European Journal of Combinatorics 72 (2018) 45–69

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Cops and Robbers on intersection graphs ☆

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ARTICLE INFO

Article history:
Received 27 July 2016
Accepted 19 April 2018
Available online 8 May 2018

ABSTRACT

The cop number of a graph G is the smallest k such that k cops win the game of cops and robber on G. We investigate the maximum cop number of geometric intersection graphs, which are graphs whose vertices are represented by geometric shapes and edges by their intersections. We establish the following dichotomy for previously studied classes of intersection graphs:

- The intersection graphs of arc-connected sets in the plane (called *string graphs*) have cop number at most 15, and more generally, the intersection graphs of arc-connected subsets of a surface have cop number at most $10g + 15$ in case of orientable surface of genus g, and at most $10g' + 15$ in case of non-orientable surface of Euler genus g'. For more restricted classes of intersection graphs, we obtain better bounds: the maximum cop number of interval filament graphs is two, and the maximum cop number of outer-string graphs is between 3 and 4.
- The intersection graphs of disconnected 2-dimensional sets or of 3-dimensional sets have unbounded cop number even in very restricted settings. For instance, it follows from known results that the cop number is unbounded on intersection graphs of two-element subsets of a line. We further show that it is also unbounded on intersection graphs of

☆ The conference versions of parts of this paper appeared in ISAAC 2013 [13] and ISAAC 2015 [14]. For a structural dynamical diagram of the results of this paper, see http://pavel.klavik.cz/orgpad/cops_on_intersection_graphs.html (supported for Firefox and Google Chrome). The third, the fourth, and the fifth authors are supported by CE-ITI (P202/12/G061 of GAČR), the first, the fourth and the fifth authors are supported by Charles University as GAUK 196213.

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https://doi.org/10.1016/j.ejc.2018.04.009
0195-6698/

3-dimensional unit balls, of 3-dimensional unit cubes or of 3-dimensional axis-aligned unit segments.

1. Introduction

The game of cops and robber on graphs has been introduced independently by Quilliot [31] and by Nowakowski and Winkler [26]. In this paper, we investigate the game on geometric intersection graphs.

Rules of the game. The first player, called *the cops*, places k cops on vertices of a graph G. Then the second player, called *the robber*, places the robber on a vertex. Then the players alternate. In the cops' move, every cop either stays on the same vertex, or moves to one of its neighbors. More cops may occupy the same vertex. In the robber's move, the robber either stays on the same vertex, or moves to a neighboring vertex. The game ends when the robber is *captured* which happens when a cop occupies the same vertex as the robber. The cops win if they are able to capture the robber. The robber wins if he is able to escape indefinitely. We note that this is a full information game, as the cops and the robber are both visible, i.e., their positions are known to both players throughout the game.

Maximum cop number. The *cop number* of a graph G, denoted by $\mathrm{cn}(G)$, is the least number k such that k cops have a winning strategy on G. For a class of graphs $\mathcal{C}$, the *maximum cop number* $\text{max-cn}(\mathcal{C})$ is the maximum cop number $\mathrm{cn}(G)$ of a connected graph $G \in \mathcal{C}$, provided the maximum does exist. Otherwise, the cop number of connected graphs in $\mathcal{C}$ is unbounded, hence we say that $\text{max-cn}(\mathcal{C})$ *is unbounded* or simply that $\mathcal{C}$ has *unbounded cop number*. The restriction to connected graphs is natural: if G has connected components $C_1, \dots, C_k$, then $\mathrm{cn}(G) = \sum_{i=1}^{k} \mathrm{cn}(C_i)$. Therefore, a graph class closed under disjoint union cannot have a bounded maximum cop number if we omit this restriction. Throughout the paper, we only work with connected graphs.

Known results. Graphs of the cop number one were characterized already by Quilliot [31] and by Nowakowski and Winkler [26]. These are the graphs that admit a *cop-win ordering* — their vertices can be linearly ordered $v_1, v_2, \dots, v_n$ so that each v_i for $i \geq 2$ is a *corner* of $G[v_1, \dots, v_i]$, i.e., v_i has a neighbor v_j for some $j < i$ such that v_j is adjacent to all other neighbors of v_i. Andreae [2] proved that the class of k-regular graphs has unbounded cop number for all $k \geq 3$.

For k part of the input, deciding whether the cop number of a graph is at most k has been shown NP-hard [11], PSPACE-hard [22] and very recently EXPTIME-complete [17], confirming a 20 years old conjecture of Goldstein and Reingold [16]. In order to test whether k cops suffice to capture the robber on an n-vertex graph, we can search the game graph which has $\mathcal{O}(n^{k+1})$ vertices to find a winning strategy for cops. In particular, if k is a fixed constant, this algorithm runs in polynomial time. On the other hand, with k as a parameter, deciding whether k cops can win on a given graph is W[2]-hard [11].

For general graphs on n vertices, it is known that at least $\sqrt{n}$ cops may be needed (e.g., for the incidence graph of a finite projective plane [29]). Meyniel's conjecture states that the cop number of a connected n-vertex graph is $\mathcal{O}(\sqrt{n})$. For more details and results, see the book [4].

Geometrically represented graphs. We want to argue that the geometry of a graph class heavily influences the maximum cop number. For instance, the classical result of Aigner and Fromme [1] shows that the maximum cop number of planar graphs is 3.

A natural generalization of planar graphs is the notion of graphs of bounded genus. The *genus* of a graph G (also known as 'orientable genus'), denoted by $\gamma(G)$, is the smallest integer g such that G has a noncrossing drawing on the orientable surface of genus g, i.e., on the sphere with g handles. Similarly, the *non-orientable genus* of G, denoted $\widetilde{\gamma}(G)$, is the smallest g such that G has a noncrossing drawing on the non-orientable surface of genus g, i.e., on the sphere with g cross-caps. Finally, the *Euler genus* $\gamma_{\mathrm{E}}(G)$ of G is the smallest g such that G has a noncrossing drawing on the orientable surface of genus $\lfloor g/2 \rfloor$ or on the non-orientable surface of genus g; in other words, we have $\gamma_{\mathrm{E}}(G) = \min\{2\gamma(G), \widetilde{\gamma}(G)\}$. It is known (see [25]) that $\widetilde{\gamma}(G) \leq 2\gamma(G) + 1$, and consequently, $\widetilde{\gamma}(G) - 1 \leq \gamma_{\mathrm{E}}(G) \leq \widetilde{\gamma}(G)$.

We let GENUS g denote the set of graphs with genus at most g, and we let EULER-GENUS g denote the set of graphs of Euler genus at most g.

Quilliot [32] has shown that the cop number is bounded on graphs of bounded genus, and his bound was later improved by Schroeder [34], who proved that

$$\mathrm{cn}(G) \le \frac{3}{2}\gamma(G) + 3. \tag{1}$$

For non-orientable surfaces, a similar result was obtained by Clarke et al. [7]:

$$\mathrm{cn}(G) \le \frac{3}{2}\widetilde{\gamma}(G) + \frac{3}{2}. \tag{2}$$

However, the exact value of the maximum cop number is not known even for toroidal graphs, i.e., graphs of genus 1.

We study *intersection representations* of a graph $G = (V, E)$, in which to each vertex $v \in V$ we associate a set φ_v in such a way that the edges of G are described by the intersections: $uv \in E \iff \varphi_u \cap \varphi_v \neq \emptyset$. The sets φ_v are usually somehow restricted to get particular classes of intersection graphs. For example, to obtain the well-known class of interval graphs, we require that every φ_v is a closed interval of the real line.

All these graph classes admit large cliques, so their genus is unbounded and the bound (1) of the maximum cop number does not apply. On the other hand, the existence of large cliques does not imply big maximum cop number since only one cop is enough to guard a maximal clique. For instance, chordal graphs, which are intersection graphs of subtrees of a tree, may have arbitrarily large cliques but their maximum cop number is 1. The fact that chordal (and therefore also interval) graphs have max-cn $= 1$ follows from the characterization of cop-win graphs [26,31]: *a simplicial ordering* of any chordal graph is a cop-win ordering.

String graphs and related classes. The class of *string graphs* (STRING) is the class of intersection graphs of *strings*: every φ_v is a bounded curve in the plane, i.e., a continuous image of the interval $[0, 1]$ into $\mathbb{R}^2$. It is known [35] that every intersection graph of arc-connected sets in the plane is a string graph, where a set $A \subseteq \mathbb{R}^d$ is *arc-connected* if for any two points $a, b \in A$ the set A contains a curve with endpoints a and b.

A *d-dimensional interval* is a Cartesian product of d intervals. The intersection graphs of d-dimensional intervals in $\mathbb{R}^d$ are known as *boxicity d graphs* (d-BOX). The boxicity 1 graphs are known as *interval graphs* (INTERVAL). Since d-dimensional intervals are clearly arc-connected, it follows that 2-BOX is a subset of STRING. Esperet and Joret [10] have shown that graphs of bounded genus also have bounded boxicity; more precisely, any graph of genus g has boxicity at most $5g + 3$.

The class of *outer-string graphs* (OUTER-STRING) consists of all string graphs having string representations with each string contained in the closed upper half-plane of the Cartesian plane, with each string φ_v having an endpoint on the x-axis and the remaining points of φ_v being strictly above the x-axis.

The class of *interval filament graphs* (INTERVAL FILAMENT), introduced by Gavril [15], consists of intersection graphs of interval filaments. An *interval filament* φ, defined on an interval $[a, b]$, is the graph of a continuous function $f : [a, b] \to \mathbb{R}$ such that $f(a) = f(b) = 0$ and $f(x) > 0$ for all $x \in (a, b)$. Observe that INTERVAL FILAMENT $\subseteq$ OUTER-STRING $\subseteq$ STRING. In fact, both these inclusions are strict: the fact that INTERVAL FILAMENT is distinct from OUTER-STRING is a consequence of Theorem 1.1(i) and (ii), while OUTER-STRING $\neq$ STRING follows, e.g., from the fact that triangle-free outer-string graphs have bounded chromatic number [33], while triangle-free string graphs have unbounded chromatic number [27].

We remark that the class of interval filament graphs includes several notable subclasses, such as chordal graphs, circle graphs, circular-arc graphs, or function graphs. We will not deal with these subclasses in our paper, and refer the interested reader to the monograph of Brandstädt et al. [5] for their definitions and basic properties.

String representations in the plane can be generalized to arbitrary surfaces in an obvious way: we say that a graph G has a string representation on a surface **S** if G has an intersection representation in which vertices are represented by bounded curves contained in **S**. We let g-GENUS STRING be the class of graphs admitting a string representation on an orientable surface of genus g, and we let g'-EULER-GENUS STRING be the class of graphs admitting a string representation on an arbitrary surface of Euler genus g'.

An easy argument shows that a graph that has a noncrossing drawing on a given surface **S** also admits a string representation on **S**. This has been pointed out by Ehrlich et al. [9] in the case when **S** is a sphere, and the argument works on any other surface as well. Therefore GENUS g is a subset of g-GENUS STRING, and similarly for the Euler genus. These inclusions are in fact strict, because each clique admits a string representation on an arbitrary surface, but only finitely many cliques admit a noncrossing drawing on a given surface.

Note that while we could work only with Euler genus for both orientable and non-orientable surfaces, the bounds on cop number obtained for orientable genus are better than for Euler genus by a factor of 2.

Intersection graphs of disconnected and higher dimensional sets. As stated above, all intersection graphs of arc-connected sets in the plane are string graphs. To obtain other classes of intersection graphs, we may consider disconnected sets in the plane, or subsets of higher-dimensional Euclidean spaces. It turns out that intersection graphs of such sets have unbounded cop number even in very restricted settings.

For a graph G, its *line-graph*, denoted by $L(G)$, is the intersection graph of the edges of G. Let LINE denote the class of all line-graphs. Observe that each line-graph can be represented as an intersection graph of two-element subsets of a line. Thus, line-graphs provide a simple example of intersection graphs of disconnected sets. As shown by Dudek et al. [8], the cop number of $L(G)$ is related to the cop number of G via the inequalities

$$\left\lceil \frac{\mathrm{cn}(G)}{2} \right\rceil \leq \mathrm{cn}(L(G)) \leq \mathrm{cn}(G) + 1.$$

In particular, the cop number of line-graphs is unbounded.

Many other geometric intersection classes can be seen as generalizations of line-graphs. Among the most studied are the *ℓ-interval graphs* (ℓ-INTERVAL) where every φ_v is a union of ℓ closed intervals in the real line. For the *ℓ-unit interval graphs* (ℓ-UNIT INTERVAL), we further consider that each of the ℓ intervals forming φ_v has unit length. Since these classes include all line-graphs whenever $\ell > 1$, their cop number is unbounded.

