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A deterministic version of the game of zombies and survivors on graphs

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ABSTRACT

We consider a variant of the pursuit–evasion game Cops and Robber, called Zombies and Survivors. The zombies, being of limited intelligence, have a very simple objective at each round: move closer to a survivor. The zombies capture a survivor if one of the zombies moves onto the same vertex as a survivor. The survivor's objective is to avoid capture for as long as possible, hopefully indefinitely. Because there may be multiple geodesics, or shortest paths, joining a zombie and its nearest survivor, the game can be considered from a probabilistic or deterministic approach. In this paper, we consider a deterministic approach to the game. In particular, we consider the worst case for the survivors; whenever the zombies have more than one possible move, they choose one that works to their advantage. This includes choice of initial position, and choosing which geodesic to move along if more than one is available. In other words, the zombies play intelligently, subject to the constraint that each zombie must move along a geodesic between itself and the nearest survivor. The zombie number of a graph G is the minimum number of zombies required to capture the survivor on G. We determine the zombie number for various graphs, examine the relationship between the zombie number and cop number of a graph, and describe some distinctions from Cops and Robber.

1. Introduction

The pursuit–evasion game *Cops and Robber* is played on a graph with two players: a set of *cops* and a single *robber*. Initially, each cop chooses a vertex to occupy, and then the robber chooses a vertex to occupy. The cops and robber then alternate moves. For the cops' move, each of the cops either moves to an adjacent vertex or stays at her current location (referred to as a *pass*). The robber's move is defined similarly. The pair of moves by the cops and the robber is referred to as a *round*, with the initial choice of positions considered to be round zero. The cops win if, after some finite number of rounds, a cop occupies the same vertex as the robber. The robber wins if he can avoid capture indefinitely. Note that, even if the cops are guaranteed a win, we assume that the robber will move so that he avoids capture as long as possible. Therefore, in a game in which the cops win, the game ends mid-round with a cop moving onto the robber.

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For a graph G, the *cop number*, $c(G)$, is the minimum number of cops required to win in G. Although the notion of cop number is quite simple, its consideration leads to many questions in structural, probabilistic, and algorithmic graph theory (see [2,4]).

Zombies and Survivors is a new variant of the well-studied game of *Cops and Robber*, in which zombies take the place of the cops and survivor(s) take the place of the robber(s). The zombies, being of limited intelligence, have a very simple objective in each round—to move closer to a survivor. Therefore, each zombie must move along some shortest path, or *geodesic*, joining itself and a nearest survivor. We say that the zombies *capture* a survivor if one of the zombies moves onto the same vertex as a survivor. For a single survivor, winning conditions for the Zombies and Survivors game are identical to those of Cops and Robber. A *zombie-play* is an ordered sequence of vertices occupied by a zombie in a particular game of Zombie and Survivors (with a single zombie). A *survivor-play* is defined similarly. A zombie-play or survivor-play is described as *winning* if it leads to either the zombies or survivors, respectively, winning the game. Note that when dealing with more than one survivor, all survivors must be captured for the zombies to win.

Because there may exist multiple geodesics between a given zombie and its nearest survivor, the game can be considered from either a *probabilistic* approach or a *deterministic* approach. In [3], the authors consider the probabilistic version: initially, the zombies randomly choose a set of vertices to occupy and during a round each zombie moves to a neighboring vertex that minimizes the distance to the survivors. However, if more than one such vertex exists, one is chosen uniformly at random. They [3] consider the minimum number of zombies needed to play such that the probability that the zombies win is strictly greater than $\frac{1}{2}$. We note that, were we to replace the $\frac{1}{2}$ in the previous statement with 0, the deterministic and probabilistic approaches would be equivalent.

In this paper, we consider a deterministic version of Zombies and Survivors played on a finite connected graph G. Most of the paper focuses on the game played with only one survivor, and this assumption should be made unless otherwise stated (such as in Section 5). The outcome of the game is largely dependent on the initial positions chosen by the zombies. To demonstrate, suppose one zombie is initially located in each partite set of $K_{n,n}$. In this case, the two zombies will capture the survivor in the first round. However, if $n-1$ zombies are initially located at $n-1$ distinct vertices in one partite set of $K_{n,n}$, then the survivor can initially occupy the remaining vertex in that partite set and avoid capture. Consequently, we are interested in the worst case for the survivors: if a winning zombie-play exists, the zombies will choose a set of initial positions which will result in capture of a survivor. Furthermore, during a particular round, if there are multiple geodesics joining a zombie and the survivors, the zombie will make a choice based on following a winning zombie-play, assuming one exists. As with cops and robbers, we assume that even if capture is inevitable, the survivor will move so that the number of rounds in the game is maximized. Therefore, the survivor will never move onto a vertex currently occupied by a zombie.

On a graph G, the minimum number of zombies required to win is denoted by $z(G)$; the parameter is certainly well-defined because $\gamma(G)$-many zombies can occupy a minimum dominating set on G and capture the survivor in the first round and thus $z(G) \leq \gamma(G)$. Moreover, noting that Zombies and Survivors is a variant of Cops and Robber whereby the cops are restricted in their movements to following a shortest path strategy, then clearly $c(G) \leq z(G)$ and any winning zombie-play is also a winning cop-play.

Lemma 1. *For any graph G, $c(G) \leq z(G)$.*

In Cops and Robber, a graph G for which $c(G) = 1$ is often referred to as a *cop-win* graph. Analogously, any graph G for which $z(G) = 1$ will be called a *zombie-win* graph.

In [1] Aigner and Fromme introduced the notion of *active* players, namely cops and robbers who are always required to move and cannot pass during any round of the game. This notion gives rise to the *active game* in which the robber and a non-empty set of cops must move on their respective turns, but where some cops are permitted to pass and remain stationary; this restricted model of Cops and Robber is also considered in [9,11]. One of the cases considered by [11] is that in which all of the cops are forever in an active state and thus are prohibited from passing during a round. Such is also the case with the zombies in the Zombies and Survivors model. However, there is a distinction: whereas an active cop has the freedom to move in any direction, including away from her nearest robber, a zombie is always required to move towards one of its nearest survivors.

The remainder of the present paper is organized as follows. In Section 2, we provide some preliminary results for the zombie number of graphs. We consider the relationship between cop-win graphs and zombie-win graphs and exhibit graphs for which the cop number and zombie number differ. Section 3 considers cop-win spanning trees and, in particular, shows that on a bridged graph, one zombie will win, given any initial position. In Section 4, we consider the zombie number for Cartesian products of graphs, noting in particular that the zombie number for the binary hypercube graph, Q_n, behaves very differently than the cop number for Q_n. In Section 5 we highlight several more distinctions between the Cops and Robber model and the Zombies and Survivors model.

In Sections 2 and 4, graph products are used to describe infinite families of graphs. We will now define the *Cartesian* and *strong products* for future use. The *Cartesian product* of graphs G and H is denoted $G \square H$, where $V(G \square H) = \{(x, y) | x \in V(G), y \in V(H)\}$, and $E(G \square H) = \{(x_1, y_1)(x_2, y_2) | x_1x_2 \in E(G) \text{ and } y_1 = y_2, \text{ or } x_1 = x_2 \text{ and } y_1y_2 \in E(H)\}$. The *strong product* of G and H is denoted $G \boxtimes H$, where $V(G \boxtimes H) = V(G \square H)$, and $E(G \boxtimes H) = E(G \square H) \cup \{(x_1, y_1)(x_2, y_2) | x_1x_2 \in E(G) \text{ and } y_1y_2 \in E(H)\}$.

