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1. Ejercicios

1.1. Ejercicio 1

Derive cada una de las siguientes funciones

$$\begin{aligned} f(x) &= e^{e^{e^{e^x}}} \\ f(x) &= \log\left(1+\log\left(1+\log\left(1+e^{1+e^{1+x}}\right)\right)\right) \\ f(x) &= (\sin x)^{\sin(\sin x)} \\ f(x) &= e^{\left(\int_0^x e^{-t^2}\ dt\right)} \\ f(x) &= x^x \end{aligned}$$

Solución

Veamos $f(x) = e^{e^{e^{e^x}}}$.
Sea $h(x) = e^x$, notemos lo siguiente:

$$\begin{aligned} h_1(x) &= (h \circ h)(x) \\ &= h(h(x)) \\ &= e^{e^x} \end{aligned}$$

Así tenemos que

$$\begin{aligned} h_1'(x) &= ((h \circ h)(x))' \\ &= h'(h(x)) \cdot h'(x) \end{aligned}$$

Sabemos que $h'(x) = e^x$, entonces

$$\begin{aligned} h_1'(x) &= h'(e^x) \cdot e^x \\ &= e^{e^x} \cdot e^x \end{aligned}$$

Veamos que

$$\begin{aligned} h_2(x) &= (h \circ h_1)(x) \\ &= h(h_1(x)) \\ &= e^{e^{e^x}} \end{aligned}$$

Así tenemos que

$$\begin{aligned} h_2'(x) &= ((h \circ h_1)(x))' \\ &= h'(h_1(x)) \cdot h_1'(x) \end{aligned}$$

Sabemos que $h'(x) = e^x$ y $h_1'(x) = e^{e^x} \cdot e^x$, entonces

$$\begin{aligned} h_2'(x) &= h'(e^{e^x}) \cdot e^{e^x} \cdot e^x \\ &= e^{e^{e^x}} \cdot e^{e^x} \cdot e^x \end{aligned}$$

Veamos que

$$\begin{aligned} h_3(x) &= (h \circ h_2)(x) \\ &= h(h_2(x)) \\ &= e^{e^{e^{e^x}}} \end{aligned}$$

Así tenemos que

$$\begin{aligned} h_3'(x) &= \left(\left(h \circ h_2\right)(x)\right)' \\ &= h'(h_2(x)) \cdot h_2'(x) \end{aligned}$$

Sabemos que $h'(x) = e^x$ y $h_1'(x) = e^{e^{x}} \cdot e^x$, entonces

$$\begin{aligned} h_3'(x) &= h'(e^{e^{e^{x}}}) \cdot e^{e^{e^{x}}} \cdot e^{e^{x}} \cdot e^x \\ &= e^{e^{e^{e^{x}}}} \cdot e^{e^{e^{x}}} \cdot e^{e^{x}} \cdot e^x \end{aligned}$$

Por lo tanto

$$\boxed{\boxed{\left(e^{e^{e^{e^{x}}}}\right)' = e^{e^{e^{e^{x}}}} \cdot e^{e^{e^{x}}} \cdot e^{e^{x}} \cdot e^x}}$$

Veamos $f(x) = \log\left(1 + \log\left(1 + \log\left(1 + e^{1+e^{1+x}}\right)\right)\right)$
Sea $g(x) = e^{1+x}$, notemos lo siguiente:

$$\begin{aligned} g_1\left(x\right) &= \left(g \circ g\right)(x) \\ &= g(g(x)) \\ &= e^{1+e^{1+x}} \end{aligned}$$

Así tenemos que

$$\begin{aligned} g_1'(x) &= \left(\left(g \circ g\right)(x)\right)' \\ &= g'(g(x)) \cdot g'(x) \end{aligned}$$

Sabemos que $g'(x) = e^{1+x}$, entonces

$$\begin{aligned} g_1'(x) &= g'(e^{1+x}) \cdot e^{1+x} \\ &= e^{e^{1+x}} \cdot e^{1+x} \end{aligned}$$

