Zombies and Survivors on Graphs

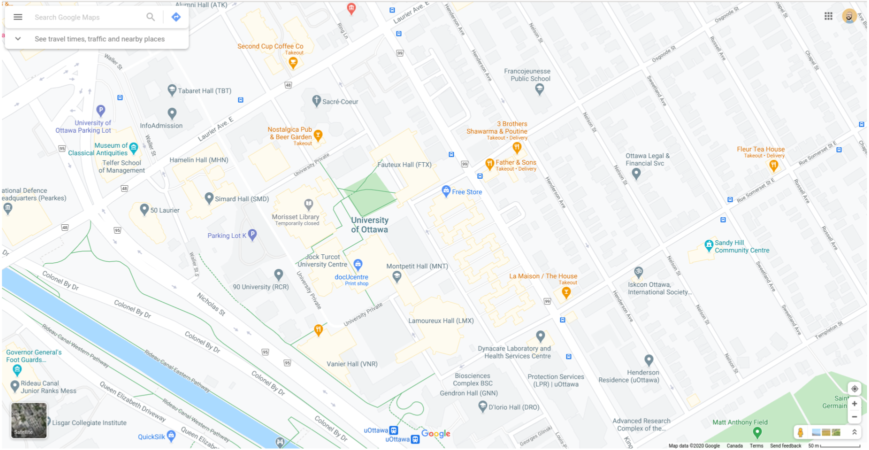
Joël Faubert

University of Ottawa

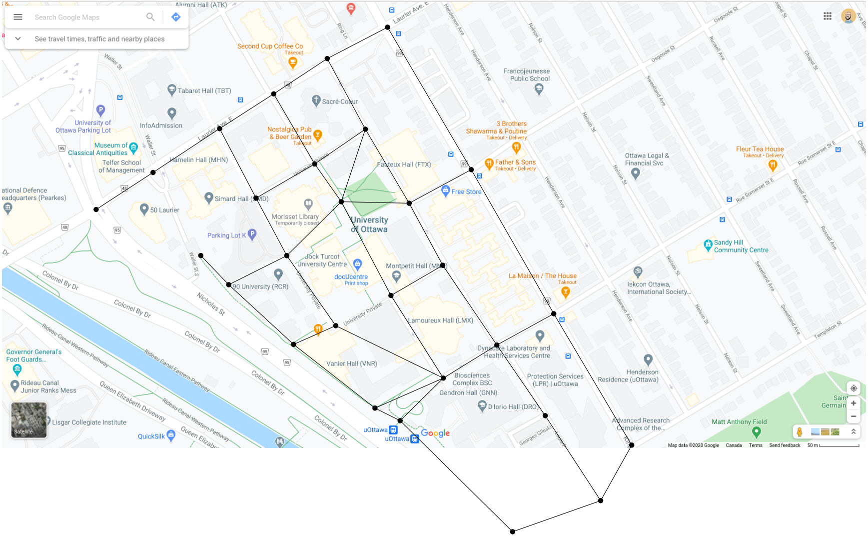
September 1, 2020

Outline

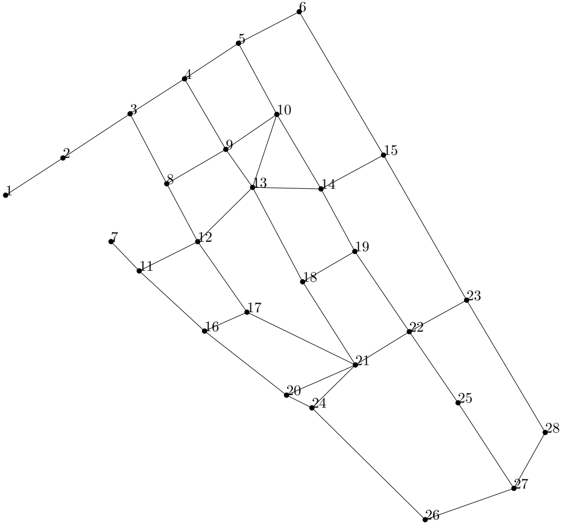
Games on Graphs



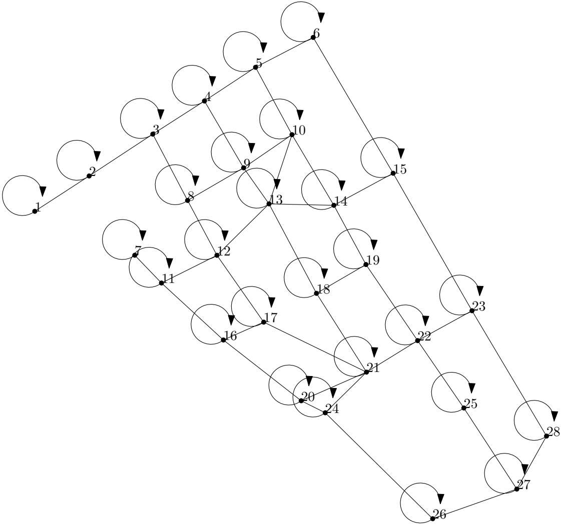
Games on Graphs



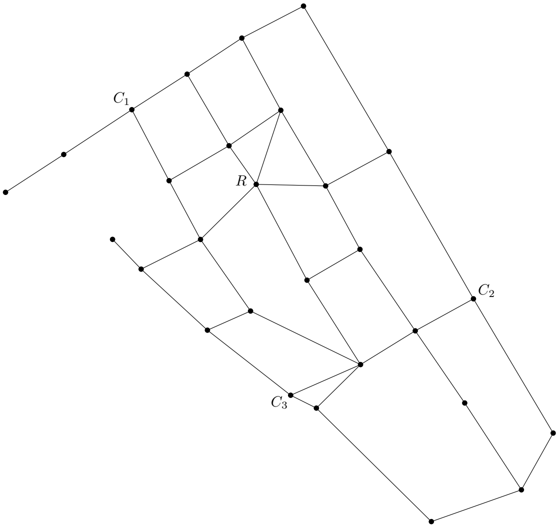
Games on Graphs



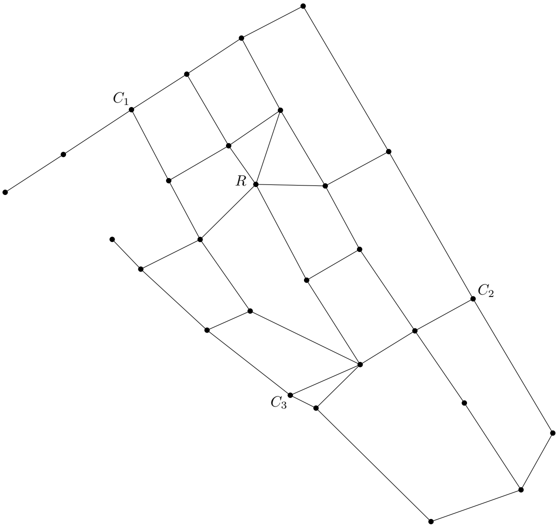
Games on Graphs



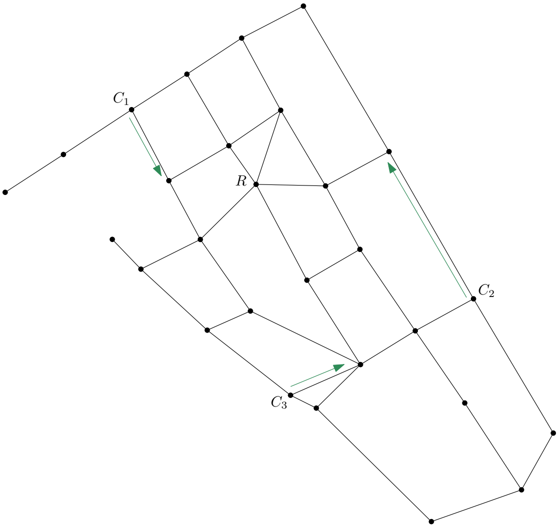
Games on Graphs



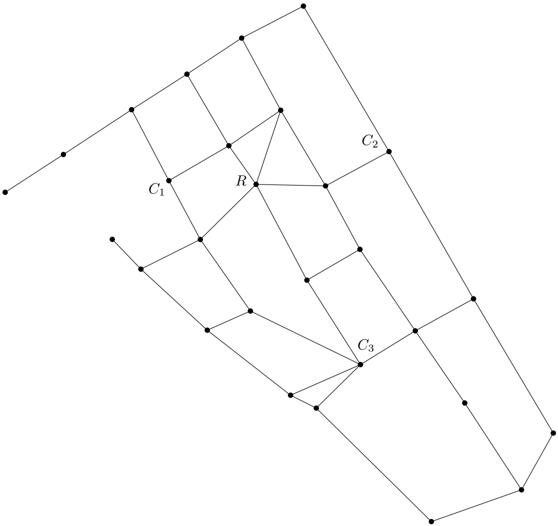
Games on Graphs



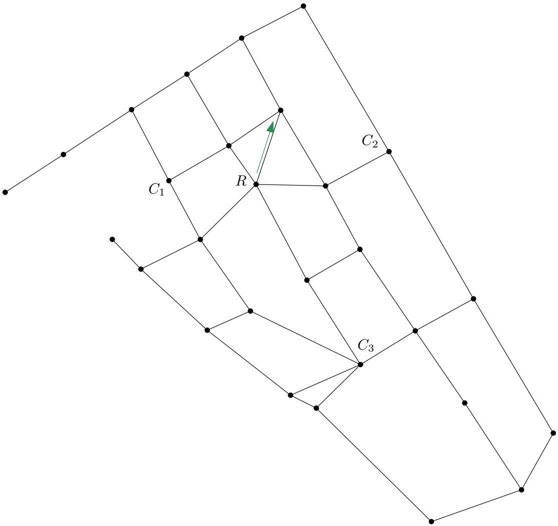
Games on Graphs



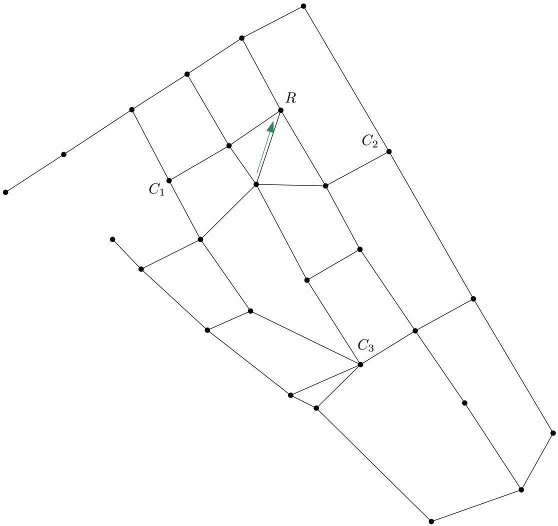
Games on Graphs



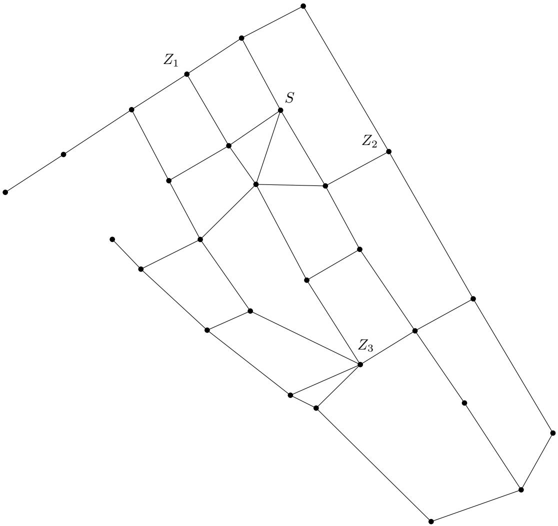
Games on Graphs



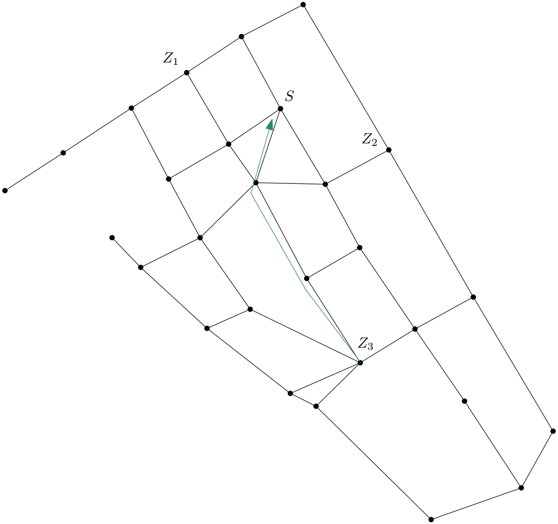
