Final Project: Optimization for Computer Vision

Iterative Shrinkage Approach to Restoration of Optical Imagery

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1 Introduction

The paper Shaked and Michailovich (2011) focuses on reconstruction of digital images degraded with Poisson noise. This noise is prevalent in optical imaging, microscopy and astronomy for example. Usually the algorithm used to recover the original image when it is blurred and contaminated with Poisson noise is the **Richardson-Lucy** algorithm (Richardson (1972) & Lucy (1974)). It uses the maximum likelihood method, however it leads to unstable and noisy result as soon as the blur matrix is poorly conditioned. There are other methods found in the scientific literature to alleviate these problems. For example, one can use approaches based on bounded variation model but it is very computationally costly. One can also use different class of reconstruction methods based on the assumption of compressibility of the true image, however it requires the original image to be sparse in order for it to work and it is not a natural assumption.

The paper from Elad Shaked and Oleg Michailovich thus explores a different method for image reconstruction based on iterative shrinkage and maximum a posteriori criterion which is general, with small assumption on the original picture, quite efficient and under certain conditions converges to a unique solution of the reconstruction problem.

2 Estimation Framework

2.1 Image Modeling

An image f is degraded via convolution with a Point Spread Function (PSF) H and Poisson noise contamination :

$$g = \mathcal{P}\{H[f]\} \tag{1}$$

where:

- $g \in \mathbb{Z}_+$ is the observed Poisson counts image,
- $f \in \mathbb{R}_+$ is the original image,
- H is the operator of convolution, with known point spread function h which is mean-preserving and positive. It represents the optical blur caused by the PSF of the optical system,
- $\mathcal{P}$ is the Poisson contamination operator.

The objective is then to recover the original image f from the observed image g by inverting the degradation process. We assume that f can be sparsely represented in the domain of a certain linear transform, meaning that :

$$f = \Phi(c) : f \mapsto \sum_{k \in \mathcal{I}} c_k \varphi_k$$

with $\{\varphi_k\}_{k\in\mathcal{I}}$ the *frame* of the original image and $c \in l_2(\mathcal{I})$ the coefficients of f in $\mathrm{Span}(\{\varphi_k\}_{k\in\mathcal{I}})$.

Then Equation (1) then becomes :

$$g = \mathcal{P}\{A[c]\},$$

$A = H \cdot \Phi$ the composition map from $l_2(\mathcal{I})$ to signal space, representing the effect of image synthesis and blur combined. Three settings are raised :

1. When the blur is negligible : $A = \Phi$, the problem is one of image **denoising**,
2. When the basis of Φ is canonical ($\Phi = I$) then $A = H$ and the problem is one of **sparse deconvolution**,
3. If neither H nor Φ is reducible or negligible then the problem is one of **sparse reconstruction**.

2.2 Maximum A Posteriori (MAP) Estimation

For generalization reasons, we do not specify the nature of A in the theoretical part. To provide the most likely solution given the observed data, we use MAP estimation. It allows us to compensate the loss of the poor conditioning of A while also taking advantage of *a priori* knowledge regarding the quantity of interest c. Denoting the prior probability distribution $P(c)$, we have

$$c_{\text{MAP}} = \text{argmax}_c P(c|g) = \text{argmax}_c P(g|c)P(c) \tag{2}$$

Knowing that the noise follows a Poisson distribution, the likelihood function for a single measure $g(x_i)$ at pixel x_i is:

$$P(g(x_i)|c) = \frac{e^{-(A[c])_i}(A[c])_i^{g(x_i)}}{g(x_i)!}$$

and so :

$$P(g|c) = \prod_{x_i} \frac{e^{-(A[c])_i}(A[c])_i^{g(x_i)}}{g(x_i)!}$$

Knowing that the coefficients $c = \{c_k\}_{k\in\mathcal{I}}$ is identically distributed according to a generalized gaussian probability law, we have:

$$P(c) = \prod_{k\in\mathcal{I}} \frac{p}{2\beta\Gamma(p^{-1})} e^{-\left(\frac{|c_k|}{\beta}\right)^p}$$

Applying the logarithm and inversing the sign of the MAP estimator we have :

$$c_{\text{MAP}} = \text{argmin}_c\{E(c)\} \tag{3}$$

$$E(c) = \langle \mathbf{1}, A[c]\rangle - \langle g, \log A[c]\rangle + \gamma||c||_p^p \tag{4}$$

with :

- $\mathbf{1}$ a $N \times M$ matrix of ones ,
- $\gamma = \dfrac{1}{\beta^p}$
- Defined on dom$E = \{c \in l_2(\mathcal{I}) \mid \Phi[c] \geq 0\}$

3 Optimization and Iterative Shrinkage Algorithm

To facilitate the optimization, they use a method called **Bound Optimization** or **Majoration, minimization** (MM) method. Instead of minimizing $E(c)$ like in the original problem, we introduce a surrogate function $Q(c, c_0)$ with $c_0 \in l_2(\mathcal{I})$:

$$Q(c, c_0) = E(c) + \Psi(c, c_0) \tag{5}$$

$$\Psi(c, c_0) = \langle g, \log(A[c]/A[c_0])\rangle - \langle A^*[g/A[c_0]], c - c_0\rangle_{l_2(\mathcal{I})} + \frac{\mu}{2}||c - c_0||_2^2 \tag{6}$$

