where two zombies start on the interior 5-cycle and the third starts on the exterior.

Case III: One zombie chooses a vertex on the interior 5-cycle, the two others choose vertices on the exterior.

We were unable to develop an argument to concisely show why the survivor wins in this case. Instead, in Appendix .3 include tables showing the first few moves of a winning survivor strategy for every possible zombie start (without loss of generality).

Case IV: All three zombies choose exterior vertices in X.

We separate this case again into sub-cases based on the number of moves required by the zombies to reach the interior cycle.

Case IV(a): All three zombies require the same number of rounds to reach the interior 5-cycle.

Suppose all the zombies have chosen vertices in X which are adjacent to vertices in C. These are vertices $Q = \{6, 10, 11, 15, 16, 20, 21, 25, 26, 30\}$. Because there are 3 zombies and 5 interior vertices, there will always be at least two vertices in the interior cycle that are not threatened in round 0. The survivor starts on one of these safe vertices.

In round 1, the zombies have no choice but to enter the interior 5-cycle since the shortest path from a vertex $q \in Q$ to $s \in C$ necessarily includes the edge qc for some $c \in C$. Thus, after their first turn, the zombies all occupy vertices in the interior 5-cycle. The survivor responds by exiting the interior 5-cycle to $s' \in Q$.

In round 2, the zombies again have no choice but to approach the survivor using vertices on the interior 5-cycle. The survivor responds by moving to some $s'' \in Y$ and we have reached a scenario just like in Case I and so the survivor has a winning strategy.

If all the zombies are at a distance of 2 from the interior 5-cycle (those vertices in Y) then the survivor can start on any vertex $s \in C$.

In round 1, the zombies approach the survivor by moving to vertices in Q. Let $q_0, q_1 \in Q \cap N(s)$ be the neighbours of the survivor which are not on the interior 5-cycle. Now, either:

1. q_0 and q_1 are occupied by zombies. In this case, there is some $c \in N(s^0) \cap C$ which is not threatened by a zombie (since two of them are adjacent to s). Therefore the survivor can safely move onto another vertex on the interior 5-cycle and, on the following round, move to an occupied vertex in Q. After another round the survivor moves to a vertex in Y and we again have reached a situation as in Case I.

2. q_0 and q_1 are not both occupied by zombies. In this case, the survivor can exit the interior 5-cycle immediately by moving to a vertex in Q. After the next round, all

three zombies are on the interior 5-cycle and the survivor moves to a vertex in Y and again we are in a situation like Case I.

If all the zombies are at a distance of 3 from the interior 5-cycle, then the survivor may start on any vertex of C and simply pass on the first round. The zombies, have no choice but to move to vertices in Y and so we find ourselves in the case described before.

Now we must deal with the cases where the zombies are at different distances from the center cycle.

Case IV(b): Two zombies start adjacent to the interior 5-cycle, and the third is at distance 2 from the interior 5-cycle.

Suppose that two of the zombies have chosen vertices in Q and the other has chosen a vertex in Y. That is, two zombies are adjacent to the interior 5-cycle while the third requires two rounds to reach the interior 5-cycle.

There are now at least three unthreatened vertices on the interior 5-cycle for the survivor to choose. The survivor can choose any unthreatened vertex on the interior 5-cycle.

In round 1, two zombies enter the interior 5-cycle and the third moves to a vertex $q \in Q$ adjacent to the interior 5-cycle. The survivor exits the interior 5-cycle to another vertex $q_0 \in Q$. This move is always available to the survivor since only one vertex in Q is occupied by a zombie and every vertex in C is adjacent to two verties in Q.

After the next turn, all three zombies are on the interior 5-cycle and the survivor is on a vertex $s^2 \in Y$ and so the survivor has a winning strategy.

Case IV(c): Two zombies start at a distance of 2 from the interior 5-cycle and the third is at a distance of 3.

The survivor may start on any of the vertices on the interior 5-cycle since none are threatened by a zombie.

In round 1, two zombies move to vertices in Q and the third moves to a vertex in Y. If the survivor is unthreatened after the first round, she may simply pass. If the survivor is threatened by one of the zombies adjacent to the interior 5-cycle, then at least one of her neighbours on the interior 5-cycle is unthreatened.

In either case, after round 1 we find ourselves in the situation described in Case IV(b).

Case IV(d): Two zombies start adjacent to the interior 5-cycle, and the third is at distance 3 from the interior 5-cycle.

This scenario is slightly more complicated as the survivor must avoid being trapped by the third zombie. Consider, for example, the start configuration $\bar{z} = (10, 26, 18)$. If the survivor chooses to start at 4, then the game plays out as follows:

Round	z_1	z_2	z_3	s
0	10	26	18	4
1	2	5	19	21
2	3	4	22	21

The survivor is cornered by the zombies approaching from the interior 5-cycle and by the third zombie which uses the edge 19-22. However, the survivor could have started at 1, in which case the game is won by the survivor as follows:

Round	z_1	z_2	z_3	s
0	10	26	18	1
1	2	5	17 or 19	6
2	1	1	16 or 20	7
3	6	6	3 or 4	29

And we see that the survivor has a winning strategy for this start configuration.

Suppose without loss of generality that the zombie at distance 3 from the interior 5-cycle has chosen vertex 18. Since there are two zombies adjacent to the interior 5-cycle, at least one of the vertices $\{1, 2, 5\}$ must be a safe start for the survivor.

We may disregard the zombies that started at a distance of 1 from the interior 5-cycle in this next analysis since the survivor's strategy will be the same as in Case IV(a) and so these zombies will not be able to capture the survivor. Having shown above that if 1 is a safe start for the survivor, it remains to show that the strategy works if only 2 or 5 are safe starts. Since they are symmetric, we show that the strategy works if 2 is a safe start for the survivor.

Round	z	s
0	18	2
1	17	10
2	16	9
3	3	8
4	2	7
5	1	29
6	30	28

Thus after 7 rounds, the survivor has succesfully baited all three zombies onto an exterior 5-path and so the game is won.

Case IV(e): One zombie starts adjacent to the interior 5-cycle, and the other two are at a distance of 2 from the interior 5-cycle.

Again, the survivor's strategy in this case is to waste time on the interior 5-cycle in order to allow all the zombies to approach. Since only one of the zombies is adjacent to the interior 5-cycle, there are four potential start vertices for the survivor. Any of these will work.

In round 1, the zombie at distance 1 from the interior 5-cycle moves onto the interior 5-cycle and the other two move to vertices $q_0, q_1 \in Q$, which are adjacent to the interior 5-cycle.

Now, either:

1. q_0 and q_1 are adjacent to s^0. In this case, the survivor moves to $s^1 \in N(s^0) \cap C$, the neighbour on the interior 5-cycle that is not occupied by the zombie that has already reached the interior 5-cycle. After the next turn, all three zombies have reached the interior 5-cycle and so the survivor can exit to some $s^2 \in Q$. Again, after another round we have returned to Case I.

2. q_0 and q_1 are not both adjacent to s^0. In this case, the survivor can exit the interior 5-cycle by moving to a vertex $s^1 \in Q$. After the next round, all three zombies are on the interior 5-cycle and we are in a situation like Case I.

In either case, the survivor has a simple winning strategy.

Case IV(f): One zombie starts at a distance of 2 from the interior 5-cycle, and the other two are at a distance of 3.

The survivor starts in the interior 5-cycle. None of the vertices on the interior 5-cycle are threatened by the zombies, since they are at a distance at least 2.

In round 1, the zombies approach the interior 5-cycle. The zombie that started at distance 2 from the interior 5-cycle is now on a vertex in Q and the other two zombies are on vertices in Y. If unthreatened, the survivor simply passes. If the survivor is threatened by the zombie that is adjacent to the interior 5-cycle, then she moves to another vertex on the interior 5-cycle. The other two zombies pose no threat in this round.

There is now one zombie at distance of 1 from the interior 5-cycle and two zombies at a distance of 2, and so we have returned to the situation describe in Case IV(e).

Case IV(g): One zombie starts at a distance of 1 from the interior 5-cycle, and the other two are at a distance of 3.

The survivor starts on one of the four safe vertices on the interior 5-cycle.

In round 1, one zombie steps onto the interior 5-cycle while the other two zombies move to vertices at distance 2 from the interior 5-cycle. Only the zombie on the interior 5-cycle can threaten the survivor at this point. If the survivor is safe, then she may pass. Otherwise, since there is only a single zombie on the interior 5-cycle, at most one of the survivor's neighbours on the interior 5-cycle is threatened. So the survivor has a safe move to a vertex on interior 5-cycle.

In round 2, the zombie on the interior 5-cycle pursues the survivor ineffectually while the other two zombies move to vertices $q_0, q_1 \in Q$ which are adjacent to the interior 5-cycle. Now, as in Case IV(e), either

1. q_0 and q_1 are adjacent to s^0. In this case, the survivor moves to $s^1 \in N(s^0) \cap C$, the neighbour on the interior 5-cycle that is not occupied by the zombie that has already reached the interior 5-cycle. After the next turn, all three zombies have reached the interior 5-cycle and so the survivor can exit to some $s^2 \in Q$. Again, after another round we have returned to Case I.

2. q_0 and q_1 are not both adjacent to s^0. In this case, the survivor can exit the interior 5-cycle by moving to a vertex $s^1 \in Q$. After the next round, all three zombies are on the interior 5-cycle and we are in a situation like Case I.

Case IV(h): The three zombies are at different distances from the interior 5-cycle.

In particular, this means that the zombies are at distances 1, 2 and 3 from the interior 5-cycle.

Observe that there is always a vertex in the interior 5-cycle that is at distance at least 3 from all zombies. This is a start position for the survivor which will allow her to survive unthreatened for at least two rounds.

In round 1, the closest zombie (more precision here - give label) moves onto the interior 5-cycle, the second closest moves to a vertex adjacent to the interior 5-cycle and the third moves to a vertex at a distance of 2 from the interior 5-cycle. The survivor remains in place.