Sea $h(x) = \log\left(1 + x\right)$, notemos lo siguiente:

$$\begin{aligned} h_1\left(x\right) &= \left(h \circ g_1\right)(x) \\ &= h(g_1(x)) \\ &= \log\left(1 + e^{1+e^{1+x}}\right) \end{aligned}$$

Así tenemos que

$$\begin{aligned} h_1'(x) &= \left(\left(h \circ g_1\right)(x)\right)' \\ &= h'(g_1(x)) \cdot g_1'(x) \end{aligned}$$

Sabemos que $h'(x) = \frac{1}{1+x}$ y $g_1'(x) = e^{e^{1+x}} \cdot e^{1+x}$, entonces

$$\begin{aligned} h_1'(x) &= h'(e^{1+e^{1+x}}) \cdot e^{e^{1+x}} \cdot e^{1+x} \\ &= \left(\frac{1}{1 + e^{1+e^{1+x}}}\right) \cdot e^{e^{1+x}} \cdot e^{1+x} \\ &= \frac{e^{e^{1+x}} \cdot e^{1+x}}{1 + e^{1+e^{1+x}}} \end{aligned}$$

Veamos que

$$\begin{aligned} h_2\left(x\right) &= \left(h \circ h_1\right)(x) \\ &= h(h_1(x)) \\ &= \log\left(1 + \log\left(1 + e^{1+e^{1+x}}\right)\right) \end{aligned}$$

Así tenemos que

$$\begin{aligned} h_2'(x) &= \left(\left(h \circ h_1\right)(x)\right)' \\ &= h'(h_1(x)) \cdot h_1'(x) \end{aligned}$$

Sabemos que $h'(x) = \frac{1}{1+x}$ y $h_1'(x) = \frac{e^{e^{1+x}} \cdot e^{1+x}}{1+e^{1+e^{1+x}}}$, entonces

$$\begin{aligned} h_2'(x) &= h'\left(\log\left(1+e^{1+e^{1+x}}\right)\right) \cdot \frac{e^{e^{1+x}} \cdot e^{1+x}}{1+e^{1+e^{1+x}}} \\ &= \frac{1}{1+\log\left(1+e^{1+e^{1+x}}\right)} \cdot \frac{e^{e^{1+x}} \cdot e^{1+x}}{1+e^{1+e^{1+x}}} \\ &= \frac{e^{e^{1+x}} \cdot e^{1+x}}{\left(1+\log\left(1+e^{1+e^{1+x}}\right)\right)\left(1+e^{1+e^{1+x}}\right)} \end{aligned}$$

Veamos que

$$\begin{aligned} h_3(x) &= \left(h \circ h_2\right)(x) \\ &= h(h_2(x)) \\ &= \log\left(1+\log\left(1+\log\left(1+e^{1+e^{1+x}}\right)\right)\right) \end{aligned}$$

Así tenemos que

$$\begin{aligned} h_3'(x) &= \left(\left(h \circ h_2\right)(x)\right)' \\ &= h'(h_2(x)) \cdot h_2'(x) \end{aligned}$$

Sabemos que $h'(x) = \frac{1}{1+x}$ y $h_2'(x) = \frac{e^{e^{1+x}} \cdot e^{1+x}}{\left(1+\log\left(1+e^{1+e^{1+x}}\right)\right)\left(1+e^{1+e^{1+x}}\right)}$, entonces

$$\begin{aligned} h_3'(x) &= h'\left(\log\left(1+\log\left(1+e^{1+e^{1+x}}\right)\right)\right) \cdot \frac{e^{e^{1+x}} \cdot e^{1+x}}{\left(1+\log\left(1+e^{1+e^{1+x}}\right)\right)\left(1+e^{1+e^{1+x}}\right)} \\ &= \frac{1}{1+\log\left(1+\log\left(1+e^{1+e^{1+x}}\right)\right)} \cdot \frac{e^{e^{1+x}} \cdot e^{1+x}}{\left(1+\log\left(1+e^{1+e^{1+x}}\right)\right)\left(1+e^{1+e^{1+x}}\right)} \\ &= \frac{e^{e^{1+x}} \cdot e^{1+x}}{\left(1+\log\left(1+\log\left(1+e^{1+e^{1+x}}\right)\right)\right)\left(1+\log\left(1+e^{1+e^{1+x}}\right)\right)\left(1+e^{1+e^{1+x}}\right)} \end{aligned}$$

