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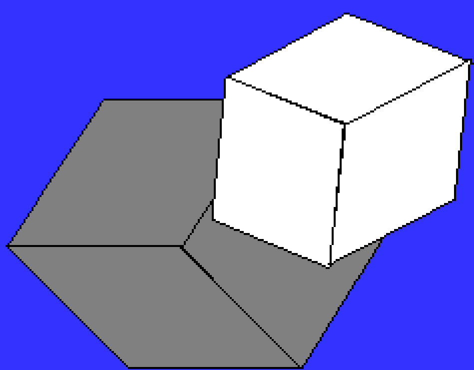

FORUM

The "Italian Journal of Pure and Applied Mathematics" cannot more take advantage of the precious collaboration of **prof. Marzio Strassoldo**, who has suddenly passed away in the beginning of January. The members of Editorial Board express their deep sorrow for this loss.

The Chief Editor, Prof. Piergiulio Corsini, wants to express his most heartfelt sympathy to prof. Strassoldo's family, for the demise of a dear friend and a colleague, who gave an important contribution for the success of our journal.

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Forum Editrice Universitaria Udinese Srl

Via Larga 38 - 33100 Udine

Tel: +39-0432-26001, Fax: +39-0432-296756

forum@forumeditrice.it

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AN UNCONDITIONALLY STABLE FINITE DIFFERENCE SCHEME FOR EQUATIONS OF CONSERVATION LAW FORM

Saeid Abbasbandy[1]
Ahmad Shirzadi

Department of Mathematics
Imam Khomeini International University
Qazvin
Iran

Abstract. This study presents a numerical scheme for solving one dimensional equations of conservation law form. The Saulyev's finite difference techniques are used to compute the solution. Although the resulting difference equation do not appear explicit, a suitable use of the equation make it explicit. It is shown that this explicit scheme is unconditionally stable. A numerical example is presented to demonstrate the accuracy and efficiency of the proposed computational procedure.
Keywords: finite difference schemes; implicit methods; explicit techniques; Saulyev's technique; Lax-Wendroff formula.

1. Introduction

Finite elements [1], [2], Finite differences [3], [4] and recently meshless methods [5], [6] are known to be powerful numerical methods to solve partial differential equations with boundary conditions. In the theory of fluid flow, the equations of motions, of continuity, and of energy can be combined into one conservation equation of the form

$$\frac{\partial U}{\partial t}+\frac{\partial F(U)}{\partial x}=0, \tag{1.1}$$

where U and F are column vectors [7, 8]. The Lax-Wendroff method, can be used to approximate Eqn. (1.1) by an explicit difference equation of second order accuracy. Consider the following conservation equation:

$$\frac{\partial u}{\partial t}+a\frac{\partial u}{\partial x}=0, \tag{1.2}$$

where a is a positive constant and u is a function of x and t. Grid points (x_i, t_j) are defined by $x_i = x_0 + ih, i = 1, 2, ...$ and $t_j = t_0 + jk, j = 1, 2, ...$ The notation $u_{i,j}$ is used for the finite difference approximations of $u(x_i, t_j)$. By Taylor's expansion and elimination of the t-derivatives using the differential equation(1.2) we obtain:

$$u_{i,j+1}=u_{i,j}-ka\left(\frac{\partial u}{\partial x}\right)_{i,j}+\frac{1}{2}k^2a^2\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j}+\cdots . \tag{1.3}$$

[1]Corresponding author. Email: abbasbandy@yahoo.com, Tel.:+98(912)1305326.

The replacement of the x-derivatives by central-difference approximations gives the explicit difference equation:

$$u_{i,j+1} = u_{i,j} - \frac{a\varphi}{2}(u_{i+1,j} - u_{i-1,j}) + \frac{a^2\varphi^2}{2}(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}), \tag{1.4}$$

where $\varphi = \frac{k}{h}$. Scheme (1.4)is named the Lax-Wendroff explicit scheme and is stable for $0 < a\varphi \le 1$ [3], [4]. In the next section we will present a scheme which is unconditionally stable.

2. Main results

2.1. Proposed finite difference scheme

A noticeable feature of the current explicit finite difference methods for the numerical solution is the restriction of the size of the time step due to stability requirements. For most problems these are impractical methods. This limitation is removed when the implicit finite difference schemes are used for the numerical solution of the equations. However, a disadvantage of these techniques is the extensive amount of CPU times utilized in determining the numerical solution compared to the explicit methods for the same selection of values of step-sizes k and h. Implicit finite difference schemes require the solution of a large number of simultaneous linear algebraic equations at each time step. The number of iterations require to achieve a modest accuracy may become large, particularly for large time increments and small space mesh size. So the need to develop unconditionally stable Saulyev's finite difference schemes is clear[9], [10]. The main advantage of these techniques is that they are unconditionally stable and are explicit in nature [3], [4]. If we use the Saulyev's A formula for (1.4) we get the following equation:

$$\begin{aligned}\left(1+\frac{a^2\varphi^2}{2}\right)u_{i,j+1} &= \frac{a^2\varphi^2}{2}u_{i-1,j+1} + \frac{a\varphi}{2}u_{i-1,j} + \left(1-\frac{a^2\varphi^2}{2}\right)u_{i,j} \\ &\quad + \frac{a\varphi}{2}(a\varphi-1)u_{i+1,j}.\end{aligned} \tag{2.1}$$

Although the above approximation does not appear explicit, because $u_{i,j+1}$ and $u_{i,j}$ are on the right-hand side, a suitable use of the equation makes it explicit. If we begin the calculation from the left to right, the only unknown is $u_{i,j+1}$.

2.2. Stability analysis

For stability analysis we consider the equation

$$\left(1+\frac{a^2\varphi^2}{2}\right)u_{i,j+1} = \frac{a^2\varphi^2}{2}u_{i-1,j+1} + \frac{a\varphi}{2}u_{i-1,j} + \left(1-\frac{a^2\varphi^2}{2}\right)u_{i,j} + \frac{a\varphi}{2}(a\varphi-1)u_{i+1,j}.$$

Substituting $u_{i,j} = z^j e^{\sqrt{-1}\lambda ih}$ and eliminating common factors from the both sides of the equation we have

$$\left(1+\frac{a^2\varphi^2}{2}\right)z = \frac{a^2\varphi^2}{2}ze^{-\sqrt{-1}\lambda h} + \frac{a\varphi}{2}e^{-\sqrt{-1}\lambda h} + \left(1-\frac{a^2\varphi^2}{2}\right) + \frac{a\varphi}{2}(a\varphi-1)e^{\sqrt{-1}\lambda h},$$

therefore,

$$\left(1+\frac{a^2\varphi^2}{2}-\frac{a^2\varphi^2}{2}\cos(\lambda h)\right)z+\sqrt{-1}\frac{a^2\varphi^2}{2}\sin(\lambda h)z$$
$$=1-\frac{a^2\varphi^2}{2}+\frac{a^2\varphi^2}{2}\cos(\lambda h)+\sqrt{-1}\left(\frac{a^2\varphi^2}{2}-a\varphi\right)\sin(\lambda h).$$

Consequently,

$$\left(1+\frac{a^2\varphi^2}{2}-\frac{a^2\varphi^2}{2}\cos(\lambda h)\right)^2+\frac{a^4\varphi^4}{4}\sin^2(\lambda h))|z|^2$$
$$=\left(1-\frac{a^2\varphi^2}{2}+\frac{a^2\varphi^2}{2}\cos(\lambda h)\right)^2+a^2\varphi^2\left(\frac{a\varphi}{2}-1\right)^2\sin^2(\lambda h),$$

so we have

$$|z|^2=\frac{\left(1-\frac{a^2\varphi^2}{2}+\frac{a^2\varphi^2}{2}\cos(\lambda h)\right)^2+a^2\varphi^2\left(\frac{a\varphi}{2}-1\right)^2\sin^2(\lambda h)}{\left(1+\frac{a^2\varphi^2}{2}-\frac{a^2\varphi^2}{2}\cos(\lambda h)\right)^2+\frac{a^4\varphi^4}{4}\sin^2(\lambda h)}.$$

For stability we should have$|z|^2\le 1$ and, therefore,

$$\left(1-\frac{a^2\varphi^2}{2}+\frac{a^2\varphi^2}{2}\cos(\lambda h)\right)^2+a^2\varphi^2\left(\frac{a\varphi}{2}-1\right)^2\sin^2(\lambda h)$$
$$\le(1+\frac{a^2\varphi^2}{2}-\frac{a^2\varphi^2}{2}\cos(\lambda h))^2+\frac{a^4\varphi^4}{4}\sin^2(\lambda h),$$

simplifying the above non-equality we have $(1-a\varphi)(1+\cos(\lambda h))\le 2$, therefore,

$$(1-a\varphi)\le\frac{2}{(1+\cos(\lambda h))},$$

since $1\le\cos(\lambda h)\le 1$, if $1-a\varphi\le 1$ or $a\varphi\ge 0$ then the above non-equality is hold. Because $a\ge 0$ and $\varphi\ge 0$, the relation is hold and, therefore, scheme (2.1) is unconditionally stable.

3. Test example

For a test example, we consider the following equation

$$\frac{\partial U}{\partial t}+\frac{\partial U}{\partial x}=0,\quad 0<x<\infty,\quad t>0, \tag{3.1}$$

with the boundary condition

$$U(0,t)=2t,\quad t>0, \tag{3.2}$$

and initial conditions

$$\begin{aligned}U(x,0)&=x(x-2),\quad 0\le x\le 2,\\ U(x,0)&=2(x-2),\quad 2\le x.\end{aligned} \tag{3.3}$$

The solution obtained via the proposed finite difference scheme are presented in some mesh points(x=1 and $t = 0.5\ i,\ i = 1, 2, ..., 8$) and are compared with the exact solution:

t	0.5	1	1.5	2	2.5	3	3.5	4
Exact	-0.75	0	1	2	3	4	5	6
approximate	-0.556	0.185	1.074	2.028	3.010	4.004	5.001	6.000

4. Conclusions

This article has outlined an approach for the study of a particular class of hyperbolic partial differential equations. The Saulyev's first kind explicit finite difference technique was applied. The Proposed finite difference method has been proved to be unconditionally stable. The new algorithm outlined here, was tested on a problem and was seen to produce good results that suggest convergence to the exact solution when h goes to zero. The new scheme discussed in this report, had the advantage of being stable and explicit. The results reveal that the method is remarkably effective.

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References

[1] Braess, D., *Finite Elements: Theory, Fast Solvers, and Applications in Solid Mechanics*, Cambridge University Press; 3 edition, 2007.

[2] Johnson, C., *Numerical Solution of Partial Differential Equations by the Finite Element Method*, Dover Publications, 2009.

[3] Lapidus, L., Pinder, G.F., *Numerical solution of partial differential equations in science and engineering*, Wiley, New York, 1982.

[4] Smith, G.D., *Numerical solution of partial differential equations*, Oxford University Press, 1985.

[5] Abbasbandy, S., Shirzadi, A., *A meshless method for two-dimensional diffusion equation with an integral condition*, Eng. Anal. Bound. Elem. 34 (12) (2010) 1031–1037.

[6] Abbasbandy, S., Shirzadi, A., *MLPG method for two-dimensional diffusion equation with Neumann's and non-classical boundary conditions*, Appl. Numer. Math. 61 (2011) 170–180.

[7] Li,X.-G., Yu, X.J., Chen, G.-N., *The third-order relaxation schemes for hyperbolic conservation laws*, J. Computat. Appl. Math., 138 (2002), 93-108.

[8] Qiu, J., Dumbser, M., C.-W., *The discontinuous Galerkin method with LaxWendroff type time discretizations*, Comput.Methods Appl.Mech.Engrg., 194 (2005), 4528-4543.

[9] Dehghan, M., *Identifying a control function in two-dimensional parabolic inverse problems*, Appl. Math. Comput., 143(2003), 375-391.

[10] Dehghan, M., *The one-dimensional heat equation subject to a boundary integral specification*, Chaos, Solitons and Fractals, 32 (2007), 661-675.

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AN AUTOMATIC SCHEME ON THE HOMOTOPY ANALYSIS METHOD FOR SOLVING NONLINEAR ALGEBRAIC EQUATIONS

Safwan Al-Shara'

Department of Mathematics
Al al-Bayt University
Mafraq 25113
Jordan

Fadi Awawdeh[1]

Department of Mathematics
Hashemite University
Zarqa 13115
Jordan
and
Mathematics and Sciences Department
Dhofar University
Salalah 211
Oman

S. Abbasbandy

Department of Mathematics
Imam Khomeini International University
Ghazvin 34149
Iran

Abstract. In this paper, an automatic scheme coupled with homotopy analysis method is presented for solving nonlinear algebraic equations. The experimental results show the potential and limitations of the new method and imply directions for future work.

Keywords: nonlinear algebraic equations; iterative method; homotopy analysis method.

1. Introduction

Many practical problems in all fields of science, engineering, or applied mathematics give rise to the need to solve nonlinear algebraic equations (NAE). Typical examples that involve NAE include: calculation of chemical equilibrium by minimization of Gibbs energy or using equilibrium constants, continuous operation of

[1]Corresponding author: Tel:+96891755035
E-mail: fawawadeh@du.edu.om

some reactors (such as continuous stirred tank reactors, CSTR), heat exchanger calculations (where logarithmic mean temperature difference is used), calculation of the parameters of activity coefficient equations (such as the Van Laar and Wilson equations), solving various equations of state for specific volume or compressibility factor and calculation of the minimum reflux ratio using the Underwood equations [13].

Many powerful algorithms and codes for solving NLE have been developed in recent years. So far, almost all iterative techniques require the prior one or more initial guesses for the desired root and may fail to converge when the initial guess in far from the required solution. Achieving convergence in an efficient manner in these situations has become a real challenge.

The Homotopy Analysis Method (HAM) is based on the classic homotopy theory and it is a general method for solving nonlinear problems. HAM was developed by Shijun Liao from 1990s to 2010s, together with contributions of many other researchers in theory and applications [7]–[11]. The basic ideas of the HAM are described in Appendix A.

The HAM method has demonstrated promise in the arena of analytical solutions of equations. Abbasbandy and et al. [3] have been successfully extended HAM to the iterative numerical solution of algebraic equations. Indeed the inherent flexibility and generality of the HAM method makes this a challenging task. The successful theoretical development of this methodology forms a core accomplishment of the work by Awawdeh [5]. Awawdeh developed the methodology for the iterative numeric solution of multivariate system of nonlinear algebraic equations.

In this work, an automatic homotopy analysis method (AHAM) is presented for a single NAE. Some problems were selected to illustrate the performance of our algorithms in solving NAE. The calculations were done using Matlab 7. Our comparison of the methods is based upon the number of iterations. We use the following stopping criterion for our computer programs:

$$|x_{k+1} - x_k| < \epsilon,$$

where $\epsilon = 2.22e - 16$ is a Matlab constant.

2. Iterative methods for solving nonlinear equations

Consider the nonlinear algebraic equation

$$f(x) = 0, \tag{1}$$

where p is a simple root of it and $f \in C^n$ on an interval containing p as $[a, b]$. Let $p_0 \in [a, b]$ be an approximation to the solution p of (1) such that $f\,'(p_0) \neq 0$.

Consider the n^{th} Taylor polynomial for $f(x)$ expanded about p_0, and evaluated at $x = p_0 - \beta$,

$$f(p_0 - \beta) = f(p_0) - \beta f'(p_0) + \frac{\beta^2}{2} f''(p_0) + \cdots + (-1)^n \frac{\beta^n}{n!} f^{(n)}(p_0) + O(\beta^{n+1}).$$

We are looking for β such that

$$0 = f(p_0 - \beta) \simeq f(p_0) - \beta f'(p_0) + \frac{\beta^2}{2} f''(p_0) + \cdots + (-1)^n \frac{\beta^n}{n!} f^{(n)}(p_0),$$

or what amounts to the same as

$$\frac{f(p_0)}{f'(p_0)} = \beta - \frac{\beta^2}{2} \frac{f''(p_0)}{f'(p_0)} + \frac{\beta^3}{6} \frac{f'''(p_0)}{f'(p_0)} + \cdots + (-1)^{n+1} \frac{\beta^n}{n!} \frac{f^{(n)}(p_0)}{f'(p_0)}. \tag{2}$$

By setting

$$\lambda_1 = \frac{f(p_0)}{f'(p_0)} \quad \textit{and} \quad \lambda_m = \frac{(-1)^{m+1}}{m!} \frac{f^{(m)}(p_0)}{f'(p_0)} \quad \text{for} \;\; m \geq 2,$$

Equation (2) reduces to

$$\mathcal{N}(\beta) = \beta + \lambda_2 \beta^2 + \lambda_3 \beta^3 + \cdots + \lambda_n \beta^n - \lambda_1 = 0.$$

We will apply the HAM to approximate β. To this end, let $\hbar \neq 0$ an auxiliary parameter and $\mathcal{L}$ an auxiliary linear operator with the property $\mathcal{L}[f(x)] = 0$ when $f(x) = 0$. Then by using $q \in [0, 1]$ as an embedding parameter, we construct such a homotopy

$$\mathcal{H}(w(q); \beta_\circ, \hbar, q) = (1-q)\mathcal{L}(w(q) - \beta_0) - q\hbar\mathcal{N}(w(q)), \tag{3}$$

where β_0 is an initial guess of β. It should be emphasized that we have great freedom to choose the initial guess β_0, the auxiliary linear operator $\mathcal{L}$ and the non-zero auxiliary parameter $\hbar$. Enforcing the homotopy (3) to be zero, i.e.,

$$\mathcal{H}(w(q); \beta_\circ, \hbar, q) = 0,$$

we have the so-called zero-order deformation equation

$$(1-q)\mathcal{L}(w(q) - \beta_0) = q\hbar\mathcal{N}(w(q)). \tag{4}$$

When $q = 0$, the zero-order deformation Equation (4) becomes

$$w(0) = \beta_0, \tag{5}$$

and when $q = 1$, it is equivalent to

$$w(1) = \beta. \tag{6}$$

Thus, according to (5) and (6), as the embedding parameter q increases from 0 to 1, $w(q)$ varies continuously from the initial approximation β_0 to the exact solution β. By Taylor's theorem, $w(q)$ can be expanded in a power series of q as follows

$$w(q) = \beta_0 + \sum_{m=1}^{\infty} \beta_m q^m, \tag{7}$$

where

$$\beta_m = \frac{1}{m!}\frac{d^m w(q)}{dq^m}|_{q=0}.$$

If the initial guess β_0, the auxiliary linear parameter $\mathcal{L}$ and the nonzero auxiliary parameter $\hbar$ are properly chosen so that the power series (7) converges at $q = 1$, then we have the series solution

$$\beta = w(1) = \beta_0 + \sum_{m=1}^{\infty}\beta_m. \tag{8}$$

The governing equation of β_m can be derived by differentiating the zero-order deformation Equation (4) m times with respective to q and then dividing by $m!$ and finally setting $q = 0$. We have the so-called m^{th}-order deformation equation

$$\begin{aligned}\mathcal{L}\left(\beta_m - \chi_m\beta_{m-1}\right) = \hbar\Big(\beta_{m-1} + \lambda_2\sum_{k=0}^{m-1}\beta_k\beta_{m-1-k} + \cdots + \lambda_n\sum_{r_1=0}^{m-1}\beta_{m-r_1-1}\sum_{r_2=0}^{r_1}\beta_{r_1-r_2}\\ \cdots\sum_{r_{n-2}=0}^{r_{n-3}}\beta_{r_{n-3}-r_{n-2}}\sum_{r_{n-1}=0}^{r_{n-2}}\beta_{r_{n-2}-r_{n-1}}\beta_{r_{n-1}}\Big) - \hbar(1-\chi_m)\lambda_1,\end{aligned}$$

where

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases}$$

Choosing the operator $\mathcal{L}$ as an identity operator, we have for $m \geqslant 1$ that

$$\begin{aligned}\beta_m = (\hbar+\chi_m)\beta_{m-1}+\hbar\lambda_2\sum_{k=0}^{m-1}\beta_k\beta_{m-1-k} + \cdots + \hbar\lambda_n\sum_{r_1=0}^{m-1}\beta_{m-r_1-1}\sum_{r_2=0}^{r_1}\beta_{r_1-r_2}\\ \cdots\sum_{r_{n-2}=0}^{r_{n-3}}\beta_{r_{n-3}-r_{n-2}}\sum_{r_{n-1}=0}^{r_{n-2}}\beta_{r_{n-2}-r_{n-1}}\beta_{r_{n-1}}] - \hbar(1-\chi_m)\lambda_1.\end{aligned} \tag{9}$$

Some cases will be discussed for various values of n:

Case [n=2] In this case, equation (9) reduces to

$$\beta_m = (\hbar + \chi_m)\beta_{m-1} + \hbar\lambda_2\sum_{k=0}^{m-1}\beta_k\beta_{m-1-k} - \hbar(1-\chi_m)\lambda_1.$$

Setting $\beta_0 = \lambda_1$, and, taking the zero-order approximation of β in (8), we can obtain

$$p = p_0 - \beta \simeq p_0 - \beta_0 = p_0 - \frac{f(p_0)}{f'(p_0)}. \tag{10}$$

We can write down the iteration form of (10) as follows

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

which is again Newton-Raphson method. Using first-order approximation of β in (8), we get that

$$p = p_0 - \beta \simeq p_0 - (\beta_0 + \beta_1) = p_0 - \frac{f(p_0)}{f'(p_0)} + \hbar \frac{f^2(p_0) f''(p_0)}{2f'^3(p_0)}. \tag{11}$$

The iteration form of (11) can be given as follows

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} + \hbar \frac{f^2(x_n) f''(x_n)}{2f'^3(x_n)}, \tag{12}$$

which is the Householder's iteration.

Case [n=3] Here, equation (9) becomes

$$\begin{aligned} \beta_m = {} & (\hbar + \chi_m)\beta_{m-1} + \hbar\lambda_2 \sum_{k=0}^{m-1} \beta_k \beta_{m-1-k} \\ & + \hbar\lambda_3 \sum_{i=0}^{m-1} \beta_{m-i-1} \sum_{j=0}^{i} \beta_j \beta_{i-j} - \hbar(1 - \chi_m)\lambda_1. \end{aligned} \tag{13}$$

Setting $\beta_0 = \lambda_1$ and taking the first-order approximation of β in (8), we get

$$p = p_0 - \beta \simeq p_0 - \beta_0 = p_0 - \frac{f(p_0)}{f'(p_0)}. \tag{14}$$

We can write down the iteration form of (14) as follows

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

which is the Newton-Raphson method. Using second-order approximation of β in (8), we obtain

$$\begin{aligned} p &= p_0 - \beta \simeq p_0 - (\beta_0 + \beta_1) \\ &= p_0 - \frac{f(p_0)}{f'(p_0)} + \hbar \frac{f^2(p_0) f''(p_0)}{2f'^3(p_0)} - \hbar \frac{f^3(p_0) f'''(p_0)}{6f'^4(p_0)}. \end{aligned} \tag{15}$$

The iteration form of (15) can be given as follows

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} + \hbar \frac{f^2(x_n)}{2f'^3(x_n)} \left(f''(x_n) - \frac{f(x_n) f'''(x_n)}{3f'(x_n)} \right), \tag{16}$$

which is the same as MADM [1] and MHPM [2] when $\hbar = -1$.

The proposed method (16) provides us with a family of iterative formulas in auxiliary parameter $\hbar$. This provides a convenient way to control convergence given by the technique. Liao [7], [11] suggested some ways to choose a fixed proper value of $\hbar$ such as the $\hbar-$curves.

The parameter $\hbar$ will not remain fixed. We will renew $\hbar$ after computing x_{n+1} by using any iterative formula for solving $f(x) = 0$. As an application to demonstrate the idea, the Chen-Li exponential iterative formula having quadratic convergence discussed in [6], [12]

$$x_{n+1} = x_n \exp(-\frac{f(x_n)}{x_n f'(x_n)}), \tag{17}$$

will be used.

Let $x_0,\ \hbar_0$ be initial guesses of p and $\hbar$. Iteration formula (16) can be written as

$$x_{n+1} = a_n + \hbar_n b_n, \tag{18}$$

where

$$\begin{aligned} a_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ b_n &= \frac{f^2(x_n)}{2f'^3(x_n)}\left(f''(x_n) - \frac{f(x_n)f'''(x_n)}{3f'(x_n)}\right). \end{aligned} \tag{19}$$

We will renew $\hbar_n$ after computing x_{n+1} by applying Chen-Li exponential iterative formula (17) on $F(\hbar) = f(a_n + \hbar b_n) = 0$ with $\hbar_0 = \exp(-\frac{f(a_0)}{f'(b_0)b_0})$ as follows

$$\begin{aligned} \hbar_{n+1} &= \hbar_n \exp(-\frac{F(\hbar_n)}{\hbar_n F'(\hbar_n)}) \\ &= \hbar_n \exp(-\frac{f(a_n + \hbar_n b_n)}{\hbar_n f'(a_n + \hbar_n b_n) b_n}). \end{aligned} \tag{20}$$

3. Numerical examples

Some problems were selected in order to demonstrate the performance of algorithm (18-20), automatic homotopy analysis method (AHAM), as a novel solver for nonlinear algebraic equations. We present the results of AHAM with the classical Newton's method (NM), Adomian method (ADM) [1] and homotopy perturbation method (HPM) [2].

Example 1. Consider the equation

$$x^2 - e^x - x + 5 = 0$$

with the solution $p = 1.906180$. The numerical results are shown in Table 1.

Table 1. Numerical results of the solutions in Example 1

method	$x_0 = 40$	$x_0 = 100$
	iter	*iter*
NM	43	102
ADM	30	70
HPM	30	70
HAM	3	19

Example 2. Consider the equation

$$\ln(x^2 + 1) - e^{0.4x} = 0$$

with the solution $p = -0.982012$. Some numerical results are listed in Table 2.

Table 2. Numerical results of the solutions in Example 2

method	$x_0 = 40$	$x_0 = 100$
	iter	*iter*
NM	37	87
ADM	11	*Divergent*
HPM	11	*Divergent*
HAM	6	6

Example 3. Consider the equation

$$5x^3 - xe^x - 1 = 0$$

with the solution $p = 4.704594$. With $x_0 = 14$ and $x_0 = 70$, we will obtain Table 3. Note that the equation in this example has another solution $p = 0.837177$, which can be obtained using algorithm (18-20), with $x_0 = 2$, after nine iterations.

Table 3. Numerical results of the solutions in Example 3

method	$x_0 = 14$	$x_0 = 70$
	iter	*iter*
NM	15	72
ADM	11	49
HPM	11	49
HAM	5	12

4. Conclusion

The AHAM algorithms are very effective and efficient which provide highly accurate results in less number of iterations as compared to some well-known existing methods. It is shown to have significant advantages over the traditional methods in terms of flexibility, convergence and possibly speed. One of the disadvantages of the algorithms that it require the computation of high derivatives. But, The practical relevance of these methods increases since computer aided formulae manipulation facilities became a common tool in numerical analysis. HAM algorithms contain the parameter $\hbar$ which can be used to ensure and accelerate the convergence. In this work, an efficient method to get the value of $\hbar$, automatically, is presented.

Appendix A: Basic idea of HAM

To show the basic idea of HAM, let us consider the following equation

$$\mathcal{N}\left(y(t)\right) = 0,$$

where $\mathcal{N}$ is a nonlinear operator, $y(t)$ is unknown function and t the independent variable. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the traditional homotopy method, Liao [7] constructs the so-called zero-order deformation equation

$$(1-q)\mathcal{L}\left(\phi(t;q) - y_0(t)\right) - qhH(t)\mathcal{N}[\phi(t;q)], \tag{21}$$

where $q \in [0,1]$ is the embedding parameter, $y_0(t)$ an initial guess of the exact solution $y(t)$, $h \neq 0$ an auxiliary parameter, $H(t) \neq 0$ an auxiliary function, $\phi(t;q)$ is a unknown function, and $\mathcal{L}$ is an auxiliary linear operator with the property $\mathcal{L}\left(y(t)\right) = 0$ when $y(t) = 0$. Then using $q \in [0,1]$ as an embedding parameter, It should be emphasized that we have great freedom to choose the initial guess $y_0(t)$, the auxiliary linear operator $\mathcal{L}$, the non-zero auxiliary parameter h, and the auxiliary function $H(t)$. Obviously, when $q = 0$ and $q = 1$, it holds

$$\phi(t;0) = y_0(t), \qquad \phi(t;1) = y(t).$$

Thus, as q increases from 0 to 1, the solution $\phi(t;q)$ varies continuously from the initial approximation $y_0(t)$ to the exact solution $y(t)$. Expanding $\phi(t;q)$ in Taylor series with respect to q, one has

$$\phi(t;q) = y_0(t) + \sum_{m=1}^{\infty} y_m(t) q^m \tag{22}$$

where

$$y_m(t) = \frac{1}{m!}\frac{\partial^m \phi(t;q)}{\partial q^m}|_{q=0}.$$

If the initial guess $y_0(t)$, the auxiliary linear parameter $\mathcal{L}$, the nonzero auxiliary parameter h, and the auxiliary function $H(t)$ are properly chosen so that the power series (22) converges at $q = 1$, one has

$$y(t) = y_0(t) + \sum_{m=1}^{\infty} y_m(t). \tag{23}$$

Define the vector

$$\overrightarrow{y}_n(t) = \{y_0(t), y_1(t), y_2(t), \ldots, y_n(t)\}.$$

Differentiating the zero-order deformation equation (21) m times with respective to q and then dividing by $m!$ and finally setting $q = 0$, we have the so-called mth-order deformation equation

$$\mathcal{L}\left(y_m(t) - \chi_m y_{m-1}(t)\right) = hH(t)\mathcal{R}_m(\vec{y}_{m-1}(t)) \tag{24}$$

where

$$\mathcal{R}_m(\vec{y}_{m-1}(t)) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\mathcal{N}\left(\phi(t;q)\right)}{\partial q^{m-1}}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases}.$$

It should be emphasized that $y_m(t)$ for $m \geq 1$ is governed by the linear equation (24) with the linear boundary conditions that come from the original problem, which can be easily solved by symbolic computation software such as Matlab and Mathematica.

References

[1] Abbasbandy, S., *Improving Newton-Raphson method for nonlinear equations by modified Adomian decomposition method*, Appl. Math. Comput., 145 (2003), 887-893.

[2] Abbasbandy, S., *Modified homotopy perturbation method for nonlinear equations and comparison with Adomian decomposition method*, Appl. Math. Comput., 172 (2006), 431-438.

[3] Abbasbandy, S., Tan, Y., Liao, S.J., *Newton-homotopy analysis method for nonlinear equations*, Appl. Math. Comput., 188 (2007), 1794-1800.

[4] Adomian, G., *Solving Frontier Problems of Physics: The Decomposition Method*, Kluwer Academic Publishers, Boston and London, 1994.

[5] Awawdeh, F., *On new iterative method for solving systems of nonlinear equations*, Numer. Algorithms, 54 (2010), 395-409.

[6] Chen, J.H., Weiguo Li, *On new exponential quadratically iterative method formulae*, Appl. Math. Comput., 180 (2006), 242-246.

[7] Liao, S.J., *Beyond Perturbation: Introduction to the Homotopy Analysis Method*, Chapman & Hall/CRC Press, Boca Raton, 2003.

[8] Liao, S.J., *Homotopy Analysis Method in Nonlinear Differential Equations*, Springer & Higher Education Press, Heidelberg, 2012.

[9] Liao, S.J., *On the homotopy analysis method for nonlinear problems*, Appl. Math. Comput., 147 (2004), 499-513.

[10] Liao, S.J., *Notes on the homotopy analysis method: Some definitions and theorems*, Commun Nonlinear Sci Numer Simul., 14 (2009), 983-997.

[11] Liao, S.J., *An optimal homotopy-analysis approach for strongly nonlinear differential equations*, Commun Nonlinear Sci Numer Simul., 15 (2010), 2003-2016.

[12] Mamta, Kanwar, V., Kukreja, V.K., Singh, S., *On a class of quadratically convergent iteration formulae*, Appl. Math. Comput., 166 (2005), 633-637.

[13] Shacham, M., Brauner, N., *Numerical solution of non-linear algebraic equations with discontinuities*, Comput. Chem. Eng., 26 (2002), 1449-1457.

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A NEW INTEGRAL TRANSFORM

B.G. Sidharth
International Institute for Applicable Mathematics & Information Sciences
Udine
Italy
and
B.M. Birla Science Centre
Adarsh Nagar, Hyderabad - 500 063
India

Abstract. Using Bauer's expansion and properties of spherical Bessel and Legender functions, we deduce a new transform and briefly indicate its use.

Using properties of spherical Bessel and Legender functions, we would now like to deduce a new Integral Transform. Our starting point is Bauer's expansion (this and a few other known results quoted can be obtained from refs. [1]-[5]):

$$e^{\imath zt} = \sum^{\infty}(2n+1)\imath^n P_n(t) j_n(z)$$

Using the orthogonality of the spherical Bessel and Legender functions, viz., the relations

$$\int_{-\infty}^{\infty} j_n(z) j_m(z) dz = \int_{-\infty}^{\infty} J_{n+\frac{1}{2}}(z) J_{m+\frac{1}{2}}(z) \frac{dz}{z} = 0$$

if

$$m \neq n$$

and

$$= \frac{2}{(2n+1)} \quad \text{if } m = n$$

$$\int_{-1}^{+1} P_n(t) P_m(t) dt = 0$$

if

$$m \neq n,$$

and

$$= \frac{2}{2n+1} \quad \text{if} \quad m = n$$

in Bauer's expansion we get

$$\int_{-1}^{1} e^{\imath zt} P_n(t) dt = 2\imath^n j_n(z) \tag{1}$$

(2) $$\int_{-\infty}^{\infty} j_n(z) e^{\imath zt} dz = 2\imath^n P_n(t)$$

We consider that z is real. We would also need the following

$$J_\nu(z) = \frac{(\frac{1}{2}z)^\nu}{\Gamma(\nu + \frac{1}{2})\Gamma(\frac{1}{2})} \int_0^\pi \cos(z \cos \Theta) \sin^{2\nu} \Theta d\Theta$$

$$\left(Re(\nu + \frac{1}{2}) > 0 \right) \quad \nu = n + \frac{1}{2}$$

whence,

$$J_\nu(-z) = (-1)^\nu J_\nu(z) = (-1)^n \imath J_\nu(z)$$

or

(3) $$j_n(-z) = (-1)^n j_n(z)$$

Let us consider a function $g(z)$ which can be expanded as an infinite linear combination of spherical Bessel functions, on the lines of Neumann's expansion in terms of ordinary Bessel functions. This can be done because of the orthogonality relations above. Similarly, we will also use the known expansion in terms of Legender functions. Thus we have,

$$\begin{aligned} g(z) &= c_n j_n(z) \\ &= \sum c_n (2\imath^n)^{-1} \int_{-1}^{1} e^{\imath zt} P_n(t) dt \end{aligned}$$

or,

(4) $$g(z) = \int_{-1}^{1} f(t) e^{\imath zt} dt$$

where

$$\begin{aligned} f(t) &= \sum \bar{c}_n P_n(t), (\bar{c}_n = c_n (2\imath^n)^{-1}) \\ &= \sum \bar{c}_n (2\imath^n)^{-1} \int_{-\infty}^{\infty} j_n(z) e^{\imath zt} dz \end{aligned}$$

or

$$\begin{aligned} f(t) &= \int \frac{1}{4} \sum c_n (-1)^n j_n(z) e^{\imath zt} dz \\ &= \frac{1}{4} \int_{-\infty}^{\infty} \sum c_n j_n(-z) e^{-\imath(-z)t} dz \end{aligned}$$

wherein we have used (3), or,

(5) $$f(t) = \frac{1}{4} \int_{-\infty}^{\infty} g(y) e^{-\imath yt} dy$$

In deducing (4) and (5), we have used (1) and (2), and the summations are infinite. Moreover we assume that for $f(t)$ and $g(z)$ derivatives of all orders exist over their domains.

So, finally,

(6) $$g(z) = \frac{1}{4} \int_{-1}^{+1} \int_{-\infty}^{\infty} g(y) e^{\imath(z-y)t} dy dt$$

Relations (4), (5) and (6) are the desired new relations. As an application, let us consider the differential equation,

$$L_{op}g(z) = h(z), \tag{7}$$

where L_{op} is a linear differential operator. Using (6) in (7), we get,

$$L_{op}g(z) = F(\frac{d}{dz})g(z) = \frac{1}{4}\int_{-1}^{+1}\int_{-\infty}^{\infty} F(\imath t)g(y)e^{\imath(z-y)t}dydt = h(z)$$

or

$$\begin{aligned} L_{op}g(z) &= A\int_{-1}^{+1} f(t)F(\imath t)e^{\imath zt}dt = h(z) \\ &= \int_{-1}^{+1} \hat{h}(t)e^{\imath zt}dt \end{aligned} \tag{8}$$

where we have used (4),

$$\begin{aligned} h(z) &= \sum d_n j_n(z), \\ \hat{h}(t) &= \sum \bar{d}_n P_n(t) \\ \bar{d}_n &= d_n(2\imath^n)^{-1} \end{aligned}$$

So we get

$$f(t)F(\imath t) = \hat{h}(t)$$

As $\hat{h}(t)$ and $F(\imath t)$ are known so is $f(t)$ known and therefore also $g(z)$. In fact,

$$f(t) = \frac{1}{4}\int_{-\infty}^{\infty} g(y)e^{-\imath yt}dy = \sum \bar{c}_n P_n(\imath t)$$

so that

$$g(z) = \sum c_n j_n(z), \;\; \bar{c}_n = (2\imath^n)^{-1}c_n$$

Remarks.

1. We note that Neumann's expansion alluded to applies for any analytical function $g(z)$:

   $$g(z) = \sum_n^{\infty} b_n J_n(z)$$

   However, the expansion in (4) is in terms of Spherical Bessel functions. As mentioned such an expansion can always be justified, as in the case of the Legender polynomial expansion of any function $f(t)$ given in (5), by using the orthogonality properties of the $j_n(z)$ and $P_n(t)$ given above.

2. The above consideration in relation (6) is to be distinguished from the so called Hankel transform. Further, it must be noted that the domains of integration in (4), (5) and (6) are $(-1, 1)$ for t and $(-\infty, \infty)$ for z.

References

[1] G.N. Watson, G.N., *Theory of Bessel Functions*, Cambridge University Press, Cambridge, 1958.

[2] Whittaker, E.T., Watson, G.N., *A Course of Modern Analysis*, Cambridge University Press, Cambridge, 1962.

[3] Morse, P.M., Feshbach, H., *Methods of Theoretical Physics*, vol. 2, McGraw Hill, New York, 1958.

[4] Copson, E.T., *Theory of Functions of a Complex Variable*, University Press, London, 1935.

[5] Bateman, H., *Higher Transcendental Functions*, Vol2. McGraw Hill, New York, 1953.

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DISJOINT J-CLASS OPERATORS

Abdelaziz Tajmouati[1]
Mohammed El Berrag

Sidi Mohamed Ben Abdellah University
Faculty of Sciences Dhar El Mahraz
Fez
Marocco
e-mails: abdelaziz.tajmouati@usmba.ac.ma
med.elberrag@gmail.com

Abstract. In this paper, we first introduce the notion of disjoint extended limit set for a tuple of bounded linear operators on a separable Banach space X, and we extend some results from a single operator to a tuple of sequences of operators.

Keywords: tuple of sequence, hypercyclic sequences, d-topologically transitive, d-J-class operators, d-J^{mix}-class operators.

1. Introduction

For a infinite-dimensional separable complex Banach space X, $\mathcal{B}(X)$ will denote the algebra of all bounded linear operators on X.

For $x \in X$, the orbit of x under $(T_n)_n$ is the set

$$Orb(T_n, x) = \{T_n x : n \in \mathbb{Z}_+\}.$$

A sequence $(T_n)_n$ of operators is called hypercyclic if there is some x whose orbit under $(T_n)_n$ is dense in X. In such a case, x is called a hypercyclic or universal vector for $(T_n)_n$. A sequence $(T_n)_n$ of operators is called topologically transitive if for every nonempty open subsets U and V of X there is some $n \geq 0$ such that

$$T_n(U) \cap V \neq \emptyset.$$

Let $T_{1,n}, T_{2,n}$ be continuous linear sequences of operators acting on an infinite dimensional separable Banach space X. For $x \in X$, the orbit of x under the pair $\mathcal{T}_n = \{(T_{1,n}, T_{2,n})\}_{n\in\mathbb{Z}_+}$ is the set

$$Orb(\mathcal{T}_n, x) = \{T_{1,n}T_{2,n}x : n \in \mathbb{Z}_+\}.$$

[1]Corresponding author.

Definition 1.1. Let $\mathcal{T}_n = \{(T_{1,n}, T_{2,n})\}_{n\in\mathbb{Z}_+}$ be a pair sequences of operators acting on an infinite dimensional Banach space X. A vector x is called a hypercyclic vector for $\mathcal{T}_n$ if $Orb(\mathcal{T}_n, x)$ is dense in X and in this case the pair $\{(T_{1,n}, T_{2,n})\}_{n\in\mathbb{Z}_+}$ is called hypercyclic.

Definition 1.2. We say that the pair of sequence of operators $\mathcal{T}_n$=$\{(T_{1,n}$,$T_{2,n})\}_{n\in\mathbb{Z}_+}$ is topologically transitive if for every nonempty open subsets U and V of X there exists $n \in \mathbb{Z}_+$ such that $T_{1,n}T_{2,n}(U) \cap V \neq \emptyset$.

Definition 1.3. We say that the pair of sequence of operators $\{(T_{1,n}, T_{2,n})\}_{n\in\mathbb{Z}_+}$ is topologically mixing if for every nonempty open subsets U and V of X there exists $n_0 \in \mathbb{Z}_+$ such that $T_{1,n}T_{2,n}(U) \cap V \neq \emptyset$, $\forall n \geq n_0$.

Definition 1.4. Let $\{n_j\}$ be a strictly increasing sequence of positive integers. We say that $\{(T_{1,n}, T_{2,n})\}_{n\in\mathbb{Z}_+}$ the pair of continuous linear sequence of operators on X satisfy the Hypercyclicity Criterion with respect to $\{n_j\}$ provided there exist two dense subsets X_0 and Y_0 in X, and mappings $S_j : Y_0 \longrightarrow X$ satisfying

1. For each $x \in X_0$, $T_{1,n_j}T_{2,n_j}x \to 0$ as $j \to \infty$.
2. For each $y \in Y_0$, $S_j y \to 0$ as $j \to \infty$.
3. For each $y \in Y_0$, $T_{1,n_j}T_{2,n_j}S_j y \to y$ as $j \to \infty$.

For some sources on these topics, see [3], [4], [14]-[18], [22].

The notion of disjoint hypercyclicity, a strengthening of hypercyclicity, concerning a tuple of linear operators, was introduced independently by Bernal [5] and by Bès and Peris [10] in 2007.

For any integer $N \geq 2$, the tuple $(T_1, T_2, ..., T_N)$ of operators, acting on the same topological vector space X is said disjoint hypercyclic, or $d-$hypercyclic for short, provided there exists $(z, ..., z)$ in X^N such that

$$\{(T_1^n z, T_2^n z, ..., T_N^n z) : n \in \mathbb{Z}_+\}$$

is dense in X^N. Such a vector z is called a $d-$hypercyclic vector for the tuple $(T_1, T_2, ..., T_N)$.

Recall that S. Shkarin in [21] gave a short proof of existence of disjoint hypercyclic tuples of operators of any given length on any separable infinite dimensional Fréchet space. Similar argument provides disjoint dual hypercyclic tuples of operators of any length on any infinite dimensional Banach space with separable dual.

We say that the operators $T_1, T_2, ..., T_N$ in $\mathcal{B}(X)$ with $N \geq 2$ are d-topologically transitive if for any non-empty open subsets $V_0, V_1, ..., V_N$ in $X,$ there exists a positive integer n so that

$$\emptyset \neq V_0 \cap T_1^{-n}(V_1) \cap T_2^{-n}(V_2) \cap ... \cap T_N^{-n}(V_N)$$

For recent results on disjoint hypercyclicity, see [6]-[9], [19]-[21].

Definition 1.5. Let $T : X \to X$ be a bounded linear operator on a Banach space X. For every $x \in X$, the sets

$J(x) = \{y \in X$: there exist a strictly increasing sequence of positive integers k_n and a sequence $x_n \subset X$ such that $x_n \to x$ and $T^{k_n} x_n \to y\}$

$J^{mix}(x) = \{y \in X$: there exist a sequence $x_n \subset X$ such that $x_n \to x$ and $T^n x_n \to y\}$

will be called the extended limit set of x under T and the extended mixing limit set of x under T respectively.

The notions of the limit and extended limit sets are well known in the theory of topological dynamics, see [11].

For $x \in X$, the orbit of x under T is the set $Orb(T, x) = \{T^n x : n \in \mathbb{N}\}$. A vector x is called a hypercyclic vector for T if $Orb(T, x)$ is dense in X and the operator T is said to be hypercyclic if there is some vector $x \in X$ such that $Orb(T, x)$ is dense in X. We say $T \in \mathcal{B}(X)$ is topologically transitive if for every open sets U, V there exists a non negative integer n such that $T^{-n}(U) \cap V \neq \emptyset$. It is well known that $T \in \mathcal{B}(X)$ is hypercyclic if and only if T is topologically transitive. For more information on the topics of the hypercyclicity, see [1], [3], [15]-[17]. It is not difficult to show that T is topologically transitive if and only if $J(x) = X$ for every $x \in X$ and that T is topologically mixing if and only if $J^{mix}(x) = X$ for every $x \in X$ see [13]. For more information on the J-class set see [2], [12], [13].

In this paper, we introduced the notion of disjoint extended limit set for a tuple of bounded linear operators on a separable Banach space X, and we extend some results known for a single operator to a tuple of sequences of operators.

2. Main results

Definition 2.6. Let $T_1, T_2, ..., T_N$ in $\mathcal{B}(X)$ with $N \geq 2$. For every $x_0 \in X$, the sets

d-$J_{(T_1,T_2,...,T_N)}(x_0)$=$\{(x_1, ..., x_N) \in X^N$: for every neighborhoods $V_0, V_1, ..., V_N$ of $x_0, x_1, ..., x_N$ respectively, there exists a positive integer n, so that $\emptyset \neq V_0 \cap T_1^{-n}(V_1) \cap T_2^{-n}(V_2) \cap ... \cap T_N^{-n}(V_N)\}$

d-$J^{mix}_{(T_1,T_2,...,T_N)}(x_0)$=$\{(x_1, ..., x_N) \in X^N$: for every neighborhoods $V_0, V_1, ..., V_N$ of $x_0, x_1, ..., x_N$ respectively, there exists a positive integer m, so that $\emptyset \neq V_0 \cap T_1^{-n}(V_1) \cap T_2^{-n}(V_2) \cap ... \cap T_N^{-n}(V_N)$ for every $n \geq m\}$

will be called the extended limit set of x_0 under $T_1, T_2, ..., T_N$ and the extended mixing limit set of x_0 under $T_1, T_2, ..., T_N$ respectively.