Games on Graphs



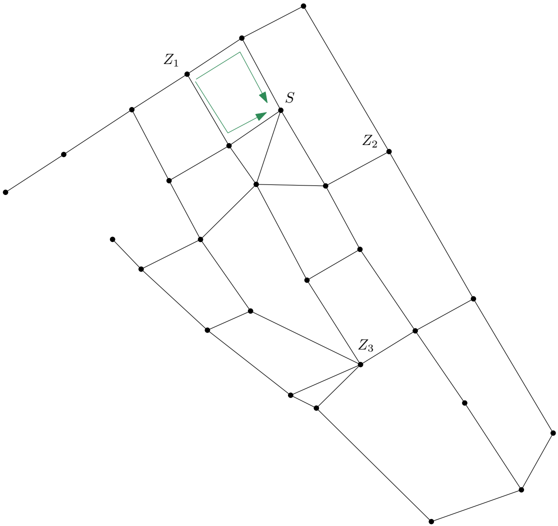
Zombies and Survivors



Zombies and Survivors



Zombies and Survivors



Cop-Number and Zombie-Number

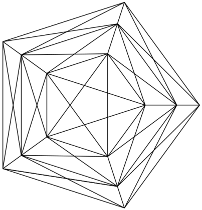


Figure: Cop-win but not zombie-win [FHMP16])

- Cop-Number $c(G)$ number of cops needed to guarantee a win on G.

Cop-Number and Zombie-Number

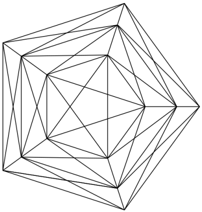


Figure: Cop-win but not zombie-win [FHMP16])

- Cop-Number $c(G)$ number of cops needed to guarantee a win on G.
- Zombie-Number $z(G)$ number of zombies needed to guarantee a win on G.

Cop-Number and Zombie-Number

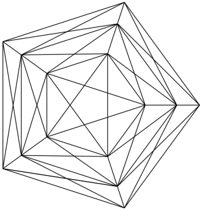


Figure: Cop-win but not zombie-win [FHMP16])

- Cop-Number $c(G)$ number of cops needed to guarantee a win on G.
- Zombie-Number $z(G)$ number of zombies needed to guarantee a win on G.

Lemma

For any graph G, $c(G) \leq z(G)$.

Cop-Number and Zombie-Number

G	$c(G)$	$z(G)$
T tree (acyclic)	1	1
C_3	1	1
C_n $(n \geq 4)$	2	2
K_n $(n \geq 1)$	1	1
$K_{n,m}$ $(n, m \geq 2)$	2	2
G planar	3	?
G outerplanar	2	?

Table: Cop and zombie number of a few graph families

Cop-Number and Zombie-Number

G	$c(G)$	$z(G)$
T tree (acyclic)	1	1
C_3	1	1
C_n $(n \geq 4)$	2	2
K_n $(n \geq 1)$	1	1
$K_{n,m}$ $(n, m \geq 2)$	2	2
G planar	3	at least 4
G outerplanar	2	at least 3

Table: Cop and zombie number of a few graph families

Zombies vs. Cops on Planar Graphs

A graph is planar if it can be *embedded in the plane*.

