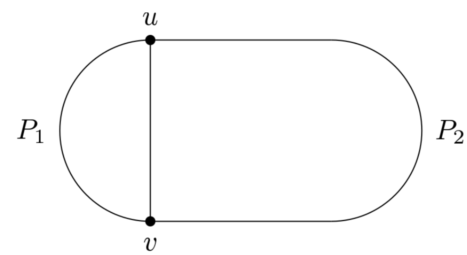
Definition 1. *We define a family of graphs we call* bifurcated cycles *and denote as $Q_{m,n}$. As the name suggests, bifurcated cycles are cycles of length $m+n$ with a single chord which divides the cycle into paths P_1 and P_2 of lengths m and n.*



$Q_{m,n}$

Theorem 1. *The Bifurcated cycle $Q_{m,n}$ is 2-zombie win if m, n are even.*

Proof. We place the two zombies on the longest half of the bifurcated cycle with z_1 at a distance of k from a chorded vertex and z_2 at a further distance of $k + \frac{n}{2}$.

Given this start configuration, we describe all winning strategies for s in terms of m, n, and k.

From that, we will show that for all m and n, there exists at least one value of k such that none of these winning strategies are viable and thus that the survivor will be captured.

Formally, let $u, v \in V(Q_{m,n})$ denote the endpoints of the chord and P_1 and P_2 denote the paths on either side of the chord.

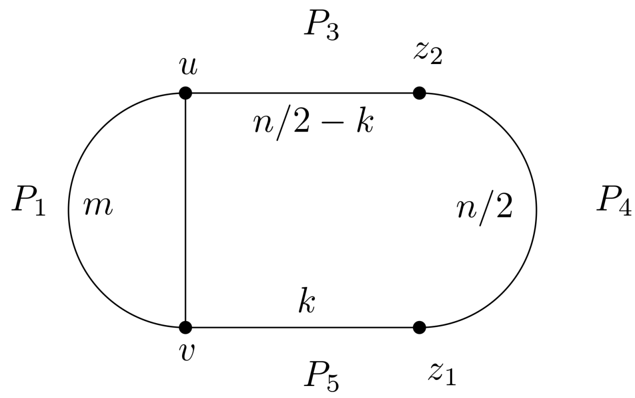
Each round of the game is composed of two turns: first the zombies' turn, followed by the survivor's turn. We denote as $z_i^{(t)} \in V(Q_{m,n})$ the position of zombie i (and $s^{(t)}$ the position of the survivor) at round t.

By construction we have $|P_1| = m$ and $|P_2| = n$ and we can assume, without loss of generality, that $m \leq n$. We also assume $m, n \geq 2$, since otherwise the construction adds parallel edges or degenerates to K_2.

Finally, let C_1 and C_2 be the subcycles of length $m+1$ and $n+1$ induced by P_1 and P_2 respectively.

Now, as mentioned above, we place the two zombies on vertices $z_1^{(0)}$ and $z_2^{(0)}$ on P_2 such that

1. The distance between the two zombies is $d(z_1^{(0)}, z_2^{(0)}) = n/2$, and
2. There is a path $P_5 = v, v_1, v_2, \ldots, v_k = z_1^{(0)}$ of length k between $z_1^{(0)}$ and the chorded vertex v. If $k = 0$, then P_5 is the trivial path v, and $z_1^{(0)} = v$.

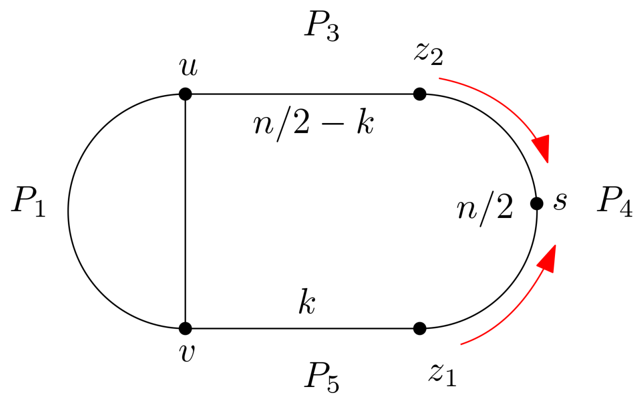


Without loss of generality, we can assume that $0 \leq k \leq n/4$, else we reflect the graph and rename the vertices.