Proposition 2.1. *An equivalent definition for the sets d-$J_{(T_1,T_2,...,T_N)}(x_0)$ and d-$J^{mix}_{(T_1,T_2,...,T_N)}(x_0)$ is the following* *d-*$J_{(T_1,T_2,...,T_N)}(x_0) = \{(x_1,...,x_N) \in X^N$*: there exist a strictly increasing sequence of positive integers* k_n *and a sequence* $(x_n) \subset X$ *such that* $x_n \to x_0$ *and* $T_i^{k_n} x_n \to x_i$ *for all* $1 \le i \le N\}$*, and*

d-$J^{mix}_{(T_1,T_2,...,T_N)}(x_0) = \{(x_1,...,x_N) \in X^N$*: there exists a sequence* $(x_n) \subset X$ *such that* $x_n \to x_0$ *and* $T_i^n x_n \to x_i$ *for all* $1 \le i \le N\}$.

Proof. We give the proof for the d-$J^{mix}_{(T_1,T_2,...,T_N)}(x_0)$, because the proof for the d-$J_{(T_1,T_2,...,T_N)}(x_0)$ is similar. Let us prove that

d-$J^{mix}_{(T_1,T_2,...,T_N)}(x_0) \subset \{(x_1,...,x_N) \in X^N$: there exists a sequence $(x_n) \subset X$ such that $x_n \to x_0$ and $T_i^n x_n \to x_i$ for all $1 \le i \le N\}$.

Let $(x_1,...,x_N) \in$ d-$J^{mix}_{(T_1,T_2,...,T_N)}(x_0)$ and consider the open balls

$V_{0,n} = B\left(x_0, \frac{1}{n}\right), V_{i,n} = B\left(x_i, \frac{1}{n}\right)$ centered at $x_0, x_i \in X$ and radius $1/n$ for $n = 1,2,...,$ and $1 \le i \le N$.

Then there exists a positive integer m so that $\emptyset \ne V_{0,n} \cap T_1^{-n}(V_{1,n}) \cap T_2^{-n}(V_{2,n}) \cap ... \cap T_N^{-n}(V_{N,n})$ for every $n \ge m$. Hence there exists $x_n \in V_{0,n} = B(x_0, \frac{1}{n})$ such that $x_n \in \bigcap_{i=1}^{N} T_i^{-n}(V_{i,n})$, this implies that $T_i^n(x_n) \in V_{i,n}$ for all $i = 1,...,N$. Therefore, there exists a sequence $(x_n) \subset X$ such that $x_n \to x_0$ and $T_i^n x_n \to x_i$ for all $1 \le i \le N$. The converse is obvious. ∎

Theorem 2.1. *Let* $T_1, T_2, ..., T_N$ *in* $\mathcal{B}(X)$ *with* $N \ge 2$*. Then the following conditions are equivalent:*

(1) $T_1, T_2, ..., T_N$ *is d-transitive.*

(2) *For every* $x_0 \in X$*, d-*$J_{(T_1,T_2,...,T_N)}(x_0) = X^N$.

Proof. We first prove that (1) implies (2). Let $x_i \in V_i$ $(0 \le i \le N)$ and $V_0, V_1, ..., V_N$ be relatively open subsets of X. There exists $n \in \mathbb{N}$ such that $\emptyset \ne V_0 \cap \bigcap_{i=1}^{N} T_i^{-n}(V_i)$. Thus $(x_1,...,x_N) \in$ d-$J_{(T_1,T_2,...,T_N)}(x_0)$, and consequently d-$J_{(T_1,T_2,...,T_N)}(x_0) = X^N$.

We will show that (2) implies (1). Let $V_0, V_1, ..., V_N$ the nonempty open. Consider $x_i \in V_i$ $(0 \le i \le N)$. Since d-$J_{(T_1,T_2,...,T_N)}(x_0) = X^N$. There exists $n \in \mathbb{N}$ such that $\emptyset \ne V_0 \cap \bigcap_{i=1}^{N} T_i^{-n}(V_i)$. By definition, $T_1, T_2, ..., T_N$ is d-transitive. ∎

Proposition 2.2. *Let* $T_1, T_2, ..., T_N$ *in* $\mathcal{B}(X)$ *with* $N \ge 2$*. Then*

$$d - J_{(\lambda_1 T_1, \lambda_2 T_2, ..., \lambda_N T_N)}(0) = d - J_{(T_1,T_2,...,T_N)}(0)$$

for every $|\lambda_i| = 1 \quad (1 \le i \le N)$.

Proof. Let $(x_1, ..., x_N) \in$ d-$J_{(\lambda_1 T_1, \lambda_2 T_2, ..., \lambda_N T_N)}(0)$. Then there exist a strictly increasing sequence of positive integers k_n and a sequence $(x_n) \subset X$ such that $x_n \to 0$ and $\lambda_i^{k_n} T_i^{k_n} x_n \to x_i$ for all $1 \leq i \leq N$. Since $|\lambda_i| = 1$ then $\lambda_i^{k_n} x_n \to 0$ and since $T_i^{k_n}(\lambda_i^{k_n} x_n) \to x_i$ it follows that $(x_1, ..., x_N) \in$ d-$J_{(T_1, T_2, ..., T_N)}(0)$.

Let $(x_1, ..., x_N) \in$ d-$J_{(T_1, T_2, ..., T_N)}(0)$. Then there exist a strictly increasing sequence of positive integers k_n and a sequence $(x_n) \subset X$ such that $x_n \to 0$ and $T_i^{k_n} x_n \to x_i$ for all $1 \leq i \leq N$. Since $|\lambda_i| = 1 \quad (1 \leq i \leq N)$, without loss of generality we can assume that $\lambda_i^{k_n} \to \mu_i$ for some $|\mu_i| = 1$. Hence $\lambda_i^{k_n} T_i^{k_n}(\frac{x_n}{\mu_i}) \to x_i$ for all $1 \leq i \leq N$ and since $\frac{x_n}{\mu_i} \to 0$ then $(x_1, ..., x_N) \in$ d-$J_{(\lambda_1 T_1, \lambda_2 T_2, ..., \lambda_N T_N)}(0)$. ∎

Proposition 2.3. *Let $T_1, T_2, ..., T_N$ in $\mathcal{B}(X)$ with $N \geq 2$ and $x_{0,n}, x_{1,n}, ..., x_{N,n}$ be the sequences in X such that $x_{0,n} \to x_0$ and $x_{i,n} \to x_i$ for some $x_0, x_i \in X$ $(1 \leq i \leq N)$.*

(1) *If $(x_{1,n}, ..., x_{N,n}) \in$ d-$J_{(T_1, T_2, ..., T_N)}(x_{0,n})$ for every $n = 1, 2, ...,$ then $(x_1, ..., x_N) \in$ d-$J_{(T_1, T_2, ..., T_N)}(x_0)$.*

(2) *If $(x_{1,n}, ..., x_{N,n}) \in$ d-$J^{mix}_{(T_1, T_2, ..., T_N)}(x_{0,n})$ for every $n = 1, 2, ...,$ then $(x_1, ..., x_N) \in$ d-$J^{mix}_{(T_1, T_2, ..., T_N)}(x_0)$.*

Proof. (1) For $n = 1$ there exists a positive integer k_1 such that

$$||x_{0,k_1} - x_0|| < \tfrac{1}{2} \quad \text{and} \quad ||x_{i,k_1} - x_i|| < \tfrac{1}{2} \ (1 \leq i \leq N).$$

Since $(x_{1,k_1}, ..., x_{N,k_1}) \in$ d-$J_{(T_1, T_2, ..., T_N)}(x_{0,k_1})$ we may find a positive integer l_1 and $z_1 \in X$ such that

$$||z_1 - x_{0,k_1}|| < \tfrac{1}{2} \quad \text{and} \quad ||T_i^{l_1} z_1 - x_{i,k_1}|| < \tfrac{1}{2} \ (1 \leq i \leq N).$$

Therefore,

$$||z_1 - x_0|| < 1 \quad \text{and} \quad ||T_i^{l_1} z_1 - x_i|| < 1 \ (1 \leq i \leq N).$$

Proceeding inductively we find a strictly increasing sequence of positive integers l_n and a sequence z_n in X such that

$$||z_n - x_0|| < \tfrac{1}{n} \quad \text{and} \quad ||T_i^{l_n} z_n - x_i|| < \tfrac{1}{n} \ (1 \leq i \leq N).$$

This completes the proof of assertion (1).

(2) For $n = 1$ there exists a positive integer k_1 such that

$$||x_{0,k_1} - x_0|| < \tfrac{1}{2} \quad \text{and} \quad ||x_{i,k_1} - x_i|| < \tfrac{1}{2} \ (1 \leq i \leq N).$$

Since $(x_{1,k_1}, ..., x_{N,k_1}) \in$ d-$J^{mix}_{(T_1, T_2, ..., T_N)}(x_{0,k_1})$ we may find a positive integer l_1 and $z_n \in X$ such that

$$||z_n - x_{0,k_1}|| < \tfrac{1}{2} \quad \text{and} \quad ||T_i^{n} z_n - x_{i,k_1}|| < \tfrac{1}{2} \ (1 \leq i \leq N).$$

For every $n \geq l_1$. Therefore,

$$||z_n - x_0|| < 1 \quad \text{and} \quad ||T_i^n z_n - x_i|| < 1 \ (1 \le i \le N).$$

For every $n \geq l_1$. Proceeding inductively we find a strictly increasing sequence of positive integers $l_2 > l_1$ and a sequence $w_n \subset X$ such that

$$||w_n - x_0|| < \tfrac{1}{2} \quad \text{and} \quad ||T_i^n w_n - x_i|| < \tfrac{1}{2} \ (1 \le i \le N).$$

For every $n \geq l_2$. Set $v_n = z_n$ for every $l_1 \leq n \leq l_2$, hence

$$||v_n - x_0|| < 1 \quad \text{and} \quad ||T_i^n v_n - x_i|| < 1 \ (1 \le i \le N).$$

Proceeding inductively we find a strictly increasing sequence of positive integers n_k and a sequence $v_n \subset X$ such that if $n \geq k$ then

$$||v_n - x_0|| < \tfrac{1}{k} \quad \text{and} \quad ||T_i^n v_n - x_i|| < \tfrac{1}{k} \ (1 \le i \le N).$$

Take any $\varepsilon > 0$. There exists a positive integer k_0 such that $\frac{1}{k_0} < \varepsilon$. Hence for every $n \geq n_{k_0}$ we get

$$||v_n - x_0|| < \tfrac{1}{k_0} < \varepsilon \quad \text{and} \quad ||T_i^n v_n - x_i|| < \tfrac{1}{k_0} < \varepsilon \ (1 \le i \le N).$$

This completes the proof of assertion (2). ∎

If $T_1, T_2, ..., T_N$ in $\mathcal{B}(X)$ with $N \geq 2$, we denote

d-$L(x_0) := \{(x_1, ..., x_N) \in X^N$: there exists a strictly increasing sequence of positive integers k_n such that $T_i^{k_n} x_0 \to x_i$ for all $1 \le i \le N\}$.

Proposition 2.4. *Let $T_1, T_2, ..., T_N$ in $\mathcal{B}(X)$ with $N \geq 2$. If T_i is power bounded for all $1 \leq i \leq N$ then d-$J_{(T_1,T_2,...,T_N)}(x_0) =$ d-$L(x_0)$ for every $x_0 \in X$.*

Proof. Since T_i is power bounded for all $1 \leq i \leq N$ there exists a positive number M such that $||T_i^n|| \leq M$ for every positive integer n and $1 \leq i \leq N$. Let $x_0 \in X$. If d-$J_{(T_1,T_2,...,T_N)}(x_0) = \emptyset$ there is nothing to prove. Therefore assume that d-$J_{(T_1,T_2,...,T_N)}(x_0) \neq \emptyset$. Since the inclusion d-$L(x_0) \subset$ d-$J_{(T_1,T_2,...,T_N)}(x_0)$ is always true, it suffices to show that d-$J_{(T_1,T_2,...,T_N)}(x_0) \subset$ d-$L(x_0)$. Take $(x_1, ..., x_N) \in$ d-$J_{(T_1,T_2,...,T_N)}(x_0)$. There exist a strictly increasing sequence of positive integers k_n and a sequence $(x_n) \subset X$ such that $x_n \to x_0$ and $T_i^{k_n} x_n \to x_i$ for all $1 \leq i \leq N$. Then we have

$$\begin{aligned} ||T_i^{k_n} x_0 - x_i|| &\leq ||T_i^{k_n} x_0 - T_i^{k_n} x_n|| + ||T_i^{k_n} x_n - x_i|| \\ &\leq M||x_0 - x_n|| + ||T_i^{k_n} x_n - x_i|| \end{aligned}$$

and, letting n goes to infinity to the above inequality, we get that $(x_1, ..., x_N) \in$ d-$L(x_0)$. ∎

From now on, let $T_{1,n}, T_{2,n}, ..., T_{2,N}$, with $N \geq 2$, be continuous linear sequences of operators acting on an infinite dimensional separable Banach space X. We extend some results from a single operator to a tuple of sequences of operators. For simplicity we state and prove our results for a pair that is a tuple with $N = 2$, and the general case follows by a similar method. Let $x \in X$, the orbit of x under the pair $\mathcal{T}_n = \{(T_{1,n}, T_{2,n})\}_{n\in\mathbb{Z}_+}$ is the set

$$Orb(\mathcal{T}_n, x) = \{T_{1,n}T_{2,n}x : n \in \mathbb{Z}_+\}.$$

In the proof of the following lemma, we use a method of the proof of [15, Theorem 1.2] to extend results for tuples. We will use $HC(\mathcal{T}_n)$ for the collection of hypercyclic vectors for the pair of sequence $\mathcal{T}_n$

Lemma 2.1. *Let X be a separable infinite dimensional Banach space and $\{(T_{1,n}, T_{2,n})\}_{n\in\mathbb{Z}_+}$ be the pair of the sequence operators $T_{1,n}$ and $T_{2,n}$. Then, $\{(T_{1,n}, T_{2,n})\}_{n\in\mathbb{Z}_+}$ is topologically transitive if and only if $HC(\mathcal{T}_n)$ is dense in X.*

Proof. Fix an enumeration $\{B_n, n \in \mathbb{Z}_+\}$ of the open balls in X with rational radii, and centers in a countable dense subset of X. By the continuity of the sequences $T_{1,n}$ and $T_{2,n}$ the sets

$$G_k = \cup\{T_{1,n}^{-1}T_{2,n}^{-1}(B_k) : n \in \mathbb{Z}_+\}$$

are open. Clearly, $HC(\mathcal{T}_n)$ is equal to

$$\cap\{G_k : k \in \mathbb{Z}_+\}.$$

Now, let $\mathcal{T}_n$ be topologically transitive and let U by an arbitrary nonempty open set in X. Then for all $k \in \mathbb{Z}_+$, there exist $n(k)$ in $\mathbb{Z}_+$ such that

$$T_{1,n(k)}T_{2,n(k)}(U) \cap B_k \neq \emptyset$$

which implies that $U \cap G_k \neq \emptyset$ for all k. Thus each G_k is dense in X and so by the Bair Category Theorem $HC(\mathcal{T}_n)$ is also dense in X.

Conversely, if $HC(\mathcal{T}_n)$ is dense in X, then each set G_k. This implies that $\mathcal{T}_n$ is topologically transitive. ∎

Theorem 2.2. (The Hypercyclicity Criterion for a Tuple of sequence) *Suppose that X is a separable infinite dimensional Banach space and $\{(T_{1,n}, T_{2,n})\}_{n\in\mathbb{Z}_+}$ is a pair of continuous linear sequence of operators on X. If there exist two dense subsets X_0 and Y_0 in X, and a strictly increasing sequence of positive integers $\{n_j\}$ and mappings $S_j : Y_0 \longrightarrow X$ such that*

(1) *For each $x \in X_0$, $T_{1,n_j}T_{2,n_j}x \to 0$ as $j \to \infty$.*

(2) *For each $y \in Y_0$, $S_j y \to 0$ as $j \to \infty$.*

(3) *For each $y \in Y_0$, $T_{1,n_j}T_{2,n_j}S_j y \to y$ as $j \to \infty$.*

Then, $\mathcal{T}_n = \{(T_{1,n}, T_{2,n})\}_{n\in\mathbb{Z}_+}$ is a hypercyclic tuple.

Proof. Let U and V are two nonempty open sets in X, then chose $x \in X_0 \cap U$ and $y \in V \cap Y_0$ and let $z_j = x + S_j y$. Then, as $j \to \infty$, $z_j \to x$ and $T_{1,n_j}T_{2,n_j}z_j = T_{1,n_j}T_{2,n_j}x + T_{1,n_j}T_{2,n_j}S_j y \to y$. Thus, for large j, we have $z_j \in U$ and $T_{1,n_j}T_{2,n_j}z_j \in V$. By Lemma 2.1, $HC(\mathcal{T}_n)$ is dense in X and this implies, clearly, that the pair $\{(T_{1,n}, T_{2,n})\}_{n\in\mathbb{Z}_+}$ is a hypercyclic pair. ∎

Proposition 2.5. *Let $\mathcal{T}_n = \{(T_{1,n}, T_{2,n})\}_{n\in\mathbb{Z}_+}$ satisfy the Hypercyclicity Criterion with respect to a sequence $\{n_j\}$. Then the pair $(T_{1,n_j}, T_{2,n_j})_{n\in\mathbb{Z}_+}$ are topologically mixing. In particular $\{(T_{1,n}, T_{2,n})\}_{n\in\mathbb{Z}_+}$ are hypercyclic.*

Proof. We show that $\{(T_{1,n_j}, T_{2,n_j})\}_{n\in\mathbb{Z}_+}$ are topologically mixing. Let X_0 and Y_0 be dense sets in X, that are given in the hypercyclicity criterion of Definition 1.4. Let U and V are two nonempty open sets in X, then choose $x \in X_0 \cap U$ and $y \in V \cap Y_0$ and $\varepsilon > 0$ such that $B(x,\varepsilon) \subset U$ and $B(y,\varepsilon) \subset V$. By Definition 1.4, there exist $j_0 \in \mathbb{N}$ so that, for all $j \geq j_0$, $||T_{1,n_j}T_{2,n_j}x|| \leq \varepsilon$, $||S_j(y)|| \leq \varepsilon$, and $||T_{1,n_j}T_{2,n_j}S_j(y) - y|| \leq \varepsilon$. Then, for each $j \geq j_0$, we have $z_j = x + S_j y \in B(x,\varepsilon) \subset U$ and $T_{1,n_j}T_{2,n_j}z_j \in B(y,\varepsilon) \subset V$. That is, $T_{1,n_j}T_{2,n_j}(U) \cap V \neq \emptyset$, $\forall j \geq j_0$. Hence, $\{(T_{1,n_j}, T_{2,n_j})\}_{n\in\mathbb{Z}_+}$ is topologically mixing. ∎

If $\{(T_{1,n}, T_{2,n})\}_{n\in\mathbb{Z}_+}$ is a sequence of $2-$tuples of continuous self-maps on a X, we denote

$$\mathbb{J} := \{(x,y) \in X \times X;\ \exists (u_n)_{n\in\mathbb{N}} \subset X : u_n \to x \text{ and } T_{1,n}T_{2,n}u_n \to y\}.$$

Proposition 2.6. *Let $\mathcal{T}_n = \{(T_{1,n}, T_{2,n})\}_{n\in\mathbb{Z}_+}$ is a sequence of $2-$tuples of continuous self-maps on a X, such that $\mathbb{J}$ is dense in $X \times X$. Then $\{(T_{1,n}, T_{2,n})\}_{n\in\mathbb{Z}_+}$ are topologically mixing.*

Proof. Let U and V are two nonempty open sets in X. Since $\mathbb{J}$ is dense in $X \times X$, we can find $x \in U$ and $y \in V$ such that $(x,y) \in \mathbb{J}$. By definition of $\mathbb{J}$, there is a sequence $(u_n)_{n\in\mathbb{N}} \subset X$ such that $u_n \to x$ and $T_{1,n}T_{2,n}u_n \to y$. Then, there exists $k_0 \in \mathbb{N}$ such that $u_n \in U$ and $T_{1,n}T_{2,n}u_n \in V$, $\forall k \geq k_0$. Hence $T_{1,n}T_{2,n}(U) \cap V \neq \emptyset$, $\forall k \geq k_0$. That is, $\mathcal{T}_n = \{(T_{1,n}, T_{2,n})\}_{n\in\mathbb{Z}_+}$ are topologically mixing. ∎

We finish this paper by the following questions:

Question 1. Let $T_1, T_2, ..., T_N$ $(n \geq 2)$ be d-hypercyclic and invertible. Must for every $x \in X$, d-$J_{(T_1,T_2,...,T_N)}(x) = X^N$?

Question 2. For a infinite-dimensional separable complex Banach space X, let $T_1, T_2, ..., T_N$ in $\mathcal{B}(X)$ with $N \geq 2$. Suppose there exists a vector $x \in X$ such that d-$J_{(T_1,T_2,...,T_N)}(x)^\circ \neq \emptyset$. It is true that d-$J_{(T_1,T_2,...,T_N)}(x) = X^N$?

Question 3. For a infinite-dimensional separable complex Banach space X, let $T_1, T_2, ..., T_N$ in $\mathcal{B}(X)$ with $N \geq 2$. Suppose that d-$J_{(T_1,T_2,...,T_N)}(x)^\circ \neq \emptyset$ for every $x \in X$. Does it follow that $T_1, T_2, ..., T_N$ is d-topologically transitive?

Question 4. Is there a relation between d-hypercyclicity see [10, Definition 1.1] and d-$J_{(T_1,T_2,...,T_N)}$-sets?

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References

[1] Ansari, S.I., *Existance of hypercyclic operatos on topological vector space,* J.F. Anal., 148 (1997), 384-390.

[2] Azimi, M.R., Muller, V., *A note on J-sets of linear operators,* Rev. R. Acad. Cienc. Exactas Fs. Nat. Ser. A Math. RACSAM, 105 (2) (2011), 449-453.

[3] Bayart, F., Matheron, E., *Dynamics of linear operators,* Cambridge University Press, 2009.

[4] Bernal-Gonzalez, L., Grosse-Erdmann, K.-G., *The hypercyclicity criterion for sequences of operators,* Studia Math., 157 (1) (2003), 17-32.

[5] Bernal-Gonzlez, L., *Disjoint hypercyclic operators,* Studia Math., 182 (2) (2007), 113-130.

[6] Bès, J., Martin, Ö., *Compositional disjoint hypercyclicity equals disjoint supercyclicity,* Houston J. Math., 38 (4) (2012), 1149-1163.

[7] Bès, J., Martin, Ö., Peris, A., *Disjoint hypercyclic linear fractional composition operators,* J. Math. Anal. Appl., 381 (2011), 843-856.

[8] Bès, J., Martin, Ö., Peris, A., Shkarin, S., *Disjoint mixing operators,* J. Funct. Anal., 263 (2012), 1283-1322.

[9] Bès, J., Martin, Ö., Sanders, R., *Weighted shifts and disjoint hypercyclicity,* J. Operator Theory, 72 (1) (2014), 15-40.

[10] Bès, J., Peris, A., *Disjointness in hypercyclicity,* J. Math. Anal. Appl., 336 (2007), 297-315.

[11] Bhatia, N.P., Szegö, G.P., *Stability theory of dynamical systems,* Die Grundlehren der Mathematischen Wissenschaften, Band 161 Springer-Verlag, New York, Berlin, 1970.

[12] Costakis, G., Manoussos, A., *J-class weighted shifts on the space of bounded sequences of complex numbers,* Integral Equations Operator Theory, 62 (2) (2008), 149-158.

[13] Costakis, G., Manoussos, A., *J-class operators and hypercyclicity,* J. Operator Theory, 67 (1) (2012), 101-119.

[14] Feldman, N.S., *Hypercyclic tuples of operators and somewhere dense orbits,* J. Math. Anal. Appl., 346 (2008), 82-98.

[15] Godefroy, G., Shapiro, J.H., *Operators with dense, invariant cyclic vector manifolds,* Journal of Functional Analysis, 98 (1991), 229-269.

[16] Goswin, K., Erdmann, G., *Universal families and hypercyclic operators,* Bulletin of the American Mathematical Society, 36 (1999), 345-381.

[17] Grosse-Erdmann, K.-G., Peris Manguillot, A., *Linear Chaos,* Springer-Verlag, London, 2011.

[18] Leon-Saavedra, F., Muller, V., *Hypercyclic sequences of operators,* Studia Math., 175 (1) (2006), 1-18.

[19] Salas, H.N., *Dual disjoint hypercyclic operators,* J. Math. Anal. Appl., 374 (1) (2011), 106-117.

[20] Sanders, R., Shkarin, S., *Existence of disjoint weakly mixing operators that fail to satisfy the Disjoint Hypercyclicity Criterion,* J. Math. Anal. Appl., 417 (2) (2014), 834-855.

[21] Shkarin,S., *A short proof of existence of disjoint hypercyclic operators,* J. Math. Anal. Appl., 367 (2) (2010), 713-715.

[22] Yousefi, B., Moghimi, G.R., *Tuple of operators and hypercyclicity,* International Journal of Pure and Applied Mathematics, 76 (2) (2012), 201-205.

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EQ-ALGEBRAS WITH PSEUDO PRE-VALUATIONS

Yongwei Yang[1]

School of Mathematics and Statistics
Anyang Normal University
Anyang 455000
China

Xiaolong Xin

School of Mathematics
Northwest University
Xi'an, 710127
China

Abstract. The concepts of (positive implicative, implicative) pseudo pre-valuations and strong pseudo pre-valuations are introduced and some related characterizations are studied. The relationships among positive implicative pseudo pre-valuations, implicative pseudo pre-valuations and pseudo pre-valuations are investigated, and conditions for a real-valued function to be a pseudo pre-valuation are also discussed. By using a congruence relation induced via a pseudo valuation, we construct a quotient structure and prove certain isomorphism theorems.

Keywords: EQ-algebra, pseudo pre-valuation, pseudo-metric space, (positive) implicative pseudo pre-valuation.

1. Introduction

Non-classical logic systems which lay logical foundation for dealing with uncertain information processing and fuzzy information, are uniquely determined by the algebraic properties of the structure of their truth values. Residuated lattices have been proposed as the most generalis algebraic counterparts of residuated systems with t-norm based semantics, where the conjunction connective is interpreted by a t-norm and the implication operator by its residuum. Some other logical algebras such as BL-algebras [10], MTL-algebras [6], lattice implication algebras [18], R_0-algebras [17] and MV-algebras [4] are considered as particular classes of residuated lattices. In a residuated lattice, the basic typical operations multiplication and implication which are closely tied by adjointness property, and the fuzzy equality (i.e., biresiduum) is derived from the operations meet and implication. Due to the algebra of truth values is no longer a residuated lattice, a new algebra for

[1]Corresponding author. E-mail: yangyw2010@126.com

the fuzzy theory which called an EQ-algebra [16] was proposed by Novák and De Baets.

EQ-algebras provide a possibility to develop fuzzy logics with the basic connective being a fuzzy equality instead of an implication. In EQ-algebras, there are three primitive operations—meet, multiplication and a fuzzy equality, and residuum and multiplication are no more closely tied by the adjunction. The implication $\to$ in EQ-algebras is defined directly from fuzzy equality $\sim$ by the formula $x \to y = (x \wedge y) \sim x$. Since the equation holds also for the biresiduum, thus each residuated lattice can be seen as an EQ-algebra but not vice versa. From these points of view, it is known that the notion of EQ-algebras generalizes that of commutative residuated lattices, so it is interesting to investigate the properties of EQ-algebras. EI-Zekey [7] introduced the notion of prelinear ordered EQ-algebras which were proved to be lattice EQ-algebras, and he also characterized the representable good EQ-algebras. As a continuation of good EQ-algebras, [8] studied the prefilters and filters of separated EQ-algebras. Ma and Hu [15] investigated the compatibility of multiplication w.r.t. the fuzzy equality in an arbitrary EQ-algebra and characterized compatible EQ-algebras by using special GLE-algebras. Since EQ-algebras are generalize form of residuated lattices, [14] extended some notions of residuated lattices to EQ-algebras, then the notions of implicative and positive implicative prefilters of EQ-algebras were proposed. Moreover, based on the fuzzy set theory it is meaningful to study the related fuzzy structures of prefilters in EQ-algebras [19, 1, 11].

Recently, Buşneag [2] defined pseudo-valuation on a Hilbert algebra, and proved that every pseudo-valuation induces a pseudo metric on a Hilbert algebra. Later he gave the notions of pseudo-valuations (valuations) on residuated lattices, and proved some theorems of extension for pseudo-valuations [3]. By using a pseudo-metric induced by a pseudo valuation on BCI-algebra [5], Ghorbani [9] introduced a congruence relation and defined the quotient algebra. Following the research of Jun et al. [12, 13], [20] investigated related characterizations of (implicative) pseudo-valuations on R_0-algebras, and showed that a pseudo-valuation on R_0-algebras is Boolean if and only if it is implicative.

In the paper, we discuss a theoretical approach of the algebraic system in EQ-algebras by using the Buşneag's model. We introduce notions of (positive implicative, implicative) pseudo pre-valuations and strong pseudo pre-valuations, then investigate some characterizations of them. A congruence relation on an EQ-algebra is constructed by using a pseudo-metric induced via a pseudo valuation. Furthermore, we construct a quotient structure related to the congruence relation and prove certain isomorphism theorems.

2. Preliminaries

In this section, we give the basic definitions and results of EQ-algebras that are useful for subsequent discussions.

An algebra $(L, \wedge, \otimes, \sim, 1)$ of type $(2, 2, 2, 0)$ is called an EQ-algebra if it satisfies the following axioms: for all $x, y, s, t \in L$, (E1) $(L, \wedge, 1)$ is a $\wedge$-semilattice with top element 1,

(E2) $(L, \otimes, 1)$ is a commutative monoid and $\otimes$ is isotone with respect to $\leq$, where $x \leq y$ if and only if $x \wedge y = x$,

(E3) $x \sim x = 1$,

(E4) $((x \wedge y) \sim s) \otimes (t \sim x) \leq s \sim (t \wedge y)$,

(E5) $(x \sim y) \otimes (s \sim t) \leq (x \sim s) \otimes (y \sim t)$,

(E6) $(x \wedge y \wedge s) \sim x \leq (x \wedge y) \sim x$,

(E7) $x \otimes y \leq x \sim y$.

In what follows, L is an EQ-algebra unless otherwise specified. For any $x, y \in L$, we put $x \to y = (x \wedge y) \sim x$ and $\tilde{x} = x \sim 1$.

Definition 2.1 [16] An EQ-algebra L is called

(1) a good EQ-algebra if $\tilde{x} = x$ for any $x \in L$;

(2) a separated EQ-algebra if $x \sim y = 1$ implies $x = y$ for any $x, y \in L$;

(3) a residuated EQ-algebra if $x \otimes y \leq z$ if and only if $x \leq y \to z$ for any $x, y, z \in L$;

(4) an involutive EQ-algebra, if L contains a bottom element 0, and $\neg\neg x = x$ holds for any $x \in L$, where $\neg x = x \sim 0 = x \to 0$.

Remark 2.2 Let $(L, \wedge, \vee, \otimes, \Rightarrow, 0, 1)$ be a residuated lattice. For any $x, y \in L$, we define $x \sim y = (x \Rightarrow y) \wedge (y \Rightarrow x)$, then $(L, \wedge, \otimes, \sim, 1)$ is a residuated EQ-algebra ([16]). In general, a residuated EQ-algebra may not be a residuated lattice ([8]), however residuated lattices are proper classes of EQ-algebras ([14]).

Proposition 2.3 [16],[8] *Let $(L, \wedge, \otimes, \sim, 1)$ be an EQ-algebra. Then the following assertions are valid: for any $x, y, z \in L$,*

(1) $x \sim y = y \sim x$, $x \sim y \leq x \to y$, $x \leq y \to x$, $x \otimes y \leq x \wedge y$;

(2) $(x \sim y) \otimes (y \sim z) \leq x \sim z$, $(x \to y) \otimes (y \to z) \leq x \to z$;

(3) $x \sim y \leq (x \sim z) \sim (y \sim z)$, $x \sim y \leq (x \wedge z) \sim (y \wedge z)$;

(4) $x \to y \leq (y \to z) \to (x \to z)$, $x \to y \leq (z \to x) \to (z \to y)$;

(5) $x \to y = x \to (x \wedge y)$, $x \to y \leq (x \wedge z) \to (y \wedge z)$;

(6) $x \leq y$ *implies* $x \to y = 1$, $x \sim y = y \to x$, $y \to z \leq x \to z$ *and* $z \to x \leq z \to y$;

(7) $(x \to y) \otimes (y \to x) \leq x \sim y$.

Lemma 2.4 [16],[14] *Let L be a good EQ-algebra. Then we have, for any $x, y, z \in L$,*

(1) $x \leq (x \to y) \to y$, $x \leq (x \sim y) \sim y$,

(2) $x \to (y \to z) = y \to (x \to z) \leq (x \otimes y) \to z$,

(3) $x \otimes (x \to y) \leq x \wedge y$.

Lemma 2.5 [16], [8], [1] *Let L be an EQ-algebra. Then the following conditions are equivalent: $x, y, z \in L$,*

(1) *L is residuated;*

(2) *L is good and $x \to y \leq (x \otimes z) \to (y \otimes z)$;*

(3) *L is separated and $(x \otimes y) \to z = x \to (y \to z)$.*

In what follows, we recall some types of prefilters in EQ-algebras.

Definition 2.6 [8, 14, 19] *Let F be a nonempty subset of L. Then F is called:*

(1) *a prefilter of L if it satisfies for any $x, y \in L$,* (F1) $1 \in F$; (F2) $x \in F$, $x \to y \in F$ *imply* $y \in F$; *if a prefilter F satisfies* (F3) $x \to y \in F$ *implies* $(x \otimes z) \to (y \otimes z) \in F$ *for any $x, y, z \in L$, then F is called a filter of L.*

(2) *a positive implicative prefilter of L if F is a prefilter of L and it satisfies* (F4) $x \to (y \to z) \in F$ *and* $x \to y \in F$ *imply* $x \to z \in F$ *for any $x, y, z \in L$.*

(3) *an implicative prefilter of L if it satisfies* (F1) *and* (F5) $z \to ((x \to y) \to x) \in F$ *and* $z \in F$ *imply* $x \in F$ *for any $x, y, z \in L$.*

Definition 2.7 [8] Let L, L' be EQ-algebras. A function $f : L \to L'$ satisfying $f(1) = 1'$ (where 1 and $1'$ are the top elements of L and L', respectively) is called a homomorphism if $f(a \boxdot b) = f(a) \boxdot' f(b)$, where $\boxdot \in \{\wedge, \otimes, \sim\}$ in L and $\boxdot' \in \{\wedge', \otimes', \sim'\}$ in L'. The order is clearly stable under homomorphism because it is defined by using meet.

3. Pseudo pre-valuations on EQ-algebras

In the section, we introduce the notion of pseudo pre-valuations, and give some characterizations of a pseudo pre-valuation on EQ-algebras. By discussing the concept of pseudo-metrics induced by pseudo pre-valuations, we obtain that the binary operations on EQ-algebras are uniformly continuous.

Definition 3.1 Let $\varphi : L \to R$ be a real-valued function, where R is the set of all real numbers. Then φ is called a pseudo pre-valuation on L if it satisfies the following conditions: for any $x, y \in L$,

(1) $\varphi(1) = 0$,

(2) $\varphi(y) \leq \varphi(x) + \varphi(x \to y)$.

If a real-valued function $\varphi : L \to R$ satisfies conditions (1) and (3) $\varphi(y) + \varphi(y \to x) = \varphi(x) + \varphi(x \to y)$, then φ is said to be a strong pseudo pre-valuation on L.

Let φ be a pseudo pre-valuation on L. φ is said to be a pseudo valuation on L if $\varphi((x \otimes z) \to (y \otimes z)) \leq \varphi(x \to y)$ for any $x, y, z \in L$. A strong pseudo pre-valuation φ is called a strong pseudo valuation if φ is a pseudo valuation. A pseudo pre-valuation φ is called a pre-valuation if $\varphi(x) = 0$ implies $x = 1$.

From the definitions of pseudo pre-valuations and strong pseudo pre-valuations, it is easy to see that a strong pseudo pre-valuation is pseudo pre-valuation, however the converse is not true in general.

Example 3.2 Let $L = \{0, a, b, 1\}$ be a chain with Cayley tables as follows:

$\otimes$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	0	b
1	0	a	b	1

$\sim$	0	a	b	1
0	1	a	0	0
a	a	1	a	a
b	0	a	1	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

One can check that $(L, \wedge, \otimes, \sim, 1)$ is an EQ-algebra. Define $\varphi : L \to R$ by $\varphi(0)=9$, $\varphi(a) = 5$, $\varphi(b) = 2$, $\varphi(1) = 0$. Routine calculation shows that φ is a pseudo pre-valuation but φ is not a strong pseudo pre-valuation since $\varphi(a) + \varphi(a \to b) = 5 \neq 7 = \varphi(b) + \varphi(b \to a)$.

The following example shows that strong pseudo pre-valuations on EQ-algebras exist.

Example 3.3 Let $L = \{0, a, b, c, d, 1\}$ be a chain with Cayley tables as follows:

$\otimes$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	0	0	a
b	0	0	a	a	a	b
c	0	0	a	0	a	c
d	0	0	a	a	a	d
1	0	a	b	c	d	1

$\sim$	0	a	b	c	d	1
0	1	0	0	0	0	0
a	0	1	d	d	d	d
b	0	d	1	d	d	d
c	0	d	d	1	d	d
d	0	d	d	d	1	1
1	0	d	d	d	1	1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	0	1	1	1	1	1
b	0	d	1	1	1	1
c	0	d	d	1	1	1
d	0	d	d	d	1	1
1	0	d	d	d	1	1

Then $(L, \wedge, \otimes, \sim, 1)$ is an EQ-algebra. Define a function $\varphi : L \to R$ as follows: $\varphi(0) = 3$, $\varphi(a) = \varphi(b) = \varphi(c) = \varphi(d) = \varphi(1) = 0$. One can check that φ is a strong pseudo pre-valuation on L.

Proposition 3.4 *Let φ be a real-valued function on L. If φ is a pseudo pre-valuation on L, then the following properties hold: for any $x, y \in L$,*

(1) $x \leq y$ *implies that* $\varphi(y) \leq \varphi(x)$;

(2) $0 \leq \varphi(x)$;

(3) $\varphi(x \to y) \leq \varphi(y)$, $\varphi(x) = \varphi(1 \to x)$.

Proof. (1) Let $x, y \in L$ such that $x \leq y$, then $x \to y = 1$, we obtain that

$$\varphi(y) \leq \varphi(x) + \varphi(x \to y) = \varphi(x) + \varphi(1) = \varphi(x) + 0 = \varphi(x).$$

(2) For any $x \in L$, we have

$$0 = \varphi(1) \leq \varphi(x \to 1) + \varphi(x) = \varphi(x).$$

(3) Since $y \leq x \to y$, thus $\varphi(x \to y) \leq \varphi(y)$. Obviously, $\varphi(1 \to x) \leq \varphi(x)$. Notice φ is a pseudo pre-valuation on L, we have $\varphi(x) \leq \varphi(1) + \varphi(1 \to x) = \varphi(1 \to x)$, thus $\varphi(1 \to x) = \varphi(x)$ ∎

Proposition 3.5 *Let L be an EQ-algebra with a bottom element 0 and φ a real-valued function on L. If φ is a strong pseudo pre-valuation on L, then we have: for any $x, y \in L$,*

(1) $x \leq y$ *implies that* $\varphi(x \sim y) = \varphi(y \to x) = \varphi(x) - \varphi(y)$,

(2) $\varphi(x) + \varphi(\neg x) = \varphi(0)$,

(3) $\varphi(\neg\neg x) = \varphi(x) = \varphi(\tilde{x})$.

Proof. (1) Assume that $x \leq y$, then $x \to y = 1$, and so $\varphi(x \to y) = 0$, therefore $\varphi(y \to x) = \varphi(x) - \varphi(y) + \varphi(x \to y) = \varphi(x) - \varphi(y)$. Notice that $y \to x = x \sim y$, thus (1) holds.

(2) Since $\neg x = x \sim 0$, then $\varphi(\neg x) = \varphi(0) - \varphi(x)$, and thus $\varphi(x) + \varphi(\neg x) = \varphi(0)$.

(3) $\varphi(\neg\neg x) = \varphi(x)$ is immediately from (2). Consider that $\tilde{x} = x \sim 1$, we get that $\varphi(\tilde{x}) = \varphi(x) - \varphi(1) = \varphi(x)$, thus (3) is valid. ∎

For any real-valued function φ on L, we consider the following set

$$\varphi_* := \{x \in L | \varphi(x) = 0\}.$$

Theorem 3.6 *Let φ be a pseudo pre-valuation on L. Then the set φ_* is a prefilter of L which is called the prefilter induced by the pseudo pre-valuation φ. If L is a residuated EQ-algebra, then φ_* is a filter.*

Proof. From $\varphi(1) = 0$, it follows that $1 \in \varphi_*$. Suppose that $x, y \in L$ such that $x, x \to y \in \varphi_*$, then we get $\varphi(x) = 0$ and $\varphi(x \to y) = 0$. Since $\varphi(y) \leq \varphi(x) + \varphi(x \to y) = 0$, therefore $\varphi(y) = 0$, and so $y \in \varphi_*$. Thus φ_* is a prefilter of L. Now let L be a residuated EQ-algebra and $x \to y \in \varphi_*$, then we get that $x \to y \leq (x \otimes z) \to (y \otimes z)$ for any $x, y, z \in L$ by Lemma 2.5, and so $\varphi((x \otimes z) \to (y \otimes z)) = 0$. Consequently, $(x \otimes z) \to (y \otimes z) \in \varphi_*$, and therefore φ_* is a filter. ∎

The following example shows that the converse of Theorem 3.6 may not be true, that is, there exist a EQ-algebra L and a real-valued function $\varphi : L \to R$ such that φ_* is a prefilter of L but φ is not a pseudo pre-valuation on L.

Example 3.7 Let $L = \{0, a, b, 1\}$ be a chain with Cayley tables as follows:

$\otimes$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
1	0	a	b	1

$\sim$	0	a	b	1
0	1	0	0	0
a	0	1	a	a
b	0	a	1	1
1	0	a	1	1

$\to$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	a	1	1
1	0	a	1	1

It is easy to see that $(L, \wedge, \otimes, \sim, 1)$ is an EQ-algebra. Define a real-valued function $\varphi : L \to R$ by $\varphi(0) = 2$, $\varphi(a) = -1$, $\varphi(b) = \varphi(1) = 0$. Then $\varphi_* = \{b, 1\}$ is a prefilter of L, but φ is not a pseudo pre-valuation since

$$\varphi(b) = 0 \not\leq \varphi(a) + \varphi(a \to b) = -1.$$

Proposition 3.8 *Let F be a prefilter of L and t a positive element of R. Define*

$$\varphi^F(x) = \begin{cases} 0, & x \in F, \\ t, & x \notin F, \end{cases}$$

then φ^F is a pseudo pre-valuation on L which is called the pseudo pre-valuation induced by the prefilter F. Moreover, $(\varphi^F)_* = F$.

Proof. It is obvious that φ^F is a pseudo pre-valuation on L.

$$(\varphi^F)_* = \{x \in L | \varphi^F(x) = 0\} = \{x \in L | x \in F\} = F.$$ ∎

In the following, we provide some conditions under which a real-valued function on L becomes to a pseudo pre-valuation.

Theorem 3.9 *Let φ be a real-valued function on L with $\varphi(1) = 0$. Then φ is a pseudo pre-valuation if and only if $x \leq y \to z$ implies $\varphi(z) \leq \varphi(x) + \varphi(y)$ for any $x, y, z \in L$.*

Proof. Assume that φ is a pseudo pre-valuation and $x \leq y \to z$. It follows that $\varphi(y \to z) \leq \varphi(x)$, and so $\varphi(z) \leq \varphi(y) + \varphi(y \to z) \leq \varphi(x) + \varphi(y)$.

Conversely, since $x \to y \leq x \to y$, thus $\varphi(y) \leq \varphi(x \to y) + \varphi(x)$, and so φ is a pseudo pre-valuation. ■

An interesting application of Theorem 3.9 is to some proofs of the following important results.

Proposition 3.10 *If φ is a pseudo pre-valuation on L, then for any $x, y, z, s, t \in L$,*

(1) $\varphi(x \wedge y) \leq \varphi(x) + \varphi(y)$;

(2) $\varphi(x \to z) \leq \varphi(x \to y) + \varphi(y \to z)$;

(3) $\varphi(x \sim z) \leq \varphi(x \sim y) + \varphi(y \sim z)$;

(4) $\varphi((x \wedge s) \sim (y \wedge t)) \leq \varphi(x \wedge y) + \varphi(s \sim t)$;

(5) $\varphi((x \to s) \sim (y \to t)) \leq \varphi(x \sim y) + \varphi(s \sim t)$,

Proof. (1) For any $x, y \in L$, we have $x \leq y \to x = y \to (x \wedge y)$ by Proposition 2.3. Using Theorem 3.9, we can get $\varphi(x \wedge y) \leq \varphi(x) + \varphi(y)$.

(2) Notice that $x \to y \leq (y \to z) \to (x \to z)$, we have $\varphi(x \to z) \leq \varphi((y \to z) \to (x \to z)) + \varphi(y \to z) \leq \varphi(x \to y) + \varphi(y \to z)$.

(3) According to Proposition 2.3, we obtain that $x \sim y \leq (y \sim z) \sim (x \sim z) \leq (y \sim z) \to (x \sim z)$, therefore $\varphi(x \sim z) \leq \varphi(x \sim y) + \varphi(y \sim z)$ by Theorem 3.9.

(4) For any $x, y, s, t \in L$, we have $x \sim y \leq (x \wedge s) \sim (y \wedge s)$ and $s \sim t \leq (y \wedge s) \sim (y \wedge t)$ by Proposition 2.3. From (3) and Proposition 3.4, it follows that $\varphi(x \wedge y) + \varphi(s \sim t) \geq \varphi((x \wedge s) \sim (y \wedge s)) + \varphi((y \wedge s) \sim (y \wedge t)) \geq \varphi((x \wedge s) \sim (y \wedge t))$.

(5) It follows immediately by the definition of $\to$ and items (3) and (4). ■

Let L be a residuated EQ-algebra. Since $x \otimes y \leq z$ if and only if $x \leq y \to z$ for any $x, y, z \in L$, then the following proposition is a consequence of Theorem 3.9.

Proposition 3.11 *Let L be a residuated EQ-algebra and φ a real-valued function on L with $\varphi(1) = 0$. Then φ is a pseudo pre-valuation if and only if $x \otimes y \leq z$ implies $\varphi(z) \leq \varphi(x) + \varphi(y)$ for any $x, y, z \in L$.*

Proposition 3.12 *Let L be a residuated EQ-algebra and φ a pseudo pre-valuation. Then for any $x, y \in L$, we have:*

(1) $\varphi(x \otimes y) \leq \varphi(x) + \varphi(y)$;

(2) $\varphi(x \wedge y) \leq \varphi(x) + \varphi(y)$;

(3) $\varphi((x \to y) \to z) \leq \varphi(x \to (y \to z))$.

Proof. (1) Since $x \otimes y \leq x \otimes y$, then $\varphi(x \otimes y) \leq \varphi(x) + \varphi(y)$ by Proposition 3.11.

(2) Notice that L is a residuated EQ-algebra, we get $x \otimes (x \to y) \leq x \wedge y$ by Lemma 2.4 and Lemma 2.5. From Proposition 3.4 and Proposition 3.11, it follows that $\varphi(x \wedge y) \leq \varphi(x) + \varphi(x \to y) \leq \varphi(x) + \varphi(y)$.