Aigner and Fromme [AF84] described a winning 3 cop strategy: enclose the robber into a shrinking *territory* using *isometric paths*.

In next section, present a planar graph where 3 zombies lose.

On Outerplanar Graphs

A graph is *outerplanar* if it can be *embedded in a cycle on the plane* such that its vertices are all adjacent to the *outer face*.

In [Cla02], Clarke showed that 2 cops suffice to win on outerplanar graphs.

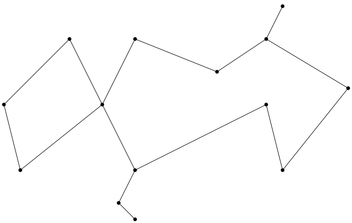


Figure: An outerplanar graph

Zombies vs Cops on Outerplanar Graphs

But 2 zombies lose on this outer-planar graph 3.

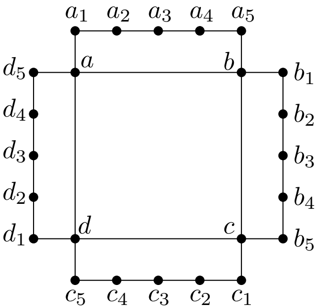


Figure: Two cops win but two zombies lose (Fig 2 of [FHMP16])

Zombies on a planar graph

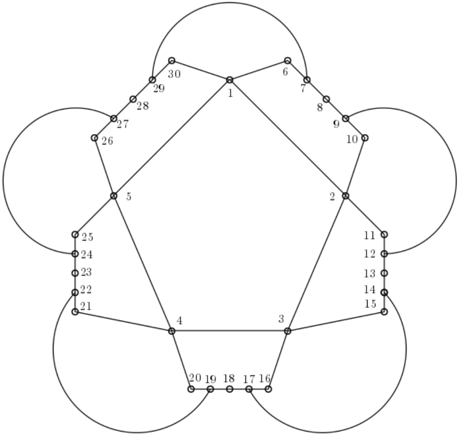


Figure: A planar graph where 3 zombies lose

Zombies on a planar graph

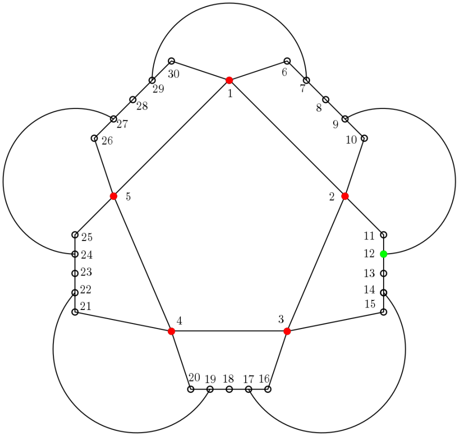


Figure: A planar graph where 3 zombies lose

Zombies on a planar graph

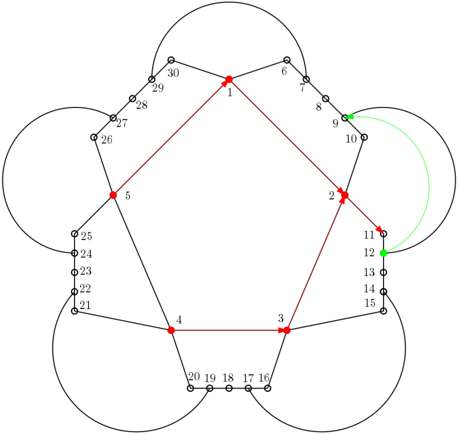


Figure: A planar graph where 3 zombies lose

Zombies on a planar graph

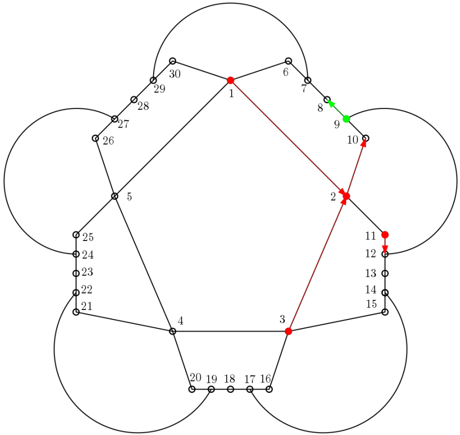


Figure: A planar graph where 3 zombies lose

Zombies on a planar graph

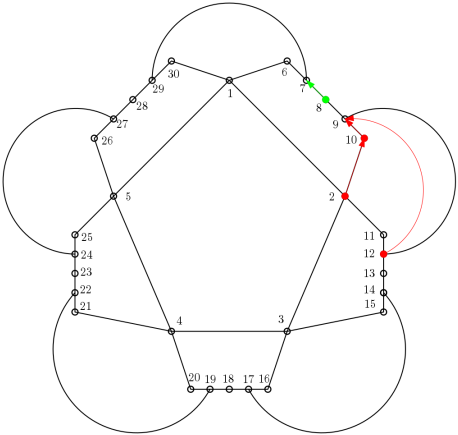


Figure: A planar graph where 3 zombies lose