Regarding graph theory notation, we generally follow that of [5], although we let $\gamma(G)$ represent the domination number of a graph G and we occasionally use $u \sim v$ to denote that a vertex u is adjacent to a vertex v.

2. Preliminary results and relationships to cop-win graphs

We begin with some elementary results about the zombie number.

Observation 2. (1) $z(T) = c(T) = 1$ *for any tree* T.
(2) $z(C_n) = c(C_n) = 2$ *for any cycle of length* $n \geq 4$.
(3) $z(K_{m,n}) = c(K_{m,n}) = 2$ *where* $m \geq n \geq 2$.
(4) $z(K_n) = c(K_n) = 1$ *for* $n \geq 1$.

Proof. By Lemma 1, it suffices to show that $c(G)$-many zombies have a winning strategy for each graph $G \in \{T, C_n, K_{m,n}, K_n\}$. For (1), as there is a unique path between any pair of vertices in a tree, a single zombie moves exactly as a single cop moves in order to win in Cops and Robber. For (2), if the zombies initially occupy vertices that are distance $diam(C_n) = \lfloor \frac{n}{2} \rfloor$ apart, then after the survivor has chosen an initial vertex to occupy, there is a unique path, P, between the zombies that contains the survivor. Furthermore, for each zombie, there is a geodesic between itself and the survivor that is contained in P. The two zombies now move exactly as two cops would choose to move in order to win in Cops and Robber. For (3), placing one zombie in each partite set will guarantee capture in the first round. (4) is obvious. ■

The structural characterization of cop-win graphs is well known from [10,12]: A graph G is cop-win if and only if its vertices can be ordered $v_1, v_2, \ldots, v_n$ so that for each v_i, where $i > 1$, there exists some v_j, where $j < i$, such that every $N[v_i] \cap \{v_1, \ldots, v_i\} \subseteq N[v_j]$. This ordering is referred to as a *cop-win ordering* of G. We also say that G is *dismantlable* whenever such an ordering exists. Let H_i be the subgraph induced on $\{v_1, v_2, \ldots, v_i\}$. The vertex v_i is referred to as a *corner* in H_i and vertex v_j is said to *dominate* v_i in H_i, whenever $N_{H_i}[v_i] \subseteq N_{H_i}[v_j]$. It follows that v_n is a corner in G, and $H_n = G$.

Consider a mapping $f_k : V(H_{k+1}) \to V(H_k)$ such that $f_k(v_i) = v_i$ for all $i = 1, \ldots, k$ and $f_k(v_{k+1}) = u$ for one choice of u, where u dominates v_{k+1} in H_{k+1}. For each $k = 1, \ldots, n-1$, let $g_k = f_k \circ \cdots \circ f_{n-1}$. If follows that $g_k : V(G) \to V(H_k)$, and under g_k, any two adjacent vertices in G are either mapped to the same vertex of H_k or adjacent vertices of H_k, and each vertex of H_k is mapped to itself. Therefore, if G is considered to be reflexive, then, by definition, H_k is a *retract* of G, and g_k is a *retraction mapping*. (Although we do not assume there are loops at all vertices, we use the terms "retract" and "retraction" to describe this type of mapping.)

When required for convenience, we will denote the retract according its highest indexed vertex, and the retraction mapping according to the image graph. Specifically, whenever $v = v_i$, we let H_v denote H_i, and whenever $H = H_i$, we let g_H denote g_i.

Although cop-win graphs can also be zombie-win, as evidenced by Observation 2, the next result shows that cop-win graphs are not necessarily zombie-win graphs. In fact, the proof gives an infinite family of graphs with this property.

Theorem 3. *If a graph is zombie-win, then it is also cop-win. However, if a graph is cop-win, then it is not necessarily zombie-win.*

Proof. From Lemma 1, we know that every zombie-win graph is also cop-win. We now construct an infinite family of graphs G_n that are cop-win, but not zombie-win. To begin, consider $P_3 \boxtimes C_n$ for $n \geq 5$. Label the vertices of C_n as $u_0, u_1, \ldots, u_{n-1}$ where u_j is adjacent to u_{j+1} (mod n). Label the vertices of P_3 as a, b, c where a and c are leaves in P_3. To construct graph G_n from $P_3 \boxtimes C_n$, we simply add an edge from each vertex with first coordinate c to every other vertex with first coordinate c. The graph G_5 is shown in Fig. 1.

We now consider the game on G_n. Let z_0 and s_0 represent the zombie's and survivor's initial positions, respectively. For the first case, suppose z_0 is either (a, u_i) or (b, u_i) and $s_0 = (a, u_{i+2})$. It follows that each z_0, s_0-geodesic has length two, and contains either (b, u_{i+1}) or (a, u_{i+1}). In round 1, the zombie moves onto one of these two vertices, and the survivor responds by moving to (a, u_{i+3}). Their positions are now equivalent to one of the two potential initial positions. Next, suppose $z_0 = (c, u_i)$, and $s_0 = (a, u_{i+2})$. Now, the only z_0, s_0-geodesic is $(c, u_i), (b, u_{i+1}), (a, u_{i+2})$. Therefore, in round 1, the zombie moves to (b, u_{i+1}). The survivor responds by moving to (a, u_{i+3}), and their positions are equivalent to one of the two potential initial positions in the first case.

Thus, regardless of the vertex initially occupied by the zombie, there is no winning zombie-play. Therefore, $z(G_n) > 1$. However, G_n has a cop-win ordering: $(c, u_0), (c, u_1), \ldots, (c, u_{n-1}), (b, u_0), (b, u_1), \ldots, (b, u_{n-1}), (a, u_0), (a, u_1), \ldots, (a, u_{n-1})$. Therefore, G_n is cop-win, but not zombie-win. ■

Theorem 3 establishes that the zombie number is not always equal to the cop number of a graph. For the cop to capture the robber on graph G_n, described in Theorem 3, an interested reader will note that during some round the cop must choose to maintain distance from the robber and not decrease distance; this is an illegal move for a zombie.

Question 4. *Does there exist a structural characterization for cop-win graphs that are also zombie-win? Or, does there exist a structural characterization of zombie-win graphs?*

As a second example, this time of a graph that is not cop-win, consider the graph H illustrated in Fig. 2. In [7], it was shown that the cop number of an outerplanar graph is at most 2. However, $z(H) = 3 > 2 = c(H)$.

To see that $z(H) > 2$, consider a cycle of length n, where $n \geq 7$. Suppose at the beginning of a round, there are two zombies and a single survivor on the cycle such that for each zombie, there is a single geodesic of length at least two between

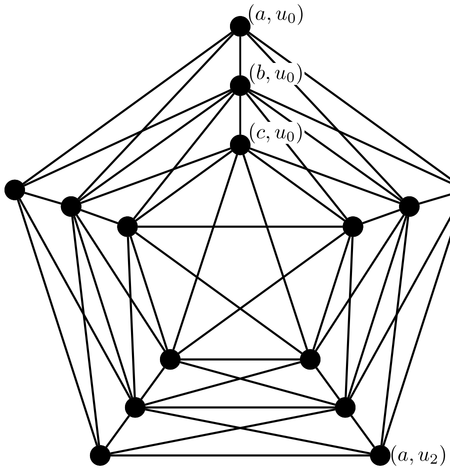

Fig. 1. A graph G_5 that is cop-win, but not zombie-win.