These zombie positions divide P_2 into sub-paths $P_3 = u \ldots z_2^{(0)}$, $P_4 = z_2^{(0)} \ldots z_1^{(0)}$, and $P_5 = v \ldots z_1^{(0)}$.

First, notice that if the survivor chooses to start on P_4, then the zombies are guaranteed to win since

$$2 \leq d(z_i^{(0)}, s^{(0)}) \leq n/2 - 2 \qquad \text{for } i = 1, 2$$



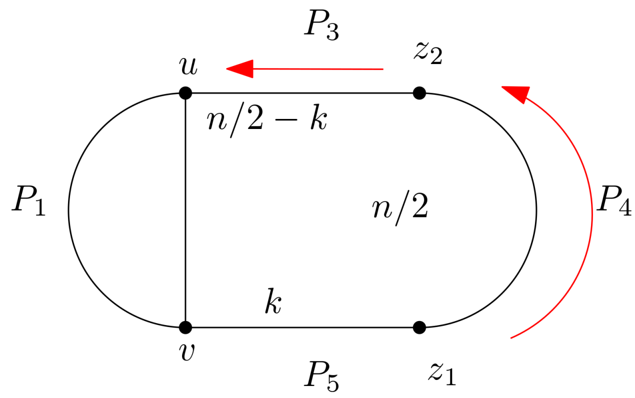
Play is effectively restricted to P_4. The zombies move in opposite directions towards the survivor and inevitably corner it.

So we can assume that the survivor does not start on P_4. We are left with three possible scenarios:

I. The zombies move in the same direction,
II. The zombies go in opposite directions,
III. z_2 may choose either direction.

Case I. One way for the survivor to win is to force the zombies to spin in the same direction from the start.

First we show by contradiction that the zombies cannot both go counterclockwise.



Clearly z_1 cannot go counterclockwise if s starts on P_5. It is impossible if the survivor starts on C_1, since all shortest zombie paths to C_1 must pass through u or v and the shortest paths to these vertices from $z_1^{(0)}$ cannot include P_4 because

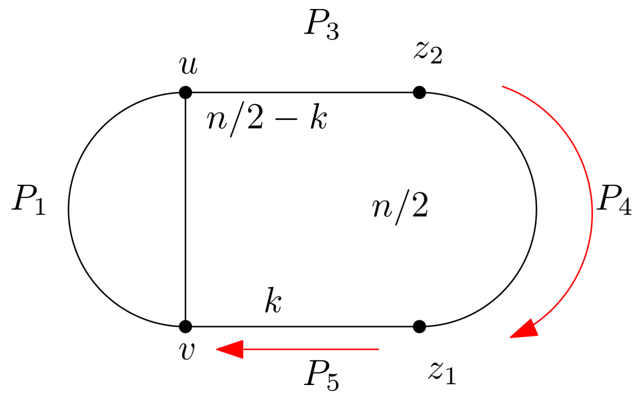
$$d(z_1^{(0)}, v) = |P_5| = k \leq \frac{n}{4} < \frac{n}{2} = |P_4| \qquad \forall n > 0$$

$$d(z_1^{(0)}, u) \leq |P_5| + 1 = k + 1 \leq \frac{n}{4} + 1 \leq \frac{n}{4} + \frac{n}{2} = \frac{3n}{4} < n - k = |P_4| + |P_3| \qquad \forall n > 0, 0 \leq k \leq \frac{n}{4}, k \in \mathbb{Z}$$

