Combinatorial Games

and fun with Unicode

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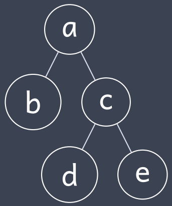
» Combinatorial games

Combinatorial games are (sequential, two-player) perfect information games.

Combinatorics: counting stuff, combinations, permutations.

Construction, existence and optimization of structures.



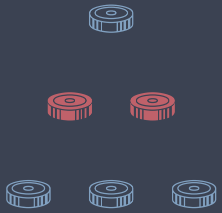
» Everything's on the table

Rules:

Two players take turns
Place 1-10 items on the table
Player to hit 100 wins

» Nim

Take as many as you like from a single pile.



* normal: last to pick up wins
* misère: last to pick up loses

» Nim solution

In normal play, the winning strategy is to finish every move with a nim-sum of 0.

1	001
2	010
3	011
0	000

Binary digital sum of previous game

See Wikipedia entry on Nim for full details.
Use nim-sum calculator or table of winning positions to defeat children.
Change size/number of heaps to keep them bewildered.

» Game Theory

- Analyzing, generating, and optimizing game objects
- Levels, strategies, likelihoods, and outcomes.

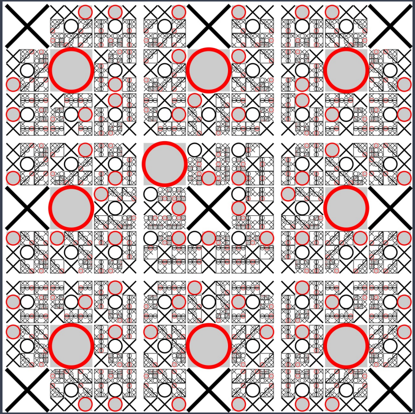
- Tic tac toe $(9-r)$ *possible moves* left
- Card games have hands
- Labyrinths have paths

$$\binom{52}{5}$$ Possible 5 card draws

$$G = (V, E) \qquad \min_{P \subset V \wedge \{P \text{ joins } a \text{ to } b\}} |P|$$

» Easy games

Tic Tac Toe 2nd player can force draw



Optimal tree for O: “A Fractal Guide to Tic Tac Toe”, Ian Stewart

» Easy (?) games

Nim (with 6 pieces) 2nd player wins; algorithm for multiple heaps

War is finite (sketchy proof I dunno).

» Connect Four

1st player has a winning strategy

Did Obama use hacks?

» Difficult (?) games

Chess

* First automaton The Turk was a hoax (!)

Go

* reinforcement learning AI quite good

 $>$ 

» How much is up to chance?

Is the game probabilistic or deterministic?

* Pure luck; playing *against* against entropy
  * lottery
  * roulette
  * dice
* Mix of chance and strategy; other players
  * poker
  * monopoly
  * dota
* Deterministic; no luck *at all*
  * Tic tac toe
  * Connect Four
  * Chess
  * Conway's Game of Life

» Player Information

Perfect information

* Tic tac toe
* Chess
* Pac-man

Imperfect information

* Poker
* Battleships
* Liar's dice

Sequential or Simultaneous

* Sequential
  * Chess, tic tac toe, checkers, monopoly
  * every player gets a turn, constituting a round
* Simultaneous
  * Bingo
  * War
  * 6 Nimmt
* Real-time
  * Dota 2
  * Hungry-hungry hippos
  * Pong

some of these last ones don't really fit; dexterity games; the challenge is in performing action

Outcomes

* Determined
  * Win
  * (or Lose)
* Undetermined
  * Possibility of draw
  * Infinite play
  * Loopy: possible to return to previous state

» Strategies and sequences

A players makes a choice, a play or a move. May allow pass/do nothing (passive).

Some games can be modeled as digraphs or state machines: move from one valid state to the next; strategies then are similar to outputs from Mealy or Moore machines

Game graphs of Nim

A.S. Fraenkel / Theoretical Computer Science 249 (2000) 265–288

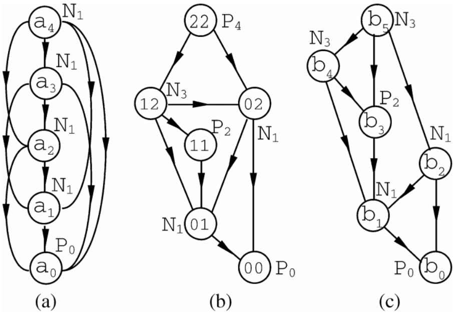


Fig. 1. Some game-graphs.