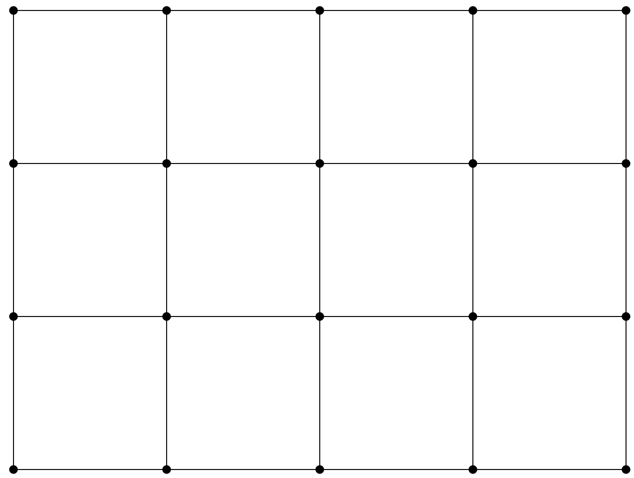


Figure 1.10: $G_{4,5}$

In round 2, the closest zombie threatens the survivor, the second closest zombie moves onto the interior 5-cycle, and the last one moves onto a vertex adjacent to the interior 5-cycle. Now, at least one of the survivor's neighbours is an unoccupied vertex in Q, which she can take to escape the interior 5-cycle.

After the next round, all three zombies are on the interior 5-cycle or one step behind the survivor and the survivor has won the game by moving to a vertex in Y as in Case I.

□

1.5 Zombies on the Grid

EDITS: We must always clearly state when an action or consequence occurs: at the end of a turn or the end of a round. (A round being composed of two turns).

Add Theorem and proof environments.

Another theorem on worst-case capture time? Diameter of grid?

The Grid graph $G_{m,n}$ is a rectangular arrangement of mn vertices in m rows and n columns. Vertices are joined by an edge if they are on the same row or column. The goal of this section is to prove

Theorem 1. The zombie number of the Grid is 2 for any m, n.

As a quick aside, note that the grid graph can be constructed by taking the Cartesian product of two paths $P_m \square P_n$. The Cartesian product of G_1 and G_2, denoted $G_1 \square G_2$ is a graph G whose vertices are all pairs of vertices of the two graphs. Two vertex pairs are connected by an edge if they are equal in one component and the other is joined in the original graph. In set notation, that is

$$V(G) = \{(u, v) \mid u \in V(G_1), v \in V(G_2)\}$$

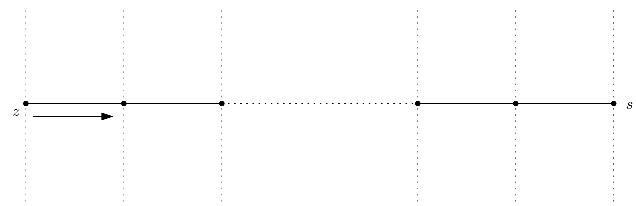


Figure 1.11: Zombie and Survivor on Same Row

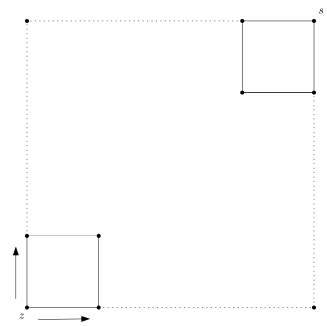


Figure 1.12: Zombie and Survivor on Different Row and Column

$$E(G) = \{\{(u_1, v_1), (u_2, v_2)\} \mid (u_1 \sim_{G_1} u_2 \wedge v_1 = v_2) \vee (u_1 = u_2 \wedge v_1 \sim_{G_2} v_2)\}$$

We claim that two zombies suffice to win on this family of graphs since they can execute a guarding strategy. To demonstrate this, we need the following observations about shortest paths on the grid.

First, if the zombie and the survivor are on the same row (or column) then there is a single shortest path and hence only one valid zombie move: the zombie moves closer to the survivor along the row (or column).

Second, if the zombie and survivor are on different rows and columns then there are at least two shortest paths joining them and exactly two possible zombie moves: horizontal or vertical. The survivor and zombie are assumed on different rows and columns so the zombie can make progress in one or the other direction.

We now show that two zombies can play a shadowing strategy which is guaranteed to capture the survivor. We mimic a proof strategy from Cops and Robbers in which we show that the Robber Territory is shrinking at every round. We analoguously define the Survivor Territory $S_j \subset V(G)$ as the set of vertices to which the survivor may move on turn j without being eaten by a zombie.

To enact the shadowing strategy, the zombies may choose any starting position (and so the set of winning zombie starts $Z_W(G_{m,n}) = V(G_{m,n})$ in Fitzpatrick's notation – so the grid belongs to the family of graphs for which any zombie starting position will work – Fitzpatrick asks if there is a characterization of graphs for which any start will win. We do not have an answer but observe that the grid belongs to this family for 2 zombies). Each zombie will shadow the survivor's position along an axis: one horizontal, the other vertical.

1.5.1 The Shadowing Strategy

Proof. Let us consider one zombie at a time, say the zombie which will shadow the survivor's vertical shadow (i.e., its column), since the other zombie's behaviour is symmetric. We show that the zombie will eventually capture the vertical shadow – that the zombie will close the horizontal distance between the survivor and the zombie – and that, once it does so, the zombie can always recapture the survivor's shadow after the survivor moves.

Assuming that the zombie and survivor are on different columns, then the zombie may move one column closer since the vertex on the same row but one column closer lies on a shortest path to the survivor. If they are on the same row, then that is the only possible zombie move. In response, the survivor may:

- Remain in place in which case the zombie has closed the horizontal distance by 1.
- Move vertically (up or down) in which case again the zombie has closed the horizontal distance by 1.
- Move horizontally towards the zombie in which case the horizontal distance is reduced by 2.
- Move horizontally away from the zombie in which case the horizontal distance is preserved.

In the first three scenarios, the horizontal distance between the two adversaries has been reduced. The fourth scenario in which the survivor moves away cannot occur indefinitely since the grid is finite. Thus, in at most n rounds (the number of columns) the zombie will capture the survivor's vertical shadow.

Suppose now that the zombie has captured the survivor's vertical shadow; that they are now on the same column. It is clear that the zombie can recapture the survivor's shadow no matter how the survivor moves:

- If the survivor moves vertically or remains in place, then the zombie must move vertically and the survivor's vertical shadow remains captured.
- If the survivor moves horizontally, then the zombie may choose to mimic the move and thereby recapture the vertical shadow.

This argument shows that after a finite number of moves a zombie may capture the survivor's vertical shadow. Now observe that once the survivor's vertical shadow has been captured, the survivor can never enter the zombie-occupied row: any attempts to go around the zombie are immediately blocked. (Expand and clarify?)

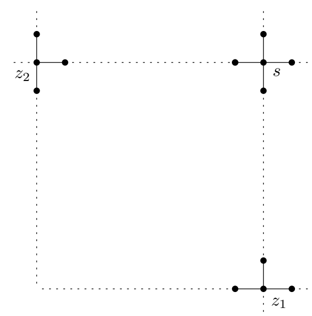


Figure 1.13: Once Both Shadows Are Captured

1.5.2 Shrinking the Survivor Territory

By the previous discussion, after $\max\{m, n\}$ rounds, both the survivor's vertical and horizontal shadows have been captured. Suppose we have reached a point in the game where the zombies have moved and captured both horizontal and vertical shadows.

The survivor has five possible moves:

- Stay in place, in which case both zombies have a single shortest path and move closer along their current row/column. So the Survivor Territory has shrunk by a column and a row.
- Move vertically, in which case the zombie capturing the vertical shadow has no choice but to move closer, while the other zombie recaptures the horizontal shadow. Here the Survivor Territory has shrunk by one row.
- Move horizontally, in which case the zombie capturing the horizontal shadow has not choice but to move closer, while the other zombie recaptures the vertical shadow. Now the Survivor Territory has shrunk by one column.

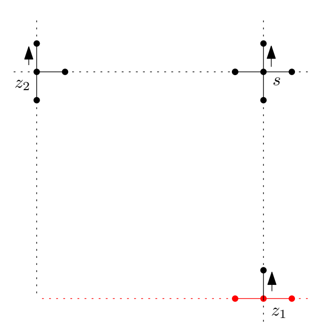


Figure 1.14: Shrinking Survivor Territory: The survivor moves up when shadowed; the zombie guarding the vertical shadow must move up, thereby eliminating the row below. The zombie guarding the horizontal shadow also moves up, so its column remains blocked.

In every scenario, at least one of the zombies is forced to move one step closer to the survivor. Since the survivor can never enter the rows and columns occupied by the shadowing zombies, this means that the Survivor Territory shrinks by at least one row or one column at every round. Since the grid is finite, the Survivor Territory is eventually empty and hence the survivor is captured.

□

1.5.3 Defining Shadowing Rigorously

Label the vertices of the grid using the integer points of the first quadrant of the plane and consider $z_1, z_2, s \in V(G_{m,n})$ as points in $[0, m-1] \times [0, n-1]$. Rows and columns are then first and second coordinates of points in a finite lattice. Say $s^j = (x_0^j, y_0^j), z_1^j = (x_1^j, y_1^j), z_2^j = (x_2^j, y_2^j)$ are the positions of the players s, z_1 and z_2 on round $j \geq 0$. The players cannot escape the bounds of the grid and so $0 \leq x_i^j \leq n-1$ and $0 \leq y_i^j \leq m-1$ for $i \in \{0, 1, 2\}$ and for $j \geq 0$.

Let's show that – after a finite number of turns – a zombie can mirror the survivor's x-coordinate. Formally, there exists a round $k \geq 0$ such that $x_0^k = x_1^{k+1}$ and that for all $j > k$, $x_0^j = x_1^{j+1}$. That is to say, from turn k onwards, the zombie can always move onto the x-projection of the survivor on its turn.