Suppose now that the survivor is on P_3 and forces the zombies to move counterclockwise. This implies that the shortest $z_1^0 s^0$-path includes P_4 and at least 2 vertices of P_3, and so has length at least $n/2 + 2$. The other path which uses P_5, the chord and then part of P_3 has length at most

$$k + 1 + (n/2 - k) - 2 = n/2 - 1$$

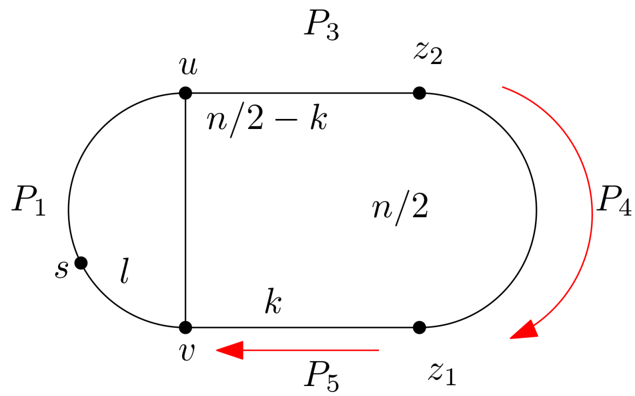
This contradicts the choice of $|P_4|$ as shortest path for z_1. So under no circumstances can the survivor force the zombies to move counterclockwise at the beginning of the game. Let us now consider if it is possible to make the zombies move clockwise.



Clearly it is not possible if the survivor starts on P_3. It is also impossible if the survivor starts on P_5 since assuming the survivor is on P_5 and z_2 moves clockwise gives

$$|P_4| + 2 = \frac{n}{2} + 2 \leq |P_3| + 1 + |P_5| - 2 = \left(\frac{n}{2} - k\right) + 1 + k - 2 = \frac{n}{2} - 1$$

A contradiction. So $z_2^{(0)}$ will not move clockwise if s starts on P_5. The other possibility is that s starts somewhere on P_1



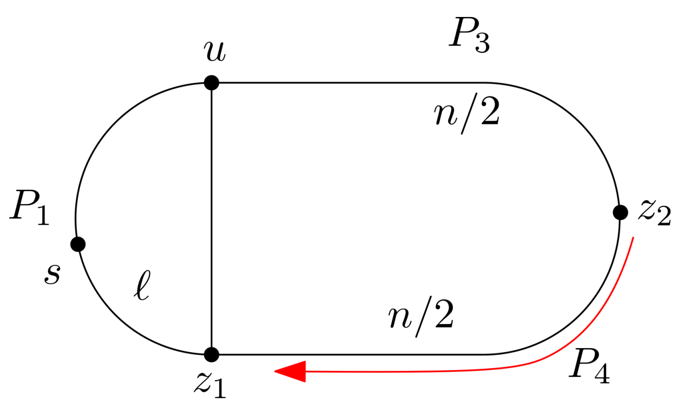
If we assume next that the survivor is on P_1 and that $z_2^{(0)}$ moves clockwise, then the shortest $z_2^{(0)}s^{(0)}$-path must pass through v and so

$$|P_4| + |P_5| = n/2 + k < |P_3| + 1 = n/2 - k + 1$$

or

$$n/2 + k \leq n/2 - k$$

which is possible only when $k = 0$. So $z_1^{(0)} = v$ and we have the following start situation:



If $\ell = m/2$, $z_2^{(0)}$ may go in either direction. So we need $\ell \leq m/2 - 1$ to force z_2 to move clockwise. We also need $\ell \geq 2$ else the survivor is captured by z_1 on the next turn.

For the next m moves, z_1 will be forced to follow s clockwise around C_1. The survivor must maintain distance at least 2 and so is forced to move around C_1. We can assume the initial distance ℓ is preserved since the survivor passing (or even reversing) is equivalent to choosing smaller initial distance of ℓ.

