Modified Acoustic, Internal and Surface Waves and Modes in the Ocean

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Abstract: Waves propagate in a free-surface ocean due to compressibility or gravity and, at much lower scale, due to surface tension. Analytical solutions have for long been derived independently for acoustic waves or internal-gravity rays (in an unbounded ocean), for surface-gravity waves (in a free-surface-ocean) and for acoustic or internal modes (in a bounded ocean). In the present study, capillarity waves and earth-rotation are neglected and a simple, unified model based on inner and boundary dispersion relations is derived for waves propagating in a compressible, stratified, free-surface ocean. Wave solutions are first identified and carefully studied geometrically in phase-space. Taylor developments are then carried out with respect to small parameters describing stratification and compressibility and are compared to numerical approximations of the intersection of the various branches of the inner and boundary dispersion surfaces. Known approximations for swell, long-surface waves, internal-gravity rays, internal modes, acoustic waves or acoustic modes are recovered and perturbations due to ocean stratification and/or compressibility are given.

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1 Introduction

Many types of waves are known to propagate in the ocean and textbooks (LeBlond and Mysak (1981); Gill (1982); Pedlosky (1982)...) have for long detailed the derivation of analytical solutions. These waves can basically be classified in several categories depending on the type of mechanisms directly involved in their propagation. Neglecting modification due to rotation for not-so-long waves, two fundamental categories are of particular interest in the present study: acoustic (or sound) waves which are a consequence of the compressibility of the ocean and gravity waves which are sustained by the gravity force. Table (1) gives a short (and necessarily incomplete) list of such waves together with their dispersion relations. Acronyms for the corresponding modified waves studied below are indicated in the last column. The names, the dispersion relations and the parametrized relations are indicated in Table (**??**). A particular wave is indeed most often described by giving one or a few space-time *dispersion* relations linking for instance its pulsation (or its period) to its wave-numbers (or to its wave-lengths). Its phase and group velocities and several physical properties such as its capacity to "disperse" wave-lengths can be derived from the dispersion relations.

In the ocean, if the wavelengths are small enough and if they are generated far enough from the surface and bottom boundaries, these waves can propagate as in any unbounded medium. However, when they are generated in the vicinity of these boundaries or when their wavelengths are large compared to the ocean depth, waves are known to take particular forms and the ocean can then be assimilated to a *wave-guide* propagating *wave-modes*. Internal (gravity) mode solutions with "long" horizontal wavelengths have for long been known to propagate in the ocean wave-guide and acoustic wave-mode solutions have lately been re-discovered by Smith (Smith (2015)). Such internal and acoustic waves are qualified as "wave modes" due to the quantification of their vertical wavelength.

The ocean free-surface is permanently shaken by a myriad of horizontally propagating waves and it is not always very clear if these waves are wave-modes or just vertically-evanescent edge-waves. Capillary waves (not studied here), swells, tidal waves, tsunamis are well-known examples.

Deriving a dispersion relation for acoustic waves or for internal wave rays in an unbounded ocean is rather straightforward. The proposed method generally includes two steps: firstly, waves are supposed to have very small amplitude and, secondly, only the specific wave-restoring mechanisms and medium characteristics are retained in an as-simple-as-possible wave dispersion model (compressibility and pressure force for acoustic waves, gravity and vertical advection of isopycnal surfaces for internal waves). The linear nature of the resulting model has two main advantages: analytical solutions can be carried out and waves can be superimposed without interacting (Lighthill (1967)). The treatment of the free-surface boundary can yet be much trickier: small-amplitudes are usually postulated and both gravity and free-surface motions are retained in the dedicated dispersion model. However, surface waves are "edge waves" propagating at the interface between the atmosphere and the ocean meaning that the surface kinematic relation (the free-surface general boundary-condition) leads to a transcendental dispersion relation with trigonometric terms. This obviously results in difficulties to derive analytical solutions for ocean surface waves and it requires, in any case, further simplifications. Numerous specific analytical solutions can then be found in the literature depending for instance on the ratio of the horizontal wavelength k_x to the ocean depth (H) (Table 1). Long gravity surface wave solutions can indeed be found with small such ratios and these waves are well-known to propagate horizontally with $\sqrt{gH}$ phase and group velocities (with g the acceleration of gravity).

Waves	Assumptions	Pulsation (Ω)	k_z
Acoustic waves	Compressible, unbounded	$\Omega_{aw}^2 = c_s^2(k_x^2 + k_z^2)$	
Internal gravity rays	Stratified, unbounded	$\Omega_{iwr}^2 = \dfrac{N^2 k_x^2}{k_x^2 + k_z^2}$	
Acoustic gravity modes	Compressible, bounded	$\Omega_{am}^2 = c_s^2(k_x^2 + k_z^2)$	$k_{z,am} = \dfrac{\pi}{2H} + \dfrac{m\pi}{H}$
Swell	Free-surface	$\Omega_{sw}^2 = g k_x \tanh(k_x H)$	$k_{z,sw} \approx k_x$
Long Surface waves	Free-surface, shallow	$\Omega_{lsw}^2 = gH\, k_x^2$	$k_{z,lsw} \approx k_x$
Internal gravity modes	Stratified, bounded	$\Omega_{im}^2 = \dfrac{N\pi}{nH}$	$k_{z,im} = \dfrac{n\pi}{H}$

Table 1: Simplified models of ocean waves and their dispersion relations in a vertical section. Ω is the pulsation of the wave, k_x and k_z are the wave-numbers, g is the acceleration of gravity, H a reference depth, N a reference Brunt-Väisälä pulsation and c_s the speed of sound. n and m are two positive integer numbers. For an unbounded ocean (top) and for a bounded ocean (bottom).

Wave solutions⟩

The derivation of mixed acoustic-gravity waves in an ocean bounded by a free surface is more tricky and is the subject of this paper. The objective is to get a more precise view of the effect of compressibility and stratification on the preceding limiting cases. One of the target is to be able to validate recent developments within the CROCO ocean model where a pseudo-compressible approach is applied to solve the non-hydrostatic equations (Auclair et al. (2018)).
Dukowicz (2013) tackled this problem and proposed a review of the *"Various approximations in atmosphere and ocean model based on an exact treatment of gravity wave dispersion"*. In the ocean, acoustic-gravity waves are shown by the author to satisfy a system of two dispersion relations and the impact of several usual assumptions often made in ocean models is evaluated. The present study builds on Dukowicz's study and more specifically focuses on the consequences of both the stratification and compressibility of the ocean on the properties of acoustic-gravity waves and wave-modes: Taylor expansions of dispersion relations and of the many resulting expressions for wavelengths and pulsation are derived in terms of both compressibility and stratification. In addition:

- A systematical graphic investigation of wave solutions is carried out in (3D) pulsation/wave-number phase-space, unfolding and investigating the dependency to the vertical wave-number. Graphic explorations give an overall idea of the possible connections between the usual acoustic, internal and surface wave solutions.

- Surface waves together with internal and acoustic wave-modes are systematically studied in the ocean.

- Long-wave solutions are investigated in details in order to better understand from which solution branch they asymptotically derive and approximate parametric relations are derived for each type of wave.

In the following section (2), a linear model of wave propagation is proposed for the ocean together with bottom and surface boundary conditions and a system of two dispersion relations is derived from this model. The inner dispersion relation is studied in details in the following section (3) and the wave solutions propagating in an unbounded ocean are investigated. Wave propagating in a bounded ocean, which also has to satisfy to boundary relation dispersion, are then studied in section (4). The resulting solutions are further discussed in a last section (5).

2 Linear model for surface and internal acoustic-gravity waves

ctionLinModels⟩

2.1 General model for a compressible, viscous ocean

ectionNSModel⟩? Ocean dynamics can be described with a small number of macroscopic variables: its velocity ($\mathbf{v}$), its pressure and density (p and ρ), its temperature and salinity (T and S). In a Cartesian framework, the general equations governing the motion of a compressible, viscous ocean are then: PAS CORIOLIS

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v}) \tag{1a}$$
NS_a

$$\frac{\partial \rho \mathbf{v}}{\partial t} = -\nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) - \nabla p + \nabla \cdot \underbrace{\Big(\mu(\nabla \mathbf{v} + \nabla \mathbf{v}^{\,T}) + \mu_2 (\nabla \cdot \mathbf{v})\, \mathbf{I}\Big)}_{\tau} + \rho \mathbf{g} \tag{1b}$$
NS_b

$$\frac{\partial T}{\partial t} = -\nabla \cdot (T \mathbf{v}) + \nabla \cdot \kappa_T \nabla T \tag{1c}$$
NS_c

$$\frac{\partial S}{\partial t} = -\nabla \cdot (S\, \mathbf{v}) + \nabla \cdot \kappa_S \nabla S \tag{1d}$$
NS_d

$$\rho = \rho(T, S, p) \tag{1e}$$
NS_e

where $\mathbf{I}$ is the identity matrix, superscript T indicates transposition, μ and μ_2 are the kinetic and bulk (or second) viscosities, κ_T and κ_S are the heat and salt diffusivities. The first equations are written in a conservative form. They specify basic conservation principles: conservation of mass for equation (1a), conservation of momentum for equation (1b) and conservation of heat and salt for equations (1c) and (1d). Equation (1e) is a functional relation describing the thermodynamic equation of state (EOS).

2.2 Surface and bottom boundary conditions

?⟨SubSectionBC⟩? At the bottom ($z = -H$) and at the surface ($z = \zeta$) of the ocean, boundary conditions must be specified for each variables (or for their derivatives). A simple condition of no penetration and no-slip at the ocean bottom surface can be written:

$$\mathbf{v}(z = -H) = \mathbf{0} \tag{2}$$
?NS_BC0?

Neglecting surface-tension pressure drop, the boundary condition for pressure at the surface of the ocean can be given by:

$$p(\mathbf{x_H}, z = \zeta, t) = p_{atm} \tag{3}$$
?NS_BC1?

with p_{atm} the atmospheric pressure imposed at the surface of the ocean. Surface capilarity waves are consequently filtered out and will not be studied in the remaining of the present study. The surface kinematic relation expresses the motion of the free-surface and relates the free-surface anomaly (ζ) to the surface velocity (v_z):

$$\frac{\mathrm{d}\zeta(\mathbf{x_H}, t)}{\mathrm{dt}} = w(z = \zeta) \tag{4}$$
?NS_BC2?

This kinetic boundary condition allows the propagation of surface gravity waves.

2.3 Pressure and density decompositions and EOS simplification

SubSectionPRHO⟩ Waves are defined as small disturbances to a motionless, thermodynamic equilibrium state and both pressure and density are decomposed into an equilibrium component and a small increment. Furthermore, the impact of the atmospheric pressure (p_{atm}) is neglected to a first approximation, it can indeed take an active part in the generation of the waves, but only play a minor role during the propagation itself.

Two usual decompositions are thus now formalized for pressure (5a) and density (5b):

$$p(\mathbf{x},t) = \underbrace{\underbrace{p_{atm}(\mathbf{x_H},t)}_{\approx 0} + g\int_z^\zeta \rho_h(\mathbf{x},t)\,dz}_{p_h(\mathbf{x},t)} + \delta p(\mathbf{x},t)$$

$$= \underbrace{\underbrace{g\int_z^\zeta \hat{\rho}_h(z)dz}_{\hat{p}_h(z)} + \underbrace{g\int_z^\zeta \left(\rho_h(\mathbf{x},t) - \hat{\rho}_h(z)\right)\,dz}_{p'_h(\mathbf{x},t)}}_{p_h(\mathbf{x},t)} + \delta p(\mathbf{x},t) \tag{5a}$$

decompoP_f

$$\rho(\mathbf{x},t) = \underbrace{\hat{\rho}_{TS}(z) + \rho'_{TS}(\mathbf{x},t)}_{\rho_{TS}(\mathbf{x},t)=\rho(T,S,p=0)} + \underbrace{\frac{1}{c_s^2}\left(\hat{p}_h(z) + p'_h(\mathbf{x},t) + \delta p(\mathbf{x},t)\right)}_{(\partial\rho/\partial p)_\eta\ p(\mathbf{x},t)} + \mathrm{O}(p^2)$$

$$\approx \underbrace{\underbrace{\hat{\rho}_{TS}(z) + \frac{\hat{p}_h(z)}{c_s^2}}_{\hat{\rho}_h(z)} + \underbrace{\rho'_{TS}(\mathbf{x},t) + \frac{p'_h(\mathbf{x},t)}{c_s^2}}_{\rho'_h(\mathbf{x},t)}}_{\rho_h(\mathbf{x},t)} + \frac{\delta p(\mathbf{x},t)}{c_s^2} \tag{5b}$$

decompor_f

with $\partial\rho/\partial p = c_s^2$ at constant entropy, $\partial\hat{p}_h/\partial z = -\hat{\rho}_h(z)g$, $\partial p'_h/\partial z = -\rho'_h(z)g$.
The first decomposition ($p = p_h + \delta p$) is based on an hydrostatic component (p_h) and a non-hydrostatic pressure increment (δp) is defined. This decomposition is based on a split of the pressure field between a slowly varying component supposedly in hydrostatic equilibrium and quickly varying non-hydrostatic component. Density can in turn be decomposed in a similar way with ρ_h the hydrostatic component and $\delta\rho = \delta p/c_s^2$ the non-hydrostatic increment. The hydrostatic component of density can be further decomposed in a depth-averaged value ($\hat{\rho}_h(z)$) and a depth-dependant increment (ρ'_h) leading to a similar decomposition for the hydrostatic component of pressure between a surface induced, barotropic component ($\hat{p}_h$) and a baroclinic increment (p'_h) .
The second decomposition focuses on compressibility: it can be viewed as a first order approximation of the EOS with respect to total pressure (p). This Taylor development is carried out in the vicinity of pressure. It is this time formulated for density. ($\rho = \rho_{TS} + p/c_s^2$) is based on a first order decomposition with respect to total pressure (p). The component ρ_{TS} is a function of temperature and salinity and p/c_s^2 is the first-order compressibility increment due to total pressure.
The approximation of the EOS (1e) in the form $\rho = \rho_{TS} + p/c_s^2$ along with the tracers's conservation equations allow to write:

$$\frac{\mathrm{d}\rho}{\mathrm{dt}} = \frac{1}{c_s^2}\frac{\mathrm{d}p}{\mathrm{dt}} \tag{6a}$$