Por lo tanto

$$\boxed{\left(\log\left(1+\log\left(1+\log\left(1+e^{1+e^{1+x}}\right)\right)\right)\right)' = \frac{e^{e^{1+x}} \cdot e^{1+x}}{\left(1+\log\left(1+\log\left(1+e^{1+e^{1+x}}\right)\right)\right)\left(1+\log\left(1+e^{1+e^{1+x}}\right)\right)\left(1+e^{1+e^{1+x}}\right)}}$$

Veamos $f(x) = (\sin x)^{\sin(\sin x)}$

Por las propiedades de la función exponencial y de la función logaritmo tenemos que

$$\begin{aligned} f(x) &= (\sin x)^{\sin(\sin x)} \\ &= e^{\sin(\sin x) \cdot \log(\sin x)} \end{aligned}$$

Así tenemos que

$$\begin{aligned} f'(x) &= \left(e^{\sin(\sin x) \cdot \log(\sin x)}\right)' \\ &= e^{\sin(\sin x) \cdot \log(\sin x)} \cdot \left(\sin(\sin x) \cdot \log(\sin x)\right)' \\ &= e^{\sin(\sin x) \cdot \log(\sin x)} \cdot \left(\left(\sin(\sin x)\right)' \cdot \log(\sin x) + \left(\log(\sin x)\right)' \cdot \sin(\sin x)\right) \\ &= e^{\sin(\sin x) \cdot \log(\sin x)} \cdot \left(\cos(\sin x) \cdot \cos x \cdot \log(\sin x) + \frac{\cos x}{\sin x} \cdot \sin(\sin x)\right) \\ &= e^{\sin(\sin x) \cdot \log(\sin x)} \cdot \left(\cos(\sin x) \cdot \cos x \cdot \log(\sin x) + \cot x \cdot \sin(\sin x)\right) \\ &= (\sin x)^{\sin(\sin x)} \left(\cos(\sin x) \cdot \cos x \cdot \log(\sin x) + \cot x \cdot \sin(\sin x)\right) \end{aligned}$$

Por lo tanto

$$\boxed{\boxed{\left((\sin x)^{\sin(\sin x)}\right)' = (\sin x)^{\sin(\sin x)}\left(\cos(\sin x)\cdot\cos x\cdot\log(\sin x)+\cot x\cdot\sin(\sin x)\right)}}$$

Veamos $f(x) = e^{\left(\int_0^x e^{-t^2}\ dt\right)}$

Sea $h(x) = e^x$ y $g(x) = \int_0^x e^{-t^2} dt$, notemos lo siguiente:

$$\begin{aligned}(h\circ g)(x) &= h(g(x))\\ &= h\left(\int_0^x e^{-t^2} dt\right)\\ &= e^{\left(\int_0^x e^{-t^2} dt\right)}\end{aligned}$$

Así tenemos que

$$\begin{aligned}f'(x) &= (h(g(x)))'\\ &= h'(g(x))\cdot g'(x)\end{aligned}$$

Sabemos que $h'(x) = e^x$ y $g'(x) = e^{-x^2}$ ya que e^{-t^2} es continua en todo $\mathbb{R}$ y por el **Primer Teorema Fundamental tenemos que** $\left(\int_0^x e^{-t^2} dt\right)' = e^{-x^2}$, entonces

$$\begin{aligned}f'(x) &= h'\left(\int_0^x e^{-t^2} dt\right)\cdot e^{-x^2}\\ &= e^{\left(\int_0^x e^{-t^2} dt\right)}\cdot e^{-x^2}\end{aligned}$$