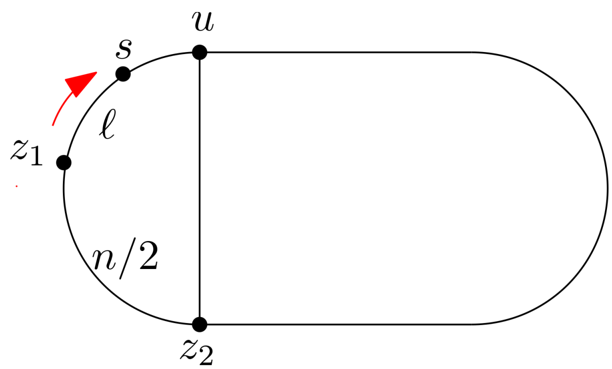
We look at the next event: when the next player attains the chord. Note that if s and z_2 reach u and v on the same round, then z_2 captures the survivor on the next turn.

So either

(A) z_2 attains the chorded vertex v before s reaches u; or, the reverse,
(B) s reaches the chord before z_2.

Subcase I(A) z_2 reaches v before s reaches u.

Since z_2 was at a distance of $n/2$, this event must occur $n/2$ rounds later and z_1 will have pursued the survivor that length around P_1.



We have supposed here that s hasn't yet reached the chord, so there exists a path of length

$$m - \ell - n/2 \geq 1$$

between s and u.

On the following round, z_2 can either follow z_1 clockwise along a hull edge or go counterclockwise using the chord edge. But since

$$m - \ell - \frac{n}{2} + 1 \leq m - 2 - \frac{n}{2} + 1 \leq n - 2 - \frac{n}{2} + 1 = \frac{n}{2} - 1 < \frac{n}{2} + \ell$$

We see that the shortest z_2s-path cannot follow the hull edge. So z_2 takes the chord and moves counterclockwise.

This allows us to conlude that if the zombies start with $k = 0$ and

$$m - \ell - \frac{n}{2} \geq 0$$

then the survivor will lose.

To avoid this scenario, the survivor must choose $\ell > m - \frac{n}{2}$, i.e. $\ell \geq m - \frac{n}{2} + 1$ while still respecting the restriction that $\ell \leq \frac{m}{2} - 1$.

In order to choose such ℓ we must have

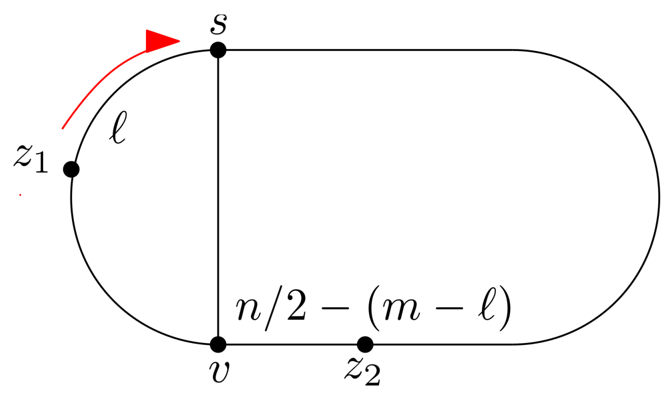
$$m - \frac{n}{2} + 1 \leq \ell \leq \frac{m}{2} - 1$$

or, simply,

$$m + 4 \leq n$$

Such choice for ℓ is impossible for the survivor whenever $m + 3 \geq n$, so we have a simple winning zombie-strategy for these configurations: choose $k = 0$.

Subcase I(B) s reaches v before z_2 reaches u.



It takes $m-\ell$ rounds for s to complete its circuit around C_1 and reach u. So we must have z_2 at distance now $n/2-(m-\ell)$ from v. This means we require