For sets in higher dimensions, notice that every graph has a representation by strings in $\mathbb{R}^3$. Therefore, to get interesting classes of graphs, we have to further restrict the geometry of the sets. Apart from the class d-BOX, described above, we consider the classes of intersection graphs of the following geometric objects in $\mathbb{R}^d$: axis parallel segments (d-GRID), d-dimensional unit cubes (d-UNIT CUBE), d-dimensional balls (d-BALL), and d-dimensional unit balls (d-UNIT BALL).

Our results. It has been asked on several occasions, last during the Banff Workshop on Graph Searching in October 2012, whether intersection-defined graph classes (other than interval graphs) have bounded maximum cop numbers. The classes in question have included circle graphs, intersection graphs of disks in the plane, graphs of boxicity 2, and others. A recent paper [3] shows that the maximum cop number of intersection graphs of unit disks (2-UNIT BALL) is between 3 and 9. We solve the general question by proving a dichotomy for previously studied classes of geometric intersection graphs in Theorem 1.1 and 1.7. For an overview of the results presented in this paper, see Fig. 1. The conference versions of parts of this paper appeared in ISAAC 2013 [13] and ISAAC 2015 [14].

Theorem 1.1. *The following bounds for the maximum cop number hold:*

(i) max-cn(INTERVAL FILAMENT) $= 2$.
(ii) $3 \leq$ max-cn(OUTER-STRING) ≤ 4.
(iii) $3 \leq$ max-cn(STRING) ≤ 15.
(iv) $g^{\frac{1}{2}-o(1)} \leq$ max-cn(g-GENUS STRING) $\leq 10g + 15$, *as* $g \to \infty$.
(v) $g'^{\frac{1}{2}-o(1)} \leq$ max-cn(g'-EULER-GENUS STRING) $\leq 10g' + 15$, *as* $g' \to \infty$.

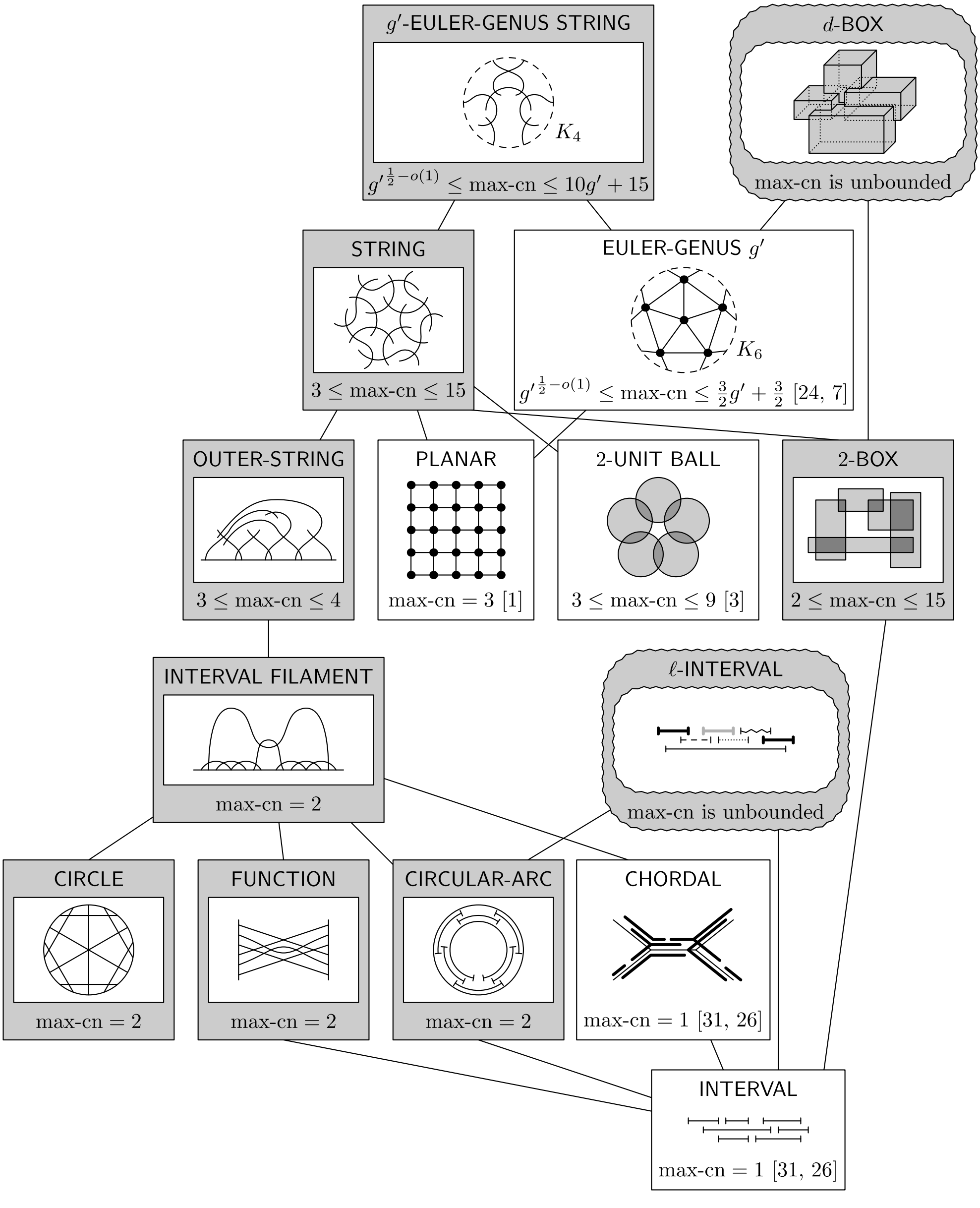

Fig. 1. The Hasse diagram of inclusions of the considered classes of graphs, together with bounds on the maximum cop number. The classes with previously known bounds are depicted in white, and the classes with the bounds proved in this paper are depicted in gray. Bounded boxicity of bounded genus graphs has been shown in [10].

We note that the strategies of cops in all upper bounds are geometric and their description is constructive, using an intersection representation of G. If only the graph G is given, we cannot generally construct these representations efficiently since recognition is NP-complete for string graphs [19] and interval filament graphs [28], and open for the other classes. Nevertheless, since the state space of the game has $\mathcal{O}(n^{k+1})$ states and the number of cops k is bounded by a constant, we can use the standard exhaustive game space searching algorithm to obtain the following:

Corollary 1.2. *There are polynomial-time algorithms computing the cop number and an optimal strategy for the cops for any interval filament graph in time $\mathcal{O}(n^3)$, outer-string graph in time $\mathcal{O}(n^5)$, string graph in time $\mathcal{O}(n^{16})$ and a string graph on a surface of a fixed genus g (resp. Euler genus g′) in time $\mathcal{O}(n^{10g+16})$ (resp. $\mathcal{O}(n^{10g'+16})$), even when representations are not given.* □

We may deduce structural information of geometrically representable graphs by combining our upper bound on cop number with the following result, due to Aigner and Fromme [1].

Fact 1.3 (*[1, Theorem 3]*)**.** *If G is a graph with all vertices of degree at least d and with no cycles of length 3 or 4, then* $\mathrm{cn}(G) \geq d$.

Fact 1.3 allows us to obtain bounds on the degeneracy of certain graphs in classes of bounded cop number. A graph is *d-degenerate* if each of its subgraphs contains a vertex of degree at most d. Observe that a d-degenerate graph has chromatic number at most $d + 1$. The *girth* of a graph is the length of its shortest cycle. A class $\mathcal{C}$ of graphs is *hereditary*, if for every $G \in \mathcal{C}$, all induced subgraphs of G are in $\mathcal{C}$. From Fact 1.3, we directly get the following consequence.

Corollary 1.4. *Let $\mathcal{C}$ be a hereditary class of graphs with* $\text{max-cn}(\mathcal{C}) = k$ *for $k \in \mathbb{N}$. Then every graph $G \in \mathcal{C}$ with girth at least 5 has a vertex of degree at most k. Consequently, G is k-degenerate and has chromatic number at most* $k + 1$.

Combining Corollary 1.4 with Theorem 1.1, we obtain, for instance, the following result on string graphs.

Corollary 1.5. *Any string graph of girth at least 5 is 15-degenerate and therefore 16-colorable.*

The fact that string graphs of girth 5 have bounded degeneracy can also be deduced from the results of Fox and Pach [12], although their results do not mention an explicit numerical bound. We do not know whether the bounds in Corollary 1.5 are tight. We note that Kostochka and Nešetřil [18] proved that the chromatic number of 1-string graphs of girth 5 is at most 6, where a 1-string graph is a graph with a string representation in which any two strings intersect at most once.

Applying Corollary 1.4 to outer-string graphs, we get the following result.

Corollary 1.6. *Every outer-string graph of girth at least 5 is 4-degenerate and therefore 5-colorable.*

Results in the spirit of Corollary 1.6 have been previously known. McGuinness [23] showed that triangle-free 1-outer-string graphs have bounded chromatic number, by a complicated argument yielding a large upper bound on the chromatic number. This result was later strengthened by Rok and Walczak [33], who proved that the chromatic number of an outer-string graph can be bounded by a function of its clique number. Again, the argument produces a rather large upper bound.

In contrast with Theorem 1.1, we will show that intersection graphs of disconnected sets, as well as intersection graphs of subsets of $\mathbb{R}^3$, have unbounded cop number even in very restricted settings. Our main tool will be a lemma stating that when we subdivide all edges of a graph G by a fixed number of new vertices, the cop number cannot decrease and it can increase by at most one. With the help of this lemma, we will deduce the following result.

Theorem 1.7. *The classes* LINE, 2-INTERVAL, 2-UNIT INTERVAL, 3-GRID, 3-BOX, 3-UNIT CUBE, 3-BALL, *and* 3-UNIT BALL have unbounded cop number.

Outline of the paper. In Section 2, we show that max-cn(INTERVAL FILAMENT) is 2. In Section 3, we show that max-cn(OUTER-STRING) is between 3 and 4. In Section 4, we show that max-cn(STRING) is at most 15. In Section 5, we combine the result on string graphs with the approach of Quilliot [32] to deduce the bounds for cop number of graphs with string representation on a given surface. In Section 6, we prove that cop numbers are unbounded for intersection graphs of disconnected or 3-dimensional sets.

Preliminaries. Let $G = (V, E)$ be a graph. For $D \subseteq V$, we let $G[D]$ denote the subgraph of G induced by D. For a vertex $v \in V$ or a set $D \subseteq V$, we write $G - v$ for the graph $G[V \setminus \{v\}]$ and $G - D$ for $G[V \setminus D]$.

For a vertex v, we use the *open neighborhood* $N(v) = \{u : uv \in E\}$ and the *closed neighborhood* $N[v] = N(v) \cup \{v\}$. Similarly for $V' \subseteq V$, we put $N[V'] = \bigcup_{v \in V'} N[v]$ and $N(V') = N[V'] \setminus V'$.

Let G be a string graph with a given string representation. Without loss of generality, we may assume that there is only a finite number of string intersections in the representation, that no string intersects itself, that strings never only touch without either also crossing each other or at least one of them ending, and that no three or more strings meet at the same point. This follows from the fact that strings can be replaced by piece-wise linear curves with finite numbers of linear segments without affecting their intersection graph. For more details see [20]. We always assume and maintain these properties.

When describing a strategy for the cops to capture a robber on a given graph, we will always assume, without specifying this explicitly, that if in a given position of the game the cops can capture the robber by a single move, they will do so. Therefore, to avoid capture, the robber must never move to a vertex which is in the closed neighborhood of a vertex occupied by a cop.

Suppose that we have a strategy for the cops. For a vertex $v \in V$, the robber *cannot safely move to* v if the strategy ensures that he is immediately captured after moving to v. Let $D, P \subseteq V$. We say that the strategy *guards* P if it ensures that the robber cannot safely move to any vertex in P. We say that the robber is *confined to* D by the strategy, if the robber occupies a vertex from D and the strategy guards $V \setminus D$. Notice that for the robber to be confined to D it is enough that he occupies a vertex in D and the strategy guards $N(D)$.

2. Catching the robber in interval filament graphs

In this section, we show that the maximum cop number of interval filament graphs is equal to two, thus establishing Theorem 1.1(i). Recall that an interval filament φ defined on an interval $[a, b]$ is the graph of a continuous function $f : [a, b] \to \mathbb{R}$ satisfying $f(a) = f(b) = 0$, and $f(x) > 0$ for every $x \in (a, b)$. The interval $[a, b]$ is the *support* of the filament φ, denoted by $\mathrm{supp}(\varphi)$. For a filament φ and $x \in \mathrm{supp}(\varphi)$, we use the notation $\varphi(x)$ to refer to the point $(x, f(x))$. In particular, $\varphi(a)$ and $\varphi(b)$ are the two endpoints of φ, with $\varphi(a)$ being the *left endpoint* and $\varphi(b)$ the *right endpoint*.