Note that have $N[z_i^j] \subseteq \{(x_i^j, y_i^j), (x_i^j \pm 1, y_i^j), (x_i^j, y_i^j \pm 1)\}$ and the inclusion is strict when z_i^j is on the boundary of the grid.

Suppose we already have $x_0^k = x_1^k$. As mentioned above, in this case the zombie has a single shortest path to $(x_1^k, y_1^k \pm 1)$ where for simplicity we will assume that the move is "upwards" to $(x_1^k, y_1^k + 1)$. The zombie has moved onto the survivor's x-coordinate. The survivor has five possible responses.

Now, we can assume that $x_0^j > x_1^j$ (the opposite being symmetric). The zombie may follow two shortest paths, one of which is to $(x_1^j + 1, y_1^j)$. ...

1.6 Zombies on a Cylinder

The Cylinder graph $C_{m,n}$ is a rectangular arrangement of mn vertices in m rows and n columns much like the Grid, except that vertices on one boundary edge are joined to vertices on the opposite side.

Again, the Cylinder graph can be considered as the Cartesian product $C_{m,n} = P_m \Box C_n$ of a cycle and a path. Note that this a planar graph for any m and n.

We claim now that three zombies suffice to win on this family of graphs since they can execute a guarding strategy similar to the one detailed in the previous section.

Place two zombies on a row such that $d(z_1, z_2) = \lfloor \frac{n}{2} \rfloor$. Now observe that if the survivor finishes its turn on a different row, the zombies may move to a vertex of the same column but closer row. If the survivor finishes its turn on the same row, then the zombies have a single zombie move on the same row but to a closer column.

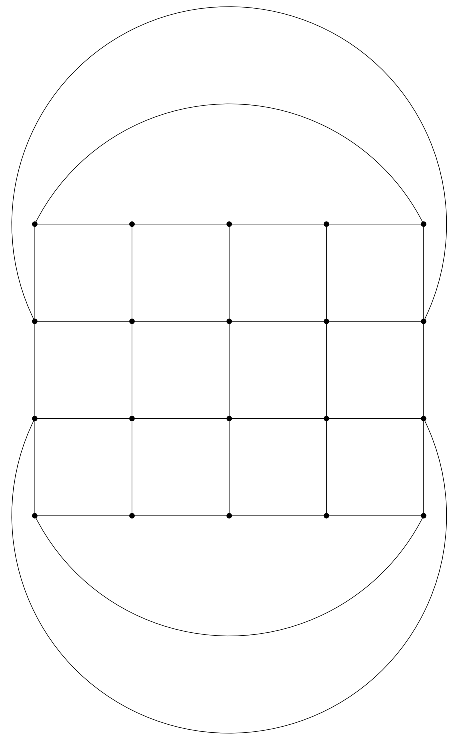


Figure 1.15: $C_{4,5}$

Thus, after a finite number of rounds, the zombies can shadow the survivor's horizontal shadow. Indeed, if we place these two zombies in the middle of the cylinder, then the survivor's horizontal shadow is captured in at most $\lceil \frac{m}{2} \rceil$ rounds.

The survivor is now trapped between these two zombies since they can always move to recapture its horizontal shadow. However, it could alternate between rows and thereby defeat the shadowing zombies.

We add another zombie to capture the survivor's vertical shadow. Once the zombie's horizontal and vertical shadows are captured, the survivor cannot remain on the same row and is unable to change row indefinitely and thus will be cornered.

1.7 Cycle With One Chord

We analyze the Game of Zombies & Survivors on a cycle with a single chord.

Definition 1. Take a cycle of length $m + n$ and add a chord which divides the cycle into paths P_m and P_n of lengths m and n. Without loss of generality $m \leq n$. We denote such a cycle as $Q_{m,n}$.

Theorem 2. The zombie number of a cycle $Q_{m,n}$ $(3 \leq m \leq n)$ with a chord dividing the cycle into paths of lengths m and n is 2.

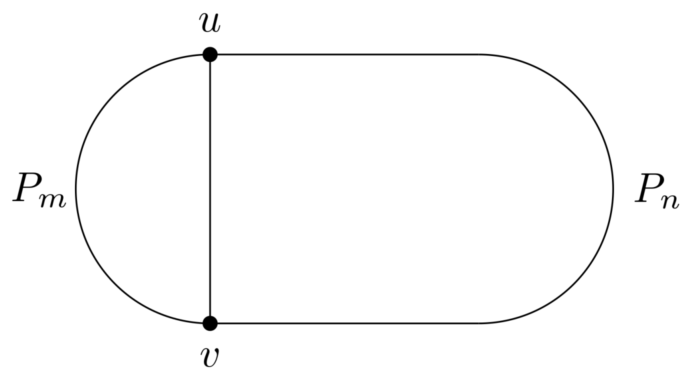


Figure 1.16: A cycle with one chord

Proof. Denote as P_m and P_n the paths of lengths m and n respectively. We think of $Q_{m,n}$ as embbeded in the plane with P_m – the shortest side – on the left. This does not limit the generality of the following and allows us to define (counter-)clockwise distance: the length of the path along a cycle with respect to this embedding.

Setting $m = n = 1$ gives K_2 with two added loops, which is zombie-win.

With $m = n = 2$ we have two adjacent cliques K_3 which are dominated by a single vertex, so it is also zombie-win.

For $m = 2$ and $n \geq 4$, 2 zombies win by starting on diametrically opposed vertices on the cycle C_{n+2}.

If $m = n = 3$ the zombie number is 2 since two zombies on the chord enpoints dominate the graph.

For $m = 3$, $n = 4$, the zombie number is also 2: placing the zombies on the endpoints of the chord divides the graph into C_4 and C_5 and the zombies clearly win from this position.

The same strategy works for $Q_{3,6}$, $Q_{4.4}$, $Q_{4,5}$ and $Q_{5,5}$ but it does not work for $Q_{3,7}$, $Q_{4,6}$ nor indeed for any $Q_{m,n}$ for $m \geq 3$ and $n \geq 6$.

We seek a winning zombie strategy (that is, a zombie start) for $m \geq 3$, $n \geq 6$. The chord is the crux of the game, so first we assume that one zombie is on the chord and another at some distance Δ while the survivor is somewhere on P_m. We know the first zombie chases the survivor around the cycle, so we need to control the arrival of the second zombie so that the survivor cannot escape, nor can it trick the second zombie into spinning the same direction as the first.

Second, we show how to position the zombies at the start of the game so that – no matter where the survivor starts – a losing position is guaranteed. Either the survivor is stuck on a path between the two zombies (so that capture is obviously inevitable) or the survivor will be pushed into the carefully orchestrated scenario described in the first part of the proof.

Lastly, we show that such a starting position is always available to the zombies for any $m \geq 3$, $n \geq 6$.

Note that if P_1 and P_2 are two possible zs-paths with distinct next moves and

$$|P_1| \leq |P_2|$$

then in the following argument we suppose that the zombie follows $|P_1|$ since that is a valid move.

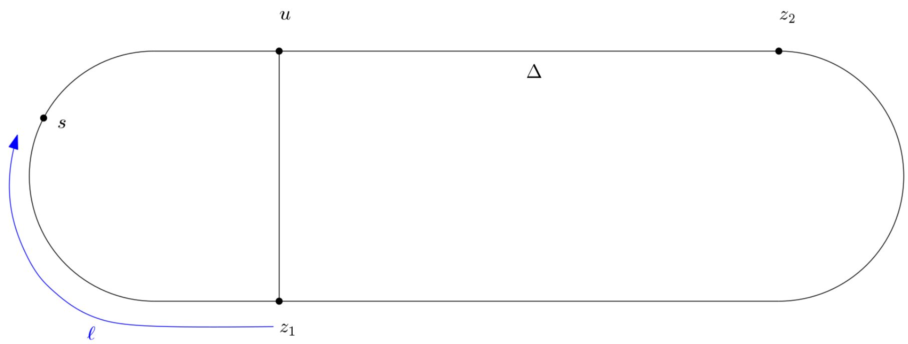


Figure 1.17: One zombie on the chord

1.7.1 Cornering the Survivor on the Smallest Cycle

Part 1. Suppose that the game has reached the following state:

- the first zombie is on an endpoint of the chord, say v
- there are Δ vertices counting clockwise from u to z_2.
- the survivor is on P_m at a distance of ℓ vertices counting clockwise from v.

By comparing the lengths of different paths, we calculate the values of Δ which guarantee that the survivor will be cornered on P_m. That is to say, the survivor will be intercepted by z_2 before it can reach any vertex in $Q_{m,n} \setminus P_m$.

Denote as ℓ the length of the clockwise path from v to s. Note that we must have $2 \leq \ell \leq m - 1$ else z_1 captures the survivor on the next round.

We can assume that once z_1 chooses a direction from v that it will continue in that direction: either the zombie has no choice or both directions around the cycle are of the same length (and so may continue in the same direction).

We can also assume that on its turn the survivor will move away from z_1 and maintain a distance of ℓ (or $m - \ell + 1$, if they are moving counter-clockwise) since a winning survivor strategy which involves waiting a turn or moving backwards is equivalent to a survivor strategy which always moves but starts with a smaller (or larger) value of ℓ.

These two assumptions allow us to "fast-forward" the game by Δ rounds and determine when the survivor is captured.