$$\frac{n}{2}-(m-\ell)\geq 1$$

This inequality allows us to bound ℓ

$$m-\frac{n}{2}+1\leq \ell \leq \frac{m-1}{2}$$

which simplifies to

$$n\geq m+3$$

Notice that the survivor has won in this scenario since

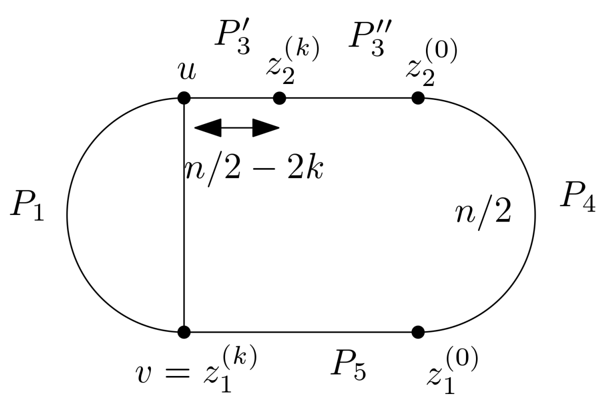
$$d(s,z_1)=\ell\leq\frac{m}{2}-1\leq\frac{n}{2}-1$$

and

$$d(s,z_2)=\frac{n}{2}-(m-\ell)+1\leq\frac{n}{2}-m+\left(\frac{m}{2}-1\right)+1=\frac{n}{2}-\frac{m}{2}<\frac{n}{2}$$

That is to say, the two zombies are now at distance less than $\frac{n}{2}$ from the survivor, so that the survivor now wins by looping around C_2.

Case II. From now on, we assume that the zombies go in different directions at the beginning of the game, so that we inevitably reach the first event of interest at round k: when z_1 reaches the chord.

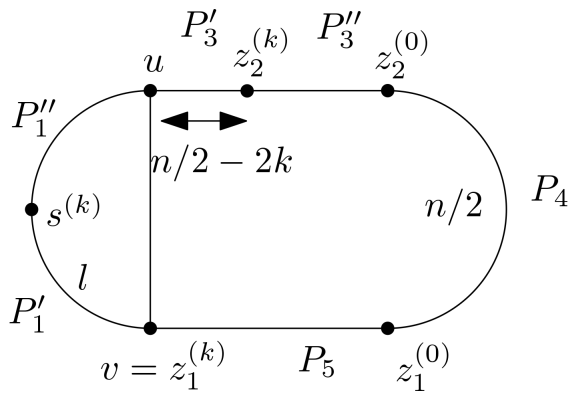


Notice that when $k = n/4$ both zombies attain the chord at the same time. (Potentially important. If the zombies attain the chord at the same time the distance between them becomes 1, which is advantageous to the survivor trying to group them. Survivor must have zombies trailing within single file lemma).

The survivor cannot be on P_3'', P_4 or P_5, as this would imply they somehow got around the zombies which are guarding these paths.

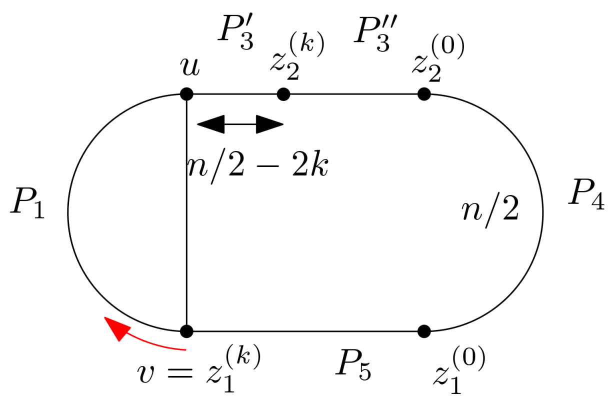
Additionally, the survivor cannot be on u, since it is adjacent to $z_1^{(k)}$. So the survivor, if still alive, must be either on $P_1 \setminus N[v]$ or $P_3' \setminus \{u\}$.

If the survivor is on $P_3' \setminus \{u\}$, then on the next turn $z_1^{(k)}$ moves to u and the survivor is surrounded on less than half of C_2 and hence loses. So we can assume further that the survivor must be on $P_1 \setminus N[v]$. Denote $P_1' : v = x_0 x_1 \dots x_\ell = s^{(k)}$, $P_1'' : s^{(k)} = y_0 y_1 \dots y_{m-\ell} = u$ the subpaths formed by the survivor's position on P_1.