For a pair of filaments φ and ψ, we say that φ is *nested below* ψ, if $\mathrm{supp}(\varphi)$ is a proper subset of $\mathrm{supp}(\psi)$ and for every $x \in \mathrm{supp}(\varphi)$ the point $\varphi(x)$ is strictly below $\psi(x)$. Note that this implies that the endpoints of $\mathrm{supp}(\psi)$ do not belong to $\mathrm{supp}(\varphi)$. We also say that φ is to the *left of* ψ (and ψ is to the *right of* φ) if $x < y$ for any $x \in \mathrm{supp}(\varphi)$, $y \in \mathrm{supp}(\psi)$. Observe that two filaments are disjoint, if and only if one is nested below the other or one is to the left of the other.

Let $G = (V, E)$ be a connected interval filament graph, and let us fix an intersection representation of G in which the vertices are represented by interval filaments. We let φ_v denote the filament representing a vertex $v \in V$ in the given representation, and let $\Phi = \{\varphi_v;\ v \in V\}$ be the set of all the filaments in the representation. We may assume that the filaments in Φ have pairwise distinct endpoints and their supports are proper intervals. We may also assume that any two filaments of Φ cross in at most finitely many points.

Let $\mathrm{supp}(\Phi)$ denote the set $\bigcup_{v \in V} \mathrm{supp}(\varphi_v)$. Note that since the graph G is connected, $\mathrm{supp}(\Phi)$ is a single closed interval, say $[l, r]$. The *leftmost filament of* Φ is the filament $\varphi \in \Phi$ such that $l \in \mathrm{supp}(\varphi)$, and similarly the *rightmost filament of* Φ is the filament $\psi \in \Phi$ such that $r \in \mathrm{supp}(\psi)$. Let $\mathbb{R}^2_+$ denote the upper half-plane of the Cartesian plane, i.e., the set of all the points with nonnegative vertical coordinate. The arc-connected components of the set $\mathbb{R}^2_+ \setminus \bigcup_{v \in V} \varphi_v$ are called *the faces* of Φ. There is a unique unbounded face, which we call the *external face of* Φ. The points of $\bigcup_{v \in V} \varphi_v$ that belong to the boundary of the external face form the *external boundary* of Φ. Note that the external boundary is arc-connected, and in fact, it is itself an interval filament whose support is precisely the interval $\mathrm{supp}(\Phi)$. A filament $\varphi \in \Phi$ that contains at least one point belonging to the external boundary is an *external filament of* Φ. Observe that the leftmost and the rightmost filament of Φ are both external filaments.

External paths. We will now consider the game of cops and robber on the graph G with a given filament representation $\Phi = \{\varphi_v;\ v \in V\}$. By a slight abuse of terminology, we will say that a cop (or robber) *occupies a filament* φ_v if the cop is placed on the vertex $v \in V$ represented by this filament. We say that a robber is *confined below* φ_v by a given cop strategy, if the strategy confines the robber to the set of vertices that are represented by filaments nested below φ_v.

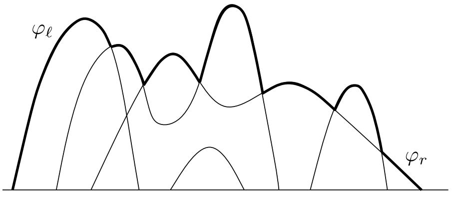

Fig. 2. A set of filaments. The external boundary is depicted in bold. The filaments representing the leftmost and rightmost vertices are marked φ_ℓ and φ_r, respectively. Note that since φ_ℓ and φ_r intersect, the two vertices ℓ and r form an external path of length 1.

Lemma 2.1. *Suppose that in a given position of the cops and robber game, a filament $\varphi_v \in \Phi$ is occupied by a cop while the robber occupies a filament nested below φ_v. Then the robber will remain confined below φ_v as long as there is a cop occupying φ_v.*

Proof. Clearly any filament intersecting a filament nested below φ_v must itself either be nested below φ_v or it must intersect φ_v. As long as there is a cop on φ_v, the robber cannot safely move to any filament intersecting φ_v and therefore remains confined below φ_v. □

Let ℓ and r denote the vertices represented by the leftmost and the rightmost filament of Φ. An *external path* of Φ is a path in G from ℓ to r whose every vertex is represented by an external filament. Note that G has at least one external path: indeed, since the external boundary of Φ is arc-connected, the vertices of G represented by external filaments induce a connected subgraph of G and therefore there is a path from ℓ to r formed by these vertices. Note that the external path may consist of a single vertex; see Fig. 2.

Lemma 2.2. *Let $p_0, p_1, \ldots, p_k$ be the sequence of vertices forming an external path in G, with $p_0 = \ell$ and $p_k = r$. Let $\pi_i \in \Phi$ denote the filament representing p_i. Suppose that in a position of the cops and robber game, there is a cop occupying a filament π_i of the external path, while the robber occupies a filament φ_v which is to the right of π_i. If the cop moves from π_i to π_{i+1} and the robber performs an arbitrary move, then in the ensuing position, the robber occupies a filament which is either to the right of π_{i+1}, or nested below π_{i+1}, or intersecting π_{i+1}. In particular, the robber cannot reach a filament to the left of π_{i+1}.*

Notice that cops can win immediately if before cops move, a cop and the robber are on intersecting filaments.

Proof. Suppose that after the cop moves from π_i to π_{i+1}, the robber moves from φ_v to φ_w (possibly with $v = w$). By assumption, supp(φ_v) is entirely to the right of supp(π_i). Moreover, supp(π_{i+1}) intersects supp(π_i), and supp(φ_w) intersects supp(φ_v). It follows that φ_w cannot be to the left of π_{i+1}. Since π_{i+1} is an external filament, it cannot be nested below φ_w. This leaves us with the possibilities that φ_w is to the right of π_{i+1}, nested below π_{i+1}, or intersecting π_{i+1}. □

A strategy for two cops. We are ready to prove that the maximum cop number of interval filament graphs is equal to two.

Proof of Theorem 1.1(i). Since a cycle of length 4 is an interval filament graph, as well as a circle graph, circular-arc graph and function graph, we see that two cops are necessary on these classes of graphs.

Let G be a connected interval filament graph, and let φ_v for $v \in V(G)$ and Φ be as above. We describe how to catch the robber with two cops. We call one cop the *guard*, and the other one the *hunter*. The strategy proceeds in phases. Every phase except the first one starts with both cops on a filament φ_u and with the robber confined below it. The guard stays on φ_u till the robber is either captured, or confined by the hunter below some filament φ_v nested below φ_u. By Lemma 2.1, the robber can only use the filaments below of φ_u. If the confinement by φ_v happens, then the guard moves to the filament φ_v occupied by the hunter, ending the phase. In the next phase the hunter proceeds with chasing the robber on the filaments nested below φ_v. In the beginning of the game, both cops are placed at the leftmost filament of Φ. During the first phase, the guard neither moves nor contributes to the confinement of the robber.

Let us describe an individual phase of the game in detail. Suppose that the ith phase of the game begins, with both cops occupying a filament $\gamma_i \in \Phi$ and the robber confined below γ_i. Let G_i be the subgraph of G induced by the vertices whose filaments are nested below γ_i, let C_i be the connected component of G_i containing the vertex occupied by the robber, and let Φ_i be the set of filaments representing the vertices of C_i. For the first phase of the game, we put $G_1 = C_1 = G$ and $\Phi_1 = \Phi$, and we leave γ_1 undefined.

Throughout the ith phase, the guard stays on γ_i, guaranteeing that the robber is confined to C_i. Meanwhile, the hunter chooses an arbitrary external path P_i of Φ_i, and moves to the leftmost filament of P_i (which is also the leftmost filament of Φ_i). The hunter proceeds to move along the path P_i towards the rightmost filament of Φ_i, until the robber is either captured or occupies a filament nested below the hunter's current position. By Lemma 2.2, the robber will never be able to occupy a filament to the left of the hunter's filament. Since P_i ends in the rightmost filament of Φ_i, the robber cannot remain to the right of the hunter indefinitely, so eventually he will be captured or confined below the hunter's filament. Once the robber takes a filament nested below the hunter's current filament, the hunter stops, the guard moves to the filament occupied by the hunter, and the ith phase ends.

Since there are only finitely many filaments nested below each other, the strategy proceeds in finitely many phases and the robber is eventually captured. □

With a slightly more careful analysis, we can in fact show that the strategy described above captures the robber in $\mathcal{O}(n)$ turns, where n is the number of vertices of G. To see this, suppose that there are p phases. Let C_i, Φ_i and γ_i be as above, and let Δ_i be the set of external filaments with respect to Φ_i. Observe that the sets $\Delta_1, \Delta_2, \dots, \Delta_p$ are pairwise disjoint, since for $i > 1$ every filament of Φ_i (and therefore every filament of $\Delta_i \cup \Delta_{i+1} \cup \dots \cup \Delta_p$) is nested below the filament $\gamma_i \in \Delta_{i-1}$, whereas none of the filaments in Δ_{i-1} can be nested below γ_i. Observe also that for every $i > 1$ there is a filament $\delta_i \in \Phi$ that simultaneously intersects γ_i and a filament in Δ_i. It follows that in the ith phase of the strategy, the hunter needs at most $|\Delta_i| + 1$ rounds to move from γ_i to the leftmost filament of Φ_i, then at most $|\Delta_i|$ rounds to walk along the path P_i, and finally the guard needs at most another $|\Delta_i| + 1$ rounds to come from γ_i to the hunter's current position. Overall, the ith phase takes at most $2 + 3|\Delta_i|$ steps, and the entire strategy takes at most $2p + 3n = \mathcal{O}(n)$ steps.

3. Catching the robber in outer-string graphs

In this section, we prove that the maximum cop number of outer-string graphs is between 3 and 4, thus establishing Theorem 1.1(ii). Our strategy is similar to the one described in Section 2.

String pairs. Let G be a connected outer-string graph on the vertex set $V = \{v_1, \dots, v_n\}$ and suppose we are given an outer-string representation in which a vertex v_i is represented by a string φ_i. Recall that in an outer-string representation, each φ_i is a curve in the upper half-plane of the Cartesian plane with exactly one endpoint on the x-axis. We will call the endpoint of φ_i touching the x-axis *the root* of φ_i. We may assume without loss of generality that the roots are pairwise distinct, and the numbering of vertices and strings is chosen so that the root of φ_i is to the left of the root of φ_{i+1}. We say that v_i is *to the left* of v_j and v_j is *to the right* of v_i if $i < j$.

We say that a string φ_k is *nested below* a pair of strings φ_i, φ_j, if $i < k < j$, the strings φ_i and φ_j intersect, and the string φ_k does not intersect any of φ_i and φ_j. Let φ_i and φ_j be a pair of intersecting strings. We say that a robber is *confined below* the pair φ_i, φ_j by a given strategy if he is confined to vertices represented by strings that are nested below the pair φ_i, φ_j. The following lemma can be proved the same way as Lemma 2.1, and we omit its proof.

Lemma 3.1. *Suppose that in a given position of the cops and robber game, two cops are occupying a pair of intersecting strings φ_i and φ_j, while the robber occupies a string φ_k nested below the pair φ_i, φ_j. Then as long as the cops remain on φ_i and φ_j, the robber will remain confined below the pair φ_i and φ_j.* □

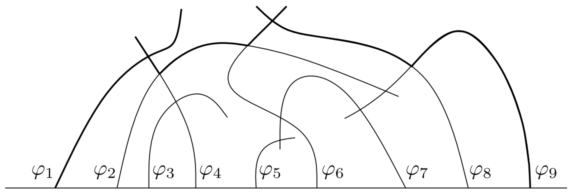

Fig. 3. An outer-string representation with the external boundary depicted in bold. A string φ_i represents the vertex v_i. The sequence of vertices v_1, v_4, v_2, v_9 forms an external path.

Let Φ be the set $\{\varphi_1, \ldots, \varphi_n\}$, and let $\mathbb{R}^2_+$ denote the closed upper half-plane of the Cartesian plane. The arc-connected components of $\mathbb{R}^2_+ \setminus \bigcup_{i=1}^n \varphi_i$ are the *faces* of Φ, and the unique unbounded face is the *external face*. The strings in Φ containing a point on the boundary of the external face are the *external strings* of Φ, and the points of these strings that appear on the boundary of the external face together form the *external boundary* of Φ. A vertex represented by an external string is an *external vertex*.