(3) Since $x \otimes y \leq x \sim y \leq x \to y$, then $(x \to y) \to z \leq (x \otimes y) \to z = x \to (y \to z)$ by Lemma 2.5. Thus $\varphi(x \to (y \to z)) \leq \varphi((x \to y) \to z)$. ∎

Proposition 3.13 *Let L be a good EQ-algebra and φ a real-valued function on L with $\varphi(1) = 0$. If φ satisfies the condition: $\varphi(x \to (x \to z)) \leq \varphi(x \to y) + \varphi(x \to (y \to z))$ for any $x, y, z \in L$, then φ is a pseudo pre-valuation.*

Proof. Notice that L is a good EQ-algebra, we obtain that $1 \to x = 1 \sim x = x \sim 1 = x$ for any $x \in L$. For any $x, y \in L$, we get $\varphi(x) + \varphi(x \to y) = \varphi(1 \to x) + \varphi(1 \to (x \to y)) \geq \varphi(1 \to (1 \to y)) = \varphi(y)$. Hence φ is a pseudo pre-valuation. ∎

Proposition 3.14 *Let φ be a real-valued function on L. Then the following statements are equivalent: for any $x, y \in L$,*

(1) *φ is a strong pseudo pre-valuation,*

(2) *$x \leq y$ implies that $\varphi(y \to x) = \varphi(x) - \varphi(y)$,*

(3) *$\varphi(y \to x) = \varphi(x \wedge y) - \varphi(y)$.*

Proof. (1) ⇒ (2) The proof follows immediately from Proposition 3.5.

(2) ⇒ (3) Observe that $x \wedge y \leq y$, we get that $\varphi(y \to x) = \varphi(y \to (x \wedge y)) = \varphi(x \wedge y) - \varphi(y)$.

(3) ⇒ (1) Suppose that (3) holds, then $\varphi(1) = \varphi(x \to x) = \varphi(x \wedge x) - \varphi(x) = 0$. For any $x, y \in L$, we have $\varphi(x) + \varphi(x \to y) = \varphi(x) + \varphi(x \wedge y) - \varphi(x) = \varphi(x \wedge y) = \varphi(y) + \varphi(x \wedge y) - \varphi(y) = \varphi(y) + \varphi(y \to x)$, thus φ is a strong pseudo pre-valuation. ∎

Let (M, d) be an ordered pair, where M is a nonempty set and $d : M \times M \to R$ is a positive function. If d satisfies the following conditions: for any $x, y, z \in M$,

(1) $d(x, x) = 0$,

(2) $d(x, y) = d(y, x)$,

(3) $d(x, z) \leq d(x, y) + d(y, z)$,

then (M, d) is called a pseudo-metric space. Moreover, if $d(x, y) = 0$ implies $x = y$, then (M, d) is called a metric space.

Theorem 3.15 *Let φ be a pseudo pre-valuation on L. Define a real-valued function on $d_\varphi : L \times L \to R$ by $d_\varphi(x, y) = \varphi(x \to y) + \varphi(y \to x)$ for any $x, y \in L$, then (M, d_φ) is a pseudo-metric space, where d_φ is called the pseudo-metric induced by the pseudo pre-valuation φ.*

Proof. It is obvious that $d_\varphi(x,y) \geq 0$, $d_\varphi(x,x) = 0$ and $d_\varphi(x,y) = d_\varphi(y,x)$ for any $x, y \in L$. According to Proposition 3.10, we have $d_\varphi(x,y) + d_\varphi(y,z) = (\varphi(x \to y) + \varphi(y \to x)) + (\varphi(y \to z) + \varphi(z \to y)) = (\varphi(x \to y) + \varphi(y \to z)) + (\varphi(z \to y) + \varphi(y \to x)) \geq \varphi(x \to z) + \varphi(z \to x) = d_\varphi(x,z)$. Hence (M, d_φ) is a pseudo-metric space. ∎

Proposition 3.16 *Let φ be a pseudo pre-valuation on L and d_φ the pseudo-metric induced by φ. Then the following inequalities hold: for any $x, y, z, \omega, \nu \in L$,*

(1) $\max\{d_\varphi(x \to z, y \to z), d_\varphi(z \to x, z \to y)\} \leq d_\varphi(x,y)$,

(2) $d_\varphi(x \to y, \omega \to \nu) \leq d_\varphi(x \to y, \omega \to y) + d_\varphi(\omega \to y, \omega \to \nu)$,

(3) $d_\varphi(x \wedge z, y \wedge z) \leq d_\varphi(x,y)$,

(4) *if φ is a pseudo valuation, then* $d_\varphi(x \otimes z, y \otimes z) \leq d_\varphi(x,y)$,

(5) $d_\varphi(x \sim z, y \sim z) \leq d_\varphi(x,y)$.

Proof. (1) For any $x, y, z \in L$, we have $y \to x \leq (x \to z) \to (y \to z)$ and $x \to y \leq (y \to z) \to (x \to z)$, thus $\varphi(y \to x) \geq \varphi((x \to z) \to (y \to z))$ and $\varphi(x \to y) \geq \varphi((y \to z) \to (x \to z))$. And so $d_\varphi(x,y) = \varphi(y \to x) + \varphi(x \to y) \geq \varphi((x \to z) \to (y \to z)) + \varphi((y \to z) \to (x \to z)) = d_\varphi(x \to z, y \to z)$. Analogously, $d_\varphi(x,y) \geq d_\varphi(z \to x, z \to y)$. Hence $\max\{d_\varphi(x \to z, y \to z), d_\varphi(z \to x,\, z \to y)\} \leq d_\varphi(x,y)$.

(2) It is trivial since d_φ is the pseudo-metric induced by φ.

(3) For any $x, y, z \in L$, we obtain $d_\varphi(x \wedge z, y \wedge z) = \varphi((x \wedge z) \to (y \wedge z)) + \varphi((y \wedge z) \to (x \wedge z))$. Notice that $x \to y \leq (x \wedge z) \to (y \wedge z)$ and $y \to x \leq (y \wedge z) \to (x \wedge z)$, we get $\varphi(x \to y) \geq \varphi((x \wedge z) \to (y \wedge z))$ and $\varphi(y \to x) \geq \varphi((y \wedge z) \to (x \wedge z))$, and so $d_\varphi(x,y) = \varphi(x \to y) + \varphi(y \to x) \geq \varphi((x \wedge z) \to (y \wedge z)) + \varphi((y \wedge z) \to (x \wedge z)) = d_\varphi(x \wedge z, y \wedge z)$.

(4) Since φ is a pseudo valuation, then we have $\varphi((x \otimes z) \to (y \otimes z)) \leq \varphi(x \to y)$ and $\varphi((y \otimes z) \to (x \otimes z)) \leq \varphi(y \to x)$ for any $x, y, z \in L$. And thus $d_\varphi(x,y) = \varphi(x \to y) + \varphi(y \to x) \geq \varphi((x \otimes z) \to (y \otimes z)) + \varphi((y \otimes z) \to (x \otimes z)) = d_\varphi(x \otimes z, y \otimes z)$.

(5) is similar to the proof of (3). ∎

Proposition 3.17 *Let d_φ be the pseudo-metric induced by a pseudo pre-valuation φ. Then $(L \times L, d^*_\varphi)$ is a pseudo-metric space, where*

$$d^*_\varphi((x,y),(\omega,\nu)) = \max\{d_\varphi(x,\omega), d_\varphi(y,\nu)\},$$

for any $(x,y), (\omega,\nu) \in L \times L$.

Proof. Consider that d_φ is a pseudo-metric on L, we have $d^*_\varphi((x,y),(x,y)) = \max\{d_\varphi(x,x), d_\varphi(y,y)\} = 0$ and $d^*_\varphi((x,y),(\omega,\nu)) = \max\{d_\varphi(x,\omega), d_\varphi(y,\nu)\} = \max\{d_\varphi(\omega,x), d_\varphi(\nu,y)\} = d^*_\varphi((\omega,\nu),(x,y))$. Now, let $(x,y), (a,b), (\omega,\nu) \in L \times L$, we get

$$
\begin{aligned}
&d^*_\varphi((x,y),(a,b)) + d^*_\varphi((a,b),(\omega,\nu)) \\
&\qquad = \max\{d_\varphi(x,a), d_\varphi(y,b)\} + \max\{d_\varphi(a,\omega), d_\varphi(b,\nu)\} \\
&\qquad \geq \max\{d_\varphi(x,a) + d_\varphi(a,\omega), d_\varphi(y,b) + d_\varphi(b,\nu)\} \\
&\qquad \geq \max\{d_\varphi(x,\omega), d_\varphi(y,\nu)\} \\
&\qquad = d^*_\varphi((x,y),(\omega,\nu)).
\end{aligned}
$$

Consequently, $(L \times L, d^*_\varphi)$ is a pseudo-metric space. ∎

Theorem 3.18 *Let φ be a pre-valuation on L and d_φ the pseudo-metric induced by φ. Then the operations $\wedge, \otimes, \sim, \rightarrow: L \times L \rightarrow L$ are uniformly continuous.*

Proof. Here we only prove $\wedge : L\times L\rightarrow L$ is uniformly continuous, other cases can be proved in a similar way. For any $x, y, \omega, \nu \in L$ and $\varepsilon > 0$, if $d^*_\varphi((x,y),(\omega,\nu)) < \frac{\varepsilon}{2}$, then $d_\varphi(x,\omega) < \frac{\varepsilon}{2}$ and $d_\varphi(y,\nu) < \frac{\varepsilon}{2}$. According to Proposition 3.16, we get $d_\varphi(x\wedge y, \omega\wedge\nu) \leq d_\varphi(x\wedge y, \omega\wedge y) + d_\varphi(\omega\wedge y, \omega\wedge\nu) \leq d_\varphi(x,\omega) + d_\varphi(y,\nu) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Hence the operation $\wedge : L \times L \rightarrow L$ is uniformly continuous. ∎

4. Quotient structures induced by pseudo valuations

In the section, we will construct a quotient EQ-algebra related to a pseudo valuation, and investigate some isomorphism theorems.

Lemma 4.1 *Let φ be a pseudo valuation on L. A relation $\approx_\varphi$ on L is defined as follows: for any $x, y \in L$,*

$$x \approx_\varphi y \text{ if and only if } \varphi(x \rightarrow y) = \varphi(y \rightarrow x) = 0,$$

Then $\approx_\varphi$ is a congruence relation on L. We denote the quotient set $L/\varphi = \{[x]_\varphi \mid x \in L\}$ induced via φ, where $[x]_\varphi$ is the equivalence class of x with respect to $\approx_\varphi$.

Proof. The proof follows immediately from Proposition 3.16. ∎

Let φ be a pre-valuation on L. It is easy to verify that $(L/\varphi, \wedge, \otimes, \sim, [1]_\varphi)$ is an EQ-algebra which is called a quotient EQ-algebra induced by the pseudo valuation φ, where the operation $\wedge$ on L/φ is defined by $[x]_\varphi \wedge [y]_\varphi = [x \wedge y]_\varphi$, and similarly for other operations. The partial order on L/F is $[x]_\varphi \leq [y]_\varphi$ if and only if $[x]_\varphi = [x]_\varphi \wedge [y]_\varphi$.

Lemma 4.2 *Let φ be a pre-valuation on L. Then $[x]_\varphi \leq [y]_\varphi$ if and only if $\varphi(x \rightarrow y) = 0$.*

Proof. Using $x \rightarrow y = x \rightarrow (x \wedge y)$, it follows that $[x]_\varphi \leq [y]_\varphi$ if and only if $[x]_\varphi = [x]_\varphi \wedge [y]_\varphi$ if and only if $\varphi(x \rightarrow y) = 0$ by Lemma 4.1. ∎

Proposition 4.3 *Let L be a residuated EQ-algebra with a bottom element* 0 *and φ a strong pseudo valuation on L. Then L/φ is an involutive EQ-algebra.*

Proof. Obviously, L/φ is an EQ-algebra, moreover L/φ is a separated EQ-algebra. Indeed, let $[x]_\varphi \sim [y]_\varphi$=$[1]_\varphi$, then $[x \sim y]_\varphi$=$[1]_\varphi$, and so $\max\{\varphi(x \to y), \varphi(x \to y)\} \leq \varphi(x \sim y) = \varphi(1 \to (x \sim y)) = 0$. Thus $\varphi(x \to y) = \varphi(x \to y) = 0$, and so $[x]_\varphi = [y]_\varphi$, consequently L/φ is a separated EQ-algebra. Next, we will show that $\neg\neg[x]_\varphi = [x]_\varphi$ for any $x \in L$.

Observe that $x \leq \neg\neg x$, we get $\varphi(\neg\neg x \to x) = \varphi(x) - \varphi(\neg\neg x)$. Due to the fact that $\varphi(\neg\neg x) + \varphi(\neg\neg x \to x) = \varphi(x) + \varphi(x \to \neg\neg x)$ and $\varphi(x) = \varphi(\neg\neg x)$, it follows that $\varphi(\neg\neg x \to x) = \varphi(x \to \neg\neg x) = 0$. Thus $\neg\neg[x]_\varphi = [\neg\neg x]_\varphi = [x]_\varphi$, and so L/φ is an involutive EQ-algebra. ∎

As an immediate consequence of the above proposition together with Lemma 4.2, we record here the following result.

Proposition 4.4 *Let L be a residuated EQ-algebra with a bottom element* 0, *and φ a strong pseudo valuation on L. Then in the involutive EQ-algebra L/φ, we have:*

(1) $[x]_\varphi \leq [y]_\varphi$ *if and only if* $\varphi(x \to y) = 0$ *if and only if* $\varphi(x) = \varphi(x \wedge y)$;

(2) $[x]_\varphi = [y]_\varphi$ *if and only if* $\varphi(x \to y) = \varphi(y \to x) = 0$ *if and only if* $\varphi(x) = \varphi(y) = \varphi(x \wedge y)$.

Moreover, the mapping $\hat{\varphi} : L/\varphi \to R$ defined by $\hat{\varphi}([x]_\varphi) = \varphi(x)$ is a strong pseudo valuation on L/φ.

Proposition 4.5 *Let φ be a pseudo valuation on L and J be a prefilter of L such that $\varphi_* \subseteq J$. Denote $J/\varphi = \{[x]_\varphi | x \in J\}$, then we have:*

(1) $x \in J$ *if and only if* $[x]_\varphi \in J/\varphi$ *for any* $x \in L$;

(2) J/φ *is a prefiter of* L/φ.

Proof. (1) Suppose that $[x]_\varphi \in J/\varphi$, then there exists $y \in J$ such that $[x]_\varphi = [y]_\varphi$. Using Lemma 4.2, we get $\varphi(y \to x) = 0$, and so $y \to x \in \varphi_* \subseteq J$. Since J is a prefilter of L, then $x \in J$. The converse is obviously.

(2) Obviously, $[1]_\varphi \subseteq J/\varphi$. Suppose that $[x]_\varphi, [x]_\varphi \to [y]_\varphi \in J/\varphi$, then we get $x, x \to y \in J$. Hence $y \in J$, and so $[y]_\varphi \in J/\varphi$. Consequently, J/φ is a prefiter of L/φ. ∎

Lemma 4.6 *Let L, L' be EQ-algebras, φ be a pseudo pre-valuation on L and $f : L \to L'$ be an epimorphism. Then $f(\varphi) : L \to R$ is a pseudo pre-valuation on L', where $f(\varphi)$ is defined by $f(\varphi)(y') = \inf\{\varphi(y) | f(y) = y', y \in L\}$ for any $y' \in L'$. Moreover, $f(\varphi_*) \subseteq (f(\varphi))_*$.*

Proof. Due to the fact that φ is a pseudo pre-valuation on L and f is an epimorphism from L to L', it follows that $f(\varphi)(1') = \inf\{\varphi(x)|f(x) = 1', x \in L\} = \varphi(1) = 0$. For any $x', y' \in L'$, $f(\varphi)(x') + f(\varphi)(x' \to y') = \inf\{\varphi(x)|f(x) = x', x \in L\} + \inf\{\varphi(z)|f(z) = x' \to y', z \in L\} \geq \inf\{\varphi(x) + \varphi(x \to y)|f(x) = x', f(y) = y', x, y \in L\} \geq \inf\{\varphi(y)|f(y) = y', y \in L\} = f(\varphi)(y')$, thus $f(\varphi)$ is a pseudo pre-valuation on L'. Now let $x \in \varphi_*$, then we get $\varphi(x) = 0$. Since $f(\varphi)(f(x)) = \inf\{\varphi(t)|f(t) = f(x), t \in L\} \leq \varphi(x) = 0$, therefore $f(\varphi)(f(x)) = 0$, that is $x \in (f(\varphi))_*$, and so $f(\varphi_*) \subseteq (f(\varphi))_*$. ∎

Lemma 4.7 *Let L, L' be EQ-algebras, φ a pseudo valuation on L' and $f : L \to L'$ a homomorphism. Then $f^{-1}(\varphi) : L' \to R$ is a pseudo pre-valuation on L, where $f(\varphi)$ is defined by $f^{-1}(\varphi)(x) = \varphi(f(x))$ for any $x \in L$. Moreover, $f^{-1}(\varphi_*) = (f^{-1}(\varphi))_*$.*

Proof. The proof is straightforward. ∎

Filters are important tools to study logical algebras, and closely related to congruence relations with which we can associate quotient algebras.

Let F be a filter in EQ-algebra L. Define a relation $\equiv$ on L as: $x \equiv y$ if and only if $x \sim y \in F$, then $\equiv$ is a congruence relation on L. Let L/F denote the quotient algebra induced by F, and $[x]^F$ denote the equivalence class of x with respect to $\equiv$, then the quotient algebra L/F is a separated EQ-algebra [16].

Proposition 4.8 *Let φ be a pseudo valuation on L. Then $L/\varphi \simeq L/\varphi_*$.*

Proof. Define a function $\Phi : L/\varphi \to L/\varphi_*$ by $\Phi([x]_\varphi) = [x]^{\varphi_*}$. We only need to prove that Φ is an isomorphism. Suppose that $[x]_\varphi, [y]_\varphi \in L/\varphi$. It is not difficult to prove that $[x]_\varphi = [y]_\varphi$ if and only if $\varphi(x \to y) = \varphi(y \to x) = 0$ if and only $[x]^{\varphi_*} = [y]^{\varphi_*}$, which implies that Φ is an one-to-one function. Obviously, Φ is subjective. Now we prove that Φ is a homomorphism. Indeed, $\Phi([1]_\varphi) = [1]^{\varphi_*}$ and $\Phi([x]_\varphi \wedge [y]_\varphi) = [x]^{\varphi_*} \wedge [y]^{\varphi_*}$; similarly for $\otimes$ and $\sim$. ∎

For the purpose of investigating homomorphism theorems of EQ-algebras based on pseudo valuations, we introduce the following notion.

Definition 4.9 Let L, L' be EQ-algebras, φ a pseudo valuation on L and $f : L \to L'$ an epimorphism. φ is called an invariant pseudo valuation with respect to f if $f(x_1) = f(x_2)$ implies $\varphi(x_1) = \varphi(x_2)$, for any $x_1, x_2 \in L$.

Proposition 4.10 *Let L, L' be EQ-algebras, $f : L \to L'$ an epimorphism and φ an invariant pseudo valuation with respect to f. Then $L/\varphi \simeq L'/f(\varphi)$.*

Proof. It is known that L/φ and $L'/f(\varphi)$ are EQ-algebras. Define a function $\psi : L/\varphi \to L'/f(\varphi)$ by $\psi([x]_\varphi) = [f(x)]_{f(\varphi)}$. (1) Suppose that $[x]_\varphi, [y]_\varphi \in L/\varphi$ such that $[x]_\varphi = [y]_\varphi$, then $\varphi(x \to y) = \varphi(y \to x) = 0$. Since f is an epimorphism, then $f(\varphi)(f(x) \to f(y)) = \inf\{\varphi(z)|f(z) = f(x) \to f(y) = f(x \to y), z \in L\} \leq \varphi(x \to y) = 0$, that is $f(\varphi)(f(x) \to f(y)) = 0$. Similarly, $f(\varphi)(f(y) \to f(x)) = 0$, and so $[x]_{f(\varphi)} = [y]_{f(\varphi)}$. Consequently ψ is well defined. (2) Now we show that ψ is a homomorphism. Due to the fact that f is a homomorphism, it follows that $\psi([1]_\varphi) = [f(1)]_{f(\varphi)} = [1']_{f(\varphi)}$; $\psi([x \boxdot y]_\varphi) = [f(x \boxdot y)]_{f(\varphi)} = [f(x)]_{f(\varphi)} \boxdot' [f(y)]_{f(\varphi)}$, where $\boxdot \in \{\wedge, \otimes, \sim\}$ in L and $\boxdot' \in \{\wedge', \otimes', \sim'\}$ in L'. (3) It is obvious that φ is subjective. (4) Suppose that $\psi([x]_\varphi) = \psi([y]_\varphi)$, that is, $[f(x)]_{f(\varphi)} = [f(y)]_{f(\varphi)}$, then $f(\varphi)(f(x) \to f(y)) = f(\varphi)(f(x \to y)) = f(\varphi)(f(y \to x)) = f(\varphi)(f(y) \to f(x)) = 0$. Moreover, $f(\varphi)(f(x \to y)) = \inf\{\varphi(z) | f(z) = f(x \to y), z \in L\} = \varphi(x \to y) = 0$, similarly for $\varphi(y \to x) = 0$, thus $[x]_\varphi = [y]_\varphi$, and so ψ is an one-to-one function. Therefore $L/\varphi \simeq L'/f(\varphi)$. ∎

Proposition 4.11 *Let L, L' be EQ-algebras, φ a pseudo valuation on L' and $f : L \to L'$ an epimorphism. Then $L/f^{-1}(\varphi) \simeq L'/\varphi$.*

Proof. Let $x_1, x_2 \in L$ such that $f(x_1) = f(x_2)$. From $f^{-1}(\varphi)(x_1) = \varphi(f(x_1)) = \varphi(f(x_2)) = f^{-1}(\varphi)(x_2)$, it follows that $f^{-1}(\varphi)$ is an invariant pseudo valuation with respect to f. Notice that f is a subjective function, it is not difficult to prove that $f(f^{-1}(\varphi)) = \varphi$. According to Proposition 4.10, we have $L/f^{-1}(\varphi) \simeq L'/\varphi$. ∎

5. Some special pseudo pre-valuations on EQ-algebras

In the section, we introduce the concepts of implicative pseudo pre-valuations and positive implicative pseudo pre-valuations, and investigate their relationships.

5.1. Positive implicative pseudo pre-valuations

Definition 5.1 *rm Let $\varphi : L \to R$ be a pseudo valuation. If φ satisfies: $\varphi(x \to z) \leq \varphi(x \to (y \to z)) + \varphi(x \to y)$ for any $x, y, z \in L$, then φ is called a positive implicative pseudo pre-valuation on L.*

The following example shows that positive implicative pseudo pre-valuations exist.

Example 5.2 *rm Let L be the EQ-algebra defined in Example 3.7. Define a real-valued function $\varphi : L \to R$ by $\varphi(0) = 2$, $\varphi(a) = 1$, $\varphi(b) = \varphi(1) = 0$, then φ is a positive implicative pseudo pre-valuation on L.*

It is obvious that every positive implicative pseudo pre-valuation on an EQ-algebra is a pseudo pre-valuation, while the converse is not true in general. In fact, let φ be a pseudo valuation on L in Example 3.2, however φ is not a positive implicative pseudo pre-valuation since $\varphi(a \to 0) = 5 > \varphi(a \to (a \to 0)) + \varphi(a \to a) = 0$.

In the following, we provide some characterizations of positive implicative pseudo pre-valuations.

Theorem 5.3 *Let $\varphi : L \to R$ be a pseudo pre-valuation on L. Then the following conditions are equivalent:*

(1) *φ is a positive implicative pseudo pre-valuation;*

(2) $\varphi((x \wedge (x \to y)) \to y) = 0$ *for any* $x, y \in L$;

(3) $\varphi(x \to y) = \varphi(x \to (x \to y))$ *for any* $x, y \in L$.

Proof. (1) $\Rightarrow$ (2) Observe that $(x\wedge(x\to y))\to x = 1$ and $(x\wedge(x\to y))\to(x\to y) = 1$, we get that $\varphi((x \wedge (x \to y)) \to y) \leq \varphi((x \wedge (x \to y)) \to x) + \varphi((x \wedge (x \to y)) \to (x \to y)) = 0$ by hypothesis, and so $\varphi((x \wedge (x \to y)) \to y) = 0$.

(2) $\Rightarrow$ (3) From $x \to y \leq x \to (x \to y)$, we get that $\varphi(x \to (x \to y)) \leq \varphi(x \to y)$ by Proposition 3.4. For the inverse inequality, consider that $x \to (x \to y) = x \to (x \wedge (x \to y))$ and $\varphi((x \wedge (x \to y)) \to y) = 0$, we get that $\varphi(x \to (x \to y)) = \varphi(x \to (x \wedge (x \to y))) + \varphi((x \wedge (x \to y)) \to y) \geq \varphi(x \to y)$, consequently (3) is valid.

(3) $\Rightarrow$ (1) Suppose that (3) hold, together with $x \to (y \to z) \leq ((y \to z) \to (x\to z))\to(x\to(x\to z))$ and $x\to y \leq (y\to z)\to(x\to z)$, we obtain that $\varphi(x\to(y\to z))+ \varphi(x\to y) \geq \varphi(((y\to z)\to(x\to z))\to(x\to(x\to z))) + \varphi((y\to z)\to(x\to z)) \geq \varphi(x \to (x \to z)) = \varphi(x \to z)$. Hence φ is a positive implicative pseudo pre-valuation. ∎

In the following, we will investigate some characterizations of positive implicative pseudo pre-valuations in some special types of EQ-algebras.

Theorem 5.4 *Let L be a residuated EQ-algebra and $\varphi : L \to R$ a pseudo pre-valuation on L. Then the following conditions are equivalent: for any $x, y, z \in L$,*

(1) *φ is a positive implicative pseudo pre-valuation;*

(2) $\varphi((x \otimes y) \to z) = \varphi((x \wedge y) \to z)$;

(3) $\varphi((x \wedge y) \to (x \otimes y)) = 0$;

(4) $\varphi((x \wedge (x \to y)) \to (x \otimes y)) = 0$;

(5) $\varphi(x \to (x \otimes x)) = 0$.

Proof. (1) $\Rightarrow$ (2) $\varphi((x \wedge y) \to z) \geq \varphi((x \otimes y) \to z)$ follows from $(x \wedge y) \to z \leq (x \otimes y) \to z$. On the other hand, since L is a residuated EQ-algebra, then $(x \otimes y) \to z \leq x \to (y \to z) \leq (x \wedge y) \to (y \wedge (y \to z))$, therefore $\varphi((x \otimes y) \to z) \geq \varphi((x \wedge y) \to (y \wedge (y \to z)))$. According to Theorem 5.3, we get that $\varphi((y \wedge (y \to z)) \to z) = 0$, and thus $\varphi((x \otimes y) \to z) \geq \varphi((x \wedge y) \to (y \wedge (y \to z))) + \varphi((y \wedge (y \to z)) \to z) \geq \varphi((x \wedge y) \to z)$. Consequently $\varphi((x \otimes y) \to z)) = \varphi((x \wedge y) \to z)$.

(2) $\Rightarrow$ (3) For any $x, y \in L$, we obtain that $\varphi((x\wedge y) \to (x\otimes y)) = \varphi((x\otimes y) \to (x \otimes y)) = 0$.

(3) $\Rightarrow$ (4) Since $x \otimes (x\to y) \leq x \wedge y$, then $(x \wedge (x\to y))\to(x \otimes (xvy)) \leq (x \wedge (x \to y)) \to (x \wedge y)$. Combination with the hypothesis, we obtain that $0 = \varphi((x \wedge (x \to y)) \to (x \otimes (x \to y))) \geq \varphi((x \wedge (x \to y)) \to (x \wedge y))$.

Hence $\varphi((x \wedge (x \to y)) \to (x \wedge y)) = 0$. According to Proposition 3.5, we have $\varphi((x \wedge (x \to y)) \to (x \otimes y)) \leq \varphi((x \wedge (x \to y)) \to (x \wedge y)) + \varphi((x \wedge y) \to (x \otimes y)) = 0$. Hence $\varphi((x \wedge (x \to y)) \to (x \otimes y)) = 0$.

(4) $\Rightarrow$ (5) Taking $y = x$, it is easy to obtain (5) from (4).

(5) $\Rightarrow$ (1) It follows immediately from Proposition 3.5 and hypothesis that $\varphi(x \to z) \leq \varphi(x \to (x \otimes x)) + \varphi((x \otimes x) \to z) = \varphi((x \otimes x) \to z) \leq \varphi((x \otimes x) \to (y \otimes (y \to z))) \leq \varphi((x \otimes x) \to (x \otimes y)) + \varphi((x \otimes y) \to (y \otimes (y \to z)))$. Obverse that L is a residuated EQ-algebra, we have $\varphi((x \otimes x) \to (x \otimes y)) \leq \varphi(x \to y)$ and $\varphi((x \otimes y) \to (y \otimes (y \to z))) \leq \varphi(x \to (y \to z))$ by Lemma 2.5. Thus $\varphi(x \to z) \leq \varphi(x \to (y \to z)) + \varphi(x \to y)$, and so φ is a positive implicative pseudo pre-valuation. ∎

Theorem 5.5 *Let L be a good EQ-algebra and φ a pseudo pre-valuation on L. Then φ is a positive implicative pseudo pre-valuation if and only if $\varphi((x \to y) \to (x \to z)) \leq \varphi(x \to (y \to z))$ for any $x, y, z \in L$.*

Proof. Since L is a good EQ-algebra, then $\varphi((x \to y) \to (x \to z)) = \varphi(x \to ((x \to y) \to z)) \leq \varphi((x \to y) \to z)$ by Lemma 2.4 and Proposition 3.4.

Conversely, we have $\varphi(x \to (y \to z)) + \varphi(x \to y) \geq \varphi((x \to y) \to (x \to z)) + \varphi(x \to y) \geq \varphi(x \to z)$ for any $x, y, z \in L$, thus φ is a positive implicative pseudo pre-valuation. ∎

5.2. Implicative pseudo pre-valuations

We now proceed to investigate particular classes of pseudo pre-valuations. For this purpose, we introduce the concept of implicative pseudo pre-valuations as follows.

Definition 5.6 Let $\varphi : L \to R$ be a real-valued function. If φ satisfies the following conditions: for any $x, y, z \in L$,

(1) $\varphi(1) = 0$,

(2) $\varphi(x) \leq \varphi(z \to ((x \to y) \to x)) + \varphi(z)$,

then φ is called an implicative pseudo pre-valuation on L.

Example 5.7 Let $L = \{0, a, b, 1\}$ be a chain with Cayley tables as follows:

$\otimes$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	a	b
1	0	a	b	1

$\sim$	0	a	b	1
0	1	0	0	0
a	0	1	b	a
b	0	b	1	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	b	1	1
1	0	a	b	1

Routine calculation shows that $(L, \wedge, \otimes, \sim, 1)$ is an EQ-algebra. Define a real-valued function $\varphi : L \to R$ by $\varphi(0) = 3$, $\varphi(a) = \varphi(b) = \varphi(1) = 0$. Then φ is an implicative pseudo pre-valuation.

Proposition 5.8 *Every implicative pseudo pre-valuation is a pseudo pre-valuation, moreover every implicative pseudo pre-valuation is a positive implicative pseudo pre-valuation.*

Proof. From $y \leq 1 \to y$, it follows that $x \to y \leq x \to (1 \to y)$. If $x \leq y$, then $1 = x \to (1 \to y) = x \to ((y \to y) \to y)$, and so $\varphi(y) \leq \varphi(x \to ((y \to y) \to y)) + \varphi(x) = \varphi(x)$, that is, $x \leq y$ implies that $\varphi(y) \leq \varphi(x)$. Now let $x, y \in L$, since $\varphi(x) + \varphi(x \to y) \geq \varphi(x) + \varphi(x \to (1 \to y)) = \varphi(x) + \varphi(x \to ((y \to 1) \to y)) \geq \varphi(y)$, thus φ is a pseudo pre-valuation. Since φ is an implicative pseudo pre-valuation on L, then $\varphi(x \to y) \leq \varphi(1 \to (((x \to y) \to y) \to (x \to y))) + \varphi(1) = \varphi(((x \to y) \to y) \to (x \to y))$ for any $x, y \in L$. On one hand, due to the fact that $x \to (x \to y) \leq ((x \to y) \to y) \to (x \to y)$, we have $\varphi(((x \to y) \to y) \to (x \to y)) \leq \varphi(x \to (x \to y))$, and so $\varphi(x \to y) \leq \varphi(x \to (x \to y))$. On the other hand, using Proposition 3.4, we get that $\varphi(x \to (x \to y)) \leq \varphi(x \to y)$, thus $\varphi(x \to (x \to y)) = \varphi(x \to y)$. By Theorem 5.3, φ is a positive implicative pseudo pre-valuation. ∎

A positive implicative pseudo pre-valuation is not an implicative pseudo pre-valuation in general. In fact, Let L be the EQ-algebra defined in Example 3.7 and $\varphi : L \to R$ a positive implicative pseudo pre-valuation defined in Example 5.2, while φ is not an implicative pseudo pre-valuation, since $\varphi(a) = 1 > \varphi(1 \to ((a \to 0) \to a)) + \varphi(1) = 0$.

The above discussing also displays that a pseudo pre-valuation is not an implicative pseudo pre-valuation in general. Nextly, we give some conditions for a pseudo pre-valuation to be an implicative pseudo pre-valuation.

Theorem 5.9 *Let $\varphi : L \to R$ be a real-valued function. If φ is a pseudo pre-valuation on L, then φ is an implicative pseudo pre-valuation if and only if $\varphi((x \to y) \to x) = \varphi(x)$ for any $x, y \in L$.*

Proof. Assume that φ is an implicative pseudo pre-valuation, then we get that $\varphi((x \to y) \to x) = \varphi(1 \to ((x \to y) \to x)) + \varphi(1) \geq \varphi(x)$ for any $x, y \in L$. To obtain the reverse inequality, notice that $x \leq (x \to y) \to x$, we have $\varphi(x) \geq \varphi((x \to y) \to x)$, and thus $\varphi((x \to y) \to x) = \varphi(x)$.

Conversely, suppose that $\varphi((x{\to}y){\to}x) = \varphi(x)$ for any $x, y \in L$. Consider that φ is a pseudo pre-valuation on L, we get that $\varphi(z{\to}((x{\to}y){\to}x)) + \varphi(z) \geq \varphi((x{\to}y){\to}x)$. From $\varphi((x{\to}y){\to}x) = \varphi(x)$, it follows that $\varphi(z{\to}((x{\to}y){\to}x)) + \varphi(z) \geq \varphi(x)$. Consequently, φ is an implicative pseudo pre-valuation. ∎

An interesting application of Theorem 5.9 is to a proof of the following important result.

Theorem 5.10 *Let L be an EQ-algebra with a bottom element* 0 *and φ a pseudo pre-valuation on L, then φ is an implicative pseudo pre-valuation if and only if $\varphi(\neg x \to x) = \varphi(x)$ for any $x \in L$.*

Proof. Suppose that φ is an implicative pseudo pre-valuation, then we have $\varphi(x) = \varphi((x \to 0) \to x) = \varphi(\neg x \to x)$ for any $x \in L$ by Theorem 5.9.

Conversely, due to the fact that $(x{\to}y){\to}x \leq \neg x{\to}x$, we have $\varphi((x{\to}y){\to}x) \geq \varphi(\neg x{\to}x) = \varphi(x)$. As for the reverse inequality, observe that $x \leq (x{\to}y){\to}x$, we get $\varphi(x) \geq \varphi((x{\to}y){\to}x)$, and thus $\varphi(x) = \varphi((x{\to}y){\to}x)$. Therefore φ is an implicative pseudo pre-valuation. ∎

Let φ be a pre-valuation on L. If $\varphi(x \to (y \to z)) = \varphi(y \to (x \to z))$ for any $x, y, z \in L$, then we say that φ has the weak exchange principle.

Theorem 5.11 *Let L be an EQ-algebra with a bottom element* 0*, and φ a pseudo pre-valuation on L with the weak exchange principle. Then the following statements are equivalent:*

(1) *φ is an implicative pseudo pre-valuation;*

(2) *$\varphi(x \to (\neg z \to y)) + \varphi(y \to z) \geq \varphi(x \to z)$ for any $x, y, z \in L$;*

(3) *$\varphi(x \to (\neg z \to z)) = \varphi(x \to z)$ for any $x, z \in L$.*

Proof. (1) $\Rightarrow$ (2) From $y \to z \leq (x \to y) \to (x \to z)$, we get that $\varphi(x \to (\neg z \to y)) + \varphi(y \to z) \geq \varphi(\neg z \to (x \to y)) + \varphi((x \to y) \to (x \to z)) \geq \varphi(\neg z \to (x \to z))$ as φ has the weak exchange principle. Notice that $\neg z \to (x \to z) \leq \neg(x \to z) \to (x \to z)$, together with Theorem 5.10, we obtain that $\varphi(\neg(x \to z) \to (x \to z)) = \varphi(x \to z) \leq \varphi(\neg z \to (x \to z))$. Therefore $\varphi(x \to (\neg z \to y)) + \varphi(y \to z) \geq \varphi(x \to z)$.

(2) $\Rightarrow$ (3) By hypothesis, we have $\varphi(x \to z) \leq \varphi(x \to (\neg z \to z)) + \varphi(z \to z) = \varphi(x \to (\neg z \to z))$ for any $x, z \in L$. To obtain the reverse inequality, obverse that $x \to z \leq x \to (\neg z \to z)$, we get $\varphi(x \to (\neg z \to z)) \leq \varphi(x \to z)$. Thus $\varphi(x \to (\neg z \to z)) = \varphi(x \to z)$.

(3) $\Rightarrow$ (1) Using the hypothesis, together with Proposition 3.4, we have $\varphi(x) = \varphi(1 \to x) = \varphi(1 \to (\neg x \to x)) = \varphi(\neg x \to x)$. By Theorem 5.10, φ is an implicative pseudo pre-valuation. ∎

Next, we further find the conditions under which a positive implicative pseudo pre-valuation is equivalent to an implicative pseudo pre-valuation.

Theorem 5.12 *Let L be an EQ-algebra with a bottom element* 0*, and φ a positive implicative pseudo pre-valuation on L. Then φ is an implicative pseudo pre-valuation if and only if $\varphi(x) \leq \varphi(\neg\neg x)$ for any $x \in L$.*

Proof. Assume that φ is an implicative pseudo pre-valuation, from $\neg\neg x = \neg x \to 0 \leq \neg x \to x$, it follows that $\varphi(\neg x \to x) \leq \varphi(\neg\neg x)$. According to Theorem 5.10, we get $\varphi(x) = \varphi(\neg x \to x) \leq \varphi(\neg\neg x)$.

Conversely, since $\neg x \to x \leq (x \to 0) \to (\neg x \to 0) = \neg x \to (\neg x \to 0)$, then $\varphi(\neg x \to (\neg x \to 0)) \leq \varphi(\neg x \to x)$. It follows immediately from Theorem 5.3 and Proposition 3.4 that $\varphi(\neg\neg x) = \varphi(\neg x \to 0) \leq \varphi(\neg x \to x) \leq \varphi(x)$. Using hypothesis, we obtain that $\varphi(\neg x \to x) = \varphi(x)$. By Theorem 5.10, φ is an implicative pseudo pre-valuation. ∎

As a consequence of Theorem 5.12 together with Lemma 2.4, we have the following result.

Corollary 5.13 *Let L be a good EQ-algebra with a bottom element 0, and φ a positive implicative pseudo pre-valuation on L. Then φ is an implicative pseudo pre-valuation if and only if $\varphi(x) = \varphi(\neg\neg x)$ for any $x \in L$.*

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References

[1] Borzooei, R.A., Ganji Saffar, B., *States on EQ-algebras,* J. Intell. Fuzzy Systems, 29 (2015), 209–221.

[2] Buşneag, D., *Hilbert algebras with valuations*, Discrete Math., 263 (2003), 11–24.

[3] Buşneag, D., *Valuations on residuated lattices*, An. Univ. Craiova Ser. Mat. Inform., 34 (2007), 21–28.

[4] Chang, C.C., *Algebraic analysis of many-valued logics*, Trans. Amer. Math. Soc., 88 (1958), 467–467.

[5] Doh, M.I., Kang, M.S., *BCK/BCI-algebras with pseudo valations*, Honam Math. J., 32 (2010), 217–226.

[6] Esteva, F., Godo, L., *Monoidal t-norm based logictowards a logic for left-continuous t-norms*, Fuzzy Sets and Systems, 124 (2001), 271–288.

[7] Ei-Zekey, M., *Representable good EQ-algebras*, Soft Comput., 14 (2010), 1011–1023.

[8] Ei-Zekey, M., NovÁK, V., Mesiar, R., *On good EQ-algebras*, Fuzzy Sets and Systems, 178 (2011), 1–23.

[9] Ghorbani, S., *Quotient BCI-algebras induced by pseudo-valuations*, Iran. J. Math. Sci. Inform., 5 (2010), 13–24.

[10] HÁJek, P., *Metamathematics of fuzzy logic*, Kluwer, Dordrecht, 1998.

[11] Jun, Y.B., Song, S.Z., *Hesitant fuzzy prefilters and filters of EQ-algebras*, Appl. Math. Sci., 9 (2015), 515–532.

[12] Jun, Y.B., Lee, K.J., Ahn, S.S., *Positive implicative pseudo-valuations on BCK-algebras*, Appl. Math. Sci., 5 (2011), 651–662.

[13] Jun, Y.B., Ahn, S.S., Roh, E.H., *BCC-algebras with pseudo-valuations*, Appl. Math. Sci., 26 (2012), 243–252.

[14] Liu, L., Zhang, X., *Implicative and positive implicative prefilters of EQ-algebras*, J. Intell. Fuzzy Systems, 26 (2014), 2087–2097.

[15] Ma, Z., Hu, B., *EQ-algebras from the point of view of generalized algebras with fuzzy equalities*, Fuzzy Sets and Systems, 236 (2014), 104–112.

[16] NovÁK, V., Baets, B.D., *EQ-algebras*, Fuzzy Sets and Systems, 160 (2009), 2956–2978.

[17] Wang, G.J., *An introduction to mathematical logic and resolution principle*, Beijing Science in China Press, 2003.

[18] Xu, Y., *Homomorphisms in lattice implication algebras*, Proceedings of 5th Symposium on Multiple Valued Logic of China, (1992), 206–211.

[19] Xin, X.L., He, P.F., Yang, Y.W., *Characterizations of some fuzzy prefilters (Filters) in EQ-Algebras*, The Scientific World Journal, 2014 (2014).

[20] Zhan, J., Jun, Y.B., *(Implicative) pseudo-valuations on R_0-algebras*, U.P.B. Sci. Bull., Series A, 75 (2013), 101–112.

A NOTE ON COMPLETENESS OF THE HAUSDORFF FUZZY METRIC SPACES

Changqing Li[1]
School of Mathematics and Statistics
Minnan Normal University
Zhangzhou, Fujian 363000
China
e-mail: helen_smile0320@163.com

Yanlan Zhang
College of Computer
Minnan Normal University
Zhangzhou, Fujian 363000
China
e-mail: zyl_1983_2004@163.com

Kexiu Zhang
School of Mathematics and Statistics
Minnan Normal University
Zhangzhou, Fujian 363000
China
kxzhang2006@163.com

Abstract. In this paper, completeness and completableness of the Hausdorff fuzzy metric spaces on the family of nonempty finite sets are explored. Also, necessary and sufficient conditions for the Hausdorff fuzzy metric spaces on the family of nonempty compact sets to be complete are found.

Keywords: fuzzy metric; Hausdorff fuzzy metric; complete; completable.
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1. Introduction

The concept of fuzzy metric, which has been introduced by many authors from different points of view [2], [3], [10], [12], is important in Fuzzy Topology. In particular, Kramosil and Michalek [12] generalized the concept of probabilistic metric and obtained the concept of fuzzy metric with the help of continuous t-norms in 1975. To make the topology induced by a fuzzy metric to be Hausdorff, George and Veeramani [3] modified the notion given by Kramosil and Michalek and gave

[1]Corresponding author.

a new notion with the help of continuous t-norms. Later, Gregori and Romaguera [7] proved that the topological space induced by the fuzzy metric is metrizable. The new version of fuzzy metric is more restrictive, but it determines the class of spaces that are tightly connected with the class of metrizable topological spaces. So it is interesting to study the new version of fuzzy metric. Many contributions to the study of fuzzy metric spaces can be found in [4]-[6], [9], [11], [13]-[17], [19], [20].

In order to explore the hyperspaces of a fuzzy metric space, Rodríguez-López and Romaguera [18] gave a construction of the Hausdorff fuzzy metric on the set of nonempty compact sets. In this paper, we study completeness and completableness of the Hausdorff fuzzy metric spaces on the family of nonempty finite sets. Moreover, we obtain necessary and sufficient conditions for the Hausdorff fuzzy metric spaces on the family of nonempty compact sets to be complete.

2. Preliminaries

In the section, we recall some concepts. Throughout the paper, the set of all natural numbers will be denoted by $\mathbb{N}$. Our basic reference for general topology is [1].

Definition 2.1.[3] A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a *continuous t-norm* if it satisfies the following conditions:

(i) $*$ is associative and commutative;
(ii) $*$ is continuous;
(iii) $a * 1 = a$ for all $a \in [0,1]$;
(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0,1]$.

Clearly, $a * b = a \cdot b$ and $a * b = \min\{a, b\}$ are two common examples of continuous t-norms.

Definition 2.2.[3] A 3-tuple $(X, M, *)$ is said to be a *fuzzy metric space* if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times (0, \infty)$ satisfying the following conditions for all $x, y, z \in X$ and $s, t \in (0, \infty)$:
(i) $M(x, y, t) > 0$;
(ii) $M(x, y, t) = 1$ if and only if $x = y$;
(iii) $M(x, y, t) = M(y, x, t)$;
(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
(v) the function $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous.

If $(X, M, *)$ is a fuzzy metric space, we will call $(M, *)$ a *fuzzy metric on X*.

Definition 2.3.[3] Let $(X, M, *)$ be a fuzzy metric space and let $r \in (0, 1), t > 0$ and $x \in X$. The set

$$B_M(x, r, t) = \{y \in X | M(x, y, t) > 1 - r\}$$

is called the *open ball with center x and radius r with respect to t*.

Clearly, $\{B_M(x,r,t)|x \in X, t > 0, r \in (0,1)\}$ forms a base of a topology in X and the topology is denoted by τ_M. In [3], it was proven that $\{B_M(x, \frac{1}{n}, \frac{1}{n})|n \in \mathbb{N}\}$ is a neighborhood base at x for the topology τ_M for every $x \in X$.

Definition 2.4.[3] Let (X,d) be a metric space. Define $a * b = a \cdot b$ for all $a, b \in [0,1]$, and let M_d be the function on $X \times X \times (0,\infty)$ defined by

$$M_d(x,y,t) = \frac{t}{t + d(x,y)}.$$

Then $(X, M_d, \cdot)$ is a fuzzy metric space and $(M_d, \cdot)$ is called *the standard fuzzy metric induced by d.*

Definition 2.5.[3] Let $(X, M, *)$ be a fuzzy metric space.