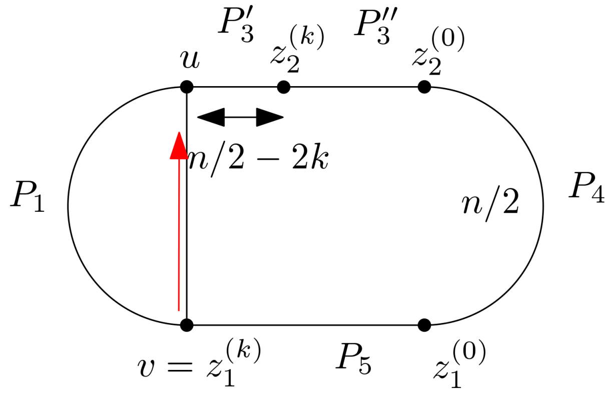


Now, either

(A) $\ell < \frac{m+1}{2}$ (or, equivalently, $\ell \leq \frac{m}{2}$), which forces z_1 to follow a hull edge onto P_1, or



(B) $\ell > \frac{m+1}{2}$ (or, equivalently, $\ell \geq \frac{m+2}{2}$), which forces z_1 take the chord edge to u.



(C) $l = \frac{m+1}{2}$, in which case the zombie can choose one or the other.

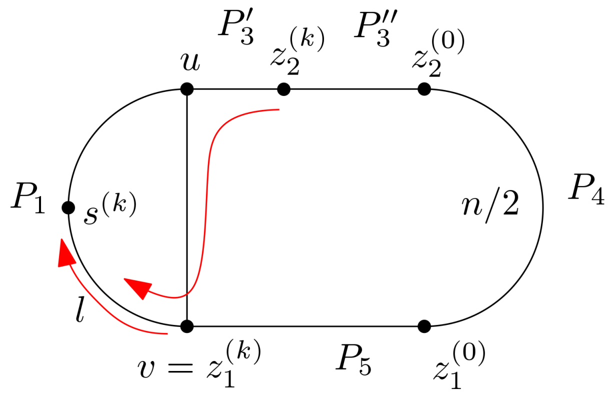
In order to win, the survivor must either:

1. Force the two zombies to follow around C_1 in the same direction, or
2. Be able to loop around C_1 with z_1 in pursuit before z_2 can reach the chord.

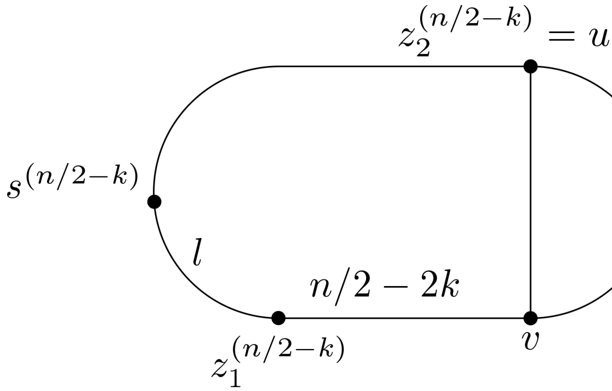
We can assume that the survivor maintains its distance from z_1, since standing still or moving towards z_1 is the same as choosing a smaller (or larger, in case (B)) initial value of ℓ. We examine both possible survivor-winning scenarios for each possible z_1 decision.

Case (A): We have $\ell < \frac{m+1}{2}$, so that z_1 follows a hull edge towards s.

Subcase (A)1. z_2 reaches the chord before the survivor. The survivor wins by forcing the zombies to follow in the same direction around C_1



We have assumed that z_1 is following s in a clockwise direction. We must consider the distances at round $n/2-k$, when z_2 attains the chord.



