Applied Time Series Assigment

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1 Theoretical Exercises

1.1 Exercise 1: *The IMA(1,1) Model*

1) Let's find the permanent component and the transitory component of an IMA(1,1) process:

$$\Delta y_t = \varepsilon_t + \theta \varepsilon_{t-1}$$

Expanding:

$$\begin{aligned} y_t - y_{t-1} &= \varepsilon_t + \theta \varepsilon_{t-1} \\ (1-L) y_t &= (1+\theta L)\varepsilon_t \end{aligned}$$

If we define $\Theta(L) = 1 + \theta L$, we have, based on the decomposition proposed by Beveridge and Nelson:

$$y_t = \Theta(1)(1-L)^{-1}\varepsilon_t + [\Theta(L) - \Theta(1)](1-L)^{-1}\varepsilon_t$$

With $\Theta(1) = 1 + \theta$, we expand:

$$\begin{aligned} y_t &= (1+\theta)(1-L)^{-1}\varepsilon_t + (1 + \theta L - 1 - \theta)(1-L)^{-1}\varepsilon_t \\ &= (1+\theta)(1-L)^{-1}\varepsilon_t - \theta\varepsilon_t \end{aligned}$$

By identification, we obtain that:

- $(1+\theta)(1-L)^{-1}\varepsilon_t$: **Permanent Component**
- $\theta\varepsilon_t$: **Transitory Component**

1.2 Exercise 2: *The trend stationary model*

Let's consider the trend stationary model $Y_t = \alpha + \beta t + \varepsilon_t$ with $\varepsilon_t \sim \mathcal{N}(0, \sigma^2)$.

1) Let's compute the variance of Y_t:

$$
\begin{aligned}
V(Y_t) &= E(Y_t^2) - E(Y_t)^2 \\
&= E((\alpha + \beta t + \varepsilon_t)^2) - (\alpha + \beta t)^2 \\
&= \alpha^2 + \beta^2 t^2 + \sigma^2 + 2\alpha\beta t + 2\alpha E(\varepsilon_t) + 2\beta t E(\varepsilon_t) - \alpha^2 - \beta^2 t^2 - 2\alpha\beta t \\
&= \sigma^2.
\end{aligned}
$$

Therefore, the variance of Y_t is time invariant.

Let's show that the effect of ε_t on Y_t dissipates asymptotically.

By the Bienaymé-Tchebychev inequality, we have:

$$
P\left(|\varepsilon_t| > (\alpha + \beta t)/\sqrt{(t)}\right) \leq \frac{\sigma^2 t}{(\alpha + \beta t)^2}.
$$

and

$$
P\left(\frac{\varepsilon_t}{\alpha + \beta t} > 1/\sqrt{t}\right) \leq \frac{\sigma^2 t}{(\alpha + \beta t)^2}
$$

Therefore:

$$
P\left(\left|\frac{\varepsilon_t}{\alpha + \beta t}\right| > 0\right) \to 0 \quad \text{as } t \to \infty.
$$

Thus, the effect of ε_t on Y_t dissipates asymptotically.

2) Let's compute the variance of the forecast error:

$$
\begin{aligned}
V(Y_{t+h} - E[Y_{t+h}|It]) &= V(\alpha + \beta(t+h) + \varepsilon_{t+h} - E[\alpha + \beta(t+h) + \varepsilon_{t+h}|It]) \\
&= V(\varepsilon_{t+h} - E[\varepsilon_{t+h}|It]) \\
&= V(\varepsilon_{t+h}) \\
&= \sigma^2.
\end{aligned}
$$

So the variance is constant.

3) Let's differentiate the data:

$$
\begin{aligned}
Y_{t+1} - Y_t &= \alpha + \beta(t+1) + \varepsilon_{t+1} - (\alpha + \beta t + \varepsilon_t) \\
&= \beta + \varepsilon_{t+1} - \varepsilon_t.
\end{aligned}
$$

The trend has been removed. The characteristic polynomial is $1 - z = 0$. So we have $z = 1 \in \{-1, 1\}$.