(a) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is called *Cauchy* if for each $r \in (0,1)$ and $t > 0$, there exists an $N \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - r$ for all $n, m \geq N$.

(b) $(X, M, *)$ is called *complete* if every Cauchy sequence in X is convergent with respect to τ_M.

Definition 2.6.[8] Let $(X_1, M_1, *_1)$ and $(X_2, M_2, *_2)$ be two fuzzy metric spaces.

(a) A mapping $f : X_1 \to X_2$ is called an *isometry* if for every $x, y \in X_1$ and $t > 0, M_1(x,y,t) = M_2(f(x), f(y), t)$.

(b) $(X_1, M_1, *_1)$ and $(X_2, M_2, *_2)$ are called *isometric* if there exists an isometry from X_1 onto X_2.

(c) A *fuzzy metric completion* of $(X_1, M_1, *_1)$ is a complete fuzzy metric space $(X_2, M_2, *_2)$ such that $(X_1, M_1, *_1)$ is isometric to a dense subspace of X_2.

(d) $(X_1, M_1, *_1)$ is said to be *completable* if it admits a fuzzy metric completion.

3. Completeness of the Hausdorff fuzzy metric on Fin(X)

Given a fuzzy metric space $(X, M, *)$, we shall denote by $\mathcal{P}(X)$, Comp(X) and Fin(X), the set of nonempty subsets, the set of nonempty compact subsets and the set of nonempty finite subsets of (X, τ_M), respectively. Let $M(a, B, t) := \sup_{b\in B} M(a,b,t)$, $M(B, a, t) := \sup_{b\in B} M(b,a,t)$ for all $a \in X$, $B \in \mathcal{P}(X)$ and $t > 0$ (see Definition 2.4 of [20]). Note that $M(a,B,t) = M(B,a,t)$. In the following, $|A|$ denotes the cardinality of A, where $A \subset X$.

Definition 3.1.[18] Let $(X, M, *)$ be a fuzzy metric space. For every $A, B \in$ Comp(X) and $t > 0$, define H_M: Comp$(X)\times$ Comp$(X) \times (0,\infty) \to [0,1]$ by

$$H_M(A,B,t) = \min\{\inf_{a\in A} M(a,B,t), \inf_{b\in B} M(A,b,t)\}.$$

Then (Comp(X),$H_M, *$) is a fuzzy metric space. $(H_M, *)$ is called *the Hausdorff fuzzy metric on Comp(X).*

Observe that $H_M(\{x\},\{y\},t) = M(x,y,t)$ for all $x,y \in X$ and $t > 0$, we can regard $(X,M,*)$ as a subspace of (Comp(X),$H_M,*$).

Lemma 3.2. [18] *Let $(X,M,*)$ be a fuzzy metric space. Then M is a continuous function on $X \times X \times (0,\infty)$.*

Lemma 3.3. [18] *Let $(X,M,*)$ be a fuzzy metric space. Then, for each $a \in X$, $B \in Comp(X)$ and $t > 0$, there exists a $b_a \in B$ such that $M(a,B,t) = M(a,b_a,t)$.*

Lemma 3.4. *In a fuzzy metric space $(X,M,*)$, the following are obtained.*

(1) *If X is a finite set or a set of isolated points, then Fin(X)=Comp(X).*
(2) *If X is a set of infinite non-isolated points, then Fin(X) is neither a closed subset nor an open subset of Comp(X).*

Proof. (1) is obviously satisfied.

(2) Suppose that X is a set of infinite non-isolated points. Then we can find a point $a \in X$ and a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X with $x_i \neq x_j$ whenever $i \neq j$ such that $\{x_n\}_{n\in\mathbb{N}}$ converges to a. Put $A = \{x_n | n \in \mathbb{N}\} \cup \{a\}$. Then $A \in$ Comp$(X)\backslash$Fin(X). Let $r_1 \in (0,1)$ and $t_1 > 0$. Then there exists an $m \in \mathbb{N}$ such that if $n \geq m$, we get that $x_n \in B_M(a, \frac{r_1}{2}, t_1)$. Since $B = \{x_1, x_2, \cdots, x_{m-1}\} \cup \{a\} \in$ Fin(X), we have that $H_M(A,B,t_1) \geq 1 - \frac{r_1}{2} > 1 - r_1$. Hence $B \in B_{H_M}(A, r_1, t_1)$. So Fin$(X)$ is not a closed subset of (Comp$(X), \tau_{H_M}$). On the other hand, let $r_2 \in (0,1)$ and $t_2 > 0$. Then there exists an $l \in \mathbb{N}$ such that if $n \geq l$, we have that $x_n \in B_M(a, \frac{r_2}{2}, t_2)$. Put $C = \{x_n | n \geq l\}$. Then $H_M(\{a\}, C, t_2) \geq 1 - \frac{r_2}{2} > 1 - r_2$, which implies that $C \in B_{H_M}(\{a\}, r_2, t_2)$. Since $\{a\} \in$ Fin(X), we deduce that Fin(X) is not an open subset of (Comp$(X), \tau_{H_M}$). This completes the proof.

Corollary 3.5. *Let $(X,M,*)$ be a fuzzy metric space. If Fin(X) is either a closed subset or an open subset of Comp(X), then Fin(X)=Comp(X).*

Lemma 3.6. [18] *Let $(X,M,*)$ be a fuzzy metric space. Then (Comp$(X), H_M, *$) is complete if and only if $(X,M,*)$ is complete.*

Lemma 3.7. *Let $(X,M,*)$ be a complete fuzzy metric space and $A \subseteq X$. Then A is a closed subset of (X,τ_M) if and only if $(A,M,*)$ is complete.*

Proof. Suppose that A is a closed subset of (X,τ_M). Let $\{x_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $(A,M,*)$. Then $\{x_n\}_{n\in\mathbb{N}}$ is also a Cauchy sequence in $(X,M,*)$. Since $(X,M,*)$ is complete, we can find an $x_0 \in X$ such that $\{x_n\}_{n\in\mathbb{N}}$ converges to x_0. Hence $x_0 \in A$. Thus, $(A,M,*)$ is complete.

Conversely, suppose that $(A,M,*)$ is complete. If A fails to be a closed subset of (X,τ_M), then there exists an $x \in X \setminus A$ such that $B_n(x, \frac{1}{n}, \frac{1}{n}) \cap A \neq \emptyset$ for all $n \in \mathbb{N}$. Take $x_n \in B_M(x, \frac{1}{n}, \frac{1}{n}) \cap A$ for every $n \in \mathbb{N}$. Then $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in $(A,M,*)$ and $\{x_n\}_{n\in\mathbb{N}}$ converges to x in X. Hence $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $(A,M,*)$. So $x \in A$, which is a contradiction. Consequently, A is a closed subset of (X,τ_M).

Theorem 3.8. *Let $(X, M, *)$ be a fuzzy metric space. If $(Fin(X), H_M, *)$ is complete, then $(X, M, *)$ is complete.*

Proof. Let $t > 0$ and $A \in \mathrm{Fin}(X)$ with $|A| \geq 2$. Take $a, b \in A$ with $a \neq b$. Put $M(a, b, 2t) = \varepsilon_0$. Then there exists a $\varepsilon_1 \in (\varepsilon_0, 1)$ such that $\varepsilon_1 * \varepsilon_1 > \varepsilon_0$. We claim that

$$B_M(a, 1 - \varepsilon_1, t) \cap B_M(b, 1 - \varepsilon_1, t) = \emptyset.$$

Indeed, if not, we can choose a $c \in B_M(a, 1 - \varepsilon_1, t) \cap B_M(b, 1 - \varepsilon_1, t)$. Hence

$$M(a, b, 2t) \geq M(a, c, t) * M(c, b, t) \geq \varepsilon_1 * \varepsilon_1 > \varepsilon_0 = M(a, b, 2t),$$

which is a contradiction. Let $x \in X$. If $x \in \overline{B_M(a, 1 - \varepsilon_1, t)}$, where $\overline{B_M(a, 1 - \varepsilon_1, t)}$ is the closure of $B_M(a, 1 - \varepsilon_1, t)$, then $x \notin B_M(b, 1 - \varepsilon_1, t)$. Hence $M(b, x, t) \leq \varepsilon_1$. Thus

$$H_M(A, \{x\}, t) \leq \inf_{y \in A} M(y, \{x\}, t) = \inf_{y \in A} M(y, x, t) \leq M(b, x, t) \leq \varepsilon_1.$$

If $x \notin \overline{B_M(a, 1 - \varepsilon_1, t)}$, then $x \notin B_M(a, 1 - \varepsilon_1, t)$, whence $M(a, x, t) \leq \varepsilon_1$. Hence $H_M(A, \{x\}, t) \leq \varepsilon_1$. So

$$\{\{x\} | x \in X\} \cap H_M(A, 1 - \varepsilon_1, t) = \emptyset,$$

which implies that X is a closed subset of $(\mathrm{Fin}(X), \tau_{H_M})$. Due to Lemma 3.7, we deduce that $(X, M, *)$ is complete.

The converse of the above theorem is false. We illustrate this with the following example.

Example 3.9. Let $X = \{0, \frac{1}{2}, \cdots, 1 - \frac{1}{n}, \cdots\} \cup \{1\}$ and d be the Euclidian metric of X. Denote $a * b = a \cdot b$ for all $a, b \in [0, 1]$. Define the function M by

$$M(x, y, t) = \frac{t}{t + d(x, y)}$$

for all $x, y \in X$ and $t > 0$. Then $(X, M, *)$ is a complete fuzzy metric space. Thanks to Lemma 3.6, we have that $(\mathrm{Comp}(X), H_M, *)$ is complete. Due to Lemma 3.4 (2), we get that $\mathrm{Fin}(X)$ is not a closed subset of $(\mathrm{Comp}(X), \tau_{H_M})$. By Lemma 3.7, we conclude that $(\mathrm{Fin}(X), H_M, *)$ fails to be complete.

Now, according to Lemma 3.4, Lemma 3.6 and Lemma 3.7, it is easy to obtain the following theorem.

Theorem 3.10. *In a complete fuzzy metric space $(X, M, *)$, the following hold:*

(1) *If X is a finite set or an isolated set, then $(Fin(X), H_M, *)$ is complete.*

(2) *If X is an infinite non-isolated set, then $(Fin(X), H_M, *)$ is not complete.*

Lemma 3.11. [18] *Let $(X, M, *)$ be a fuzzy metric space. Then (Comp$(X), H_M, *)$ is completable if and only if $(X, M, *)$ is completable.*

Theorem 3.12. *Let $(X, M, *)$ be a fuzzy metric space. Then (Fin$(X), H_M, *)$ is completable if and only if $(X, M, *)$ is completable.*

Proof. Assume that $(\mathrm{Fin}(X), H_M, *)$ is completable. Then there exists an isometry $i : (\mathrm{Fin}(X), H_M, *) \to (\widetilde{\mathrm{Fin}(X)}, \widetilde{H_M}, \star)$ such that $i(\mathrm{Fin}(X))$ is dense in $\widetilde{\mathrm{Fin}(X)}$. Observe that for each $x \in X, \{x\} \in \mathrm{Fin}(X)$, the restriction i_X of i is an isometry between $(X, M, *)$ and $(i_X(X), \widetilde{H_M}, \star)$. Let $\overline{i_X(X)}$ be the closure of $i_X(X)$. By Lemma 3.7, we have that $(\overline{i_X(X)}, \widetilde{H_M}, \star)$ is a complete fuzzy metric space that has $i_X(X)$ as a dense subspace. Thus $(X, M, *)$ is completable.

Conversely, assume that $(X, M, *)$ is completable. Then, by Lemma 3.11, $(\mathrm{Comp}(X), H_M, *)$ is completable. Hence there exists an isometry $i : (\mathrm{Comp}(X), H_M, *) \to (\widetilde{\mathrm{Comp}(X)}, \widetilde{H_M}, \star)$ such that $i(\mathrm{Comp}(X))$ is dense in $\widetilde{\mathrm{Comp}(X)}$.

Note that the restriction $i_{\mathrm{Fin}(X)}$ of i is an isometry between $(\mathrm{Fin}(X), H_M, *)$ and $(i_{\mathrm{Fin}(X)}(\mathrm{Fin}(X)), \widetilde{H_M}, \star)$. Let $\overline{i_{\mathrm{Fin}(X)}(\mathrm{Fin}(X))}$ be the closure of $i_{\mathrm{Fin}(X)}(\mathrm{Fin}(X))$. It follows from Lemma 3.7 that $(\overline{i_{\mathrm{Fin}(X)}(\mathrm{Fin}(X))}, \widetilde{H_M}, \star)$ is a complete fuzzy metric space that has $i_{\mathrm{Fin}(X)}(\mathrm{Fin}(X))$ as a dense subspace. So $(\mathrm{Fin}(X), H_M, *)$ is completable.

4. Completeness of the Hausdorff fuzzy metric on Comp(X)

In the section, we will give necessary and sufficient conditions for the Hausdorff fuzzy metric spaces on Comp(X) to be complete.

In a fuzzy metric space $(X, M, *)$, put

$$\mathrm{Comp}_K(X) = \{K' \in \mathrm{Comp}(X) | K \subseteq K'\}$$

for every $K \in \mathrm{Comp}(X)$.

Lemma 4.1. *Let $(X, M, *)$ be a fuzzy metric space and $K \in Comp(X)$. Then $Comp_K(X)$ is a closed subset of $(Comp(X), \tau_{H_M})$.*

Proof. Let $A \in \mathrm{Comp}(X) \setminus \mathrm{Comp}_K(X)$. Then there exists an $b \in K$ such that $b \notin A$. Let $t > 0$. Put $M(b, A, t) = \varepsilon_0$. Since $A \in \mathrm{Comp}(X)$, then, by Lemma 3.3, there exists an $a_b \in A$ such that $M(b, a_b, t) = M(b, A, t)$. Hence $0 < \varepsilon_0 < 1$. Let $B \in \mathrm{Comp}_K(X)$. Then $b \in B$. Since

$$\begin{aligned} H_M(A, B, t) &= \min\left\{\inf_{x\in A} M(x, B, t), \inf_{y\in B} M(A, y, t)\right\} \\ &\leq \inf_{y\in B} M(A, y, t) \leq M(A, b, t) \\ &= \varepsilon_0, \end{aligned}$$

we have that $B \notin B_{H_M}(A, 1-\varepsilon_0, t)$. It follows that

$$B_{H_M}(A, 1-\varepsilon_0, t) \cap \mathrm{Comp}_K(X) = \emptyset.$$

So $\mathrm{Comp}_K(X)$ is a closed subset of $(\mathrm{Comp}(X), \tau_{H_M})$.

Theorem 4.2. *Let $(X, M, *)$ be a fuzzy metric space. Then $(X, M, *)$ is complete if and only if $(Comp_K(X), H_M, *)$ is complete for all $K \in Comp(X)$.*

Proof. Assume that $(X, M, *)$ is complete. Then, by Lemma 3.6, we have that $(\mathrm{Comp}(X), H_M, *)$ is complete. Let $K \in \mathrm{Comp}(X)$. Due to Lemma 4.1, we obtain that $\mathrm{Comp}_K(X)$ is a closed subset of $\mathrm{Comp}(X)$. Consequently, according to Lemma 3.7, $(\mathrm{Comp}_K(X), H_M, *)$ is complete.

Conversely, assume that $(\mathrm{Comp}_K(X), H_M, *)$ is complete for all $K \in \mathrm{Comp}(X)$. Let $\{a_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $(X, M, *)$. Take $a \in X$. For each $n \in \mathbb{N}$, put $A_n = \{a, a_n\}$. Then $A_n \in \mathrm{Comp}_{\{a\}}(X)$. Let $r \in (0, 1)$ and $t > 0$. Then there exists an $N \in \mathbb{N}$ such that $M(a_n, a_m t) > 1 - r$ for all $n, m \geq N$.

Note that

$$\begin{aligned} H_M(A_n, A_m, t) &= \min\left\{\inf_{x\in A_n} M(x, A_m, t), \inf_{y\in A_m} M(A_n, y, t)\right\} \\ &= \min\{M(a_n, A_m, t), M(A_n, a_m, t)\}. \end{aligned}$$

Since $M(a_n, A_m, t) \geq M(a_n, a_m, t)$ and $M(A_n, a_m, t) \geq M(a_n, a_m, t)$, we get that

$$H_M(A_n, A_m, t) \geq M(a_n, a_m, t) > 1 - r,$$

which means that $\{A_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathrm{Comp}_{\{a\}}(X)$. So $\{A_n\}_{n\in\mathbb{N}}$ converges to $A \in \mathrm{Comp}_{\{a\}}(X)$. If $|A| \geq 3$, then we can choose an $A' \subseteq A$ with $|A'| = 3$. Put $\varepsilon' = \max\{M(x, y, 2t) | x, y \in A'\}$. Then there exists a $\varepsilon_0 \in (\varepsilon', 1)$ such that $\varepsilon_0 * \varepsilon_0 > \varepsilon'$. Put $\mathcal{A} = \{A_n | n \in \mathbb{N}\}$. Then $B_{H_M}(A, 1-\varepsilon_0, t) \cap \mathcal{A} = \emptyset$, which contradicts that $\{A_n\}_{n\in\mathbb{N}}$ converges to A. In fact, if there exists a $B \in B_{H_M}(A, 1-\varepsilon_0, t) \cap \mathcal{A}$, then

$$H_M(A, B, t) = \min\{\inf_{x\in A} M(x, B, t), \inf_{y\in B} M(A, y, t)\} > \varepsilon_0.$$

Hence $\inf_{x\in A} M(x, B, t) > \varepsilon_0$. Since $A' \subseteq A$, we have that $\inf_{x\in A'} M(x, B, t) > \varepsilon_0$. Therefore $M(x, B, t) > \varepsilon_0$ for all $x \in A'$. Thus, due to Lemma 3.3, for each $x \in A'$, there exists a $y_x \in B$ such that $M(x, y_x, t) = M(x, B, t) > \varepsilon_0$. Hence $x \in B_M(y_x, 1-\varepsilon_0, t)$ for all $x \in A'$. So

$$A' \subseteq \bigcup_{y_x\in B} B_M(y_x, 1-\varepsilon_0, t) \subseteq \bigcup_{y\in B} B_M(y, 1-\varepsilon_0, t).$$

Since $|B| = 2$ and $|A'| = 3$, there exist $x_1, x_2 \in A'$ and $y_1 \in B$ such that $x_1, x_2 \in B_M(y_1, 1-\varepsilon_0, t)$. We get that

$$M(x_1, x_2, 2t) \geq M(x_1, y_1, t) * M(y_1, x_2, t) \geq \varepsilon_0 * \varepsilon_0 > \varepsilon' \geq M(x_1, x_2, 2t),$$

which is a contradiction. So $|A| \leq 2$. Let $r \in (0,1)$ and $t > 0$. Then there exists an $N_1 \in \mathbb{N}$ such that

$$H_M(A, A_n, t) = \min\{\inf_{x \in A} M(x, A_n, t), \inf_{y \in A_n} M(A, y, t)\} > 1 - r$$

for all $n \geq N_1$. In case $A = \{a\}$ we have that

$$\begin{aligned} H_M(A, A_n, t) &= \min\{M(a, A_n, t), \inf_{y \in A_n} M(a, y, t)\} \\ &= \min\{1, M(a, a_n, t)\} = M(a, a_n, t). \end{aligned}$$

Hence $M(a, a_n, t) > 1 - r$. Consequently, $\{a_n\}_{n \in \mathbb{N}}$ converges to a. Let $A = \{a, b\}$. Set $M(b, a, t) = \varepsilon_1$. Then there exists $r_1 \in (0, r)$ such that $1 - r_1 > \varepsilon_1$. Then there exists an $N_2 \in \mathbb{N}$ such that

$$\begin{aligned} H_M(A, A_n, t) &= \min\{\inf_{x \in A} M(x, A_n, t), \inf_{y \in A_n} M(A, y, t)\} \\ &= \min\{M(b, A_n, t), M(A, a_n, t)\} \\ &> 1 - r_1 \end{aligned}$$

for all $n \geq N_2$. Hence

$$M(b, A_n, t) = \max\{M(b, a, t), M(b, a_n, t)\} > 1 - r_1.$$

Since $M(b, a, t) = \varepsilon_1 < 1 - r_1$, we get that $M(b, a_n, t) > 1 - r_1 > 1 - r$. Therefore, $\{a_n\}_{n \in \mathbb{N}}$ converges to b. So $(X, M, *)$ is complete.

From Lemma 3.6 and Theorem 4.2 we immediately deduce the next corollary.

Corollary 4.3. *Let $(X, M, *)$ be a fuzzy metric space. Then the following are equivalent.*

(i) *$(X, M, *)$ is complete.*

(ii) *$(Comp(X), H_M, *)$ is complete.*

(iii) *$(Comp_K(X), H_M, *)$ is complete for all $K \in Comp(X)$.*

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References

[1] Engelking, R., *General Topology*, PWN-Polish Science Publishers, Warsaw, 1977.

[2] Erceg, M.A., *Metric spaces in fuzzy set theory*, J. Math. Anal. Appl., 69 (1979), 205–230.

[3] George, A., Veeramani, P., *On some resules in fuzzy metric spaces*, Fuzzy Sets and Systems, 64 (1994), 395–399.

[4] George, A., Veeramani, P., *Some theorems in fuzzy metric spaces*, J. Fuzzy Math., 3 (1995), 933–940.

[5] George, A., Veeramani, P., *On some resules of analysis for fuzzy metric spaces*, Fuzzy Sets and Systems, 90 (1997), 365–368.

[6] Gregori, V., Mascarell, J.A., Sapena, A., *On completion of fuzzy quasi-metric spaces*, Topology and its Applications, 153 (2005), 886–899.

[7] Gregori, V., Romaguera, S., *Some properties of fuzzy metric spaces*, Fuzzy Sets and Systems, 115 (2000), 485–489.

[8] Gregori, V., Romaguera, S., *On completion of fuzzy metric spaces*, Fuzzy Sets and Systems, 130 (2002), 399–404.

[9] Gregori, V., Romaguera, S., *Characterizing completable fuzzy metric spaces*, Fuzzy Sets and Systems 144 (2004) 411–420.

[10] Kaleva, O., Seikkala, S., *On fuzzy metric spaces*, Fuzzy Sets and Systems, 12 (1984), 215–229.

[11] Kočinac, Lj.D.R., *Selection properties in fuzzy metric spaces*, Filomat, 26 (2) (2012), 305–312.

[12] Kramosil, I., Michalek, J., *Fuzzy metric and statistical metric spaces*, Kybernetika, 11 (1975), 326–334.

[13] Li, C.Q., *On some results of metrics induced by a fuzzy ultrametric*, Filomat, 27 (6) (2013), 1133–1140.

[14] Li, C.Q., Yang, Z.Q., *Fuzzy ultrametrics based on idempotent probability measures*, J. Fuzzy Math., 22 (2) (2014), 463–476.

[15] Li, C.Q., *Some properties of intuitionstic fuzzy metric spaces*, Journal of Computational Analysis and Applications, 16 (4) (2014), 670–677.

[16] Park, J.H., Park, Y.B., Saadati, R., *Some results in intuitionistic fuzzy metric spaces*, Journal of Computational Analysis and Applications, 10 (4) (2008), 441-451.

[17] Repovš, D., Savchenko, A., Zarichnyi, M., *Fuzzy Prokhorov metric on the of probability measures*, Fuzzy Sets and Systems, 175 (2011), 96–104.

[18] Rodríguez-López, J., Romaguera, S., *The Hausdorff fuzzy metric on compact sets*, Fuzzy Sets and Systems, 147 (2004), 273–283.

[19] Savchenko, A., Zarichnyi, M., *Fuzzy ultrametrics on the set of probability measures*, Topology, 48 (2009), 130–136.

[20] Veeramani, P., *Best approximation in fuzzy metric spaces*, J. Fuzzy Math., 9 (2001), 75–80.

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ON NEARLY CAP-EMBEDDED SUBGROUPS OF FINITE GROUPS

Yong Xu
School of Mathematics and Statistics
Southwest University
Chongqing, 400715
China
School of Mathematics and Statistics
Henan University of Science and Technology
Luoyang, Henan 471023
China
e-mail: xuy_2011@163.com

Guiyun Chen[1]
School of Mathematics and Statistics
Southwest University
Chongqing, 400715
China
e-mail: gychen1963@163.com

Abstract. We introduce a new subgroup embedding property of a finite group called nearly CAP-embedded subgroup. Using this subgroup property, we determine the structure of finite groups with some nearly CAP-embedded subgroups of Sylow subgroups. Our results unify and generalize some recent theorems on p-nilpotency and supersolvability of finite groups.

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1. Introduction

In this paper, all groups considered are finite. Let $\pi(G)$ stand for the set of all prime divisors of the order of a group G. Let $\mathcal{F}$ denote a formation, $\mathcal{U}$ the class of supersolvable groups. $H\ Char\ G$ means that H is a characteristic subgroup of G. The other notations and terminology are standard (see[9]).

Let H be a subgroup of G, and A/B be a G-chief factor. We say that H covers A/B if $HA = HB$; and H avoids A/B if $H \cap A = H \cap B$. H is said to have cover-avoiding property in G, in brevity, H is a CAP-subgroup of G, if H either covers or avoids any G-chief factor. In 1962, Gaschütz[5] introduced a certain conjugacy class of subgroups of a solvable group called the pre-Frattini subgroups. These subgroups have cover-avoidance property. Thereafter, many

[1]Corresponding author.

authors devoted to find some kind of subgroups of a solvable group having this property, for example, Gillam[6] and Tomkinson[14]. In 1993, Ezquerro[4] considered the converse questions, he gave some characterizations for a group G to be p-supersolvable and supersolvable based on the assumption that all maximal subgroups of some subgroups of G are CAP-subgroups. Asaad in [1] obtained further results within the framework of formation theory. As a generalization of CAP-subgroups, Guo and Guo in[7] introduced CAP-embedded subgroups. A subgroup H of G is said to have the CAP-embedded property in G or is called a CAP-embedded subgroup of G if, for each prime p dividing the order of H, there exists a CAP subgroup K of G such that a Sylow p-subgroup of H is also a Sylow p-subgroup of K. Moreover, they presented some conditions for a finite group to be p-nilpotent and supersolvable under the condition that some subgroups of Sylow subgroup are CAP-embedded.

In recent years, it has been of interest to use some supplemented properties of subgroups to determine the structure of a group. For example, Wang in [15] introduced the concept of c-normal subgroups. A subgroup H of G is c-normal in G if there is a normal subgroup K_1 of G such that $G = HK_1$ and $H \cap K_1 \leq H_G = Core_G(H)$. As applications, he gave some criteria for the solvability and supersolvability of groups.

We provide examples in Section 2 to show that CAP-embedded property and c-normality cannot imply from one to the other one. In this paper, we will try an attempt to unify the two concepts and introduce a new subgroup embedding property of a finite group called nearly CAP-embedded subgroup. As applications, we study the influence of nearly CAP-embedded subgroups on the structure of finite groups. We present some sufficient conditions for a group to be p-nilpotent, p-supersolvable and supersolvable.

2. Basic definitions and preliminary results

When we recall the concepts of a $c-$normal subgroup and a CAP-embedded subgroup, it is easy to see that a normal subgroup N of G is both c-normal and CAP-embedded. The following examples show that c-normal and CAP-embedded are different properties:

Example 2.1. Let $G = A_5$, the alternative group of degree 5. Then all Sylow subgroups of G are CAP-embedded subgroups of G, but every Sylow subgroup is not a c-normal subgroup of G.

Example 2.2. Let A_4 be the alternative group of degree 4 and $C = \langle c \rangle$ be a cyclic group of order 2. Let $G = C \times A_4$. Then $A_4 = [K_4]C_3$, where $K_4 = \langle a, b \rangle$ is the Klein Four Group with generators a and b of order 2 and C_3 is the cyclic group of order 3. Take $H = \langle ac \rangle$ be the cyclic subgroup of order 2 of G. Then $G = HA_4$ and $H \cap A_4 = 1$. By definition, H is c-normal in G. However, H is not a CAP-embedded subgroup of G, if not, then there exists a CAP-subgroup B of G such that $H \in Syl_2(B)$, so B covers or avoids $(C \times K_4)/C$, it is impossible.

In the c-normal case, $G = HK_1$, if we let $K_2 = H_G K_1$, then $G = HK_2$ and $H \cap K_2 = H_G$; $H \cap K_2$ is, of course, a CAP-embedded subgroup of G. Based on the observation, we introduce the following:

Definition 2.3. A subgroup H of a group G is said to be nearly CAP-embedded in G if there are a subnormal subgroup T of G and a CAP-embedded subgroup H_{ce} of G contained in H such that $G = HT$ and $H \cap T \leq H_{ce}$.

If H is a CAP-embedded subgroup of G, taking $T = G$, we get H is a nearly CAP-embedded subgroup of G. Hence nearly CAP-embedded subgroup is a real uniform generalization of a c-normal subgroup and a CAP-embedded subgroup.

For the sake of convenience, we list here some known results which will be useful in the sequel.

Lemma 2.4 (([7, Lemma 1])). *Suppose that U is CAP-embedded in a group G and $N \trianglelefteq G$. Then UN/N is CAP-embedded in G/N.*

Lemma 2.5. ([19, Lemma 2.4]) *Let H be a normal subgroup of a group G such that G/H is p-nilpotent and let P be a Sylow p-subgroup of H, where p is a prime divisor of $|G|$. If $|P| \leq p^2$ and one of the following conditions holds, then G is p-nilpotent:*

(1) $(|G|, p-1) = 1$ *and* $|P| \leq p$;

(2) *G is A_4-free if* $p = min\pi(G)$;

(3) $(|G|, p^2 - 1) = 1$.

Lemma 2.6. ([20, Theorem 3.1]) *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, and G a group with a normal subgroup N such that $G/N \in \mathcal{F}$. If all Sylow subgroups of $F^*(N)$ are cyclic, then $G \in \mathcal{F}$.*

Lemma 2.7. ([17, Theorem 3.1]) *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, G a group with a solvable normal subgroup H such that $G/H \in \mathcal{F}$. If for any maximal subgroup M of G, either $F(H) \leq M$ or $F(H) \cap M$ is a maximal subgroup of $F(H)$, then $G \in \mathcal{F}$. The converse also holds, in the case where $\mathcal{F} = \mathcal{U}$.*

Lemma 2.8. *Let U be a nearly CAP-embedded subgroup and N a normal subgroup of a group G. Then*

(1) *If $N \leq U$, then U/N is nearly CAP-embedded in G/N.*

(2) *If $(|U|, |N|) = 1$, then UN/N is nearly CAP-embedded in G/N.*

Proof. By the hypotheses, there are a subnormal subgroup T of G and a CAP-embedded subgroup U_{ce} of G contained in U such that $G = UT$ and $U \cap T \leq U_{ce}$.

(1) $G/N = (U/N)(TN/N)$, $TN/N \lhd\lhd G/N$ by [3, Chap A, Lemma 14.1(b)], and $(U/N) \cap (TN/N) = (U \cap TN)/N = (U \cap T)N/N \leq (U_{ce}N)/N$. By Lemma 2.4, $(U_{ce}N)/N$ is CAP-embedded in G/N. Hence U/N is nearly CAP-embedded in G/N.

(2) Let π be the set of all prime divisors of $|U|$, then N is a normal π'-subgroup and U is a π-subgroup of G. Since $|G|_{\pi'} = |T|_{\pi'} = |TN|_{\pi'}$, we have that $|T \cap N| =$

$|T \cap N|_{\pi'} = |N|_{\pi'} = |N|$ and hence $N \leq T$. Therefore, $G/N = (UN/N)(T/N)$, $T/N \lhd \lhd G/N$ by [3, Ch. A, Lemma 14.1(b)], and $(UN/N) \cap T/N = (U \cap T)N/N \leq (U_{ce}N)/N$. By Lemma 2.4, we have $(U_{ce}N)/N$ is CAP-embedded in G/N. Hence, $(UN)/N$ is nearly CAP-embedded in G/N. ∎

3. Main results and their proofs

Theorem 3.1. *Let G be a group, N a normal subgroup of G such that G/N is p-nilpotent and P a Sylow p-subgroup of N, where $p \in \pi(G)$ with $(|G|, p-1) = 1$. If all maximal subgroups of P are nearly CAP-embedded in G, then G is p-nilpotent.*

Proof. Assume that the result is false. Let G be a minimal counterexample with least $|N| + |G|$.

(1) G has a unique minimal normal subgroup L contained in N, G/L is p-nilpotent and $L \nleq \Phi(G)$.

Let L be a minimal normal subgroup of G contained in N. Consider the factor group $\overline{G} = G/L$. Clearly $\overline{G}/\overline{N} \cong G/N$ is p-nilpotent and $\overline{P} = PL/L$ is a Sylow p-subgroup of $\overline{N}$, where $\overline{N} = N/L$. Now let $\overline{P_1} = P_1L/L$ be a maximal subgroup of $\overline{P}$. We may assume that P_1 is a maximal subgroup of P. Then $P_1 \cap L = P \cap L$ is a Sylow p-subgroup of L. By the hypothesis, there are a subnormal subgroup T of G and a CAP-embedded subgroup $(P_1)_{ce}$ contained in P_1 of G such that $G = P_1T$ and $P_1 \cap T \leq (P_1)_{ce}$. Clearly $TL/L \lhd \lhd G/L$. Now we let $\pi(G) = \{p_1, p_2, \cdots p_n\}$ where $p_1 = p$, and T_{p_i} be a Sylow p_i-subgroup of T $(i = 2, \cdots, n)$. Then T_{p_i} is also a Sylow p_i-subgroup of G, hence $T_{p_i} \cap L$ is a Sylow p_i-subgroup of L $(i = 2, \cdots, n)$. Write $V = \langle L \cap T_{p_2}, \cdots, L \cap T_{p_n} \rangle$, then $V \leq T \cap L$. Note that $(|L : P_1 \cap L|, |L : V|) = 1$, $L = (P_1 \cap L)V$, thus $P_1L \cap TL = (P_1L \cap T)L = (P_1V \cap T)L = (P_1 \cap T)VL = (P_1 \cap T)L$. It follows from Lemma 2.4 that $(P_1L/L) \cap (TL/L) = (P_1 \cap T)L/L \leq (P_1)_{ce}L/L$ and $(P_1)_{ce}L/L$ is CAP-embedded in G/L. Therefore $\overline{P_1}$ is nearly CAP-embedded in $\overline{G}$. The choice of G implies that $\overline{G}$ is p-nilpotent. Since the class of p-nilpotent groups is a saturated formation, L is a unique minimal normal subgroup of G contained in N and $L \nleq \Phi(G)$.

(2) $O_{p'}(G) = 1$.

If $E = O_{p'}(G) \neq 1$, we consider $\overline{G} = G/E$. Clearly, $\overline{G}/\overline{N} \cong G/NE$ is p-nilpotent because G/N is, where $\overline{N} = NE/E$. Let $\overline{P_1} = P_1E/E$ be a maximal subgroup of PE/E. We may assume that P_1 is a maximal subgroup of P. Since P_1 is nearly CAP-embedded in G, P_1E/E is nearly CAP-embedded in G/E by Lemma 2.8 (2). The minimality of G yields that G is p-nilpotent, therefore G is p-nilpotent, a contradiction.

(3) $O_p(N) = 1$ and so L is not p-nilpotent.

If not, then by (1), $L \leq O_p(N)$ and, there is a maximal subgroup M of G such that $G = LM$ and $L \cap M = 1$. Since $M_p < P$, where $M_p \in Syl_p(M)$, we may let P_1 be a maximal subgroup of P containing M_p. Because P_1 is a nearly CAP-embedded subgroup of G, there are a subnormal subgroup T of G and a CAP-embedded subgroup $(P_1)_{ce}$ contained in P_1 of G such that $G = P_1T$ and $P_1 \cap T \leq (P_1)_{ce} \in Syl_p(K)$, where K is a CAP subgroup of G. If K covers

$L/1$, then $L \leq K$. It follows from $(P_1)_{ce} \in Syl_p(K)$ that $L \leq P_1$, thus $P = LM_p = LP_1 = P_1$, a contradiction. So K must avoids $L/1$, i.e., $K \cap L = 1$, hence $P_1 \cap T \cap L = 1$. Consequently, $|T \cap L| \leq p$. Since $T/L \cap T \cong TL/L \leq G/L$, $T/L \cap T$ is p-nilpotent. It follows that T is p-nilpotent by Lemma 2.5. Let $T_{p'}$ be the normal p-complement of T. Then $T_{p'}$ is a Hall p'-subgroup of G and $T_{p'}$ $Char$ $T \trianglelefteq \trianglelefteq G$, so $T_{p'} \trianglelefteq G$, contrary to $O_{p'}(G) = 1$.

If L is p-nilpotent, then $L_{p'}$ $Char$ $L \trianglelefteq N$, so $L_{p'} \leq O_{p'}(N) \leq O_{p'}(G) = 1$ by (2). Thus L is a p-group, $L \leq O_p(N) = 1$, a contradiction. Hence (3) holds.

(4) The final contradiction.

If $P \cap L \leq \Phi(P)$, then L is p-nilpotent by Tate's theorem [9, IV, Th 4.7], contrary to (3). Consequently, there exists a maximal subgroup P_1 of P such that $P = (L \cap P)P_1$. Let T be a subnormal supplement of P_1 in G, we have $P_1 \cap T \leq (P_1)_{ce} \in Syl_p(K)$, where K is a CAP subgroup of G. If K covers $L/1$, then $L \leq K$. It follows from $(P_1)_{ce} \in Syl_p(K)$ that $P_1 \cap K = (P_1)_{ce} \in Syl_p(K)$, then $P_1 \cap L \in Syl_p(L)$. Thus $L \cap P = L \cap P_1$. We obtain $P = (L \cap P)P_1 = P_1$, a contradiction. So K must avoids $L/1$, i.e., $K \cap L = 1$, hence $P_1 \cap T \cap L = 1$. Consequently, $|P \cap T \cap L| \leq p$. Since $T/L \cap T \cong TL/L \leq G/L$, $T/L \cap T$ is p-nilpotent. It follows that T is p-nilpotent by Lemma 2.5. Let $T_{p'}$ be the normal p-complement of T. Then $T_{p'}$ is a Hall p'-subgroup of G and $T_{p'}$ $Char$ $T \trianglelefteq \trianglelefteq G$, so $T_{p'} \trianglelefteq G$, contrary to $O_{p'}(G) = 1$. This contradiction completes the proof. ∎

Theorem 3.2. *Let p be a prime dividing the order of the group G and let N be a p-solvable normal subgroup of G such that G/N is p-supersolvable. If there exists a Sylow p-subgroup P of N such that every maximal subgroup of P is nearly CAP-embedded in G, then G is p-supersolvable.*

Proof. Assume that the result is false and let G be a counterexample of minimal order. Now, arguing as in the proof of Theorem 3.1, the following statements (1) and (2) about G are true.

(1) G has a unique minimal normal subgroup L contained in N, G/L is p-supersolvable and $L \nleq \Phi(G)$.

(2) $O_{p'}(G) = 1$.

Since G is p-solvable and $O_{p'}(G) = 1$, L is a p-group and $L \leq P$. If $L \leq \Phi(P)$, by [12, Theorem 5.2.13], $L \leq \Phi(G)$, a contradiction. Consequently, there exists a maximal subgroup P_1 of P such that $P_1L = P$. Since P_1 is a nearly CAP-embedded subgroup of G, there are a subnormal T of G and a CAP-embedded subgroup $(P_1)_{ce}$ of G such that $G = P_1T$ and $P_1 \cap T \leq (P_1)_{ce} \in Syl_p(K)$, where K is a CAP subgroup of G. If K covers $L/1$, then $L \leq K$. It follows from $(P_1)_{ce} \in Syl_p(K)$ that $L \leq P_1$, thus $P = LP_1 = P_1$, a contradiction. So K must avoids $L/1$, i.e., $K \cap L = 1$, hence $P_1 \cap T \cap L = 1$. Consequently, $|T \cap L| \leq p$. Noting that G/T_G is p-group, so $N \cap T_G \neq 1$. If not, then $G = G/N \cap T_G \lesssim G/N \times G/T_G$ is p-supersolvable, a contradiction. So $L \leq N \cap T_G$ by (1). Hence $|L| = |T \cap L| = p$. The p-supersolvability of G/L implies that G is p-supersolvable, final contradiction. ∎

Remark 3.3. The hypothesis that N is p-solvable in Theorem 3.2 is essential. For example, if we let G be the alternating group A_5 of degree 5, $N = G$ and $p = 3$, then it is clear that the statement of Theorem 3.2 does not hold.

Theorem 3.4. *Let G be a group. Then G is supersolvable if and only if there exists a normal subgroup N such that G/N is supersolvable and all maximal subgroups of any Sylow subgroup of N are nearly CAP-embedded in G.*

Proof. The necessity part can be obtained if we let $N = G$ and apply a result due to Ezquerro[4]. So we need to prove the sufficiency part.

Let p be the smallest prime divisor of $|G|$. The supersolvability of G/N implies that G/N is p-nilpotent. By Theorem 3.1, G is p-nilpotent. Furthermore G is solvable. Applying Theorem 3.2, it is easy to see that G is supersolvable. ∎

Theorem 3.5. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a normal subgroup N such that $G/N \in \mathcal{F}$. If all maximal subgroups of any Sylow subgroup of N are nearly CAP-embedded in G, then $G \in \mathcal{F}$.*

Proof. Let G be a minimal counterexample. With similar arguments as in the proof of Theorem 3.1, we have the following claim (1).

(1) G has a unique minimal normal subgroup L contained in N such that $G/L \in \mathcal{F}$ and $L \nleq \Phi(G)$.

(2) L is an elementary abelian p-group for some prime p.

Let q be the smallest prime divisor of $|N|$, Q a Sylow q-subgroup of N. If $Q \cap L \nleq \Phi(Q)$, then there exists a maximal subgroup Q_1 of Q such that $Q = (Q \cap L)Q_1$. By the hypotheses, there are a subnormal T of G and a CAP-embedded subgroup $(Q_1)_{ce}$ of G such that $G = Q_1T$ and $Q_1 \cap T \leq (Q_1)_{ce} \in Syl_q(K)$, where K is a CAP subgroup of G. If K covers $L/1$, then $L \leq K$. It follows from $(Q_1)_{ce} \in Syl_q(K)$ that $L \cap Q_1 = L \cap Q$, thus $Q = (Q \cap L)Q_1 = (Q_1 \cap L)Q_1 = Q_1$, a contradiction. So K must avoids $L/1$, i.e., $K \cap L = 1$, hence $Q_1 \cap T \cap L = 1$. Consequently, $|T \cap L| \leq q$. Noting that G/T_G is q-group, so $N \cap T_G \neq 1$. If not, then $G = G/N \cap T_G \lesssim G/N \times G/T_G$ belongs to $\mathcal{F}$, a contradiction. So $L \leq N \cap T_G$ by (1). Hence $|L| = |T \cap L| = q$. By applying Lemma 2.6, we obtain $G \in \mathcal{F}$, a contradiction. Therefore, $Q \cap L \leq \Phi(Q)$, then L is q-nilpotent by Tate's theorem [9, IV, Th 4.7] and, by the Odd Order Theorem, L is solvable, statement (2) is true.

(3) A final contradiction.

From (1) and (2), there exists a maximal subgroup M of G such that $G = LM$ and $L \cap M = 1$. Let P be a Sylow p-subgroup of N. Then $P = LM_p$ where $M_p \in Syl_p(G)$. Since $M_p < P$, we may let P_1 be a maximal subgroup of P such that $M_p \leq P_1$. By the hypotheses, there are a subnormal T of G and a CAP-embedded subgroup $(P_1)_{ce}$ of G such that $G = P_1T$ and $P_1 \cap T \leq (P_1)_{ce} \in Syl_p(K)$, where K is a CAP subgroup of G. If K covers $L/1$, then $L \leq K$. It follows from $(P_1)_{ce} \in Syl_p(K)$ that $L \leq P_1$, thus $P = LM_p \leq P_1$, a contradiction. So K must avoids $L/1$, i.e., $K \cap L = 1$, hence $P_1 \cap T \cap L = 1$. Consequently, $|T \cap L| \leq p$. Noting that G/T_G is p-group, so $N \cap T_G \neq 1$. If not, then $G = G/N \cap T_G \lesssim G/N \times G/T_G$

belongs to $\mathcal{F}$, a contradiction. So $L \leq N \cap T_G$ by (1). Hence $|L| = |T \cap L| = q$. By applying Lemma 2.6, we obtain $G \in \mathcal{F}$, final contradiction. We are done. ∎

Theorem 3.6. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let N be a solvable normal subgroup of G such that $G/N \in \mathcal{F}$. If all maximal subgroups of any Sylow subgroup of $F(N)$ are nearly CAP-embedded subgroups of G, then $G \in \mathcal{F}$.*

Proof. Assume that the result is false and let G be a counterexample of minimal order. First we have $\Phi(G) = 1$. Suppose that $\Phi(G) \neq 1$ and take a prime p dividing $|\Phi(G)|$. Denote $D = O_p(\Phi(G)) \neq 1$. Clearly $D \trianglelefteq G$. Let $F(ND/D) = L/D$. By L/D *Char* $ND/D \trianglelefteq G/D$, $L/D \trianglelefteq G/D$. Hence $L \trianglelefteq G$. Since L/D is a normal nilpotent subgroup of G/D and $D \leq \Phi(G)$, applying a result due to Gaschütz[9, III, Theorem 3.5] , we have that L is a normal nilpotent subgroup of ND. Thus $L \leq F(ND)$. Consequently $F(ND/D) = F(ND)/D = L/D$. By [2, Lemma 3.1], $F(ND/D) = F(N)D/D$. It is clear that $(G/D)/(ND/D) \cong G/ND \cong (G/N)/(ND/N)$ belongs to $\mathcal{F}$. Now, by Lemma 2.8(1), the hypotheses of the theorem hold in G/D. By the minimality of G, $G/D \in \mathcal{F}$. Since $\mathcal{F}$ is saturated, $G \in \mathcal{F}$, a contradiction. We obtain $\Phi(N) \leq \Phi(G) = 1$. Let M be a maximal subgroup of G such that $F(N) \nleq M$. Then there exists a prime p such that $O_p(N) \nleq M$. It follows that $G = O_p(N)M$. Clearly, $O_p(N) \cap M < O_p(N)$, so we may take a maximal subgroup P_1 of $O_p(N)$ containing $O_p(N) \cap M$. Then $P_1 \cap M = O_p(N) \cap M \trianglelefteq G$, therefore $P_1 \cap M \leq (P_1)_G$. If $(P_1)_G M = G$, then $O_p(N) = O_p(N) \cap (P_1)_G M = (P_1)_G(O_p(N) \cap M) = (P_1)_G$, a contradiction. Thus $(P_1)_G M < G$, so $(P_1)_G \leq O_p(N) \cap M$ and $P_1 \cap M = O_p(N) \cap M = (P_1)_G$. Let $O_p(N)/K$ be a chief factor of G with $O_p(N) \cap M \leq K$. Then $O_p(N) \cap M = K \cap M$. If $KM = G$, then $O_p(N) = O_p(N) \cap KM = K(O_p(N) \cap M) = K$, a contradiction. Thus $KM < G$, so $K \leq M$ and $K = O_p(N) \cap M = (P_1)_G$. Since P_1 is a nearly CAP-embedded subgroup of G, there are a subnormal T of G and a CAP-embedded subgroup $(P_1)_{ce}$ of G such that $G = P_1 T$ and $P_1 \cap T \leq (P_1)_{ce} \in Syl_p(B)$, where B is a CAP subgroup of G. Clearly $(P_1)_G(O_p(N) \cap T)$ is normal in G. From the fact that $O_p(N)/(P_1)_G$ is a G-chief factor, we know that either $(P_1)_G = (P_1)_G(O_p(N) \cap T)$ or $(P_1)_G(O_p(N) \cap T) = O_p(N)$. If the former holds, then $O_p(N) \cap T \leq (P_1)_G$. Furthermore, $O_p(N) \cap T = P_1 \cap T$ and $O_p(N) = P_1$ as $P_1 T = O_p(N)T = G$, a contradiction. So $(P_1)_G(O_p(N) \cap T) = O_p(N)$, we obtain $O_p(N) \leq (P_1)_G T$. Thus $G = P_1 T = (P_1)_G T$. Noting that B is a CAP subgroup of G. If B covers $O_p(N)/(P_1)_G$, then $O_p(N) \leq B(P_1)_G$. It follows from $(P_1)_{ce} \in Syl_p(B)$ that $O_p(N) \leq P_1$, a contradiction. So B must avoids $O_p(N)/(P_1)_G$, i.e., $(P_1)_{ce} = B \cap O_p(N) = B \cap (P_1)_G$, hence $(P_1)_{ce} \leq (P_1)_G$. Consequently $(P_1)_G \cap T = P_1 \cap T$, we have $P_1 = (P_1)_G = O_p(N) \cap M$. Therefore $|G : M| = |O_p(N) : O_p(N) \cap M| = p$. By Lemma 2.7, we get $G \in \mathcal{F}$, a final contradiction. ∎

Remark 3.7. The hypothesis that N is solvable in Theorem 3.6 cannot be removed. For example, if we let $G = SL(2, 5)$ and $N = G$, then $F(N)$ is a group of order 2. Thus all maximal subgroups of any Sylow subgroup of $F(N)$ have the nearly CAP-embedded property in G, but G is not supersolvable.