Por lo tanto

$$\boxed{\boxed{\left(e^{\left(\int_0^x e^{-t^2} dt\right)}\right)' = e^{\left(\int_0^x e^{-t^2} dt\right)}\cdot e^{-x^2}}}$$

Veamos $f(x) = x^x$ Por las propiedades de la función exponencial y de la función logaritmo tenemos que

$$\begin{aligned}f(x) &= (\sin x)^{\sin(\sin x)}\\ &= e^{x\log x}\end{aligned}$$

Así tenemos que

$$\begin{aligned}f'(x) &= \left(e^{x\log x}\right)'\\ &= e^{x\log x}(x\log x)'\\ &= e^{x\log x}\left(x\cdot(\log x)'+\log x\cdot x'\right)\\ &= e^{x\log x}\left(x\cdot\left(\frac{1}{x}\right)+\log x\cdot 1\right)\\ &= e^{x\log x}(1+\log x)\\ &= x^x(1+\log x)\end{aligned}$$

Por lo tanto

$$\boxed{\boxed{(x^x)' = x^x(1+\log x)}}$$

1.2. Ejercicio 2

1. Compruebe que $f'(x) = f(x)\left(\log\left(f(x)\right)\right)'$, sí $f > 0$ y derivable
2. Halle $f'(x)$ para cada una de las siguientes funciones

$$f(x) = (1+x)\left(1+e^{x^2}\right)$$
$$f(x) = \frac{(3-x)^{\frac{1}{3}}x^2}{(1-x)(3+x)^{2^3}}$$
$$f(x) = \frac{e^x - e^{-x}}{e^{2x}(1+x^3)}$$

Solución

Veamos (1)
Sabemos que $f(x) > 0\ \forall x \in D_f$ entonces $\log f(x)$ está bien definido, también tenemos por hipótesis que $f'(x)$ existe, también sabemos que $\log x$ es derivable así $\log\left(f(x)\right)$ es derivable. Consecuentemente:

$$\begin{aligned}\left(\log\left(f(x)\right)\right)' &= \frac{1}{f(x)}\cdot f'(x)\\ &= \frac{f'(x)}{f(x)}\end{aligned}$$

Despejando $f'(x)$ tenemos que

$$f'(x) = \left(\log\left(f(x)\right)\right)'\cdot f(x)$$

Veamos (2)
Veamos $f(x) = (1+x)\left(1+e^{x^2}\right)$.
Obteniendo $\log$ de ambos lados

$$\begin{aligned}\log\left(f(x)\right) &= \log\left((1+x)\left(1+e^{x^2}\right)\right)\\ &= \log\left(1+x\right) + \log\left(1+e^{x^2}\right)\end{aligned}$$

Derivando de ambos lados

$$\begin{aligned}\frac{f'(x)}{f(x)} &= \frac{1}{1+x} + \frac{1}{1+e^{x^2}}\cdot e^{x^2}\cdot 2x\\ &= \frac{1}{1+x} + \frac{2x\cdot e^{x^2}}{1+e^{x^2}}\end{aligned}$$

Los anterior es equivalente a

$$f'(x) = f(x)\left(\frac{1}{1+x} + \frac{2x\cdot e^{x^2}}{1+e^{x^2}}\right)$$
$$f'(x) = (1+x)\left(1+e^{x^2}\right)\left(\frac{1}{1+x} + \frac{2x\cdot e^{x^2}}{1+e^{x^2}}\right)$$