Since G is connected, the external boundary is arc-connected, and in particular, the external vertices induce a connected subgraph in G. Note also that the leftmost string φ_1 and the rightmost string φ_n are both external. An *external path* is any path in G from v_1 to v_n whose every vertex is external. See Fig. 3 for an example. The next lemma is an outer-string analogue of Lemma 2.2.

Lemma 3.2. *Let $p_0, p_1, \ldots, p_k$ be the sequence of vertices forming an external path in G, with $p_0 = v_1$ and $p_k = v_n$. Let $\pi_i \in \Phi$ denote the string representing p_i. Suppose that two cops are occupying two consecutive strings π_{i-1} and π_i of the external path, for some $i \in \{2, \ldots, k-1\}$, while the robber occupies a string φ_j which is to the right of both π_{i-1} and π_i and is disjoint from both these strings.*

Suppose that from this position, one cop moves from π_{i-1} to π_i while the other moves from π_i to π_{i+1}. The robber then performs an arbitrary move. In the ensuing position, the robber occupies a string which either intersects one of the strings occupied by the two cops, or is to the right of both the strings occupied by the cops, or is nested below the pair π_i, π_{i+1}. In particular, the robber cannot reach a string to the left of π_i without being captured immediately.

Proof. Suppose that after the two cops perform their move, the robber moves from φ_j to φ_k (possibly with $j = k$). If φ_k intersects π_i or π_{i+1}, we are done, so suppose this is not the case. If φ_k is to the left of π_i, then π_i is nested below the pair φ_k, φ_j (recall that by assumption, φ_j is to the right of π_i and disjoint from it). This is impossible since π_i is an external string, and therefore cannot be nested below a pair of other strings. We conclude that φ_k is to the right of π_i, and therefore it is either nested below π_i, π_{i+1}, or it is to the right of both π_i and π_{i+1}. $\square$

Strategies for outer-string graphs. We are ready to prove that the maximum cop number of outer-string graphs is equal to three or four.

Proof of Theorem 1.1(ii). To see that two cops are not enough for outer-string graphs, consider the 3-by-5 toroidal grid depicted in Fig. 4. On this graph, a robber may evade two cops forever by simply choosing an initial vertex that is not in the closed neighborhood of any cop, and then applying the following 'lazy' strategy: if the robber occupies a vertex v and there is no cop in $N[v]$, the robber stays on v. If there is a cop in $N[v]$, the robber moves to a vertex $w \in N(v)$ such that $N[w]$ contains no cop. One easily verifies that such a choice of w is always possible. We remark that it is easy to check that three cops can catch the robber on the 3-by-5 toroidal grid; see Luccio and Pagli [21].

We will now describe a winning strategy for four cops. Let G be a connected outer-string graph, with $v_1, \ldots, v_n$ and $\varphi_1, \ldots, \varphi_n$ having the same meaning as before. The strategy follows the same basic idea as the two-cop strategy for interval filament graphs, except that the four cops are now divided into two *guards* and two *hunters*. In the beginning of the game, all the cops occupy the leftmost

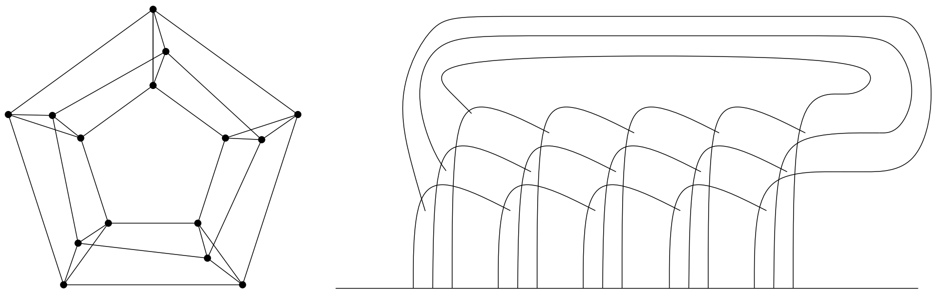

Fig. 4. The 3-by-5 toroidal grid and its outer-string representation.

string φ_1. The strategy is divided into phases. During each phase except the first one, the two guards occupy a pair of intersecting strings λ and ρ, confining the robber below them.

Suppose that the ith phase is about to begin. Let λ_i and ρ_i be the pair of intersecting strings occupied by the guards, confining the robber. The two guards will occupy these two strings throughout the ith phase. Let G_i be the subgraph induced by the strings nested below λ_i and ρ_i, and let C_i be the connected component of G_i containing the vertex with the robber. For $i = 1$, we let $G_1 = C_1 = G$, and we leave λ_1 and ρ_1 undefined. By Lemma 3.1, the robber is confined to C_i for the duration of the ith phase. Let P be an external path of C_i, and let the vertices of P be represented by strings $\pi_1, \pi_2, \ldots, \pi_k$, in the order in which they appear on P, with π_1 corresponding to the leftmost string of C_i and π_k the rightmost one. The two hunters will traverse P from π_1 to π_k in such a way that after each move, they will occupy a pair of intersecting strings π_i, π_{i+1}. By Lemma 3.2, eventually they will either catch the robber, or the robber will be confined below the two strings occupied by the hunters. When such confinement happens, the hunters stop moving, the guards move to the two strings occupied by the hunters, and a new phase begins.

Since there are only finitely pairs of intersecting strings nested in each other, the strategy proceeds in finitely many phases and the robber is captured. □

Similarly as in Section 2, it can be shown that the strategy requires at most a linear number of moves.

4. Catching the robber in string graphs

In this section, we show that the maximum cop number of string graphs is at most 15. Before we describe the strategy, we introduce several auxiliary lemmas. Our approach is inspired by a lemma of Aigner and Fromme [1] and its subsequent extension by Chiniforooshan [6].

Let G be a graph, let x and y be two distinct vertices, and let P be a shortest path from x to y in G. Aigner and Fromme [1] have shown that a single cop has a strategy which guarantees, after a finite number of moves, that the robber will not be able to enter any vertex of P. They have used this result to show that the cop-number of any planar graph G is at most three, by describing a strategy in which three cops, by occupying various shortest paths in G, can surround the robber and confine him to a steadily shrinking region in the planar drawing of G.

Our strategy for string graphs is inspired by the strategy for 3 cops in planar graphs. Consider a string graph G with a fixed string representation. As in the case of planar drawings, we may associate a path in G with a curve in the plane formed by the corresponding strings, and we want to use these curves to confine the robber to a certain region of the plane.

However, unlike in the case of planar drawings, in the string representation a robber may "cross" the curve formed by the strings of a path P without actually occupying a vertex of P: it is enough for the robber to occupy a vertex whose string crosses a string representing a vertex of P. Thus, our strategy should prevent the robber not only from occupying a vertex of a shortest path P, but also from occupying any vertex of $N[P]$. To this end, we use an idea of Chiniforooshan [6], who showed that five cops have a strategy which allows them, after a finite number of moves, to prevent the robber from entering any vertex of $N[P]$, where P is a shortest path. Hence, our strategy will require $3 \cdot 5 = 15$ cops.

Unfortunately, we are not able to use Chiniforooshan's result directly, since the paths we use in our argument are not necessarily shortest paths of the underlying graph. In fact, they may not be paths at all, but rather walks. However, these walks possess certain technical properties that allow us to apply a modified version of Chiniforooshan's idea on them.

Patrolling walks with no shortcuts. Let $G = (V, E)$ be a connected graph. A *walk of length k* in G is a sequence of vertices $v_0, v_1, \ldots, v_k$ such that any two consecutive vertices are connected by an edge. Let $W = v_0, v_1, \ldots, v_k$ be a walk of length k in a graph G.

Our strategy uses the notion of distance between a vertex of W and a vertex of $G - W$. However, we will not use the standard graph distance (i.e., the length of a shortest path). Instead we will define a distance (in fact, several notions of distance) by referring to the topological properties of a string representation. To avoid describing repeatedly the same ideas for different notions of distance, we first introduce an abstract notion of distance for which we prove the key properties to be used later.

Let $W = v_0, v_1, \ldots, v_k$ be a walk in G, and let A be a set of vertices in G. A *distance function* for W and A is a function $\mathrm{d} : \{0, 1, \ldots, k\} \times A \to \mathbb{N}_0 \cup \{\infty\}$ such that if $\mathrm{d}(i, r) \leq 1$ for some i and r, then $r \in N[v_i]$.

Intuitively, the set A will denote the vertices available to the robber in a given position of the cops-and-robber game, and $\mathrm{d}(i, r)$ will correspond to the number of moves the robber must make to move from r to v_i given a particular strategy of the cops. Note however, that with our notion of distance function, even when v_i and v_j represent the same vertex of G, we may have $\mathrm{d}(i, r) \neq \mathrm{d}(j, r)$ for some $r \in A$.

We say that the walk W admits a *shortcut* through a vertex $r \in A$ with respect to a distance function d, if there are two distinct indices i and j with the property

$$\mathrm{d}(i, r) + \mathrm{d}(j, r) < |i - j|.$$

For a vertex $r \in A$, we say that r is a *near vertex* (with respect to d) if $\mathrm{d}(i, r) \leq 1$ for at least one $i \in \{0, \ldots, k\}$. Otherwise we say that r is a *distant vertex*. We let $N_{\mathrm{d}}(W)$ and $D_{\mathrm{d}}(W)$ denote the sets of near vertices and distant vertices, respectively. Note that $N_{\mathrm{d}}(W)$ and $D_{\mathrm{d}}(W)$ form a partition of A.

We will describe a general strategy for five cops to guard the set $N_{\mathrm{d}}(W)$ for a given walk with no shortcut, assuming suitable initial conditions. We call one of the cops *the sheriff*, and the remaining four are *the deputies*. For a walk $W = v_0, \ldots, v_k$, we say that in a given position of the game the five cops are *patrolling* the walk W, if these conditions are satisfied:

- The sheriff is on a vertex $v_s \in W$, with $0 \leq s \leq k$, and the deputies are on the four adjacent vertices $v_{s-2}, v_{s-1}, v_{s+1}, v_{s+2}$, where we use the convention $v_{-2} = v_{-1} = v_0$ and $v_{k+2} = v_{k+1} = v_k$.
- The robber is on a vertex $r \in D_{\mathrm{d}}(W)$.
- For every index $i \in \{0, \ldots, k\}$, we have $\mathrm{d}(i, r) \geq |s - i|$.

Lemma 4.1. *Let $W = v_0, \ldots, v_k$ be a walk, let A be a set of vertices, and let* d *be a distance function for W and A, satisfying the following properties:*

(1) *If $r \in D_{\mathrm{d}}(W)$ and $r' \in N_{\mathrm{d}}(W)$ are adjacent vertices, then there is an index i such that* $\mathrm{d}(i, r) = 2$ *and* $\mathrm{d}(i, r') = 1$.
(2) *For any two adjacent vertices r and r' in $D_{\mathrm{d}}(W)$ and any index i, we have* $|\mathrm{d}(i, r) - \mathrm{d}(i, r')| \leq 1$.
(3) *W admits no shortcut through any distant vertex.*

Suppose that five cops are patrolling W, and it is the robber's turn to move. Then the cops have a strategy ensuring that, as long as the robber only uses vertices from A, if the robber ever moves to a near vertex, he will be captured immediately by the cops, and moreover, until the robber is captured, the cops will be patrolling W before every future robber move.

Proof. Let v_s be the vertex occupied by the sheriff in the given position, and let r be the vertex occupied by the robber. Recall that r is a distant vertex, by definition of patrolling. Suppose that the robber moves from r to r'. If $r = r'$, i.e., if the robber stands still, the cops stand still as well and they continue to patrol W. Suppose that $r \neq r'$, and r' is in $N_{\mathrm{d}}(W)$. By assumption, there is an index i such that $\mathrm{d}(i, r) = 2$ and $\mathrm{d}(i, r') = 1$. In particular, $|i - s| \leq 2$ and therefore v_i is occupied by a cop, who is now able to capture the robber on r'.

Now, suppose $r \neq r'$ and $r' \in D_{\mathrm{d}}(W)$. If after the robber's move the cops still patrol the walk, then they do nothing. Suppose that this is not the case, i.e., there is a vertex $v_i \in W$ such that $\mathrm{d}(i, r') < |s-i|$. From our assumptions on d, we know $\mathrm{d}(i, r') \geq \mathrm{d}(i, r) - 1$, and therefore $\mathrm{d}(i, r') = |s-i| - 1$. Suppose, without loss of generality, that $i < s$. Then for every $j > s$ we must have $\mathrm{d}(j, r') \geq |s - j| + 1$, otherwise W would have a shortcut from v_i to v_j through r'. The sheriff now moves from v_s to v_{s-1}, and the deputies move accordingly, so that after their move they again patrol W. □

To use the strategy of Lemma 4.1, the cops must be able to first reach a position in which they patrol the corresponding walk. The next lemma addresses this task.