{?}

or using mass conservation

$$-\rho\nabla\cdot u = \frac{1}{c_s^2}\frac{\mathrm{d}p}{\mathrm{dt}} \tag{7a}$$

EqEOSd

Without loss of generality, the present study can now be restricted to the (O, x, z) vertical plan to simplify notations. Equation (7a) can then be expanded to:

$$-\rho\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) = \frac{1}{c_s^2}\left(\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + w\frac{\partial p}{\partial z}\right) \tag{8a}$$

EqEOSd1

A Taylor expansion of model equations can now be carried out in the vicinity of the reference profiles ($\hat{p}_h(z)$, $\hat{\rho}_h(z)$). Small amplitude wave-induced increments are given by $\delta V = (p'_h + \delta p,\ \rho'_h + \delta p/c_s^2,\ u,\ w)$.

At first order in δV, conservation of mass and vertical advection of pressure and density can be rewritten: the left hand side of (8a) can be rewritten

$$-\rho\left(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right)=-\left(\frac{\partial\hat{\rho}_h u}{\partial x}+\frac{\partial\hat{\rho}_h w}{\partial z}-w\frac{\partial\hat{\rho}_h}{\partial z}\right)+\mathrm{O}(\delta V^2),$$

while the right hand side of (8a) can be rewritten

$$\frac{1}{c_s^2}\left(\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}+w\frac{\partial p}{\partial z}\right)=\frac{1}{c_s^2}\left(\frac{\partial p}{\partial t}+w\frac{\partial\hat{\rho}_h}{\partial z}\right)+\mathrm{O}(\delta V^2)=\frac{1}{c_s^2}\left(\frac{\partial p}{\partial t}-\hat{\rho}_h g w\right)+\mathrm{O}(\delta V^2),$$

leading to

$$\frac{\partial p}{\partial t}=-c_s^2\left(\frac{\partial\hat{\rho}_h u}{\partial x}+\frac{\partial\hat{\rho}_h w}{\partial z}\right)+\underbrace{\left(g+\frac{c_s^2\partial\hat{\rho}_h(z)/\partial z}{\hat{\rho}_h(z)}\right)}_{-c_s^2N^2/g}\hat{\rho}_h w+\mathrm{O}(\delta V^2) \tag{9}$$
?Eq_EOS1?

where we have introduced $N^2(z)=-g\left(1/\hat{\rho}_h(z)\,\partial\hat{\rho}_h(z)/\partial z+g/c_s^2\right)$ the Brunt-Väisälä frequency for a compressible ocean (Gill (1982), p169).
The free-surface variations are introduced through the surface boundary condition for pressure:

$$p(z=0)=\hat{\rho}_h(0)g\zeta+\mathrm{O}(\delta V^2)$$

and the kinematic boundary condition can be written for pressure:

$$\frac{\mathrm{dp}}{\mathrm{dt}}(z=0)=g\hat{\rho}_h(0)w(z=0)+\mathrm{O}(\delta V^2)$$

2.4 Linear, inviscid wave model

ctionLinModel⟩? Based on the pressure and density decompositions proposed in subsection (2.3), the simpler, inviscid, linear, rotation-less $(p-\rho)$ model can be used to model acoustic, internal and surface waves. At first order in wave-induced increments δV, the conservations of horizontal and vertical momentums and mass and the EOS are written as:

$$\frac{\partial\hat{\rho}_h u}{\partial t}=-\frac{\partial p}{\partial x} \tag{10a}$$
WM_a

$$\frac{\partial\hat{\rho}_h w}{\partial t}=-\frac{\partial p}{\partial z}-\rho g \tag{10b}$$
?WM_b?

$$\frac{\partial\rho}{\partial t}=-\left(\frac{\partial\hat{\rho}_h u}{\partial x}+\frac{\partial\hat{\rho}_h w}{\partial z}\right) \tag{10c}$$
?WM_c?

$$\frac{\partial p}{\partial t}=-c_s^2\left(\frac{\partial\hat{\rho}_h u}{\partial x}+\frac{\partial\hat{\rho}_h w}{\partial z}\right)-\left(c_s^2N^2/g\right)\hat{\rho}_h w \tag{10d}$$
WM_d

with the (flat) bottom and surface relations:

$$w(z=-H)=0 \tag{11a}$$
WM_bc_a

$$\frac{\partial p}{\partial t}(z=0)=\hat{\rho}_h g w(z=0) \tag{11b}$$
WM_bc_b

As in Dukowicz (2013), in the following, we will express part of the results in term of $D(z)$ a vertical length scale associated to the stratification and defined by $\dfrac{1}{D(z)}=\dfrac{N^2(z)}{g}+\dfrac{g}{c_s^2}$. Indeed the background stratification satisfies $\dfrac{\partial\hat{\rho}_h(z)}{\partial z}=-\dfrac{1}{D(z)}\hat{\rho}_h(z)$.

2.5 General propagation equation and polarization

Form of the wave solutions:
Dispersion relations can be derived by postulating and specifying the form of the waves. Horizontally-propagating surface waves, wave-modes propagating in the ocean wave-guide, internal wave-rays and acoustic waves all satisfy the following "polarization" relations:

SectionPropag⟩?

$$\begin{pmatrix} \hat{\rho}_h u \\ \hat{\rho}_h w \\ \rho \\ p \end{pmatrix} = \begin{pmatrix} \widetilde{U}(z) \\ \widetilde{W}(z) \\ \widetilde{\rho}(z) \\ \widetilde{p}(z) \end{pmatrix} e^{i(k_x x - \Omega t)} \tag{12a}$$ {?}

where k_x and Ω are respectively the horizontal wave-number and the pulsation of the wave.

Inner dispersion relation:
These relations can be substituted in the propagation model (10a) to (10d). After some developments, an ordinary differential equation can be obtained for $\widetilde{W}(z)$:

$$\widetilde{W}''(z) + \frac{1}{D(z)}\widetilde{W}'(z) + \left(k_x^2 \frac{N^2 - \Omega^2}{\Omega^2} + \frac{\Omega^2}{c_s^2} - \frac{D'(z)}{D^2(z)}\right)\widetilde{W}(z) = 0 \tag{13}$$?eqwfirst?

with the following polarization relations:

⟨POL_c⟩

$$\widetilde{U}(z) = -ik_x \frac{(c_s^2 - gD(z))\widetilde{W}(z) + c_s^2 D(z)\widetilde{W}'(z)}{D(z)(\Omega^2 - c_s^2 k_x^2)} \tag{14a}$$ {?}

$$\widetilde{\rho}(z) = -i\frac{k_x^2(c_s^2 - gD(z))\widetilde{W}(z) + \Omega^2 D(z)\widetilde{W}'(z)}{D(z)\Omega(\Omega^2 - c_s^2 k_x^2)} \tag{14b}$$ {?}

$$\widetilde{p}(z) = -i\Omega\frac{(c_s^2 - gD(z))\widetilde{W}(z) + c_s^2 D(z)\widetilde{W}'(z)}{D(z)(\Omega^2 - c_s^2 k_x^2)} \tag{14c}$$ {?}

To obtain this relation and in particular to eliminate the pressure from this expression, one needs to exclude the possibility that:

$$\Omega^2 = c_s^2 k_x^2 \tag{15}$$?lamb0?

which is the dispersion relation for acoustic Lamb waves (Apel, 1987) in the atmosphere. As for the atmospheric case, it can be shown that these waves are characterized by $W(z) = 0, P(z) = P_0 e^{-gz/c_s^2}$. However, the surface boundary condition (11b) constraints P_0 to be zero and only the trivial null solution is obtained. As a consequence, Lamb waves does not exist in the ocean and as several authors point out, they are replaced by the surface gravity waves discussed later.
Stratification-induced change of variables:
To remove first order term and simplify future developments, the following change of variable can be made:

$$\widetilde{W}(z) = e^{\int_z^0 \frac{\mathrm{d}z_1}{2D(z_1)}} F(z) \tag{16}$$ CVF

The unknown function $F(z)$ satisfies:

$$F''(z) + \left(k_x^2 \frac{N^2 - \Omega^2}{\Omega^2} + \frac{\Omega^2}{c_s^2} - \frac{1 + 2D'(z)}{4D(z)^2}\right)F(z) = 0 \tag{17}$$?eqF?

$F(z)$ differs from the vertical momentum $W(z)$ by the attenuation factor $e^{\int_z^0 \frac{\mathrm{d}z_1}{2D(z_1)}}$. This factor reduces the vertical extent of the wave anomalies based on the length scale $D(z)$. The stronger the stratification, the shorter $D(z)$ and the shorter this vertical extent.

Boundary dispersion relation:
The polarization relations must also be substituted in the surface boundary condition (11b) leading to:

$$-i\Omega\widetilde{p}(0) = g\widetilde{W}(0) \tag{18}$$
{?}

or, using (14),

$$\widetilde{W}'(0) + \left(\frac{1}{D(0)} - \frac{gk_x^2}{\Omega^2}\right)\widetilde{W}(0) = 0. \tag{19}$$
surfacedisperr

Note that (19) can be checked to be identical to Eq. 71 of Dukowicz (2013) where the author suggests than the first term in parentheses ($1/D(0)$) can only appear when the equations are formulated with a Lagrangian vertical coordinate in order to properly take into account the surface boundary condition. To our opinion, this is not correct: the first term appears when none of the incompressibility or Boussinesq approximations are done.
In term of the unknown function $F(z)$, this surface boundary relation writes:

$$F'(0) + \left(\frac{1}{2D(0)} - \frac{gk_x^2}{\Omega^2}\right)F(0) = 0 \tag{20}$$
?eqFbc?

The bottom boundary condition (11a) simply leads to:

$$F(-H) = 0 \tag{21}$$
?eqFbc2?

Constant Brunt-Väisälä pulsation:
In the following sections, $D(z)$ is assumed to be constant $D(z) = D_0$ (or equivalently $N^2 = N_0^2$) and relation (16) can be rewritten:

$$\widetilde{W}(z) = e^{-\frac{z}{2D_0}}F(z) \tag{22}$$
?CVF2?

and $\hat{\rho}_h(z)$ is given by $\hat{\rho}_h(z) = \hat{\rho}_h(0)e^{-z/D_0}$. The general expression of the vertical velocity perturbation profile for a constant scale height D_0 is:

$$w(x,z,t) = \frac{1}{\hat{\rho}_h(z)}\widetilde{W}(z)e^{i(k_x-\Omega t)} = \frac{1}{\hat{\rho}_h(0)}e^{z/2D_0}F(z)e^{i(k_x-\Omega t)} \tag{23}$$
{?}

2.6 Inner and boundary dispersion relations

ubSectionDisp⟩? ?⟨EqDimF⟩? When the Brunt-Väisälä frequency N is constant, the solution is obtained by solving the system:

$$F''(z) + \overbrace{\left(k_x^2\frac{N^2-\Omega^2}{\Omega^2} + \frac{\Omega^2}{c_s^2} - \frac{1}{4D_0^2}\right)}^{\equiv k_z^2}F(z) = 0 \tag{24a}$$
EqDimFa

$$F'(0) + \left(\frac{1}{2D_0} - \frac{gk_x^2}{\Omega^2}\right)F(0) = 0 \tag{24b}$$
EqDimFb

$$F(-H) = 0 \tag{24c}$$
EqDimFc

The square of the vertical wave-number k_z is defined in (24a) and is a function of k_x and Ω. This consequently leads to a first dispersion relation in dimensional form:

$$k_z^2 + k_x^2\left(1 - \frac{N^2}{\Omega^2}\right) - \frac{\Omega^2}{c_s^2} + \frac{1}{4D_0^2} = 0 \tag{25}$$
EqDispRefInner

This relation does not take into account the boundary conditions at the surface nor at the bottom and consequently only deals with the propagation of waves in the inner ocean. It shall now be referred to as the *inner* dispersion relation.
The general solution of (24a) which satisfies the bottom boundary condition (24c) is

$$F(z) = \frac{\sin\left(k_z(H+z)\right)}{\sin(k_z H)}. \tag{26}$$

?EqDispRef?