On a cycle with one chord

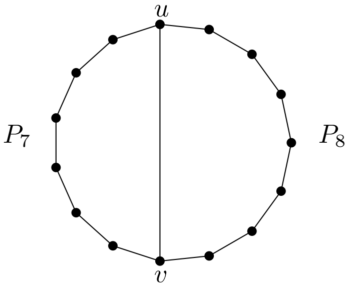


Figure: $Q_{7,8}$

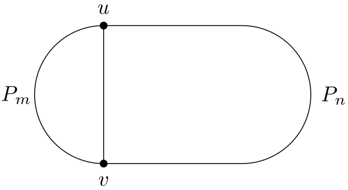


Figure: $Q_{m,n}$

2-zombie-win strategy

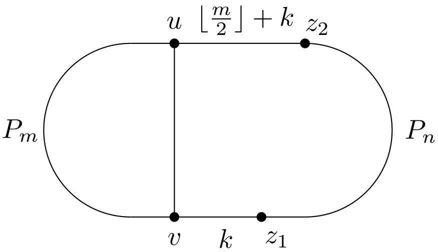


Figure: A winning 2-zombie start for a cycle with one chord.

where

$$k = \left\lfloor \frac{n - 2\left\lfloor \frac{m}{2} \right\rfloor + 3}{4} \right\rfloor.$$

Proof: Cornering the survivor on C_{m+1}

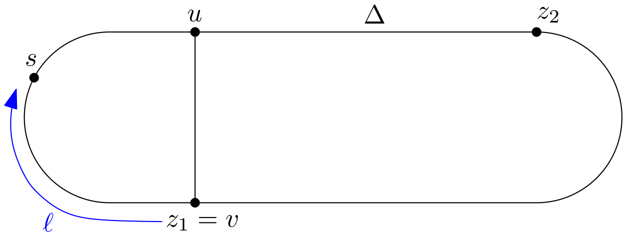


Figure: z_1 on v, z_2 is Δ away from u, and s on P_m

Which way do the Zombies turn?

Depends on value of $2 \leq \ell \leq m - 1$.

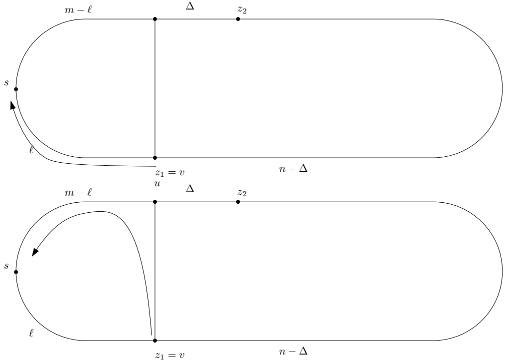
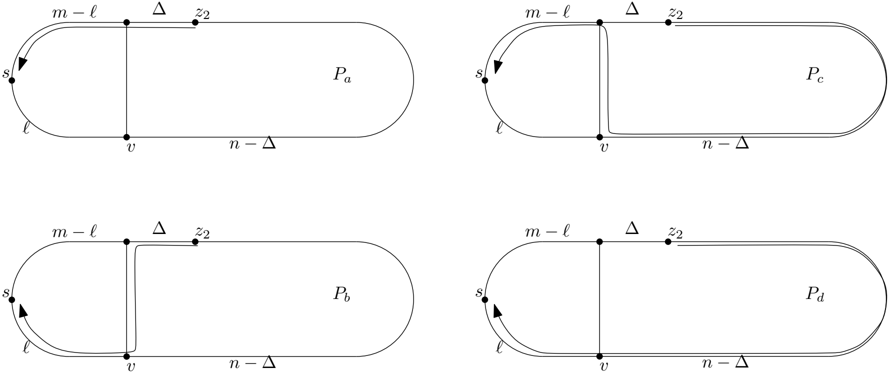


Figure: Clockwise or Counter-clockwise

Possible z_2s-paths

Depends on value of ℓ and Δ.



Path lengths

Four possible paths for z_2:

- $|P_a| = \Delta + (m - \ell)$,
- $|P_b| = \Delta + 1 + \ell$,
- $|P_c| = (n - \Delta) + 1 + (m - \ell)$,
- $|P_d| = (n - \Delta) + \ell$,

but only two possible choices: CW or CCW.

Gives four possible cases

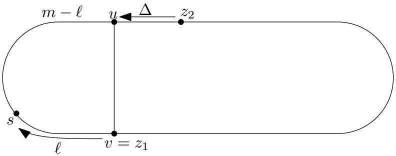


Figure: Case I.A.

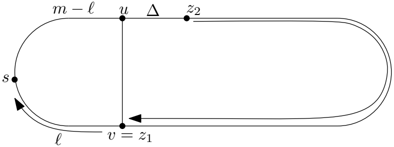


Figure: Case II.A.

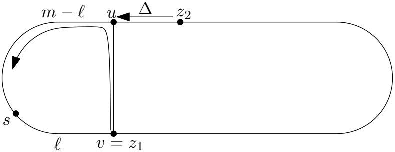


Figure: Case I.B.

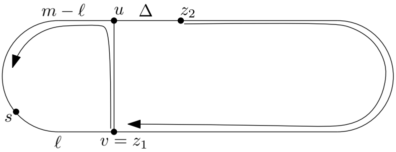


Figure: Case II.B.