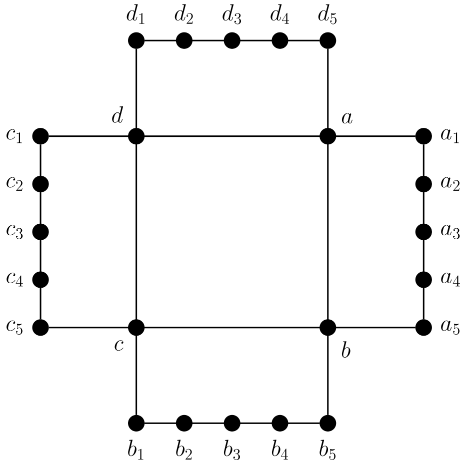

Fig. 2. A graph that has cop number 2 and zombie number 3.

itself and the survivor and these two geodesics intersect. It follows that one of the zombies is on the geodesic that joins the survivor and the other zombie, and all three participants are on the same "half" of the cycle. As a result, the survivor will never be captured.

In the graph H, the survivor will occupy one of the outside 7-cycles so that the zombies will be lured to the same "half" of the cycle. If the zombies initially occupy the same or adjacent vertices on the interior 4-cycle, the survivor positions himself accordingly on a 7-cycle containing these vertices. If the zombies initially occupy b and d, the survivor would choose b_2. This leads the zombies to both occupy c. The survivor then moves around the 7-cycle containing $b_1, b_2, \ldots, b_5$, avoiding capture indefinitely. For other initial positions, the survivor can move on the interior 4-cycle until capture is imminent. At that point, it is mid-round, the zombies are in positions equivalent to a and c_5, respectively, and the survivor is on either b or c. The survivor then moves onto the 7-cycle containing b and c, and avoids capture indefinitely.

While two zombies cannot win, it is straightforward to see that three zombies, using a, b, and c_3 as their initial positions, can win. Thus the zombie number and cop number differ by one. This leads to the following question:

Question 5. *How large can the difference between $z(G)$ and $c(G)$ be?*

In Section 4 we show that for the binary hypercube graphs Q_n the difference $z(G) - c(G)$ can be arbitrarily large and so we then ask about how large the ratio $\frac{z(G)}{c(G)}$ can possibly be.

3. Cop-win spanning trees and bridged graphs

A *cop-win spanning tree* T of G was introduced in [7]. Given a spanning tree T of G, and a cop-win ordering $\mathcal{O} = v_1, v_2 \ldots, v_n$, T is a cop-win spanning tree associated with $\mathcal{O}$ if for each $v_iv_j \in E(T)$ such that $j < i$, v_j dominates v_i in

$G[\{v_1, \ldots, v_i\}]$. We note that every cop-win graph has at least one associated cop-win spanning tree. It also follows that if T is a cop-win spanning tree of G, then $T[\{v_1, \ldots, v_i\}]$ is a cop-win spanning tree of $G[\{v_1, \ldots, v_i\}]$. Furthermore, if we draw T as a rooted tree with v_1 as the root, we see that the parent of each vertex v_i is a dominating vertex of v_i in $G[\{v_1, \ldots, v_i\}]$. It was shown in [7] that, in the game of Cops and Robber, the cop-win spanning tree provides a strategy in which the cop could win in at most $|V(G)| - 1$ moves.

It was shown by Chepoi [6] that any ordering resulting from a breadth-first search of a bridged graph is a cop-win ordering. (A graph is *bridged* if it contains no isometric cycle of length greater than three.) Moreover, the spanning tree resulting from this breadth-first search is also a cop-win spanning tree. Motivated by this result, we will consider all graphs in which the spanning tree resulting from some breadth-first search is also a cop-win spanning tree. However, we first define terms associated with search trees.

For a graph G, let $\mathcal{O} = v_1, v_2, \ldots, v_n$ represent an ordering of the vertices produced by a breadth-first search of G, and T represent the associated breadth-first search tree. We will write $v_j \prec v_i$ whenever $j < i$. For each vertex v in G, let $\ell(v) = d(v_1, v)$. That is, $\ell(v)$ denotes the distance from v to the root of T. We say the vertex y is an *ancestor* of vertex x in T if $\ell(y) < \ell(x)$ and there exists a y, x-path in T of length $\ell(x) - \ell(y)$. If $\ell(x) - \ell(y) = 1$, then y is also said to be the *parent* of x, and x is a *child* of y. Note that for any vertices u and v, if some u, v-path P has length $|\ell(v) - \ell(u)|$, then P is a geodesic in G. Also note that whenever $\ell(u) < \ell(v)$, $u \prec v$.

Theorem 6. *If there exists a breadth-first search of a graph G such that the associated spanning tree is also a cop-win spanning tree, then G is zombie-win.*

Proof. Suppose there is a breadth-first search of graph G such that the associated spanning tree is also a cop-win spanning tree. It follows that the ordering, $\mathcal{O}$, from that breadth-first search is a cop-win ordering.

Let H_k and g_k be defined for each $k = 1, \ldots, n-1$ as in Section 2. It follows that the retraction mapping $g_k : V(G) \to V(H_k)$ has the property that for each $x \in \{v_{k+1}, \ldots, v_n\}$, $g_k(x) = v_j$ where j is the greatest integer such that $1 \le j \le k$ and v_j is an ancestor of x.

We now play the game of Zombies and Survivors on G. Let z_i and s_i denote the positions of the zombie and survivor, respectively at the end of round i, where the initial choice of position is considered to be round zero. For any round in the game, we refer to the midpoint of the round as the point in time in which the zombie has completed its move, but the survivor is yet to move. Therefore, at the midpoint of round of i, where $i \ge 1$, the zombie and survivor occupy vertices z_i and s_{i-1}, respectively.

Assume that G is survivor-win. It follows that there is no k such that z_k and s_k are adjacent. We now claim that there is a zombie-play such that for every $k \ge 1$, z_k is an ancestor of s_{k-1} and $z_0 \prec z_1 \prec \cdots \prec z_k$. We prove the claim by induction on the round number.

In round zero, assume that the zombie chooses vertex v_1 ($z_0 = v_1$) and the survivor chooses vertex s_0. The zombie then moves to a child of z_0 that is also an ancestor of s_0. Therefore, the zombie has moved on a z_0, s_0-geodesic. Furthermore, $z_0 \prec z_1$.

Suppose we are at the midpoint of round k, $k \ge 1$. Assume that $z_0 \prec z_1 \prec \cdots \prec z_k$ and z_k is an ancestor of s_{k-1}. To complete round k, the survivor moves from vertex s_{k-1} to s_k. We now show that the zombie can move along a z_k, s_k-geodesic to a vertex z_{k+1} such that z_{k+1} is an ancestor of s_k and $z_k \prec z_{k+1}$. Note that, since z_k is an ancestor of s_{k-1}, $\ell(s_{k-1}) > \ell(z_k)$. Since s_k and s_{k-1}, are adjacent, it follows that $|\ell(s_k) - \ell(s_{k-1})| \le 1$, and $\ell(s_k) \ge \ell(z_k)$.

Suppose $\ell(s_k) = \ell(z_k)$. Let $H = H_{z_k}$ if $s_k \prec z_k$. Otherwise, let $H = H_{s_k}$. Therefore, H contains both z_k and s_k. Furthermore, H is a retract of G, where s_{k-1} is mapped to z_k and s_k is mapped to itself, under g_H. Since $s_{k-1}s_k$ is an edge in G, $z_k s_k$ is an edge in H. Therefore, z_k and s_k are adjacent in G, which is a contradiction.