Since z_1 is already on the same cycle as the survivor, it has two options:

A. z_1 goes clockwise if $\ell \leq 1 + m - \ell$. Combined with the bounds on ℓ, this gives $4 \leq 2\ell \leq m + 1$

B. z_1 goes counter-clockwise if $1 + m - \ell \leq \ell$. Combined with the bounds on ℓ, we obtain $m + 1 \leq 2\ell \leq 2m - 2$

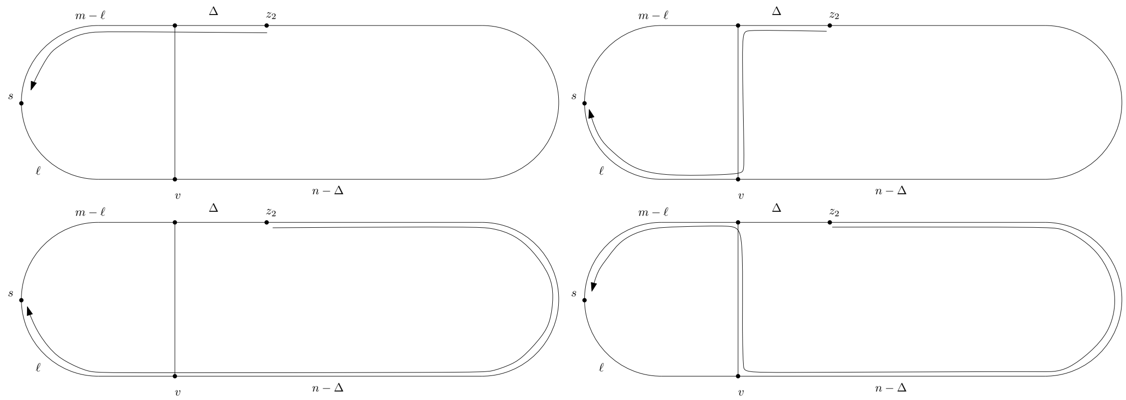


Figure 1.18: Four possible outcomes

There are four possible shortest paths for z_2 to the survivor:

- P_a of length $\Delta + (m-\ell)$
- P_b of length $\Delta + 1 + \ell$
- P_c of length $(n-\Delta) + 1 + (m-\ell)$
- P_d of length $(n-\Delta) + \ell$

Comparing path lengths we see that:

I. z_2 moves counter-clockwise if either $|P_a| \leq \min\{|P_c|, |P_d|\}$ or $|P_b| \leq \min\{|P_c|, |P_d|\}$.

II. z_2 goes clockwise if either $|P_c| \leq \min\{|P_a|, |P_b|\}$ or $|P_d| \leq \min\{|P_a|, |P_b|\}$.

We will examine all combinations of the possible decisions made by the zombies from this configuration:

- I. z_2 goes counter-clockwise
- II. z_2 goes clockwise.
- A. z_1 goes clockwise
- B. z_1 goes counter-clockwise

Case I.A We have the following constraint on ℓ from assumption A.

$$4 \leq 2\ell \leq m+1$$

and the following constraints on Δ from assumption I.

$$\begin{aligned} \Delta + (m-\ell) &\leq n - \Delta + 1 + m - \ell && \text{and} \\ \Delta + (m-\ell) &\leq n - \Delta + \ell \end{aligned}$$

or

$$\begin{aligned} \Delta + 1 + \ell \le& n - \Delta + 1 + m - \ell && \text{and} \\ \Delta + 1 + \ell \le& n - \Delta + \ell \end{aligned}$$

These can be simplified with a bit of algebra and assumption A:

$$\begin{aligned} 2\Delta \le& n + 1 && \text{and} \\ 2\Delta \le& n - m + 2\ell \le n + 1 \end{aligned}$$

or

$$\begin{aligned} 2\Delta \le& n + m - 2\ell && \text{and} \\ 2\Delta \le& n - 1 \le n + m - 2\ell \end{aligned}$$

So for z_2 to follow either P_a or P_b and go counter-clockwise we must have

$$\begin{aligned} 2\Delta \le& n - m + 2\ell && \text{or} \\ 2\Delta \le& n - 1 \end{aligned}$$

Next we consider: which of s or z_2 reaches u first? If $\Delta = m - \ell$ both z_2 and s reach u on the same round, with the survivor moving onto the zombie-occupied vertex (and losing). If we have $\Delta = m - \ell + 1$, then s reaches u first but is caught by z_2 on the following round. So, to guarantee the survivor has not escaped P_m we need

$$\Delta \le m - \ell + 1$$

otherwise the survivor can reach the chord at least two rounds before z_2 can move to block. We wish to prevent this scenario since the survivor could then take the chord and possibly escape, pulling both zombies into a loop either on C_m or C_n. This constraint on Δ guarantees that the survivor cannot escape C_m before z_2's arrival in Case I.A.

That is not sufficient, however. We must also ensure that z_2 moves counter-clockwise (opposite to z_1) once it reaches u in order to trap the survivor. So we need

$$m - \ell - \Delta \le 1 + \Delta + \ell$$

Or, in terms of Δ,

$$2\Delta \ge m - 2\ell - 1$$

When we combine all the restrictions we obtain
Case I.A. Summary
z_1 goes clockwise:

$$4 \leq 2\ell \leq m+1$$

and z_2 goes counter-clockwise

$$\begin{aligned} 2\Delta \leq& n-m+2\ell && \text{or} \\ 2\Delta \leq& n-1 \end{aligned}$$

the zombies win:

$$\begin{aligned} 2\Delta \leq& 2m-2\ell+2 && \text{and} \\ m-2\ell-1 \leq& 2\Delta \end{aligned}$$

Case I.B From assumption B and the constraint on ℓ, we must have

$$m+1 \leq 2\ell \leq 2m-2$$

and the constraints on Δ from assumption I are again:

$$\begin{aligned} \Delta+(m-\ell) \leq& n-\Delta+1+m-\ell && \text{and} \\ \Delta+(m-\ell) \leq& n-\Delta+\ell \end{aligned}$$

or

$$\begin{aligned} \Delta+1+\ell \leq& n-\Delta+1+m-\ell && \text{and} \\ \Delta+1+\ell \leq& n-\Delta+\ell \end{aligned}$$

These can be simplified using assumption B:

$$\begin{aligned} 2\Delta \leq& n+1 \leq n-m+2\ell && \text{and} \\ 2\Delta \leq& n-m+2\ell \end{aligned}$$

or

$$\begin{aligned} 2\Delta \leq& n+m-2\ell \leq n-1 && \text{and} \\ 2\Delta \leq& n-1 \end{aligned}$$

So for z_2 to go counter-clockwise in this case we must have

$$\begin{aligned} 2\Delta \leq& n+1 && \text{or} \\ 2\Delta \leq& n+m-2\ell \end{aligned}$$

Again we must consider who reaches the chord first. We have assumed that z_1 is going counter-clockwise. If $\ell = \Delta$, then z_2 reaches u and s reaches v on the same round, and therefore s will be caught on the next. Therefore, to guarantee the survivor has not escaped P_m in this scenario we need

$$\Delta \leq \ell$$

otherwise the survivor reaches the chord before z_2 and could escape.

Then, to ensure that z_2 traps the survivor by going clockwise once it reaches u we need

$$\begin{aligned} 1 + \ell - \Delta \leq & \Delta - 1 + m - \ell + 1 \\ 2\ell - m + 1 \leq & 2\Delta \end{aligned}$$

Case I.B. Summary

z_1 goes counter-clockwise:

$$m + 1 \leq 2\ell \leq 2m - 2$$

and z_2 goes counter-clockwise

$$\begin{aligned} 2\Delta \leq & n + 1 && \text{or} \\ 2\Delta \leq & n + m - 2\ell \end{aligned}$$

the zombies win:

$$\begin{aligned} 2\Delta \leq & 2\ell \\ 2\ell - m + 1 \leq & 2\Delta \end{aligned}$$

Case II.A We have the following constraint on ℓ from assumption A.

$$4 \leq 2\ell \leq m + 1$$

and the following constraints on Δ from assumption II.

$$\begin{aligned} n - \Delta + \ell \leq & \Delta + (m - \ell) && \text{and} \\ n - \Delta + \ell \leq & \Delta + 1 + \ell \end{aligned}$$

or

$$\begin{aligned} n - \Delta + 1 + m - \ell \leq & \Delta + (m - \ell) && \text{and} \\ n - \Delta + 1 + m - \ell \leq & \Delta + 1 + \ell \end{aligned}$$

These can be simplified with a bit of algebra:

$$\begin{aligned} n - m + 2\ell \leq & 2\Delta && \text{and} \\ n - 1 \leq & 2\Delta \end{aligned}$$

or

$$\begin{aligned} n+1 &\leq 2\Delta && \text{and} \\ n+m-2\ell &\leq 2\Delta \end{aligned}$$

These inequalites are of the form

$$\begin{aligned} n-x &\leq 2\Delta && \text{and} \\ n-1 &\leq 2\Delta \end{aligned}$$

or

$$\begin{aligned} n+x &\leq 2\Delta && \text{and} \\ n+1 &\leq 2\Delta \end{aligned}$$

Where $x = m - 2\ell$.

Supposing $x \geq 0$, we have

$$\begin{aligned} n-x &\leq n+x \leq 2\Delta && \text{and} \\ n-1 &< n+1 \leq 2\Delta \end{aligned}$$

and take the lowest bounds because of the disjunction, so that $2\Delta \geq n - x = n - m + 2\ell$ and $2\Delta \geq n - 1$ suffices.

Since assumption A gives $m - 2\ell \geq -1$, supposing $x < 0$ reduces the inequalites to

$$\begin{aligned} n+1 &\leq 2\Delta && \text{and} \\ n-1 &\leq 2\Delta \end{aligned}$$

which is satisfied by $2\Delta \geq n - x = n - m + 2\ell$ and $2\Delta \geq n - 1$.

Thus z_2 will go clockwise under assumption A if

$$\begin{aligned} 2\Delta &\geq n-m+2\ell && \text{and} \\ 2\Delta &\geq n-1 \end{aligned}$$

We have assumed that z_1 is going clockwise. If $m - \ell = n - \Delta$, then z_2 reaches v and s reaches u on the same round and s will be caught on the next. Therefore, to guarantee the survivor has not escaped P_m in this scenario we need

$$\begin{aligned} n-\Delta &\leq m-\ell \\ \Delta &\geq n-m+\ell \end{aligned}$$

otherwise the survivor could reach the chord before z_2.