Assumptions I or II, A or B

$$2 \leq \ell \leq m - 1 \tag{1}$$

z_1 goes clockwise if $\ell \leq 1 + m - \ell$. Combined with 1, we have

$$4 \leq 2\ell \leq m + 1. \tag{A}$$

z_1 goes counter-clockwise if $1 + m - \ell \leq \ell$. Combined 1, we obtain

$$m + 1 \leq 2\ell \leq 2m - 2. \tag{B}$$

z_2 moves counter-clockwise if either

$$|P_a| \leq \min\{|P_c|, |P_d|\} \qquad \text{or} \qquad |P_b| \leq \min\{|P_c|, |P_d|\}. \tag{I}$$

z_2 goes clockwise if either

$$|P_c| \leq \min\{|P_a|, |P_b|\} \qquad \text{or} \qquad |P_d| \leq \min\{|P_a|, |P_b|\}. \tag{II}$$

Case I.A.

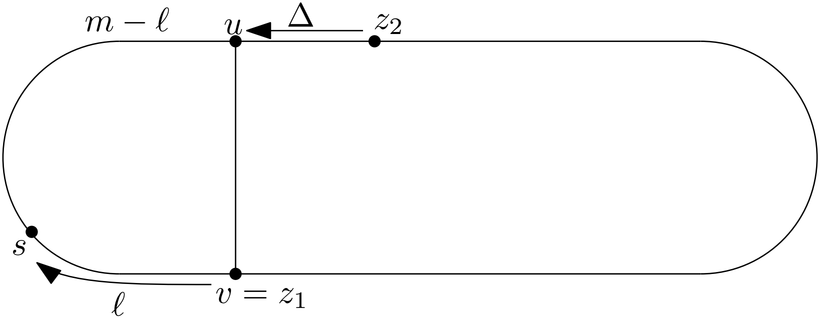


Figure: Case I.A.

Case I.A. Inequalities

Constraints on ℓ from A

$$4 \leq 2\ell \leq m + 1$$

Constraints on Δ from I

$$\Delta + (m - \ell) \leq n - \Delta + 1 + m - \ell \qquad \text{and}$$
$$\Delta + (m - \ell) \leq n - \Delta + \ell$$

or

$$\Delta + 1 + \ell \leq n - \Delta + 1 + m - \ell \qquad \text{and}$$
$$\Delta + 1 + \ell \leq n - \Delta + \ell.$$

Combined and simplified

For z_2 to go counter-clockwise (following either P_a or P_b) we must have

$$2\Delta \leq n - m + 2\ell$$

or

$$2\Delta \leq n - 1.$$

Outcome

What happens Δ turns later?

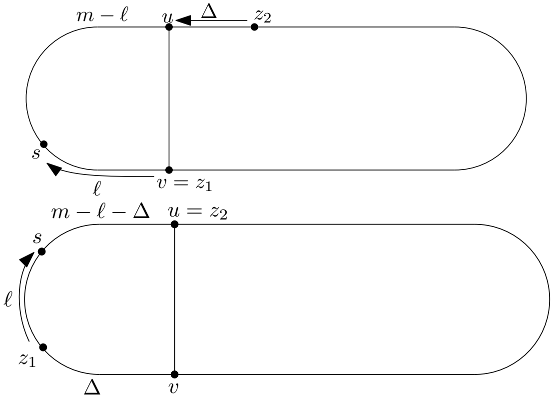


Figure: Case I.A. before/after Δ rounds

Winning Condition

If $\Delta = m - \ell$ both z_2 and s reach u on the same round, with the survivor moving onto the zombie-occupied vertex (and losing).

If we have $\Delta = m - \ell + 1$, then s reaches u first but is caught by z_2 on the following round.

So, to guarantee the survivor has not escaped P_m we need

$$\Delta \leq m - \ell + 1.$$

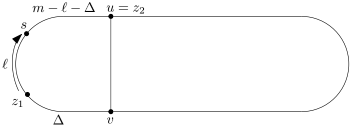


Figure: Case I.A. after Δ rounds

Win Condition II

Must also ensure that z_2 moves counter-clockwise (opposite to z_1) once it reaches u in order to trap the survivor. So we also need

$$m - \ell - \Delta \leq 1 + \Delta + \ell$$

Or, in terms of Δ,

$$2\Delta \geq m - 2\ell - 1.$$

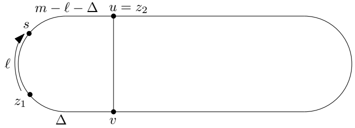


Figure: Case I.A. after Δ rounds

Case I.A.

z_1 clockwise: $4 \leq 2\ell \leq m + 1$
z_2 counter-clockwise:
$2\Delta \leq n - m + 2\ell$ or $2\Delta \leq n - 1$
zombies win:
$2\Delta \leq 2m - 2\ell + 2$ and
$m - 2\ell - 1 \leq 2\Delta$

Case II.A.

z_1 clockwise: $4 \leq 2\ell \leq m + 1$
z_2 clockwise:
$n - m + 2\ell \leq 2\Delta$ and $n - 1 \leq 2\Delta$
zombies win:
$2\Delta \geq 2n - 2m + 2\ell$ and
$2\Delta \leq 2n + 2\ell - m - 1$

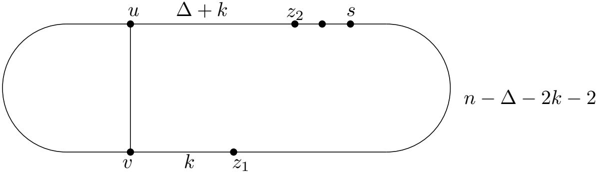
Case I.B.