Note that in the limiting $k_z \to 0$ case, the vertical profile of $F(z)$ is linear and given by $F(z) = 1 + z/H$.
The surface boundary condition (24b) leads then to the *boundary* dispersion relation:

$$\Omega^2 = \frac{g k_x^2 \tan(H k_z)}{k_z + \frac{\tan(H k_z)}{2D_0}} = \frac{g k_x^2}{\frac{1}{2D_0} + k_z \operatorname{cotan}(H k_z)} \tag{27}$$

EqDispRefs

A wave propagating in a "bounded ocean" must satisfy both the *inner* and *boundary* (dimensional) dispersion relation (25) and (27).
Note that the traditional inner and boundary dispersion relations for a Boussinesq incompressible fluid Gill (1982) can be recovered from (25) and (27) by setting $c_s \to +\infty$ (incompressibility) and then $D_0 \to +\infty$ (Boussinesq approximation), leading to

$$\Omega^2 = N^2 \frac{k_x^2}{k_x^2 + k_z^2} = g k_x^2 \frac{\tan(H k_z)}{k_z}$$

2.6.1 Dimensionless dispersion relations

As in Dukowicz (2013), two parameters are now defined to obtain dimension-less dispersion relations:

$$\epsilon_i^2 = \frac{N^2 H}{g},\ \epsilon_a^2 = \frac{gH}{c_s^2} \tag{28}$$

{?}

ϵ_i is thus a small gravity parameter defined as the ratio of the order of magnitude of internal wave mode one (NH) to the velocity of long surface waves ($\sqrt{gH}$). ϵ_a is a small acoustic parameter defined as the ratio of the velocity of long surface waves ($\sqrt{gH}$) to the one of sound waves c_s. Three dimensionless variables are further defined:

$$\omega = \Omega\sqrt{\frac{H}{g}},\ \delta_x = k_x H,\ \delta_z = k_z H \tag{29}$$

{?}

The constant scale-depth can be rewritten: $D_0 = H/(\epsilon_i^2 + \epsilon_a^2)$. The inner (25) and boundary (27) dispersion relation can be written in terms of the dimensionless parameters as:

?⟨EqFullDisper⟩?

$$\delta_x^2 + \delta_z^2 = \epsilon_i^2 \frac{\delta_x^2}{\omega^2} + \epsilon_a^2 \omega^2 - \frac{(\epsilon_a^2 + \epsilon_i^2)^2}{4} \tag{30a}$$

EqFullDispera

$$\omega^2 = \frac{\delta_x^2 \tan(\delta_z)}{\delta_z + \frac{\epsilon_a^2 + \epsilon_i^2}{2}\tan(\delta_z)} \tag{30b}$$

EqFullDisperb

In a free-surface ocean, wave solutions must satisfy simultaneously relations (30a) and (30b). This means that only one of the parameters among the pulsation ω and the horizontal and vertical wave-numbers (δ_x and δ_z) can be settled by forcing, the remaining two adjusted so that the wave satisfies the two dispersion relations. For short vertical-wave-number and far for the bottom and surface boundaries, wave solutions can satisfied only the inner dispersion relation and be dynamically consistent. Purely acoustic waves or pure internal-gravity wave-rays are for instance known to propagate in the inner ocean as in an unbounded ocean.

The resulting set of two equations (30a),(30b) of three variables ($\delta_x,\ \delta_z,\ \omega$) and two parameters ϵ_a, ϵ_i is non-linear and simple general solution cannot be carried out analytically. Much insight can yet be gained in these ocean waves investigating geometrically the inner and boundary surfaces associated to the corresponding dispersion relations. The surfaces associated respectively to the inner and boundary relations will respectively be named the *inner and boundary dispersion surfaces*.
In the following, we will use the fact that δ_z is either real $\delta_z \in \mathbb{R}$ or pure imaginary $\delta_z \in i\mathbb{R}$ and thus that ω^2 is real and positive. This is demonstrated in appendix (A). It is important to note that only the boundary dispersion relation constrains ω to be real and that others solutions are possible when considering only the inner dispersion relation. These solutions will not be discussed in the following, even in section (3) on waves in an unbounded ocean.
The rest of the paper is organized as follows: the inner dispersion relation corresponding to waves in an unbounded ocean is studied in section (3). Additional constraints related to the boundary dispersion relation and so a bounded ocean are added in section (4).
When not explicitly mentioned, the standard values of the parameters used in the rest of the paper are mentioned in table (2).

Gravity	g	9.8 m.s^{-2}
Sound speed	c_s	1500 m.s^{-1}
Depth	H	4000 m
Brunt-Väisälä frequency	N	10^{-3} s^{-1}
Acoustic small parameter	$\epsilon_a = \dfrac{\sqrt{gH}}{c_s}$	≈ 0.132
Internal small parameter	$\epsilon_i = \sqrt{\dfrac{N^2 H}{g}}$	≈ 0.02020
Scale depth	$D_0 = 1/\left(\frac{N^2}{g} + \frac{g}{c_s^2}\right) = \frac{H}{\epsilon_a^2+\epsilon_i^2}$	≈ 224 km

Table 2: Main parameters used to plot dispersion relations.

ableParameters⟩

3 Inner dispersion relation & waves in an unbounded ocean

⟨SectionInner⟩ The inner dispersion relation (30a) must be satisfied by any type of ocean waves whether the ocean can or cannot be locally considered as an unbounded medium (far from its surface and bottom). We shall show in the present section that (i) in ($\delta_x,\ \delta_z,\ \omega$) phase-space, the inner dispersion relation leads to a three-branch dispersion surface, (ii) two acoustic and stratification factorization functions $\omega_a(\delta_x,\ \delta_z)$ and $\omega_i(\delta_x,\ \delta_z)$ can be good approximations of the branches, (iii) the upper and lower branches of the inner dispersion surface (for respectively large and low pulsations) correspond to acoustic waves and internal rays propagating in an unbounded ocean and (iv) the bounded central branch of the inner dispersion surface corresponds to vertically vanishing waves which are referred to as surface waves in the following.

3.1 Factorizing functions and roots of the inner dispersion relation

tionFactoDisp⟩? 3.1.1 Acoustic and stratification factorizing functions ($\omega_a,\ \omega_i$)

bsectionFacto⟩? To gain insight in the physics of the wave solutions, the inner dispersion relation (30a) is rewritten below:

$$\delta_x^2 + \delta_z^2 = \epsilon_i^2 \frac{\delta_x^2}{\omega^2} + \epsilon_a^2 \omega^2 - \frac{(\epsilon_a^2 + \epsilon_i^2)^2}{4} \tag{31}$$ eqomegaparam2

which can, as in Tolstoy (1963), be reformulated in the simpler form:

$$\frac{\omega^2}{\omega_a^2} + \frac{\omega_i^2}{\omega^2} = 1 \tag{32}$$ eqlink

where ω_i and ω_a are functions of the horizontal and vertical wave-numbers and can be defined by
?⟨EqDispSols⟩?

$$\omega_a^2(\delta_x,\ \delta_z) = \frac{1}{\epsilon_a^2}\left(\delta_x^2 + \delta_z^2 + \frac{(\epsilon_a^2+\epsilon_i^2)^2}{4}\right) \tag{33a}$$?SolAcous?

$$\omega_i^2(\delta_x,\ \delta_z) = \frac{\delta_x^2\ \epsilon_i^2}{\delta_x^2 + \delta_z^2 + (\epsilon_a^2+\epsilon_i^2)^2/4} \tag{33b}$$?SolGrav?

These two functions are not roots of the inner dispersion equation (31) but they play an important role in the approximation of the roots of this equation and more specifically in their physical interpretation. Indeed

- If $\epsilon_a^2\omega^2 \gg \frac{\epsilon_i^2\delta_x^2}{\omega^2}$ (for instance for large pulsations) then (31) simplifies to $\omega^2 \approx \omega_a^2$. A compressible, homogeneous (unstratified) ocean satisfies this assumption.

- If now $\epsilon_a^2\omega^2 \ll \frac{\epsilon_i^2\delta_x^2}{\omega^2}$ (for instance for small pulsations) then (31) simplifies to $\omega^2 \approx \omega_i^2$. An incompressible, stratified ocean satisfies this assumption.

ω_a can therefore be interpreted as the factorizing *acoustic function* accounting for the compressibility content of the inner dispersion relation and ω_a is the solution of the *inner* dispersion relation for an homogeneous ocean. ω_i plays an equivalent role for the ocean stratification: it can be interpreted as the factorizing *stratification function* and ω_i is the solution of the *inner* dispersion relation for an incompressible ocean.
Recall that ω^2 is constrained to be real positive (this is actually enforced by the boundary relation dispersion, see (A))) and that δz is either real or pure imaginary. ω_a^2 and ω_i^2 have the same sign since their product is equal to $\delta_x^2\frac{\epsilon_i^2}{\epsilon_a^2} > 0$. And so (32) implies $\omega_a^2 \geq 0, \omega_i^2 \geq 0$ and for a given (δ_x, δ_z) we always have

$$\omega_i^2(\delta_x,\delta_z) \leq \omega^2(\delta_x,\delta_z) \leq \omega_a^2(\delta_x,\delta_z)$$

Thus the pulsation is always bounded by ω_i and ω_a.
Let us define $R^2(\delta_x,\delta_z)$ the ratio of the acoustic to the stratification function:

$$R^2(\delta_x,\delta_z) = \frac{\omega_i^2}{\omega_a^2} = \frac{\epsilon_a^2\epsilon_i^2\delta_x^2}{\left(\delta_x^2+\delta_z^2+\frac{(\epsilon_a^2+\epsilon_i^2)^2}{4}\right)^2} \tag{34}$$ eqratio

$R(\delta_x,\ \delta_z)$ plays a central role in the location of the roots of the inner dispersion relation. The condition $R^2 = \frac{\omega_i^2}{\omega_a^2} \leq \frac{1}{4}$ is required for Eq. (32) to have two real roots, and thus for ω^2 to be real. These roots can be formulated for the square of the pulsation:

$$\omega_\pm^2 = \frac{\omega_a^2}{2}\left(1 \pm \sqrt{1 - 4\ \underbrace{\frac{\omega_i^2}{\omega_a^2}}_{\equiv R^2}}\right) \tag{35}$$ solseq

When R^2 is small the two roots are well separated and correspond to $\omega_+ \approx \omega_a, \omega_- \approx \omega_i$.
It is also important to see that the product of the two roots is always equal to $\omega_a^2\omega_i^2 = \frac{\epsilon_i^2}{\epsilon_a^2}\delta_x^2$ and thus we have:

$$\omega_i^2(\delta_x,\delta_z) \leq \omega_-^2(\delta_x,\delta_z) \leq \frac{\epsilon_i}{\epsilon_a}\delta_x \leq \omega_+^2(\delta_x,\delta_z) \leq \omega_a^2(\delta_x,\delta_z) \tag{36}$$ boundedomega2

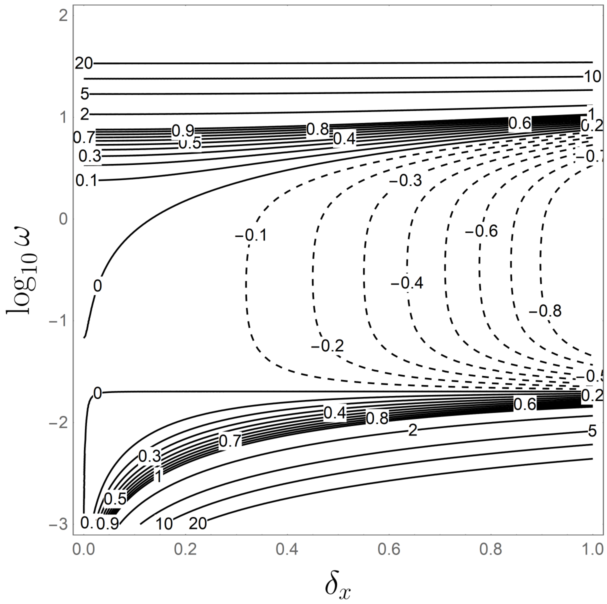


Figure 1: *contours of* $\delta_z^2(\delta_x,\ \omega)$. Plain lines correspond to positive values of δ_z^2 ($\delta_z \in \mathbb{R}$) whereas dashed lines are associated to negative values ($\delta_z \in i\mathbb{R}$).