For the remainder of the proof, assume $\ell(s_k) > \ell(z_k)$. If z_k is an ancestor of s_k, then there is a $z_k s_k$-path in T that is also a geodesic in G. This is due to the fact that T is obtained from a breadth-first search of G. Since z_k and s_k are not adjacent, the zombie moves on this path to a vertex that is both a child of z_k and an ancestor of s_k. Hence, $z_k \prec z_{k+1}$.

Assume that z_k is not an ancestor of s_k. Since $\ell(z_k) < \ell(s_k)$, there is some vertex x such that x is an ancestor of s_k and $\ell(x) = \ell(z_k)$. We now consider two cases depending on whether $x \prec z_k$ or $z_k \prec x$.

Case 1: $x \prec z_k$.

Recall that $\ell(s_k) \ge \ell(z_k) + 1$. If $\ell(s_k) \ge \ell(z_k) + 2$, let v be the ancestor of s_k such that v is the child of x. If $\ell(s_k) = \ell(z_k) + 1$, let $v = s_k$. It follows that H_v is a retract of G containing both v and z_k, but none of their children. Therefore, under g_{H_v} s_{k-1} is mapped to z_k and s_k is mapped to v. Consequently z_k and v are adjacent, and $\ell(z_k) < \ell(v)$. Recall that z_k and s_k are not adjacent, so it follows that $v \ne s_k$. Since the path P, consisting of the edge $z_k v$ followed by the v, s_k-path in T, has length $1 + \ell(s_k) - \ell(v) = \ell(s_k) - \ell(z_k)$, it follows that P is a z_k, s_k-geodesic of length at least two. Therefore, v serves as the required z_{k+1}.

Case 2: $z_k \prec x$.

In this case, we consider the retract H_x. Under g_{H_x}, s_{k-1} is mapped to z_k and s_k is mapped to x. Since s_{k-1} is adjacent to s_k, it follows that z_k is adjacent to x. It now suffices to show that x is on some z_k, s_k-geodesic.

Let Q be the x, s_k-path in T. Since Q has length $\ell(s_k) - \ell(x)$, and $\ell(x) = \ell(z_k)$, there is an z_k, s_k-path of length $\ell(s_k) - \ell(z_k) + 1$ containing x (specifically, the edge $z_k x$ followed by Q). Suppose, however, that such a path is not a geodesic. Since any z_k, s_k-path has length at least $\ell(s_k) - \ell(z_k)$, it follows that a z_k, s_k-geodesic has length $\ell(s_k) - \ell(z_k)$. Let Q' be a z_k, s_k-geodesic. It follows that if y is the ith vertex in Q', then $\ell(y) = \ell(z_k) + i - 1$. Since $z_k \prec x$, this implies that for every

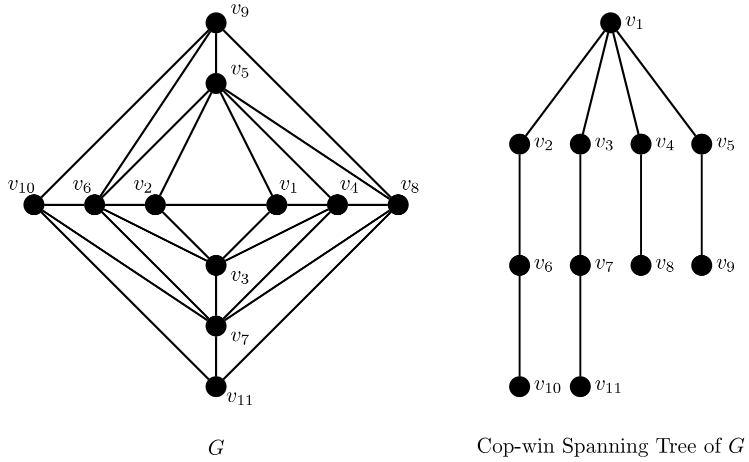

Fig. 3. A graph that has a breadth-first search cop-win ordering.

vertex a in Q' and b in Q such that $\ell(a) = \ell(b)$, $a \prec b$. However, this leads to $s_k \prec s_k$ which is a contradiction. Therefore, x is on a $z_k s_k$-geodesic and x serves as the required z_{k+1}.

Hence, we have found a sequence of zombie moves $z_0, z_1, z_2, \ldots$ having the property that $z_0 \prec z_1 \prec z_2 \prec \cdots$. However, this implies that the zombie never revisits a vertex, and the sequence of moves is finite. This contradicts the assumption that G is survivor-win. ■

In addition to the fact that any breadth-first search ordering $\mathcal{O}$ of a bridged graph is a cop-win ordering, Chepoi [6] also showed that for each vertex v_i in the resulting tree T, its parent dominates v_i in H_i. It follows that T is a cop-win spanning tree.

Corollary 7. *If G is a bridged graph, then G is zombie-win. Furthermore, for any $v \in V(G)$, there is a winning zombie-play with v as the zombie's initial position.*

To distinguish the set of graphs which satisfy the hypothesis of Theorem 6 from bridged graphs, consider the example in Fig. 3. The graph on the left has cop-win ordering $v_1, v_2, \ldots, v_{11}$, and an associated cop-win spanning tree is on the right. It is straightforward to verify that this tree is also a breadth-first search tree of G starting at v_1. Therefore, the graph G in Fig. 3 is zombie-win. However, G is not a bridged graph, which is evident from the 4-cycle induced on the set $\{v_8, v_9, v_{10}, v_{11}\}$.

4. Cartesian products of graphs

In this section, we consider the zombie number of the Cartesian product of graphs. We first provide an infinite class of graphs $P_2 \,\square\, C_n$ (for n odd and $n \geq 5$) for which the zombie number differs from the cop number. We determine the zombie number exactly for $P_m \,\square\, C_n$ and provide an upper bound for $G \,\square\, T$ where T is a tree. We also comment on translating the upper bound from [13] on the cop number of graph products to the zombie number.

Lemma 8. *For n odd and $n \geq 5$, $z(P_2 \,\square\, C_n) > 2$.*

Proof. Let $n \geq 5$ be odd where $C_n = \{v_0, v_1, \ldots, v_{n-1}\}$ with $v_i \sim v_{i+1} \pmod{n}$. Let $P_2 = \{a, b\}$. Suppose we now play the game with two zombies, Z_1 and Z_2, and a survivor S. The proof that follows makes use of two special winning positions for the survivor, as illustrated in Fig. 4. We begin by showing that if the zombies and survivor find themselves in either of these two positions at the beginning of a round, the survivor can evade capture indefinitely.

Positions of Fig. 4(a): In this case, Z_2 must move to (b, v_{i+1}) in the current round, and there are two possible moves for Z_1. If Z_1 moves to (b, v_{i+1}), then clearly the survivor can evade capture indefinitely. If Z_1 moves to (a, v_{i+2}), then S moves to (a, v_{i+3}) and the locations are equivalent to those at the beginning of this round.

Positions of Fig. 4(b): In this case, each of the zombies has two possible moves available. If the zombies both move to (b, v_i), then clearly the survivor can evade capture indefinitely. If the zombies both move to (a, v_{i+1}), then S moves to (b, v_{i+2}) and the locations are equivalent to those at the beginning of the round. Finally, if the zombies move to (b, v_i) and (a, v_{i+1}), then S moves to (b, v_{i+2}) and the positions are equivalent to those of Fig. 4(a).