4. Some applications

Since many relevant families of subgroups, such as normal subgroups, c-normal subgroups, CAP subgroups, CAP-embedded subgroups and $c^\sharp$-normal subgroups, enjoy the nearly CAP-embedded property, a lot of nice results can be obtained according to our theorems.

Recall first the concept of $c^\sharp$-normal subgroups mentioned above. Let H be a subgroup of G. We call H a $c^\sharp$-normal subgroup of G if there exists a normal subgroup T of G such that $G = HT$ and $H \cap T$ is a CAP subgroup of G (see[16]).

Now, we here list special cases of our theorems which can be found in the literature.

Theorem 3.1 immediately implies:

Corollary 4.1. ([7, Theorem 3.1]) *Let p be a prime dividing the order of the group G with $(|G|, p-1) = 1$ and let H be a normal subgroup of G such that G/H is p-nilpotent. If there exists a Sylow p-subgroup P of H such that P is cyclic or every maximal subgroup of P is CAP-embedded in G, then G is p-nilpotent.*

Proof. If P is a cyclic group, by [9, p. 420, Theorem 2.8], we have G is p-nilpotent. So every maximal subgroup of P has the CAP-embedded property in G. Hence G is p-nilpotent by Theorem 3.1. ∎

Corollary 4.2. ([8, Theorem 3.4]) *Let p be the smallest prime number dividing the order of a group G and P a Sylow p-subgroup of G. If every maximal subgroup of P is c-normal in G, then G is p-nilpotent.*

Corollary 4.3. ([16, Theorem 3.1]) *Let G be a group, H a normal subgroup of G such that G/H is p-nilpotent and P a Sylow p-subgroup of H, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If all maximal subgroups of P are $c^\sharp$-normal in G, then G is p-nilpotent. In particular, G is p-supersolvable.*

From Theorem 3.2 we obtain:

Corollary 4.4. ([7, Theorem 4.1]) *Let p be a prime dividing the order of the group G and let H be a p-solvable normal subgroup of G such that G/H is p-supersolvable. If there exists a Sylow p-subgroup P of H such that every maximal subgroup of P is CAP-embedded in G, then G is p-supersolvable.*

Corollary 4.5. ([16, Theorem 3.4]) *Let G be a p-solvable group, H a normal subgroup of G such that G/H is p-supersolvable and P a Sylow p-subgroup of H, where p is a prime. If all maximal subgroups of P are $c^\sharp$-normal in G, then G is p-supersolvable.*

By Theorem 3.5 we have:

Corollary 4.6. ([13, Theorem 1]) *If the maximal subgroups of the Sylow subgroups of G are normal in G, then G is supersoluble.*

Corollary 4.7. ([11, Theorem 3.5]) *Assume that G/H is supersolvable and all maximal subgroups of the Sylow subgroups of H are normal in G. Then G is supersolvable.*

Corollary 4.8. ([15, Theorem 4.1]) *If the maximal subgroups of the Sylow subgroups of G are c-normal in G, then G is supersoluble.*

Corollary 4.9. ([16, Theorem 4.1]) *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathcal{F}$. If all maximal subgroups of any Sylow subgroup of H are $c^{\sharp}$-normal in G, then $G \in \mathcal{F}$.*

As immediate corollaries of Theorem 3.6, we have the following:

Corollary 4.10. ([11, Theorem 3.1]) *Assume that G is solvable and every maximal subgroup of the Sylow subgroups of $F(G)$ is normal in G. Then G is supersolvable.*

Corollary 4.11. [7, Theorem 4.3] *Let G be a group. Then G is supersolvable if and only if there exists a solvable normal subgroup H such that G/H is supersolvable and all maximal subgroups of any Sylow subgroup of $F(H)$ have the CAP-embedded property in G.*

Corollary 4.12. ([10, Theorem 2]) *Let G be a group and E a soluble normal subgroup of G such that G/E is supersoluble. If all maximal subgroups of the Sylow subgroups of $F(E)$ are c-normal in G, then G is supersoluble.*

Corollary 4.13. [1, Theorem 4.4] *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a solvable group with a normal subgroup H such that $G/H \in \mathcal{F}$. If all maximal subgroups of all Sylow subgroups of $F(H)$ are CAP-subgroups of G, then $G \in \mathcal{F}$*

Corollary 4.14. ([18, Theorem 1]) *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a soluble normal subgroup H such that $G/H \in \mathcal{F}$. If all maximal subgroups of all Sylow subgroups of $F(H)$ are c-normal in G, then $G \in \mathcal{F}$.*

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References

[1] Asaad, M., *On maximal subgroups of Sylow subgroups of finite groups*, Comm. Algebra, 26 (11) (1998), 3647-3652.

[2] Asaad, M., Ramadan, M., Shaalan, A., *Influence of π-quasinormality on maximal subgroups of Sylow subgroups of Fitting subgroups of a finite group*, Arch. Math., 56 (1991), 521-527.

[3] Doerk, K., Hawkes, T., *Finite Solvable Groups*, Walter de Gruyter, Berlin, New York, 1992.

[4] Ezquerro, L.M., *A contribution to the theory of finite supersolvable groups*, Rend. Semin. Mat. Univ. Padova, 89 (1993), 161-170.

[5] Gaschütz, W., *Praefrattini gruppen*, Arch. Math., 13 (1962), 418-426.

[6] Gillam, J.D., *Cover-avoid subgroups in finite solvable groups*, J. Algebra, 29 (1974), 324-329.

[7] Guo, P.F., Guo, X.Y., *On CAP-Embedded Subgroups in Finite Groups*, Journal of Mathematical Research & Exposition, 29 (6) (2009), 992-998.

[8] Guo, X.Y., Shum, K.P., *On c-normal maximal and minimal subgroups of Sylow p-subgroups of finite groups*, Arch. Math., 80 (2003), 561-569.

[9] Huppert, B., *Endliche Gruppen* I, Springer, New York, Berlin, 1967.

[10] Li, D.Y., Guo, X.Y., *The influence of c-normallity of subgroups on the structure of finite groups*, II. Comm. Algebra, 26 (1998), 1913-1922.

[11] Ramadan, M., *Influence of normality on maximal subgroups of Sylow subgroups of finite groups*, Acta Math. Hungar, 73 (1996), 335-342.

[12] Robinson, D.J.S., *A Course in the Theory of Groups*, Springer-Verlag, New York, 1993.

[13] Srinivasan, S., *Two sufficient conditions for supersolvability of finite groups*, Israel J. Math., 35 (3) (1980), 210-214.

[14] Tomkinson, M.J., *Cover-avoidance properties in finite soluble groups*, Can. Math. Bull., 19 (2) (1976), 213-216.

[15] Wang, Y.M., *c-normality of groups and its properties*, J. Algebra, 180 (1996), 954-965.

[16] Wang, Y.M., Wei, H.Q., *$c^\sharp$-Normality of Groups and its Properties*, Algebras and Representation Theory, 16 (1) (2013), 193-204.

[17] Wang, Y.M., Wei, H.Q. Li, Y.M., *A generalization of a theorem of kramer and its appications*, Bull. Austral. Math. Soc., 65 (2002), 467-475.

[18] Wei, H.Q., *On c-normal maximal and minimal subgroups of Sylow subgroups of finite groups*, Comm. Algebra, 29 (2001), 2193-2200.

[19] Wei, H.Q., Wang, Y.M., *On CAS-subgroups of finite groups*, Israel J. Math., 159 (2007), 175-188.

[20] Xu, Y., Zhao, T., Li, X.H., *On CISE-normal subgroups of finite groups*, Turish J. Math., 36 (2012), 231-243.

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SOURCE TERM IDENTIFICATION IN SEMIDIFFERENTIAL EQUATIONS[1]

Mohammad F. Al-Jamal[2]
E.A. Rawashdeh

Department of Mathematics
Yarmouk University
Irbid 21163
Jordan
e-mails: mfaljamal@yu.edu.jo
edris@yu.edu.jo

Abstract. In this paper we propose a numerical method for the source term identification in semidifferential equations form noisy data. Our method employ a mollification technique to stabilize (regularize) the inverse solution. We prove convergence results for both the continuous and discretized problems. Numerical examples are provided to validate the effectiveness of the proposed approach.

Keywords: inverse problems, regularization, source term, mollification, semidifferential, Bagley-Torvik.
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1. Introduction

Let a, b, c be constants, and $\alpha \in (0, 2)$. Consider the semidifferential equation

$$aD^2y(t) + bD_*^\alpha y(t) + cy(t) = f(t), \;\; 0 < t < 1, \tag{1.1}$$

subject to the initial conditions

$$y(0) = y_0, \;\; y'(0) = y'_0, \tag{1.2}$$

where D^2 stands for the second derivative operator, while D_*^α denotes the Caputo fractional differential operator of order α defined by

$$D_*^\alpha y(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} D^n y(s)ds, \;\; n-1 < \alpha < n, \;\; n \in \mathbb{N}^+,$$

which reduces to the ordinary derivative when α is a positive integer; see [1]–[4] for more about fractional calculus and its applications.

Recently, the initial value problem (1.1)–(1.2) has found many applications in physics and mechanics. For instance, when $\alpha = 0.5$ it models the motion of a single degree-of-freedom spring-mass-damper system where in this case a, b, and

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[2]Corresponding author

c represent the mass, damping coefficient, and stiffness, respectively, and $f(t)$ is the externally applied force. For $\alpha = 1.5$, (1.1) reduces to the Bagley-Torvik equation which models, for example, the motion of a rigid thin-plate immersed in a Newtonian fluid, where a is the mass, c is the stiffness, b is a constant related to the area of the plate and fluid viscosity, and f is the loading force. For the numerical and analytical treatment of (1.1) and general semidifferential equations, we refer the reader to [5]–[7] and references therein.

In many practical applications, it is desirable to determine the source term f (loading force) from noisy observations y^ϵ of the displacement y. Therefore, we propose the important

Inverse Problem. *Estimate f from noisy observation y^ϵ satisfying $\|y^\epsilon - y\|_\infty \leq \epsilon$.* This problem is ill-posed as it can be demonstrated by the following example.

Example 1.1. Let $b = c = 0$ in (1.1), and take $y^\epsilon(t) = y(t) + \epsilon \sin(\epsilon^{-1}t)$. Then

$$\|y^\epsilon - y\|_\infty \leq \epsilon \to 0, \text{ while } \|f^\epsilon - f\|_\infty \leq a/\epsilon \to \infty,$$

as $\epsilon \to 0$, showing the instability behavior of the inverse problem.

A consequence of the ill-posedness of the inverse problem is that standard analytical and numerical methods will fail to produce stable solutions regardless of how small the perturbations in the data is. Therefore, to tackle the instability issue, one needs to regularize the problem using, for instance, the Tikhonov regularization; see Engl et al. [8] for more about the theory of regularization and ill-posed problems.

To overcome the instability issue indicated above, we employ the mollification method to smooth out the given noisy data, then we compute the solution using equation (1.1). More precisely, let $y^{\epsilon,\delta}$ denotes the mollified (smoothed) data, then the source term f is approximated by the function

$$f^{\epsilon,\delta}(t) = aD^2 y^{\epsilon,\delta}(t) + bD_*^\alpha y^{\epsilon,\delta}(t) + cy^{\epsilon,\delta}(t),$$

where $\delta > 0$ is a parameter that controls the degree of smoothing. In practice only discrete data is available and therefore equation (1.1) must be discretized using appropriate finite-difference schemes. We will prove convergence results and provide error bounds for both the continuous and discretized problems.

The rest of this article proceeds as follows. In Section we introduce some definitions and preliminary results, in Section we present the mollification approach for the proposed inverse problem and prove the main results, numerical examples are given in Section .

2. Mollification technique

In this section we give a short review of mollification theory and some auxiliary results related to the source term inverse problem. Then we introduce a regularization scheme based on the mollification method for handling the source term identification problem. We prove convergence results for both the continuous and discretized schemes.

2.1. Preliminaries

We use the notation $\|g\|_{\infty,K}$ to denote the uniform norm of the function g over the set K.

Let $\delta > 0$ and define $A = \int_{-3}^{3} \exp(-s^2)\,ds$. The δ-mollifier, denoted by ρ_δ, is defined by

$$\rho_\delta(t) = \frac{1}{A\delta}\begin{cases} \exp\left(-t^2/\delta^2\right), & |t| \leq 3\delta, \\ 0, & |t| > 3\delta. \end{cases}$$

The function ρ_δ is nonnegative, $C^\infty(-3\delta, 3\delta)$ satisfies the normalization property

$$\int_{t-3\delta}^{t+3\delta} \rho_\delta(t-s)ds = \int_{-3\delta}^{3\delta} \rho_\delta(s)ds = 1.$$

Let $I = [0,1]$ and set $I_\delta = [3\delta, 1-3\delta]$ for $\delta < 1/6$, we have the following definition.

Definition 2.1. The δ-mollification of a function $g \in L^1(I)$ on I_δ, denoted by $J_\delta g$, is given by

$$(J_\delta g)(t) = \int_{-3\delta}^{3\delta} \rho_\delta(s)g(t-s)ds = \int_{t-3\delta}^{t+3\delta} \rho_\delta(t-s)g(s)ds, \qquad t \in I_\delta.$$

Figure 1 shows the δ-mollification of the function $g(t) = |t-0.5|$ for various values of δ. It is evident that δ represents a smoothing parameter; the larger the value of δ the more smoothing effect.

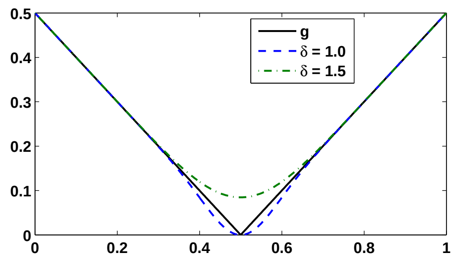


Figure 1: The function $g(t)=|x-0.5|$ and its δ-mollification for several values of δ.

The following result shows the convergence behavior for the δ-mollification.

Lemma 2.1. *Let $\|g^\epsilon - g\|_{\infty,I} \leq \epsilon$. There exists a constant C independent of δ and ϵ such that:*

(a) *If $g \in C(I)$, then $\|J_\delta g^\epsilon - g\|_{\infty,I_\delta} \leq C(\delta + \epsilon)$.*

(b) *If $g \in C^2(I)$, then $\|D_*^\alpha(J_\delta g^\epsilon) - D_*^\alpha g\|_{\infty,I_\delta} \leq C\left(\delta + \frac{\epsilon}{\delta^r}\right)$, where $r = 1$ if $\alpha \in (0,1]$, and $r = 2$ if $\alpha \in (1,2]$.*

Proof. cf. [9], [10]. ∎

The definition of the δ-mollification can be extended for discretized functions. To this end, let $K : 0 \leq t_1 < t_2 < \cdots < t_n \leq 1$ be uniform partition of $[0,1]$ and $G = \{G_1, G_2, \ldots, G_n\}$ be some discrete data.

Definition 2.2. The discrete δ-mollification of the data G is defined by

$$(J_\delta G)(t) = \sum_{i=1}^{n} \left(\int_{s_{i-1}}^{s_i} \rho_\delta(t-s)ds \right) G_i,$$

where $s_0 = 0$, $s_n = 1$, and $s_i = (t_i + t_{i+1})/2$ for $i = 1, \dots, n-1$.

Let $G^\epsilon = \{G_1^\epsilon, \dots, G_n^\epsilon\}$ be a perturbed version of the data $G = \{G_i = g(t_i) \mid t_i \in K\}$ satisfying $\|G^\epsilon - G\|_\infty \le \epsilon$, and set $\Delta t = 1/(n-1)$. We have the following results [9], [10]:

Lemma 2.2.
(a) *If $g \in C(I)$, then there exists a constant C independent of ϵ and δ such that*

$$\|J_\delta G^\epsilon - g\|_{\infty, I_\delta} \le C\left(\epsilon + \delta + \Delta t\right).$$

(b) *If $g \in C^2(I)$, then there exists a constant C independent of ϵ and δ such that*

$$\|D_*^\alpha \left(J_\delta G^\epsilon\right) - D_*^\alpha g\|_{\infty, I_\delta} \le C\left(\delta + \frac{\epsilon}{\delta^r} + \Delta t\right).$$

where $r = 1$ if $\alpha \in (0,1)$ and $r = 2$ if $\alpha \in (1,2)$. Moreover, there exists a constant C_δ independent of ϵ and Δt such that

$$\|\boldsymbol{D}^2(J_\delta G^\epsilon) - g''\|_{\infty, I_\delta} \le C\left(\delta + \frac{\epsilon + \Delta t}{\delta^2}\right) + C_\delta \left(\Delta t\right)^2,$$

where $\boldsymbol{D}^2$ denotes the centered difference approximation for the second derivative.

2.2. Error analysis

As indicated in the introduction of this paper, our strategy for stabilizing the source term problem is to replace the noisy data y^ϵ and its discrete version Y^ϵ by their mollifications $y^{\epsilon,\delta} = J_\delta y^\epsilon$ and $Y^{\epsilon,\delta} = J_\delta Y^\epsilon$. Then the continuous and discrete approximations to the source term f, denoted by $f^{\epsilon,\delta}$ and $F^{\epsilon,\delta}$, are computed respectively as

$$f^{\epsilon,\delta}(t) = aD^2 y^{\epsilon,\delta}(t) + bD_*^\alpha y^{\epsilon,\delta}(t) + cy^{\epsilon,\delta}(t), \quad F^{\epsilon,\delta} = a\mathbf{D}^2 Y^{\epsilon,\delta} + bD_*^\alpha Y^{\epsilon,\delta} + cY^{\epsilon,\delta}.$$

Assume $\|y^\epsilon - y\|_{\infty, I} \le \epsilon$, then we have the following main result:

Theorem 2.1. *If $y \in C^2(I)$, then there exists constant C independent of ϵ and δ such that*

$$\|f^{\epsilon,\delta} - f\|_{\infty, I_\delta} \le C\left(\delta + \epsilon + \frac{\epsilon}{\delta^2}\right),$$

$$\|F^{\epsilon,\delta} - f\|_{\infty, I_\delta} \le C\left(\delta + \epsilon + \Delta t + \frac{\epsilon + \Delta t}{\delta^2}\right) + C_\delta(\Delta t)^2.$$

Proof. Using Lemma 2.1 and the Triangle inequality, we have

$$\begin{aligned} \|f^{\epsilon,\delta}-f\|_{\infty,I_\delta} &\leq |a| \left\|D^2y^{\epsilon,\delta}-D^2y\right\|_{\infty,I_\delta} + |b| \left\|D^\alpha y^{\epsilon,\delta}-D^\alpha y\right\|_{\infty,I_\delta} + |c| \left\|y^{\epsilon,\delta}-y\right\|_{\infty,I_\delta} \\ &\leq C\left(\delta+\epsilon+\frac{\epsilon}{\delta^2}\right). \end{aligned}$$

The proof of the second bound is similar but using Lemma 2.2 instead. ■

Remark 2.1.
(i) If δ is chosen so that $\delta = O(\epsilon^\mu)$ for some $0 < \mu < 0.5$, then we obtain the convergence result $\|f - f^{\epsilon,\delta}\|_{\infty,I_\delta} \to 0$, as $\epsilon \to 0$.
(ii) If δ is chosen so that $\delta = O\left((\epsilon + \Delta t)^\mu\right)$ for some $0 < \mu < 0.5$, then we obtain the convergence result $\|F - F^{\epsilon,\delta}\|_{\infty,I_\delta} \to 0$, as $\epsilon, \Delta t \to 0$.

3. Numerical experiments

In this section, we present numerical examples to test the feasibility and validity of the proposed algorithm.

In the experiments below, we take $\Delta t = 0.01$. The noisy data Y^ϵ is computed according to the formula

$$Y_i^\epsilon = y(t_i) + \gamma u_i, \quad i = 1, \ldots, 101,$$

where u_i is a uniformly distributed random number in $[-1, 1]$. Here the number γ determines the (percentage) noise level ϵ which is traditionally defined as

$$\epsilon = \frac{\|Y^\epsilon - Y\|}{\|Y\|} \times 100\%,$$

where $\|\cdot\|$ is the usual Euclidean norm. To assess the quality of the approximations, we use the relative root-mean-square error (RES) given by

$$\text{RES} = \frac{\sqrt{\sum_{i=1}^{101}\left[f(t_i) - f^{\delta,\epsilon}(t_i)\right]^2}}{\sqrt{\sum_{i=1}^{101}\left[f(t_i)\right]^2}},$$

which is the discrete version of the relative L^2-error. The mollification parameter δ is determined by the Principle of Generalized Cross Validation as described in [11], where as the discretization of the Caputo fractional derivative is computed using the algorithm described in [9].

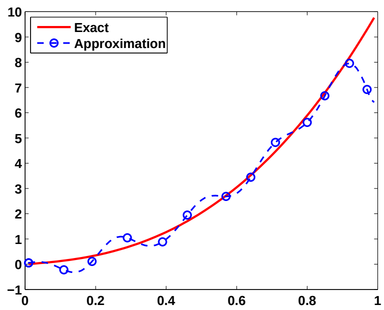
Example 3.2. Consider the semidifferential equation

$$D^2y(t) + 15\sqrt{\pi}D_*^{1/2}y(t) + 6y(t) = t^3 + t + 8t^{5/2}, \;\; y(0) = 0, \; y'(0) = 0.$$

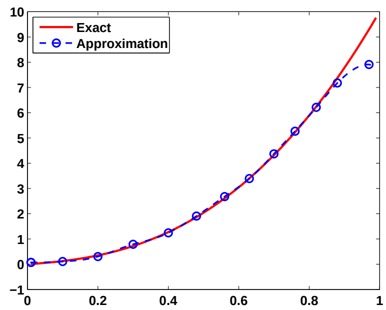
Here the solution is $y(t) = t^3/6$. The RES computed for different noise levels are presented in Table 1. Plots comparing the exact and approximated source term are shown in Figure 2.

Table 1: Errors measured by the RES for Example 3.2.

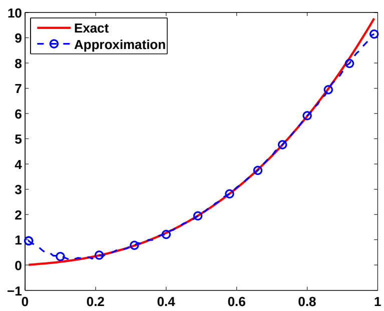
Noise Level	0.1	0.02	0.004	0.0008
RES	0.1509	0.0902	0.0526	0.0098



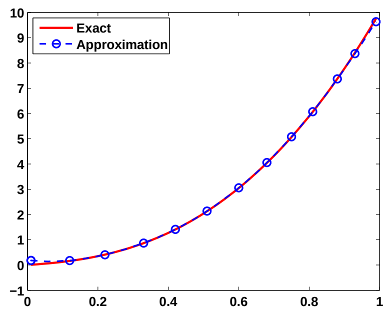
(a) $\epsilon = 0.1$



(b) $\epsilon = 0.02$



(c) $\epsilon = 0.004$



(d) $\epsilon = 0.0008$

Figure 2: Exact and approximated source term f in Example 3.2 for different noise levels.

Example 3.3. Consider the semidifferential equation

$$D^2 y(t) + 5D_*^{3/2} y(t) + 3y(t) = f(t), \;\; y(0) = -15/16, \; y'(0) = -1/2.$$

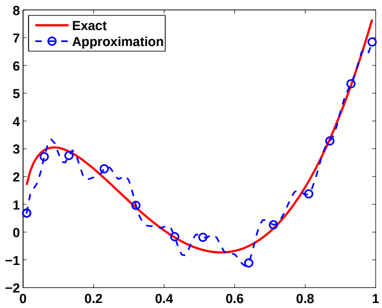
Here the solution is $y(t) = \left(t - \frac{1}{2}\right)^4 - 1$, whereas the exact source term is

$$f(t) = 3t^4 - 6t^3 + \frac{33t^2}{2} - \frac{27t}{2} + \frac{30\sqrt{t}}{\sqrt{\pi}} - \frac{80t^{3/2}}{\sqrt{\pi}} + \frac{64t^{5/2}}{\sqrt{\pi}} + \frac{3}{16}.$$

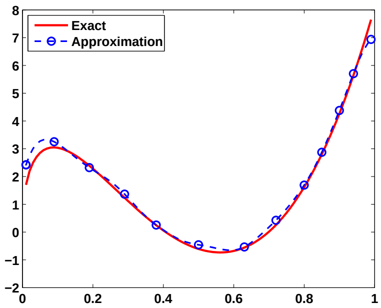
The RES computed for different noise levels are presented in Table 2. Plots comparing the exact and approximated source term are shown in Figure 3.

Table 2: Errors measured by the RES for Example 3.3.

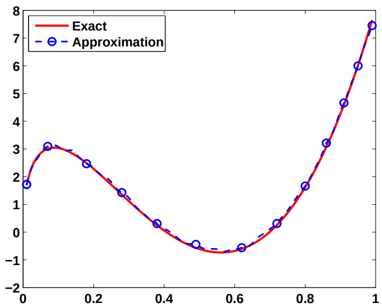
Noise Level	0.1	0.02	0.004	0.0008
RES	0.1617	0.0735	0.0291	0.0256



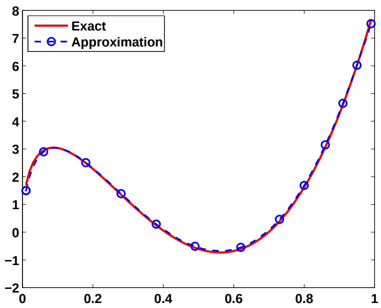
(a) $\epsilon = 0.1$



(b) $\epsilon = 0.02$



(c) $\epsilon = 0.004$



(d) $\epsilon = 0.0008$

Figure 3: Exact and approximated source term f in Example 3.3 for different noise levels.

4. Conclusions and future work

We investigated the possibility of recovering the source term in semidifferential equations from noisy observed data. We proposed a regularization scheme based on the mollification method. We provided convergence results for both the continuous and discrete data. Our numerical experiments showed noteworthy results. We look forward to generalize our approach for systems of fractional differential equations; we defer these investigations for a sequel to this paper.

References

[1] Podlubny, I., *Fractional Differential Equations*, Academic Press, San Diego, USA, 1991.

[2] Diethelm, K., *The Analysis of Fractional Differential Equations*, Springer, Berlin, Germany, 2010.

[3] Kilbas, A.A., Srivastava, H.M., Trujillo, J.J., *Theory and Applications of Fractional Differential Equations*, Elsevier Science Inc, New York, USA, 2006.

[4] Uchaikin, V.V., *Fractional Derivatives for Physicists and Engineers: Background and Theory*, Springer, Berlin, Germany, 2013.

[5] Hamarsheh, M.H., Rawashdeh, E.A., *A numerical method for solution of semi differential equations*, Matematicki Vesnik., 62 (2010), 117–126.

[6] Diethelm, K., Ford, N.J., *Numerical solution of the Bagley-Torvik equation*, BIT, 42 (2002), 490–507.

[7] Yuan, L., Agrawal, O.P., *A numerical scheme for dynamic systems containing fractional derivatives*, J. Vibration Acoustics, 124 (2002), 321–324.

[8] Engl, H.W., Hanke, M., Neubauer, A., *Regularization of Inverse Problems*, Kluwer Academic, Dordrecht, Netherlands, 1996.

[9] Murio, D.A., *On the stable numerical evaluation of Caputo fractional derivatives*, Comput. Math. Appl., 51 (2006), 1539–1550.

[10] Murio, D.A., Mejla, C.E., Zhan, S., *Discrete mollification and automatic numerical differentiation*, Comput. Math. Appl., 35 (1998), 1–16.

[11] Murio, D.A., *Mollification and space marching*, in: K. Woodbury (Ed.), *Inverse Engineering Handbook*, CRC Press, Florida, USA, 2003.

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REGULAR MULTIPLICATIVE TERNARY HYPERRING

Md. Salim
T.K. Dutta

Department of Pure Mathematics
University of Calcutta
35, Ballygunge Circular Road
Kolkata – 700019
India
e-mails: smpmath746@yahoo.in
duttatapankumar@yahoo.co.in

K.P. Shum

Institute of Mathematics
Yunnan University
Kunming, 650091
People Republic of China
e-mail: kpshum@ynu.edu.cn

Abstract. Regular multiplicative ternary hyperring are introduced and considered. Some properties of regular multiplicative ternary hyperring are studied. Several characterization theorems of the above ternary hyperrings in terms of its hyperideals are obtained. In addition, regular hyperideals in a multiplicative ternary hyperring are particularly considered and investigated. Finally, we explore the relationships between the regular multiplicative ternary hyperrings and the hyperideals of a multiplicative ternary hyperring.

Keywords and phrases: regular multiplicative ternary hyperring, idempotent hyperideal, regular hyperideal.
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1. Introduction

Algebraic structures play an important role in mathematics with wide applications in many disciplines such as theoretical physics, computer science, information science and coding theory etc. The theory of algebraic hyperstructures (or hypersystem) is a well established branch of classical algebraic theory. The theory of hyperstructure was introduced by F. Marty [6] in 1934. He first studied the hypergroups and analyzed their properties and then applied them to groups and rational algebraic functions. Nowadays there has been a remarkable growth of hyperstructure theory. Many researchers have observed that the theory of hyperstructure has many applications in pure and applied science. In [2], P. Corsini and V. Leoreanu have collected numerous applications of algebraic hyperstructures. The notion of multiplicative hyperring has been introduced by R. Rota [8] in which the addition is a binary operation and multiplication is a binary hyperoperation. M.A. Krasner also introduced the notion of hyperring, called Krasner hyperring [4]. In a Krasner hyperring $(R, +, \cdot)$, '$+$' is a binary hyperoperation and '$\cdot$' is a binary operation, in which the zero element is absorbing.

In 1971, W.G. Lister [5] introduced the notion of ternary ring and study some important properties of it. According to W.G. Lister [5], a ternary ring is an algebraic system consisting of a nonempty set R together with a binary operation, called addition and a ternary multiplication, which forms a commutative group relative to addition, a ternary semigroup relative to multiplication and left, right, lateral distributive laws hold.

In 2014, J.R. Castillo and Jocelyn S. Paradero-Vilela [1] have introduced a special kind of ternary hyperrings, called the Krasner ternary hyperring. In a Krasner ternary hyperring $(R, +, \cdot)$, '$+$' is a binary hyperoperation and '$\cdot$' is a ternary multiplication.

In [11], we have studied the multiplicative ternary hyperrings. Our notion of multiplicative ternary hyperring in this paper differs from the notion of Krasner multiplicative ternary hyperring. In our multiplicative ternary hyperring $(R, +, \circ)$, '$+$' is a binary operation and '$\circ$' is always a ternary hyper operation, in which the zero element is an absorbing zero (i.e., $0_R \circ x \circ y = x \circ 0_R \circ y = x \circ y \circ 0_R = \{0_R\}$ for all $x, y \in R$). We will introduce the notion of regular multiplicative ternary hyperring and to give some characterization theorems for such ternary hyperrings. We prove that if a multiplicative ternary hyperring R is regular, then for an hyperideal I of R, both I and R/I are regular.

Conversely, if R is a multiplicative ternary hyperring and if there exists an hyperideal I of R such that both I and R/I are regular, then R is regular. We also consider the idempotent hyperideals and prove that if a commutative multiplicative ternary hyperring R is regular then every hyperideal I of R is idempotent and conversely.

Finally, we consider the regular hyperideal and prove that a multiplicative ternary hyperring is regular if and only if $\{0\}$ is regular hyperideal of R.

2. Preliminaries

Definition 2.1. By a ternary hype roperation '$\circ$' on a nonempty set H, we mean a mapping $\circ$: $H \times H \times H \to P^*(H)$, where $P^*(H)$ is the set of all nonempty subsets of H. For $x, y, z \in H$, the image of the element $(x, y, z) \in H \times H \times H$ under the mapping '$\circ$' will be denoted by $x \circ y \circ z$ (which is called the ternary hyperproduct of x, y, z).

Definition 2.2. A multiplicative ternary hyperring $(R, +, \circ)$ is an additive commutative group $(R, +)$ endowed with a ternary hyperoperation '$\circ$' such that the following conditions hold :

(i) $(a \circ b \circ c) \circ d \circ e = a \circ (b \circ c \circ d) \circ e = a \circ b \circ (c \circ d \circ e)$;

(ii) $(a + b) \circ c \circ d \subseteq a \circ c \circ d + b \circ c \circ d$;
$a \circ (b + c) \circ d \subseteq a \circ b \circ d + a \circ c \circ d$;
$a \circ b \circ (c + d) \subseteq a \circ b \circ c + a \circ b \circ d$;

(iii) $(-a) \circ b \circ c = a \circ (-b) \circ c = a \circ b \circ (-c) = -(a \circ b \circ c)$ for all $a, b, c \in R$;

(iv) $0_R \circ x \circ y = x \circ 0_R \circ y = x \circ y \circ 0_R = \{0_R\}$ for all $x, y \in R$ (absorbing property of 0_R),

for all $a, b, c, d, e \in S$. We remark here that, if the inclusions in (ii) are replaced by equalities, then the multiplicative ternary hyperring is called a strongly distributive multiplicative ternary hyperring.

Definition 2.3. A multiplicative ternary hyperring $(R, +, \circ)$ is called commutative if $a_1 \circ a_2 \circ a_3 = a_{\sigma(1)} \circ a_{\sigma(2)} \circ a_{\sigma(3)}$, where σ is a permutation of $\{1, 2, 3\}$ for all $a_1, a_2, a_3 \in R$.

Definition 2.4. A nonempty finite subset $\varepsilon = \{(e_i, f_i)\}_{i=1}^{n}$ of $R \times R$ where R is a multiplicative ternary hyperring is called a left (resp. lateral, right) identity set of R if

$$\text{for any } a \in R,\ a \in \sum_{i=1}^{n} e_i \circ f_i \circ a \ \left(\text{resp. } a \in \sum_{i=1}^{n} e_i \circ a \circ f_i,\ a \in \sum_{i=1}^{n} a \circ e_i \circ f_i\right)$$

An element e of a multiplicative ternary hyperring $(R, +, \circ)$ is called a unital element of R if $a \in (e \circ e \circ a) \cap (e \circ a \circ e) \cap (a \circ e \circ e)$.

Definition 2.5. Let $(S, +, \circ)$ and $(S', +, \circ)$ be two multiplicative ternary hyperrings. Then a mapping$f : S \to S'$ is called a homomorphism(a good homomorphism) if $f(a + b) = f(a) + f(b)$ and $f(a \circ b \circ c) \subseteq f(a) \circ f(b) \circ f(c)$ (resp. $f(a \circ b \circ c) = f(a) \circ f(b) \circ f(c)$).

Definition 2.6. Let $(R, +, \circ)$ be a multiplicative ternary hyperring. An additive subgroup I of R is called

(i) a left hyperideal of R if $r, r' \in R, \Rightarrow r \circ r' \circ x \subseteq I$, for all $x \in I$;

(ii) a right hyperideal of R if $r, r' \in R, \Rightarrow x \circ r \circ r' \subseteq I$, for all $x \in I$;

(iii) a lateral hyperideal of R if $r, r' \in R, \Rightarrow r \circ x \circ r' \subseteq I$, for all $x \in I$;

(iv) a two sided hyperideal of R if I is both a left and a right hyperideal of R;

(v) an hyperideal of R if I is a left, a right, and a lateral hyperideal of R.

Based on the above definitions, we have the following resullts.

Remark 2.7. Let R be a multiplicative ternary hyperring. If I, J and K are three hyperideals of R, then

$$I \circ J \circ K = \cup \left\{ \sum_{finite} a_i \circ b_i \circ c_i : a_i \in I, b_i \in J, c_i \in K \right\}$$

is a hyperideal of R.

Remark 2.8. If $(R, +, \circ)$ is a multiplicative ternary hyperring and $a \in R$, then

$$\langle a \rangle_r = a \circ R \circ R + na$$

(where n is an integer) is the right hyperideal of R generated by a.

Similarly if $(R, +, \circ)$ is a multiplicative ternary hyperring and $a \in R$, then the following equations hold:

$$\langle a \rangle_m = R \circ a \circ R + R \circ R \circ a \circ R \circ R + na$$

(where n is an integer) is the lateral hyperideal of R generated by a, and

$$\langle a \rangle_l = R \circ R \circ a + na$$

(where n is an integer) is the left hyperideal of R generated by a.

Remark 2.9. If $(R, +, \circ)$ is a multiplicative ternary hyperring $a \in R$ with a unital element e, then the following equation holds:

$$\langle a \rangle_r = a \circ R \circ R$$

is the right hyperideal of R generated by a.

Similarly,

$$\langle a \rangle_m = R \circ a \circ R + R \circ R \circ a \circ R \circ R$$

is the lateral hyperideal of R generated by a. And

$$\langle a \rangle_l = R \circ R \circ a$$

is the left hyperideal of R generated by a.

3. Regular ternary hyperring

Definition 3.1. Let $(R, +, \circ)$ be a multiplicative ternary hyperring. Then, an element $a \in R$ is called a regular element if there exists an element $x \in R$ such that $a \in a \circ x \circ a$The multiplicative ternary hyperring $(R, +, \circ)$ is called regular if all of its elements are regular.

Example 3.2. [13] Let $(R, +, \cdot)$ be a regular ternary ring with a unital element e. Let A be a subset of R containing e. Now we define a multiplicative ternary hyperoperation '$\circ$' on R as follows:

$$a \circ b \circ c = \{a \cdot x \cdot b \cdot y \cdot c : x, y \in A\}.$$

Then $(R, +, \circ)$ is a multiplicative ternary hyperring. We denote this ring by $(R_A, +, \circ)$. Let $a \in R_A = R$. Since R is a regular ring, there exists an element $b \in R$ such that $aba = a$. Now, we derive that

$$a = a \cdot e \cdot b \cdot e \cdot a \in \{a \cdot x \cdot b \cdot y \cdot a : x, y \in A\} = a \circ b \circ a,$$

because $e \in A$. Thus, we have shown that $(R_A, +, \circ)$ is a regular multiplicative ternary hyperring.

Theorem 3.3. *Let f be a homomorphism from a regular multiplicative ternary hyperring R onto a multiplicative ternary hyperring T. Then T is a regular ternary hyperring. we observe that the homomorphic image of a regular multiplicative ternary hyperring is still a regular multiplicative ternary hyperring.*

Proof. The proof is obvious and hence we omit the proof. ∎

In the following proposition, we characterize the regular multiplicative ternary hyperrings.

Proposition 3.4. *A multiplicative ternary hyperring $(R, +, \circ)$ is regular if and only if for each $a \in R$, there exist $x_1, x_2, y_1, y_2, z_1, z_2 \in R$ such that*

$$a \in (a \circ x_1 \circ x_2) \circ (y_1 \circ a \circ y_2) \circ (z_1 \circ z_2 \circ a).$$

Proof. Let R be a regular multiplicative ternary hyperring and $a \in R$. Then there exists $x \in R$ such that $a \in a \circ x \circ a$ which implies that $a \in a \circ x \circ a \circ x \circ a \subseteq a \circ x \circ a \circ x \circ a \circ x \circ a \subseteq a \circ x \circ a \circ x \circ a \circ x \circ a \circ x \circ a$. Consequently, $a \in (a \circ x_1 \circ x_2) \circ (y_1 \circ a \circ y_2) \circ (z_1 \circ z_2 \circ a)$ for some $x_1, x_2, y_1, y_2, z_1, z_2 \in R$. Conversely, let a an arbitrary element of a multiplicative ternary hyperring $(R, +, \circ)$ and $a \in (a \circ x_1 \circ x_2) \circ (y_1 \circ a \circ y_2) \circ (z_1 \circ z_2 \circ a)$ for some $x_1, x_2, y_1, y_2, z_1, z_2 \in R$. This implies that $a \in a \circ (x_1 \circ x_2 \circ y_1 \circ a \circ y_2 \circ z_1 \circ z_2) \circ a \Rightarrow a \in a \circ s \circ a$ for some $s \in x_1 \circ x_2 \circ y_1 \circ a \circ y_2 \circ z_1 \circ z_2$. This proves that a and hence R is regular. ∎

It is noted that a left and a right hyperideal of a regular multiplicative ternary hyperring may not be regular; however, for a lateral hyperideal of a regular multiplicative ternary hyperring, we have the following results.

Proposition 3.5. *If I is a lateral hyperideal of a regular multiplicative ternary hyperring, then I is regular as a multiplicative ternary hyperring.*

Proof. Let I be a lateral hyperideal of a multiplicative ternary hyperring R. Let $a \in I$. Since R is regular, there exists $b \in R$ such that $a \in a \circ b \circ a \subseteq (a \circ b \circ a) \circ b \circ a = a \circ (b \circ a \circ b) \circ a$ where $b \circ a \circ b \subseteq I$. Thus I is regular. ∎

Corollary 3.6. *Every hyperideal of a regular multiplicative ternary hyperring is regular as a multiplicative ternary hyperring.*

Theorem 3.7. *If I and J are regular hyperideals of a multiplicative ternary hyperring $(R, +, \circ)$, then $I \cap J$ is regular as a multiplicative ternary hyperring.*

Proof. Obviously $I \cap J$ is an hyperideal of R. Let $a \in I \cap J$. Then there exist $x \in I$ and $y \in J$ such that $a \in a \circ x \circ a$ and $a \in a \circ y \circ a$. we hence deduce that $a \in a \circ x \circ a \subseteq (a \circ x \circ a) \circ x \circ (a \circ y \circ a) = a \circ (x \circ a \circ x \circ a \circ y) \circ a$. Now, $x \circ a \circ x \circ a \circ y \subseteq I \cap J$. Consequently, $I \cap J$ is regular as a multiplicative ternary hyperring. ■

Definition 3.8. [Proposition 3.8, [12]] Let $(R, +, \circ)$ be a multiplicative ternary hyperring and I be an hyperideal of R. Then the multiplicative ternary hyperring $R/I = \{a + I : a \in R\}$ is called a quotient multiplicative ternary hyperring of $(R, +, \circ)$ by I, where $(a+I)+(b+I) = (a+b)+I$ and $(a+I)\circ(b+I)\circ(c+I) = \{p + I : p \in a \circ b \circ c\}$ for any $a, b, c \in R$.

For the hyperideals in a multiplicative ternary hyperring, we have the following results.

Theorem 3.9. *Let R be a regular strongly distributive multiplicative ternary hyperring and I an hyperideal of R. Then I (as a multiplicative ternary hyperring) and the quotient multiplicative ternary hyperring R/I are regular. Conversely, if $(R, +, \circ)$ is a multiplicative ternary hyperring and if there exists an hyperideal I of R such that both I (as a multiplicative ternary hyperring) and R/I are regular, then R is regular.*

Proof. The first part of the theorem follows directly from Proposition 3.5 and Theorem 3.3, since R/I is a homomorphic image of R.

Conversely, suppose that R is a multiplicative ternary hyperring and there exists an hyperideal I of R such that both I and R/I are regular. Let $a \in R$. Then $a + I \in R/I$. Since R/I is regular, there exists an element $b + I \in R/I$ where $b \in R$, such that $a + I \in (a + I) \circ (b + I) \circ (a + I)$. Then $a + I = z + I$ for some $z \in a \circ b \circ a$. Since $a + I = z + I$, again $a - z \in I$. Since I is regular

$$\begin{aligned} a - z &\in (a - z) \circ y \circ (a - z) \text{ for some } y \in I \\ &= a \circ y \circ a - a \circ y \circ z - z \circ y \circ a + z \circ y \circ z \\ &\subseteq a \circ y \circ a - a \circ y \circ a \circ b \circ a - a \circ b \circ a \circ y \circ a + a \circ b \circ a \circ y \circ a \circ b \circ a \\ &= a \circ (y - y \circ a \circ b - b \circ a \circ y + b \circ a \circ y \circ a \circ b) \circ a \end{aligned}$$

i.e., $a - z \in a \circ s \circ a$ for some $s \in y - y \circ a \circ b - b \circ a \circ y + b \circ a \circ y \circ a \circ b$. Thus $a - z = t$ for some $t \in a \circ s \circ a$. Then $a = t + z \subseteq a \circ s \circ a + a \circ b \circ a = a \circ (s + b) \circ a$. So a and hence R is regular. ■

Theorem 3.10. [Theorem 3.18, [12]] *Let I and J be two hyperideals of a multiplicative ternary hyperring $(R, +, \circ)$. Then $I/(I \cap J) \cong (I + J)/J$.*

Theorem 3.11. *Let $(R, +, \circ)$ be a strongly distributive multiplicative ternary hyperring and I and J are hyperideals of R as multiplicative ternary hyperring. If I and J are regular, then $I + J$ is regular.*

Proof. Since I and J are regular, by Theorem 3.7 $I \cap J$ is regular. Again by Theorem 3.9, $I/(I \cap J)$ is regular. Since $(I + J)/J$ is a homomorphic image of $I/(I \cap J)$ and so $(I + J)/J$ is regular. Since J and $(I + J)/J$ are regular, by Theorem 3.9, the hyperideal $I + J$ is regular. ∎

In the following theorems, we further investigate the properties of the multiplicative ternary hyperrings.