z_1 counter-clockwise:
$m + 1 \leq 2\ell \leq 2m - 2$
z_2 counter-clockwise:
$2\Delta \leq n + 1$ or $2\Delta \leq n + m - 2\ell$
zombies win:
$\Delta \leq 2\ell$ and
$2\ell - m + 1 \leq 2\Delta$

Case II.B.

z_1 counter-clockwise:
$m + 1 \leq 2\ell \leq 2m - 2$
z_2 clockwise:
$n + 1 \leq 2\Delta$
zombies win:
$n - \Delta \leq \ell + 1$ and
$2\Delta \leq 2n + m - 2\ell + 1$

Guarding C_{n+1}



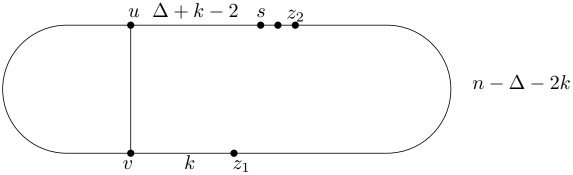


Figure: Preventing the zombies from turning in same direction on C_{m+1}

Guarding C_{n+1} II

To ensure zombies turn in opposite directions, we need:

$$n - \Delta - 2k - 2 \leq \Delta + k + 1 + k + 2 \qquad \text{and}$$
$$\Delta + 2k - 1 \leq n - \Delta - 2k + 2.$$

Solving for k gives

$$n - 2\Delta - 5 \leq 4k \leq n - 2\Delta + 3$$

How the strategy works

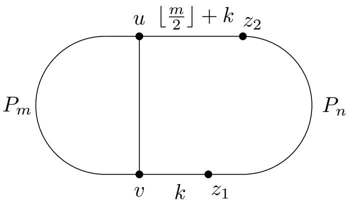


Figure: A winning 2-zombie start for a cycle with one chord.

where

$$k = \left\lfloor \frac{n - 2\left\lfloor \frac{m}{2} \right\rfloor + 3}{4} \right\rfloor .$$

- If s starts between the two zombies on C_{n+1} – on subpath that does not include the chord – then survivor loses.

How the strategy works

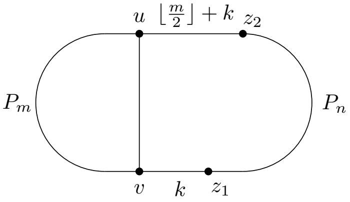


Figure: A winning 2-zombie start for a cycle with one chord.

where

$$k = \left\lfloor \frac{n - 2\left\lfloor \frac{m}{2} \right\rfloor + 3}{4} \right\rfloor.$$

- If s starts between the two zombies on C_{n+1} – on subpath that does not include the chord – then survivor loses.
- Otherwise, s must be on other half of C_{n+1} or on C_{m+1}.

How the strategy works

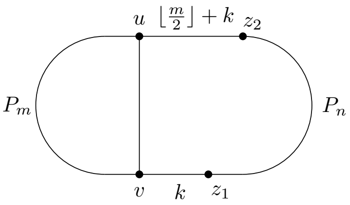


Figure: A winning 2-zombie start for a cycle with one chord.

where

$$k = \left\lfloor \frac{n - 2\left\lfloor \frac{m}{2} \right\rfloor + 3}{4} \right\rfloor .$$

- If s starts between the two zombies on C_{n+1} – on subpath that does not include the chord – then survivor loses.
- Otherwise, s must be on other half of C_{n+1} or on C_{m+1}.
- Pushed by zombies onto P_m.

How the strategy works

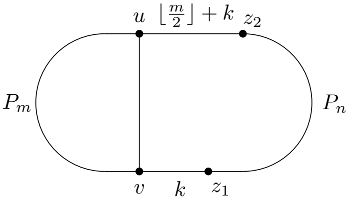


Figure: A winning 2-zombie start for a cycle with one chord.

where

$$k = \left\lfloor \frac{n - 2\left\lfloor \frac{m}{2} \right\rfloor + 3}{4} \right\rfloor .$$

- If s starts between the two zombies on C_{n+1} – on subpath that does not include the chord – then survivor loses.
- Otherwise, s must be on other half of C_{n+1} or on C_{m+1}.
- Pushed by zombies onto P_m.
- Falls into Δ trap after k turns.

Research Directions: Planar Zombies

- Is there an upper bound on the zombie-number for planar graphs?

Research Directions: Planar Zombies

- Is there an upper bound on the zombie-number for planar graphs?
- Is it possible to construct increasingly elaborate graphs (while still being planar) which would always provide the survivor with a winning strategy?

Research Directions: Zombies and Cycles

- Can this type of strategy be applied to any two cycles with a shared path?
- What about bowtie graphs?
- Can our results in $Q_{m,n}$ be used to determine the probabilistic zombie number?

Research Directions: On Visibility Graphs

- Recently shown that the visibility graphs of simple polygons are cop-win.
- Are they also cop-win?

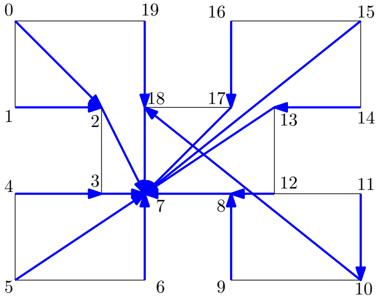


Figure: A Polygon Inscribed with a BFS Cop-win Tree

Cop-win Graphs and Dismantlings

- Study Cops and Robbers first attributed to Quilliot [Qui78], and Nowakowski and Winkler [NW83].