Therefore, if the survivor finds himself, with the zombies, in the positions described by Fig. 4(a) or (b), he can evade capture indefinitely. (If it is not the beginning of the round, the survivor can pass.) We now show that the survivor can either evade capture indefinitely or, at the end of some round, the zombies and survivor find themselves in positions equivalent to those illustrated in Fig. 4(a) or (b).

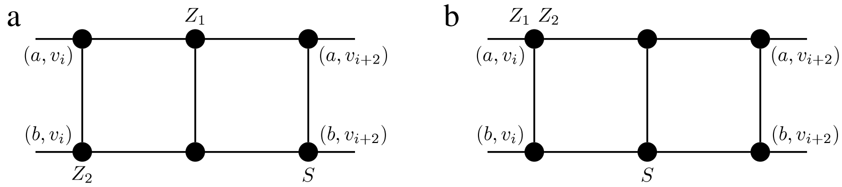

Fig. 4. Positions (a) (left) and (b) (right).

Without loss of generality, assume Z_1 initially occupies vertex (a, v_0). We consider two cases, depending on whether Z_2 initially occupies a vertex with first coordinate a or b.

Case 1: Zombie Z_2 is located at vertex (a, v_j) for some $0 \le j \le n-1$. Since n is odd, for $j \in \{1, 2, \dots, n-1\}$, one v_0v_j-path on C_n is of odd length, and one is of even length. Without loss of generality, let $v_0, v_1, \dots, v_j$ form a v_0v_j-path on C_n of even length. Then j is even.

If $j = 0$, then S can occupy (a, v_2) and avoid capture indefinitely, just as he would when playing against a single zombie on C_n. We may, therefore, assume that $j \ge 2$.

(i) If $j = 2$, then S initially occupies (b, v_1). During round 1, if the zombies both change their first coordinate, then S changes his first coordinate and at the end of round 1, the locations of the zombies and survivor are equivalent to the end of round 0. During round 1, if the zombies both move to (a, v_1), then S moves to (b, v_2) and the positions are equivalent to those of Fig. 4(b). If the zombies move to (b, v_0) and (a, v_1), then S moves to (b, v_2) and the positions are equivalent to those of Fig. 4(a). Finally, if the zombies move to (a, v_1) and (b, v_2), then the survivor moves to (b, v_0) and the positions are equivalent to those of Fig. 4(a).

(ii) If $j \ge 4$, then S initially occupies $(a, v_{j/2})$. Note that there is exactly one geodesic joining the vertices occupied by Z_1 and S. The same is true for Z_2 and S. For rounds 1 through $j/2 - 2$, S will pass. As a result, for rounds 1 through $j/2 - 1$, Z_1 and Z_2 will each have a single move available to them, just as in round 1. Hence, at the beginning of round $j/2 - 1$, Z_1 will move to $(a, v_{(j/2)-1})$ and Z_2 will move to $(a, v_{(j/2)+1})$. The response by S is to move to $(b, v_{j/2})$, and the positions at the end of round $j/2 - 1$ are equivalent to those of round 0 in (i).

Case 2: Suppose Z_2 is initially located at (b, v_j). If $j = 0$, then S initially occupies (b, v_2). During round 1, Z_2 moves to (b, v_1) and Z_1 either moves to (a, v_1) or (b, v_0). If Z_1 moves to (a, v_1), then S moves to (b, v_3) and the locations are equivalent to those at the end of round 0. If Z_1 moves to (b, v_0), then S will subsequently move to (b, v_3). If $n \ge 6$, S can then evade capture forever by moving around the cycle induced on the vertices with first coordinate b. If $n = 5$, then Z_1 and Z_2 move to (b, v_4) and (b, v_2), respectively, and S moves to (a, v_3). Now, if Z_1 and Z_2 move to (a, v_4) and (a, v_2), S moves to (b, v_3) and the positions are equivalent to those at the end of the previous round. Therefore, without loss of generality, we assume that Z_1 and Z_2 move to (b, v_3) and (a, v_2). Now, S responds by moving to (a, v_4), and the positions of Z_1, Z_2 and S are equivalent to those in Fig. 4(a).

We now assume $j \ge 1$. Since n is odd, for $j \in \{1, 2, \dots, n-1\}$ there is one v_0v_j path in C_n of odd length, and one of even length. Without loss of generality, let $v_0, v_1, \dots, v_j$ form a v_0v_j path on C_n of odd length. Then j is odd.

(i) If $j = 1$, then S will occupy (a, v_2), and the positions will be equivalent to those of Fig. 4(a).

(ii) If $j = 3$, then S chooses (b, v_1) as his initial position. At the end of round 0 there are multiple geodesics joining Z_1 and S, and Z_1 can move to either (b, v_0) or (a, v_1). There is a single geodesic joining Z_2 and S. Therefore, Z_2 moves to (b, v_2). If, in round 1, Z_1 moves to (a, v_1), then S moves to (b, v_0) and the positions are equivalent to those in Fig. 4(a). Otherwise, Z_1 moves to (b, v_0). With Z_1 and Z_2 on (b, v_0) and (b, v_2), respectively, S will move to (a, v_1). The zombie and survivor positions are now identical to those in Case 1 (i). Therefore, after the next round, the zombies and survivor will have positions equivalent to one of the configurations in Fig. 4.

(iii) If $j \ge 5$, then S chooses $(b, v_{(j-1)/2})$ as his initial position. It follows that Z_1 moves to either (b, v_0) or (a, v_1), and Z_2 moves to (b, v_{j-1}). If, in round 1, Z_1 moves to (b, v_0), then Z_1 and Z_2 are on (b, v_0) and (b, v_{j-1}), and S will choose to stay at $(b, v_{(j-1)/2})$. Since $j-1$ is even, we now have positions at the end of round 1 equivalent to one of the initial positions in Case 1. If, in round 1, Z_1 moves to (a, v_1) then Z_1 and Z_2 are on (a, v_1) and (b, v_{j-1}), and S will choose to stay at $(b, v_{(j-1)/2})$. The positions at the end of round 1 are now equivalent to the initial positions of round 0 in Case 2, but with a smaller value of j.

Therefore, in any case, the survivor can either evade capture indefinitely or, at the end of some round, the zombies and survivor will be in positions equivalent to those described in Fig. 4(a) or (b). Therefore, there is a winning survivor-play when playing against two zombies. ■

From [9] it is known that $c(P_m \square C_n) = 2$ for n odd, $n \ge 5$, and $m \ge 2$. Combining this with the results of Lemma 8 yields a family of graphs for which the zombie number and cop number differ by exactly one. To see that $z(P_2 \square C_n) = 3$ for n odd and $n \ge 5$, initially place zombies at (a, v_0), (b, v_0), and $(a, v_{(n-1)/2})$. Using the following strategy, three zombies Z_1, Z_2 and Z_3 can capture a survivor S, regardless of the initial location of S. Zombie Z_1 will move along a geodesic joining itself and S so that Z_1's position always has a as its first coordinate. Similarly, Z_2 will move so that it always occupies a vertex with b as its first coordinate. Finally, Z_3 will move so that, midway through the round, the vertices occupied by S and Z_3 have the same first coordinate.

