Bonus Problem 1. Let $f : X \to Y$ be a function. For any $U \subseteq X$, define

$$f[U] = \{f(x) \mid x \in U\}.$$

For any $V \subseteq Y$, define

$$f^{-1}[V] = \{x \in X \mid f(x) \in V\}.$$

Prove the following statements: for any $U_0, U_1 \subseteq X$ and $V, V_0, V_1 \subseteq Y$,

(i) $f[U_0 \cup U_1] = f[U_0] \cup f[U_1]$

(ii) $f^{-1}[V_0 \cup V_1] = f^{-1}[V_0] \cup f^{-1}[V_1]$

(iii) $f^{-1}[Y \backslash V] = X \backslash f^{-1}[V]$

(iv) $f^{-1}[V_0 \cap V_1] = f^{-1}[V_0] \cap f^{-1}[V_1]$

Is it true that $f[U_0 \cap U_1] = f[U_0] \cap f[U_1]$?

Bonus Problem 2. Let $f : X \to Y$, $g : Y \to Z$, and $h = g \circ f : X \to Z$.

(i) If h and g are both injective, then f is injective.

(ii) If h and f are both surjective, then g is surjective.

(iii) Are either (i) or (ii) true if we swap f and g? If so, then prove it. If not, then come up with a counterexample.