After $n - \Delta$ rounds, we have (insert diagram)

Then, to ensure that z_2 goes counter-clockwise once it reaches v, we need

$$\begin{aligned}1 + m - \ell - (n - \Delta) \leq& n - \Delta + \ell \\ 2\Delta \leq& 2n + 2\ell - m - 1\end{aligned}$$

All together this gives *Case II.A. Summary*
z_1 goes clockwise:

$$4 \leq 2\ell \leq m + 1$$

and z_2 goes clockwise

$$\begin{aligned}n - m + 2\ell \leq& 2\Delta \qquad \text{and} \\ n - 1 \leq& 2\Delta\end{aligned}$$

the zombies win:

$$\begin{aligned}2\Delta \geq& 2n - 2m + 2\ell \\ 2\Delta \leq& 2n + 2\ell - m - 1\end{aligned}$$

Case II.B We have the following constraint on ℓ from assumption B.

$$m + 1 \leq 2\ell \leq 2m - 2$$

and the following constraints on Δ from assumption II.

$$\begin{aligned}n - \Delta + \ell \leq& \Delta + (m - \ell) \qquad \text{and} \\ n - \Delta + \ell \leq& \Delta + 1 + \ell\end{aligned}$$

or

$$\begin{aligned}n - \Delta + 1 + m - \ell \leq& \Delta + (m - \ell) \qquad \text{and} \\ n - \Delta + 1 + m - \ell \leq& \Delta + 1 + \ell\end{aligned}$$

These can be simplified further with a bit of algebra:

$$\begin{aligned}n - m + 2\ell \leq& 2\Delta \qquad \text{and} \\ n - 1 \leq& 2\Delta\end{aligned}$$

or

$$\begin{aligned}n + 1 \leq& 2\Delta \qquad \text{and} \\ n + m - 2\ell \leq& 2\Delta\end{aligned}$$

We have

$$\begin{aligned}n - \Delta + \ell \leq& \Delta + (m - \ell) \qquad \text{and} \\ n - \Delta + \ell \leq& \Delta + 1 + \ell\end{aligned}$$

or

$$\begin{aligned} n - \Delta + 1 + m - \ell \leq& \Delta + (m - \ell) && \text{and} \\ n - \Delta + 1 + m - \ell \leq& \Delta + 1 + \ell \end{aligned}$$

These can be simplified further with a bit of algebra:

$$\begin{aligned} n - m + 2\ell \leq& 2\Delta && \text{and} \\ n - 1 \leq& 2\Delta \end{aligned}$$

or

$$\begin{aligned} n + 1 \leq& 2\Delta && \text{and} \\ n + m - 2\ell \leq& 2\Delta \end{aligned}$$

These inequalites are of the form

$$\begin{aligned} n - x \leq& 2\Delta && \text{and} \\ n - 1 \leq& 2\Delta \end{aligned}$$

or

$$\begin{aligned} n + 1 \leq& 2\Delta && \text{and} \\ n + x \leq& 2\Delta \end{aligned}$$

Where $x = m - 2\ell$. Now since assumption B gives $m - 2\ell \leq -1$, we see that

$$n - 1 \leq n - x \leq 2\Delta$$
or
$$n + x \leq n + 1 \leq 2\Delta$$

Now we consider: which of s or z_2 reaches v first? If $n - \Delta = \ell$, then they both reach u at the same time, with the survivor moving onto the z_2-occupied vertex (and losing). If we have $n - \Delta = \ell + 1$, then s reaches u first but is caught by z_2 on the following round. So, to guarantee the survivor has not escaped P_m we need

$$n - \Delta \leq \ell + 1$$

otherwise the survivor reaches the chord before z_2 can move to block. If the survivor reaches the chord first, then it could take the chord and possibly escape. (more detail??)

Then, to ensure that z_2 takesgoes clockwise once it reaches v, we need

$$\begin{aligned} \ell - (n - \Delta) \leq& 1 + (n - \Delta - 1) + (m - \ell + 1) \\ 2\Delta \leq& 2n + m - 2\ell + 1 \end{aligned}$$

Case II.B. Summary
z_1 goes counter-clockwise:

$$m+1 \leq 2\ell \leq 2m-2$$

and z_2 goes clockwise

$$n+1 \leq 2\Delta$$

the zombies win:

$$\begin{aligned} n-\Delta &\leq \ell+1 \\ 2\Delta &\leq 2n+m-2\ell+1 \end{aligned}$$

1.7.2 Forcing the Survivor into a Losing Position

Part 2. We now consider the game on this graph in general and show how we can guarantee the survivor will be caught.

Given m, n and Δ as computed below, we place the zombies on C_{n+1} so that the zombies move in opposite direction wherever the survivor may start. We need only consider the cycle C_{n+1} since, if the survivor starts on $C_{m+1} \setminus \{u, v\}$, then the zombies play as though the survivor is on u or v.

We choose k such that positioning

1. z_2 at $\Delta + k$ clockwise from u
2. z_1 at k counter-clockwise from v

forces the survivor into a losing position: it is either immediately sandwiched on C_{n+1}, or falls into the trap described above on C_{m+1}.

The survivor cannot start next to the zombies else it loses right away. So we choose k such that, even if the survivor is as far away from one of the zombies as possible on C_n, then the zombies still move in opposite directions. This leads to the following inequalities

$$\begin{aligned} n-\Delta-2k-2 &\leq \Delta+k+1+k+2 \qquad \text{and} \\ \Delta+2k-1 &\leq n-\Delta-2k+2x` \end{aligned}$$

Solving for k gives

$$n-2\Delta-5 \leq 4k \leq n-2\Delta+3$$

Such k guarantees that the zombies start on vertices such that they must move in opposite directions if the survivor plays on C_n.

If the survivor starts between the zombies such that access to the chord is blocked, then clearly it has lost. Otherwise, the zombies must move towards the chord and in k rounds we reach the scenario described in Part 1 when z_1 reaches the chord and z_2 is Δ away. With suitable Δ, then, the survivor cannot win.

Part 3. Computing the Winning Zombie Start

Given m and n, we choose Δ so that whenever we reach the scenario described in the first part, the survivor will be cornered. Such Δ must satisfy the following constraints for any possible value of ℓ.

Case I.A. Summary

z_1 goes clockwise:

$$4 \leq 2\ell \leq m+1$$

and z_2 goes counter-clockwise

$$\begin{aligned} 2\Delta &\leq n-m+2\ell && \text{or} \\ 2\Delta &\leq n-1 \end{aligned}$$

the zombies win:

$$\begin{aligned} 2\Delta &\leq 2m-2\ell+2 && \text{and} \\ m-2\ell-1 &\leq 2\Delta \end{aligned}$$

Case I.B. Summary

z_1 goes counter-clockwise:

$$m+1 \leq 2\ell \leq 2m-2$$

and z_2 goes counter-clockwise

$$\begin{aligned} 2\Delta &\leq n+1 && \text{or} \\ 2\Delta &\leq n+m-2\ell \end{aligned}$$

the zombies win:

$$\begin{aligned} 2\Delta &\leq 2\ell \\ 2\ell-m+1 &\leq 2\Delta \end{aligned}$$

Case II.A. Summary

z_1 goes clockwise:

$$4 \leq 2\ell \leq m+1$$

and z_2 goes clockwise

$$\begin{aligned} n-m+2\ell &\leq 2\Delta && \text{and} \\ n-1 &\leq 2\Delta \end{aligned}$$

the zombies win:

$$\begin{aligned}2\Delta \geq& 2n - 2m + 2\ell\\ 2\Delta \leq& 2n + 2\ell - m - 1\end{aligned}$$

Case II.B. Summary

z_1 goes counter-clockwise:

$$m + 1 \leq 2\ell \leq 2m - 2$$

and z_2 goes clockwise

$$n + 1 \leq 2\Delta$$

the zombies win:

$$\begin{aligned}n - \Delta \leq& \ell + 1\\ 2\Delta \leq& 2n + m - 2\ell + 1\end{aligned}$$

A simple algorithm to calculate possible values of Δ loops over $0 \leq \Delta \leq n$ and over $2 \leq \ell \leq m - 1$ and tests, for each Δ and each ℓ, to determine which of the four cases is applicable and, if in one of the cases, whether the zombie-win constrains are satisfied. A value of Δ is accepted if, for every value of ℓ, the zombies win.

Once we have obtained possible Δ, we can then determine k by calculating the bounds

$$n - 2\Delta - 5 \leq 4k \leq n - 2\Delta + 3$$

□

1.7.3 Existence of Winning Start

We wish to show that, for any m, n, there exist Δ and k which guarantee the survivor is caught. First we show that $\Delta = \lfloor \frac{m}{2} \rfloor$ always works for the cornering strategy.

Note that

$$2\Delta = 2\left\lfloor \frac{m}{2} \right\rfloor = \begin{cases} m & \text{if } m \text{ is even} \\ m-1 & \text{if } m \text{ is odd} \end{cases}$$

and so $m - 1 \leq 2\lfloor \frac{m}{2} \rfloor \leq m$.

Suppose that we are in Case I. A. and $\Delta = \lfloor \frac{m}{2} \rfloor$. Case I. A is characterized by the following constraints:

$$4 \leq 2\ell \leq m + 1$$

and

$$2\Delta \leq n - m + 2\ell$$

or

$$2\Delta \leq n - 1$$

The zombies win if

$$\begin{aligned} 2\Delta \leq& 2m - 2\ell + 2 \qquad \text{and} \\ m - 2\ell - 1 \leq& 2\Delta \end{aligned}$$

So if we are in Case I. A. and $\Delta = \lfloor \frac{m}{2} \rfloor$ the zombies win since

$$\begin{aligned} 2\Delta = 2\lfloor \frac{m}{2} \rfloor \leq m < 2m - (m+1) + 2 \leq 2m - 2\ell + 2 \qquad \text{and} \\ m - 2\ell - 1 \leq m - 5 < 2\lfloor \frac{m}{2} \rfloor = 2\Delta \end{aligned}$$

Which shows that the zombie-win requirements are met.