Corollary 9. *For n odd and $n \geq 5$, $z(P_2 \,\square\, C_n) = 3 > 2 = c(P_2 \,\square\, C_n)$.*

From [13] it is known that $c(G \square H) \leq c(G) + c(H)$. The proof, given by Tošić, utilizes the idea of capturing the *shadow* of the robber. By *shadow* we mean the projection of the robber's position in $G \square H$ onto a copy of G according to a mapping $f : V(G \square H) \to \{(u, v_1) | u \in V(G)\}$, where $f((u, v)) = (u, v_1)$ for some particular $v_1 \in V(H)$. In Tošić's proof the cops first capture the shadow of the robber in a copy of G in the product graph, and then one cop will maintain capture of the shadow of the robber in that copy of G, while the other $c(G) - 1$ cops (plus an additional cop) transition to capture the shadow of the robber in another copy of G in the product graph. However, the movements of the cops in the transition are independent of the robber's moves and cannot be directly translated to the Zombies and Survivor game.

Question 10. *Is $z(G \square H) \leq z(G) + z(H)$ for all graphs G and H?*

While the above question remains open, in results that follow, we show specific cases for which $z(G \square H) \leq z(G) + z(H)$ is true. We begin by demonstrating more general upper bounds.

Theorem 11. *For any graph G and $n \geq 4$, $z(G \square C_n) \leq 3z(G)$.*

Proof. Let $C_n = \{v_0, v_1, \dots, v_{n-1}\}$ where $v_i \sim v_{i+1} \pmod n$. In $G \square C_n$, there are n copies of G, labeled $G_0, G_1, \dots, G_{n-1}$ where G_j contains the vertices with second coordinate v_j and (u_a, v_j) in G_j is adjacent to (u_a, v_{j+1}) in $G_{j+1} \pmod n$. Let $u_1, u_2, \dots, u_{z(G)}$ be a set of starting positions for zombies in G with a winning zombie-play in G.

In $G \square C_n$, initially place zombies at:

- $(u_1, v_0), (u_2, v_0), \dots, (u_{z(G)}, v_0)$ and label the set of zombies $\mathcal{H}$.
- $(u_1, v_{\lfloor n/2 \rfloor}), (u_2, v_{\lfloor n/2 \rfloor}), \dots, (u_{z(G)}, v_{\lfloor n/2 \rfloor})$ and label the set of zombies $\mathcal{H}'$.
- $(u_1, v_0), (u_2, v_0), \dots, (u_{z(G)}, v_0)$ and label the set of zombies $\mathcal{H}''$.

Initially, the zombies of $\mathcal{H}$ and $\mathcal{H}'$ aim to capture the shadow of S in G_0 and $G_{\lfloor n/2 \rfloor}$ respectively (*i.e.*, capture the first coordinate of the position of S), while the zombies of set $\mathcal{H}''$ aim to capture the second coordinate of S. Clearly, the zombies of $\mathcal{H}$ and $\mathcal{H}'$ will not only achieve their goals, but achieve their goals during the same time step.

($\star$) During round t, the zombies of $\mathcal{H}$ and $\mathcal{H}'$ achieve their goals. For all subsequent steps, if S changes his first coordinate in round $t^* \geq t$, the zombies of $\mathcal{H}$ and $\mathcal{H}'$ will change their first coordinate so that $d_{t^*+1}(Z, S) = d_{t^*}(Z, S)$ for each $Z \in \mathcal{H} \cup \mathcal{H}'$. Since these zombies have captured the shadow of S, they can continue to recapture the shadow of S every turn. If S changes his second coordinate during round t^*, then suppose zombies Z and Z' are located at vertices of G_i and G_{i+k}, respectively, and S moves from G_x to G_y. In C_n, both v_x and v_y lie on a path between vertices v_i and v_{i+k} with $k \leq n/2$. So Z moves from G_i to G_{i+1} and Z' moves from G_{i+k} to G_{i+k-1}. Then the zombies in one of $\mathcal{H}$ or $\mathcal{H}'$ have decreased their distance to S since the last round and the zombies in the other set have maintained their distance since the last round. Without loss of generality, $d_{t^*+1}(Z, S) = d_{t^*}(Z, S)$ for all $Z \in \mathcal{H}$ and $d_{t^*+1}(Z, S) = d_{t^*}(Z, S) - 2$ for all $Z \in \mathcal{H}'$. Note that if $y = i + 1$ or $y = i + k - 1$, then the zombies capture S in this round.

Once the zombies of $\mathcal{H}$ and $\mathcal{H}'$ have achieved their goal, S is restricted in the second coordinate to a path, rather than a cycle. Clearly then, the zombies of $\mathcal{H}''$ will achieve their goals of each capturing the second coordinate of S.

Suppose $\mathcal{H}''$ has achieved its goal. Then if during round t', S changes his second coordinate (*i.e.*, his copy of G), then the zombies of $\mathcal{H}''$ change their second coordinate (to remain in the same copy of G as S), maintaining distance to S: $d_{t'+1}(Z, S) = d_{t'}(Z, S)$ for every $Z \in \mathcal{H}''$. However, the zombies of $\mathcal{H}$ and $\mathcal{H}'$ will continue to follow the sequence of moves described in ($\star$) and one of $\mathcal{H}$ or $\mathcal{H}'$ will maintain distance to S and the other will decrease distance to S.

Whenever S changes his first coordinate (*i.e.*, his copy of C_n), then the zombies of $\mathcal{H}$ and $\mathcal{H}'$ will continue to follow the sequence of moves described in ($\star$) and maintain their distance to S. Suppose S is located in G_k. Then the zombies of $\mathcal{H}''$ change their first coordinate, following a winning zombie-play in G_k. This will either maintain or decrease their distance to S. Note that S cannot indefinitely change his first coordinate because the zombies of $\mathcal{H}''$ would capture him. ∎

Since $z(P_m) = 1$ and $z(C_n) = 2$, we obtain the following corollary.

Corollary 12. *For n odd, $n \geq 5$, and $m \geq 2$, $z(P_m \square C_n) \leq z(P_m) + z(C_n)$.*

Theorem 13. *If T is a finite tree, then for any graph G, $z(G \square T) \leq 2z(G)$.*

Proof. Since the result is trivial when T has one vertex, let T be a finite tree with $|V(T)| = n \geq 2$ and label the vertices of T as $v_1, v_2, \dots, v_n$ where v_1 is a leaf in T and v_2 is adjacent to v_1 in T. For each $i \in \{1, 2, \dots, n\}$, let G_i be the subgraph of $G \square T$ induced by the vertices of the form (u, v_i) for every $u \in V(G)$; note $G_i \cong G$.

Let $u_1, u_2, \dots, u_{z(G)}$ be initial starting vertices for the zombies when they use a winning strategy in G. In $G \square T$, initially place zombies at $(u_1, v_1), (u_2, v_1), \dots, (u_{z(G)}, v_1)$ and also at $(u_1, v_2), (u_2, v_2), \dots, (u_{z(G)}, v_2)$. In $G \square T$, suppose the survivor is initially located at (u_x, v_y).

The zombies initially aim to capture the second coordinate of S. If at the end of round $t \geq 0$, the zombies are located in G_i and G_j and S is located at (u_a, v_b) where $b \notin \{i, j\}$ (i.e. the zombies have not yet captured the second coordinate of S) then let P be the unique path from v_i to v_b in T (note that P contains v_j) and in T and let v_k ($k \neq i$) be the vertex adjacent to v_j on

P. Then during round $t+1$, the zombies in G_j move to G_k and the zombies in G_i move to G_j and since T is a tree, the zombies will eventually catch the second coordinate of S in $G \square T$.

Suppose the zombies have captured the second coordinate of S in $G \square T$. Let S be located at (u_r, v_q) and the zombies be located in G_p, G_q (where $p \sim q$ in T). Now the zombies aim to capture the first coordinate of S while maintaining capture of his second coordinate.