Here z_1 must continue in the same direction. In order for the survivor to win, we must have z_2 forced to take the chord on the next move and follow in clockwise direction. This implies that

$$\begin{aligned} 1 + n/2 - 2k + \ell &< m - \ell - (n/2 - 2k) \\ 2\ell &< m - n + 4k - 1 \\ 2\ell &\leq m - n + 4k - 2 \\ \ell &\leq \frac{m - n + 4k - 2}{2} \end{aligned}$$

Since we know $\ell \geq 2$, this allows us to bound ℓ:

$$2 \leq \ell \leq \frac{m - n + 4k - 2}{2}$$

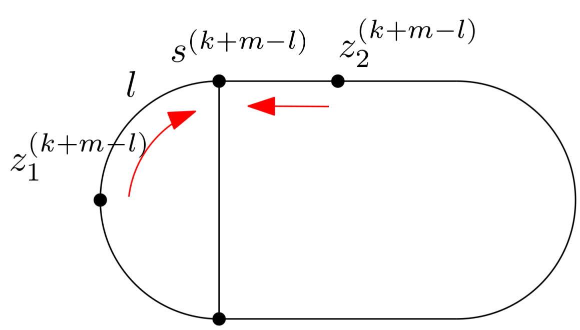
In order to be able to choose ℓ, we must then have

$$2 \leq \frac{m - n + 4k - 2}{2}$$

or

$$k \geq \frac{n - m + 6}{4}$$

Subcase (A)2. The survivor is able to reach the chord before z_2 closes in.



In order for the survivor to win in this scenario, we must have s able to reach the chord before z_2 gets to u's neighbour on P_2. This implies that

$$\frac{n}{2} - 2k - (m - l) \geq 2$$
$$l \geq m + 2k - \frac{n}{2} + 2$$

Now since $l < \frac{m+1}{2}$, or $l \leq \frac{m}{2}$ we have

$$m + 2k - \frac{n}{2} + 2 \leq l \leq \frac{m}{2}$$

So to be able to choose ℓ to make this strategy viable we require

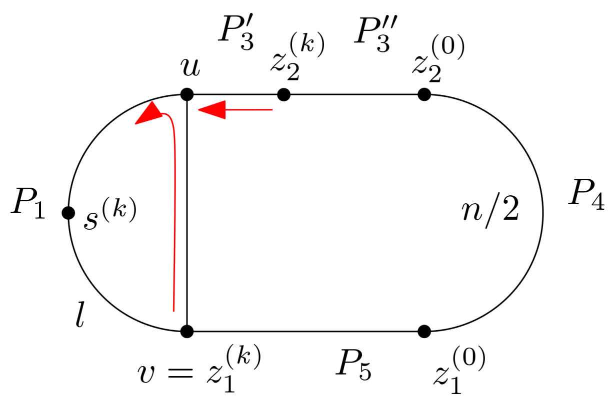
$$m + 2k - \frac{n}{2} + 2 \leq \frac{m}{2}$$

And solving for k gives

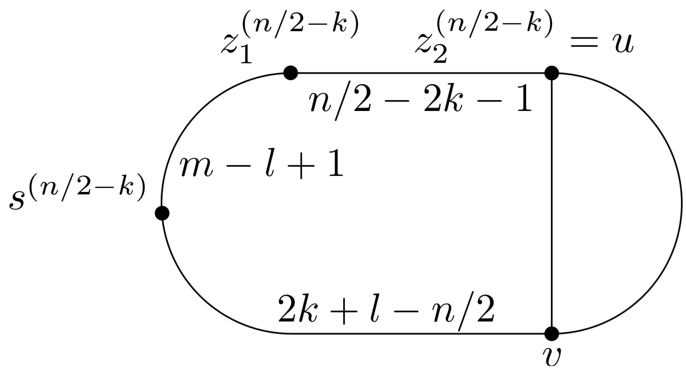
$$k \leq \frac{n - m - 4}{4}$$

Case (B): We have $l > \frac{m+1}{2}$, so that z_1 follows the chord edge towards s.

Subcase (B)1. z_2 reaches the chord before s. The survivor wins by forcing the zombies to follow in the same (counterclockwise) direction around C_1.



We have assumed that z_1 is following s in a counterclockwise direction around C_1. We again look at the next decision point: when z_2 attains the chord.



We have assumed that $k \leq \frac{n}{4}$. If we have equality, then the zombies reach the chord at the same time and the survivor has won since the zombies are on less than half of C_1 and will move in same direction.