Suppose now that we are not in Case 1. A. Negating the constraints of Case I. A. gives

$$2\Delta \geq n - m + 2\ell + 1$$

and

$$2\Delta \geq n - 1 + 1$$

or

$$m + 1 \leq 2\ell \leq 2m - 2$$

If we assume that m is odd and $2\Delta \geq n$ then we obtain a contradiction since

$$2\Delta = 2\lfloor \frac{m}{2} \rfloor = m - 1 \geq n$$

and we have assumed that $m \leq n$.

If m even, $m = n$ and $2\Delta \geq n - m + 2\ell + 1$ then

$$\begin{aligned} 2\Delta &\geq n - m + 2\ell + 1 \\ m &\geq m - m + 2\ell + 1 \\ m &\geq 2\ell + 1 \\ 2\ell &\leq m - 1 \end{aligned}$$

So, if $m = n$ and they are even, then we are in Case 1. A unless $2\ell \leq m - 1$.

To recap: If we set $\Delta = \lfloor \frac{m}{2} \rfloor$, we are in Case 1.A unless

$$m = n \qquad \text{and they are even}$$

$$\Delta = \lfloor \frac{m}{2} \rfloor = \frac{m}{2}$$

$$4 \leq 2\ell \leq m - 1$$

Now, can we be in Case 1. B? Case 1. B is described by the following constraints:

$$m + 1 \leq 2\ell \leq 2m - 2$$

and

$$2\Delta \leq n + 1$$

or

$$2\Delta \leq n + m - 2\ell$$

The negation of which is:

$$2\Delta \geq n + 1 + 1$$

and

$$2\Delta \geq n + m - 2\ell + 1$$

or

$$4 \leq 2\ell \leq m + 1$$

But this leads to the contradiction:

$$n \geq m \geq 2\Delta \geq n + 2$$

In remains to check if we win in Case 2. A.

Assuming still that

$$m = n \qquad \text{they are even}$$

$$\Delta = \frac{m}{2}$$

$$4 \leq 2\ell \leq m - 1$$

The win conditions require

$$\begin{aligned}
2n - 2m + 2\ell \leq 2\Delta &\leq 2n + 2\ell - m - 1 \\
2m - 2m + m - 1 \leq 2\Delta &\leq 2m + 4 - m - 1 \\
m - 1 \leq 2\Delta &\leq m + 3
\end{aligned}$$

Which holds for $\Delta = \frac{m}{2}$.

1.8 Visiblity Graphs, Cop-win Trees, Zombie Trees

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.1 $Q_{m,n}$ Appendices

.1 Simplifying z_2's inequalities for Case II.A.

We have

$$\begin{aligned} n - \Delta + \ell \leq& \Delta + (m - \ell) && \text{and} \\ n - \Delta + \ell \leq& \Delta + 1 + \ell \end{aligned}$$

or

$$\begin{aligned} n - \Delta + 1 + m - \ell \leq& \Delta + (m - \ell) && \text{and} \\ n - \Delta + 1 + m - \ell \leq& \Delta + 1 + \ell \end{aligned}$$

These can be simplified further with a bit of algebra:

$$\begin{aligned} n - m + 2\ell \leq& 2\Delta && \text{and} \\ n - 1 \leq& 2\Delta \end{aligned}$$

or

$$\begin{aligned} n + 1 \leq& 2\Delta && \text{and} \\ n + m - 2\ell \leq& 2\Delta \end{aligned}$$

These inequalites are of the form

$$\begin{aligned} n - x \leq& 2\Delta && \text{and} \\ n - 1 \leq& 2\Delta \end{aligned}$$

or

$$\begin{aligned} n + x \leq& 2\Delta && \text{and} \\ n + 1 \leq& 2\Delta \end{aligned}$$

Where $x = m - 2\ell$.
Supposing $x \geq 0$, we have

$$\begin{aligned} n - x \leq& n + x \leq 2\Delta && \text{and} \\ n - 1 \leq& n + 1 \leq 2\Delta \end{aligned}$$

Whereas if $x < 0$, then from assumption A we must have $m - 2\ell = -1$, so that our constraints reduce to

$$\begin{aligned} n + 1 \leq& 2\Delta && \text{and} \\ n - 1 \leq& 2\Delta \end{aligned}$$

.2 Simplifying z_2's inequalities for Case II.B.

We have

$$\begin{aligned} n - \Delta + \ell \leq& \Delta + (m - \ell) && \text{and} \\ n - \Delta + \ell \leq& \Delta + 1 + \ell \end{aligned}$$

or

$$\begin{aligned} n - \Delta + 1 + m - \ell \leq& \Delta + (m - \ell) && \text{and} \\ n - \Delta + 1 + m - \ell \leq& \Delta + 1 + \ell \end{aligned}$$

These can be simplified further with a bit of algebra:

$$\begin{aligned} n - m + 2\ell \leq& 2\Delta && \text{and} \\ n - 1 \leq& 2\Delta \end{aligned}$$

or

$$\begin{aligned} n + 1 \leq& 2\Delta && \text{and} \\ n + m - 2\ell \leq& 2\Delta \end{aligned}$$

These inequalites are of the form

$$\begin{aligned} n - x \leq& 2\Delta && \text{and} \\ n - 1 \leq& 2\Delta \end{aligned}$$

or

$$\begin{aligned} n + 1 \leq& 2\Delta && \text{and} \\ n + x \leq& 2\Delta \end{aligned}$$

Where $x = m - 2\ell$. Now since assumption B gives $m - 2\ell \leq -1$, we see that

$$\begin{aligned} n - 1 \leq& n - x \leq 2\Delta \\ &\text{or} \\ n + x \leq& n + 1 \leq 2\Delta \end{aligned}$$

.3 Planar Zombies Counterexample Case III

Here are all the possible start configurations (without loss of generality) of Case III with the first few moves demonstrating that the survivor wins.

Round	z_1	z_2	z_3	s
0	1	6	11	3
1	2	1	2	4
2	3	5	3	20
3	4	4	4	19

Round	z_1	z_2	z_3	s
0	1	6	12	3
1	2	1	11	4
2	3	5	2	20
3	4	4	3	19

Round	z_1	z_2	z_3	s
0	1	6	13	3
1	2	1	14	4
2	3	5	15	20
3	4	4	3	19

Round	z_1	z_2	z_3	s
0	1	6	14	4
1	5	1	15	21
2	4	5	3	22

Round	z_1	z_2	z_3	s
0	1	6	15	4
1	5	1	3	21
2	4	5	4	22

Round	z_1	z_2	z_3	s
0	1	7	11	3
1	2	6	2	4
2	3	1	3	21
3	4	5	4	22

Round	z_1	z_2	z_3	s
0	1	7	12	3
1	2	6	11	4
2	3	1	2	21
4	4	5	3	22

Round	z_1	z_2	z_3	s
0	1	7	13	3
1	2	6	14	4
2	3	1	15	21
3	4	5	3	22

Round	z_1	z_2	z_3	s
0	1	7	14	3
1	2	6	15	4
2	3	1	3	21
3	4	5	3	22

Round	z_1	z_2	z_3	s
0	1	7	15	4
1	5	6	3	21
2	4	1	4	22

Note: 23 as next move leads to loss.

Round	z_1	z_2	z_3	s
0	1	8	11	3
1	2	9	2	4
2	3	10	3	5
3	4	2	4	26
4	5	1	5	27
5	26	5	26	28

Round	z_1	z_2	z_3	s
0	1	8	12	3
1	2	9	11	4
2	3	10	2	5
3	4	2	1	26
4	5	1	5	27

Round	z_1	z_2	z_3	s
0	1	8	13	3
1	2	9	14	4
2	3	10	15	5
3	4	2	3	26
4	5	1	4	27
5	26	5	5	28
6	27	26	26	29

Round	z_1	z_2	z_3	s
0	1	8	14	3
1	2	9	15	4
2	3	10	3	5
3	4	2	4	26
4	5	1	5	27

Round	z_1	z_2	z_3	s	
0	1	8	15	4	
1	5	7 or 9	3	21	
2	4	6 or 10	4	22	
3	21	1 or 2	21	23	caught!
4	22	5 or 1 or 3	22	24	
5	23	25 or 5 or 2 or 4	23	27	
6	24	24 or 26 or 1 or 5	24	28	
7	27	27 or 27 or 30 or 26	29		

Round	z_1	z_2	z_3	s	
0	1	8	15	17	
1	2	9	14	18	Note: center strategy fails
2	3	12	17	19	
3	16	13	18	22	

Round	z_1	z_2	z_3	s
0	1	9	11	3
1	2	10	2	4
2	3	2	3	5
3	4	1	4	26
4	5	5	5	27

Round	z_1	z_2	z_3	s
0	1	9	12	3
1	2	10	11	4
2	3	2	2	5
3	4	1	1	26
4	5	5	5	27

Round	z_1	z_2	z_3	s
0	1	9	13	3
1	2	10	14	4
2	3	2	15	5
3	4	1	3	26
4	5	5	4	27
5	26	26	5	28
6	27	27	26	29

Round	z_1	z_2	z_3	s
0	1	9	14	3
1	2	10	15	4
2	3	2	3	5
3	4	1	4	26
4	5	5	5	27

Round	z_1	z_2	z_3	s
0	1	9	15	4
1	5	10	3	21
2	4	2	4	22
3	21	3	21	23
4	22	4	22	24
5	23	5	23	27
6	24	26	24	28

Round	z_1	z_2	z_3	s
0	1	10	11	3
1	2	2	2	16
2	3	3	3	17

Round	z_1	z_2	z_3	s
0	1	10	12	3
1	2	2	11	4
2	3	3	2	5

Round	z_1	z_2	z_3	s
0	1	10	13	3
1	2	2	14	4
2	3	3	15	5
3	4	4	3	1

Round	z_1	z_2	z_3	s
0	1	10	14	3
1	2	2	15	4
2	3	3	3	5

Round	z_1	z_2	z_3	s
0	1	10	15	4
1	5	2	3	20
2	4	3	4	19

Round	z_1	z_2	z_3	s
0	1	6	16	4
1	5	1	3	21
2	4	5	4	22
3	21	4	21	23
4	22	21	22	24

Round	z_1	z_2	z_3	s
0	1	6	17	4
1	5	1	16	21
2	4	5	3	22
3	21	4	4	23
4	22	21	21	24

Round	z_1	z_2	z_3	s
0	1	6	18	4
1	5	1	19	3 (21 leads to loss)
2	4	2	20	16
3	3	3	4	17
4	16	16	3	18
5	17	17	16	19