Cop-win Graphs and Dismantlings

- Study Cops and Robbers first attributed to Quilliot [Qui78], and Nowakowski and Winkler [NW83].
- Characterize graphs where the cop always wins, now known as *cop-win* graphs.

Cop-win Graphs and Dismantlings

- Study Cops and Robbers first attributed to Quilliot [Qui78], and Nowakowski and Winkler [NW83].
- Characterize graphs where the cop always wins, now known as *cop-win* graphs.
- Exist ordering of the vertices called a *dismantling*.

Cop-win Graphs and Dismantlings

- Study Cops and Robbers first attributed to Quilliot [Qui78], and Nowakowski and Winkler [NW83].
- Characterize graphs where the cop always wins, now known as *cop-win* graphs.
- Exist ordering of the vertices called a *dismantling*.
- Successive deletion of *corners* resulting in a single vertex.

Meyniel's Conjecture

Upper bound on the cop-number [Fra87]

$$\mathcal{O}(\sqrt{|V(G)|})$$

.
Incremental progress has been made on special classes of graphs as well as for graphs in general [GHL16][p. 31].

Cop-Number and Minimum Degree

Cop-Number and Genus of a Graph

In 2001, Schroeder conjectured [BM17] that for a graph of genus g, the cop-number is at most $g+3$.

Currently, the best known bound [Sch01] for graph G of genus g is

$$c(G) \leq \left\lfloor \frac{3}{2} g \right\rfloor + 3$$

.

References I

- Oswin Aichholzer, Greg Aloupis, Erik D Demaine, Martin L Demaine, Vida Dujmovic, Ferran Hurtado, Anna Lubiw, Günter Rote, André Schulz, Diane L Souvaine, et al., *Convexifying polygons without losing visibilities.*, CCCG, 2011.
- Martin Aigner and Michael Fromme, *A game of cops and robbers*, Discrete Applied Mathematics **8** (1984), no. 1, 1–12.
- Andrew Beveridge, Andrzej Dudek, Alan Frieze, and Tobias Müller, *Cops and robbers on geometric graphs*, arXiv preprint arXiv:1108.2549 (2011).
- Alessandro Berarducci and Benedetto Intrigila, *On the cop number of a graph*, Advances in Applied Mathematics **14** (1993), no. 4, 389–403.

References II

- Anthony Bonato and Bojan Mohar, *Topological directions in cops and robbers*, arXiv preprint arXiv:1709.09050 (2017).
- Anthony Bonato, Dieter Mitsche, Xavier Pérez-Giménez, and Paweł Prałat, *A probabilistic version of the game of zombies and survivors on graphs*, Theoretical Computer Science **655** (2016), 2–14.
- A. Bonato and R.J. Nowakowski, *The game of cops and robbers on graphs*, Student mathematical library, American Mathematical Society, 2011.
- Nancy Elaine Blanche Clarke, *Constrained cops and robber.*

References III

- Thomas Erlebach and Jakob T Spooner, *A game of cops and robbers on graphs with periodic edge-connectivity*, International Conference on Current Trends in Theory and Practice of Informatics, Springer, 2020, pp. 64–75.
- SL Fitzpatrick, J Howell, ME Messinger, and DA Pike, *A deterministic version of the game of zombies and survivors on graphs*, Discrete Applied Mathematics **213** (2016), 1–12.
- Shannon L Fitzpatrick, *The game of zombies and survivors on the cartesian products of trees*, arXiv preprint arXiv:1806.04628 (2018).
- Peter Frankl, *Cops and robbers in graphs with large girth and cayley graphs*, Discrete Applied Mathematics **17** (1987), no. 3, 301–305.

References IV

- Tomáš Gavenčiak, Przemysław Gordinowicz, Vít Jelínek, Pavel Klavík, and Jan Kratochvíl, *Cops and robbers on intersection graphs*, European Journal of Combinatorics **72** (2018), 45–69.
- Ralucca Gera, Stephen Hedetniemi, and Craig Larson, *Graph theory: Favorite conjectures and open problems-1*, Springer, 2016.
- Ilya Gromovikov, William B Kinnersley, and Ben Seamone, *Fully active cops and robbers*, arXiv preprint arXiv:1808.06734 (2018).
- Gena Hahn and Gary MacGillivray, *A characterisation of k-cop-win graphs and digraphs*.
- Anna Lubiw, Jack Snoeyink, and Hamideh Vosoughpour, *Visibility graphs, dismantlability, and the cops and robbers game*, Computational Geometry **66** (2017), 14–27.

References V

- Richard Nowakowski and Peter Winkler, *Vertex-to-vertex pursuit in a graph*, Discrete Mathematics **43** (1983), no. 2-3, 235–239.
- Paweł Prałat, *How many zombies are needed to catch the survivor on toroidal grids?*, Theoretical Computer Science **794** (2019), 3–11.
- Alain Quilliot, *Jeux et pointes fixes sur les graphes*, Ph.D. thesis, Ph. D. Dissertation, Université de Paris VI, 1978.
- Bernd SW Schröder, *The copnumber of a graph is bounded by [3/2 genus (g)]+ 3*, Categorical perspectives, Springer, 2001, pp. 243–263.