Veamos $f(x) = \frac{(3-x)^{\frac{1}{3}}x^2}{(1-x)(3+x)^{2^3}} = \frac{(3-x)^{\frac{1}{3}}x^2}{(1-x)(3+x)^8}$
Obteniendo $\log$ de ambos lados

$$\begin{aligned}
\log(f(x)) &= \log\left(\frac{(3-x)^{\frac{1}{3}}x^2}{(1-x)(3+x)^8}\right) \\
&= \log\left((3-x)^{\frac{1}{3}}x^2\right) - \log\left((1-x)(3+x)^8\right) \\
&= \log\left((3-x)^{\frac{1}{3}}\right) + \log\left(x^2\right) - \left(\log(1-x) + \log\left((3+x)^8\right)\right) \\
&= \frac{1}{3}\log(3-x) + 2\log(x) - (\log(1-x) + 8\log(3+x)) \\
&= \frac{1}{3}\log(3-x) + 2\log(x) - \log(1-x) - 8\log(3+x)
\end{aligned}$$

Derivando de ambos lados

$$\begin{aligned}
\frac{f'(x)}{f(x)} &= \frac{1}{3}\left(-\frac{1}{3-x}\right) + 2\left(\frac{1}{x}\right) - \left(-\frac{1}{1-x}\right) - 8\left(\frac{1}{3+x}\right) \\
&= -\frac{1}{3(3-x)} + \frac{2}{x} + \frac{1}{1-x} - \frac{8}{3+x}
\end{aligned}$$

Lo anterior es equivalente a

$$\begin{aligned}
f'(x) &= f(x)\left(-\frac{1}{3(3-x)} + \frac{2}{x} + \frac{1}{1-x} - \frac{8}{3+x}\right) \\
f'(x) &= \left(\frac{(3-x)^{\frac{1}{3}}x^2}{(1-x)(3+x)^8}\right)\left(-\frac{1}{3(3-x)} + \frac{2}{x} + \frac{1}{1-x} - \frac{8}{3+x}\right)
\end{aligned}$$

Veamos $f(x) = \frac{e^x - e^{-x}}{e^{2x}(1+x^3)}$
Obteniendo $\log$ de ambos lados

$$\begin{aligned}
\log(f(x)) &= \log\left(\frac{e^x - e^{-x}}{e^{2x}(1+x^3)}\right) \\
&= \log\left(e^x - e^{-x}\right) - \log\left(e^{2x}(1+x^3)\right) \\
&= \log\left(e^x - e^{-x}\right) - \left(\log\left(e^{2x}\right) + \log\left(1+x^3\right)\right) \\
&= \log\left(e^x - e^{-x}\right) - \log\left(e^{2x}\right) - \log\left(1+x^3\right) \\
&= \log\left(e^x - e^{-x}\right) - 2x - \log\left(1+x^3\right)
\end{aligned}$$

Derivando de ambos lados

$$\begin{aligned}
\frac{f'(x)}{f(x)} &= \left(\frac{1}{e^x - e^{-x}}\right)\left(e^x + e^{-x}\right) - 2 - \left(\frac{1}{1+x^3}\right)\left(3x^2\right) \\
&= \frac{e^x + e^{-x}}{e^x - e^{-x}} - 2 - \frac{3x^2}{1+x^3}
\end{aligned}$$

Lo anterior es equivalente a

$$\begin{aligned}
f'(x) &= f(x)\left(\frac{e^x + e^{-x}}{e^x - e^{-x}} - 2 - \frac{3x^2}{1+x^3}\right) \\
f'(x) &= \left(\frac{e^x - e^{-x}}{e^{2x}(1+x^3)}\right)\left(\frac{e^x + e^{-x}}{e^x - e^{-x}} - 2 - \frac{3x^2}{1+x^3}\right)
\end{aligned}$$

1.3. Ejercicio 3

Hallar los siguientes límites mediante la regla de L'Hôpital

$$\lim_{x\to 0} \frac{\sin x - x + \frac{x^3}{6}}{x^3}$$
$$\lim_{x\to 0} \frac{\sin x - x + \frac{x^3}{6}}{x^4}$$
$$\lim_{x\to 0} \frac{\cos x - 1 + \frac{x^2}{2}}{x^2}$$