Theorem 3.12. *Let $(R, +, \circ)$ be a multiplicative ternary hyperring. Then the following statements are equivalent:*

(i) *R is a regular multiplicative ternary hyperring;*

(ii) *For any right hyperideal I, lateral hyperideal J, and left hyperideal K of R, $I \circ J \circ K = I \cap J \cap K$;*

(iii) *For $a, b, c \in R$, $\langle a\rangle_r \circ \langle b\rangle_m \circ \langle c\rangle_l = \langle a\rangle_r \cap \langle b\rangle_m \cap \langle c\rangle_l$;*

(iv) *$\langle a\rangle_r \circ \langle a\rangle_m \circ \langle a\rangle_l = \langle a\rangle_r \cap \langle a\rangle_m \cap \langle a\rangle_l$, for each element $a \in R$.*

Proof. (i) $\Rightarrow$ (ii). Suppose that R is a regular multiplicative ternary hyperring. Let I, J and K be a right hyperideal, a lateral hyperideal and a left hyperideal of R respectively. Obviously $I \circ J \circ K \subseteq I \cap J \cap K$ (1). Now let $a \in I \cap J \cap K$. Then we have $a \in a \circ x \circ a$ for some $x \in R$. This implies that $a \in a \circ x \circ a \subseteq (a \circ x \circ a) \circ (x \circ a \circ x) \circ (a \circ x \circ a) \subseteq I \circ J \circ K$. Thus we have $I \cap J \cap K \subseteq I \circ J \circ K$ (2). From (1) and (2), it follows that $I \circ J \circ K = I \cap J \cap K$.

Clearly, (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv).

(iv) $\Rightarrow$ (i). Let $a \in R$. Then $a \in \langle a\rangle_r \cap \langle a\rangle_m \cap \langle a\rangle_l = \langle a\rangle_r \circ \langle a\rangle_m \circ \langle a\rangle_l = (a \circ R \circ R + na) \circ (R \circ a \circ R + R \circ R \circ a \circ R \circ R + na) \circ (R \circ R \circ a + na) \subseteq a \circ R \circ a$ i.e. $a \in a \circ y \circ a$ for some $y \in R$. Hence, we have proved that a and R are regular. ∎

Theorem 3.13. *The following conditions on a multiplicative ternary hyperring R are equivalent:*

(i) *R is regular,*

(ii) *$I \cap J = I \circ R \circ J$ for every right hyperideal I and every left hyperideal J of R,*

(iii) *For $a, b \in R$, $\langle a\rangle_r \cap \langle b\rangle_l = \langle a\rangle_r \circ R \circ \langle b\rangle_l$,*

(iv) *For $a \in R$, $\langle a\rangle_r \cap \langle a\rangle_l = \langle a\rangle_r \circ R \circ \langle a\rangle_l$.*

Proof. Since R is a lateral hyperideal of itself, by (ii) of Theorem 3.12, (i) $\Rightarrow$ (ii) follows.

Clearly, (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv).

(iv) $\Rightarrow$ (i). Let $a \in R$. Then $a \in \langle a\rangle_r \cap \langle a\rangle_l = \langle a\rangle_r \circ R \circ \langle a\rangle_l = (a \circ R \circ R + na) \circ R \circ (R \circ R \circ a + na) \subseteq a \circ R \circ a$. Thus $a \in a \circ R \circ a$. Thus there exists an element $x \in R$ such that $a \in a \circ x \circ a$. Hence, a and R are regular. ∎

Definition 3.14. An element a of a multiplicative ternary hyperring $(R, +, \circ)$ is called idempotent if $a \circ a \circ a = a$.

Definition 3.15. An hyperideal I of a multiplicative ternary hyperring $(R, +, \circ)$ is called idempotent if $I \circ I \circ I = I$.

In the following theorem, we characterize the idempotent multiplicative ternary hyperrings.

Theorem 3.16. *Let $(R, +, \circ)$ be a commutative multiplicative ternary hyperring. Then R is regular if and only if every hyperideal I of R is idempotent.*

Proof. Let R be a regular multiplicative ternary hyperring and I be any hyperideal of R. Then $I \circ I \circ I \subseteq R \circ R \circ I \subseteq I$. Let $a \in I$. Then there exists $x \in R$ such that $a \in a \circ x \circ a \subseteq (a \circ x \circ a) \circ x \circ a = a \circ (x \circ a \circ x) \circ a \subseteq I \circ I \circ I$. Thus $I \subseteq I \circ I \circ I$ and hence $I \circ I \circ I = I$, i.e., I is idempotent.

Conversely, suppose that every hyperideal of R is idempotent. Let I, J and K be three hyperideals of R. Obviously $I \circ J \circ K \subseteq I \cap J \cap K$. Again, we have $I \cap J \cap K = (I \cap J \cap K) \circ (I \cap J \cap K) \circ (I \cap J \cap K) \subseteq I \circ J \circ K$. Thus $I \cap J \cap K = I \circ J \circ K$. Therefore, by Theorem 3.12, we have proved that R is a regular multiplicative ternary hyperring. ∎

4. Regular hyperideal

Let $(R, +, \circ)$ be a regular multiplicative ternary hyperring. Then for any right hyperideal S, lateral hyperideal M and left hyperideal L we have $S \circ M \circ L = S \cap M \cap L$. Hence, we have $(0) + S \circ M \circ L = S \cap M \cap L$. Also for an hyperideal T of R, we have $T + S \circ M \circ L = S \cap M \cap L$ ....(1), where S is a right hyperideal containing T, M is a lateral hyperideal containing T and L is a left hyperideal containing T. Based on relation (1), we now give the following definition of a regular hyperideal.

Definition 4.1. An hyperideal I of a multiplicative ternary hyperring $(R, +, \circ)$ is called a regular hyperideal of R if $I + S \circ M \circ L = S \cap M \cap L$ for any right hyperideal $S \supseteq I$, lateral hyperideal $M \supseteq I$ and left hyperideal $L \supseteq I$.

Based on the above definition, we have the following propostions concerning the regular hyperideals of a multipllicative ternary hyperrings.

Proposition 4.2. *Let $(R, +, \circ)$ be a multiplicative ternary hyperring and I be a regular hyperideal of R. If T is an hyperideal of R containing the regular hyperideal I, then T is also regular.*

Proof. Let S be a right hyperideal containing T, M a lateral hyperideal containing T and L a left hyperideal containing T. Then $S \supset T \supset I$, $M \supset T \supset I$ and $S \supset T \supset I$. Since I is regular, $I + S \circ M \circ L = S \cap M \cap L$. Now, we deduce that $T + S \circ M \circ L \supseteq I + S \circ M \circ L = S \cap M \cap L$. Again, $T \subseteq S \cap M \cap L$ and $S \circ M \circ L \subseteq S \cap M \cap L$ and hence $T + S \circ M \circ L \subseteq S \cap M \cap L$. Thus $T + S \circ M \circ L = S \cap M \cap L$. Hence, we have shown that T is regular. ■

Proposition 4.3. *A multiplicative ternary hyperring $(R, +, \circ)$ is a regular multiplicative ternary hyperring if and only if (0) is a regular hyperideal of R.*

Proof. $(R, +, \circ)$ be a regular multiplicative ternary hyperring
$\Leftrightarrow S \circ M \circ L = S \cap M \cap L$ for any right hyperideal S, lateral hyperideal M and left hyperideal L of R (by Proposition 3.12)
$\Leftrightarrow (0) + S \circ M \circ L = S \cap M \cap L$ for any right hyperideal $(0) \subseteq S$, lateral hyperideal $(0) \subseteq M$ and left hyperideal $(0) \subseteq L$
$\Leftrightarrow (0)$ is a regular hyperideal of R. ■

Corollary 4.4. *Let $(R, +, \circ)$ be a regular multiplicative ternary hyperring and T be an hyperideal of R. Then the hyperideal T is regular.*

Let N be the intersection of all nonzero hyperideals of R, N_r the intersection of all nonzero right hyperideals of R, N_m the intersection of all nonzero lateral hyperideals of R and N_l the intersection of all nonzero left hyperideals of R. If $N = \{0\}$, then obviously we have $N = N_r = N_m = N_l = (0)$.

Finally, we give some theorems related to regular multiplicative ternary hyperrings and their regular hyperideals.

Theorem 4.5. *Let $(R, +, \circ)$ be a multiplicative ternary hyperring and $N = N_r = N_m = N_l$. Then R is a regular multiplicative ternary hyperring if and only if N is a regular hyperideal of R.*

Proof. Let R be a regular multiplicative ternary hyperring. If $N = N_r = N_m = N_l = \{0\}$, then the proof follows from Proposition 4.3. Hence, we may assume that $N = N_r = N_m = N_l \neq \{0\}$. By Proposition 4.3, it follows that $\{0\}$ is a regular hyperideal of R. Now, $\{0\} \subseteq N$ implies that N is a regular hyperideal of R, by Proposition 4.2.

Conversely, let N be a regular hyperideal of R. Let S be a right hyperideal, M a lateral hyperideal and L a left hyperideal of R. Then $N{=}N_r \subseteq S$, $N{=}N_m \subseteq M$ and $N = N_l \subseteq L$. Since N is a regular hyperideal of R, $N + S \circ M \circ L = S \cap M \cap L$. Since $N \circ N \circ N$ is a right hyperideal of R, $N = N_r \subseteq N \circ N \circ N \subseteq S \circ M \circ L$ and $N + S \circ M \circ L = S \circ M \circ L$. Consequently, $N + S \circ M \circ L = S \cap M \cap L$ implies that $S \circ M \circ L = S \cap M \cap L$. Hence, from Theorem 3.12, it follows that R is a regular multiplicative ternary hyperring. ■

Theorem 4.6. *Let R be a multiplicative ternary hyperring and $N = N_r = N_m = N_l \neq (0)$. Then R is regular multiplicative ternary hyperring if and only if every nonzero hyperideals of R is regular.*

Proof. Let R be a regular multiplicative ternary hyperring and I be a nonzero hyperideal of R. Since R is regular, (0) is a regular hyperideal of R, by Proposition 4.3. Agai,n since $I \supseteq (0)$, by Proposition 4.2, we see that I is a regular hyperideal of R.

Conversely, assume that every nonzero hyperideals of R is regular and $N = N_r = N_m = N_l \neq (0)$. Then N is a regular hyperideal of R. Now, by Theorem 4.5, R is a regular ring. ∎

Lemma 4.7. *Let $(R, +, \circ)$ be a multiplicative ternary hyperring and I be an hyperideal of R. Then, the following conditions are equivalent:*

(i) $\langle a\rangle_r \cap \langle a\rangle_m \cap \langle a\rangle_l = \langle a\rangle_r \circ \langle a\rangle_m \circ \langle a\rangle_l$ *for any* $a \in R$,

(ii) $S \cap M \cap L = S \circ M \circ L$, *for any right hyperideal S, lateral hyperideal M and left hyperideal L.*

Proof. (i) $\Rightarrow$ (ii) Let $x \in S\cap M\cap L$. Now, $x \in \langle x\rangle_r \cap \langle x\rangle_m \cap \langle x\rangle_l = \langle x\rangle_r \circ \langle x\rangle_m \circ \langle a\rangle_l$ by (i) $\subseteq S \circ M \circ L$. Therefore, we have $S \cap M \cap L \subseteq S \circ M \circ L$. Obviously, $S \circ M \circ L \subseteq S \cap M \cap L$. Hence $S \cap M \cap L = S \circ M \circ L$.

Obviously, (ii) $\Rightarrow$ (i) holds. ∎

Theorem 4.8. *The following conditions are equivalent in a multiplicative ternary hyperring $(R, +, \circ)$.*

(i) *I is a regular hyperideal of R;*

(ii) *For $a, b, c \in R$, $I + \langle a\rangle_r \circ \langle b\rangle_m \circ \langle c\rangle_l = I + \langle a\rangle_r \cap \langle b\rangle_m \cap \langle c\rangle_l$;*

(iii) *For each $a \in R$, $I + \langle a\rangle_r \circ \langle a\rangle_m \circ \langle a\rangle_l = I + (\langle a\rangle_r \cap \langle a\rangle_m \cap \langle a\rangle_l)$;*

(iv) *For each $a \in R\backslash I = I^c$, $a = x + \sum_{i=1}^{n} a \circ p_i \circ a \circ q_i \circ a + \sum_{i=1}^{n} a \circ r_i \circ s_i \circ a \circ u_i \circ v_i \circ a$, for some $x \in I$ and $p_i, q_i, r_i, s_i, u_i, v_i \in R$.*

Proof. (i) $\Rightarrow$ (ii). Suppose that I is a regular hyperideal of R. For any $a, b, c \in R$, $I \subseteq (I + \langle a\rangle_r)$, $(I + \langle b\rangle_m)$ and $(I + \langle c\rangle_l)$. Now, we deduce that

$$\begin{aligned}
&I + \langle a\rangle_r \cap \langle b\rangle_m \cap \langle c\rangle_l \subseteq (I + \langle a\rangle_r) \cap (I + \langle b\rangle_m) \cap (I + \langle c\rangle_l) \\
&= I + (I + \langle a\rangle_r) \circ (I + \langle b\rangle_m) \circ (I + \langle c\rangle_l) \text{ (since } I \text{ is regular)} \\
&\subseteq I + I \circ I \circ I + I \circ \langle b\rangle_m \circ I + I \circ \langle b\rangle_m \circ \langle c\rangle_l + I \circ I \circ \langle c\rangle_l + \langle a\rangle_r \circ I \circ I \\
&+ \langle a\rangle_r \circ I \circ \langle c\rangle_l + \langle a\rangle_r \circ \langle b\rangle_m \circ I + \langle a\rangle_r \circ \langle b\rangle_m \circ \langle c\rangle_l \subseteq I + \langle a\rangle_r \circ \langle b\rangle_m \circ \langle c\rangle_l
\end{aligned}$$

Again $\langle a\rangle_r \circ \langle b\rangle_m \circ \langle c\rangle_l \subseteq \langle a\rangle_r \cap \langle b\rangle_m \cap \langle c\rangle_l$ implies that

$$I + \langle a\rangle_r \circ \langle b\rangle_m \circ \langle c\rangle_l \subseteq I + \langle a\rangle_r \cap \langle b\rangle_m \cap \langle c\rangle_l.$$

Hence $I + \langle a\rangle_r \circ \langle b\rangle_m \circ \langle c\rangle_l = I + \langle a\rangle_r \cap \langle b\rangle_m \cap \langle c\rangle_l$.

Obviously, (ii) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (iv). We first observe that $\langle I + \langle a \rangle_r \rangle_r = I + \langle a \rangle_r = I + \langle a \rangle_r \cap R \cap R = I + \langle a \rangle_r \circ R \circ R =$ (by Lemma 4.7) $I + (a \circ R \circ R + na) \circ R \circ R \subseteq I + a \circ R \circ R$. Obviously, $I + a \circ R \circ R \subseteq \langle I + \langle a \rangle_r \rangle_r$, and hence, $\langle I + \langle a \rangle_r \rangle_r = I + a \circ R \circ R$. Similarly, we have $\langle I + \langle a \rangle_m \rangle_m = I + R \circ a \circ R + R \circ R \circ a \circ R \circ R$ and $\langle I + \langle a \rangle_l \rangle_l = I + R \circ R \circ a$. Now, we have

$$
\begin{aligned}
&\langle a \rangle_r \cap \langle a \rangle_m \cap \langle a \rangle_l \subseteq I + \langle I + \langle a \rangle_r \rangle_r \cap \langle I + \langle a \rangle_m \rangle_m \cap \langle I + \langle a \rangle_l \rangle_l \\
&= I + \langle I + \langle a \rangle_r \rangle_r \circ \langle I + \langle a \rangle_m \rangle_m \circ \langle I + \langle a \rangle_l \rangle_l \text{ (by Lemma 4.7)} \\
&= I + (I + a \circ R \circ R) \circ (I + R \circ a \circ R + R \circ R \circ a \circ R \circ R) \circ (I + R \circ R \circ a) \\
&\subseteq I + (a \circ R \circ R \circ R \circ a \circ R \circ R \circ R \circ a \\
&+ a \circ R \circ R \circ R \circ R \circ a \circ R \circ R \circ R \circ R \circ a) \\
&\subseteq I + a \circ R \circ a \circ R \circ a + a \circ R \circ R \circ a \circ R \circ R \circ a.
\end{aligned}
$$

Since $a \in \langle a \rangle_r \cap \langle a \rangle_m \cap \langle a \rangle_l$, there exist elements $x \in I$ and $p_i, q_i, r_i, s_i, u_i, v_i \in R$ such that

$$a = x + \sum_{i=1}^{n} a \circ p_i \circ a \circ q_i \circ a + \sum_{i=1}^{n} a \circ r_i \circ s_i \circ a \circ u_i \circ v_i \circ a.$$

(iv) $\Rightarrow$ (i). Let S, M and L be any right, lateral and left hyperideal of R respectively such that $S \supseteq I$, $M \supseteq I$ and $L \supseteq I$. Then, obviously, $I + S \circ M \circ L \subseteq S \cap M \cap L$. Again, let $a \in S \cap M \cap L$. Then, by condition (iv), we have $a = x + \sum_{i=1}^{n} a \circ p_i \circ a \circ q_i \circ a + \sum_{i=1}^{n} a \circ r_i \circ s_i \circ a \circ u_i \circ v_i \circ a$ for some $x \in I$ and $p_i, q_i, r_i, s_i, u_i, v_i \in R$. Since $\sum_{i=1}^{n} a \circ p_i \circ a \circ q_i \circ a, \sum_{i=1}^{n} a \circ r_i \circ s_i \circ a \circ u_i \circ v_i \circ a \in S \circ M \circ L$. $a \in I + S \circ M \circ L$ and hence $S \cap M \cap L \subseteq I + S \circ M \circ L$. Thus $S \cap M \cap L = I + S \circ M \circ L$. Consequently, I is a regular hyperideal. ∎

References

[1] Castillo, J.R., Paradero-Vilela, J.S., *Quotient and homomorphism in ternary hyperrings*, International Journal of Mathematical Analysis, 8 (58) (2014), 2845-2859.

[2] Corsini, P., Leoreanu, V., *Applications of hyperstructure theory*, Advances in Mathematics, vol. 5, Kluwer Academic Publishers, 2003.

[3] Dutta, T.K., Kar, S., *A Note on regular ternary semirings*, Kyungpook Math. J., 46 (2006), 357-365.

[4] Krasner, M., *A class of hyperrings and hyperfields*, Internat. J. Math, Sci., 6 (1983), 307-311.

[5] Lister, W.G., *Ternary rings*, Trans. Amer. Math. Soc., 154 (1971), 37-55.

[6] Marty, F., *Sur une generalization de la notion de group*, 8^{th} Congrès des Math. Scandinaves, Stockholm, 1934, 45-49.

[7] Rosenfeld, A., *Some applications of n-ary group theory*, Belaruskaya Navuka, Minsk, 1998.

[8] Rota, R., *Strongly distribute multiplicative hyperrings*, Journal of Geometry, 39 (1) (1990).

[9] Salim, Md., Dutta, T.K., *A class of general type of regularities and generalized semiprime ideals in ternary ring*, The Southeast Asian Bulletin Mathematics (to appear).

[10] Salim, Md., Dutta, T.K., *On right strongly prime ternary rings,* East-West Journal of Mathematics (to appear).

[11] Salim, Md., Chanda, T., Dutta, T.K., *Regular equivalence and strongly regular equivalence on multiplicative ternary hyperring*, Journal of hyperstructures, 4 (1) (2015), 20-36.

[12] Salim, Md., Dutta, T.K., *Correspondence between hyperideals and regular congruence relations of a multiplicative ternary hyperring.* (communicated)

[13] Tamas, V., *Regular ternary rings*, An. Stiin. Univ. Al. I. Cuza. Iaşi, Sec. Ia Mat., 33 (2) (1987), 89-92.

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NUMERICAL TREATMENT OF NEUTRAL FRACTIONAL VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS WITH INFINITE DELAY

E.A. Rawashdeh

Department of Mathematics
Yarmouk University
Irbid
Jordan
e-mail: edris@yu.edu.jo

Abstract. This paper deals with the numerical solution of fractional integro-differential equations with infinite delay. We applied the continuous spline collocation methods to approximate the solution. A new technique for evaluating the Caputo fractional derivative of the spline polynomials based on the Lagrange coefficients is obtained. Some numerical examples are provided to test the convergence of the method.

Keywords: fractional integro-differential equations, infinite delay, spline collocation method.
2000 AMS Subject Classification: 26A33, 45J05, 65M70.

1. Introduction

Consider the fractional integro-differential equation with infinite delay

$$\text{(1.1)} \qquad \begin{cases} D_*^\alpha y(t) = f(t, y(t)) + \int_{-\infty}^t k(t, s, y(t), y(s))ds, & t \in [0, T], \\ y(t) = \phi(t), & t \in (-\infty, 0]. \end{cases}$$

It will be assumed that functions f, ϕ and k are sufficiently smooth, moreover D_*^α denotes the fractional differential operator of order $\alpha \in (0, 1)$ in the sense of Caputo and is defined by

$$\text{(1.2)} \qquad D_*^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{y'(s)}{(t-s)^\alpha} ds.$$

For more properties of the operators D_*^α, one can see [16].

In this paper, we study the numerical solution of an equivalent form of equation (1.1) which is given by the initial value problem

$$\text{(1.3)} \quad D_*^\alpha y(t) = f(t, y(t)) + (\psi y)(t) + \int_0^t k(t, s, y(t), y(s))ds, \; y(0) = \phi(0), \; t \in [0, T],$$

where

$$(\psi y)(t) = \int_{-\infty}^{0} k(t, s, y(t), \phi(s))ds, \quad t \in [0, T].$$

The existence and uniqueness of equation (1.3) have received reasonable attention in the last few years (cf. [3] and [5]).

Recently, it turns out that fractional derivatives and integrals can be a valuable tools in the modelling of many phenomena in applied sciences, therefore they have been investigated by many researchers (cf, e.g., [1], [2], [4], [9] and references therein). In particular, fractional integro-differential equations appear in electromagnetic, acoustics, viscoelasticity, heat conduction in materials with memory, fluid dynamics, biological models, chemical kineticsand and many other phenomena (cf, e.g., [12],[13], [14], [17] and references therein).

Numerical solution of integro-differential equations with infinite delay has been attracted by few mathematicians in the literature. Brunner [7] and [8] applied the collocation method to solve integro-differential with finite and infinite delays. In his papers he mentioned two important applications; the Volterra's population equation and integro-differential equations of polymer rheology. Jaradat et al. [11] used the homotopy analysis method to solve the population growth model. In this paper we extend the spline collocation method to fractional order integro-differential with infinite delay. To our best of knowledge, equation (1.1) has received little attention in the literature (cf, e.g., [15] and references therein).

This paper is organized as follows: In Section 2, we define the continuous spline space. In Section 3, a system of equations based on the collocation method is derived. Numerical examples are given in Section 4.

2. Polynomial spline space

For $N \in \mathbb{N}$, let

$$Z_N = \{t_0, t_1, \ldots, t_N : \ 0 = t_0 < t_1 < \ldots < t_N = T\} \tag{2.1}$$

be a partition of the interval $[0, T]$, given by the grid points

$$t_n = nh, \text{ with } h = \frac{T}{N} \text{ and } n = 0, 1, \ldots, N. \tag{2.2}$$

Let

$$\sigma_n = [t_n, t_{n+1}], \ (n = 0, ..., N-1).$$

For given integer $m \geq 1$, let $S_m^{(0)}(Z_N)$ be the spline space of continuous polynomials on the grid (2.1) and (2.2)

$$S_m^{(0)}(Z_N) = \{u(t) : u(t)\Big|_{t \in \sigma_n} = u_n(t) \in \pi_m \text{ on } \sigma_n (n = 0, ..., N-1)\}$$

with

$$u_{n-1}(t_n) = u_n(t_n) \quad t_n \in Z_N - \{0, T\},$$

where π_m denotes the set of all real polynomials of degree not exceeding m. The dimension of $S_m^{(0)}(Z_N)$ is given by

$$\dim\, S_m^{(0)}(Z_N) = mN + 1.$$

3. Derivation of the collocation method

In every subinterval $\sigma_n = [t_n, t_{n+1}]$, $(n = 0, ..., N-1)$, we introduce m interpolation points (called collocation points) $t_{n,1} < \ldots < t_{n,m,}$ with

$$t_{n,j} = t_n + c_j h : j = 1, ..., m; n = 0, ..., N-1,$$

where $c_{1,} \ldots, c_m$ do not depend on n and N and satisfy

$$0 \leq c_1 < ... < c_m \leq 1.$$

Let

$$X(N) = \bigcup_{n=0}^{N-1} X_n \text{ with } X_n = \{t_{n,j} = t_n + c_j h : j = 1, ..., m\} \subset \sigma_n.$$

The exact solution y of (1.3) will be approximated on I by an element $u \in S_m^{(0)}(Z_N)$ (called the collocation solution) satisfying on the set $X(N)$

$$\text{(3.1)}\quad D_*^\alpha u(t) = f(t, u(t)) + (\psi u)(t) + \int_0^t k(t, s, u(t), u(s))ds,\ u(0) = \phi(0),\ t \in X(N),$$

where

$$(\psi u)(t) = \int_{-\infty}^0 k(t, s, u(t), \phi(s))ds, \quad t \in X(N).$$

Now, to evaluate the fractional derivative of the spline function $D_*^\alpha u$, we need the following lemma.

Lemma 3.1. *Let $\alpha \in (0, 1)$, then*

$$D_*^\alpha u(t_{n,j}) = \sum_{i=0}^{n-1} \sum_{k=1}^m D_{i,k} Y_{i,k} + \sum_{k=1}^m D_{n,k}(c_j) Y_{n,k}$$

where

$$\begin{aligned} Y_{n,k} &= u_n'(t_{n,k}), \\ D_{i,k} &= \frac{h}{\Gamma(1-\alpha)} \int_0^1 \frac{L_k(v)}{(t_{n,j} - t_i - vh)^\alpha} dv, \\ D_{n,k}(c_j) &= \frac{h}{\Gamma(1-\alpha)} \int_0^{c_j} \frac{L_k(v)}{(t_{n,j} - t_n - vh)^\alpha} dv, \\ L_k(v) &= \prod_{l=1, l\neq k}^m (v - c_l)/(c_k - c_l). \end{aligned}$$

Proof. On the interval σ_n, u_n' is a polynomial of degree $m-1$, thus it can be written in the form

$$u_n'(t_n+vh)=\sum_{k=1}^{m}L_k(v)Y_{n,k} \tag{3.2}$$

where $Y_{n,k}=u_n'(t_{n,k})$ and $L_k(v)=\prod_{l=1,l\neq k}^{m}(v-c_l)/(c_k-c_l)$.

If we substitute $s=t_i+vh$ in formula (1.2), we obtain

$$\begin{aligned}
D_*^\alpha u(t_{n,j}) &= \frac{1}{\Gamma(1-\alpha)}\int_0^{t_{n,i}}\frac{u'(s)}{(t_{n,j}-s)^\alpha}ds\\
&= \frac{1}{\Gamma(1-\alpha)}\Big(\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\frac{u_i'(s)}{(t_{n,j}-s)^\alpha}ds+\int_{t_n}^{t_{n,j}}\frac{u_n'(s)}{(t_{n,j}-s)^\alpha}ds\Big)\\
&= \frac{h}{\Gamma(1-\alpha)}\Big(\sum_{i=0}^{n-1}\int_0^1\frac{\sum_{k=1}^m L_k(v)Y_{i,k}}{(t_{n,j}-t_i-vh)^\alpha}dv+\int_0^{c_j}\frac{\sum_{k=1}^m L_k(v)Y_{n,k}}{(t_{n,j}-t_n-vh)^\alpha}dv\Big)\\
&= \frac{h}{\Gamma(1-\alpha)}\Big(\sum_{i=0}^{n-1}\sum_{k=1}^{m}\big(\int_0^1\frac{L_k(v)}{(t_{n,j}-t_i-vh)^\alpha}dv\big)Y_{i,k}\\
&+ \sum_{k=1}^{m}\big(\int_0^{c_j}\frac{L_k(v)}{(t_{n,j}-t_n-vh)^\alpha}dv\big)Y_{n,k}\Big)\\
&= \sum_{i=0}^{n-1}\sum_{k=1}^{m}D_{i,k}Y_{i,k}+\sum_{k=1}^{m}D_{n,k}(c_j)Y_{n,k}.
\end{aligned}$$

■

Note that Lemma 3.1 provides a new technique for finding $D_*^\alpha u$, for other techniques one can see [6] and [10].

Now, from equation (3.2), we get

$$u_n(t_n+vh)=y_n+h\sum_{k=1}^{m}a_k(v)Y_{n,k} \tag{3.3}$$

where $y_n=u_n(t_n)=u_{n-1}(t_n)$, $y_0=y(0)=\phi(0)$, and $a_k(v)=\int_0^v L_k(z)dz$.

Applying these representation in the collocation equation (3.1) and using Lemma 3.1, we obtain the following nonlinear system

$$\begin{aligned}
\sum_{i=0}^{n-1}\sum_{k=1}^{m}D_{i,k}Y_{i,k}+\sum_{k=1}^{m}D_{n,k}(c_j)Y_{n,k} &= f(t_{n,j},W_{n,j})+(\psi u)(t_{n,j})\\
&+h\sum_{i=0}^{n-1}\int_0^1 k\Big(t_{n,j},t_i+vh,W_{n,j},y_i+h\sum_{k=1}^{m}a_k(v)Y_{i,k}\Big)dv\\
&+h\int_0^{c_j}k\Big(t_{n,j},t_n+vh,W_{n,j},y_n+h\sum_{k=1}^{m}a_k(v)Y_{n,k}\Big)dv,
\end{aligned} \tag{3.4}$$

where

$$W_{n,j} = u_n(t_{n,j}) = y_n(t_n) + h\sum_{k=1}^{m} a_k(c_j)Y_{n,k}$$

and

$$(\psi u)(t_{n,j}) = \int_{-\infty}^{0} k(t_{n,j}, s, W_{n,j}, \phi(s))ds.$$

For each $n = 0, 1, \ldots, N-1$, (3.4) represents a nonlinear system with the variables

$$Y_n = \begin{pmatrix} Y_{n,1} \\ \vdots \\ Y_{n,m} \end{pmatrix},$$

where $k, j = 1, 2, \ldots, m$. Once the vectors Y_n are known, the collocation solution $u \in S_m^{(0)}(Z_n)$ that is given by (3.3) is completely determined.

4. Numerical illustration

We solve the nonlinear system (3.4) in the space $S_2^{(0)}(Z_N)$ for a couple of examples:

1) Consider the following fractional integro-differential equation with infinite delay

$$\begin{cases} D_*^{0.25}y(t) = f(t, y(t)) + \int_{-\infty}^{t} e^{4t}sy(s)ds, & t \in [0,1], \\ y(t) = \frac{t}{t^4+2}, & t \in (-\infty, 0], \end{cases} \tag{4.1}$$

where f have been chosen in such a way that the exact solution of (4.1) is $y(t) = t^3$.

2) Consider the following fractional integro-differential equation with infinite delay

$$\begin{cases} D_*^{0.5}y(t) = f(t, y(t)) + \int_{-\infty}^{t}(t+s)y(s)ds, & t \in [0,1], \\ y(t) = e^t, & t \in (-\infty, 0], \end{cases} \tag{4.2}$$

where

$$f(t, y(t)) = \sqrt{2}e^{2t}erf(\sqrt{2t}) - te^{2t} - \frac{4te^{2t} - e^{2t} - 2t - 3}{4} - te^{-2t}y(t)$$

and the exact solution of (4.2) is $y(t) = e^{2t}$. Here erf is the error function, and is defined by

$$erf(t) = \frac{2}{\sqrt{\pi}}\int_0^t e^{-\tau^2}d\tau.$$

In both examples we choose the collocation parameters to be the Gauss points ($c_1 = \frac{\sqrt{3}+1}{2\sqrt{3}}$ $c_2 = \frac{\sqrt{3}-1}{2\sqrt{3}}$) and the Radau II points ($c_1 = \frac{1}{3}$ and $c_2 = 1$).

The absolute error $|y(t_n) - u(t_n)|$ where $u \in S_2^{(0)}(Z_N)$ at certain values of $t \in [0,1]$ is listed in Table 4.1 and Table 4.2.

Table 4.1 for Example 4.1

t_n	N	Gauss points $\lvert y(t_n) - u(t_n)\rvert$	Radau II points $\lvert y(t_n) - u(t_n)\rvert$
0.1	10	0.881×10^{-4}	0.577×10^{-4}
	20	0.154×10^{-4}	0.939×10^{-4}
	40	0.224×10^{-5}	0.149×10^{-5}
0.5	10	0.117×10^{-3}	0.326×10^{-3}
	20	0.191×10^{-4}	0.541×10^{-4}
	40	0.290×10^{-5}	0.873×10^{-5}
1	10	0.71×10^{-2}	6.02
	20	0.399×10^{-2}	0.876
	40	0.103×10^{-2}	0.131

Table 4.2 for Example 4.2

t_n	N	Gauss points $\lvert y(t_n) - u(t_n)\rvert$	Radau II points $\lvert y(t_n) - u(t_n)\rvert$
0.1	10	0.101×10^{-3}	0.998×10^{-4}
	20	0.154×10^{-4}	0.171×10^{-4}
	40	0.207×10^{-5}	0.293×10^{-5}
0.5	10	0.279×10^{-3}	0.378×10^{-3}
	20	0.305×10^{-4}	0.646×10^{-4}
	40	0.497×10^{-5}	0.110×10^{-5}
1	10	0.757×10^{-3}	0.137×10^{-2}
	20	0.104×10^{-3}	0.238×10^{-3}
	40	0.139×10^{-4}	0.410×10^{-4}

It is clear that in both examples, the spline collocation method is convergent to the exact solution and the method is much better if we choose the collocation parameters to be the Gauss points. This is because the m-point interpolatory quadrature formula has the highest degree of precision $2m-1$ on the interval $[0, 1]$.

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References

[1] Agarwal, R.P., de Andrade, B., Cuevas, C., *On type of periodicity and ergodicity to a class of fractional order differential equations*, Advances in Difference Equations, 2010 (2010) Article ID 179750.

[2] Ahmed, H.M., *Boundary controllability of nonlinear fractional integro-differential systems*, Advances in Difference Equations, 2010 (2010) Article ID 279493.

[3] Aissani, K., Benchohra, M., *Semlinear Fractional order Integro-Differential Equations with Infinite Delay in Banach Spaces*, Archivum Mathematicum, 49 (2013), 105-117.

[4] Alsaedi, A., Ahmad, B., *Existence of solutions for nonlinear fractional integro-differential equations with three-point non local fractional boundary conditions*, Advances in Difference Equations, 2010 (2010) Article ID 691721.

[5] Benchohra, M., Hendersonb, J., Ntouyasc, S.K., Ouahaba, A., *Existence results for fractional order functional differential equations with infinite delay*, Journal of Mathematical Analysis and Applications, 338 (2) (2008), 1340-1350.

[6] Blank, L., *Numerical treatment of differential equation of fractional order*, Numerical Analysis Report 287. Manchester Center for Numerical Computational Mathematics, 1996.

[7] Brunner, H., *The numerical solution of neutral Volterra integro-differential equations with delay arguments*, Ann. Numer. Math., 1 (1994), 309-322.

[8] Brunner, H., *Collocation Methods for Nonlinear Volterra Integro-Differential Equations with Infinite Delay*, Math., 35 (488) (1989), 571-587.

[9] El-Borai, M.M., *Some probability densities and fundamental solutions of fractional evolution equations*, Chaos, Solitons and Fractals, 14 (2) (2002), 433-440.

[10] Hamarsheh, M.H., Rawashdeh, E.A., *A Numerical Method for Solution of Semidifferential Equations,* Matematicki Vesnik, 62 (2010), 117-126.

[11] Jaradat, H.M., Rawashdeh, E.A., Awaedeh, F., *On Volteera's Population Growth Models*, Annals of the University of Craiova, Mathematics and Computer Science Series, 38 (2) (2011), 18-25.

[12] Liang, J., Xiao, T.-J., van Casteren, J., *A note on semilinear abstract functional differential and integro-differential equations with infinite delay*, Applied Mathematics Letters, 17 (4) (2004), 473- 477.

[13] Liang, J., Xiao, T.-J., *Semilinear integro-differential equations with non local initial conditions*, Computers & Mathematics with Applications, 47 (6-7) (2004), 863-875.

[14] Liang, J., Liu, J.H., Xiao, T.-J., *Non local problems for integro-differential equations*, Dynamics of Continuous, Discrete & Impulsive Systems, Series A 15 (6) (2008), 815-824.

[15] Lil, F., Zhang, J., *Existence of Mild Solutions to Fractional Integro-differential Equations of Neutral Type with Infinite Delay*, Advances in Difference Equations, 2011 (2011) Article ID 963463.

[16] Podlubny, I., Fractional differential equations: An introduction to fractional Derivatives, Fractional Differential Equations, Some Methods of their solution and some of their applications, Mathematics in Science and Engineering, vol.198, Academic Press, San Diego, 1999.

[17] Xiao, T.-J., Liang, J., *Blow-up and global existence of solutions to integral equations with infinite delay in Banach spaces*, Nonlinear Analysis: Theory, Methods & Applications, 71 (12) (2009), 1442-1447.

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A BRIEF COMPARISON OF G-CONTRACTION CONDITIONS AND A GENERALIZED FIXED POINT THEOREM

T. Phaneendra[1]
S. Saravanan

Department of Mathematics
School of Advanced Sciences
VIT University
Vellore-632014, Tamil Nadu
India
e-mails: drtp.indra@gmail.com
sarodhana87@gmail.com

Abstract. Let (X, G) be a G-metric space and f denote a self-map on X. A brief comparison of some G-contraction conditions is made, and a new generalized fixed point theorem is obtained by employing a wider inequality.

Keywords: G-metric space, G-Cauchy sequence, fixed point, G-contractive fixed point.
(2010) Mathematical Subject Classification: 54H25.

1. Introduction

Let X be a nonempty set and $G : X \times X \times X \to \mathbb{R}$ be such that

(G1) $G(x, y, z) \geq 0$ for all $x, y, z \in X$ with $G(x, y, z) = 0$ if $x = y = z$,

(G2) $G(x, x, y) > 0$ for all $x, y \in X$ with $x \neq y$,

(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$,

(G4) $G(x, y, z) = G(x, z, y) = G(y, x, z) = G(z, x, y) = G(y, z, x) = G(z, y, x)$ for all $x, y, z \in X$,

(G5) $G(x, y, z) \leq G(x, w, w) + G(w, y, z)$ for all $x, y, z, w \in X$.

Then G is called a G-metric on X and the pair (X, G) denotes a G-metric space. Axiom (G5) is referred to as the rectangle inequality (of G). This notion was introduced by Mustafa and Sims [3] in 2006.

[1]Corresponding author

It is easily seen that

$$G(x,y,y) \le 2G(x,x,y) \text{ for all } x,y \in X. \tag{1.1}$$

We use the following notions, developed in [3]:

Let (X,G) be a G-metric space. A G-ball in X is defined by

$$B_G(x_0,r) = \{y \in X : G(x_0,y,y) < r\}.$$

It is easy to see that the family of all G-balls forms a base topology, called the G-metric topology $\tau(G)$ on X. A sequence $\langle x_n\rangle_{n=1}^{\infty} \subset X$ is said to be *G-convergent* with limit $p \in X$, if it converges to p in $\tau(G)$. A sequence $\langle x_n\rangle_{n=1}^{\infty}$ in a G-metric space (X,G) is said to be G-Cauchy, if $\lim\limits_{n,m\to\infty} G(x_n,x_m,x_m) = 0$. The space X is said to be G-complete, if every G-Cauchy sequence in X converges in it.

Definition 1.1 Let (X,G) be a G-metric space. A set $S \subset X$ is said to be G-bounded or simply bounded, if there exists a positive number M such that $G(x,y,z) < M$ for all $x,y,z \in S$. Note that if S is G-bounded, then its diameter $\delta(S) = \sup\{G(x,y,z) : x,y,z \in S\}$ is finite.

Definition 1.2 A self-map f on a G-metric space (X,G) is G-continuous at a point $p \in X$, if $f^{-1}(B_G(fp,r)) \in \tau(G)$ for all $r > 0$, and f is G-continuous on X, if it is G-continuous at every $p \in X$.

Lemma 1.1 *A self-map f on a G-metric space (X,G) is G-continuous at a point $p \in X$ if and only if the sequence $\langle fp_n\rangle_{n=1}^{\infty} \subset X$ G-converges to fp whenever $\langle p_n\rangle_{n=1}^{\infty}$ is a sequence in X which G-converges to fp.*

2. A brief comparison

Mohanta [1] proved a pair of results which are given below:

Theorem 2.1 *Let (X,G) be a complete G-metric space and f, a self-map on X satisfying*

$$\begin{aligned} G(fx,fy,fz) \le{} & aG(x,y,z) + bG(x,fx,fx) + cG(y,fy,fy) + dG(z,fz,fz) \\ & + e\max\{G(x,fy,fy), G(y,fx,fx), G(y,fz,fz), G(z,fy,fy), \\ & G(z,fx,fx), G(x,fz,fz)\} \text{ for all } x,y,z \in X, \end{aligned} \tag{2.1}$$

where $a,b,c,d,e \ge 0$ with $a+b+c+d+2e < 1$. Then f has a unique fixed point p and f is G-continuous at p.

Theorem 2.2 *Let (X,G) be a complete G-metric space and f, a self-map on X satisfying*

$$\begin{aligned} G(fx,fy,fz) \le{} & a[G(x,fy,fy) + G(y,fx,fx)] + b[G(y,fz,fz) + G(z,fy,fy)] \\ & + c[G(x,fz,fz) + G(z,fx,fx)] + dG(x,y,z) \\ & + e\max\{G(x,fx,fx), G(y,fy,fy), G(z,fz,fz)\} \\ & \text{for all } x,y,z \in X, \end{aligned} \tag{2.2}$$

where $a.b,c,d,e \ge 0$ with $2a+2b+2c+d+2e < 1$. Then f has a unique fixed point p and f is G-continuous at p.

The case $e = 0$ of Theorem 2.1 gives that of Mustafa et al [2], while with $d = e = 0$ and $a+b+c = k$, Theorem 2.2 reduces to the result proved by Mustafa and Sims [4].

Theorem 2.3 *Let (X, G) be a complete G-metric space, f a self-map on X satisfying*

$$\begin{aligned} G(fx, fy, fz) \leq k \max\big\{ &[G(x, fy, fy) + G(y, fx, fx) + G(z, fz, fz)], \\ &[G(y, fz, fz) + G(z, fy, fy) + G(x, fx, fx)], \\ &[G(z, fx, fx) + G(x, fz, fz) + G(y, fy, fy)]\big\} \\ &\qquad\qquad\qquad \textit{for all } x, y, z \in X, \end{aligned} \tag{2.3}$$

where $0 < k < 1/3$. Then f will have a unique fixed point p and f will be G-continuous at p.

and Vats et al [5] proved

Theorem 2.4 *Suppose that (X, G) is a complete G-metric space, f a self-map on X satisfying the condition*

$$\begin{aligned} G(fx, fy, fz) \leq k \max\big\{ &G(x, fx, fx) + G(y, fy, fy) + G(z, fz, fz), \\ &G(x, fy, fy) + G(y, fx, fx) + G(z, fy, fy), \\ &G(x, fz, fz) + G(y, fz, fz) + G(z, fx, fx)\big\} \\ &\qquad\qquad\qquad \textit{for all } x, y, z \in X, \end{aligned} \tag{2.4}$$

where $0 < k < 1/4$. Then f will have a unique fixed point p and f is continuous at p.

Since $\alpha + \beta + \gamma \leq 3\max\{\alpha, \beta, \gamma\}$ for $\alpha \geq 0$, $\beta \geq 0$ and $\gamma \geq 0$, one can obtain the following generalizations respectively:

Theorem 2.5 *Let (X, G) be a complete G-metric space and f denote a self-map on X satisfying*

$$\begin{aligned} G(fx, fy, fz) \leq k \max\big\{ &G(x, fy, fy), G(y, fx, fx), G(z, fz, fz), \\ &G(y, fz, fz), G(z, fy, fy), G(x, fx, fx), \\ &G(z, fx, fx), G(x, fz, fz), G(y, fy, fy)]\big\} \\ &\qquad\qquad\qquad \textit{for all } x, y, z \in X, \end{aligned} \tag{2.5}$$

where $0 \leq k < 1/9$. Then f has a unique fixed point p, and f is G-continuous at p.

Theorem 2.6 *Let (X, G) be a complete G-metric space, and f denote a self-map on X satisfying*

$$\begin{aligned} G(fx, fy, fz) \leq k \max\big\{ &G(x, fx, fx), G(y, fy, fy), G(z, fz, fz), \\ &G(x, fy, fy), G(y, fx, fx), G(z, fy, fy), \\ &G(x, fz, fz), G(y, fz, fz), G(z, fx, fx)\} \\ &\qquad\qquad\qquad \textit{for all } x, y, z \in X, \end{aligned} \tag{2.6}$$

where $0 \leq k < 1/12$. Then $f$has a unique fixed point p, and f is G-continuous at p.

In this paper, a nice generalization of Theorem 2.5 and Theorem 2.6 is obtained by employing a wider inequality.

3. Main result

Given $x, y, z \in X$, define $D(x, y, z) = \max E_f(x, y, z)$, where

$$E_f(x,y,z) = \left\{ G(f^i p, f^j q, f^k r) : 0 \le i, j, k \le 1; p, q, r \in \{x, y, z\} \right\}. \tag{3.1}$$

It may be noted that $E_f(x, y, z)$ has 36 elements.

Our main result is

Theorem 3.1 *Let (X, G) be a G-metric space and $f : X \to X$ satisfying the following inequality:*

$$G(fx, fy, fz) \le c \max D(x, y, z) \text{ for all } x, y, z \in X, \tag{3.2}$$

where $0 < c < 1/3$. If X is G-complete, then f has a unique fixed point p and f is G-continuous at p.