Aviezri S. Fraenkel, *Recent results and questions in combinatorial game complexities*

» Progressively finite

* if game digraph has no cycles
* (cannot return to previous positions)
* the game ends in finite turns.

» Kernel of a game

Let D be the game graph of a progressively finite game and $K \subset V(D)$. The set K is called the Kernel of D if it satisfies the following three properties:

1. all the winning vertices are in K
2. there is no edge from any vertex in K to any (other) vertex in K
3. from every vertex not in K, there is an edge to some vertex in K.

Kernel of the Table game:

$$K = \{100, 89, 78, 67, 56, 45, 34, 23, 12, 1\}$$

Portable Game Notation

```
[Event "F/S Return Match"]
[Site "Belgrade, Serbia JUG"]
[Date "1992.11.04"]
[Round "29"]
[White "Fischer, Robert J."]
[Black "Spassky, Boris V."]
[Result "1/2-1/2"]
```

```
1. e4 e5 2. Nf3 Nc6 3. Bb5 a6
{This opening is called the Ruy Lopez.}
4. Ba4 Nf6 5. O-O Be7 ...
```

A tournament match in Portable Game Notation

» Winner/Gagnant

A strategy can be described by a sequence of moves. A winning strategy is a way for one player to win no matter what the opponent may do.

$$\exists k \;:\; \exists x_1 \forall y_1 \exists x_2 \forall y_2 \ldots \forall y_{k-1} \exists x_k$$

such that x_k is a win for player x.

» Finding the best move

Searching decision trees

$$\underline{v_i} = \max_{a_i} \min_{a_{-i}} v_i(a_i, a_{-i})$$

Minimax algorithm

» Uh oh factorial

Problem: Counting all possible choices explodes.

Factorial is bad; solving for many games is known to NP-Complete or EXPtime-Complete (chess, go).

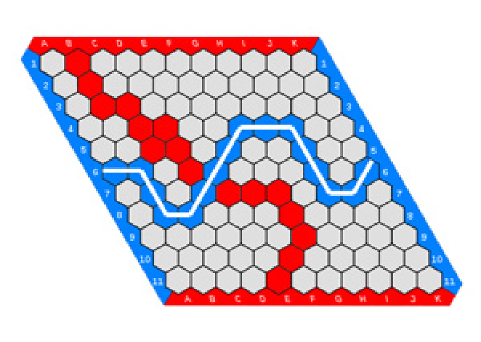
Algorithms compensate by trying to explore decision trees intelligently, pruning bad moves.

» Shannon switching game

Distinguished vertices *a* and *b*.

Player short can reinforce edges.

Player cut can remove (non-reinforced) edges.



Game of Hex (similar to shortcut)

» Contagion and Fire

Some nodes start on fire maybe.

Deploy *k* firepeople to put out he fire.

Firepeople

- cut links of the graph,
- put out nodes,
- or protect nodes from catching

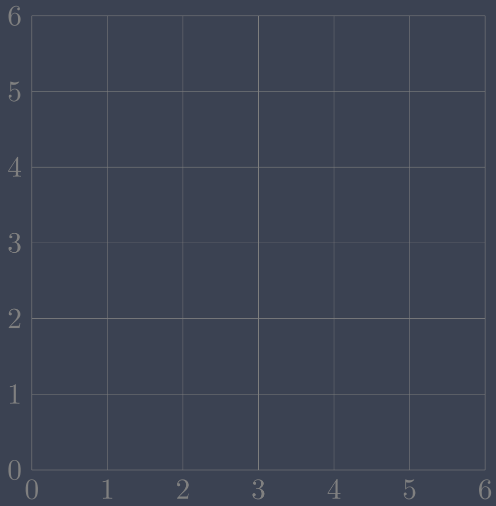
Fire/virus power

- ability to spread
- eventually die out/recover

» Firefighter problem

- Invented in 1995
- Fire starts at node s, every round firefighters added to protect non-fire nodes.
- Goal is to contain 🔥, maximizing saved nodes
- NP-hard on bipartite graphs and trees with $\delta \leq 3$
- Even aproximation is NP-hard.