Therefore, there is a moving average unit root.

1.3 Exercise 3: *The MA(2) Model*

Consider the process $Y_t \sim MA(2)$, $Y_t = \varepsilon_t - \theta_1\varepsilon_{t-1} - \theta_2\varepsilon_{t-2}$, where $\varepsilon_t \sim WN(0, \sigma^2)$.

1) Let's compute the mean of Y_t:

$$\begin{aligned} E[Y_t] &= E[\varepsilon_t - \theta_1\varepsilon_{t-1} - \theta_2\varepsilon_{t-2}] \\ &= 0. \end{aligned}$$

Let's compute the variance of Y_t:

$$\begin{aligned} V(Y_t) &= E(Y_t^2) - [E(Y_t)]^2 \\ &= E(\varepsilon_t^2) + \theta_1^2 E(\varepsilon_{t-1}^2) + \theta_2^2 E(\varepsilon_{t-2}^2) + E[\text{cross terms}] - 0 \\ &= \sigma^2 + \theta_1^2\sigma^2 + \theta_2^2\sigma^2 \\ &= \sigma^2(1 + \theta_1^2 + \theta_2^2). \end{aligned}$$

2) Let's compute the autocorrelation function for this process:

$$\begin{aligned} \gamma_0 &= V(Y_t) = \sigma^2(1 + \theta_1^2 + \theta_2^2), \\ \gamma_1 &= \text{Cov}(Y_{t+1}, Y_t) \\ &= \text{Cov}(\varepsilon_{t+1} - \theta_1\varepsilon_t - \theta_2\varepsilon_{t-1}, \varepsilon_t - \theta_1\varepsilon_{t-1} - \theta_2\varepsilon_{t-2}) \\ &= -\theta_1\sigma^2 + \theta_1\theta_2\sigma^2, \\ \gamma_2 &= \text{Cov}(Y_{t+2}, Y_t) \\ &= \text{Cov}(\varepsilon_{t+2} - \theta_1\varepsilon_{t+1} - \theta_2\varepsilon_t, \varepsilon_t - \theta_1\varepsilon_{t-1} - \theta_2\varepsilon_{t-2}) \\ &= -\theta_2\sigma^2, \\ \gamma_k &= 0 \quad \text{for } k \geq 3. \end{aligned}$$

So we have:

$$\tau_k = \begin{cases} \frac{\gamma_1}{\sigma^2(1+\theta_1^2+\theta_2^2)}, & \text{if } k = 1, \\ \frac{\gamma_2}{\sigma^2(1+\theta_1^2+\theta_2^2)}, & \text{if } k = 2, \\ 0, & \text{if } k \geq 3. \end{cases}$$

3) Let's assume that $\theta_1 = \frac{5}{6}$ and $\theta_2 = \frac{1}{6}$:

$$\begin{aligned} \gamma_0 &= \frac{38}{36}\sigma^2, \\ \gamma_1 &= \frac{-5}{36}\sigma^2, \\ \gamma_2 &= \frac{-1}{6}\sigma^2. \end{aligned}$$

So:

$$\begin{aligned} \tau_1 &= \frac{38\sigma^2 \times \frac{-6}{\sigma^2}}{36} = \frac{-5}{6}, \\ \tau_2 &= \frac{-5\sigma^2 \times \frac{-6}{\sigma^2}}{36} = \frac{5}{6}. \end{aligned}$$

Let's assume that $\theta_1 = -1$ and $\theta_2 = 6$:

$$\gamma_0 = 38\sigma^2,$$
$$\gamma_1 = -5\sigma^2,$$
$$\gamma_2 = -6\sigma^2.$$

We get:

$$\tau_1 = \frac{-5}{38},$$
$$\tau_2 = \frac{-6}{38}.$$

2 Empirical Exercises