Proof. Writing $x = x_{n-1}$, $y = z = x_n$ in (3.2) and then simplifying, we get

$$G(x_n, x_{n+1}, x_{n+1}) = G(fx_{n-1}, fx_n, fx_n) \le cD(x_{n-1}, x_n, x_n) = cM, \tag{3.3}$$

where

$$\begin{aligned} M = \max\{ & G(x_{n-1}, x_{n-1}, x_n), G(x_{n-1}, x_{n-1}, x_{n+1}), \\ & G(x_{n-1}, x_n, x_n), G(x_{n-1}, x_n, x_{n+1}), G(x_{n-1}, x_{n+1}, x_{n+1}) \\ & G(x_n, x_n, x_{n+1}), G(x_n, x_{n+1}, x_{n+1})\}. \end{aligned} \tag{3.4}$$

On the other hand, taking $x = y = x_{n-1}$, $z = x_n$ in (3.2), one obtains similarly

$$G(x_n, x_n, x_{n+1}) \le cM, \tag{3.5}$$

Therefore from (3.3) and (3.5), it follows that

$$\max\{G(x_n, x_{n+1}, x_{n+1}), G(x_n, x_n, x_{n+1})\} \le cM \text{ for all } n \ge 1. \tag{3.6}$$

Now the following cases arise:

Case (a). $M = G(x_{n-1}, x_{n-1}, x_n)$. Then (3.6) implies that

$$G(x_n, x_n, x_{n+1}) \le cG(x_{n-1}, x_{n-1}, x_n)$$

which, by induction, gives

$$G(x_n, x_n, x_{n+1}) \le c^n G(x_0, x_0, x_1) \text{ for all } n \ge 1.$$

Then for $m > n$, from the repeated application of the rectangle inequality, it follows that

$$\begin{aligned} G(x_n,&x_n, x_m) \\ &\le \underbrace{G(x_n, x_n, x_{n+1}) + G(x_{n+1}, x_{n+1}, x_{n+2}) + ... + G(x_{m-1}, x_{m-1}, x_m)}_{m-n \text{ terms}} \\ &\le c^n \left(1 + c + \cdots + c^{m-n-1}\right) G(x_0, x_0, x_1) \\ &\le \frac{c^n}{1-c} \cdot G(x_0, x_0, x_1) \to 0 \text{ as } m, n \to \infty. \end{aligned}$$

Case (b). $M = G(x_{n-1}, x_{n-1}, x_{n+1})$. Then (3.6) implies that

$$\begin{aligned} G(x_n, x_n, x_{n+1}) &\le cG(x_{n-1}, x_{n-1}, x_{n+1}) \\ &\le c[G(x_{n-1}, x_{n-1}, x_n) + G(x_n, x_n, x_{n+1})] \\ &\le \left(\tfrac{c}{1-c}\right) G(x_{n-1}, x_{n-1}, x_n) \\ &\dots \\ &\le \left(\tfrac{c}{1-c}\right)^n G(x_0, x_0, x_1) \text{ for all } n \ge 1, \end{aligned}$$

Then for $m > n$, by the rectangle inequality, it follows that

$$G(x_n,x_n, x_m) \le \tfrac{1}{1-c} \left(\tfrac{c}{1-c}\right)^n G(x_0, x_0, x_1) \to 0 \text{ as } m, n \to \infty.$$

Case (c). $M = G(x_{n-1}, x_n, x_n)$. Then (3.6) implies that

$$G(x_n, x_{n+1}, x_{n+1}) \le cG(x_{n-1}, x_n, x_n) \le \cdots \le c^n G(x_0, x_1, x_1) \text{ for all } n \ge 1.$$

Then for $m > n$, we have

$$G(x_n,x_m, x_m) \le \left(\tfrac{c}{1-c}\right)^n \cdot G(x_0, x_1, x_1) \to 0 \text{ as } m, n \to \infty.$$

Case (d). $M = G(x_{n-1}, x_n, x_{n+1})$. Then (3.6) and induction imply that

$$G(x_n, x_n, x_{n+1}) \le \left(\tfrac{2c}{1-c}\right)^n G(x_0, x_0, x_1) \text{ for all } n \ge 1,$$

Then for $m > n$, it follows that

$$G(x_n,x_n, x_m) \le \tfrac{1}{1-c} \left(\tfrac{2c}{1-c}\right)^n G(x_0, x_0, x_1) \to 0 \text{ as } m, n \to \infty.$$

Case (e). $M = G(x_{n-1}, x_{n+1}, x_{n+1})$ is similar to Case (b).
For $M{=}G(x_n, x_n, x_{n+1})$, we see that $G(x_n, x_n, x_{n+1}){=}0$, while $G(x_n, x_{n+1}, x_{n+1}){=}0$ if $M = G(x_n, x_{n+1}, x_{n+1})$ for $n \ge 1$. Thus, from all these cases, we find that $\langle x_n \rangle_{n=1}^{\infty}$ is G-Cauchy.

Since X is G-complete. We can find a point $p \in X$ such that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} f^n x_0 = p. \tag{3.7}$$

Now, writing $x = x_{n-1}$, $y = z = p$ in (3.2), we obtain

$$G(x_n, fp, fp) = G(fx_{n-1}, fp, fp) \leq cD(x_{n-1}, p, p)\cdot \tag{3.8}$$

Applying the limit as $n \to \infty$ in (3.8) and then using (3.7) and (1.1), it follows that

$$G(p, fp, fp) \leq c \max\{G(p, p, fp), G(p, fp, fp)\} \leq 2c\, G(p, fp, fp)$$

If $p \neq fp$ in this, then (G2) would give

$$0 < G(p, fp, fp) \leq 2c\, G(p, fp, fp) < G(p, fp, fp),$$

which is a contradiction, since $0 < 2c < 1$. Therefore $fp = p$, showing that p is a fixed point of f. The uniqueness of the fixed point follows easily from (3.2), (G2) and (1.1).

Continuity at p: Suppose that $\langle p_n \rangle_{n=1}^{\infty}$ is such that

$$\lim_{n\to\infty} p_n = p. \tag{3.9}$$

Writing $x = p_n$ with $y = z = p$ in (3.2), and using (G5), we get

$$G(fp_n, p, p) = G(fp_n, fp, fp) \leq cD(p_n, p, p) \leq cP, \tag{3.10}$$

where

$$P = \max\{G(p_n, p_n, fp_n), G(p_n, p_n, p), G(p_n, fp_n, fp_n), \\ G(p_n, fp_n, p), G(p_n, p, p), G(fp_n, fp_n, p), G(fp_n, p, p)\}\cdot$$

Here again we discuss different cases:

Case (a). Suppose that $P = G(p_n, p_n, fp_n)$. Then (3.10) and (G5) imply that

$$G(fp_n, fp, fp) \leq cG(p_n, p_n, fp_n) \leq c[G(p_n, p_n, p) + G(p, p, fp_n)] \leq \tfrac{c}{1-c} G(p_n, p_n, p)$$

Proceeding the limit as $n \to \infty$ in this, it follows that $G(fp_n, fp, fp) \to 0$. In other words, $\lim_{n\to\infty} fp_n = fp = p$, proving that f is continuous at p.

Case (b). Suppose that $P = G(p_n, fp_n, fp_n)$. Then

$$G(fp_n, fp, fp) \leq cG(p_n, fp_n, fp_n) \leq 2cG(p_n, p_n, fp_n)$$

and the continuity follows from Case (a).

Case (c). Suppose that $P = G(p_n, fp_n, p)$. Then

$$G(fp_n, fp, fp) \leq cG(p_n, fp_n, p) \leq c[G(p_n, p, p) + G(p, fp_n, p)] \leq \tfrac{c}{1-c} G(p_n, p, p).$$

and the continuity follows as in Case (a).

Case (d). Suppose that $P = G(fp_n, fp_n, p)$. Then

$$G(fp_n, fp, fp) \le cG(fp_n, fp_n, p) \le 2cG(fp_n, p, p)$$

and the continuity follows again from Case (a).

If $P = G(p_n, p_n, p)$ or $G(p_n, p, p)$, it easily follows that $G(fp_n, fp, fp) \to 0$ as $n \to \infty$, in view of (3.9). Finally, if $P = G(fp_n, p, p)$, the continuity is obvious. ∎

4. Corollaries and discussion

First, we have

Corollary 4.1 (Mohanta [1], Theorem 3.7) *Let (X, G) be a complete G-metric space and f be a self-map on X satisfying*

$$G(fx, fy, fz) \le k \max A_f(x, y, z) \text{ for all } x, y, z \in X, \tag{4.1}$$

where

$$\begin{aligned} A_f(x, y, z) = \big\{&G(x, fx, fx), G(y, fy, fy), G(z, fz, fz), G(x, fy, fy), \\ &G(y, fz, fz), G(z, fx, fx), G(x, fz, fz), G(y, fx, fx), \\ &G(z, fy, fy), G(x, fy, fz), G(y, fz, fx), G(z, fx, fy), \\ &G(x, y, fz), G(y, z, fx), G(z, x, fy), G(x, y, z)\big\} \end{aligned}$$

and $0 < k < 1/3$. Then f has a unique fixed point p and f is G-continuous at p

Since $A_f(x, y, z) \subset E_f(x, y, z)$, the corollary follows from Theorem 3.1.

Theorem 4.1 (Vats et al. [5]) *Suppose that (X, G) is a complete G-metric space and f, a self-map on X satisfying*

$$G(fx, fy, fz) \le k \max B_f(x, y, z) \text{ for all } x, y, z \in X, \tag{4.2}$$

where

$$\begin{aligned} B_f(x, y, z) = \big\{&G(x, fx, fx), G(x, fy, fy), \\ &G(x, fz, fz), G(y, fy, fy), G(y, fx, fx), \\ &G(y, fz, fz), G(z, fz, fz), G(z, fx, fx) \end{aligned}$$

and $0 \le k < 1/2$. Then f will have a unique fixed point p and f is G-continuous at p.

Since $B_f(x, y, z) \subset E_f(x, y, z)$, this corollary also follows from Theorem 3.1.

We conclude the paper with the following remarks:

Remark 4.1 The right hand side terms in (2.5) and (2.6) are some among those of $E_f(x, y, z)$. Hence (2.5) and (2.6) imply (3.2). Thus the conclusions of Theorem 2.5 and Theorem 2.6 follow from that of Theorem 3.1.

Remark 4.2 If we choose a, b, c, d and e such that $a+b+c+d+2e < 1/3$, then the right hand side of (2.1) is less than or equal to the right hand side of (3.2). Thus the conclusion of Theorem 2.1 follows from that of Theorem 3.1.

Remark 4.3 If we choose a, b, c, d and e such that $2a+2b+2c+d+2e < 1/3$, then the right hand side of (2.2) is less than or equal to the right hand side of (3.2). Thus the conclusion of Theorem 2.2 follows from that of Theorem 3.1.

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References

[1] Mohanta, S.K., *Some fixed point theorems in G-Metric spaces*, An. St. Univ. Ovidius Constanta, 20 (1) (2012), 285-306.

[2] Mustafa, Z., Obiedat, H., Awawdeh, F., *Some fixed point theorems for mapping on complete G-Metric Spaces*, Fixed Point Theory and Applications (2008), Article ID 189870, 1-12.

[3] Mustafa, Z., Sims, B., *A new approach to generalized metric spaces*, Journal of Nonlinear and Convex Anal., 7 (2) (2006), 289-297.

[4] Mustafa, Z., Sims, B., *Fixed point theorems for contractive mappings in complete G-metric spaces*, Fixed Point Theory and Appl., 2009 Article ID 917175, 10 pp.

[5] Vats, R.K., Kumar, S., Sihag,V., *Fixed point theorems in complete G-metric space*, Fasc. Math., 47 (2011), 127-138.

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EXISTENCE SOLUTION FOR WEIGHTED $p(x)$-LAPLACIAN EQUATION

S.R. Mousaviankhatir

M. Alimohammady[1]

Department of Mathematics
University of Mazandaran
P.O. Box 47416-95447, Babolsar
Iran

H. Jafari

Department of Mathematics
University of Mazandaran
P.O. Box 47416-95447, Babolsar
Iran
and
Department of Mathematical Sciences
University of South Africa
UNISA 0003
South Africa

Abstract. This paper deals with the existence solution for the following type of boundary value problems:

$$\begin{cases} \Delta\left(|x|^{p(x)}\,|\Delta u|^{p(x)-2}\,\Delta u\right) = \lambda\,|u|^{q(x)-2}\,u, & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\Re^N$. It is established for a negative λ, there exists at least one weak solution. Our approach relies on the variable exponent theory of generalized Lebesgue-Sobolev spaces and a variant of the Mountain Pass theorem.

Keywords: $p(x)$-biharmonic, variable exponent Lebesgue space, variable exponent Sobolev space.
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1. Introduction

The literature of various mathematical problems with variable exponent have received a lot of attention in recent years [1], [12]. Fourth order equations appears

[1]Corresponding author. E-mail: amohsen@umz.ac.ir

in many contexts. Some of these problems come from different areas of applied mathematics and physics such as Micro Electro-Mechanical systems, surface diffusion on solids, flow in Hele-Shaw cells [5]. In addition, this type of equation can describe the static from change of beam or the sport of rigid body. There are many authors have pointed out that type of non linearity furnishes models to study travelling waves in suspension bridges [2], [8].

Given a bounded smooth domain $\Omega \subseteq \Re^N$, we recall some definitions and basic properties of the variable exponent Lebesgue space and Sobolev space $L^{p(x)}(\Omega)$ and $W_0^{k,p(x)}(\Omega)$, where $p(x) : \bar{\Omega} \to (1, \infty)$ is a continuous function. For further information in this regards we refer to [7] and [10]. On the other hand, regarding applications of variable exponent Lebesgue and Sobolev spaces to PDEs we refer to [6] while for some physical motivations of such problems we remember the contributions of Rajagopal and Ruzicka [11], Ruzicka [12] and Zhikov[14]. For any continuous function $h : \overline{\Omega} \to (1, \infty)$, set

$$h^+ = \sup_{x \in \Omega} h(x) \quad \text{and} \quad h^- = \inf_{x \in \Omega} h(x).$$

Given $p(x) \in C\left(\overline{\Omega}, (1, \infty)\right)$, the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) := \left\{ \begin{array}{c} u;\ u \text{ is measurable real-valued function on } \Omega \\ \text{such that } \displaystyle\int_\Omega |u(x)|^{p(x)} dx < \infty \end{array} \right\}.$$

$L^{p(x)}(\Omega)$ endowed with the Luxemburg norm

$$|u|_{p(x)} = inf \left\{ \mu > 0; \int_\Omega \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \leq 1 \right\} \tag{1.1}$$

is a reflexive Banach space [4].

It is well know that if $p_1(x) \leq p_2(x)$ almost everywhere in Ω then there exists a continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$, whose norm does not exceed $|\Omega| + 1$. We denote by $L^{q(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$ the Holder type inequality valid [4]

$$\left| \int_\Omega uvdx \right| \leq \left(\frac{1}{p^-} + \frac{1}{q^-} \right) |u|_{p(x)} |v|_{q(x)} \leq 2 |u|_{p(x)} |v|_{q(x)}. \tag{1.2}$$

An important role in manipulating for generalized Lebesgue-Sobolev spaces is played by the modular of the $L^{p(x)}(\Omega)$ space, which is the mapping $\rho_{p(x)} : L^{p(x)}(\Omega) \to \Re$ defined by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)} dx.$$

If (u_n) and u are a sequence and an element respectively in $L^{p(x)}(\Omega)$ and $p^+ < \infty$ then the following relations hold [4]:

$$|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^+}, \tag{1.3}$$

$$|u|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p^-} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^-}, \tag{1.4}$$

$$|u_n - u|_{p(x)} \to 0 \Leftrightarrow \rho_{p(x)}(u_n - u) \to 0, \tag{1.5}$$

$$|u_n|_{p(x)} \to \infty \Leftrightarrow \rho_{p(x)}(u_n) \to \infty. \tag{1.6}$$

As usual $W_0^{1,p(x)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\| = \|\nabla u\|_{p(x)}.$$

Set

$$C_+(\bar{\Omega}) = \left\{ p \in C(\bar{\Omega}) ; \inf_{x\in\bar{\Omega}} p(x) > 1 \right\}.$$

For any $p(x) \in C_+(\bar{\Omega})$, denote by $p_k^*(x) = \frac{Np(x)}{N-kp(x)}$ if $p(x) < \frac{N}{k}$ and $p_k^*(x) = +\infty$ if $p(x) \ge \frac{N}{k}$. Define the variable exponent Sobolev space $W^{k,p(x)}(\Omega)$ by

$$W^{k,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega), |\alpha| \le k \right\}.$$

where $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1{}^{\alpha_1} \ldots \partial x_N{}^{\alpha_N}}$ with $\alpha = (\alpha_1, \alpha_2, ..., \alpha_N)$ is multi-index and $|\alpha| = \sum_{i=1}^{N} \alpha_i$. The space $W^{k,p(x)}(\Omega)$ endowed with norm $\|u\| = \sum_{|\alpha|\le k} \|D^\alpha u\|_{p(x)}$ is a separable reflexive Banach space. For $p, r \in C_+(\bar{\Omega})$ in which $r(x) < p_k^*(x)$ for all $x \in \bar{\Omega}$, there is a continuous and compact embedding $W^{k,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega)$. We denote by $W_0^{k,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$ with respect to the norm $\|u\|$. For more information one can refer to [7]-[9].

2. Existence solution

In this section, we denote by $D_0^{2,p(x)}(\Omega)$ the closure of $C_c^2(\Omega)$ endowed with the norm

$$\|u\| = ||x| |\Delta u||_{p(x)}$$

$\left(D_0^{2,p(x)}(\Omega), \|.\|\right)$ is a reflexive Banach space. Assume that $q : \bar{\Omega} \to (1, \infty)$ is a continuous function satisfying $1 < q^- \le q^+ < \frac{2Np^-}{2N+p^-}$. We investigate the existence of solutions for the following problem

$$\begin{cases} \Delta\left(|x|^{p(x)} |\Delta u(x)|^{p(x)-2} \Delta u(x)\right) = \lambda |u(x)|^{q(x)-2} u(x) & \text{for } x \in \Omega, \\ u(x) = \Delta u = 0 & \text{for } x \in \partial\Omega, \end{cases} \tag{2.1}$$

where λ is a negative constant. The above equation called weighted $p(x)$-Laplacian equation. If

$$\int_\Omega |x|^{p(x)} |\Delta u|^{p(x)-2} \Delta u \Delta v dx - \int_\Omega |u|^{q(x)-2} uvdx = 0, \qquad \forall v \in D_0^{2,p(x)}(\Omega),$$

then $u \in D_0^{2,p(x)}(\Omega)$ is a weak solution of problem (2.1).

Now, we show the following existence result for (2.1).

Theorem 1. *For each $\lambda < 0$, (2.1) has a nontrivial weak solution.*

Proof. For each $\lambda < 0$, we consider the energy functional associated with problem (2.1), $J_\lambda : D_0^{2.p(x)}(\Omega) \to \Re$ by

$$J_\lambda(u) = \int_\Omega \frac{|x|^{p(x)}}{p(x)} |\Delta u|^{p(x)} dx - \lambda \int_\Omega \frac{1}{q(x)} |u|^{q(x)} dx$$

for each $u \in D_0^{2,p(x)}(\Omega)$. Standard arguments show that $J_\lambda \in C^1\left(D_0^{2,p(x)}(\Omega), \Re\right)$ and its derivative is given by

$$\left\langle \acute{J}_\lambda(u), v\right\rangle = \int_\Omega |x|^{p(x)} |\Delta u|^{p(x)-2} \Delta u \Delta v dx - \lambda \int_\Omega |u|^{q(x)-2} uvdx,$$

for all $u \in D_0^{2,p(x)}(\Omega)$. We infer that u is a solution of problem (2.1) if and only if it is a critical point of J_λ. Consequently, we concentrate our efforts on finding critical points for J_λ. In this context we prove the following assertions:

(a) J_λ is weakly lower semi-continuous.

(b) J_λ is bounded from below and coercive.

(c) There exists $\psi \in D_0^{2,p(x)}(\Omega) - \{0\}$ such that $J_\lambda(\Psi) < 0$.

The arguments to prove $(a), (b)$ and (c) are detailed as below.

(a) First, we prove that the functional $\Lambda : D_0^{2,p(x)}(\Omega) \to \Re$ defined by

$$\Lambda(u) = \int_\Omega \frac{|x|^{p(x)}}{p(x)} |\Delta u|^{p(x)} dx,$$

is convex. Indeed, since the function

$$[0, \infty) \ni t \to t^\theta,$$

is convex for any $\theta > 1$, so for each $x \in \Omega$

$$\left|\frac{\xi + \psi}{2}\right|^{p(x)} \le \left(\frac{|\xi| + |\psi|}{2}\right)^{p(x)} \le \frac{1}{2}|\xi|^{p(x)} + \frac{1}{2}|\Psi|^{p(x)}, \ \forall \xi, \psi \in \Re^N,$$

holds. Using this inequality,

$$\left|\frac{\Delta u+\Delta v}{2}\right|^{p(x)} \leq \left(\frac{|\Delta u|+|\Delta v|}{2}\right)^{p(x)} \leq \frac{1}{2}|\Delta u|^{p(x)}+\frac{1}{2}|\Delta v|^{p(x)},$$

$\forall u, v \in D_0^{2,p(x)}(\Omega)$.

Multiplying with $\frac{|x|^{p(x)}}{p(x)}$ and integrating over Ω we obtain:

$$\Lambda\left(\frac{u+v}{2}\right) \leq \frac{1}{2}\Lambda(u)+\frac{1}{2}\Lambda(v), \forall u, v \in D_0^{2,p(x)}(\Omega).$$

Next, we show that Λ is weakly lower semi continuous on $D_0^{2,p(x)}(\Omega)$. Taking into account that Λ is convex, by Corollary 3.8 in [3] it is enough to show that Λ is strongly lower semi-continuous on $D_0^{2,p(x)}(\Omega)$. We fix $u \in D_0^{2,p(x)}(\Omega)$, $\epsilon > 0$ and $v \in D_0^{2,p(x)}(\Omega)$. Since Λ is convex and holder type inequality (1.2) holds true,

$$\begin{aligned}
\Lambda(v) &\geq \Lambda(u)+\langle \Lambda'(u), v-u\rangle \\
&\geq \Lambda(u)-\int_\Omega |\Delta u|^{p(x)-1}|\Delta(v-u)|\,dx \\
&\geq \Lambda(u)-D_1\left||\Delta u|^{p(x)-1}\right|_{\frac{p(x)}{p(x)-1}}|\Delta(u-v)|_{p(x)} \\
&\geq \Lambda(u)-D_2\|u-v\|_{p(x)} \\
&\geq \Lambda(u)-\epsilon,
\end{aligned}$$

for all $v \in D_0^{2,p(x)}(\Omega)$ with

$$|u-v|_{p(x)} < \frac{\epsilon}{||\Delta u|^{p(x)-1}|_{\frac{p(x)}{p(x)-1}}}.$$

We have denoted by D_1, D_2 two positive constants. It follows that Λ is strongly lower semi continuous and convex, so (by corollary 3.8 in [3]) Λ is weakly lower semi continuous. Finally, if $\{u_n\} \subset D_0^{2,p(x)}(\Omega)$ is a sequence weakly converges to u in $D_0^{2,p(x)}(\Omega)$, then $\{u_n\}$ converges strongly to u in $L^{q(x)}(\Omega)$. Thus, J_λ is weakly lower semi continuous and proof of (a) is complete.

(b) It is clear that for any $u \in D_0^{2,p(x)}(\Omega)$

$$\begin{aligned}
J_\lambda(u) &\geq \frac{1}{p^+}\int_\Omega |x|^{p(x)}|\Delta u|^{p(x)}dx-\frac{\lambda}{q^-}\int_\Omega |u|^{q(x)}dx, \\
&\geq \frac{1}{p^+}\int_\Omega |x|^{p(x)}|\Delta u|^{p(x)}dx-\frac{\lambda}{q^-}\left(|u|_{q(x)}^{q^-}+|u|_{q(x)}^{q^-}\right).
\end{aligned}$$

Since $1 < q^- \leq q \leq q^+ < \frac{2Np^-}{2N+p^-} < p^-$, by Theorem 2 in [9] there is a continuous compact embedding of $D_0^{2,p(x)}(\Omega)$ into $L^{q(x)}(\Omega)$ such that,

$$\exists c > 0 : |u|_{q(x)} \leq c\,\|u\|\,, \forall u \in D_0^{2,p(x)}(\Omega)\,.$$

If $\|u\| > 1$, using this inequality,

$$J_\lambda(u) \geq \frac{1}{p^+}\|u\|^{p^-} - \frac{k\lambda}{q^-}\left(\|u\|^{q^-} + \|u\|^{q^+}\right),$$

where k is a positive constant. For $\lambda < 0$, $\lim\limits_{\|u\|\to\infty} J_\lambda(u) = \infty$, means that J_λ is coercive. It is clear that for any $u \in D_0^{2,p(x)}(\Omega)$

$$J_\lambda(u) \geq \frac{1}{p^+}\min\left\{\|u\|^{p^+}, \|u\|^{p^-}\right\} - \frac{k\lambda}{q^-}\left(\|u\|^{q^-} + \|u\|^{q^+}\right).$$

We deduce that J_λ is bounded from below.

(c) Suppose that $\varphi \in C_c^2(\Omega), \varphi \neq 0$. Then, for each $t \in (0.1)$,

$$\begin{aligned} J_\lambda(t\varphi) &= \int_\Omega \frac{|x|^{p(x)}\, t^{p(x)}}{p(x)} |\Delta\varphi|^{p(x)}\, dx - \lambda \int_\Omega \frac{t^{q(x)}}{q(x)} |\varphi|^{q(x)}\, dx \\ &\leq t^{p^-} \int_\Omega \frac{|x|^{p(x)}}{p(x)} |\Delta\varphi|^{p(x)}\, dx - \lambda t^{q^+} \int_\Omega \frac{1}{q(x)} |\varphi|^{q(x)}\, dx \\ &\leq \frac{t^{p^-}}{p^-}\max\left\{\|\varphi\|^{p^-}, \|\varphi\|^{p^+}\right\} - \frac{\lambda t^{q^+}}{q^+}\left(|\varphi|_{q(x)}^{q^+} + |\varphi|_{q(x)}^{q^-}\right). \end{aligned}$$

Set

$$L_1 = \frac{1}{p^+}\max\left\{|\varphi|^{p^-}, |\varphi|^{p^+}\right\}$$

and

$$L_2 = \frac{\lambda}{q^+}\left(|\varphi|_{q(x)}^{q^+} + |\varphi|_{q(x)}^{q^-}\right).$$

Then

$$\begin{aligned} J_\lambda(t\varphi) &\leq L_1 t^{p^-} - L_2 t^{q^+},\ q^+ < p^- \\ &= L_1 t^{p^-}\left(1 - \frac{L_2}{L_1} t^{q^+ - p^-}\right). \end{aligned}$$

Since L_1 is a positive constant, from this inequality

$$\begin{aligned} J_\lambda < 0 &\iff 1 - \frac{L_2}{L_1}\, t^{q^+ - p^-} < 0 \\ &\iff t < \left(\frac{L_2}{L_1}\right)^{\frac{1}{p^- - q^+}}. \end{aligned}$$

We infer that, for any $t \in (0, \min\{1, (\frac{L_2}{L_1})^{\frac{1}{p^- - q^+}}\})$,

$$J_\lambda(t\varphi) < 0.$$

These facts together with Theorem (1) of [13] implies the existence of $u_\lambda \in D_0^{2,p(x)}(\Omega)$ as a global minimum point of J_λ. Moreover, since (c) hold true it follows that $u_\lambda \neq 0$. ∎

References

[1] Acerbi, E., Mingion, G., *Regularrity results for a class of functional with non-standard growth*, Arch. Rational Mech. Anal., 156 (2001), 121-140.

[2] Bonanno, G., Bella, B.Di., *A boundary value problem for forth-order elastic beam equations*, J. Math. Anal. Appl., 343 (2) (2008), 1116-1176.

[3] Brezis, H., *Analise functional Theorie Methodes et Applications*, Masson Paris, 1992.

[4] Denisa, S.D., Mihailescu, M., Radulescu, V., *A Caffarrelli-Kohn-Nirenberg.type in equality with variable exponent and applications to PDEs*, Complex Variables and Elliptic Equations, 56 (2011), 659-669.

[5] Ferrero, A., Warnault, G., *On a solutions of second and forth order elliptic with power type nonlinearities*, Nonlinear Anal. T.M.A, 70 (2009), 2889-2902.

[6] Harjulehto, P., Hasto, P., Le, U.V., Nuortio, M., *Overview of differential equations with non-standard growth*, Nonlinear Anal., 72, (2010), 4551-4574.

[7] Kovacik, O., Rakosnik, J., *On spaces $L^{p(x)}$ $W^{1,p(x)}$*, Czechoslovak Math. J., 41 (1991), 592-618.

[8] Micheletti, A.M., Pistoia, A., *Multiplicity results for a fourth-order semilinear elliptic problem*, Nonlinear Anal., 31 (7) (1998), 895-908.

[9] Mihaillescu, M., Radulescu, V., *A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids*, Proc. R. Soc. Lond. Ser. A, 462 (2006), 2625-2641.

[10] Musielak, J., *Orlicz spaces and modular spaces*, Lecture notes in mathematics, Springer, Berlin, vol. 1034, 1983.

[11] Rajagopal, K.R., Ruzicka, M.,

Mathematical modeling of electrorheological fluids, Mech. Thermodyn., 13 (2001), 59-78.

[12] Ruzicka, M., *Electrorheological fluids modeling and mathematical theory*, Springer-Verlag, Berlin, 2002.

[13] Struwe, M., *Variational methods: Applications to nonlinear partial differential equations and Hamiltonian systems*, Spinger, Heidelberg, 1996.

[14] Zhikov, V., *Averaging of functional of the calculus of variations elasticity theory*, Math. USSR. IZV, 29 (1987), 33-66.

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ON REVERSE WEIGHTED ARITHMETIC-GEOMETRIC MEAN INEQUALITIES FOR TWO POSITIVE OPERATORS

Jianming Xue
Oxbridge College
Kunming University of Science and Technology
Kunming, Yunnan 650106
P.R. China
e-mail: xuejianming104@163.com

Abstract. Let A, B be positive operators on a Hilbert space with $0 < m \leq A, B \leq M$. Then for every positive unital linear map Φ,

$$\begin{aligned}
(A\nabla_\mu B)^2 &\leq \left[\frac{(M+m)^2}{4Mm}\right]^2 (A\sharp_\mu B)^2, && 0 \leq \mu \leq 1,\\
\Phi^2(A\nabla_\mu B) &\leq \left[\frac{(M+m)^2}{4Mm}\right]^2 \Phi^2(A\sharp_\mu B), && 0 \leq \mu \leq 1,\\
\Phi^2(A\nabla_\mu B) &\leq \left[\frac{(M+m)^2}{4Mm}\right]^2 [\Phi(A)\sharp_\mu \Phi(B)]^2, && 0 \leq \mu \leq 1.
\end{aligned}$$

Keywords: operator inequalities, weighted arithmetic-geometric mean inequalities, positive linear maps.

(2010) Mathematical Subject Classification: 47A63; 47A30.

1. Introduction

Throughout this paper, let M, m be scalars, I be the identity operator and $\mathcal{B}(\mathcal{H})$ be the set of all bounded linear operators on a Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$. The operator norm is denoted by $\|\cdot\|$. We write $A \geq 0$ if the operator A is positive. If $A - B \geq 0$, then we say that $A \geq B$. For $A, B > 0$, we use the following notation:

$$A\nabla_\mu B = (1-\mu)A + \mu B, \quad A\sharp_\mu B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\mu A^{\frac{1}{2}}, \quad \text{where } 0 \leq \mu \leq 1.$$

When $\mu = \frac{1}{2}$, we write $A\nabla B$ and $A\sharp B$ for brevity for $A\nabla_{\frac{1}{2}} B$ and $A\sharp_{\frac{1}{2}} B$, respectively, see Kubo and Ando [1].

Let A and B be positive operators on a Hilbert space with $0 < m \leq A$, $B \leq M$. Tominaga [2] showed that the following reverse AM-GM inequality with Specht ratio:

$$A\nabla_\mu B \leq S(h) A\sharp_\mu B, \tag{1.1}$$

where $S(h) = \frac{h^{\frac{1}{h-1}}}{e \log h^{\frac{1}{h-1}}}$ for $h = \frac{M}{m} \geq 1$. Indeed,

$$S(h) \leq \frac{(M+m)^2}{4Mm} \leq S^2(h) \ (h \geq 1) \tag{1.2}$$

was observed by Lin [3, (3.3)].

Inequality (1.1) can be regarded as a counterpart to operator AM-GM inequality which says

$$A\nabla_\mu B \geq A\sharp_\mu B. \tag{1.3}$$

By (1.1) and (1.2), we have the following inequalities:

$$A\nabla_\mu B \leq \frac{(M+m)^2}{4Mm} A\sharp_\mu B, \tag{1.4}$$

$$\Phi(A\nabla_\mu B) \leq \frac{(M+m)^2}{4Mm} \Phi(A\sharp_\mu B). \tag{1.5}$$

It is well known that for two positive operators A, B,

$$A \geq B \nRightarrow A^2 \geq B^2.$$

Can inequalities (1.4) and (1.5) be squared? This is a main motivation for the present paper.

In this paper, the main results are that inequalities (1.4) and (1.5) can be squared, which we will present in the next section.

2. Main results

We begin this section with the following lemmas.

Lemma 2.1. [4] *Let $A, B > 0$. Then the following norm inequality holds:*

$$\|AB\| \leq \frac{1}{4} \|A+B\|^2 . \tag{2.1}$$

Lemma 2.2. [5] *Let A be a positive operator on a Hilbert space. Then for every positive unital linear map Φ,*

$$\Phi(A^{-1}) \geq \Phi^{-1}(A). \tag{2.2}$$

Theorem 2.3. *Let $0 < m \leq A, B \leq M$. Then*

$$(A\nabla_\mu B)^2 \leq \left[\frac{(M+m)^2}{4Mm}\right]^2 (A\sharp_\mu B)^2, \quad 0 \leq \mu \leq 1, \tag{2.3}$$

or equivalently

$$\left\|(A\nabla_\mu B)(A\sharp_\mu B)^{-1}\right\| \le \frac{(M+m)^2}{4Mm}, \quad 0 \le \mu \le 1. \tag{2.4}$$

Proof. By Lemma 2.1, inequality (2.4) is true if

$$A\nabla_\mu B + Mm(A\sharp_\mu B)^{-1} \le M+m.$$

By $(A\sharp_\mu B)^{-1} = A^{-1}\sharp_\mu B^{-1}$ and inequality (1.3), we have

$$\begin{aligned} A\nabla_\mu B + Mm(A\sharp_\mu B)^{-1} &\le A\nabla_\mu B + MmA^{-1}\nabla_\mu B^{-1} \\ &= (1-\mu)A + \mu B + Mm[(1-\mu)A^{-1} + \mu B^{-1}] \\ &\le M+m, \end{aligned}$$

where the last inequality is by [6, (2.3)]. This proves (2.4). ∎

Theorem 2.4. *Let $0 < m \le A, B \le M$. Then for every positive unital linear map Φ,*

$$\Phi^2(A\nabla_\mu B) \le \left[\frac{(M+m)^2}{4Mm}\right]^2 \Phi^2(A\sharp_\mu B), \qquad 0 \le \mu \le 1, \tag{2.5}$$

and

$$\Phi^2(A\nabla_\mu B) \le \left[\frac{(M+m)^2}{4Mm}\right]^2 [\Phi(A)\sharp_\mu \Phi(B)]^2, \quad 0 \le \mu \le 1. \tag{2.6}$$

Proof. Inequality (2.5) is equivalent to

$$\left\|\Phi(A\nabla_\mu B)\Phi^{-1}(A\sharp_\mu B)\right\| \le \frac{(M+m)^2}{4Mm}. \tag{2.7}$$

Compute

$$\begin{aligned} &\left\|\Phi(A\nabla_\mu B)Mm\Phi^{-1}(A\sharp_\mu B)\right\| \\ &\quad\le \frac{1}{4}\left\|\Phi(A\nabla_\mu B) + Mm\Phi^{-1}(A\sharp_\mu B)\right\|^2 && (\text{by } (2.1)) \\ &\quad\le \frac{1}{4}\left\|\Phi(A\nabla_\mu B) + Mm\Phi[(A\sharp_\mu B)^{-1}]\right\|^2 && (\text{by } (2.2)) \\ &\quad= \frac{1}{4}\left\|\Phi(A\nabla_\mu B) + Mm\Phi[(A^{-1}\sharp_\mu B^{-1})]\right\|^2 \\ &\quad\le \frac{1}{4}\left\|\Phi(A\nabla_\mu B) + Mm\Phi[(A^{-1}\nabla_\mu B^{-1})]\right\|^2 && (\text{by } (1.3)) \\ &\quad\le \frac{1}{4}(M+m)^2. && (\text{by } [6, (2.3)]) \end{aligned}$$

That is,

$$\left\|\Phi(A\nabla_\mu B)\Phi^{-1}(A\sharp_\mu B)\right\| \leq \frac{(M+m)^2}{4Mm}.$$

Thus, (2.7) holds. The proof of (2.6) is similar, we omit the details. ∎

Remark 2.5. When $\mu = \frac{1}{2}$, by (2.5) and (2.6) we obtain [3, (2.1)] and [3, (2.2)], respectively. Thus, (2.5) and (2.6) are generalizations of [3, (2.1)] and [3, (2.2)], respectively.

References

[1] Kubo, F., Ando,T., *Means of positive linear operators*, Math. Ann., 246 (1980), 205-244.

[2] Tominaga, M., *Specht's ratio in the Young inequality*, Sci. Math. Japon., 55 (2002), 583-588.

[3] Lin, M., *Squaring a reverse AM-GM inequality*, Stud. Math., 215 (2013), 187-194.

[4] Bhatia, R., Kittaneh, F., *Notes on matrix arithmetic-geometric mean inequalities*, Linear Algebra Appl., 308 (2000), 203-211.

[5] Bhatia, R., *Positive definite matrices*, Princeton University Press, Princeton, 2007.

[6] Lin, M., *On an operator Kantorovich inequality for positive linear maps*, J. Math. Anal. Appl., 402 (2013), 127-132.

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SOME INCLUSION PROPERTIES OF STARLIKE AND CONVEX FUNCTIONS ASSOCIATED WITH HOHLOV OPERATOR. II

M. Kasthuri
K. Vijaya [1]
K. Uma

School of advanced Sciences
VIT University
Vellore – 632014
India

Abstract. A new subclass $\mathbf{K}(\lambda, \alpha)$ involving Hohlov Operator is introduced and some inclusion relations and distortion bounds are obtained for $f \in \mathbf{K}(\lambda, \alpha)$.

Keywords and Phrases: Gaussian hypergeometric functions, Convex functions, Starlike functions, Hadamard product, Carlson-Shaffer operator, Hohlov operator.

2000 Mathematics Subject Classification: 30C45.

1. Introduction

Let $\mathcal{A}$ be the class of functions f normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

As usual, we denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions which are also univalent in $\mathbb{U}$. The well known subclasses of $\mathcal{S}$ are the class of starlike functions($\mathcal{ST}$) and convex functions($\mathcal{CV}$). A function $f(z) \in \mathcal{S}$ is starlike of order $\alpha(0 \leq \alpha < 1)$ denoted by $\mathcal{ST}(\alpha)$, if $\text{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha$ and it is convex of order $\alpha(0 \leq \alpha < 1)$ denoted by $\mathcal{CV}(\beta)$, if $\text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta$. It is an established fact that $f \in \mathcal{CV}(\alpha) \Longleftrightarrow zf' \in \mathcal{ST}(\alpha)$.

[1] Corresponding author. E-mail: kvijaya@vit.ac.in

For functions $f \in \mathcal{A}$ given by (1.1) and $g \in \mathcal{A}$ given by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad (z \in \mathbb{U}). \tag{1.2}$$

Let $\mathcal{T}$ denote the subclass of $\mathcal{A}$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0;\ z \in \mathbb{U}). \tag{1.3}$$

The class $\mathcal{T}$ was introduced by Silverman [10]. We denote by $\mathcal{T}^*(\alpha)$ and $\mathcal{C}(\alpha)$ denote the class of functions of the form (1.3) which are, respectively, starlike of order α and convex of order α with $0 \leq \alpha < 1$.

The Gaussian hypergeometric function $F(a, b; c, z)$ given by

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n \quad (z \in \mathbb{U}) \tag{1.4}$$

where, a, b, c are complex numbers such that $c \neq 0, -1, -2, -3, \ldots,$ $(a)_0 = 1$ for $a \neq 0$ and for each positive integer n, $(a)_n = a(a+1)(a+2)\ldots(a+n-1)$ is the Pochhammer symbol, and is the solution of the homogenous hypergeometric differential equation

$$z(1-z)w''(z) + [c - (a+b+1)z]w'(z) - abw(z) = 0$$

has rich applications in various fields such as conformal mappings, quasi conformal theory, continued fractions and so on. The Gauss Summation theorem

$$F(a, b; c; 1) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} = \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)} \quad \text{for } \mathrm{Re}(c-a-b) > 0 \tag{1.5}$$

and the function $F(a, b; c; 1)$ is bounded if $\mathrm{Re}(c-a-b) > 0$ and has a pole at $z = 1$ if $\mathrm{Re}(c-a-b) \leq 0$.

For $f \in \mathcal{A}$, we recall the operator $I_{a,b,c}(f)$ of Hohlov [5] which maps $\mathcal{A}$ into itself defined by means of Hadamard product as

$$I_{a,b,c}(f)(z) = zF(a, b; c; z) * f(z) \tag{1.6}$$

Therefore, for a function f defined by (1.1), we have

$$I_{a,b,c}(f)(z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1} (b)_{n-1}}{(c)_{n-1} (1)_{n-1}} a_n z^n. \tag{1.7}$$

$$\Phi(n) = \frac{(a)_{n-1} (b)_{n-1}}{(c)_{n-1} (1)_{n-1}} \quad (a, b > 0;\ n \geq 2). \tag{1.8}$$

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}^\tau(A,B)$, $(\tau \in \mathbb{C}\backslash\{0\},\ -1 \le B < A \le 1)$, if it satisfies the inequality

$$\left|\frac{f'(z)-1}{(A-B)\tau - B[f'(z)-1]}\right| < 1 \quad (z \in \mathbb{U}).$$

The class $\mathcal{R}^\tau(A,B)$ was introduced earlier by Dixit and Pal [3]. If we put

$$\tau = 1,\ A = \beta \ \text{ and } \ B = -\beta \ (0 < \beta \le 1),$$

we obtain the class of functions $f \in \mathcal{A}$ satisfying the inequality

$$\left|\frac{f'(z)-1}{f'(z)+1}\right| < \beta \quad (z \in \mathbb{U}; 0 < \beta \le 1)$$

which was studied by (among others) Padmanabhan [8] and Caplinger and Causey [2], (see also [12]). We recall the following lemma relevant for our discussions.

Lemma 1.1 [3] *If $f \in \mathcal{R}^\tau(A,B)$ is of form* (1.1)*, then*

$$|a_n| \le (A-B)\frac{|\tau|}{n}, \quad n \in \mathbb{N}\setminus\{1\}. \tag{1.9}$$

The result is sharp for the function

$$f(z) = \int_0^z \left(1 + \frac{(A-B)\tau z^{n-1}}{1+Bz^{n-1}}\right) dz, \qquad (n \ge 2;\ z \in \mathbb{U}).$$

In this paper, we consider the following subclass of $\mathcal{S}$ due to Kamali et al. [7] as given below:

For some $\alpha(0 \le \alpha < 1)$ and $\lambda(0 \le \lambda < 1)$, we let $\mathbf{K}(\lambda,\alpha)$ be a new subclass of $\mathcal{S}$ consisting of functions of the form (1.1) satisfying the analytic criteria

$$\text{Re}\left(\frac{\lambda z^3 f'''(z) + (2\lambda+1)z^2 f''(z) + zf'(z)}{\lambda z^2 f''(z) + zf'(z)}\right) > \alpha,\ z \in \mathbb{U}.$$

We recall the following lemma due to Kamali et al. [7] to prove the main results.

Lemma 1.2 *A function $f \in \mathcal{T}$ belongs to the class $\mathbf{K}(\lambda,\alpha)$ if and only if*

$$\sum_{n=2}^{\infty} n(n-\alpha)(1+n\lambda-\lambda)|a_n| \le 1-\alpha. \tag{1.10}$$

Lemma 1.3 [10] *A function f of the form* (1.3) *is in $\mathcal{T}^*(\alpha)$ if and only if*

$$\sum_{n=2}^{\infty}(n-\alpha)a_n \le 1-\alpha \quad (0 \le \alpha < 1)$$

and is in $\mathcal{C}(\alpha)$ if and only if

$$\sum_{n=2}^{\infty} n(n-\alpha)a_n \le 1-\alpha \quad (0 \le \alpha < 1).$$

Motivated by the earlier works on hypergeometric functions studied recently in [9], [11]–[14], we will study the action of the hypergeometric function on the class $\mathbf{K}(\lambda, \alpha)$.

2. Main results

Theorem 2.1 [14] *Let $a, b \in \mathbb{C} \setminus \{0\}$, and c be a real number. If $f \in \mathcal{ST}$ and the inequality*

$$\begin{aligned}(2.11)\quad &\lambda \frac{|a||b|(1+|a|)(1+|b|)(2+|a|)(2+|b|)(3+|a|)(3+|b|)}{c(1+c)(2+c)(3+c)}\ {}_2F_1(4+|a|, 4+|b|; 4+c, 1)\\ &+[1-\lambda(\alpha-9)]\frac{|a||b|(1+|a|)(1+|b|)(2+|a|)(2+|b|)}{c(1+c)(2+c)}\ {}_2F_1(3+|a|, 3+|b|; 3+c, 1)\\ &+[6-\lambda(5\alpha-19)-\alpha]\frac{|a||b|(1+|a|)(1+|b|)}{c(1+c)}\ {}_2F_1(2+|a|, 2+|b|; 2+c, 1)\\ &+(1-\alpha)\frac{|ab|}{c}\ {}_2F_1(1+|a|, 1+|b|; 1+c, 1) \leq 1-\alpha\end{aligned}$$

is satisfied, then $I_{a,b,c}(f) \in \mathbf{K}(\lambda, \alpha)$.

Theorem 2.2 [14] *Let $a, b \in \mathbb{C} \setminus \{0\}$ and let c be a real number. If $f \in \mathcal{CV}$ and the inequality*

$$\begin{aligned}(2.12)\quad &\lambda \frac{|a||b|(1+|a|)(1+|b|)(2+|a|)(2+|b|)}{c(1+c)(2+c)}\ {}_2F_1(3+|a|, 3+|b|; 3+c, 4; 1)\\ &+[1-\lambda(\alpha-5)]\frac{|a||b|(1+|a|)(1+|b|)}{c(1+c)}\ {}_2F_1(2+|a|, 2+|b|; 2+c, 3; 1)\\ &+[3-2\lambda(\alpha-2)-\alpha]\frac{|ab|}{c}\ {}_2F_1(1+|a|, 1+|b|; 1+c, 2; 1)\\ &+(1-\alpha)\ {}_2F_1(|a|, |b|; c, 1; 1) \leq 2(1-\alpha)\end{aligned}$$

is satisfied, then $I_{a,b,c}(f) \in \mathbf{K}(\lambda, \alpha)$.

Theorem 2.3 *Let $a, b \in \mathbb{C} \setminus \{0\}$ and let c be a real number such that $c > |a| + |b| + 1$. If $f \in \mathcal{R}^{\tau}(A, B)$ and if the inequality*

$$\begin{aligned}(2.13)\quad &\lambda \frac{|a||b|(1+|a|)(1+|b|)}{c(1+c)} F(2+|a|, 2+|b|, 2+c; 1)\\ &+[1-\lambda(\alpha-2)]\frac{|ab|}{c} F(1+|a|, 1+|b|, 1+c; 1) + (1-\alpha) F(|a|, |b|, c; 1)\\ &\leq (1-\alpha)\left(\frac{1}{(A-B)|\tau|} + 1\right)\end{aligned}$$

is satisfied, then $I_{a,b,c}(f) \in \mathbf{K}(\lambda, \alpha)$.