⟨FigContourdz⟩

3.2 Three regions in $(\delta_x,\ \delta_z,\ \omega)$ phase-space

SectionDeltaz⟩? Figure (1) shows the variations of the square of the vertical wave-number (δ_z^2) as a function of $(\delta_x,\ \omega)$. Negative values are encountered for medium-range pulsations ($10^{-0.7} < \omega < 10^{0.7}$) and large enough horizontal wave-numbers ($\delta_x \geq 0.1 - 0.2$). This region is limited by two regions of positive δ_z^2, one for large pulsations and the other for small pulsation. The transition lines between these regions are given by $\delta_z = 0$.

- If $\omega \gg 1$ and $\epsilon_a \neq 0$ then $\omega^2 \approx \omega_a^2$ and the transition line is given by $\epsilon_a^2\omega^2 \approx \delta_x^2 + \frac{(\epsilon_a^2+\epsilon_i^2)^2}{4}$. This line is a parabola and the pulsation is not bounded when δ_x increases.
- If $\omega \ll 1$ and $\epsilon_i \neq 0$ then $\omega^2 \approx \omega_i^2$, the equation of the transition line is $\omega^2 \approx \delta_x^2\epsilon_i^2/(\delta_x^2+(\epsilon_a^2+\epsilon_i^2)^2/4)$. This line has an upper bound $\omega_{c,i} = \epsilon_i$. In dimensional form, this latter bound can be rewritten $\Omega \leq N$ and can be related to the well-known cut-off pulsation for internal gravity waves.

Dispersion relations (30a - 30b) therefore authorize three types of wave solutions: two with real vertical wave-numbers ($\delta_z^2 \geq 0$) and one with pure-imaginary wave-numbers ($\delta_z^2 < 0$).

3.2.1 Real vertical wave-number ($\delta_z \in \mathbb{R}$)

realwavenumber⟩ The ratio of the factorizing functions $R^2(\delta_x, \delta_z)$ (34) depends on two variables $(\delta_x,\ \delta_z)$ and two parameters $(\epsilon_i,\ \epsilon_a)$. A study of its variations for $(\delta_x,\ \delta_z) \in \mathbb{R}^2$ shows that, for non-vanishing $(\epsilon_i,\ \epsilon_a)$, it has an upper bound:

$$\max R^2(\delta_x, \delta_z) = \frac{1}{4}\frac{\epsilon_a^2\epsilon_i^2}{\delta_z^2 + \frac{(\epsilon_a^2+\epsilon_i^2)^2}{4}}. \tag{37}$$?boundR2?

This maximum value is attained for $\delta_x^2 = \delta_z^2 + \frac{(\epsilon_a^2+\epsilon_i^2)^2}{4}$. R^2 can be closed to $1/4$ (and thus the two roots can coincide) for a) a close to depth-independent vertical profile (i.e. $\delta_z \approx 0$) b) $\epsilon_a \approx \epsilon_i$. For other cases, and so in the vast majority of cases, the two roots are well separated and given by:

$$\omega_-(\delta_x,\ \delta_z) \approx \omega_i(\delta_x,\ \delta_z), \quad \omega_+(\delta_x,\ \delta_z) \approx \omega_a(\delta_x,\ \delta_z) \tag{38}$$ {?}

Modified internal waves (MIW) The traditional dispersion relation for dispersive internal gravity wave rays in the context of a Boussinesq incompressible fluid (Gill (1982), see also Table 1 above) is

$$\omega_{iwr}^2 = \epsilon_i^2 \frac{\delta_x^2}{\delta_x^2 + \delta_z^2} \tag{39}$$?DispRays?

A Taylor development of the gravity-wave root ω_-^2 of (35) with respect to small parameters ϵ_a and ϵ_i leads to

$$\frac{\omega_-^2}{\omega_{iwr}^2} = 1 - \frac{(\epsilon_i^2 + \epsilon_a^2)^2}{4(\delta_x^2 + \delta_z^2)} + \frac{(\epsilon_i^2 \epsilon_a^2)^2}{(\delta_x^2 + \delta_z^2)^2}\delta_x^2 + \mathrm{O}((\epsilon_i^2 + \epsilon_a^2)^4), \tag{40}$$ DispRaysDT

while the development of ω_i^2 leads to

$$\frac{\omega_i^2}{\omega_{iwr}^2} = 1 - \frac{(\epsilon_i^2 + \epsilon_a^2)^2}{4(\delta_x^2 + \delta_z^2)} + \mathrm{O}((\epsilon_i^2 + \epsilon_a^2)^4).$$

Thus ω_{iwr}^2 is a second order approximation of ω_-^2 with respect to $(\epsilon_i^2 + \epsilon_a^2)$. In comparison to ω_i^2, ω_-^2 includes corrective terms linked to the fact the the two roots of the inner dispersion relation are not fully separated. The combined effect of compressibility and stratification is always a reduction of the frequency in comparison with the approximated value ω_{iwr}. Indeed the two corrective terms on the right hand side of (40) can be rewritten as $-\dfrac{(\epsilon_i^2 + \epsilon_a^2)^2\delta_z^2 + (\epsilon_a^2 - \epsilon_i^2)^2\delta_x^2}{4(\delta_x^2 + \delta_z^2)^2} \leq 0$.
The ocean waves satisfying (40) shall now be referred to as *Modified Internal Waves (MIW)*.
AJOUTER PLOTS DE $\omega_-^2/\omega_{iwr}^2$ pour differentes valeus de epsi, epsa

Modified acoustic waves (MAW) The well-known dispersion relation for acoustic waves in an homogeneous fluid is thus recovered at fourth order zero:

$$\epsilon_a^2 \omega_{aw}^2 = \delta_x^2 + \delta_z^2 \tag{41}$$?DispAcous?

A second-order Taylor development of the acoustic root (ω_+) with respect to ϵ_a and ϵ_i leads this time to:

$$\frac{\omega_+^2}{\omega_{aw}^2} = 1 + \frac{(\epsilon_i^2 + \epsilon_a^2)^2}{4(\delta_x^2 + \delta_z^2)} - \frac{(\epsilon_i^2 \epsilon_a^2)^2}{(\delta_x^2 + \delta_z^2)^2}\delta_x^2 + \mathrm{O}((\epsilon_i^2 + \epsilon_a^2)^4), \tag{42}$$ DispAcousDT

while the development of ω_a^2 leads to

$$\frac{\omega_a^2}{\omega_{aw}^2} = 1 + \frac{(\epsilon_i^2 + \epsilon_a^2)^2}{4(\delta_x^2 + \delta_z^2)} + \mathrm{O}((\epsilon_i^2 + \epsilon_a^2)^4).$$

Thus ω_{aw}^2 is a second order approximation of ω_+^2 with respect to $(\epsilon_i^2 + \epsilon_a^2)$. In comparison to ω_a^2, ω_+^2 includes corrective terms linked to the fact the the two roots of the inner dispersion relation are not fully separated. The combined effect of compressibility and stratification is always a increase of the frequency in comparison with the approximated value ω_{iwr}. Indeed the two corrective terms on the right hand side of (42) can be rewritten as $\dfrac{(\epsilon_i^2 + \epsilon_a^2)^2\delta_z^2 + (\epsilon_a^2 - \epsilon_i^2)^2\delta_x^2}{4(\delta_x^2 + \delta_z^2)^2} \geq 0$.
Ocean waves satisfying (42) will be called *Modified Acoustic Waves (MAW)* in the following.
Obviously the modifications to the traditional internal and acoustic waves dispersion relations by compressibility and stratification effects are connected. Indeed, it is trivial to see that

$$\frac{\omega_-^2}{\omega_{iwr}^2}\frac{\omega_+^2}{\omega_{aw}^2} = 1$$

AJOUTER PLOTS DE ω_+^2/ω_{aw}^2 pour differentes valeus de epsi, epsa

3.2.2 Pure-imaginary vertical wave-number ($\delta_z \in i\,\mathbb{R}$)

ubsubsectioniR⟩ If now δ_z is a pure-imaginary complex, it can be written $\delta_z = i\delta_{z,i}$ with $\delta_{z,i} \in \mathbb{R}$ and wave-solutions are vertically-evanescent. The inner dispersion relation keeps the same form as (30a) :

$$\delta_x^2 - \delta_{z,i}^2 = \epsilon_i^2 \frac{\delta_x^2}{\omega^2} + \epsilon_a^2\omega^2 - \frac{(\epsilon_a^2 + \epsilon_i^2)^2}{4} \tag{43}$$ EqFullDisperai

When the horizontal and vertical wavenumbers are close to each other, this shows the left-hand-side and thus the right-hand-side both vanish. This means that the influence of the stratification ($\epsilon_i^2\delta_x^2/\omega^2 - (\epsilon_a^2 + \epsilon_i^2)^2/4$) and that of the compressibility ($\epsilon_a^2\omega_a^2$) are cancelled out. In other words, differences between the horizontal and vertical wavenumbers are an indication of the influence of the stratification and/or of the compressibility of the ocean. In an incompressible, homogeneous (unstratified) ocean, vertical and horizontal wave-numbers are equal.

The developments of ω_-^2, ω_+^2 for small ϵ_i, ϵ_a coincide with the development for real vertical wave-numbers (40), (42) just replacing δ_z^2 by $-\delta_{z,i}^2$.
The remaining question is that of the separation of the roots when $\delta_z \in i\,\mathbb{R}$. Unlike when $\delta_z \in \mathbb{R}$ (previous sub-section), the ratio $R^2(\delta_x,\ i\,\delta_{z,i})$ can be equal to 1/4 even when δ_z is not small. Relation (43) imposes $\delta_{z,i}^2 \le \delta_x^2 + \frac{(\epsilon_a^2+\epsilon_i^2)^2}{4}$ and in this range of values, R^2 is an increasing function of $\delta_{z,i}^2$. The value of $R^2 = 1/4$ is attained for

$$\delta_x^2 - \delta_{z,i}^2 = 2\epsilon_a\epsilon_i\delta_x - \frac{(\epsilon_i^2 + \epsilon_a^2)^2}{4} \tag{44}$$?deltazi?

for which the inner dispersion relation has a double root $\omega_+^2 = \omega_-^2 = \dfrac{\epsilon_i}{\epsilon_a}\delta_x$. When δ_{z_i} is decreased from this maximum value, the two roots become well separated.
Even if (43) has two roots, the two corresponding branches are always connected as shown above and thus form one family of ocean waves. Ocean waves satisfying (43) will be called *Modified Surface Waves (MSW)* in the following.

3.3 Summary: waves solutions in an unbounded ocean

tionUsualDisp⟩? In the preceding subsection, three types of waves have been identified. There are summarized in table (3) for the internal and acoustic waves and in table (4) for the surface waves.

	Internal Waves	Acoustic Waves
a) ($\epsilon_i = \epsilon_a = 0$)	X	X
b) ($\epsilon_i = 0, \epsilon_a \neq 0$)	X	$\dfrac{\omega_+^2}{\omega_{aw}^2} = 1 + \dfrac{\epsilon_a^4}{4(\delta_x^2 + \delta_z^2)}$
c) ($\epsilon_i \neq 0, \epsilon_a = 0$)	$\dfrac{\omega_-^2}{\omega_{iwr}^2} = 1 - \dfrac{\epsilon_i^4}{4(\delta_x^2 + \delta_z^2)}$	X
d) ($\epsilon_i \neq 0, \epsilon_a \neq 0$)	$\dfrac{\omega_-^2}{\omega_{iwr}^2} \approx 1 - \dfrac{(\epsilon_i^2 + \epsilon_a^2)^2}{4(\delta_x^2 + \delta_z^2)} + \dfrac{(\epsilon_i^2\epsilon_a^2)^2}{(\delta_x^2 + \delta_z^2)^2}\delta_x^2$	$\dfrac{\omega_+^2}{\omega_{aw}^2} \approx 1 + \dfrac{(\epsilon_i^2 + \epsilon_a^2)^2}{4(\delta_x^2 + \delta_z^2)} - \dfrac{(\epsilon_i^2\epsilon_a^2)^2}{(\delta_x^2 + \delta_z^2)^2}\delta_x^2$

Table 3: Modified Internal and Acoustic waves in an unbounded ocean. H: homogeneous, I: Incompressible, S: Stratified
y_unbounded_IA⟩

Figure (2) shows plots of the inner dispersion surface respectively for a stratified and compressible ocean while figure (3) shows the limiting cases. The values of the parameters ϵ_a, ϵ_i are those of table (2).
In a more realistic bounded ocean, their existences are guaranteed only if their vertical scales is (much) smaller than the ocean depth ($|\delta_z| \gg 1$) and if they do not interfere with the bottom or the surface of the ocean. The subject of the next section, "Wxaves in a bounded ocean", is to look at the impact of adding the boundary dispersion relation (30b).

