**Homework 2**
**ECSE 556**
Gian Favero — 261157396

December 1st, 2023

# 1 Some Graph Theory

We can consider a simple graph network to help facilitate the proof:

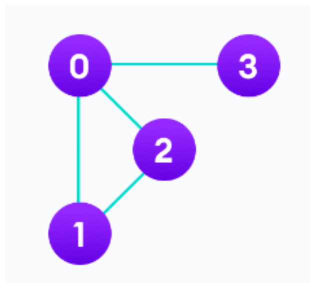


Figure 1: A simple undirected graph

This graph has a corresponding adjacent matrix:

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

We can prove by induction that in an unweighted undirected graph, the $(i, j)$ entry of the $m^{th}$ power of adjacency matrix $A$ counts the number of walks of length $m$ from $i$ to $j$.

We begin with $m = 2$, and consider nodes 1 and 2, $i = 1, j = 2$. Visually, we can perceive that there is only 1 walk between node 1 and node 2 than can be done in 2 lengths. This is the walk $1 \to 0 \to 2$.

$$A^2 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & \mathbf{1} & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 2 & \mathbf{1} & 1 \\ 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

Clearly, this is correct. Similarly, we can see the assumption is correct for all other nodes as well. We now suppose that the assumption is still correct for $m = k$. We can show that it will hold for $m = k + 1$.

With $m = k$, we have that the number of walks is $(A_{ij})^k$. When $m = k + 1$, we can consider a third vertex, $p$, which is a 1 length walk from $j$. Thus, the number of walks from $i$ to $j$ of length $k + 1$ is the sum of the number of walks from $i$ to $p$ of length $k$ multiplied by the number of walks from $p$ to $j$ of length 1.

Thus,

$$(A_{ij})^{k+1} = \sum_{p \in V(G)} (A_{ip})^k (A_{pj})$$
$$LS = RS$$

By induction, we have proven that the $(i, j)$ entry of the $m^{th}$ power of adjacency matrix $A$ counts the number of walks of length $m$ from $i$ to $j$.

## 2 Random Walks

A simple random walk with the following transition probabilities is given by:

$$P_{ij} = Pr(X_{k+1} = j | X_k = i) = \begin{cases} \frac{1}{d(i)} & \text{if } ij \in E \\ 0 & \text{otherwise} \end{cases}$$

where $d(i)$ is the degree of node $i$, or how many neighbours it has. A stationary distribution (row vector) of a random walk is a distribution $\pi$ such that $\pi P = \pi$.

If $\rho_k$ is a row vector giving the probability distribution of $X_k$,

$$\rho_k(i) = P(X_k = i), i \in V(G)$$
$$\rho_{k+1} = \rho_k P$$

For a walk of length $k$, we have that $\rho_k = \rho_0 P^k$, where $\rho_0$ is the initial distribution.

The Perron-Frobenius theorem implies the existence of a stationary distribution $\pi$ which is a positive left eigenvector of $P$ corresponding to the eigenvalue 1. This means that:

$$\pi P = \pi$$
$$\pi(i) > 0$$
$$\sum_{i \in V(G)} \pi(i) = 1$$

If the initial vertex of the walk is chosen according to $\pi$, then the distribution at time $k$ is also $\pi$. Hence,

$$\rho_k = \pi P^k = \pi$$
$$P(X_k = i) = \pi(i)$$

To find a probability distribution $\pi$ that satisfies the above conditions, we solve:

$$\pi(i) p_{ij} = \pi(j) p_{ji}$$

And when this condition is satisfied:

$$\pi(i) = \sum_{j \in V(G)} \pi(i) p_{ij} = \sum_{i \in V(G)} \pi(j) p_{ji}$$

With the transition probabilities given in the question, we have:

$$\frac{\pi(i)}{d(i)} = \frac{\pi(j)}{d(j)} = k$$

Exploring further:

$$1 = \sum_{i \in j} \pi(i) = k \sum_{i \in V(G)} d(i) = k(2m)$$

Thus,

$$\pi(i) = kd(i)$$
$$\pi(i) = \frac{d(i)}{2m} = \frac{d(i)}{n}$$

The stationary distribution for each node is clearly proportional to its degree.

## 3 Random Walks without a Restart