Canceling the gradient of Q with respect to c, we find that :

$$c = S_{p,\gamma,\mu}\left(c_0 + \frac{1}{\mu}A^*[g/A[c_0] - 1]\right)$$

with $S_{p,\gamma,\mu}$ being the inverse of the function $c \mapsto c + \frac{p\gamma}{\mu}|c|^{p-1}\mathrm{sgn}(c)$ for $p > 1$. For $p = 1$ (the most natural and the most used), it corresponds to the **soft-thresholding operator**:

$$S_{1,\gamma,\mu} = \begin{cases} \left(|c| - \frac{\gamma}{\mu}\right)\mathrm{sgn}(c) & \text{if } |c| \geq \frac{\gamma}{\mu} \\ 0 & \text{otherwise} \end{cases}$$

It gives us the following iterative shrinkage update :

$$c^{t+1} = S_{p,\gamma,\mu}\left(c^t + \frac{1}{\mu}A^*\left[\frac{g - A[c^t]}{A[c^t]}\right]\right)$$

4 Convergence Analysis

Convergence is guaranteed if the following conditions hold:

$$\begin{cases} Q(c, c_t) \geq E(c) & \forall c \in \mathrm{domE} \\ Q(c, c_t) = E(c_t) \end{cases}$$

with c_t arbitrarily fixed in domE.

Furthermore, the surrogate function Q used in Section 3 satisfies the previous condition. Indeed, in this context, it is proved that we only need the Hessian of Ψ : $\nabla^2\Psi$ to be positive definite for the condition to be fulfilled. Thus, as long as the coefficient μ is chosen verifying :

$$\mu \geq ||g/(A[c])^2||_\infty \lambda_{\max}(A^*A)$$

(with $\lambda_{\max}(A^*A)$ the maximum eigenvalue of the matrix A^*A), the algorithm converges.

In the paper, there is also an algorithm presented which enables to find the smallest μ ensuring: $E(c_{t+1}) \leq E(c_t)$ which we won't explain here (can be found in the paper).

5 Results and Comparison

5.1 Methods

We tried to reproduce the results of comparison of the algorithm with the Richardson-Lucy (RL) algorithm. The link to the source code available in the paper was a dead link so we coded the method (*see the notebook*). In the same way as in the paper, we compared the methods using two different metrics :

1. **NMSE** (Normalized Mean Squared Error) which is defined as :

$$\mathrm{NMSE} = \mathbb{E}\left[\frac{||f - \tilde{f}||_F^2}{||f||_F^2}\right]$$

with f the true image, $\tilde{f}$ the estimation of f and $||\ .||_F^2$ the Froebenius norm.

2. **SSIM quality index** (Structural Similarity) which is a metric denoting the closeness in structure of two pictures, inspired by how close to the human eye the two pictures would probably appear.

5.2 Results

For the experiment we chose an astronomical picture `cosmo.nyu.edu/hogg/visualization/` because the original image has to be sparse according to the paper. We conducted the following experiment:

- First we zoomed, gray scaled and applied a blur and a Poisson noise to the original picture;
- Then we reconstructed the image both with the Richardson-Lucy and the Poisson Iterative Shrinkage algorithms to compare their performance on NMSE and SSIM metrics. The resulting images can be seen in Figure (1);

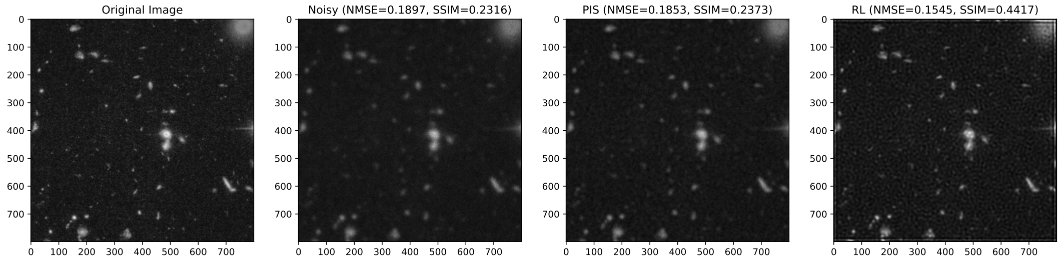

Figure 1: Comparison of the PIS and RL methods on an astronomical image

- We omitted the algorithm of finding a suitable μ because it was failing the code to converge (might be coming from a misunderstanding of the algorithm from me but I could not compare to the source code because it was not available anymore);
- Contrary to the results in the paper, our PIS results are not better than RL results. We observe a slight improvement both in NMSE and SSIM for the PIS method whereas a more noticeable improvement is observed using the RL method;
- The difference in performance might be coming from the fact that we did not use the algorithm to find the suitable scaling parameter. Maybe in the paper the authors also chose more suitable pictures to test their codes because we also experimented on the Shepp-Logan Phantom image (from `skimage.data`), which they also use in the paper and observed better results from the PIS than the RL algorithm.

6 Discussion and Conclusion

In this project, we implemented and tested the Poisson Iterative Shrinkage Algorithm, for image deblurring and denoising and compared it to the results of Richardson-Lucy algorithm. We applied both methods to an astronomical image that we chose for its sparsity.

Contrary to the paper's results, PIS did not outperform RL, neither in NMSE nor in SSIM. The algorithm for finding the scaling parameter μ failed to converge which might have affected the performance of the PIS method.

Limitations and future work

- The problem of convergence of the algorithm for finding the scaling parameter μ should be addressed;
- Nowadays, there exists some methods who have stronger performance with no hyperparameter tuning. For instance, Zhang et al. (2017) used CNN to denoise images which achieve good performance.

References

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