	Surface Waves
a) $(\epsilon_i = \epsilon_a = 0)$	ω not specified and $\delta_{z,i} = \delta_x, \omega$
b) $(\epsilon_i = 0, \epsilon_a \neq 0)$	$\frac{\omega_+^2}{\omega_{aw}^2} = 1 + \frac{\epsilon_a^4}{4(\delta_x^2 + \delta_z^2)}$
c) $(\epsilon_i \neq 0, \epsilon_a = 0)$	$\frac{\omega_+^2}{\omega_{aw}^2} = 1 + \frac{\epsilon_a^4}{4(\delta_x^2 + \delta_z^2)}$
d) $(\epsilon_i \neq 0, \epsilon_a \neq 0)$	$\frac{\omega_+^2}{\omega_{aw}^2} = 1 + \frac{\epsilon_a^4}{4(\delta_x^2 + \delta_z^2)}$

Table 4: Modified Surface waves in an unbounded ocean. H: homogeneous, I: Incompressible, S: Stratified

ry_unbounded_S⟩

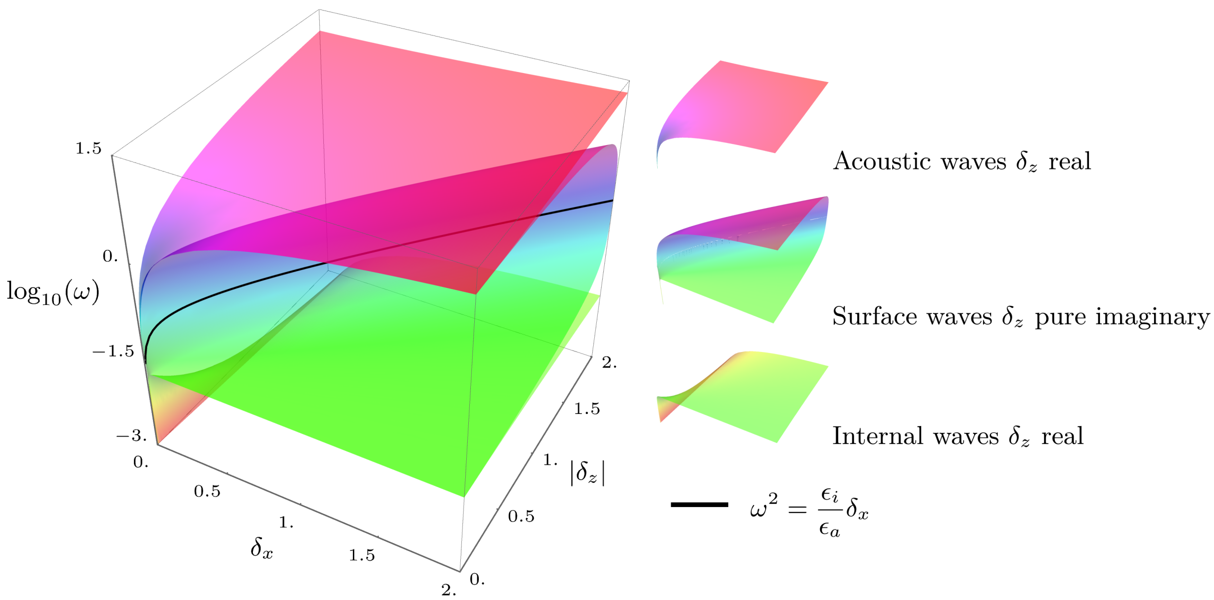


Figure 2: Inner dispersion surfaces in $(\delta_x, \delta_z, \log_{10}(\omega))$ space for a stratified and compressible ocean.

edcompressible⟩

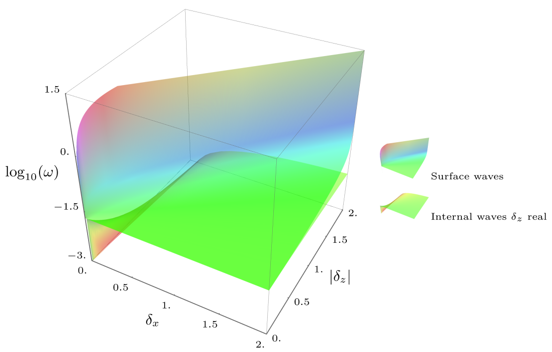
4 Waves in a bounded ocean

SectionGraphic⟩

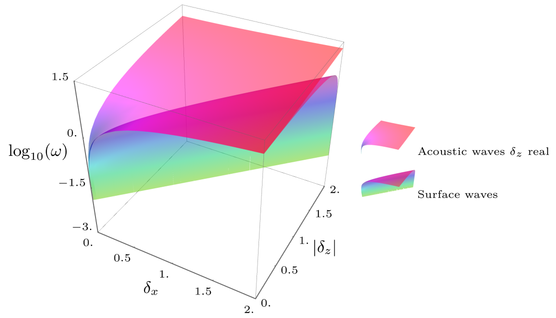
4.1 Graphical investigation of MSW, MAM & MIM

ionPotBranches⟩ The compressible and stratified ocean is now supposed to be bounded. Wave solutions must consequently satisfy both the inner (30a) and boundary (30b) dispersion relations. In phase space, they must lie at the intersection of the inner and boundary dispersion surfaces.

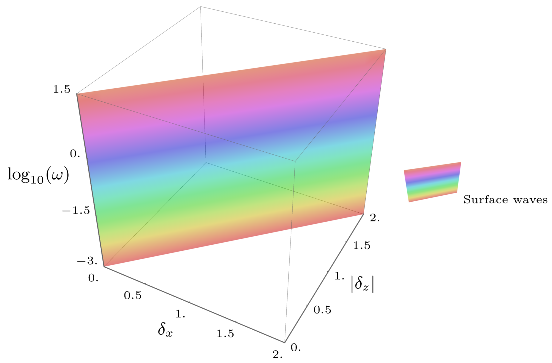
Branches of the boundary dispersion surface:
Figure (4) displays now simultaneously the inner and boundary dispersion surfaces for the same reference values of (ϵ_i) and (ϵ_a) (table 2).
For real vertical wave-numbers, the boundary dispersion-surface (light-blue surface) is a piece-wise vertical surface. It includes several branches: piece-wise tangent-like vertical branches for $\delta_z \approx n\pi$ (small values of ω) and for $\delta_z = \pi/2 + m\pi$ (large values of ω), with m and n two non-vanishing integer numbers. For pure imaginary wave-numbers now, the light-blue surface looks like an horizontal hyperbolic-tangent-like surface. Figure 4.d shows that real and pure-imaginary boundary-dispersion surfaces are very close for small vertical wave-numbers and cannot be distinguished graphically. This is a consequence of the fact that the tangent and hyperbolic-tangent functions have similar first-order Taylor expansion (with respect to δ_z or $\delta_{z,i}$).



(a) $\epsilon_a = 0$



(b) $\epsilon_i = 0$

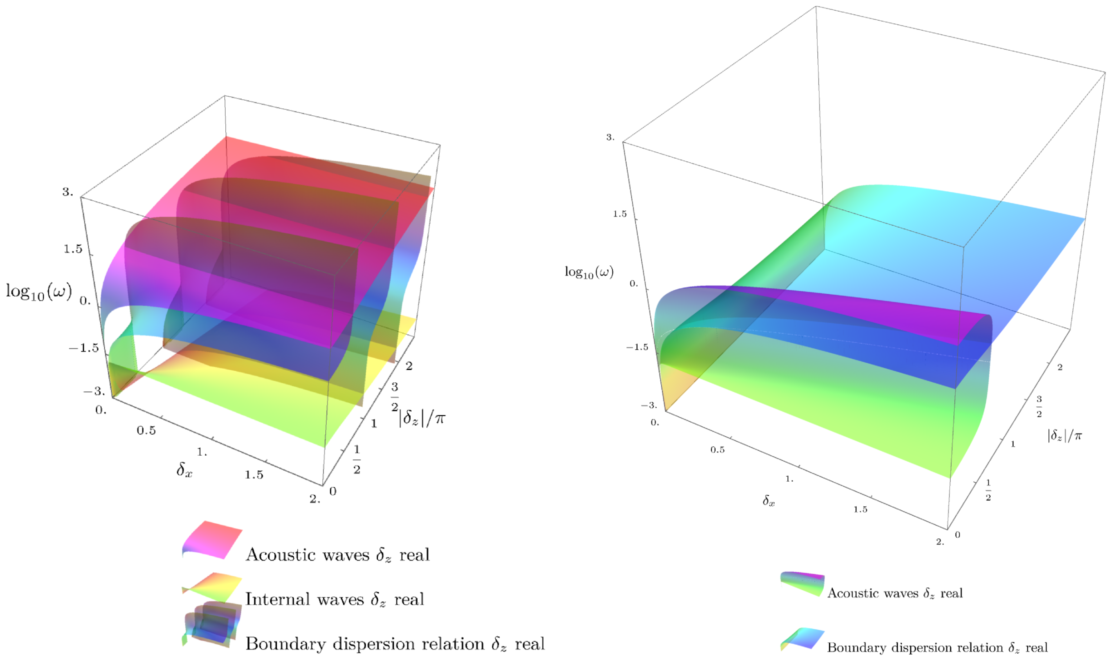


(c) $\epsilon_a = \epsilon_i = 0$

Figure 3: Inner dispersion surfaces for the limiting cases for the limiting cases of (a) an incompressible ocean (b) an homogeneous ocean and (c) an incompressible and homogeneous ocean.
implifiedcases⟩

Figure (4) confirms for this model of compressible, stratified, free-surface ocean, the existing of three types of intersections between the inner and boundary dispersion surfaces and, as a consequence, three regions of phase-space where waves can propagate in a bounded ocean. These intersections are shown by three lines of color points. The black-point intersection corresponds to wave-solutions propagating with a pure imaginary vertical wave-number in the middle-range pulsation region (figures 4.b, c and d). These are surface (edge) vertically-vanishing waves propagating in a compressible, stratified ocean, hereafter named Modified Surface Waves (MSW). Figure 4.d shows that they occupy a region of phase-space where the inner dispersion surface is approximately vertical and tangent to the $\delta_x \approx \delta_{z,i}$ plane meaning that the influence of compressibility and stratification (gravity) are both small.
The red and blue lines of points indicate that two remaining wave-solutions are possible, this time with real vertical wave-numbers. One (blue points) is an intersection of the boundary dispersion surface with the upper acoustic branch of the inner dispersion surface (figures 4.a, c and d), the other (red points) is an intersection of the same boundary dispersion surface with the lower stratification branch of the inner dispersion surface (figures 4.a and d). Since in both cases, vertical wave-numbers are quantified ($\delta_z = \pi/2 + m\pi$ and $\delta_z \approx n\pi$), the resulting wave solutions can be associated to respectively Modified Acoustic Modes (MAM) and Modified Internal Modes (MIM), these modes being modified by both compressibility and stratification (gravity).

Long-wave solutions:
In the vicinity of the origin for ($|\delta_x| \ll 1$ and $|\delta_z| \ll 1$) and for ($|\omega| \ll 1$) (i.e. for large wave-lengths and large periods), the upper and lower branches of the inner boundary surface on one side and the boundary dispersion surfaces for ($\delta_z \in \mathbb{R}$) and ($\delta_z \in i\,\mathbb{R}$) on the other side intersect for $\delta_z = 0$ in the small-pulsation region. The MAW and the MSW branches of the inner dispersion surface cannot be distinguished when δ_x and δ_z simultaneously tend toward 0 for larger pulsations (not shown in the phase-space region above figure (4.c).



(a) (b)

Figure 4: CORRIGER LEGENDE PLOT b)*Dispersion surfaces in* $(\delta_x,\ \delta_z,\ Log_{10}(\omega)$ *space and wave solutions.*
(a) Wave solutions with real δ_x*. Polychrome surfaces: Inner dispersion surface for real* δ_z *(acoustic and internal branches). Blue: Boundary dispersion surface. Black points: acoustic wave (upper branch) and internal wave solutions (lower branch).*
(b) Wave solutions with pure imaginary δ_x*. Polychrome surface: Inner dispersion surface for pure imaginary* δ_z *(surface gravity-wave branch). Blue: Boundary dispersion surface. Red points: surface wave solutions.*

gDispSolutions⟩

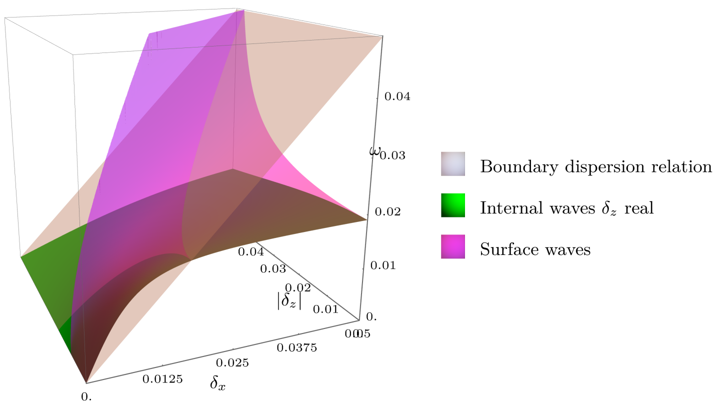


Figure 5: *Dispersion surfaces in* $(\delta_x,\ \delta_z,\ Log_{10}(\omega)$ *or* $\omega)$ *space and wave solutions.*
Wave solutions in the vicinity of the origin. Polychrome surface: Lower (respectively upper) δ_x*: inner dispersion surface for real* δ_z *(pure imaginary* δ_z*). Blue: Boundary dispersion surface. Black (red) points: surface wave solutions.*

ongpSolutions⟩**?**

In this figure (4.c), the black-point intersection is the continuation for small wave-numbers of the MSW intersection in figure (4.b) whereas the red-point intersection is not connected to the MIM intersection shown in (4.a). This continuous-by-part intersection leads to wave-solutions whatever δ_x in this region. For small values of δ_x the solution is given by the MIM branch (red points) whereas for larger values of δ_x it is given by the MSW branch (black points) and the vertical wave-number vanishes in between these solutions. Interestingly enough and contrary to the conclusions given by Dukowicz (2013), the barotropic mode pulsation cannot saturate at the buoyancy pulsation for increasing horizontal wave-number (δ_x) but transforms into a vertically evanescent surface-wave while its vertical number switches from real to pure imaginary. The resulting branch (in red) is an extension of the vertically evanescent wave solution presented in the previous section.