Solución

Veamos $\lim\limits_{x\to 0} \frac{\sin x - x + \frac{x^3}{6}}{x^3}$

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin x - x + \frac{x^3}{6}}{x^3} &= \frac{\lim\limits_{x\to 0}\left(\sin x - x + \dfrac{x^3}{6}\right)}{\lim\limits_{x\to 0} x^3} \\
&= \frac{\lim\limits_{x\to 0}\sin x - \lim\limits_{x\to 0} x + \lim\limits_{x\to 0}\dfrac{x^3}{6}}{\lim\limits_{x\to 0} x^3} \\
&= \frac{0}{0}
\end{aligned}$$

Entonces podemos aplicar L'Hôpital

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin x - x + \frac{x^3}{6}}{x^3} &= \lim_{x\to 0} \frac{\cos x - 1 + \frac{x^2}{2}}{3x^2} \\
&= \frac{\lim\limits_{x\to 0}\left(\cos x - 1 + \dfrac{x^2}{2}\right)}{\lim\limits_{x\to 0} 3x^2} \\
&= \frac{\lim\limits_{x\to 0}\cos x - \lim\limits_{x\to 0} 1 + \lim\limits_{x\to 0}\dfrac{x^2}{2}}{\lim\limits_{x\to 0} 3x^2} \\
&= \frac{1 - 1 + 0}{0} \\
&= \frac{0}{0}
\end{aligned}$$

Podemos aplicar L'Hôpital

$$\begin{aligned}
\lim_{x\to 0} \frac{\cos x - 1 + \frac{x^2}{2}}{3x^2} &= \lim_{x\to 0} \frac{-\sin x + x}{6x} \\
&= \frac{\lim\limits_{x\to 0}(-\sin x + x)}{\lim\limits_{x\to 0} 6x} \\
&= \frac{-\lim\limits_{x\to 0}\sin x + \lim\limits_{x\to 0} x}{\lim\limits_{x\to 0} 6x} \\
&= \frac{-0 + 0}{0} \\
&= \frac{0}{0}
\end{aligned}$$

Podemos aplicar L'Hôpital

$$\begin{aligned}\lim_{x\to 0}\frac{-\sin x + x}{6x} &= \lim_{x\to 0}\frac{-\cos x + 1}{6}\\ &= \frac{-\cos 0 + 1}{6}\\ &= \frac{-1+1}{6}\\ &= \frac{0}{6}\\ &= 0\end{aligned}$$

Por lo tanto

$$\boxed{\boxed{\lim_{x\to 0}\frac{\sin x - x + \frac{x^3}{6}}{x^3} = 0}}$$

Veamos $\lim\limits_{x\to 0}\dfrac{\sin x - x + \frac{x^3}{6}}{x^4}$

$$\begin{aligned}\lim_{x\to 0}\frac{\sin x - x + \frac{x^3}{6}}{x^4} &= \frac{\lim\limits_{x\to 0}\left(\sin x - x + \dfrac{x^3}{6}\right)}{\lim\limits_{x\to 0} x^4}\\ &= \frac{\lim\limits_{x\to 0}\sin x - \lim\limits_{x\to 0} x + \lim\limits_{x\to 0}\dfrac{x^3}{6}}{\lim\limits_{x\to 0} x^4}\\ &= \frac{0}{0}\end{aligned}$$

Entonces podemos aplicar L'Hôpital

$$\begin{aligned}\lim_{x\to 0}\frac{\sin x - x + \frac{x^3}{6}}{x^4} &= \lim_{x\to 0}\frac{\cos x - 1 + \frac{x^2}{2}}{4x^3}\\ &= \frac{\lim\limits_{x\to 0}\left(\cos x - 1 + \dfrac{x^2}{2}\right)}{\lim\limits_{x\to 0} 4x^3}\\ &= \frac{\lim\limits_{x\to 0}\cos x - \lim\limits_{x\to 0} 1 + \lim\limits_{x\to 0}\dfrac{x^2}{2}}{\lim\limits_{x\to 0} 4x^3}\\ &= \frac{1-1+0}{0}\\ &= \frac{0}{0}\end{aligned}$$