Lemma 4.2. *Let $W = v_0, \ldots, v_k$ be a walk, A a set of vertices, and* d *a distance function for A and W, with these properties:*

1. *For any two adjacent vertices r and r' in A and any index i, we have $|\mathrm{d}(i, r) - \mathrm{d}(i, r')| \leq 1$.*
2. *W admits no shortcut through any vertex in A.*

There is a strategy for five cops with a finite number of steps which allows, as long as the robber only moves in A, to either catch the robber or reach a position in which the five cops patrol W, and it is the robber's turn to move.

Proof. In the beginning, the cops occupy the first five vertices $v_0, \ldots, v_4$ of W, with the sheriff on v_2. Then, before each cops' move, the cops check whether they are already patrolling W. If they are, they stand still, otherwise each cop moves from his current position v_i to v_{i+1}. We will show that with this approach, the cops will eventually catch the robber or reach a position in which they patrol W.

In a position when the sheriff occupies a vertex v_s and the robber occupies a vertex r, define *the score* of a vertex v_i to be the difference $\mathrm{d}(i, r) - |s-i|$. Recall that the cops patrol W if in a given position all vertices of W have nonnegative score, and the robber is on a distant vertex. Observe that in every move, the score of a vertex changes by at most one.

We claim that at some point, all vertices will have nonnegative score after the cops' move. Suppose that this is not the case. Then at some point, the game reaches the position when there is an index $i < s$ such that v_i has negative score. Suppose that this first happens after t moves of the game (the cops' move and a robber move both count as one separate move). If $t = 0$, i.e., the above situation already occurs in the initial position, then the robber is at a distance at most 1 from a vertex occupied by one of the deputies, and he is captured immediately. Assume therefore that $t > 0$. We claim that after $t - 1$ moves, all vertices had a nonnegative score. Suppose they did not. Then there must have been an index $j > s$ such that v_j had a negative score. It follows that after t moves, v_j had a nonpositive score, and W has a shortcut from v_i to v_j through r. This is a contradiction.

It remains to show that after all vertices have nonnegative score, the robber occupies a distant vertex. Suppose the robber occupies a near vertex r, with $\mathrm{d}(i, r) = 1$. Since v_i has nonnegative score, the sheriff occupies one of the vertices v_{i-1}, v_i, v_{i+1}. If it is the cops' turn, they can immediately capture the robber. If it is the robber's turn, then the cops could have captured the robber in their preceding move, as before this move some cop occupied v_i. □

Segments, faces and regions. We now translate the cop strategies introduced above to a more geometric setting which will be useful in the context of string graphs. For the rest of this section, let $G = (V, E)$ be a string graph with a fixed string representation where a vertex $v \in V$ is represented by a string φ_v. Let $\Phi = \{\varphi_v;\ v \in V\}$ be the set of strings in the representation. Recall that Φ is the set of strings in a string representation of G. The topological arc-connected components of $\mathbb{R}^2 \setminus \bigcup_{\varphi \in \Phi} \varphi$ are called *faces* and their topological closures are the *closed faces*; every face is an open set.

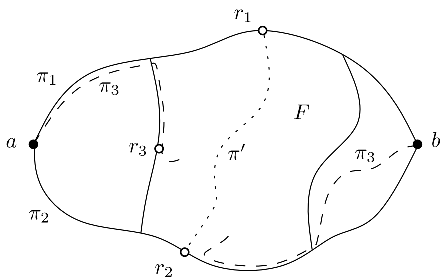

Fig. 5. Illustration of the proof of Lemma 4.3: there cannot be both a point r_1 private for π_1 and a point r_2 private for π_2 on the boundary of F.

A *segment* of a string $\varphi \in \Phi$ is a maximal arc-connected subset of φ not containing any intersection with another string in Φ. A *region* of Φ is a closed subset of $\mathbb{R}^2$ obtained as the closure of a union of some of the faces. We say that a curve $\gamma \subseteq \mathbb{R}^2$ is *supported by Φ if γ is a subset of* $\bigcup_{\varphi\in\Phi}\varphi$. Recall that we assume that the number of intersections of Φ is finite, therefore the number of faces and the number of segments is finite as well. For $X \subseteq \mathbb{R}^2$, let $\overline{X}$ denote the topological closure of X, $\operatorname{int}(X)$ the topological interior of X, and ∂X the boundary of X; in particular, $\partial X = \overline{X} \setminus \operatorname{int}(X)$. We say that a vertex v of G is *contained in* X if $\varphi_v \subseteq \operatorname{int}(X)$. We let G_X denote the subgraph of G induced by the vertices contained in X.

Two curves π_1 and π_2 are said to be *internally disjoint* if every point in $\pi_1 \cap \pi_2$ is a common endpoint of the two curves. We say that two simple curves π_1 and π_2 are *bounding* a set $R \subseteq \mathbb{R}^2$ if π_1 and π_2 are internally disjoint, share the same pair of endpoints, and R has boundary $\pi_1 \cup \pi_2$. In such case, if additionally π_1 and π_2 are supported by Φ and R is a closed set, then R is a region of Φ. Note that by Jordan's theorem, a pair of simple internally disjoint curves with common endpoints is bounding two closed sets, one of which is bounded (the *internal region*) and the other unbounded (the *external region*).

Lemma 4.3. *Let π_1 and π_2 be two internally disjoint simple curves supported by Φ, with two distinct common endpoints a and b. Suppose π_1 and π_2 are bounding a region R. Let $\pi_3 \subseteq R$ be a simple curve from a to b supported by Φ, containing at least one interior point of R. Then every arc-connected component F of $R \setminus (\pi_1 \cup \pi_2 \cup \pi_3)$ is bounded by two simple internally disjoint curves π_i' and π_3', where $\pi_i' \subseteq \pi_i$ for some $i \in \{1, 2\}$, and $\pi_3' \subseteq \pi_3$.*

Proof. Without loss of generality, we may assume that R is the internal region bounded by π_1 and π_2, otherwise we can apply circular inversion. Note that F is an open arc-connected set, and that ∂F is a simple closed Jordan curve. We say that a point $p \in \partial F$ is *private to* π_i, if p belongs to π_i but does not belong to π_j for any $j \in \{1, 2, 3\} \setminus \{i\}$. Note that π_3 has at least one private point $r_3 \in \partial F$, otherwise we would have $\partial F = \pi_1 \cup \pi_2$, so $\overline{F} = R$, which contradicts that π_3 intersects $\operatorname{int}(R)$.

We claim that at most one of the two curves π_1 and π_2 may have a private point on ∂F. Suppose for contradiction that there exist both $r_1 \in \partial F \setminus (\pi_2 \cup \pi_3)$ and $r_2 \in \partial F \setminus (\pi_1 \cup \pi_3)$. In such case, there is a curve $\pi' \subseteq (F \cup \{r_1, r_2\})$ from r_1 to r_2 separating a from b in R, not intersecting π_3. By Jordan curve theorem, this contradicts that π_3 is a curve from a to b through F; see Fig. 5. On the other hand, at least one of π_1 and π_2 must have a private point on ∂F, otherwise ∂F would be a subset of π_3, contradicting the fact that π_3 is a simple curve. Without loss of generality, we assume that r_1 is a private point of π_1, r_3 is a private point of π_3, and π_2 has no private point on ∂F. In particular, ∂F is a subset of $\pi_1 \cup \pi_3$.

Define the set π_3' as $\partial F \cap \pi_3$. Observe that if π_3' were empty or if it contained just a single point, it would contradict the fact that π_1 is a simple curve. We now show that π_3' is a connected set. Suppose that this is not the case. Then π_3' contains a pair of points p, q such that both connected components of $\partial F \setminus \{p, q\}$ contain a point not belonging to π_3. Let γ_1 and γ_2 be the two components of $\partial F \setminus \{p, q\}$, and let s_1 and s_2 be two points satisfying $s_1 \in \gamma_1 \setminus \pi_3$ and $s_2 \in \gamma_2 \setminus \pi_3$. Then there is a curve $\pi'' \subseteq (F \cup \{s_1, s_2\})$

from s_1 to s_2 separating p from q in R and not intersecting π_3, which is impossible, since p and q both belong to the curve π_3, which is connected. Therefore π_3' is connected, i.e., it is a subcurve of π_3. Let c and d be the endpoints of π_3'. Define $\pi_1' = (\partial F \setminus \pi_3') \cup \{c, d\}$. We see that π_1' is a curve with endpoints c, d. Since no internal point of π_1' belongs to π_3, we have $\pi_1' \subseteq \pi_1$. The lemma follows. □

Patrolling curves. Let $X = x_0, x_1, \ldots, x_k$ be a walk of length k in G. We say that a curve α *represents* the walk X, if α is the concatenation of $k+1$ curves $\alpha_0, \alpha_1, \ldots, \alpha_k$, where α_i is a subcurve of the string representing the vertex x_i, and moreover, for every $i < k$, the curves α_i and α_{i+1} intersect in exactly one point, which is their common endpoint. We call the subcurves $\alpha_0, \ldots, \alpha_k$ *the parts* of the representation α. Note that the two endpoints of α can be any two points belonging to φ_{x_0} and φ_{x_k}, respectively. Note also that we allow the possibility that a part α_i consists of a single point.

The curve α representing a walk X need not be a simple curve, i.e., it may contain self-intersections. We say that α is a *simple representation* of X if α is a simple curve representing X. Note that some walks may not admit any simple representation, while others may admit multiple simple representations, even when the two endpoints of the representing curve are prescribed. Let X and Y be a pair of walks represented by simple curves α and β bounding a region R. We say that a curve γ is *attached* to α in R, if γ has exactly one endpoint on α and all the other points of γ are in the interior of R. The point of $\alpha \cap \gamma$ is then *the attachment* of γ to α.

Recall that φ_z is the string representing a vertex z. We say that a vertex z is attached to α in R if φ_z has a subcurve of positive length attached to α. We let $\mathrm{Att}_R(\alpha)$ denote the set of vertices attached to α in R, and $\mathrm{Att}_R(\alpha \cup \beta)$ the set of vertices attached to α or β. Observe that if z is attached to α, then z is adjacent to a vertex of X or z itself belongs to X. On the other hand, not every vertex adjacent to X is necessarily attached to α; indeed, a vertex adjacent to X may well be represented by a string contained entirely in the interior of R; see, e.g., the string φ_s in Fig. 6.

Let $X = x_0, \ldots, x_k$ be again a walk with simple representation α, and let $\alpha_0, \alpha_1, \ldots, \alpha_k$ be the parts of α. Let Y, β and R be as above. A curve γ is *internal* to R, if all the points of γ are in the interior of R, except possibly its endpoints.

Recall that G_R is the subgraph of G induced by the vertices whose strings are contained in the interior of R. Let A be the set of vertices whose strings intersect the interior of the region R. Note that each $z \in A$ is either attached to α or β, or belongs to G_R.

For a vertex $z \in A$ and part α_i of α, an *R-internal walk from z to α_i* is a walk in G from z to x_i that admits a simple representation γ which is internal to R, and has the property that one endpoint of γ belongs to α_i, and the other to φ_z. For a vertex $z \in A$, let $\mathrm{d}_R^\alpha(i, z)$ denote the length of the shortest R-internal walk from z to α_i, and write $\mathrm{d}_R^\alpha(i, z) = \infty$ if no such walk exists. Similarly, $\mathrm{d}_R^\beta(i, z)$ is the length of the shortest R-internal walk from z to the ith part of β, or ∞ if no such walk exists. Notice that d_R^α and d_R^β are distance functions with respect to the walks X and Y and the set A. Notice also that any vertex attached to α is a near vertex with respect to the distance function d_R^α; however, there might be near vertices for d_R^α not attached to α, such as the vertex s on Fig. 6.

Lemma 4.4. *Let X and Y be a pair of walks represented by simple curves α and β bounding a region R. Suppose that X admits no shortcut through any vertex of G_R with respect to the distance function d_R^α, and similarly for Y and d_R^β. Suppose that on each of the two walks there is a team of five cops patrolling the walk with respect to the corresponding distance function d_R^α or d_R^β.*

In this position, the ten cops have a strategy ensuring that the robber is captured immediately after taking any vertex in $\mathrm{Att}_R(\alpha \cup \beta)$, and until the robber is captured, each of the two groups of five cops will patrol their walk before every robber's move. In particular, the cops' strategy confines the robber to the subgraph G_R.