Searching the grid



» Searching on the line

Place a robot on the real line at zero.

The target is somewhere on the right ($x > 0$) or on the left ($x < 0$).

How far do you go in one direction before deciding to turn around?

Is there an optimal strategy? (yes)

» Cops and Robbers

Cop-number $c(G)$ of a graph: How many cops do you need to guaranteed robber will be caught?

Aigner and Fromme showed that 3 cops have a winning strategy on a planar graph (genus 0).

For arbitrary g, it has been shown that:

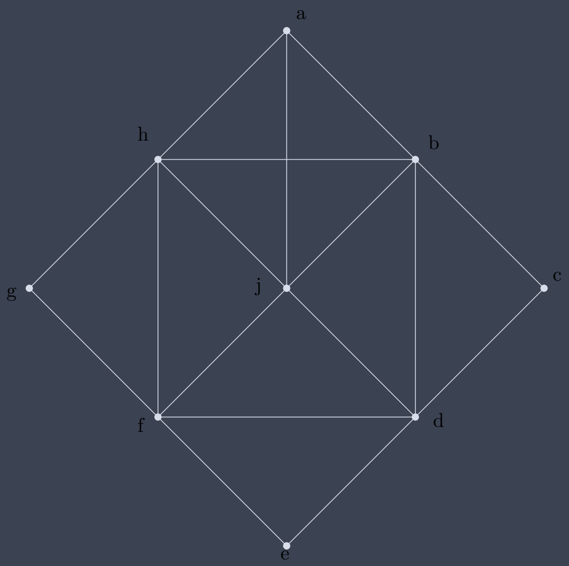
$$c(G) \leq \left\lfloor \frac{3}{2} g \right\rfloor + 3$$

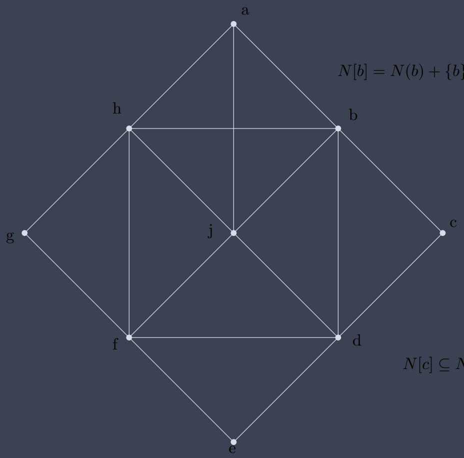
If G has girth at least 5, then

$$c(G) \geq \delta(G)$$

where $\delta(G)$ is the minimum degree of G.

allowframebreaks





$N[b] = N(b) + \{b\} = \{a, h, j, c, d, b\}$

$N[c] = N(c) + c = \{b, d, c\}$

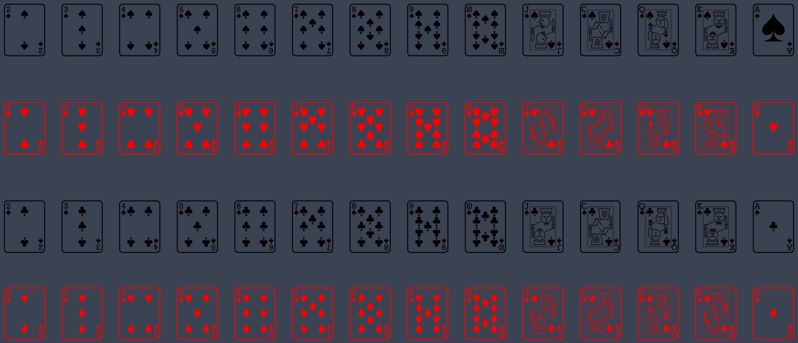
$N[c] \subseteq N[b]$

Cop-win graph

» Zombies

What if cops were zombies instead?

» Gardner's Second Favourite Puzzle



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