

CSCI 341 Workshop 8

NP

November 19, 2025

Problem 1 (Word Acceptance Problem for Finite Automata). Consider the following decision problem: recall that a finite automaton is a quadruple $\mathcal{A} = (Q, A, \delta, F)$ with a set Q of states, a transition relation $\delta \subseteq Q \times A \times Q$, and a set of accepting states $F \subseteq Q$. The *word acceptance problem for finite automata* is the decision problem

$$WAcc = \{(\mathcal{A}, x, w) \mid \mathcal{A} \text{ is a finite automaton, } x \in Q, \text{ and } w \in \mathcal{L}(\mathcal{A}, x)\}$$

We are going to show that this language is in NP.

- (1) Start by considering the related problem

$$DWacc = \{(\mathcal{A}, x, w) \mid \mathcal{A} \text{ is a deterministic finite automaton, } x \in Q, \text{ and } w \in \mathcal{L}(\mathcal{A}, x)\}$$

- (a) Assume you have a class (like in Python or Java or C++) that stores an automaton and allows you to observe the set $\delta(x, a)$ for any $x \in Q$ and $a \in A$. Write some pseudocode that decides whether $(\mathcal{A}, x, w) \in DWacc$ for any instance (\mathcal{A}, x, w) of the problem.
 - (b) Come up with a faithful string representation of the deterministic word acceptance problem.
Hint: how would you describe a finite automaton with a basic text editor?
 - (c) Show that (the string representation of) $DWacc$ is in P by adapting your pseudocode from part (a) to run on a Turing machine (do you need multiple tapes?).
- (2) Adapt your work from (1) to describe a verifier that solves $Wacc$ in nondeterministic polynomial time.
 - (3) Doesn't this contradict Rice's theorem?

Problem 2. [Graph Reachability Problem] A *directed graph* is a pair $\mathcal{G} = (X, \rightarrow)$ consisting of a set X of *nodes* and a relation $\rightarrow \subseteq X \times X$ of *edges*. Given nodes $x, y \in X$, y is *reachable from* x if there are $x_1, \dots, x_n \in X$ such that

$$x \rightarrow x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_n \rightarrow y$$

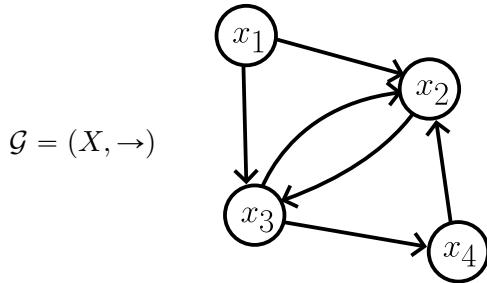
Define

$$\mathbf{DGr} = \{(\mathcal{G}, x, y) \mid \mathcal{G} = (X, \rightarrow) \text{ is a directed graph and } x, y \in X\}$$

Then the *reachability problem* is

$$Rch = \{(\mathcal{G}, x, y) \in \mathbf{DGr} \mid y \text{ is reachable from } x \text{ in } \mathcal{G}\} \subseteq \mathbf{DGr}$$

- (1) Write down the sets X and \rightarrow of nodes and edges in the directed graph below.



- (a) Is (\mathcal{G}, x_1, x_2) in Rch ?
 - (b) Is (\mathcal{G}, x_2, x_1) in Rch ?
 - (c) Is (\mathcal{G}, x_1, x_4) in Rch ?
 - (d) Is (\mathcal{G}, x_4, x_1) in Rch ?
- (2) Come up with a faithful string representation of the reachability problem, and write down the string representation of (\mathcal{G}, x_1, x_4) above.

Hint: try describing the directed graph above with a basic text editor.

- (3) Now we'll show that the reachability problem is in NP. Remember that the goal is to describe a general Turing program that either verifies or disproves that a particular instance of the problem lies in (the string representation of) Rch in polynomial time.
- a Write some pseudocode that describes a nondeterministic algorithm for solving the problem. What is a "guess" in this scenario? Write down the definition of "verification in polynomial time" and check that your pseudocode does, in fact, run in polynomial time.
 - b Now let's think about how to do this with a Turing machine. Describe a few different ways that you might set up the tape to store all of the necessary information about a string representation of (\mathcal{G}, x, y) and the information you need to keep track of as the program runs. Decide which one you like best as a team.
 - c Use your pseudocode in (a) to describe a Turing machine that implements your algorithm. Check that it runs in nondeterministic polynomial time.

Problem 3 (Strongly Connected Problem). A directed graph $\mathcal{G} = (X, \rightarrow)$ is *strongly connected* if for any $x, y \in X$, x is reachable from y and y is reachable from x . Show that the problem

$$SCon = \{\mathcal{G} \mid \mathcal{G} \text{ is a strongly connected directed graph}\}$$

is in NP by cooking up a reduction $r: SCon \leq Rch$ that runs in $\mathcal{O}(n^2)$ -time.

If you don't quite get to this problem, don't worry, we will go over it briefly in class on Friday.

Problem 4 (3SAT). The set *Form* of all (*three-propositional formulas*) is generated by the grammar

$$F \rightarrow p \mid q \mid r \mid (F \wedge F) \mid (F \vee F) \mid (\neg F)$$

where p, q, r are the *basic propositional formulas*. An *assignment* is a function $\alpha: \{p, q, r\} \rightarrow \{0, 1\}$, indicating whether p, q , and r are taken to be "false" or "true". The *truth value* of a formula is computed from an assignment α recursively with the truth table below:

φ_1	φ_2	$\varphi_1 \wedge \varphi_2$	$\varphi_1 \vee \varphi_2$	$\neg \varphi_1$
1	1	1	1	0
0	1	0	1	1
1	0	0	1	0
0	0	0	0	1

- Consider the following decision problem:

$$FTrue = \{(\varphi, \alpha) \mid \varphi \in Form, \alpha: \{p, q, r\} \rightarrow \{0, 1\}\}$$

Show that (a string representation of) *FTrue* is in P.

- A formula $\varphi \in Form$ is *satisfiable* if there exists an assignment $\alpha: \{p, q, r\} \rightarrow \{0, 1\}$ that makes $\varphi = 1$ (i.e., true). We obtain the decision problem

$$\mathcal{3SAT} = \{\varphi \in Form \mid \varphi \text{ is satisfiable}\}$$

Use your polynomial time algorithm in the previous part to show that (a string representation of) $\mathcal{3SAT}$ is in NP.