Chordal Graphs

http://www.arindam.info

In algorithmic applications of Graph Theory, one of the most important classes of graphs is the class of *chordal graphs*. A graph is chordal if it has no induced cycles of length 4 or more.

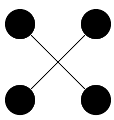
Introduction

Definition 1 (Chordal Graph)**.** A graph is called chordal if any cycle in it of length 4 or more has a chord.

Chordal graphs are sometimes also referred to as *triangulated* graphs.

Observation 1. The class chordal graphs is not closed under the minor relation or the usual subgraph relation. It is however (trivially) closed under the induced subgraph relation.

Example 1. Consider the following graph.



It contains no cycles, so it is chordal. But its complement is a 4-cycle without a chord. Complements of chordal graphs are not in general chordal.

Lemma 1. *Let $G = (V, E)$ be a chordal graph and S be a minimal separator in G. The graph $G[S]$ is complete.*

Proof. Let $u, v \in S$. We prove the statement by showing that $uv \in E$. In the graph $G[V \setminus S]$, let a and b be two vertices which are in distinct components, A and B, respectively. Since S is a minimal separator in G, it is also a minimal a—b separator. Thus there are u—v paths $ux_1 \cdots x_s v$ and $uy_1 \cdots y_t v$, such that $x_1, \ldots, x_s \in A$ and $y_1, \ldots, y_t \in B$. Choose these paths such that s and t are minimum.

By the minimality of s and t, $x_i x_j \notin E$ and $y_i y_j \notin E$ for $|i - j| > 1$. Since S is a separator and $A, B \subseteq V \setminus S$, there are no edges between $\{x_1, \ldots, x_s\}$ and $\{y_1, \ldots, y_t\}$.

The two u—v paths together form a cycle of length at least 4, and since G is chordal, the cycle must contain a chord. The only chord possible is uv. Thus for any two vertices $u, v \in S$, $uv \in E$, i.e. $G[S]$ is complete. □

Observation 2. Let S be a minimal a—b separator in G and let A and B be components in $G[V \setminus S]$ containing a and b respectively. Consider the graph $G' = G[V(A) \cup V(B)]$. Since S is a minimal a—b separator, it is a minimal separator in G'. If G is chordal, then so is G', and by the above lemma, $G'[S]$ is a clique. Thus any minimal a—b separator in a chordal graph G induces a clique.

Characterizations of Chordal Graphs

In this section, we look at two different characterizations of chordal graphs. Let $G = (V, E)$ be a graph of order n. A vertex $v \in V$ is called *simplicial* if the set of neighbours of v, $N(v)$ induces a complete graph.

Observation 3. In a complete graph, every vertex is simplicial.

Now we are ready to prove the following lemma.

Lemma 2. *Let G be a chordal graph. If G is incomplete, then it has two nonadjacent simplicial vertices. If it is a clique, then all its vertices are simplicial.*

Proof. We prove the statement by induction on $|G|$. The statement is clearly true when $|G| = 1$. Now assume that the statement is true for all incomplete graphs of order less than n and let G be an incomplete graph of order n. Let S be a minimal separator in G and let A and B be two connected components of $G - S$. Let $G_{A+S} = G[V(A) \cup S]$ and $G_{B+S} = G[V(A) \cup S]$.

Since $|G_{A+S}| < n$, if it is incomplete, then it has two nonadjacent simplicial vertices, by the induction hypothesis. By Lemma 1, $G[S]$ is a clique, so at least one of these vertices must be in A. If G_{A+S} is complete, every vertex in it is a simplicial vertex. In either case, there is a vertex v in A which is simplicial in G_{A+S}. Since $N(v) \subseteq V(A) \cup S$, v is simplicial in the entire graph. Similarly, there is a vertex in B which is simplicial in G_{B+S}, and therefore also simplicial in G.

Since A and B are distinct components of $G - S$, the two vertices are nonadjacent. Thus G has two nonadjacent simplicial vertices. □

Let G be a graph and $v_1, \ldots, v_n$ be an ordering of its vertices. The ordering is called a *perfect elimination scheme* if for each $1 \leq i \leq n-1$, v_i is simplicial in $G[v_i, v_{i+1}, \ldots, v_n]$. The following theorem gives two additional characterizations of chordal graphs.

Theorem 1. *Let G be a graph of order n. The following statements are equivalent:*

1. *G is chordal.*
2. *There is a perfect elimination scheme for G.*
3. *Any minimal separator in G induces a clique.*

Proof. We first prove $\boxed{1 \implies 2}$. Since G is chordal, it has a simplicial vertex, by Lemma 2 Let v_1 be such a vertex. Now consider $G - v_1$. It is also chordal and thus has a simplicial vertex v_2. Continuing this way, we can obtain an ordering $v_1, \ldots, v_n$ of V such that for each $1 \leq i \leq n-1$, v_i is simplicial in $G[v_i, \ldots, v_n]$, i.e. $v_1, \ldots, v_n$ is a perfect elimination scheme.

We now prove $\boxed{2 \implies 1}$. If G has no cycle of length at least 4, then we are done. So let C be a cycle of length at least 4 in G and let $v_1, \ldots, v_n$ be a perfect elimination scheme. Let v_k be the first vertex in the scheme which is from $V(C)$. Then $C \leq G' = G[v_k, \ldots, v_n]$ and v_k's neighbours in C, u and w, are also its neighbours in G'. Since v_k is simplicial in G', u and w are adjacent, i.e. uw is a chord of C. Thus G is chordal.

$\boxed{1 \implies 3}$ has already been proved (Lemma 1). It remains to be shown that $\boxed{3 \implies 1}$ holds. Suppose C is a cycle of length at least 4 in G. Let u, v and w be three consecutive vertices of C and let S be a minimal u—w separator. It must contain v and some other vertex $v' \in V(C) \setminus \{u, v, w\}$ (since $len(C) \geq 4$, such a vertex exists). By Observation 2, S induces a clique, and thus v and v' are adjacent, i.e. vv' is a chord of C. $\square$

Perfection

For any graph G, the quantity $\chi(G)$ denotes the smallest size of any partition P of $V(G)$ such that each component of the partition is an independent set in G. Partitioning V in such a way is equivalent to colouring the vertices such no two adjacent vertices have the same colour. $\chi(G)$ is called the *chromatic number* of G.

Let c be the largest integer such that K_c (the complete graph of order c) is a subgraph of G. c is called the *clique number* of G, and denoted by $\omega(G)$.

It is easy to see that $\omega(G) \leq \chi(G)$. Graphs in which these two quantities are equal for every induced subgraph are called *perfect*.

Definition 2 (Perfect Graph)**.** A graph G is called *perfect* if for any induced subgraph $H \sqsubseteq G$, $\omega(H) = \chi(H)$.

Theorem 2. *Chordal graphs are perfect.*