Summary:
Three types of wave solutions spreading on the three branches of the inner dispersion surface have thus been identified graphically: internal gravity (in a stratified ocean), acoustic (in a compressible ocean) and surface waves (in a free-surface ocean) are now successively investigated. Analytical expressions are systematically derived using Taylor developments of general roots $\omega_\pm$ with respect to small parameters (ϵ_i, ϵ_a), simple approximations of wave dispersion relations. When necessary, asymptotic relations are derived with respect to δ_x, δ_z or ω. The Taylor developments additionally give indications on how usual wave solutions can be modified by gravity and the ocean stratification (ϵ_i) and/or by the ocean compressibility (ϵ_a). Table (5) additionally gives orders of magnitudes of the pulsation and length scales of the various wave solutions.

	Notation	*Reference*	*10-m-deep*	$N = 10^{-2}\ s^{-1}$
Parameters	ϵ_a	0.13	$6.6\ 10^{-3}$	0.13
	ϵ_i	$2.0\ 10^{-2}$	$1.0\ 10^{-3}$	0.20
Acoustic cut-off	$2\pi\sqrt{H/g}/\omega_{c,a}$	30 *mn*	30 *mn*	9.6 *mn*
Internal cut-off	$2\pi\sqrt{H/g}/\omega_{c,i}$	1.7 *h*	1.7 *h*	10.5 *mn*
LMIM-LMSW cut-off	$2\pi H/\delta_{x,0}$	1367 *km*	1000 *km*	123 *km*
	$2\pi\sqrt{H/g}\omega_{x,0}$	1.9 *h*	1.7 *h*	10.5 *mn*
LMIM-$\delta_z(0)$	$2\pi H/\delta_{z,0}$	1379 *km*	62 *km*	125 *km*
	$2\pi\sqrt{H/g}\omega_{z,0}$	∞	∞	∞
MSW-neutral point	$2\pi H/\delta_{x,*}$	161 *km*	407 *m*	123 *km*
	$2\pi H/\delta_{z,*}$	164 *km*	407 *m*	2148 *km*
	$2\pi\sqrt{H/g}\omega_*$	12 *mn*	41.3 *s*	10.5 *mn*

Table 5: orders of magnitude of various scales. Notations refer to non-dimensional variables whereas orders of magnitude are given for dimensional quantities. Parameters for the "*Reference*" ocean are given in Table (2). "*10-m-deep*" ocean is a 10-m-deep *Reference* ocean and "$N = 10^{-2}\ s^{-1}$" refers to a *"Reference"* ocean with $N = 10^{-2}\ s^{-1}$.
TableOrdersMag⟩

4.2 Real δ_z: modified internal and acoustic waves

As shown in (3.2.1), as long as δ_z is not close to zero, the upper (acoustic) and lower (gravity) branches of the inner dispersion surface for real δ_z are well-separated and the MAM and MIM solutions can thus be studied independently for real vertical wave-numbers. We will investigate the case $\delta_z \approx 0$ later in this paragraph.

4.2.1 Development of internal-gravity modes modified by compressibility (MIM)

ionGraphicMIW⟩? Waves can propagate horizontally between the bottom and surface of the ocean as in a wave guide. Internal gravity modes are well-known examples (Gill (1982)). In the previous section, graphical inspections of wave solutions confirmed that gravity waves with quantified vertical wave-numbers could be found at the intersection of the inner and boundary dispersion surfaces (numerical solution shown with a red line on figure (4.a). We have also shown in section (3.2.1) that the root of the inner dispersion relation corresponding to internal gravity waves is well-approximated by ω_i^2. This root can be substituted in the boundary dispersion relation (30b) to obtain

$$\omega^2(\delta_x) \approx \omega_i^2 \implies \frac{\delta_x^2}{\delta_z/\tan(\delta_z) + \frac{\epsilon_i^2+\epsilon_a^2}{2}} \approx \frac{\epsilon_i^2\delta_x^2}{\delta_x^2 + \delta_z^2 + \frac{1}{4}\left(\epsilon_i^2+\epsilon_a^2\right)^2} \tag{45}$$?DispSysIntMode

Since the right-hand-side is small (bounded by ϵ_i^2), $\delta_z/\tan(\delta_z)$ has to be large and thus the vertical wave-number δ_z has to be close to $n\pi$ with n a non-zero integer. This agrees with the internal gravity wave solution found graphically in subsection (4.1).
In order to get a more precise approximation, we insert the surface dispersion relation (30b) into the non-approximated inner dispersion relation (30a) and proceed to a Taylor development with respect to δ_z in the vicinity of $\delta_{z,n} = n\pi$. We obtain an analytical relation of the form $\delta_z(\delta_x)$ and the pulsation can finally be expressed as a function of δ_x:

$$\delta_z(\delta_x) = \delta_{z,mim}(\delta_x) = \underbrace{\delta_{z,n}}_{\delta_{z,im}}\left(1 + \frac{\epsilon_i^2}{\delta_x^2+\delta_{z,n}^2} + \frac{(\delta_x^2-\delta_{z,n}^2)}{(\delta_x^2+\delta_{z,n}^2)^3}\epsilon_i^4\right) + \mathrm{O}((\epsilon_i^2+\epsilon_a^2)^3) + \mathrm{O}((\delta_z-\delta_{z,n})^3) \tag{46a}$$?ParamallMIM1?

$$\omega^2(\delta_x) = \omega_{mim}^2(\delta_x) = \epsilon_i^2\left[\frac{\delta_x^2}{\delta_x^2+\delta_{z,n}^2}\left(1 - \frac{2\epsilon_i^2\delta_{z,n}^2}{(\delta_x^2+\delta_{z,n})^2}\right) + \mathrm{O}((\epsilon_i^2+\epsilon_a^2)^2)\right] \tag{46b}$$?ParamallMIM2?

where $\delta_{z,im}$ and ω_{im} are defined in dimensional form in Table (1). This shows that the internal gravity modes are robust to compressibility since there is no dependency to ϵ_a at orders lower than 4. This agrees with the separation of the roots $\omega_\pm$ of the inner dispersion relation for real vertical wave-numbers.

4.2.2 Development of acoustic Modes modified by gravity (MAM)

ionGraphicMAW⟩? Here we use the fact that the root of the inner dispersion relation corresponding to internal gravity waves is well-approximated by ω_a^2. This root can be substituted in the boundary dispersion relation (30b) to obtain

$$1/\omega^2(\delta_x) \approx 1/\omega_a^2 \implies \frac{\delta_z/\tan(\delta_z) + \frac{\epsilon_i^2+\epsilon_a^2}{2}}{\delta_x^2} \approx \frac{\epsilon_a^2}{\delta_x^2 + \delta_z^2 + \frac{1}{4}\left(\epsilon_i^2+\epsilon_a^2\right)^2} \tag{47}$$?DispSysIntMode

Since the right-hand-side decreases fast with δ_z, $\delta_z/\tan(\delta_z)$ has to be small and thus the vertical wave-number δ_z has to be close to $\pi/2 + m\pi$ with m a non-zero integer.
Again in order to get a more precise approximation, we insert the surface dispersion relation (30b) into the non-approximated inner dispersion relation (30a) and proceed to a Taylor development with respect to δ_z in the vicinity of $\delta_{z,m} = \pi/2 + m\pi$. We obtain an analytical relation of the form $\delta_z(\delta_x)$ and the pulsation can finally be expressed as a function of δ_x:

$$\delta_z(\delta_x) = \delta_{z,mam}(\delta_x) = \underbrace{\delta_{z,m}}_{\delta_{z,am}} - \frac{(\delta_x^2-\delta_{z,m}^2)}{2\delta_{z,m}(\delta_x^2+\delta_{z,m}^2)}\epsilon_a^2 + \frac{\epsilon_i^2}{2\delta_{z,m}^2} + \mathrm{O}((\epsilon_i^2+\epsilon_a^2)^2) \tag{48a}$$ ParamallMAM1

$$\omega^2(\delta_x) = \omega_{mam}^2(\delta_x) = \frac{1}{\epsilon_a^2}\left[\delta_x^2 + \delta_{z,m}^2 - \frac{(\delta_x^2-\delta_{z,m}^2)}{(\delta_x^2+\delta_{z,m}^2)}\epsilon_a^2 + \epsilon_i^2 + \mathrm{O}((\epsilon_i^2+\epsilon_a^2)^2)\right] \tag{48b}$$ ParamallMAM2

(48c) {?}

The stratification has a first order contribution to vertical wavenumber and the wave frequency. However, it is clear that the associated impact is very small since, because we have $\delta_{z,m} \geq \pi/2$, ϵ_i^2 is negligible in front of $\delta_{z,m}^2$ in (48a), (48b).

4.2.3 Long waves

ongWavesrealdz⟩ For close to depth-independent waves (i.e. $\delta_z \approx 0$), as shown in (3.2.1), we cannot ensure that the two roots of the inner dispersion relations are well separated and thus are well approximated by ω_i^2, ω_a^2. This is particularly true when ϵ_i is close to ϵ_a which implies $R^2 \approx 1/4$. A specific development is needed. Inserting the boundary dispersion relation (30b) into the inner dispersion relation (30a) and making a second order Taylor development in δ_z leads to:

$$\delta_z^2 = \left(\delta_{x,0}^2 - \delta_x^2\right) \frac{(1 - \epsilon_a^2/2 + \epsilon_i^2/2)}{\left(1 + (\epsilon_a^2 + \epsilon_i^2)/2\right)\left(1 + \epsilon_i^2/3 - \frac{\delta_x^2}{3(1+(\epsilon_a^2+\epsilon_i^2)/2)^2}\epsilon_a^2\right)} \tag{49}$$ deltazsurface

with

$$\delta_{x,0}^2 = \frac{(1 + (\epsilon_a^2 + \epsilon_i^2)/2)(-\epsilon_a^4/4 + \epsilon_i^2(1 + \epsilon_i^2/4))}{(1 - \epsilon_a^2/2 + \epsilon_i^2/2)} \approx \epsilon_i^2$$

and the corresponding frequency is well approximated by

$$\omega^2 = \delta_x^2\left(1 + \frac{1}{6}\left(-\epsilon_i^2 + \epsilon_a^2(-3 + \epsilon_i^2)\right)\right)$$

The existence of these long waves can only appear for $\delta_x \leq \delta_{x,0} \approx \epsilon_i$. This implies $\omega^2 \approx \delta_x^2 \leq \epsilon_i\delta_x \leq \frac{\epsilon_i}{\epsilon_a}\delta_x$ (since $\epsilon_a < 1$). And thus using the bounds on ω_-^2, ω_+^2 given by (36), this shows that these waves are always from the internal waves branch (ω_-^2), and not from the acoustic wave branch (ω_+^2). That is to say that there is no intersection between the boundary dispersion relation and the (real) acoustic waves branch for δ_z close to zero.

For $\delta_x \geq \delta_{x,0}$, long waves are from the surface waves branch as shown in the next subsection.

4.3 Pure imaginary δ_z: surface acoustic-gravity waves

ionGraphicMSW⟩? "Surface waves" generally refer to wave propagating horizontally as anomalies of the ocean free-surface Gill (1982). In the vertical direction, these surface wave anomalies are "evanescent" meaning that, with the notation chosen in the present study, the vertical wave-number (δ_z) is a purely imaginary complex number. An alternative way to introduce "surface waves" is to introduce them as a limiting case of internal gravity mode (Duckowicz, 2015): they are then referred to as a barotropic mode with mode-number "n = 0" (using the notations of Section (4)).