(1) If S moves to (u_o, v_q) during round t, then the zombies move within G_p, G_q under the assumption that S moved to $(u_o, v_p), (u_o, v_q)$ (respectively). Then $d_{t+1}(Z, S) = d_t(Z, S)$ for each zombie Z in G_p and $d_{t+1}(Z, S) < d_t(Z, S)$ for each zombie Z in G_q.

(2) If S moves to (u_r, v_p) during round t, then the zombies move within G_p, G_q under the assumption that S moved to $(u_r, v_p), (u_r, v_q)$ (respectively). Then $d_{t+1}(Z, S) < d_t(Z, S)$ for each zombie Z in G_p and $d_{t+1}(Z, S) = d_t(Z, S)$ for each zombie Z in G_q.

(3) If S moves to (u_r, v_m) where $m \notin \{p, q\}$, then the zombies move from G_q to G_m and from G_p to G_q. Then $d_{t+1}(Z, S) = d_t(Z, S)$ for each zombie Z. However, we note that since T is a tree, eventually S will not be able to make such a move (*i.e.*, when his second coordinate is a leaf in T) and will be forced to move in (1) or (2).

As T is finite, eventually the zombies will capture S as during each of (1), (2) the distance from some zombies to S decreases. We also note that in each move of the zombies, they have moved along a shortest path to S. ∎

Theorem 14. *Let H be a graph with m vertices and at least one vertex of degree $m-1$. For any graph G, $z(G \square H) \le z(G) + 1$.*

Proof. Denote the zombies in $G \square H$ by $Z_1, Z_2, \dots, Z_{z(G)+1}$. Let G_1 be the copy of G in $G \square H$ that corresponds to the vertex of H of degree $m-1$.

Initially place all zombies in G_1, arranged so that the zombies of $\mathcal{H} = \{Z_2, \dots, Z_{z(G)+1}\}$ have a winning zombie-play for catching a survivor in G_1. Regardless of where the survivor S is located within $G \square H$, let S_1 denote his shadow in G_1. $\mathcal{H}$ now pursues S_1 within G_1, and independently Z_1 pursues S_1 in G_1 (that is, unless the survivor positions himself to be adjacent to some zombie (potentially in another copy of G), in which case he gets caught).

Each time that S moves between copies of G, Z_1 is able to get closer to S_1. So if S can be repeatedly forced to move from one copy of G to another, then Z_1 will eventually share its location with S_1 in G_1.

Regardless of the moves taken by S, the zombies in $\mathcal{H}$ can successfully capture S_1 in G_1. However, immediately prior to the move in which S_1 could be occupied by a zombie, the zombies of $\mathcal{H}$ change their strategy. Instead of continuing to pursue S_1 in G_1, they now move to the same copy of G in which S is located. If, on his next move, S remains in that copy of G, he will then be captured by the zombies of $\mathcal{H}$ and so he is forced to move to a different copy of G. The zombies of $\mathcal{H}$ henceforth maintain this strategy, thereby forcing S to move to a new copy of G in each remaining round of the game. As a result, Z_1 is able to reach S_1, and in the next round the survivor is captured by one of the zombies. ∎

Corollary 15. *For any graph G, $z(G \square K_m) \le z(G) + 1$.*

We now turn to the binary hypercube Q_n which itself is a Cartesian product: $Q_1 = K_2$ and for each $n \ge 2$, $Q_n = Q_{n-1} \square K_2$. In [8], Maamoun and Meyniel proved that the cop number of the product of n trees is exactly $\lceil \frac{n+1}{2} \rceil$ provided that the robber is not permitted to indefinitely stay at rest, and hence $c(Q_n) = \lceil \frac{n+1}{2} \rceil$ for all $n \ge 1$.

More recently, Offner and Ojakian have shown that within a model whereby each of the cops is active, and so is the robber, then the hypercube Q_n has cop number $\lceil \frac{2n}{3} \rceil$ [11]. In the context of zombies and a survivor, each zombie is active, but does not necessarily have the full range of motion as in the model considered by Offner and Ojakian (in their model, a cop can choose to increase her distance to the robber). Additionally, for our Zombies and Survivor model we permit the survivor to choose to rest (*i.e.*, he is not required to be always active). Nevertheless, the Offner and Ojakian result shows that an active robber can successfully avoid being caught by a force of fewer than $\lceil \frac{2n}{3} \rceil$ active cops, and hence a survivor can follow a similar strategy to avoid being captured by any horde of fewer than $\lceil \frac{2n}{3} \rceil$ zombies. Thus we have:

Theorem 16. *For each integer $n \ge 1$, $z(Q_n) \ge \lceil \frac{2n}{3} \rceil$.*

For small values of n, it is easy to determine the zombie number of Q_n exactly: $z(Q_1) = 1$, $z(Q_2) = 2$ and $z(Q_3) = 2$. This in turn enables us to establish an upper bound:

Lemma 17. *For each integer $n \ge 3$, $z(Q_n) \le n - 1$.*

Proof. For $n = 3$ the result follows from the established values of $z(Q_n)$. For $n \ge 4$ note that $Q_n = Q_3 \square K_2 \square K_2 \square \cdots \square K_2$ (there are $n-3$ copies of K_2 in this product) and by Corollary 15 it follows that $z(Q_n) \le 2 + (n-3) = n - 1$. ∎

Despite the wide gap between the bounds of Theorem 16 and Lemma 17, we conjecture that the lower bound of Theorem 16 is in fact the zombie number itself:

Conjecture 18. *For each integer $n \ge 1$, $z(Q_n) = \lceil \frac{2n}{3} \rceil$.*

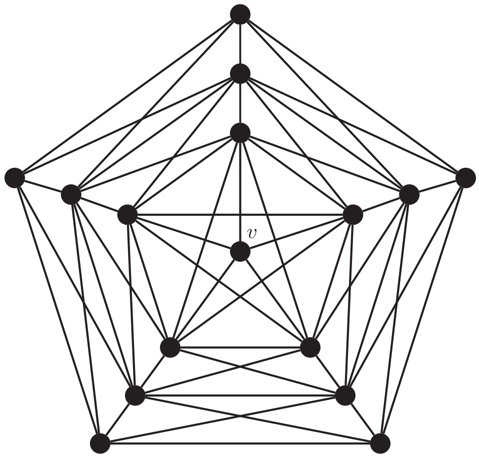

Fig. 5. G'_5.

Note that the lower bound for $z(Q_n)$ given by Theorem 16 diverges from the cop number of $\lceil \frac{n+1}{2} \rceil$ established by Maamoun and Meyniel, and thus the binary hypercubes serve as an example of a class of graphs G for which the difference $z(G) - c(G)$ forever widens as the graphs get larger, thereby providing some insight into Question 5. Instead of asking about the difference between the two parameters, it might be more useful to inquire whether their ratio is bounded.

Question 19. *How large can the ratio* $\frac{z(G)}{c(G)}$ *be?*

For hypercubes this ratio appears to be approximately $\frac{4}{3}$, assuming that Conjecture 18 is valid. For other types of graphs, recall that Fig. 2 illustrates a graph for which the ratio is $\frac{3}{2}$, while the graph in Fig. 1 has a ratio of precisely 2. As yet we have not observed any graph with a ratio that exceeds 2.

5. Additional distinctions from cops and robber

A key characteristic of the Cops and Robber model is that the initial location(s) of the cop(s) is irrelevant to determining whether a robber can be captured in a connected graph G. Although their initial placement may affect the capture time, the $c(G)$ cops that are needed to capture a robber can be positioned arbitrarily at the start of the game.