Round	z_1	z_2	z_3	s
0	1	6	19	4
1	5	1	20	3
2	4	2	4	16
3	3	3	3	17

Round	z_1	z_2	z_3	s
0	1	6	20	3
1	2	1	4	16
2	3	2	3	17
3	16	3	16	18
4	17	16	17	19

Round	z_1	z_2	z_3	s
0	1	7	16	4
1	5	6	3	21
2	4	1	4	22
3	21	5	21	19 (23 leads to loss)
4	22	4	22	18
5	19	20	19	17

Round	z_1	z_2	z_3	s
0	1	7	17	3
1	2	6	16	4
2	3	1	3	21
3	4	5	4	22
4	21	4	21	23
5	22	21	22	24

Round	z_1	z_2	z_3	s
0	1	7	18	3
1	2	6	17	4
2	3	1	16	21
3	4	5	3	22

Round	z_1	z_2	z_3	s
0	1	7	19	4
1	5	6	20	3
2	4	1	4	15
3	3	2	3	14

Round	z_1	z_2	z_3	s
0	1	7	20	3
1	2	6	4	15
2	3	1	3	14

Round	z_1	z_2	z_3	s
0	1	8	16	4
1	5	7 or 9	3	21
2	4	6 or 10	4	22
3	21	1 or 2	21	23
4	22	5 or 1	22	24
5	23	25 or 5	23	27
6	24	24 or 26	24	28
7	27	27	27	29

Round	z_1	z_2	z_3	s
0	1	8	17	3
1	2	9	16	4
2	3	10	3	5
3	4	2	4	26
4	5	1	5	27

Round	z_1	z_2	z_3	s
0	1	8	18	3
1	2	9	17	4
2	3	10	16	5
3	4	2	3	26
4	5	1	4	27

Round	z_1	z_2	z_3	s
0	1	8	19	4
1	5	7 or 8	20	3
2	4	6 or 10	4	15
3	3	1 or 2	3	14

Round	z_1	z_2	z_3	s
0	1	8	19	4
1	5	7 or 8	20	3
2	4	6 or 10	4	15
3	3	1 or 2	3	14

Round	z_1	z_2	z_3	s
0	1	8	20	14
1	2	9	4	14
2	3	12	3	14
3	15	13	3	17
4	14	14	16	18
5	17	17	17	19

Round	z_1	z_2	z_3	s
0	1	9	16	4
1	5	10	3	21
2	4	2	4	22

Round	z_1	z_2	z_3	s
0	1	9	17	3
1	2	10	16	4
2	3	2	3	5
3	4	1	4	26
4	5	5	5	27

Round	z_1	z_2	z_3	s
0	1	9	18	3
1	2	10	17	4
2	3	2	16	5
3	4	1	3	26
4	5	5	4	27

Round	z_1	z_2	z_3	s
0	1	9	19	3
1	2	10	20	16
2	3	2	4	17

Round	z_1	z_2	z_3	s
0	1	9	20	3
1	2	10	4	16
2	3	2	3	17

Round	z_1	z_2	z_3	s
0	1	10	16	4
1	5	2	3	21
2	4	3	4	22

Round	z_1	z_2	z_3	s
0	1	10	17	3
1	2	2	16	4
2	3	3	3	5

Round	z_1	z_2	z_3	s
0	1	10	18	3
1	2	2	17	4
2	3	3	16	5
3	4	4	3	26
4	5	5	4	27

Round	z_1	z_2	z_3	s
0	1	10	19	3
1	2	2	20	15
2	3	3	4	14

Round	z_1	z_2	z_3	s
0	1	10	20	3
1	2	2	4	15
2	3	3	3	14

Round	z_1	z_2	z_3	s
0	1	6	21	3
1	2	1	4	16
2	3	2	3	17

Round	z_1	z_2	z_3	s
0	1	6	22	4
1	5	1	21	3
2	4	2	4	16
3	3	3	3	17

Round	z_1	z_2	z_3	s
0	1	6	23	4
1	5	1	22	3
2	4	2	21	16
3	3	3	4	17

Round	z_1	z_2	z_3	s
0	1	6	24	4
1	5	1	25	3
2	4	2	5	16
3	3	3	4	17

Round	z_1	z_2	z_3	s
0	1	6	25	4
1	5	1	5	3
2	4	2	4	16
3	3	3	3	17

Round	z_1	z_2	z_3	s
0	1	7	21	3
1	2	6	4	16
2	3	1	3	17

Round	z_1	z_2	z_3	s
0	1	7	22	4
1	5	6	21	3
2	4	1	4	16
3	3	2	3	17

Round	z_1	z_2	z_3	s
0	1	7	23	4
1	5	6	22	3
2	4	1	21	16
3	3	2	4	17

Round	z_1	z_2	z_3	s
0	1	7	24	4
1	5	6	25	3
2	4	1	5	16
3	3	2	4	17

Round	z_1	z_2	z_3	s
0	1	7	25	4
1	5	6	5	3
2	4	1	4	16
3	3	2	3	16

Round	z_1	z_2	z_3	s
0	1	8	21	3
1	2	9	4	16
2	3	10	3	17
3	16	2	16	14
4	17	3	17	13
5	14	15	14	12

Round	z_1	z_2	z_3	s
0	1	8	22	3
1	2	9	21	16
2	3	10	4	17
3	16	2	3	14
4	17	3	15	13
5	14	15	14	12

Round	z_1	z_2	z_3	s
0	1	8	23	3
1	2	9	22	4
2	3	10	21	5
3	4	2	4	26
4	5	1	5	27

Round	z_1	z_2	z_3	s
0	1	8	24	3
1	2	9	25	4
2	3	10	5	21
3	4	2	4	22

Round	z_1	z_2	z_3	s
0	1	8	25	3
1	2	9	5	16
2	3	10	4	17
3	16	2	3	14
4	17	3	15	13
5	14	15	14	12

Round	z_1	z_2	z_3	s
0	1	9	21	3
1	2	10	4	16
2	3	2	3	17

Round	z_1	z_2	z_3	s
0	1	9	22	4
1	5	10	21	3
2	4	2	4	16
3	3	3	3	17

Round	z_1	z_2	z_3	s
0	1	9	23	4
1	5	10	22	3
2	4	2	21	16
3	3	3	4	17

Round	z_1	z_2	z_3	s
0	1	9	24	4
1	5	10	25	3
2	4	2	5	16
3	3	3	4	17

Round	z_1	z_2	z_3	s
0	1	9	25	4
1	5	10	5	3
2	4	2	4	16
3	3	3	3	17

Round	z_1	z_2	z_3	s
0	1	10	21	3
1	2	2	4	16
2	3	3	3	17

Round	z_1	z_2	z_3	s
0	1	10	22	3
1	2	2	21	16
2	3	3	4	17

Round	z_1	z_2	z_3	s
0	1	10	23	3
1	2	2	22	15
2	3	3	21	14
3	15	15	4	17
4	14	14	3	18
5	17	17	16	19

Round	z_1	z_2	z_3	s
0	1	10	24	3
1	2	2	25	4
2	3	3	5	20
3	4	4	4	19

Round	z_1	z_2	z_3	s
0	1	10	25	3
1	2	2	5	16
2	3	3	4	17

Round	z_1	z_2	z_3	s
0	1	6	26	4
1	5	1	5	3
2	4	2	4	16
3	3	3	3	17

Round	z_1	z_2	z_3	s
0	1	6	27	4
1	5	1	26	3
2	4	2	5	16
3	3	3	4	17

Round	z_1	z_2	z_3	s
0	1	6	28	4
1	5	1	27	3
2	4	2	26	16
3	3	3	5	17

Round	z_1	z_2	z_3	s
0	1	6	29	4
1	5	1	30	3
2	4	2	1	16
3	3	3	2	17

Round	z_1	z_2	z_3	s
0	1	6	30	4
1	5	1	1	3
2	4	2	2	16
3	3	3	3	17

Round	z_1	z_2	z_3	s
0	1	7	26	4
1	5	6	5	3
2	4	1	4	16
3	3	2	3	17

Round	z_1	z_2	z_3	s
0	1	7	27	4
1	5	6	26	3
2	4	1	5	16
3	3	2	4	17

Round	z_1	z_2	z_3	s
0	1	7	28	4
1	5	6	27	3
2	4	1	26	16
3	3	2	5	17

Round	z_1	z_2	z_3	s
0	1	7	29	4
1	5	6	30	3
2	4	1	1	16
3	3	2	2	17

Round	z_1	z_2	z_3	s
0	1	7	30	4
1	5	6	1	3
2	4	1	2	16
3	3	2	3	17

Round	z_1	z_2	z_3	s
0	1	8	26	3 (start at 3 prevents branching from 8)
1	2	9	5	16
2	3	10	4	17
3	16	2	3	18
4	17	3	16	19
5	18	4	17	22
6	19	21	18	23

Round	z_1	z_2	z_3	s
0	1	8	27	3
1	2	9	26	4
2	3	10	5	21
3	4	2	4	22

Round	z_1	z_2	z_3	s
0	1	8	28	3
1	2	9	27 or 29	4
2	3	10	26 or 30	20
3	4	2	5 or 1	19