Proof. Let us write d for d_R^α and d' for d_R^β. Each of the two groups of cops pursues independently the strategy of Lemma 4.1. However, to ensure that this approach succeeds, we cannot invoke Lemma 4.1 directly, since the two distance functions may fail to satisfy the first and the second property in the statement of Lemma 4.1. For instance, it might happen that there is a vertex $r' \in \mathrm{Att}_R(\alpha)$, and a vertex $r \in D_{\mathrm{d}}(X)$ adjacent to r', but there is no R-internal walk from r to any part of α. See, e.g., Fig. 6, where r, despite being adjacent to the vertex x_1, has no R-internal walk to any part of α. However, we will

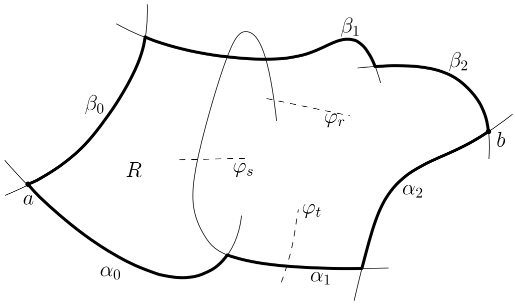

Fig. 6. A region R bounded by a pair of curves $\alpha = \alpha_0 \cup \alpha_1 \cup \alpha_2$ and $\beta = \beta_0 \cup \beta_1 \cup \beta_2$. Let $X = x_0, x_1, x_2$ be the walk represented by α. Let r, s and t be vertices represented by the strings φ_r, φ_s and φ_t. Then $\mathrm{Att}_R(\alpha) = \{x_0, x_1, t\}$. Note that $\mathrm{d}^\alpha_R(1, t) = \mathrm{d}^\alpha_R(1, s) = 1$ while $\mathrm{d}^\alpha_R(1, r) = \infty$.

show that in such case there is an R-internal walk of length at most 2 from r to a part of β, so the cops patrolling Y will catch the robber anyway.

Suppose that the robber moves from a vertex $r \in G_R$ to a vertex $r' \in \mathrm{Att}_R(\alpha \cup \beta)$. Let p be a point where the string φ_r intersects the string $\varphi_{r'}$. Let q be a point where $\varphi_{r'}$ intersects $\alpha \cup \beta$, chosen so that the subcurve of $\varphi_{r'}$ joining p and q has no other intersections with $\alpha \cup \beta$. Suppose, without loss of generality, that q belongs to a part α_i of α. Then either x_i is adjacent to r' or $x_i = r'$, and in any case, we have $\mathrm{d}^\alpha_R(i, r) \le 2$. Since there were five cops patrolling X before the robber's move, by the definition of patrolling, there was a cop on x_i, and this cop can now capture the robber on r'.

If the robber moves from a vertex $r \in G_R$ to a vertex $r' \in G_R$ (possibly with $r = r'$), each team of five cops will follow the strategy of Lemma 4.1. Since r and r' are in G_R, the corresponding strings φ_r and $\varphi_{r'}$ do not intersect $\alpha \cup \beta$, and we easily see that for any part α_i of α, any R-internal walk from r to α_i of length k can be modified into an R-internal walk from r' to α_i of length at most $k + 1$. The same applies to parts of β as well. In particular, we have $|\mathrm{d}(i, r) - \mathrm{d}(i, r')| \le 1$, and similarly for d', that is, the two vertices r and r' satisfy the assumptions of Lemma 4.1. Therefore, the strategy of Lemma 4.1 guarantees that the two groups of cops will continue to patrol their walks after their move.

Since the robber can never take a vertex of $\mathrm{Att}_R(\alpha \cup \beta)$, he is confined to G_R. □

We remark that the argument in the proof of Lemma 4.4 can actually establish a slightly stronger conclusion, namely that the cops are able to guard the set $N_{\mathrm{d}}(X) \cup N_{\mathrm{d}'}(Y)$, as opposed to just guarding the subset $\mathrm{Att}_R(\alpha \cup \beta)$. However, we will not need this extra strength.

Informally speaking, in the strategy of Lemma 4.4, if for a string $\varphi \in \Phi$ the intersection $\varphi \cap R$ is disconnected, then every connected component of the intersection is treated as a different string by the cops. The robber cannot exploit the fact that all the components of $\varphi \cap R$ belong to a single string, because he is captured immediately after taking this string.

Proof of Theorem 1.1(iii). We are ready to prove that the maximum cop number of string graphs is at least 3 and at most 15. The lower bound of 3 cops follows from the graph in Fig. 4. In the rest of the section, we argue that there exists a strategy using 15 cops. Our strategy proceeds in phases, monotonously shrinking the confinement of the robber. In the beginning of each phase, the robber is confined to a set $D \subseteq V$ either (A) by a single cop guarding a vertex whose neighborhood separates D from the rest of the graph, or (B) by ten cops patrolling two curves bounding a region containing D. The core of our argument is formulated in the following Lemma.

Lemma 4.5. *Let $G = (V, E)$ be a connected string graph with representation Φ. Let D be a set of vertices inducing a connected subgraph of G, and suppose that the robber occupies one of the vertices in D. Suppose further that the robber is confined to the set D by one of the following two situations:*

(A) *There is a cop occupying a vertex $x \in V$ and D forms a connected component of $G - N[x]$.*

(B) *G contains two walks X and Y represented respectively by two internally disjoint simple curves α and β bounding a region R, with a group of five cops on each of the two walks, satisfying the assumptions of Lemma 4.4. D is the connected component of G_R containing the vertex occupied by the robber.*

From such a position, 15 cops have a strategy to capture the robber, with the robber being confined to D throughout the strategy.

Proof. We prove the claim by induction on $|D|$ and the number of segments in the interior of the region R. More precisely, we will show that the case (A) can be reduced either to another position of type (A) with a strictly smaller set D, or a position of type (B) with the same or smaller set D, while any position of type (B) can be reduced to another position of type (B) with a strictly smaller region R (in sense of number of segments) and the same or smaller set D, or a position of type (A) with a strictly smaller set D. The lemma then follows by induction as its statement is obviously true when $|D| \leq 1$. For the other cases, we treat the two situations (A) and (B) separately.

Case (A). Without loss of generality, we may assume that V is the disjoint union of $N[x]$ and D. Otherwise we could apply induction to the subgraph of G induced by $N[x] \cup D$, observing that any strategy obtained from the lemma on this subgraph would work for the whole graph G as well. For the same reason, we may assume that any vertex in $N(x)$ has a neighbor in D, otherwise the vertex might be removed.

Suppose first that the vertex x has a neighbor y such that $N[x]$ is a subset of $N[y]$. Since y has a neighbor in D by the argument of the previous paragraph, we know that $N[x]$ is a proper subset of $N[y]$. We may now place a cop on y, and define D' to be the connected component of $D \setminus N[y]$ containing the robber's vertex. We are then back in case (A), with y taking the role of x, and D' the role of D.

Suppose now that x has no such neighbor y; in particular, x has at least two neighbors. Let α be a minimal subcurve of φ_x with the property that for any $y \in N(x)$, the string φ_y intersects α. Let a and b be the endpoints of α. By minimality of α, we know that a and b are the respective intersection points of φ_x with two distinct strings φ_y and φ_z, for some $y, z \in N(x)$. The minimality of α also implies that neither φ_y nor φ_z has any other intersection with α. Let X be the walk consisting of the single vertex x; clearly α is a representation of X.

Let Y be a shortest walk that can be represented by a simple curve β with endpoints a and b which is internally disjoint from α. Note that at least one such walk exists: since both y and z are adjacent to D, and since D is connected, y and z can be connected by an induced path (possibly a single edge) whose every internal vertex is in D. Such a path can be represented by a curve β with the required properties.

Our goal is to reach a position described in case (B) of the lemma, with α and β being as defined above, and R being one of the two regions bounded by $\alpha \cup \beta$. For this, it remains to describe a strategy for the cops to start patrolling the walks X and Y. First we add four more cops to x, so that the team of five cops is trivially patrolling the trivial walk X. Next, we must position the cops on Y so that they can confine the robber to a region R bounded by $\alpha \cup \beta$ and patrol the walk Y.

Let $y_0, y_1, \ldots, y_k$ be the walk Y, and let $\beta_0, \ldots, \beta_k$ be the corresponding parts of β. For a vertex $r \in D$, define $\mathrm{d}(i, r)$ to be the length of the shortest walk from r to y_i that can be represented by a simple curve that has one endpoint in φ_r, the other endpoint in β_i, and which is internally disjoint with α. We put $\mathrm{d}(i, r) = \infty$ if no such walk exists. Note that the walks we consider in the definition of $\mathrm{d}(i, r)$ may contain vertices from $N[x]$ and hence not belonging to D. On the other hand, we only define the function $\mathrm{d}(i, r)$ for vertices r of D, since we know that when the vertex x is occupied by a cop, the robber cannot leave D. Observe that d is a distance function for the walk Y and the set D.

We will show that d satisfies the two assumptions of Lemma 4.2, with the set D playing the role of A in the statement of Lemma 4.2, and Y playing the role of W. To verify the first assumption, choose any two adjacent vertices $r, r' \in D$ and a part β_i of β. We claim that $|\mathrm{d}(i, r) - \mathrm{d}(i, r')| \leq 1$. Without loss of generality, we may assume that $\mathrm{d}(i, r) \leq \mathrm{d}(i, r')$, and we show that $\mathrm{d}(i, r') \leq \mathrm{d}(i, r) + 1$. By definition of $\mathrm{d}(i, r)$, there is a simple curve γ internally disjoint from α connecting a point of φ_r to a point of β_i, representing a walk of length $\mathrm{d}(i, r)$. Since r and r' are in D, both φ_r and $\varphi_{r'}$ are disjoint from α. Let γ' be a simple curve starting in a point of $\varphi_r \cap \varphi_{r'}$, following along φ_r towards a point of γ, then following γ towards a point of β_i. Then γ' represents a walk of length at most $\mathrm{d}(i, r) + 1$, showing that $\mathrm{d}(i, r') \leq \mathrm{d}(i, r) + 1$. This verifies the first assumption of Lemma 4.2.

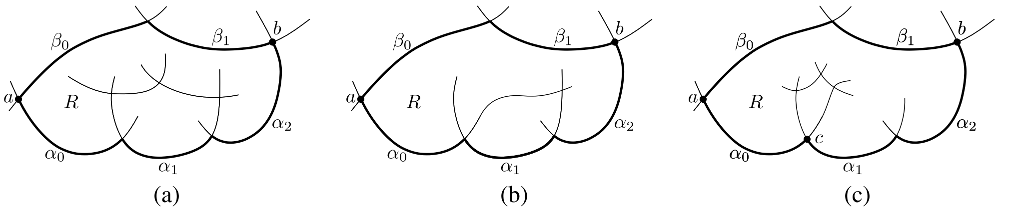

Fig. 7. Three examples of a possible situation arising in case (B) of Lemma 4.5. A region R is bounded by a pair of curves $\alpha = \alpha_0 \cup \alpha_1 \cup \alpha_2$ and $\beta = \beta_0 \cup \beta_1$. The goal is to find, if possible, a simple curve γ from a to b in R, with at least one point in int(R), representing a shortest possible walk. In example a), the only such curve will not represent a path but rather a general walk. In example b), the only such curve will represent the same walk as α. In example c), no such curve γ exists.

We now show that Y has no shortcut with respect to d through any vertex $r \in D$. Suppose there is such a shortcut from β_i to β_j through $r \in D$, with $i < j$. Let γ and γ' be two simple curves internally disjoint with α, connecting φ_r to β_i and β_j, and representing walks of respective lengths $\mathrm{d}(i, r)$ and $\mathrm{d}(j, r)$. It follows that the set $\gamma \cup \varphi_r \cup \gamma'$ contains a simple curve δ connecting a point $p \in \beta_0 \cup \beta_1 \cup \cdots \cup \beta_i$ to a point $q \in \beta_j \cup \beta_{j+1} \cup \cdots \cup \beta_k$, with δ representing a walk of length at most $\mathrm{d}(i, r) + \mathrm{d}(j, r)$. We choose p and q in order to make δ as short as possible. By replacing the subcurve of β between p and q with the curve δ, we obtain a simple curve from a to b internally disjoint with α and representing a shorter walk than Y, contradicting the choice of Y. We conclude that d satisfies the assumptions of Lemma 4.2.

By Lemma 4.2, we may position five cops to patrol Y. Let R be the region bounded by $\alpha \cup \beta$ containing the robber's vertex after the cops start patrolling Y. Let D' be the connected component of G_R containing the robber's vertex. Observe that $D' \subseteq D$. Moreover, defining $\mathrm{d}_R^{\beta}(i, r)$ as in Lemma 4.4, we observe that $\mathrm{d}_R^{\beta}(i, r) \geq \mathrm{d}(i, r)$ for any $r \in G_R$, and in particular, Y has no shortcut through a vertex in G_R with respect to d_R^{β}. We have thus reduced the position to case (B), with D' taking the role of D.