Podemos aplicar L'Hôpital

$$\begin{aligned}\lim_{x\to 0}\frac{\cos x - 1 + \frac{x^2}{2}}{4x^3} &= \lim_{x\to 0}\frac{-\sin x + x}{12x^2}\\ &= \frac{\lim\limits_{x\to 0}(-\sin x + x)}{\lim\limits_{x\to 0} 12x^2}\\ &= \frac{-\lim\limits_{x\to 0}\sin x + \lim\limits_{x\to 0} x}{\lim\limits_{x\to 0} 12x^2}\\ &= \frac{-0+0}{0}\\ &= \frac{0}{0}\end{aligned}$$

Podemos aplicar L'Hôpital

$$\begin{aligned}
\lim_{x\to 0} \frac{-\sin x + x}{12x^2} &= \lim_{x\to 0} \frac{-\cos x + 1}{24x} \\
&= \frac{\lim\limits_{x\to 0} (-\cos x + 1)}{\lim\limits_{x\to 0} 24x} \\
&= \frac{-\lim\limits_{x\to 0} \cos x + \lim\limits_{x\to 0} 1}{\lim\limits_{x\to 0} 24x} \\
&= \frac{-1+1}{0} \\
&= \frac{0}{0}
\end{aligned}$$

Podemos aplicar L'Hôpital

$$\begin{aligned}
\lim_{x\to 0} \frac{-\cos x + 1}{24x} &= \lim_{x\to 0} \frac{\sin x}{24} \\
&= \frac{0}{24} \\
&= 0
\end{aligned}$$

Por lo tanto

$$\boxed{\boxed{\lim_{x\to 0} \frac{\sin x - x + \frac{x^3}{6}}{x^4} = 0}}$$

Veamos $\lim\limits_{x\to 0} \dfrac{\cos x - 1 + \frac{x^2}{2}}{x^2}$

$$\begin{aligned}
\lim_{x\to 0} \frac{\cos x - 1 + \frac{x^2}{2}}{x^2} &= \frac{\lim\limits_{x\to 0} \left(\cos x - 1 + \dfrac{x^2}{2}\right)}{\lim\limits_{x\to 0} x^2} \\
&= \frac{\lim\limits_{x\to 0} \cos x - \lim\limits_{x\to 0} 1 + \lim\limits_{x\to 0} \dfrac{x^2}{2}}{\lim\limits_{x\to 0} x^2} \\
&= \frac{1-1+0}{0} \\
&= \frac{0}{0}
\end{aligned}$$

Podemos aplicar L'Hôpital

$$\begin{aligned}
\lim_{x\to 0} \frac{\cos x - 1 + \frac{x^2}{2}}{x^2} &= \lim_{x\to 0} \frac{-\sin x + x}{2x} \\
&= \frac{\lim\limits_{x\to 0} (-\sin x + x)}{\lim\limits_{x\to 0} 2x} \\
&= \frac{-\lim\limits_{x\to 0} \sin x + \lim\limits_{x\to 0} x}{\lim\limits_{x\to 0} 2x} \\
&= \frac{-0+0}{0} \\
&= \frac{0}{0}
\end{aligned}$$

Podemos aplicar L'Hôpital

$$\begin{aligned}
\lim_{x \to 0} \frac{-\sin x + x}{2x} &= \lim_{x \to 0} \frac{-\cos x + 1}{2} \\
&= \lim_{x \to 0} \frac{-\cos x + 1}{2} \\
&= \frac{-\cos 0 + 1}{2} \\
&= \frac{-1 + 1}{2} \\
&= \frac{0}{2} \\
&= 0
\end{aligned}$$