A MSW defined by its triplet ($\delta_x,\ \delta_z,\ \omega$) must satisfy both the inner (30a) and boundary (30b) dispersion relations for $\delta_z\ =\ i\ \delta_{z,i}$. We recall below the inner-dispersion relation (43):

$$\delta_x^2 - \delta_{z,i}^2 = \epsilon_i^2\frac{\delta_x^2}{\omega^2} + \epsilon_a^2\omega^2 - \frac{(\epsilon_a^2 + \epsilon_i^2)^2}{4} \tag{50}$$?eqinternecompl

and the surface boundary relation (30b):

$$\omega^2 = \frac{\delta_x^2}{\frac{\delta_{z,i}}{\tanh(\delta_{z,i})} + \frac{\epsilon_a^2+\epsilon_i^2}{2}} \tag{51}$$ eqsurfacecompl

In an homogeneous, incompressible ocean ($\epsilon_i = \epsilon_a = 0$), we get $\delta_{z,i} = \delta_x$ and the boundary dispersion relation simplifies to the usual relation $\omega^2 = \omega_{sw}^2 = \delta_x \tanh(\delta_x)$.

In the more general case relation (51) helps us to bound the magnitude of the right hand side of the inner-dispersion relation. Indeed, using $\dfrac{\delta_{z,i} + 1}{2} \leq \dfrac{\delta_{z,i}}{\tanh(\delta_{z,i})} \leq \delta_{z_i} + 1$, we have

$$\epsilon_i^2\frac{\delta_x^2}{\omega^2} \leq \epsilon_i^2\left(\delta_{z,i} + 1 + \frac{\epsilon_a^2 + \epsilon_i^2}{2})\right), \qquad \epsilon_a^2\omega^2 \leq 2\epsilon_a^2\delta_x^2$$

and then, using the smallness of the parameters ϵ_a, ϵ_i, we conclude that $\delta_{z,i}$ is close to δ_x. This relation is often postulated in textbooks to reduce the number of variables. Vertical polarization relations are then functions of δ_x only (Gill (1982)) and, as a consequence, the only-remaining dispersion relation is the boundary dispersion relation (30b) for pure imaginary vertical wave-number ($\delta_z = i\delta_x$) or an approximation of this relation $\omega^2 = \delta_x \tanh \delta_x$. In this case, $\delta_{z,i}$ is just the vertical length-scale for wave decrease downward from the surface and, for (very) long waves $\delta_x \gg 1$, the surface wave is approximately depth-independent.

4.3.1 Swell-like approximation of MSW

The crude ($\delta_{z,i} \approx \delta_x$) assumption is sufficiently accurate to recover usual swell-like approximations (table 1), i.e. for sufficiently large δ_x (or δ_z). A more accurate expression can be given by

$$\delta_z = \delta_x - \frac{\epsilon_i^2 + \epsilon_a^2}{2}, \qquad (52) \quad \boxed{\texttt{longMSW}}$$

which has been deduced just by replacing $\tanh(\delta_z)$ by 1 in the boundary dispersion relation and by performing a first order Taylor development in $\epsilon_i^2, \epsilon_a^2$. but it does not embrace the behavior of long MSW observed in Section (4). A higher order approximation of $\delta_{z,i}$ is consequently required.
For long waves and close to depth-independent vertical profile (δ_z close to zero), as in subsection (4.2.3), we perform a Taylor development in δ_z around zero to get a similar, reverse, expression:

$$\delta_z^2 = \left(\delta_x^2 - \delta_{x,0}^2\right) \frac{(1 - \epsilon_a^2/2 + \epsilon_i^2/2)}{\left(1 + (\epsilon_a^2 + \epsilon_i^2)/2\right)\left(1 + \epsilon_i^2/3 - \frac{\delta_x^2}{3(1+(\epsilon_a^2+\epsilon_i^2)/2)^2}\epsilon_a^2\right)} \qquad (53) \quad \boxed{\texttt{deltazsurface}}$$

with

$$\delta_{x,0}^2 = \frac{(1 + (\epsilon_a^2 + \epsilon_i^2)/2)(-\epsilon_a^4/4 + \epsilon_i^2(1 + \epsilon_i^2/4))}{(1 - \epsilon_a^2/2 + \epsilon_i^2/2)} \approx \epsilon_i^2$$

The corresponding frequency is obtained by inserting (52) or (53) in the surface dispersion relation (30b). For $\delta_{z,i}^2$ to remain positive, (53) implies that we must have

$$\delta_x^2 \geq \delta_{x,0}^2 \approx \epsilon_i^2$$

Orders of magnitudes:
For the 4000-m-deep reference ocean (Table 2), the horizontal length scale associated to the transformation of the MSW into the long MIM ($\lambda_{x,lmsw} = 2\pi/(\delta_{x,lmsw}/4000)$) reaches 1367 *km* against 62 *km* for the same 10-m-deep ocean. When δ_x keeps on decreasing below $\delta_{x,lmsw}$, δ_z increases monotically to a maximum value $\delta_{z,lmsw}(\delta_x = 0)$. This vertical length scale reaches 1379 *km* for the 4000-m-deep reference ocean and 62 *km* for the same 10-m-deep ocean. The longer the horizontal length scale of the oscillation, the shorter the vertical length scale and the weaker the stratification (vanishing ϵ_i) the longer the horizontal length-scale $\lambda_{x,lmsw}$. This long MIM solution is a low-frequency oscillation of the ocean due gravity and associated to the stratification of the ocean. It disappears when the stratification vanishes and the ocean can be assimilated to an homogeneous layer of water. It does persist in an incompressible ocean but is slightly modified by compressibility.

4.4 Summary: waves solutions in a bounded ocean

Main differences between the classical theory.
The most important difference is related to long waves (i.e. close to depth-independent vertical profile).

Internal Waves	Acoustic Waves	Surface Waves
$\delta_{z,n} = n\pi, \quad n \geq 1$	$\delta_{z,m} = \pi/2 + m\pi, \quad m \geq 0$	$\delta_z \geq 1$
$\delta_z = \delta_{z,n}\left(1 + \frac{\epsilon_i^2}{\delta_x^2+\delta_{z,n}^2} + \frac{(\delta_x^2-\delta_{z,n}^2)}{(\delta_x^2+\delta_{z,n}^2)^3}\epsilon_i^4\right)$	$\delta_z = \delta_{z,m} - \frac{(\delta_x^2-\delta_{z,m}^2)}{2\delta_{z,m}(\delta_x^2+\delta_{z,m}^2)}\epsilon_a^2 + \frac{\epsilon_i^2}{2\delta_{z,m}^2}$	$\delta_z = \delta_x - \frac{\epsilon_i^2+\epsilon_a^2}{2}$
$\omega^2 = \frac{1}{\epsilon_a^2}\left[\delta_x^2 + \delta_{z,m}^2 - \frac{(\delta_x^2-\delta_{z,m}^2)}{(\delta_x^2+\delta_{z,m}^2)}\epsilon_a^2 + \epsilon_i^2\right]$	$\omega^2 = \epsilon_i^2\left[\frac{\delta_x^2}{\delta_x^2+\delta_{z,n}^2}\left(1 - \frac{2\epsilon_i^2\delta_{z,n}^2}{(\delta_x^2+\delta_{z,n}^2)^2}\right)\right]$	$\omega^2 = \delta_x^2\frac{\tanh(\delta_z)}{\delta_z} \approx \delta_x\tanh(\delta_x)$

Table 6: Modified Internal and Acoustic waves in an unbounded ocean. H: homogeneous, I: Incompressible, S: Stratified

ry_bounded_IA⟩?

	Surface Waves
a) $(\epsilon_i = \epsilon_a = 0)$	ω not specified and $\delta_{z,i} = \delta_x$, ω
b) $(\epsilon_i = 0, \epsilon_a \neq 0)$	$\dfrac{\omega_+^2}{\omega_{aw}^2} = 1 + \dfrac{\epsilon_a^4}{4(\delta_x^2+\delta_z^2)}$
c) $(\epsilon_i \neq 0, \epsilon_a = 0)$	$\dfrac{\omega_+^2}{\omega_{aw}^2} = 1 + \dfrac{\epsilon_a^4}{4(\delta_x^2+\delta_z^2)}$
d) $(\epsilon_i \neq 0, \epsilon_a \neq 0)$	$\dfrac{\omega_+^2}{\omega_{aw}^2} = 1 + \dfrac{\epsilon_a^4}{4(\delta_x^2+\delta_z^2)}$

Table 7: Modified Surface waves in an unbounded ocean. H: homogeneous, I: Incompressible, S: Stratified

ary_bounded_S⟩?

5 Discussion, conclusion

tionDiscussion⟩ Dukowicz's acoustic, gravity and surface wave Lagrangian model based on two dispersion relations Dukowicz (2013) has been re-derived with in a fully Eulerian context. Not that this later approach is not physically more coherent but its derivation is just simpler. Acoustic and internal wave rays propagating in an unbounded ocean have first been re-visited with a single dispersion equation. Smith's acoustic modes (Smith (2015)) have been recovered with this model and a both a short and a long-wave approximation of these acoustic modes has been proposed. Well-known internal modes have also been revisited in a compressible, stratified, free-surface ocean. Surface waves (edge waves) have been systematically investigated in a compressible and stratified ocean. Long surface waves have also been revisited questioning their classification as barotropic modes. Dukowicz effort to give a coherent and complete framework for the description of geophysical waves is thus carried on further integrating new wave solutions and clarifying the description of key regions of phase-space such as long waves.

$(\delta_x,\ \delta_z,\ \omega)$ phase-space has been explored geometrically to identify possible ocean waves and modes: intersections of the inner and boundary dispersion surfaces are localized numerically and are then used as references to derive wave approximations. This original investigation in phase-space when associated with systematic Taylor developments with respect to small compressibility and stratification parameters, provides an adapted approach to circumvent the non-linearity and the transcendental character of the boundary dispersion relation and the (high) fourth-order dependency in the pulsation ω of the inner dispersion relation. In $(\delta_x,\ \delta_z,\ \omega)$ phase-space, the inner dispersion surface has been decomposed into three distinct branches with a one-to-one correspondence along the ω axis. For large pulsations, the *acoustic branch* is well described by the simple factorizing function ω_a and is bounded for vanishing $(\delta_x,\ \delta_z)$ by the acoustic cut-off pulsation $(\omega_{MAM,-})$. Acoustic waves propagating in the ocean as in an unbounded medium (MAW) belongs to this branch together with acoustic modes modified by gravity (MAM) which can be found at the intersection of this acoustic branch with the dispersion boundary surface given by the transcendental relation (30b).
For low pulsations (long waves), the *internal gravity branch* of the inner dispersion surface is in turn well approximated by the internal factorizing function (ω_i). The pulsation of the internal waves belong-

ing to this branch is bounded by the internal parameter ϵ_i (the cut-off pulsation is N, the Brunt-Väisälä). Internal gravity rays belong to this surface. At the intersection of the lower-pulsation gravity branch with the *boundary dispersion surface*, are found internal gravity modes (MIM).
Compressibility-induced perturbations to MIM and gravity-induced perturbations to MAM have been shown to be high-ordered perturbations in small parameters ϵ_i and ϵ_a. These wave-modes have been shown to be well-separated graphically and analytically. They are well consequently well approximated respectively by the stratification and acoustic factorizing functions ω_i and ω_a. The situation is somehow different for MSW. This latter wave solution is a linear combination of the real roots ($\omega_\pm$), but these two roots might not be well-separated for waves with pure-imaginary vertical wave-numbers. A consequence is that unlike for real vertical wave-numbers, the contributions of the acoustic factorizing function ω_a and of the stratification factorizing function ω_i cannot be meaningfully separated.
Well-know relations for internal wave-rays or acoustic waves in an unbounded ocean and for internal-gravity or acoustic wave-modes in a bounded ocean have been recovered. Lower-order perturbations due to stratification and gravity (ϵ_i) and to compressibility (ϵ_a) have been proposed for each type of waves. Acoustic Lamb-waves have been examined as a particular case: they are solution of the proposed (first-order) wave-model only under an additional "rigid-lid" assumption.