Case (B). We may assume that the vertex set of G is the union of $X \cup Y \cup \mathrm{Att}_R(\alpha \cup \beta) \cup D$. We may also assume that every vertex y in $\mathrm{Att}_R(\alpha \cup \beta)$ either belongs to $X \cup Y$ or is adjacent to a vertex in D. Otherwise, such a vertex y could be removed, and the strategy obtained from induction on the graph $G - y$ would also work in G.

Let us first suppose that G has a walk $Z = z_0, z_1, \ldots, z_k$ which is represented by a simple curve $\gamma \subseteq R$ with endpoints a and b and with the property that at least one point of γ is in the interior of R. Let us fix Z to be the shortest such walk, and γ its corresponding simple representation satisfying the properties above. Note that the walk Z is not necessarily a path; see Fig. 7a. Also, we do not necessarily assume that Z contains a vertex not belonging to $X \cup Y$. In fact, Z might for instance be identical to X, with γ and α being merely different representations of X; see Fig. 7b.

Let $\gamma_0, \gamma_1, \ldots, \gamma_k$ be the parts of γ. For an index $i \in \{0, \ldots, k\}$ and a vertex $r \in D$, define the function $\mathrm{d}(i, r)$ to be the length of the shortest walk from r to z_i that can be represented by a simple curve π internal to R that connects a point of φ_r to a point of γ_i. We observe that d is a distance function for the walk Z and set D. By the same arguments as in the proof of case (A), we note that this distance function satisfies the assumptions of Lemma 4.2. Applying Lemma 4.2 for Z, D and d, we can position the five remaining cops to patrol the walk Z.

Let F be the arc-connected component of $\mathbb{R}^2 \setminus (\alpha \cup \beta \cup \gamma)$ that contains the vertex of the robber after the cops start patrolling Z. By Lemma 4.3, F is bounded by two curves σ and τ, where σ is a subcurve of γ, and τ is a subcurve of α or of β. Suppose without loss of generality that τ is a subcurve of β. Let R' be the closure of F, and let D' be the vertex set of the connected component of $G_{R'}$ that contains the vertex occupied by the robber. The five cops patrolling β with respect to the distance function d_R^{β} are also patrolling τ with respect to $\mathrm{d}_{R'}^{\tau}$, since d_R^{β} is a lower bound for $\mathrm{d}_{R'}^{\tau}$. This also shows that τ has no shortcuts through D' with respect to $\mathrm{d}_{R'}^{\tau}$, since any such shortcut would also be a shortcut for β with respect to d_R^{β}. Similarly, σ has no shortcuts through $G_{R'}$ in $\mathrm{d}_{R'}^{\sigma}$, and the five cops patrolling γ are also patrolling σ. The five cops patrolling X are no longer needed to confine the robber. We have thus reached the situation of case (B), with the role of R being played by the strictly smaller region R', and the role of D by the subset D'.

It remains to deal with a situation where there is no walk Z admitting a simple representation by a curve in R with at least one point in the interior of R. Recall that D is a connected component of G_R, and that every vertex in $\mathrm{Att}_R(\alpha \cup \beta)$ not belonging to $X \cup Y$ is adjacent to a vertex of D. Since there is no simple curve from a to b containing at least one interior point of R, we see that $\alpha \cup \beta$ must contain a single point c with the property that any R-internal path from a vertex in D to a vertex in $\alpha \cup \beta$ is represented by a curve whose attachment is c; see Fig. 7c.

Two possibilities may occur: if the point c is in the interior of one of the parts of α or β, then there is a vertex z whose string φ_z is attached to α or β in the point c, and every path from D to $X \cup Y$ in G contains the vertex z. We then place one cop on z, where he starts guarding $N[z]$, and define D' to be the connected component of $G - N[z]$ containing the vertex with the robber. Observe that D' is a proper subset of D, since at least one vertex of D must be adjacent to z. The cops patrolling α and β are then no longer needed, and can be dismissed. This reduces the situation to case (A) with D' taking the role of D, and we can apply induction.

Suppose now that the point c is a common endpoint of two adjacent parts of $\alpha \cup \beta$; this situation is depicted in Fig. 7c. Note that this includes the possibility that c is equal to a or to b. Let x and y be the two vertices of $X \cup Y$ whose strings meet in c. In particular, any path connecting a vertex in D with a vertex out of D must contain at least one of x and y. Note that at least one of x and y must have a neighbor in D, otherwise G would not be connected. Suppose without loss of generality that x has a neighbor z in D. We then place one cop on x and let D' be the connected component of $G - N[x]$ containing the robber's vertex. D' is a proper subset of D (since D' does not contain z), and we have obtained an instance of case (A) where the role of D is played by the strictly smaller set D'. This proves the lemma. □

Theorem 1.1(iii) is then a direct consequence of the previous lemma: given a connected string graph G, we can place one cop on any vertex $x \in V$ and let D be the connected component of $G - N[x]$ containing the robber. This yields a position corresponding to case (A) of the lemma, and therefore 15 cops have a winning strategy from this position.

5. Catching the robber in string graphs on bounded genus surfaces

In this section, we generalize the results of the previous section to graphs having a string representation on a fixed surface, and we prove Theorem 1.1(iv).

Definitions. We assume familiarity with basic topological concepts related to curves on surfaces, such as genus, Euler genus, non-contractible closed curves, the fundamental group of surfaces and graph embeddings. A suitable treatment of these notions can be found in [25,30].

Recall that a walk of length k in a graph G is a sequence $W = w_0, w_1, \dots w_k$ of vertices where w_i and w_{i+1} adjacent. A walk is called *closed* if $w_0 = w_k$. Let $|W|$ denote the length k of W. For walks $W = w_0, w_1, \dots, w_k$ and $W' = w'_0, w'_1, \dots, w'_\ell$ with $w_k = w'_0$, we let $W + W'$ denote their concatenation $w_0, w_1, \dots, w_k, w'_1, w'_2, \dots, w'_\ell$. Let $-W$ be the reversal $w_k, w_{k-1}, \dots, w_0$ of W, and let us write $W_1 - W_2$ for $W_1 + (-W_2)$.

So far, we considered a curve to be the image of a continuous function $f : [0, 1] \to R^2$, where f is known as the *parametrization* of the curve. As in this section, we need to work with oriented curves, we use a slightly different, but equivalent approach: henceforth, a curve π itself is a continuous function from the interval $[0, 1]$ to the surface. The curve π is *closed* if $\pi(0) = \pi(1)$. The concatenation of curves $\pi_1 + \pi_2$ is defined naturally whenever $\pi_1(1) = \pi_2(0)$, and similarly $-\pi$ is the reversal and $\pi_1 - \pi_2 = \pi_1 + (-\pi_2)$. We use the following topological lemma, following from the properties of the fundamental group; see [30].

Lemma 5.1 ([30]). *Let π_1, π_2 and π_3 be three curves on a surface* **S** *from a to b. If the closed curve $\pi_1 - \pi_2$ is non-contractible, then at least one of $\pi_1 - \pi_3$ and $\pi_2 - \pi_3$ is non-contractible.*

Consider a string representation of G on a surface $\mathbf{S}$, where φ_v is the string representing a vertex v and Φ is the set $\{\varphi_v;\ v \in V\}$. We represent the combinatorial structure of Φ by an auxiliary multigraph $A(\Phi)$ embedded on $\mathbf{S}$ defined as follows. The vertices of $A(\Phi)$ are the endpoints of the strings of Φ and the intersection points of pairs of strings of Φ. The edges of $A(\Phi)$ correspond to segments of strings of Φ, i.e., to subcurves connecting pairs of vertices appearing consecutively on a string of Φ. By representing Φ by the graph $A(\Phi)$, we can use the well-developed theory of graph embeddings on surfaces.

Walks with non-contractible representations. As in the previous section, we say that a walk $W = w_0, w_1, \dots w_k$ in G is *represented* by a curve $\pi \subseteq \mathbf{S}$ on the surface $\mathbf{S}$ if π can be partitioned into a sequence of consecutive subcurves $\pi_0, \pi_1, \dots, \pi_k$ such that $\pi = \sum_{i=0}^{k} \pi_i$ and $\pi_i \subseteq \varphi_{w_i}$. A closed walk W *has a non-contractible representation* if there is a non-contractible closed curve π representing W.

Lemma 5.2. *Let Φ be a string representation of G on an orientable (resp. non-orientable) surface $\mathbf{S}$ of genus $g > 0$ (resp. Euler genus $g' > 0$) and let W be a closed walk in G with non-contractible representation. Then every connected component of $G - N[W]$ has a string representation on a surface of genus at most $g - 1$ (resp. Euler genus at most $g' - 1$).*

Proof. Note that the proof and the arguments are the same for orientable genus and Euler genus.

If $A(\Phi)$ has an embedding on a surface of genus $g - 1$ (resp. Euler genus $g' - 1$), then G has a string representation on this surface and we are done. Suppose then that this is not the case, i.e., $A(\Phi)$ is a graph of genus g (resp Euler genus g'). Therefore its embedding on $\mathbf{S}$ is a 2-cell embedding, i.e., every face of $A(\Phi)$ is homeomorphic to an open disk.

Let π be the non-contractible closed curve representing W. The curve π traces a closed walk W' in $A(\Phi)$. Since π is non-contractible, W' contains a non-contractible simple cycle C of $A(\Phi)$. By standard results on 2-cell embeddings (see [25, Chapter 4.2]), the genus (resp. the Euler genus) of every connected component of $A(\Phi) - C$ is strictly smaller than the genus of $A(\Phi)$.

Let Φ' be the set $\{\varphi_v : v \in V \setminus N[W]\}$. Note that Φ' is the string representation of $G - N[W]$. The auxiliary multigraph $A(\Phi')$ is a minor of $A(\Phi) - C$, and hence each of its connected components has an embedding on a surface of genus $g - 1$ (resp. Euler genus $g' - 1$). This embedding corresponds to a string representation of a connected component of $G - N[W]$ on a surface of genus $g - 1$ (resp. Euler genus $g' - 1$). □

Lemma 5.3. *If G has no string representation in the plane, then for every string representation Φ of G on a surface $\mathbf{S}$ there is a closed walk W in G with a non-contractible representation.*

Proof. Since $A(\Phi)$ is not planar, the embedding of $A(\Phi)$ contains a non-contractible cycle (see [25, Chapter 4.2]), which corresponds to a non-contractible closed curve on $\mathbf{S}$. This curve represents a closed walk W of G. □

Lemma 5.4. *Let Φ be a string representation of G on a surface $\mathbf{S}$, let $u, v \in V$ be two vertices, and let W_1, W_2, W_3 be three walks from u to v. If $W_1 - W_2$ has a non-contractible representation, then at least one of $W_1 - W_3$ and $W_2 - W_3$ has a non-contractible representation.*

Proof. Let π_{12} be a non-contractible closed curve representing $W_1 - W_2$. Looking at the consecutive subcurves of π_{12} corresponding to vertices of $W_1 - W_2$, we have that $\pi_{12} = \pi_1 - \pi_2$ with π_1 representing W_1 and π_2 representing W_2. Let $a \in \varphi_u$ be the first point of π_1 and $b \in \varphi_v$ be the last point of π_1.

Now let π_3 be any curve with endpoints a and b representing W_3. Observe that $\pi_1 - \pi_3$ represents $W_1 - W_3$ and $\pi_2 - \pi_3$ represents $W_2 - W_3$. By Lemma 5.1, at least one of $\pi_1 - \pi_3$ and $\pi_2 - \pi_3$ is non-contractible and the lemma follows. □

Lemma 5.5. *On a graph G with a string representation Φ on a surface $\mathbf{S}$ and a shortest closed walk W with a non-contractible representation, 10 cops have a strategy to guard $N[W]$ after a finite number of initial moves.*

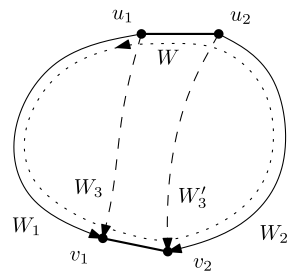

Fig. 8. The situation in the proof of Lemma 5.5.

Proof. If $|W| \leq 10$, the cops may occupy every vertex of W for the rest of the game and we are done. Otherwise we divide W into two walks W_1 and W_2, with $|W_1| \geq |W_2| \geq |W_1| - 1$, where W_i has endvertices u_i to v_i, and u_1u_2, v_1v_2 are edges. Note that $W = W_1 + v_1v_2 - W_2 - u_1u_2$; see Fig. 8. Note also that $|W| = |W_1| + |W_2| + 2$.