We can assume that the survivor preserves its distances of $m-l+1$ from z_1, since moving back or staying still is equivalent to choosing a larger initial value of l. In order for the survivor to win, we must have z_2 forced to follow in the same direction. This implies that

$$\begin{aligned} \frac{n}{2} - 2k - 1 + (m - l + 1) &< 1 + 2k + l - \frac{n}{2} \\ n + m &< 1 + 4k + 2l \\ 2l &> n + m - 4k - 1 \\ 2l &\geq n + m - 4k \\ l &\geq \frac{n + m - 4k}{2} \end{aligned}$$

Since $l \leq m - 1$, we see that

$$\frac{n + m - 4k}{2} \leq l \leq m - 1$$

So in order to choose ℓ to enact this strategy we need

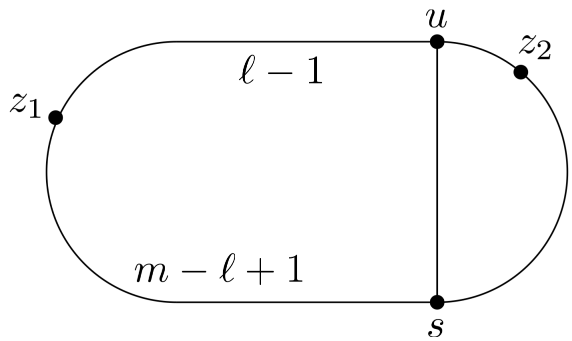
$$\frac{n + m - 4k}{2} \leq m - 1$$

Which allows us to conclude that

$$k \geq \frac{n - m + 2}{4}$$

Subcase (B)2. z_1 follows the chord edge and s reaches the chord before z_2

We start with the same scenario as in (B)1; z_1 is forced to take the chord edge since $\ell > \frac{m+1}{2}$. Or, equivalently since $\ell \in \mathbb{Z}$, $\ell \geq \frac{m+2}{2}$.



z_2 was at a distance of $n/2 - 2k$ from the chorded vertex u and s requires ℓ turns in order to reach v. Thus, in order for the survivor to escape we must have

$$\frac{n}{2} - 2k - \ell \geq 1$$

Solving for ℓ gives

$$\ell \leq \frac{n}{2} - 2k - 1$$

Combined with our lower bound for ℓ this gives

$$\frac{m+2}{2} \leq \ell \leq \frac{n}{2} - 2k - 1$$

So to be able to choose ℓ to make this strategy viable we need

$$\frac{m+2}{2} \leq \frac{n}{2} - 2k - 1$$

Solving for k gives

$$k \leq \frac{n-m-4}{4}$$

All together now, we have the following constraints for the different survivor-win scenarios:

II(A)1. $k \geq \frac{n-m+6}{4}$
II(A)2. $k \leq \frac{n-m-4}{4}$
II(B)1. $k \geq \frac{n-m+2}{4}$
II(B)2. $k \leq \frac{n-m-4}{4}$

If any of these conditions on k are true, then the surivor has a winning strategy. So, to guarantee that none of these strategies will work, we must choose k such that

II(A)1. $k < \frac{n-m+6}{4}$
II(A)2. $k > \frac{n-m-4}{4}$
II(B)1. $k < \frac{n-m+2}{4}$

II(B)2. $k > \frac{n-m-4}{4}$

Are all satisfied. Or, equivalently,

II(A)1. $k \leq \frac{n-m+5}{4}$
II(A)2. $k \geq \frac{n-m-3}{4}$
II(B)1. $k \leq \frac{n-m+1}{4}$
II(B)2. $k \geq \frac{n-m-3}{4}$

Now because

$$\frac{n-m-3}{4} < \frac{n-m+1}{4} < \frac{n-m+5}{4}$$

We must choose $k \in [\frac{n-m-3}{4}, \frac{n-m+1}{4}]$. We know there exists such an integer k since:

$$\frac{n-m+1}{4} - \frac{n-m-3}{4} = 1$$

$\square$