In contrast, with the Zombies and Survivors model, the initial placement of the zombies is critical to their ability to capture a survivor. As an example, take the graph G_n described in the proof of Theorem 3 and create a new graph G'_n by adding a new vertex v and making it adjacent to each of $(c, u_0), \ldots, (c, u_{n-1})$; the graph G'_5 is illustrated in Fig. 5. Then $z(G'_n) = 1$, since a single zombie that is initially placed at v will capture a survivor in at most three moves, regardless of the initial position chosen by the survivor. However, if the zombie is initially located at one of (a, u_i), (b, u_i) or (c, u_i) for some i then the survivor can successfully evade capture by starting at (a, u_{i+2}) and then following a survivor-play similar to that described in the proof of Theorem 3.

With this revelation that in a zombie-win graph G the fate of the survivor is a function of the initial placement of the zombie, we let $Z_W(G) \subseteq V(G)$ denote the set of vertices which, if selected as the zombie's initial position, will result in the capture of the survivor. A natural question arises:

Question 20. *For which zombie-win graphs G is $Z_W(G) = V(G)$?*

We note that Corollary 7 asserts that finite bridged graphs constitute a family of graphs which satisfy Question 20. These are not the only graphs which satisfy the property in Question 20. For example, consider a single zombie playing on any graph with a universal vertex. If the zombie starts on a universal vertex, he will win the game in the next round. If he starts on a non-universal vertex, v, the survivor will choose a vertex u that is not adjacent to v. However, there is a u, v-geodesic of length two that includes a universal vertex. Therefore, the zombie can move onto a universal vertex in round 1, and win in round 2. It follows that we can construct a non-bridged graph G such that $Z_W(G) = V(G)$ by taking any graph H with an induced cycle of length four and adding a universal vertex.

An infinite family of graphs with this property is obtained by letting $H = Q_n$ for any $n \geq 2$. This is demonstrated with $n = 2$ in Fig. 6.

Another intrinsic property of the Cops and Robber model is that adding additional robbers is inconsequential. That is, in a connected graph G, $c(G)$ cops can capture any number of robbers by following a strategy of dedicating all efforts to catching one robber at a time, letting the others run amok while a single robber of interest is pursued. However, in the context of the Zombies and Survivors model, in which each zombie's motion is restricted to the pursuit of one of its closest survivors, it is

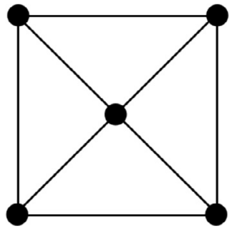

Fig. 6. A graph G with $Z_W(G) = V(G)$.

not immediately apparent whether $z(G)$ zombies can successfully capture an arbitrary number of survivors; the survivors might be able to divide the attention of the zombies and thereby devise a winning strategy for themselves.

Define $z_k(G)$ to be the minimum number of zombies that are necessary in order to capture k survivors in the graph G. We operate under the assumption that a survivor is eliminated from play when captured (for a different scenario, one might assume that survivors are transformed into zombies upon their capture). Clearly $z_1(G) = z(G)$.

Assuming that there is no upper bound on the number of survivors that can occupy a single vertex, then it is not hard to show that the number of zombies needed to capture all of the survivors does not decrease when the number of survivors increases:

Lemma 21. *For any graph G, $z_{k+1}(G) \geq z_k(G)$ for all $k \geq 1$.*

Proof. By way of contradiction, suppose that G is a connected graph such that $z_{k+1}(G) < z_k(G)$ for some k. A potential strategy that might be adopted by a set of $k+1$ survivors would be for two of them, say S_1 and S_2 to stay united, always occupying a common vertex, while the other $k-1$ survivors are permitted to travel independently. Such a strategy is, in effect, a strategy for k survivors, which by the definition of $z_k(G)$ cannot be captured by any fewer than $z_k(G)$ zombies. Given that $z_{k+1}(G) < z_k(G)$ then such a strategy would enable the $k+1$ survivors to successfully evade capture by $z_{k+1}(G)$ zombies, in contradiction to the definition of $z_{k+1}(G)$. ■

Potentially $z_{k+1}(G) > z_k(G)$ for some k and some G. As an example, consider the cycle C_n where n is a sufficiently large odd number. Clearly $z(C_n) = 2$, and so it takes two zombies to capture a single survivor. But what if there are two survivors? As it happens, there is no initial placement of two zombies that will ensure that both survivors are eventually captured.

Lemma 22. *For all $n \geq 9$, $z_2(C_n) > z_1(C_n)$. Moreover, if $n \geq 13$ two zombies fail to capture either survivor.*

Proof. First consider $n \geq 13$. Let d denote the initial distance between the two zombies. If $d \geq 5$ then with respect to an ordering (say clockwise) of the cycle, place S_1 (resp. S_2) two vertices ahead of Z_1 (resp. Z_2). For $d \leq 4$ we may assume, without loss of generality, that Z_1 is d vertices ahead of Z_2, and then place both survivors two vertices ahead of Z_1. In each of these cases the zombies will perpetually run around the cycle chasing the survivors.

For $9 \leq n \leq 12$, the possible winning positions for two zombies and one survivor have $d = 3$ or $d = 4$. Positioning one (unlucky) survivor on the shorter path between the zombies and the other appropriately on the longer path, forces the zombies closer together to catch the unlucky survivor. The zombies then fail to catch the other survivor. ■

Although we have seen that $z_{k+1}(G)$ can exceed $z_k(G)$, for those graphs characterized by Question 20 there is no increase in the number of zombies required to capture additional survivors.

Lemma 23. *If G is a zombie-win graph for which $Z_W(G) = V(G)$, then $z_k(G) = 1$ for all $k \geq 1$.*

Proof. Since $z_1(G) = z(G) = 1$, it suffices to prove that $z_k(G) = 1$ for all $k \geq 2$. Given k survivors, say $S_1, \ldots, S_k$, let $d_t(S_i)$ denote the distance between survivor S_i and the zombie Z at the start of round t. Let δ_t be the minimum of the k elements of $\{d_t(S_i) : i \in \{1, \ldots, k\}\}$. Although it is possible for $d_{t+1}(S_i)$ to exceed $d_t(S_i)$, such as when the zombie is closer to some other survivor(s) than it is to S_i, it is impossible for δ_{t+1} to exceed δ_t (except when $\delta_t = 1$ and the zombie is about to capture a survivor, in which case δ_{t+1} is the minimum distance from Z to the remaining survivors).

Consider a survivor S for whom $d_t(S) = \delta_t$. This survivor has attracted the attention of the zombie Z. Since G is a zombie-win graph, and also that there is no vertex of G on which Z could be located so as to allow S to devise a winning strategy (supposing that S were the sole survivor) then the zombie will ultimately capture S. That is, unless the zombie's attention is diverted by some other survivor. The only way for Z to lose interest in S is for some other survivor, S^*, to deliberately approach Z so that $d_{t^*}(S) > d_{t^*}(S^*)$ at some time $t^* > t$. In such a case we have $\delta_{t^*} < \delta_t$. The situation now iterates: either Z ultimately captures S^* or δ is further reduced at some subsequent step in time (by having a survivor other than S^* approach Z). Since δ can only be reduced a finite number of times without capturing a survivor, then some survivor will indeed be captured, at which time δ is reset accordingly and the zombie begins pursuing one of the remaining survivors to which it is now closest. ■

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