We claim that both W_1 and W_2 are shortest paths in G. If W_1 is not a shortest path, let W_3 be a shortest path from u_1 to v_1, so $|W_3| < |W_1|$. Then both closed walks $W_1 - W_3$ and $W_3 + v_1v_2 - W_2 - u_1u_2$ would be shorter than W:

$$|W_1 - W_3| = |W_1| + |W_3| < 2|W_1| \leq |W_1| + |W_2| + 1 < |W|,$$

$$|W_3 + v_1v_2 - W_2 - u_1u_2| = |W_2| + |W_3| + 2 < |W_1| + |W_2| + 2 = |W|.$$

By Lemma 5.4, at least one of them is non-contractible which contradicts the assumption. Similarly, there exists no path W_3' from u_2 to v_2 with $|W_3'| < |W_2|$.

Therefore, we may use Lemmas 4.1 and 4.2 with the distance function being the standard graph distance in G and A equal to V, to guard $N[W_1]$ and $N[W_2]$ with ten cops, after a finite number of initial moves. □

Proof of Theorem 1.1(iv,v). We are ready to prove that the maximum cop number of g-GENUS STRING graphs is at least $g^{\frac{1}{2}-o(1)}$ and at most $10g + 15$, and of g'-EULER-GENUS STRING graphs is at least $g'^{\frac{1}{2}-o(1)}$ and at most $10g' + 15$.

Proof of Theorem 1.1(iv,v). The lower bounds follows from the results of Mohar [24], who proved that for any g there is a graph G embeddable on a surface of genus g with $\mathrm{cn}(G) = g^{\frac{1}{2}-o(1)}$ as $g \to \infty$. Since any graph embeddable on a given surface also has a string representation on that surface, the lower bounds follow.

For the upper bound, we proceed by induction on the genus g in case of orientable surfaces, resp. by the Euler genus g' in case of non-orientable surfaces. The proof here is the same for both parameters. Note that a non-orientable surface may become orientable after removing a non-contractible curve (as in Lemma 5.2), the proof for Euler genus does not depend on the underlying surface being non-orientable.

The cases $g = 0$ and $g' = 0$ are proved by Theorem 1.1(iii). Suppose now that $g > 0$ (resp. $g' > 0$) and fix a string representation Φ of G on a surface of genus g (resp. Euler genus g'). Let W be a shortest closed walk in G with non-contractible representation. By Lemma 5.5, 10 cops prevent the robber from entering $N[W]$ till the end of the game; see Fig. 9.

Thus, after a finite number of moves the robber will remain confined to a connected component G' of $G - N[W]$. By Lemma 5.2, G' has a string representation on a surface of genus at most $g - 1$ (resp. Euler genus $g' - 1$). By the induction hypothesis, $15 + 10(g - 1)$ cops (resp. $15 + 10(g' - 1)$ cops) have a strategy to capture the robber on G' and, together with 10 cops guarding W, also on G. □

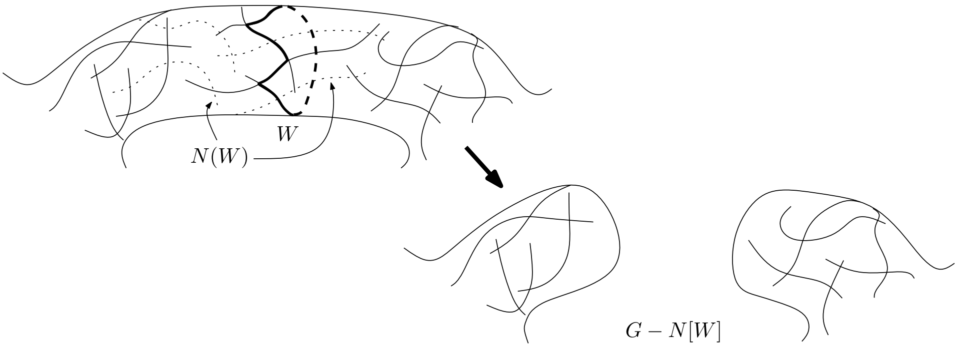

Fig. 9. Cutting a surface handle after guarding $N[W]$ where W is a closed walk represented by a non-contractible curve. Note that our proof is general and works for any non-contractible curve.

6. Unbounded cop number of intersection graphs of disconnected or 3-dimensional sets

In this section, we prove Theorem 1.7 stating that the cop number is unbounded even for very simple intersection classes of disconnected or 3-dimensional sets.

Cop number of subdivisions. For a graph $G = (V, E)$ and an integer $d \geq 1$, let $G^{(d)}$ denote the graph obtained from G by replacing each edge $e = xy \in E$ by a path P_e of length d connecting x and y. In other words, $G^{(d)}$ is obtained from G by subdividing each edge of G by $d - 1$ new vertices. The vertices of $G^{(d)}$ that subdivide an edge of G are the *subdividing vertices*, while the vertices of $G^{(d)}$ belonging to G are the *branching vertices*. The path P_e is the *edge-path corresponding to e*.

Lemma 6.1. *For a connected graph $G = (V, E)$ and an integer $d \geq 1$, we have*

$$\mathrm{cn}(G) \leq \mathrm{cn}(G^{(d)}) \leq \mathrm{cn}(G) + 1.$$

Proof (*Part 1: The Inequality* $\mathrm{cn}(G) \leq \mathrm{cn}(G^{(d)})$)**.** On $G^{(d)}$, we consider the (d, d)-*game* which is a modification of the standard game of cops and robber in which both the robber and the cops are allowed to make d consecutive moves in each turn, instead of just one move. In other words, in the (d, d)-game the robber and the cops are allowed to move, in each turn, to any vertex at distance at most d from their current position. Note that if the cops have a winning strategy for the standard game on a graph H, then they also have a winning strategy for the (d, d)-game on H: a d-fold move of the robber can be interpreted as a sequence of d simple moves, and each can be reacted according to the winning strategy for the standard game.

We say that a vertex $\bar{v}$ of the graph G *approximates* a vertex v of $G^{(d)}$ if the following holds: either v is a branching vertex and $\bar{v} = v$, or v is a subdividing vertex belonging to an edge-path P_e and $\bar{v} \in e$. Notice that if u and v are two vertices at distance at most d in $G^{(d)}$, and if $\bar{u}$ is a vertex of G approximating u, then there is a vertex $\bar{v} \in V$ approximating v such that $\bar{u}$ and $\bar{v}$ have distance at most one in G.

Let $\mathcal{S}$ be a winning strategy for k cops in the (d, d)-game on $G^{(d)}$. We now describe a winning strategy for k cops playing the standard game on G. Each cop playing on G is identified with one cop in $\mathcal{S}$. When the strategy $\mathcal{S}$ moves the cop to a vertex v in $G^{(d)}$, the corresponding cop moves to a vertex $\bar{v}$ of G that approximates v. As argued above, this is always possible, so each move in $G^{(d)}$ corresponds to an approximating move in G. When the robber moves from u to v in G, this move can be translated into a d-fold move from u to v in $G^{(d)}$. Therefore, the strategy $\mathcal{S}$ can be used to find the corresponding response on G, approximating the response on $G^{(d)}$. Since $\mathcal{S}$ captures the robber, the cops win the game in G.

Part 2: the inequality $\mathrm{cn}(G^{(d)}) \leq \mathrm{cn}(G) + 1$. Suppose that k cops have a winning strategy $\mathcal{S}$ for the standard cops and robber game on G. We use $k + 1$ cops on $G^{(d)}$. The first k cops, called the *regular* *cops*, are identified with the k cops of $\mathcal{S}$, while the remaining cop, called the *tracker*, follows a special strategy. In the beginning of the game, the tracker follows a shortest path towards the vertex initially occupied by the robber. As soon as the tracker reaches a vertex previously occupied by the robber, he only moves along the edges previously used by the robber. More precisely, if the tracker stands in a vertex x, he moves through the edge that was used by the robber during his most recent departure from x.

Suppose that the robber moves from u to v. We say that this move is a *hesitant move* if either the robber stands still (i.e., $u = v$), or the robber retraces an edge (i.e., his immediately preceding move was from v to u). When the robber makes a hesitant move, his distance to the tracker decreases. Thus, the tracker ensures that the robber can only make a limited number of hesitant moves without getting captured.

The strategy of the regular cops works as follows. Recall that each regular cop corresponds to a cop in $\mathcal{S}$. For simplicity, we first assume that the robber never makes a hesitant move. In the beginning of the game, the regular cops occupy the initial positions prescribed by $\mathcal{S}$. They wait at these positions until the robber first reaches a branching vertex u; the robber makes no hesitant moves so he must reach a branching vertex within the first d moves. Suppose that the robber moves from the branching vertex u to a subdividing vertex of an edge-path P_e, where $e = uv$ is an edge of G. Since the robber makes no hesitant moves, he moves along P_e all the way to v. After the robber enters P_e, each cop looks up in $\mathcal{S}$ the prescribed response to the robber's move from u to v. If $\mathcal{S}$ says a cop should move from a vertex x to a vertex y along an edge e', the corresponding cop spends d moves moving from x to y along $P_{e'}$. Thus, after d rounds of the play, both the regular cops and the robber will again occupy branching vertices, and the sequence of d rounds corresponded to a single round of $\mathcal{S}$. The regular cops then repeat the same process, imitating the moves of $\mathcal{S}$, until the robber is caught.

It remains to deal with the robber's hesitant moves. If the robber stays in a vertex, all regular cops stay in their vertices as well. If the robber retraces an edge, all regular cops also retrace their last used edges. So if the robber starts moving from u to v along P_e in $G^{(d)}$ and then starts moving back, the regular cops mimic him: if a regular cop moves from x to y along $P_{e'}$, he keeps the same distance on $P_{e'}$ to x as the robber on P_e to u. (Notice that each change in the direction of the robber's moves on P_e is a hesitant move.) Since the robber has a limited number of hesitant moves to avoid getting captured by the tracker, the strategy $\mathcal{S}$ applies and he is captured by one of the regular cops. □

Proof of Theorem 1.7. We are ready to prove that the cop number of intersection graphs of disconnected or higher dimensional sets is unbounded.

Proof (*Theorem 1.7*)**.** As mentioned in the introduction, the class LINE of line graphs has unbounded cop number by results of Dudek et al. [8]. Moreover, each line graph can be represented as the intersection graph of two-element subsets of the real line, and therefore the classes 2-UNIT INTERVAL and 2-INTERVAL contain LINE as a subclass. It follows that these classes have unbounded cop number as well.

Let us now deal with geometric intersection classes of higher-dimensional objects. The class 3-GRID (and thus also 3-BOX) contains $G^{(3)}$ for all graphs G: the vertices are represented by long parallel segments, say, in the direction of the z-axis, having pairwise different x and y coordinates. Each edge is represented by an L-shape (consisting of two segments), connecting the parallel segments representing the corresponding vertices. We may even assume that each segment of the representation has unit length.

For 3-CUBE and 3-BALL, draw any graph G in space without crossing of edges in such a way that all edge-curves have the same length. Also ensure that around every vertex there is a ball containing only the initial parts of the incident edge-curves and that these parts are straight segments. Let a be the minimum of all the diameters of these balls and the distances between edge-curves outside of the balls, note that $a > 0$.

Now replace every branching vertex by a cube/ball of size sufficiently smaller than a and notice that then there are disjoint tubular corridors of a positive diameter around every edge-curve outside the vertex cubes/balls. Therefore there exists a suitable value of d (depending on G and the curve representation) such that $G^{(d)}$ can be represented by chains of sufficiently small cubes/balls within these corridors.

For 3-UNIT CUBE and 3-UNIT BALL a similar construction works for cubic graphs G where we additionally require that the angles of edge-curves at the branching vertices are 120°. The rest of the construction is analogous. □

7. Conclusions

In this paper, we have determined the maximum cop number of circle, circular arc, function and interval filament graphs, and we gave bounds for outer-string graphs, string graphs, and string graphs on both orientable and non-orientable bounded genus surfaces. The following open problems remain.

Problem 1. Improve lower and upper bounds for the maximum cop number of string graphs, outer string graphs and other intersection graphs of arc-connected sets in the plane such as 2-dimensional segments, boxes, disks, unit disks, convex sets, etc.

We note that Beveridge et al. [3] show that the maximum cop number of unit disk graphs is at most 9. Their strategy, like our strategy for string graphs, is inspired by the approach of Aigner and Fromme [1]. The difference is that in the case of intersections of unit disks, the geometric structure allows to patrol a shortest path with just 3 cops, as opposed to the 5 cops needed to patrol a walk in general graphs.

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