Between these upper and lower branches of the inner dispersion surface, waves can propagate with middle-range pulsations only if their vertical wave-number is a pure-imaginary complex. The intersection with the *boundary dispersion surface* hosts surface waves (MSW) which can be "modified" by compressibility and by gravity (stratification). The medium-range branch is indeed "folded" both by the ocean compressibility and by the vertical density stratification. In both cases the surface is bounded: an upper bound for vanishing wave-numbers due to the acoustic cut-off an a lower bound for large pulsations due to the Brunt-Väisälä pulsation cut-off. When pulsation is increased from very low pulsations, MSW intersections can change from surface waves modified by stratification to surface waves modified by compressibility. In the transition region, the two wave solutions merge into a single "neutral" solution at the limit of the region where pulsation are pure imaginary complex numbers. In the long-wave limit now, a modified "n=0" gravity mode (MIM-0) exists only in a stratified ocean. It can be associated to a low-frequency oscillation of the stratified water layer. It is part of the small-pulsation branch of the real-δ_z inner dispersion surface. It thus cannot be the asymptotic limit of the shorter swell-like MSW branch. Usual approximations for long surface waves ($\delta_z = \delta_x$ and $\omega = \delta_x$) can be recovered from the results of the present studies in two ways. It can either be introduced as a long-wave approximation of MSW for homogeneous (non-stratified) ocean or as a long-wave approximation of mode-0 MIM. In this latter case, a vanishing vertical wave-number is recovered only for $\epsilon_i = \epsilon_a = 0$. MSW are thus primarily edge wave: the free-surface anomaly is one way or the other translated into a pressure anomaly by gravity, wave propagation is then achieved by a simple compensation mechanism based on the conservation of mass and momentum. If the ocean is homogeneous and incompressible, the pressure is an harmonic function and the horizontal and vertical length-scales are equal ($\delta_x = \delta_z$). Compressibility and stratification modifies this and as shown in Section (3.2.2), in this case, the difference between the horizontal and vertical wave-number increases. The vertical wave-number decreases consequently faster and the MSW variations become very small over the vertical. If δ_x is further decreased, δ_z finally vanishes for small but finite δ_x.
If the horizontal wave-number keeps on decreasing, a surface wave can only propagate as a mode-0 MIM. In this case, the vertical wave-number must increase for a decreasing horizontal wave-number due to the stratification barrier: the stronger the stratification, the shallower the in-depth penetration of the surface wave.

In a stratified ocean (whether compressible or not), long surface waves (LMSW in the vicinity of the origin in phase-space) cannot propagate and are replaced by (mode-0) MIM solutions. The longest surface wave that can propagate is barotropic (depth-independent): its horizontal wave-number is $\delta_{x,lmsw}$ and its vertical wave-number is zero. When the stratification weakens, $\delta_{x,lmsw}$ tends toward 0 and the

resulting long waves approaches LSW (with $\sqrt{gH}$ phase and group velocity and $\delta_x = \delta_z$). Propagation in an homogeneous ocean can be studied with the present model by setting to zero the stratification parameter ϵ_i together with the last (advective) term on the right-hand-side of the inner dispersion relation (30a): $-(\epsilon_i^2 + \epsilon_a^2)^2/4$.
Further inspection of MSW waves has shown that there exists a particular triplet of properties $(\delta_x,\ \delta_z,\ \omega)$ for which the pair of real roots (35) merges and the discriminant of the second-order polynomial equation in the pulsation vanishes. This double root is located in the region where the inner dispersion surface is vertical and contributions due to compressibility and stratification are smaller. This MSW solution is very peculiar in the sense that it is located at the edge of the region of $(\delta_x,\ \delta_z,\ \omega)$ phase-space where wave solutions are divergent as time goes on. Ocean waves originating in this region of phase space might have singular behaviour.

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A δ_z real or pure imaginary

⟨kzreal⟩ Starting from

$$\begin{aligned} F''(z) + k_z^2 F(z) &= 0 \\ F'(0) + \left(\frac{1}{2D_0} - \frac{g k_x^2}{\Omega^2}\right) F(0) &= 0 \\ F(-H) &= 0 \end{aligned}$$

with $k_z^2 = \left(k_x^2 \dfrac{N^2 - \Omega^2}{\Omega^2} + \dfrac{\Omega^2}{c_s^2} - \dfrac{1}{4D_0^2}\right)$.

Let $s = \dfrac{z+H}{H}$ and $G(s(z)) = F(z)$ we write:

$$\begin{aligned} G''(s) + (Hk_z)^2 G(s) &= 0 \\ G'(1) + \left(\frac{H}{2D_0} - \frac{gHk_x^2}{\Omega^2}\right) G(1) &= 0 \\ G(0) &= 0 \end{aligned}$$

Introducing $\epsilon_i^2 = \dfrac{N^2 H}{g}, \epsilon_a^2 = \dfrac{gH}{c_s^2}, \delta_x = k_x H, \delta_z = k_z H, \omega = \Omega\sqrt{\dfrac{H}{g}}$ and using $\dfrac{H}{D_0} = \epsilon_i^2 + \epsilon_a^2$

$$G''(s) + \delta_z^2 G(s) = 0 \tag{54}$$ sturm1

$$G'(1) + \left(\frac{\epsilon_i^2 + \epsilon_a^2}{2} - \frac{\delta_x^2}{\omega^2}\right) G(1) = 0 \tag{55}$$ sturm2

$$G(0) = 0 \tag{56}$$ sturm3

with $\delta_z^2 = \left(\delta_x^2 \frac{\epsilon_i^2 - \omega^2}{\omega^2} + \epsilon_a^2\omega^2 - \frac{(\epsilon_i^2+\epsilon_a^2)^2}{4}\right)$
Multiplying (54) by $\overline{G(s)}$ and integrating over [0, 1] we get:

$$\int_0^1 G''(s)\overline{G(s)} + \delta_z^2 \int_0^1 |G(s)|^2 = 0$$

$$-\int_0^1 |G'(s)|^2 + \delta_z^2 \int_0^1 |G(s)|^2 + G'(1)\overline{G(1)} - G'(0)\overline{G(0)} = 0$$

and using (55), (56)

$$\delta_z^2 \int_0^1 |G(s)|^2 + \left(\frac{\delta_x^2}{\omega^2} - \frac{\epsilon_i^2+\epsilon_a^2}{2}\right)|G(1)|^2 = \int_0^1 |G'(s)|^2$$

Using the Poincaré inequality $\int_0^1 |G(s)|^2 \leq \int_0^1 |G'(\sigma)|^2 d\sigma$ and $|G(1)|^2 \leq \int_0^1 |G'(\sigma)|^2 d\sigma$, we obtain:

$$\delta_z^2\mu + \left(\frac{\delta_x^2}{\omega^2} - \frac{\epsilon_i^2+\epsilon_a^2}{2}\right)\nu = 1 \tag{57}$$ sturm4

with $\mu \leq 1$ and $\nu \leq 1$.
Taking the imaginary part of (57)

$$\mu\left(\epsilon_a^2\Im[\omega^2] + \delta_x^2\epsilon_i^2\Im[1/\omega^2]\right) + \nu\delta_x^2\Im[1/\omega^2] = 0$$

or, using $\Im[1/\omega^2] = -\Im[\omega^2]/|\omega|^4$,

$$\Im[\omega^2]\left(\mu\left(\epsilon_a^2 - \epsilon_i^2\frac{\delta_x^2}{|\omega|^4}\right) - \nu\frac{\delta_x^2}{|\omega|^4}\right) = 0$$

Let's suppose that $\Im[\omega^2] \neq 0$. We will show below that, in this case, solutions can exist only for non physical values of ϵ_i, ϵ_a satisfying $\max(\epsilon_i, \epsilon_a) > \sqrt{2}$.
If $\Im[\omega^2] \neq 0$, we deduce that

$$\mu\left(\epsilon_a^2\frac{|\omega|^4}{\delta_x^2} - \epsilon_i^2\right) - \nu = 0 \tag{58}$$ eqmunu1

Now taking the real part:

$$\mu\left(-\delta_x^2 + \delta_x^2\epsilon_i^2\Re[1/\omega^2] - \frac{(\epsilon_i^2+\epsilon_a^2)^2}{4} + \epsilon_a^2\Re[\omega^2]\right) + \nu\left(\delta_x^2\Re[\frac{1}{\omega^2}] - \frac{\epsilon_i^2+\epsilon_a^2}{2}\right) = 1$$

or, using $\Re[1/\omega^2] = \Re[\omega^2]/|\omega|^4$,

$$\mu\left(-\delta_x^2 - \frac{(\epsilon_i^2+\epsilon_a^2)^2}{4} + (\frac{\delta_x^2\epsilon_i^2}{|\omega|^4} + \epsilon_a^2)\Re[\omega^2]\right) + \nu\left(\frac{\delta_x^2}{|\omega|^4}\Re[\omega^2] - \frac{\epsilon_i^2+\epsilon_a^2}{2}\right) = 1$$

or using (58)

$$\mu\left(-\delta_x^2 - \frac{(\epsilon_i^2+\epsilon_a^2)^2}{4} + 2\epsilon_a^2\Re[\omega^2]\right) - \nu\left(\frac{\epsilon_i^2+\epsilon_a^2}{2}\right) = 1 \tag{59}$$ eqmunu2

Note that (59) implies $\Re[\omega^2] \geq 0$.
Eqs (59) and (58) are summarized in

$$\begin{aligned} \alpha\mu - \beta\nu &= 1 \\ \gamma\mu - \nu &= 0 \end{aligned} \tag{60}$$ eqmunu

with

$$\alpha = -\delta_x^2 - \frac{(\epsilon_i^2 + \epsilon_a^2)^2}{4} + 2\epsilon_a^2 \Re[\omega^2], \beta = \frac{\epsilon_i^2 + \epsilon_a^2}{2}, \gamma = \epsilon_a^2 \frac{|\omega|^4}{\delta_x^2} - \epsilon_i^2$$

Using $\beta > 0$, it is easy to check that (60) has solutions (μ, ν) positive and with magnitude less than one only if

$$\gamma > 1 \text{ and } \alpha > (1+\beta)\gamma$$

or

$$0 \leq \gamma \leq 1 \text{ and } \alpha > 1 + \beta\gamma$$

And so in order to have $\Im[\omega^2] \neq 0$, the conditions above immediately exclude the cases $\epsilon_a = 0$ (which leads to $\gamma \leq 0$) and $\Re[\omega^2] \leq 0$ (which leads to $\alpha \leq 0$). They will not be considered below.

- First case: $0 \leq \gamma \leq 1$
  This implies:

$$|\omega|^4 \leq \frac{\delta_x^2}{\epsilon_a^2}(1 + \epsilon_i^2) \tag{61}$$ maj1

  $\alpha > 1 + \beta\gamma$ writes:

$$\Re[\omega^2] > \frac{1}{2\epsilon_a^2}\left[1 + \delta_x^2 + \frac{(\epsilon_a^2 + \epsilon_i^2)^2}{4} + \frac{\epsilon_i^2 + \epsilon_a^2}{2}\left(\epsilon_a^2 \frac{|\omega|^4}{\delta_x^2} - \epsilon_i^2\right)\right]$$

  Now using the inequalities $|\Re[\omega^2]|^2 \leq |\omega|^4$ and (61), we get

$$\frac{\delta_x^2}{\epsilon_a^2}(1 + \epsilon_i^2) \geq |\omega|^4 \geq \left(\frac{1}{2\epsilon_a^2}\left[1 + \delta_x^2 + \frac{(\epsilon_a^2 + \epsilon_i^2)^2}{4} + \frac{\epsilon_i^2 + \epsilon_a^2}{2}\left(\epsilon_a^2 \frac{|\omega|^4}{\delta_x^2} - \epsilon_i^2\right)\right]\right)^2 \tag{62}$$ systeme1

  Using a computing algebra software to simplify the technical exercice, we can prove that (62) has solutions if and only if

$$\epsilon_i^2 > \sqrt{4 + \epsilon_a^4}$$

  which requires $\epsilon_i > \sqrt{2}$.

- Second case: $\gamma > 1$
  This implies:

$$|\omega|^4 \geq \frac{\delta_x^2}{\epsilon_a^2}(1 + \epsilon_i^2) \tag{63}$$ maj2

  $\alpha > (1+\beta)\gamma$ can be written as:

$$\gamma < \frac{\alpha}{1+\beta}$$

  or

$$|\omega|^4 \leq \frac{\delta_x^2}{\epsilon_a^2}\left[\epsilon_i^2 + \frac{1}{1 + \frac{\epsilon_a^2+\epsilon_i^2}{2}}\left(-\delta_x^2 - \frac{(\epsilon_i^2 + \epsilon_a^2)^2}{4} + 2\epsilon_a^2 \Re[\omega^2]\right)\right]$$

Now using $0 \leq \Re[\omega^2] \leq |\omega|^2$ and adding (63), we get:

$$\frac{\delta_x^2}{\epsilon_a^2}(1+\epsilon_i^2) \leq |\omega|^4 \leq \frac{\delta_x^2}{\epsilon_a^2}\left[\epsilon_i^2 + \frac{1}{1+\frac{\epsilon_a^2+\epsilon_i^2}{2}}\left(-\delta_x^2 - \frac{(\epsilon_i^2+\epsilon_a^2)^2}{4} + 2\epsilon_a^2|\omega|^2\right)\right]$$

which has non trivial solutions if and only if

$$\epsilon_a^2 > 2 + \epsilon_i^2$$

which requires $\epsilon_a > \sqrt{2}$.

This concludes the proof. Adding the condition $\epsilon_a \neq 0$, we get. If $\epsilon_a = 0$ or $\max(\epsilon_i, \epsilon_a) \leq \sqrt{2}$, then $\Im[\omega^2]$ is zero. This also leads to $\Im[\delta_z^2] = 0$ and so that δ_z is either real or pure imaginary.