Round	z_1	z_2	z_3	s
0	1	8	29	3
1	2	9	30	4
2	3	10	1	20
3	4	2	5	19

Round	z_1	z_2	z_3	s
0	1	8	30	3
1	2	9	1	4
2	3	10	5	20
3	4	2	4	19

Round	z_1	z_2	z_3	s
0	1	9	26	3
1	2	10	5	16
2	3	2	4	17

Round	z_1	z_2	z_3	s
0	1	9	27	3
1	2	10	26	4
2	3	2	5	20
3	4	3	4	19

Round	z_1	z_2	z_3	s
0	1	9	28	4 (start at 4 to prevent branching from 28)
1	5	10	27	3
2	4	2	26	16
3	3	3	5	17

Round	z_1	z_2	z_3	s
0	1	9	29	4
1	5	10	30	3
2	4	2	1	16
3	3	3	2	17

Round	z_1	z_2	z_3	s
0	1	9	30	4
1	5	10	1	3
2	4	2	2	16
3	3	3	3	17

Round	z_1	z_2	z_3	s
0	1	10	26	4
1	5	2	5	20
2	4	3	4	19

Round	z_1	z_2	z_3	s
0	1	10	27	4
1	5	2	26	20
2	4	3	5	19

Round	z_1	z_2	z_3	s
0	1	10	28	4
1	5	2	27	20
2	4	3	26	19
3	20	4	5	18
4	19	20	4	17
5	18	19	3	14
6	17	18	15	13
7	14	17	14	12

Round	z_1	z_2	z_3	s
0	1	10	29	4
1	2	2	30	3
2	3	3	1	20
3	4	4	5	19

Round	z_1	z_2	z_3	s
0	1	10	30	4
1	5	2	1	20
2	4	3	5	19

Round	z_1	z_2	z_3	s
0	1	11	16	4
1	5	2	3	20
2	4	3	4	21

Round	z_1	z_2	z_3	s
0	1	11	17	3
1	2	2	16	4
2	3	3	3	20

Round	z_1	z_2	z_3	s
0	1	11	18	3
1	2	2	17	4
2	3	3	16	20

Round	z_1	z_2	z_3	s
0	1	11	19	4
1	5	2	20	21
2	4	3	4	22

Round	z_1	z_2	z_3	s
0	1	11	20	3
1	2	2	4	16
2	3	3	3	17

Round	z_1	z_2	z_3	s
0	1	12	16	4
1	5	11	3	20
2	4	2	4	19

Round	z_1	z_2	z_3	s
0	1	12	17	3
1	2	11	16	4
2	3	2	3	20
3	4	3	4	19

Round	z_1	z_2	z_3	s
0	1	12	18	3
1	2	11	17	4
2	3	2	16	20
3	4	3	3	19

Round	z_1	z_2	z_3	s
0	1	12	19	4
1	5	11	20	3
2	4	2	4	16
3	3	3	3	17

Round	z_1	z_2	z_3	s
0	1	12	20	3
1	2	11	4	16
2	3	2	3	17

Round	z_1	z_2	z_3	s
0	1	13	16	4
1	5	14	3	21 (move to 20 causes branching)
2	4	15	4	22
3	21	3	21	23
4	22	4	22	24
5	23	5	23	27
6	24	26	24	28

Round	z_1	z_2	z_3	s
0	1	13	17	3
1	2	14	16	4
2	3	15	3	5
3	4	3	4	26
4	5	2	5	27

Round	z_1	z_2	z_3	s
0	1	13	18	3
1	2	14	17	4
2	3	15	16	5
3	4	3	3	26
4	5	4	4	27

Round	z_1	z_2	z_3	s
0	1	13	19	4
1	5	14	20	21
2	4	15	4	22
3	21	3	21	23
4	22	4	22	24
5	23	5	23	27
6	24	26	24	28

Round	z_1	z_2	z_3	s
0	1	13	20	9
1	2	12	4	8
2	10	9	3 or 5	7
3	9	8	2 or 1	29
4	8	7	1 or 30	28
5	7	29	30 or 29	27
6	29	28	29 or 28	24

Note: this is a tricky case. luring to center does not work!

Round	z_1	z_2	z_3	s
0	1	14	16	4
1	5	15	3	21
2	4	3	4	22

Round	z_1	z_2	z_3	s
0	1	14	17	4
1	5	15	16	21
2	4	3	3	22

Round	z_1	z_2	z_3	s
0	1	14	18	3
1	2	15	17	4
2	3	3	16	5
3	4	4	3	26
4	5	5	4	27

Round	z_1	z_2	z_3	s
0	1	14	19	4
1	5	15	20	21
2	4	3	4	22

Round	z_1	z_2	z_3	s
0	1	14	20	3
1	2	15	4	16
2	3	3	3	17

Round	z_1	z_2	z_3	s
0	1	15	16	4
1	5	3	3	21
2	4	4	4	22

Round	z_1	z_2	z_3	s
0	1	15	17	4
1	5	3	16	21
2	4	4	3	22

Round	z_1	z_2	z_3	s	
0	1	15	18	12	
1	2	14	17	9	Note: here again, center strategy does not work
2	10	13	14	8	
3	9	12	13	7	

Round	z_1	z_2	z_3	s
0	1	15	19	4
1	5	3	20	21
2	4	4	4	22

Case 2–4

Round	z_1	z_2	z_3	s
0	1	11	21	3
1	2	2	4	16
2	3	3	3	17

Round	z_1	z_2	z_3	s
0	1	11	22	3
1	2	2	21	16
2	3	3	4	17

Round	z_1	z_2	z_3	s
0	1	11	23	3
1	2	2	22	4
2	3	3	21	20
3	4	4	4	19

Round	z_1	z_2	z_3	s
0	1	11	24	3
1	2	2	25	4
2	3	3	5	20
3	4	4	4	19

Round	z_1	z_2	z_3	s
0	1	11	25	3
1	2	2	5	16
2	3	3	4	17

Round	z_1	z_2	z_3	s
0	1	12	21	3
1	2	11	4	16
2	3	2	3	17

Round	z_1	z_2	z_3	s
0	1	12	22	3
1	2	11	21	16
2	3	2	4	17

Round	z_1	z_2	z_3	s
0	1	12	23	3
1	2	11	22	4
2	3	2	21	20
3	4	3	4	19

Round	z_1	z_2	z_3	s
0	1	12	24	3
1	2	11	25	4
2	3	2	5	20
3	4	3	4	19

Round	z_1	z_2	z_3	s
0	1	12	25	3
1	2	11	5	16
2	3	2	4	17

Round	z_1	z_2	z_3	s
0	1	13	21	9
1	2	12	4	8
2	10	9	3 or 5	7
3	9	8	2 or 1	29
4	8	7	1 or 30	28
5	7	29	30 or 29	27
6	29	28	29 or 28	24

Note: center strategy fails here. symmetric to 1, 15, 23

Round	z_1	z_2	z_3	s
0	1	13	22	4
1	5	14	21	3
2	4	15	4	16
3	3	3	3	17

Note: starting at 3 fails. starting at 4 and moving to 20 also fails

Round	z_1	z_2	z_3	s
0	1	13	23	3
1	2	14	22	4
2	3	15	21	20
3	4	3	4	19

Round	z_1	z_2	z_3	s
0	1	13	24	3
1	2	14	25	4
2	3	15	5	20
3	4	3	4	19

Round	z_1	z_2	z_3	s
0	1	13	25	4
1	5	14	5	3
2	4	15	4	2
3	3	3	3	10

Round	z_1	z_2	z_3	s
0	1	14	21	3
1	2	15	4	16
2	3	3	3	17

Round	z_1	z_2	z_3	s
0	1	14	22	3
1	2	15	21	16
2	3	3	4	17

Round	z_1	z_2	z_3	s
0	1	14	23	3
1	2	15	22	4
2	3	3	21	20
3	4	4	4	19

Round	z_1	z_2	z_3	s
0	1	14	24	3
1	2	15	25	4
2	3	3	5	20
3	4	4	4	19

Round	z_1	z_2	z_3	s
0	1	14	25	3
1	2	15	5	16
2	3	3	4	17

Round	z_1	z_2	z_3	s
0	1	15	21	13
1	2	14	4	12
2	11	13	3	9
3	12	12	2	8
4	9	9	10	7

Note: zombies cover center at start

Round	z_1	z_2	z_3	s
0	1	15	22	4
1	5	3	21	20
2	4	4	4	19

Round	z_1	z_2	z_3	s
0	1	15	23	27
1	5	3	24	28
2	26	2 or 4	27	29
3	27	1 or 5	28	7
4	28	6 or 1	29	8
5	29	7 or 6	7	9
6	7	8 or 7	8	12

Note: center strategy fails

Round	z_1	z_2	z_3	s
0	1	15	24	4
1	5	3	25	20
2	4	4	5	19

Round	z_1	z_2	z_3	s
0	1	15	25	4
1	5	3	5	20
2	4	4	4	19

.4 Case IV

Round	z_1	z_2	z_3	s
0	6	11	21	3
1	1	2	4	15
2	2	3	3	14

Round	z_1	z_2	z_3	s
0	6	11	22	3
1	1	2	21	15
2	2	3	4	14

Round	z_1	z_2	z_3	s
0	6	11	23	3
1	1	2	22	4
2	5	3	21	20
3	4	4	4	19

Round	z_1	z_2	z_3	s
0	6	11	24	5
1	1	2	25	4
2	5	3	5	20
3	4	4	4	19

Round	z_1	z_2	z_3	s
0	6	11	25	3
1	1	2	5	16
2	2	3	4	17

Round	z_1	z_2	z_3	s
0	6	12	21	3
1	1	11	4	16
2	2	2	3	17

Round	z_1	z_2	z_3	s
0	6	12	22	3
1	1	11	21	16
2	2	2	4	17

Round	z_1	z_2	z_3	s
0	6	12	23	5
1	1	11	24	4
2	5	2	25	20
3	4	3	5	19

Round	z_1	z_2	z_3	s
0	6	12	24	5
1	1	11	25	4
2	5	2	5	20
3	4	3	4	19

Round	z_1	z_2	z_3	s
0	6	12	25	4
1	1	11	5	20
2	5	2	4	19

Round	z_1	z_2	z_3	s
0	6	13	21	