Proof. Let f be of the form (1.1) belong to the class $\mathcal{R}^\tau(A,B)$. By virtue of Lemma 1.1, it suffices to show that

$$\sum_{n=2}^{\infty} n(n-\alpha)(1+n\lambda-\lambda)\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n\right| \le 1-\alpha. \tag{2.14}$$

Taking into account the inequality (1.9) and the relation $|(a)_{n-1}| \le (|a|)_{n-1}$, we deduce that

$$\begin{aligned}
&\sum_{n=2}^{\infty} n(n-\alpha)(1+n\lambda-\lambda)\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n\right| \\
&\quad \le (A-B)|\tau|\left(\lambda\sum_{n=2}^{\infty}(n-1)(n-2)\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right|\right. \\
&\quad \left. + [1-\lambda(\alpha-2)]\sum_{n=2}^{\infty}(n-1)\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right| + (1-\alpha)\ \sum_{n=2}^{\infty}\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right|\right) \\
&\quad \le (A-B)|\tau|\left(\lambda\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-3}} + [1-\lambda(\alpha-2)]\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-2}}\right. \\
&\quad \left. + (1-\alpha)\ \sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right) \\
&\quad = (A-B)|\tau|\left(\lambda\frac{(|a|_2)(|b|_2)}{(c)_2}\sum_{n=2}^{\infty}\frac{(2+|a|)_{n-3}(2+|b|)_{n-3}}{(2+c)_{n-3}(1)_{n-3}}\right. \\
&\quad \left. + [1-\lambda(\alpha-2)]\frac{|ab|}{c}\sum_{n=2}^{\infty}\frac{(1+|a|)_{n-2}(1+|b|)_{n-2}}{(1+c)_{n-2}(1)_{n-2}} + (1-\alpha)\ \sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right) \\
&\quad = (A-B)|\tau|\left(\lambda\frac{|a||b|(1+|a|)(1+|b|)}{c(1+c)}F(2+|a|,2+|b|,2+c;1)\right. \\
&\quad \left. + [1-\lambda(\alpha-2)]\frac{|ab|}{c}F(1+|a|,1+|b|,1+c;1) + (1-\alpha)\left(F(|a|,|b|,c;1)-1\right)\right),
\end{aligned}$$

where we use the relation $(a)_n = a(a+1)_{n-1}$.

The proof now follows by an application of the Gauss summation theorem and (1.5). ∎

Next, we prove the following properties for the operator $I_{a,b;c}(f)$, when a function f belongs to the class $\mathbf{K}(\lambda,\alpha)$.

Theorem 2.4 *Let $a,b>0$, $c \ge \max\{0, a+b-1, (1/2)(ab+a+b-1)\}$ and let a function f of the form* (1.3) *be in $\mathbf{K}(\lambda,\alpha)$. Then*

$$|z| - \frac{(1-\alpha)}{2(2-\alpha)(1+\lambda)}\ \frac{ab}{c}\,|z|^2 \le |I_{a,b;c}f(z)| \le |z| + \frac{(1-\alpha)}{2(2-\alpha)(1+\lambda)}\ \frac{ab}{c}\,|z|^2 \tag{2.15}$$

and

$$1-\frac{(1-\alpha)}{(2-\alpha)(1+\lambda)}\,\frac{ab}{c}\,|z| \le |(I_{a,b;c}f(z))'| \le 1+\frac{(1-\alpha)}{(2-\alpha)(1+\lambda)}\,\frac{ab}{c}\,|z|. \tag{2.16}$$

The results are sharp.

Proof. We note that

$$I_{a,b;c}f(z)=\Big(zF(a,b;c;z)*f\Big)(z)=z-\sum_{n=2}^{\infty}\Phi(n)a_nz^n,$$

where

$$\Phi(n)=\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\quad (a,b>0;\ n\ge 2)$$

and $0<\Phi(n+1)\le\Phi(n)$ $(n\ge 2)$ under the assumption for c. Since $f\in\mathbf{K}(\lambda,\alpha)$, by Lemma 1.2, we have

$$2(2-\alpha)(1+\lambda)\sum_{n=2}^{\infty}a_n \le \sum_{n=2}^{\infty}n(n-\alpha)(1+n\lambda-\lambda)a_n \le 1-\alpha. \tag{2.17}$$

Therefore, by using (2.17), we obtain

$$\begin{aligned}
|I_{a,b;c}(f)| &\le |z|+\sum_{n=2}^{\infty}\Phi(n)a_n|z|^n\\
&\le |z|+\Phi(2)|z|^2\sum_{n=2}^{\infty}a_n\\
&\le |z|+\frac{(1-\alpha)}{2(2-\alpha)(1+\lambda)}\frac{ab}{c}|z|^2
\end{aligned}$$

and

$$\begin{aligned}
|I_{a,b;c}(f)| &\ge |z|-\sum_{n=2}^{\infty}\Phi(n)a_n|z|^n\\
&\ge |z|-\Phi(2)|z|^2\sum_{n=2}^{\infty}a_n\\
&\ge |z|-\frac{(1-\alpha)}{2(2-\alpha)(1+\lambda)}\frac{ab}{c}|z|^2.
\end{aligned}$$

From (2.17), we note that

$$\sum_{n=2}^{\infty}na_n \le \frac{(1-\alpha)}{(2-\alpha)(1+\lambda)}. \tag{2.18}$$

By using (2.18), we obtain (2.16). The results are sharp for the function $f(z)=z-\frac{1-\alpha}{2(2-\alpha)(1+\lambda)}z^2$. ∎

Now, we find the order β $(0\le\beta<1)$ for which the operator $I_{a,b;c}(f)$ belongs to the classes $\mathcal{T}^*(\beta)$ and $\mathcal{C}(\beta)$ when a function f belongs to the class $\mathbf{K}(\lambda,\alpha)$.

Theorem 2.5 *Let* $a, b > 0$, $\max\{2ab/3, a+b-1, (1/2)(ab+a+b-1)\} \le c \le ab$ *and let a function* f *of the form* (1.3) *be in* $\mathbf{K}(\lambda, \alpha)$. *Then* $I_{a,b;c}(f) \in \mathcal{T}^*(\beta)$, *where*

$$\beta = 1 - \frac{\Phi(2)(1-\alpha)}{2(2-\alpha)(1+\lambda) - \Phi(2)(1-\alpha)}. \tag{2.19}$$

Proof. Let $f \in \mathbf{K}(\lambda, \alpha)$. Consider the operator

$$I_{a,b;c}f(z) = z + \sum_{n=2}^{\infty} \Phi(n) a_n z^n,$$

where

$$\Phi(n) = \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \quad (a, b > 0;\ n \ge 2).$$

Since $\Phi(n)$ is a decreasing function for n, by Lemma 1.3, we need to find β $(0 \le \beta < 1)$ that

$$\Phi(2) \sum_{n=2}^{\infty} \frac{n-\beta}{1-\beta} a_n \le 1.$$

Since $f \in \mathbf{K}(\lambda, \alpha)$, by Lemma 1.2, we have

$$\sum_{n=2}^{\infty} n(n-\alpha)(1+n\lambda-\lambda)a_n \le 1-\alpha.$$

To complete the proof, it suffices to find β such that

$$\frac{n-\beta}{1-\beta}\Phi(2) \le \frac{n(n-\alpha)(1+n\lambda-\lambda)}{1-\alpha}. \tag{2.20}$$

From (2.20), we obtain

$$\beta \le \Psi(n),$$

where

$$\Psi(n) = \frac{n(n-\alpha)(1+n\lambda-\lambda) - n\Phi(2)(1-\alpha)}{n(n-\alpha)(1+n\lambda-\lambda) - \Phi(2)(1-\alpha)}\,. \tag{2.21}$$

By the assumption of the theorem, it is easy to see that $\Psi(n)$ is an increasing function for n $(n \ge 2)$. Setting $n = 2$ in (2.21), we have

$$\beta = \frac{2(2-\alpha)(1+\lambda) - 2\Phi(2)(1-\alpha)}{2(2-\alpha)(1+\lambda) - \Phi(2)(1-\alpha)},$$

hence we get(2.19). Therefore we complete the proof of Theorem 2.5. ∎

Theorem 2.6 *Let* $a, b > 0$, $\max\{2ab/3, a+b-1, (1/2)(ab+a+b-1)\} \le c \le ab$ *and let a function* f *of the form* (1.3) *be in* $\mathbf{K}(\lambda, \alpha)$. *Then* $I_{a,b;c}(f) \in \mathcal{C}^*(\beta)$, *where*

$$\beta = 1 - \frac{\Phi(2)(1-\alpha)}{(2-\alpha)(1+\lambda) - \Phi(2)(1-\alpha)}\,. \tag{2.22}$$

Proof. Let $f \in \mathbf{K}(\lambda, \alpha)$. Consider the operator

$$I_{a,b;c}f(z) = z + \sum_{n=2}^{\infty} \Phi(n) a_n z^n,$$

where

$$\Phi(n) = \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \quad (a, b > 0;\ n \geq 2).$$

Since $\Phi(n)$ is a decreasing function for n, by Lemma1.3, we need to find β $(0 \leq \beta < 1)$ that

$$\Phi(2) \sum_{n=2}^{\infty} n \frac{n-\beta}{1-\beta} a_n \leq 1.$$

Since $f \in \mathbf{K}(\lambda, \alpha)$, by Lemma 1.2, we have

$$\sum_{n=2}^{\infty} n(n-\alpha)(1+n\lambda-\lambda)a_n \leq 1-\alpha.$$

To complete the proof, it suffices to find β such that

$$n\frac{n-\beta}{1-\beta}\Phi(2) \leq \frac{n(n-\alpha)(1+n\lambda-\lambda)}{1-\alpha}. \tag{2.23}$$

From (2.23), we obtain

$$\beta \leq \Upsilon(n),$$

where

$$\Upsilon(n) = \frac{(n-\alpha)(1+n\lambda-\lambda) - n\Phi(2)(1-\alpha)}{(n-\alpha)(1+n\lambda-\lambda) - \Phi(2)(1-\alpha)}. \tag{2.24}$$

By the assumption of the theorem, it is easy to see that $\Psi(n)$ is an increasing function for n $(n \geq 2)$. Setting $n = 2$ in (2.21), we have

$$\beta = \frac{(2-\alpha)(1+\lambda) - 2\Phi(2)(1-\alpha)}{(2-\alpha)(1+\lambda) - \Phi(2)(1-\alpha)},$$

hence we get (2.19). Therefore we complete the proof of Theorem 2.6. ∎

3. Concluding remarks

If $a = 1, b = 1+\delta$, $c = 2+\delta$ with $\mathrm{Re}(\delta) > -1$, then the convolution operator $I_{a,b,c}(f)$ turns into a Bernardi operator

$$B_f(z) = [I_{a,b,c}(f)](z) = \frac{1+\delta}{z^\delta} \int_0^1 t^{\delta-1} f(t)\mathrm{d}t.$$

Further, $I_{1,1,2}(f)$ and $I_{1,2,3}(f)$ are known as Alexander and Libera operators, respectively. Further, note that, when $|b| = 1$, we get $I_{a,1,c}(f) = \mathcal{L}(a,c)f(z) = \left(z + \sum\limits_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} z^n\right) * f(z) = z + \sum\limits_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} a_n z^n$, the Carlson-Shaffer operator and also for $a = \delta + 1(\delta > -1),\ b = 1,\ c = 1$ the Ruscheweyh derivative operator

$$\mathcal{D}^{\delta} f(z) = \frac{z}{(1-z)^{\delta+1}} * f(z) = z + \sum_{n=2}^{\infty} \begin{pmatrix} \delta + n - 1 \\ n-1 \end{pmatrix} a_n z^n,$$

hence one can deduce various interesting results for the function class defined by these operator as a corollary, we omit the details involved.

References

[1] Carlson, B.C., Shaffer, D.B., *Starlike and prestarlike hypergeometric functions*, SIAM J. Math. Anal., 15 (1984), 737-745.

[2] Caplinger, T.R., Causey, W.M., *A class of univalent functions*, Proc. Amer. Math. Soc., 39 (1973), 357–361.

[3] Dixit, K.K., Pal, S.K., *On a class of univalent functions related to complex order*, Indian J. Pure Appl. Math., 26 (9) (1995), 889-896.

[4] Goodman, A.W., *On uniformly convex functions*, Ann. Polon. Math., 56 (1991), 87-92.

[5] Hohlov, Y.E., *Operators and operations in the class of univalent functions*, Izv. Vysš. Učebn. Zaved. Matematika, 10 (1978), 83–89 (in Russian).

[6] Kanas, S., Wiśniowska, A., *Conic regions and k-starlike functions*, Rev. Roumaine Math. Pures Appl., 45 (2000), 647-657.

[7] Kamali, M., Akbulut, S., *On a subclass of certain convex functions with negative coefficients*, App. Math. and Comput., 145 (2003), 341-350.

[8] Padmanabhan, K.S., *On a certain class of functions whose derivatives have a positive real part in the unit disc*, Ann. Polon. Math., 23 (1970), 73–81.

[9] Srivastava, H.M., Mishra, A.K., *Applications of fractional calculus to parabolic starlike and uniformly convex functions*, Comput. Math. Appl., 39 (3/4) (2000), 57-69.

[10] Silverman, H., *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc., 51 (1975), 109–116.

[11] Silverman, H., *Starlike and convexity properties for hypergeometric functions*, J. Math. Anal. Appl., 172 (3) (1993), 574–581.

[12] Srivastava, H.M., G. Murugusundaramoorthy, G., Sivasubramanian, S., *Hypergeometric functions in the parabolic starlike and uniformly convex domains*, Integral Transform. Spec. Funct., 18 (2007), 511–520.

[13] Srivastava, H.M., Karlson, P.W., *Multiple Gaussian Hypergeometric Series*, Halsted Press (Ellies Horwood Limited, Chichester), John Wiley and Sons, New York, 1985.

[14] Vijaya, K., Kasthuri, M., *Some Inclusion properties of starlike and convex functions associated with Hohlov Operator.* I, International Conference on Mathematics And its Applications, University College of Engineering, Villupuram, Tamilnadu, India, 2014.

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BOUBAKER PIVOTAL ITERATION SCHEME (BPIS)

Amjed Zraiqat

Al-Zaytoonah University of Jordan
Amman
Jordan

Abstract. In this paper, we present a numerical scheme for the solution of fourth-order boundary value problems with two-point boundary conditions. The Boubaker Pivotal Iteration Scheme (BPIS) is applied to construct the numerical solution. This approach provides the solution in the form of analytical function and not at grid points. Some examples are displayed to demonstrate the computational efficiency of the method.

Keywords: fourth-order BVPs; Boubaker polynomials expansion scheme; pivotal function; numerical solution.

1. Introduction

Numerical methods are becoming more and more important in mathematical and engineering applications not only because of the difficulties encountered in finding exact analytical solutions, but also because of the ease with which numerical techniques can be used in conjunction with modern high-speed digital computers. There exist a large number of fourth-order BVPs in science and engineering, whose solutions cannot easily be obtained by the well-known analytical methods. For such problems, we can obtain approximate solutions for the given problems using numerical methods under the given boundary conditions. The objective of numerical methods is to solve complex numerical problems using only the simple operations of arithmetic, to develop and evaluate methods for computing numerical results from given data. The methods of computation are called algorithms. An algorithm is a finite sequence of rules for performing computations on a computer such that at each instant the rules determine exactly what the computer has to do next. Numerical methods tend to emphasize the implementation of the algorithms. Thus, numerical methods are methods for solving problems on computers by numerical calculations, often giving a table of numbers and/or graphical representations or figures.

This work considers numerical approximation for fourth-order nonlinear boundary value problem of the form

$$u^{(4)}(x) + \sum_{i=0}^{3} f_i(x)u^{(i)}(x) = f(x, u(x)), \quad 0 \le x \le 1 \tag{1.1}$$

with the boundary conditions

$$u(0) = \alpha_0, \quad u(1) = \alpha_1, \quad u'(0) = \beta_0, \quad u'(1) = \beta_1 \tag{1.2}$$

where α_i, and β_i (i=0, 1) all are real constants, $f(x, u)$ is a continuous real valued function, and $f_i(x)$ (i=0, 1, 2, 3, 4) are all continuous functions on the interval $[0, 1]$.

Two-point boundary value problems have been extensively studied in the literature. These problems generally arise in the mathematical modeling of visco-elastic and inelastic flows, deformation of beams, plate deflection theory, and other branches of mathematical, physical and engineering sciences, see [5]-[7]. Theorems which discuss the conditions for the existence and uniqueness of solutions of such problems can be found in Agarwal's book [8]. Exact solutions of such problems can be found only in very rare cases. Various numerical methods such as Finte difference method [9], Spline techniques [10], [11], B-spline technique [12], [13], and others have been employed to solve fourth-order boundary value problems.

In this paper, we present a novel technique to solve (1.1) and (1.2) by using BPIS where the Boubaker Polynomials Expansion Scheme (BPES) constitutes the base of our method. The Boubaker Polynomials Expansion Scheme BPES is a resolution protocol which has been successfully applied to several applied-physics and mathematics problems. Solutions have been proposed through the BPES in many fields such as numerical analysis [14]-[19], theoretical physics [16]-[21], mathematical algorithms [18], heat transfer [22], [23], and material characterization [24]. The rest of the paper is organized as follows. In the next section some properties of Boubaker Polynomials which gave the fundamentals of BPIS is introduced. The solution of (1.1) and (1.2) using BPIS introduced in Section 3. The numerical Examples are presented in Section 4. Section 5 ends this paper with a brief conclusion.

2. Properties of Boubaker polynomials

In this section, we start with some notations, definitions and basic results that are useful for the proposed method. In this paper, the Boubaker Polynomials will be used to investigate the fourth-order boundary value problems. In recent years, a lot of attention has been devoted to the study of Boubaker Polynomials Expansion Scheme to investigate various scientific models, using only the subsequence $\{B_{4k}(x)\}_{k=1}^{\infty}$. The monic Boubaker polynomials are defined as:

$$B_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} \frac{n-4k}{n-k} x^{n-2k}, \ n \geq 1, \quad \left\lfloor \frac{n}{2} \right\rfloor = \frac{2n + (-1)^n - 1}{4} \tag{2.1}$$

and $B_0(x) = 1$. Such polynomials satisfy the relations

$$\begin{aligned} B_n(-x) &= (-1)^n B_n(x), && n \geq 0 \\ B_n(x) &= xB_{n-1}(x) - B_{n-2}(x), && n = 3, 4, ... \end{aligned} \tag{2.2}$$

In particular, the $4k$-Boubaker polynomials satisfy the relation

$$B_{4(k+1)} = (x^4 - 4x^2 + 2)B_{4k} - \beta_k B_{4(k-1)}(x), \quad k \geq 1 \tag{2.3}$$

with $B_0(x) = 1$ and $B_4(x) = x^4 - 2$ where $\beta_0 = 0, \beta_1 = -2$ and $\beta_k = 1$ for $k \geq 2$, see [**?**]. For example, for $k \leq 4$ we have

$$\begin{aligned} B_4(x) &= x^4 - 2 \\ B_8(x) &= x^8 - 4x^6 + 8x^2 - 2 \\ B_{12}(x) &= x^{12} - 8x^{10} + 18x^8 - 35x^4 + 24x^2 - 2 \\ B_{16}(x) &= x^{16} - 12x^{14} + 52x^{12} - 88x^{10} + 168x^6 - 168x^4 + 48x^2 - 2. \end{aligned} \tag{2.4}$$

As a well known powerful properties of the $4k$-Boubaker polynomials their values at 0 and r_k, where r_k designates the $4k$-Boubaker polynomial minimal positive root. Some of these properties are listed below:

$$\sum_{k=1}^{N} B_{4k}(x)\big|_{x=0} = -2N, \qquad \sum_{k=1}^{N} B_{4k}(x)\big|_{x=r_k} = 0, \tag{2.5}$$

also first derivatives values:

$$\begin{aligned} &\sum_{k=1}^{N} \frac{dB_{4k}(x)}{dx}\Big|_{x=0} = 0, \qquad \sum_{k=1}^{N} \frac{dB_{4k}(x)}{dx}\Big|_{x=r_k} = \sum_{k=1}^{n} H_k, \\ &H_n = B'_{4n}(r_n) = \frac{4r_n(2 - r_n^2) \sum\limits_{k=1}^{n} B_{4k}^2(r_n))}{B_{4(n+1)}(r_n)} + 4r_n^3. \end{aligned} \tag{2.6}$$

Finally, we close this section by stating one the most noticed results, see [25]:

Theorem 2.1 *Every polynomial $B_n(x)$, $n \geq 2$, has two conjugate complex roots $\pm i\sqrt{\gamma_n}$, $\gamma_n > 0$, and other zeros are real and symmetrically distributed in $(-2, 2)$, where $\lim\limits_{n\to\infty} \gamma_n = \frac{4}{3}$.*

3. Analysis of the method

In this section, we employ our technique of **BPIS** to find out an analytical solution of the general fourth-order boundary value problem (1.1) and (1.2). We first formulate and analyze **BPIS** for solving such problems.

Definition 3.1 The boundary conditions $u(1) = \alpha_1$, and $u'(0) = \beta_0$ in (1.2) are called pivotal conditions of the boundary value problem (1.1) and (1.2).

Definition 3.2 A function $\psi(x) \in C^4[0, 1]$ is said to be a pivotal function of the fourth-order boundary value problem (1.1) and (1.2) if it satisfies the conditions $\psi(1) = -\alpha_1$, and $\psi'(0) = -\beta_0$.

Remark 3.3 It can be noted that the boundary value problem (1.1) and (1.2) has infinitely many pivotal functions. For example, each term in the sequence $\{(\beta_0-\alpha_1)x^n-\beta_0 x\}_{n=2}^{\infty}$ is a pivotal function.

For the convenience of the reader, we consider a fourth-order boundary value problem of the form

$$Lu+Nu=g \tag{3.1}$$

with boundary conditions (1.2), where L is the linear differential operator which is given by

$$L=\frac{d^4}{dx^4}+\sum_{i=1}^{3} f_i(x)\frac{d^i}{dx^i}, \tag{3.2}$$

Nu represents the nonlinear term, and $g(x)$ is a source term. The BPIS starts by transforming the above problem using the transformation

$$v(x)=u(x)+\psi(x) \tag{3.3}$$

where $\psi(x)$ is an arbitrary pivotal function. Thus we obtain a new fourth-order boundary value problem of the form

$$Lv+N(v-\psi)=g(x)+L\psi \tag{3.4}$$

with boundary conditions

$$v(0)=\alpha_0+\psi(0),\quad v(1)=0,\quad v'(0)=0,\quad v'(1)=\beta_1+\psi'(1) \tag{3.5}$$

Assume that the approximation v_N of the solution of (3.4) and (3.5) has the form

$$v_N(x)=\frac{1}{2N}\sum_{k=1}^{N}\lambda_k B_{4k}(r_k x) \tag{3.6}$$

N is a preassigned integer, $r_k|_{k=1,\ldots,N}$ are $4k$-Boubaker minimal positive roots, and the constants $\lambda_k|_{k=1,\ldots,N}$ are to be determined. It is worth noting that the function $v_N(x)$ satisfies the transformed pivotal conditions.

For the determination of the constants $\lambda_k|_{k=1,\ldots,N}$, an appropriate N-sampling is carried out inside the interval $[0,1]$ by setting

$$x_k=\frac{k-1}{N-1},\quad k=1,2,\ldots,N. \tag{3.7}$$

Then we seek for the values of $\lambda_k|_{k=1,\ldots,N}$ that minimizes the objective function

$$\sum_{k=1}^{N}\Big(L(v_N)|_{x_k}-F(x_k)\Big)^2, \tag{3.8}$$

where

$$F(x_k) = g(x_k) + L(\psi)|_{x_k} - N\left(\frac{1}{2N} B_{4k}(r_k x) - \psi\right)\Big|_{x_k} \tag{3.9}$$

subject to the constraints

$$\sum_{k=1}^{N} \lambda_k = -N(\alpha_0 + \psi(0)), \qquad \frac{dv_N}{dx}\Big|_{x=1} = \beta_1 + \psi'(1). \tag{3.10}$$

Thus $u_N(x) = v_N(x) - \psi(x)$ is an approximate solution of the solution of the boundary value problem (3.1) and (1.2). This algorithm will be clarified with a famous example in the next section where the approximate solution is in good agreement with the exact one even for small values of N. The above analysis leads us to a very important linearization process which comes from the flexibility in choosing the pivotal functions. The second example in the next section illustrate the linearization process.

4. Numerical examples

In this section, we present and discuss the numerical results by employing the BPIS for two examples. Results demonstrate that the present method is remarkably effective.

Example 1. Scott and Watts [1] considered the following special fourth-order boundary value problem:

$$u^{(4)}(x) - (1+c)u'' + cu(x) = \frac{1}{2}cx^2 - 1 \tag{4.1}$$

with the boundary conditions

$$u(0) = 1, \quad u'(0) = 1, \quad u(1) = 1.5 + \sinh(1), \quad u'(1) = 1 + \cosh(1). \tag{4.2}$$

The exact solution of the above problem is

$$u(x) = 1 + \frac{1}{2}x^2 + \sinh(x). \tag{4.3}$$

The solution of (4.1) is independent of the parameter c. Scott and Watts have solved (4.1) with boundary conditions (4.2) for large values of c with orthonormalization process. It has been shown that each time the solutions started to lose their linear independence, one has to perform orthonormalization. In fact, as c got bigger it required more normalization. Momani and Noor obtained the solution of problem (4.1) with boundary conditions (4.2) by differential transform method (DTM) and the solution is accurate for $c < 10^6$ [2]. Noor and Mohyud-Din solved the same boundary value problem by using variational iteration method [3]. A new reproducing kernel Hilbert space method (RKHSM) introduced in [4] to solve (4.1) and (4.2) for large values of c. Equation (4.1) can be rewritten as

$$[u^{(4)}(x) - u^{(2)}(x) + 1] - c\left[u^{(2)}(x) - u(x) + \frac{1}{2}x^2\right] = 0. \tag{4.4}$$

The solution of the fourth-order boundary value problem is also a solution of

$$u^{(2)}(x) - u(x) + \frac{1}{2}x^2 = 0, \tag{4.5}$$

which is just the term in brackets multiplying c in (4.4). The remaining term in brackets in (4.4) is just the second derivative of (4.5). This unusual behavior results in the solution of the original problem being independent of the constant c. Using the new technique, we discuss the solution of this problem from different points of view. First, we show the vital rule played by pivotal functions in obtaining the approximation $u_N(x)$ of $u(x)$. Second, We examine our approach using different values of c. Finally, we compare our results with previous methods. Consider the following pivotal functions:

$$\psi_1(x) = (-.5 - \sinh(1))x^2 - x \tag{4.6}$$

$$\psi_2(x) = e^{-x} - e^{-1} - 1.5 - \sinh(1) \tag{4.7}$$

$$\psi_3(x) = -\sinh(x) - 1.5 - \sinh(1) \tag{4.8}$$

According to BPIS, one can obtain the approximation $u_N(x)$ of $u(x)$. We have computed the absolute errors $\psi_i^{(4)}$ $(i = 1, 2, 3)$ corresponding to the proposed pivotal functions ψ_1, ψ_2, and ψ_3 with $N = 4$. Numerical results are given in Table 1.

Table 1. Absolute errors $\psi_i^{(4)}$ $(i = 1, 2, 3)$ corresponding to the proposed pivotal functions ψ_1, ψ_2, and ψ_3.

x	True Solution	$\psi_1^{(4)}$	$\psi_2^{(4)}$	$\psi_3^{(4)}$
0.0	1.0000000000	0.0000	0.0000	0.0000
0.1	1.1051667500	5.4e-4	5.7e-5	1.6e-7
0.2	1.2213360025	1.6e-3	2.2e-4	5.5e-7
0.3	1.3495202934	2.7e-3	4.5e-4	9.4e-7
0.4	1.4907523258	3.4e-3	6.9e-4	1.0e-6
0.5	1.6460953055	3.5e-3	8.8e-4	7.5e-7
0.6	1.8166535821	3.1e-3	9.5e-4	1.6e-7
0.7	2.0035837018	2.3e-3	8.5e-4	4.2e-7
0.8	2.2081059822	1.3e-3	5.7e-4	6.1e-7
0.9	2.4315167257	4.1e-4	2.1e-4	3.2e-7
1.0	2.6752011936	0.0000	0.0000	0.0000

Table 1 clearly shows the improvement we achieved using different pivotal functions.

Next, we consider the pivotal function $\psi_3(x)$ and calculate the absolute errors associated with $c = 10^{-6},\ c = 10,\ c = 10^6$.

The table below exhibits the computed results and shows that the absolute errors were not affected by changing the value of the constant c even for very small values which is never discussed before.

Table 2. Absolute errors for $c = 10^{-6},\ c = 10,\ c = 10^6$ when $N = 7$.

x	True Solution	$\psi_3^{(7)}(c = 10^{-6})$	$\psi_3^{(7)}(c = 10)$	$\psi_3^{(7)}, (c = 10^6)$
0.0	1.0000000000	0.0000	0.0000	0.0000
0.1	1.1051667500	1.2e-12	2.3e-12	6.6e-13
0.2	1.2213360025	1.6e-12	2.5e-12	9.2e-13
0.3	1.3495202934	1.8e-12	2.5e-12	3.6e-13
0.4	1.4907523258	3.3e-12	3.6e-12	2.0e-13
0.5	1.6460953055	6.9e-12	6.8e-12	1.5e-12
0.6	1.8166535821	1.1e-11	1.0e-11	2.5e-12
0.7	2.0035837018	1.3e-11	1.3e-11	2.1e-12
0.8	2.2081059822	1.7e-11	1.7e-11	3.5e-12
0.9	2.4315167257	1.7e-11	1.7e-11	6.8e-12
1.0	2.6752011936	0.0000	0.0000	0.0000

Table 3 reveals a comparison between the errors obtained by using the methods mentioned in [4] and our approach. Examining this table closely shows the improvement obtained by the proposed scheme.

Table 3. Numerical results for Example 1 when $c = 10^6$.

x	True Solution	ADM[4]	HPM[4]	DTM[4]	RHKSM[4]	BPIS($\psi_3^{(7)}$)
0.0	1.0000000000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1	1.1051667500	6.5E+3	6.5E+3	1.5E-10	7.5E-11	6.6e-13
0.2	1.2213360025	8.8E+4	8.8E+4	3.7E-8	2.1E-10	9.2e-13
0.3	1.3495202934	3.6E+5	3.6E+5	9.0E-7	3.4E-10	3.6e-13
0.4	1.4907523258	9.1E+5	9.1E+5	8.5E-6	4.1E-10	2.0e-13
0.5	1.6460953055	1.6E+6	1.6E+6	4.8E-5	4.1E-10	1.5e-12
0.6	1.8166535821	2.3E+6	2.3E+6	1.9E-4	3.5E-10	2.5e-12
0.7	2.0035837018	2.6E+6	2.6E+6	6.4E-4	2.3E-10	2.1e-012
0.8	2.2081059822	2.1E+6	2.1E+6	1.7E-3	1.1E-10	3.5e-12
0.9	2.4315167257	9.1E+5	9.1E+5	4.2E-3	2.6E-11	6.8e-12
1.0	2.6752011936	0.0000	0.0000	0.0000	0.0000	0.0000

Example 2. We next consider the nonlinear boundary value problem:

$$u^{(4)} - e^x u'' + u + \sin(u) = f(x) \tag{4.9}$$

subject to the boundary conditions

$$u(0) = 1, \quad u'(0) = 1, \quad u(1) = 1 + \sinh(1), \quad u'(1) = \cosh(1) \tag{4.10}$$

where

$$f(x) = 1 + \sin(1 + \sinh(x)) - (-2 + e^x)\sinh(x) \tag{4.11}$$

The exact solution of (4.9) and (4.10) is given by

$$u(x) = 1 + \sinh(x) \tag{4.12}$$

This is an interesting problem considered by Fashan [4]. A linearization process applied to the problem above using two pivotal functions and the results compared with [4], see table 5. Consider the pivotal functions

$$\psi_1(x) = -x - \sinh(1) \tag{4.13}$$

$$\psi_2(x) = -e^x + e - 1 - \sinh(1). \tag{4.14}$$

Let $N = 4$. Applying BPIS with respect to the pivotal function ψ_1, we obtain the approximation $U_4(x)$ of $u(x)$, where

$$\begin{aligned} U_4(x) = &-0.00000000000980353850353216475911390986832 5x^{16} \\ &+0.0000000023542610058198409258849496537761x^{14} \\ &+0.00000004252939950281279857297540062468 2x^{12} \\ &-0.0000134338202380548758700682774821 86x^{10} \\ &+0.00045005860626105410201065313102606x^8 \\ &-0.0049453446305244210781767729908711x^6 \\ &+0.10493314578309954311980631268 05x^4 \\ &+0.07477672283134646247231534779930 7x^2 + x \\ &+1.000000000000000006938893903907 2. \end{aligned} \tag{4.15}$$

Substituting (4.15) in the nonlinear term $\sin(u(x))$, we obtain the linearized form of the boundary value problem (4.9) and (4.10) as follows:

$$u^{(4)}(x) - e^x u''(x) + u(x) = f(x) - \sin(u_4(x)) \tag{4.16}$$

with boundary conditions

$$u(0) = 1, \quad u'(0) = 1, \quad u(1) = 1 + \sinh(1), \quad u'(1) = \cosh(1) \tag{4.17}$$

Applying BPIS to the linearized problem using the pivotal function $\psi_2(x)$, we have the approximation

$$
\begin{aligned}
u_4^{(2)}(x) = e^x &- 0.50000417574041429870090557455166x^2 \\
&-0.041638055568834480012148060896809x^4 \\
&-0.0014425473149225348443037016434657x^6 \\
&+0.000013311147131641983031454006923302x^8 \\
&-0.0000090876855913124668338091573609605x^{10} \\
&-0.000000083424445094660028827396146693082x^{12} \\
&+0.0000000037876040459617077946991317998246x^{14} \\
&-0.000000000015772219821391087485678341815356x^{16} \\
&+0.000000000000000013877787807814456755295395851135
\end{aligned}
\tag{4.18}
$$

Again, one can get the approximation $u_5^{(3)}(x)$ of $u(x)$ by substituting $u_4^{(2)}(x)$ in the nonlinear term $\sin(u(x))$, which is given by

$$
\begin{aligned}
u_5^{(3)}(x) = e^x &- 0.49999989830481488172581934346248x^2 \\
&-0.041668056606074833445932563442428x^4 \\
&-0.0013839852672572706244782813768148x^6 \\
&-0.000031972982740266785270422060761747x^8 \\
&+0.0000044279090765597957835121606922403x^{10} \\
&-0.0000011698051952654471291813237324335x^{12} \\
&+0.000000020219576882570412793447981920844x^{14} \\
&+0.000000000023074475795622385833449540653001x^{16} \\
&-0.00000000000089655271029356142341740348491118x^{18} \\
&+0.000000000000001699758430998214551186562586755x^{20} \\
&+0.0000000000000003386180225106727448292076587677
\end{aligned}
\tag{4.19}
$$

Continuing the linearization process, the absolute errors associated with the approximation $u_4^{(2)}(x)$, $u_5^{(3)}(x)$, $u_6^{(4)}(x)$, and $\bar{\phi}_7^{(5)}(x)$ are listed in Table 4.

Table 4. Absolute errors associated with $u_4^{(2)}(x)$, $u_5^{(3)}(x)$, $u_6^{(4)}(x)$, and $u_7^{(5)}(x)$ for Example 2.

x	True Solution	$u_4^{(2)}(x)$	$u_5^{(3)}(x)$	$u_6^{(4)}(x)$	$u_7^{(5)}(x)$
0.0	1.0000000000	0.0000	0.0000	0.0000	0.0000
0.1	1.1051667500	3.9e-8	8.8e-10	1.3e-11	5.5e-13
0.2	1.2213360025	1.2e-7	2.1e-9	1.1e-10	7.2e-13
0.3	1.3495202934	1.8e-7	1.0e-9	3.6e-10	8.2e-13
0.4	1.4907523258	1.3e-7	3.5e-9	6.8e-10	1.5e-12
0.5	1.6460953055	4.6e-8	8.5e-9	8.4e-10	3.1e-12
0.6	1.8166535821	2.9e-7	9.3e-9	7.9e-10	4.7e-12
0.7	2.0035837018	4.6e-7	3.6e-9	7.3e-10	5.9e-12
0.8	2.2081059822	4.2e-7	3.7e-9	7.9e-10	7.7e-12
0.9	2.4315167257	1.8e-7	4.2e-9	5.3e-10	7.4e-12
1.0	2.6752011936	0.0000	0.0000	0.0000	0.0000

Finally, Table 5 shows that is powerful and effective compared with [4].

Table 5. Numerical results for Example 2.

x	True Solution	RHKSM [4]	BPEI($u_7^{(5)}(x)$)
0.0	1.000000000	0.0000	0.0000
0.1	1.1051667500	2.78E-8	5.5E-13
0.2	1.2213360025	8.09E-8	7.2E-13
0.3	1.3495202934	1.20E-7	8.2E-13
0.4	1.4907523258	1.25E-7	1.5E-12
0.5	1.6460953055	9.56E-8	3.1E-12
0.6	1.8166535821	4.82E-8	4.7E-12
0.7	2.0035837018	7.38E-9	5.9E-12
0.8	2.2081059822	1.07E-8	7.7E-12
0.9	2.4315167257	7.08E-9	7.4E-12
1.0	2.6752011936	0.0000	0.0000

5. Conclusion

The computations associated with the two examples discussed above were performed by using Matlab R2012a. The existence and uniqueness of the solution is guaranteed by Agarwals book [8]. The proposed algorithm using BPIS, produced a reliable computational method for handling boundary value problems. Comparing the obtained results with other works, the BPIS was clearly reliable if compared with grid points techniques where the solution is defined at grid points only. Moreover, numerical methods based on the approach we used would require considerably less computational effort.

References

[1] Scott, M.R., Watts, H.A., suport – *A computer code for two-point boundary-value problems via orthonormalization*, sand75-0198, Sandia Laboratories, Albuquerque, NM, 1975.

[2] Momani, S., Noor, M.A., *Numerical comparison of methods for solving a special fourth-order boundary value problem*, Applied Mathematics and Computation, 191 (2007), 218-224.

[3] Noor, M.A., Mohyud-Din, S.T., *Variational iteration technique for solving higher order boundary value problems*, Applied Mathematics and Computation, 189 (2) (2007), 1929-1942.

[4] Geng, F.Z., *A new reproducing kernel Hilbert space method for solving nonlinear fourth-order boundary value problems*, Applied Mathematics and Computation, 213 (2009), 163-169.

[5] Momani, S.M., *Some problems in non-Newtonian fluid mechanics*, Ph.D. Thesis, Walse University, United Kingdom, 1991.

[6] Ma, T.F., Silva, J., *Iteration solution for a beam equation with nonlinear boundary conditions of third order*, Applied Mathematics and Computation, 159 (1) (2004), 11-18.

[7] Chawla, M.M., Katti, C.P., *Finite difference methods for two-point boundary-value problems involving higher order differential equation*, BIT 19 (1979), 27-33.

[8] Agarwal, R.P., *Boundary Value Problems for High Ordinary Differential Equations*, World Scientific, Singapore, 1986.

[9] Rashidinia, J., *Finite difference methods for a class of two point boundary value problems*, IUST International Journal of Engineering Science, 19 (2008), 67-72.

[10] Fyfe, D.J., *The use of cubic splines in the solution of certain fourth order boundary value problems*, The Computer Journal, 13 (1970), 204-205.

[11] Usmani, R.A., *Smooth spline approximations for the solution of a boundary value problem with engineering applications,* Journal of Computational and Applied Mathematics, 6 (1980), 93-98.

[12] Sakai, M., Usmani, R., *Spline solutions for nonlinear fourth order two point boundary value problems*, Publications of the Research Institute for Mathematical Sciences, Kyoto University, 19 (1983), 135-144.

[13] Gupta, Y., Kumar, M., *B-spline method for solution of linear fourth order boundary value problem*, Canadian Journal on Computing in Mathematics, Natural Sciences, Engineering and Medicine, 2 (2011), 166-169.

[14] Kumar, A.S., *An Analytical Solution to Applied Mathematics-Related Love's Equation Using the Boubaker Polynomials Expansion Scheme*, Journal of the Franklin Institute, 347 (9) 2010, 1755-1761.

[15] Belhadj, A., Bessrour, J., Bouhafs, M., Barrallier, L., *Experimental and theoretical cooling velocity profile inside laser welded metals using keyhole approximation and Boubaker polynomials expansion*, Journal of Thermal Analysis and Calorimetry, 97 (3) 2009, 911-920.

[16] Barry, P., Hennessy, A., *Meixner-Type Results for Riordan Arrays and Associated Integer Sequences*, Section 6: *The Boubaker Polynomials*, Journal of Integer Sequences, 13 (2010), 1-34.

[17] Agida, M., Kumar, A.S., *A Boubaker Polynomials Expansion Scheme Solution to Random Love Equation in the Case of a Rational Kernel*, El, Journal of Theoretical Physics, 7 (2010), 319-326.

[18] Yildirim, A., Mohyud-Din, T., Zhang, D.H., *Analytical Solutions to the Pulsed Klein-Gordon Equation Using Modified Variational Iteration Method* (MVIM) *and Boubaker Polynomials Expansion Scheme* (BPES), Computers and Mathematics with Applications, 59 (8) 2010, 2473-2477.

[19] Ghanouchi, J., Labiadh, H., Boubaker, K., *An attempt to solve the heat transfer equation in a model of pyrolysis spray using* 4*q-order m-Boubaker polynomials*, International Journal of Heat and Technology, 26 (2008), 49-53.

[20] Awojoyogbe, O.B., Boubaker, K., *A solution to bloch NMR flow equations for the analysis of hemodynamic functions of blood flow system using m-Boubaker polynomials*, Current Applied Physics, 9 (1) (2009), 278-288.

[21] Slama, S., Bessrour, J., Bouhafs, M., Ben Mahmoud, K.B., *Numerical distribution of temperature as a guide to investigation of melting point maximal front spatial evolution during resistance spot welding using Boubaker polynomials*, Numerical Heat Transfer, Part A, 55 (4) (2009), 401-404.

[22] Slama, S., Bouhafs, M., Ben Mahmoud, K.B., *A Boubaker polynomials solution to heat equation for monitoring A3 point evolution during resistance spot welding*, International Journal of Heat and Technology, 26 (2) (2008), 141-146.

[23] Tabatabaei, S., Zhao, T., Awojoyogbe, O., Moses, F., *Cut-off cooling velocity profiling inside a keyhole model using the Boubaker polynomials expansion scheme*, International Journal of Heat and Mass Transfer, 45 (10) (2009), 1247-1255.

[24] Fridjine, S., Amlouk, M., *A new parameter: an* ABACUS *for optimizing functional materials using the Boubaker polynomials expansion scheme*, Modern Physics Letters B, 23 (17) (2009), 2179-2182

[25] Milovanovic, G.V., Joksimovic, D., *Some properties of Boubaker polynomials and applications*, AIP Conf. Proc., 1479 (2012), 1050-1053.

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THE STABILITY AND HOPF BIFURCATION OF THE DENGUE FEVER MODEL WITH TIME DELAY[1]

Jinlan Guan [2]
Basic Courses Department
Guangdong Agricultural Industry Business Polytechnic
Guangzhou 510507
P.R. China

Lizhi Wu
The Network Center
Guangdong Construction Polytechnic
Guangzhou 510440
P.R. China

Minna Chen
Basic Courses Department
Guangdong Polytechnic of Environmental Protection Engineering
Foshan 528216
P.R. China

Xiaoqian Dong
Huiping Tang
Zhiqi Chen
Basic Courses Department
Guangdong Agricultural Industry Business Polytechnic
Guangzhou 510507
P.R. China

Abstract. This paper studied the dengue fever model with time delay. This paper divided the time delay into four cases: (1) $\tau_1 = \tau_2 = 0$, (2) $\tau_1 = \tau$, $\tau_2 = 0$, (3) $\tau_1 = 0$, $\tau_2 = \tau$, (4) $\tau_1 = \tau$, $\tau_2 = \tau$, and studied the stability and Hopf bifurcation of the model on these three cases. At the end of this paper, we simulated the dengue model with time delay by using Matlab software, and gained the numerical condition of this model which appearing periodic solutions and Hopf bifurcation. On the first case $\tau_1 = \tau, \tau_2 = 0$, the time delay threshold is $\tau_0 = 0.6155$; on the second case $\tau_1 = 0, \tau_2 = \tau$, the time delay threshold is $\tau_0 = 0.0490$; on the third case $\tau_1 = \tau, \tau_2 = \tau$, the time delay threshold is $\tau_0 = 3.5454$.

Keywords: Dengue fever model; time delay; Hopf bifurcation; phase diagram.
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[2]Corresponding author. E-mail address: jinlan128@163.com

1. Introduction

Dengue fever and dengue hemorrhagic fever is a kind of acute mosquito borne infectious diseases spread rapidly around the globe. It is one of the most serious diseases infecting humans[1]. It has the characteristics of spreading rapidly and high incidence which can cause large scale epidemic in an area. According to WHO estimates, it has the risk of 25 billion people suffering from the dengue fever, and 5000 million to 1 billion people infect dengue fever. There are 50 million dengue fever hospitalization, of which 2 million died from dengue fever and dengue shock syndrome. Dengue fever has been a threat to 1/3 of the world population's health and safety[2].

Dengue is transmitted to humans through the bite of infected Aedes aegypti and A. albopictus mosquitoes. It is understood that four closely related serotypes of DENV exist-DENV-1, DENV-2, DENV-3 and DENV-4 and these four serotypes cause infections of varying severity in humans [1]-[2]. The infected individual usually suffers from acute febrile illness called Dengue Fever (DF) which is cleared by a complex immune response in a short time of approximately 7 days after onset of fever. We note that, though there is a huge effort going on to develop an effective vaccine against dengue infections, commercial dengue vaccines are not yet available [3]-[7]. In this context, it is important to understand the biological mechanisms and dynamical processes involved during this infection. Also these complex non-linear biological processes lead to dynamic models that are interesting for their varied and rich dynamics. The epidemiology of dengue in different populations have been studied previously using improved or extended versions of the basic SIR model [8]-[16].

At present, people put forward various kinds of dengue fever model, for example, the SEIRS model considering the effects of using pesticides on dengue fever epidemic [17]; the population model only consider a virus and assume the number of susceptible people and patients is constant [18]; the model with the exponential growth population and constant infection rate [19]; the model with two kinds of serum virus and variable population [20].

So far, there is few research in dengue fever model considering time delay. The time delay plays a very important role on studying the dynamics of dengue virus transmission behavior. It explains the dynamics behavior of sick people "susceptible people" the carrying-virus aedes mosquitoes and the no-carrying-virus aedes mosquitoes from the angle of mathematics, which can help people understand the spread of dengue fever model law better, so as to control the spread of dengue virus better.

2. The dengue fever model with time delay

Let $S_1(t), I_1(t), R_1(t), S_2(t), I_2(t)$ represents the number of susceptible people, patient, remove people, no-viruses-carrying mosquitoes, viruses-carrying mosquitoes at time t. In reference[21], Tewa et.al study the following dengue fever model:

$$(1)\qquad \begin{cases} S_1' = \mu_1 N_1 - \dfrac{b\beta_1}{N_1+m} S_1 I_2 - \mu_1 S_1, \\ I_1' = \dfrac{b\beta_1}{N_1+m} S_1 I_2 - (\mu_1+\gamma_1) I_1, \\ R_1' = \gamma_1 I_1 - \mu_1 R_1, \\ S_2' = A - \dfrac{b\beta_1}{N_1+m} S_2 I_1 - \mu_2 S_2, \\ I_2' = \dfrac{b\beta_1}{N_1+m} S_2 I_1 - \mu_2 I_2 N_1 = S_1 + R_1 + I_1. \end{cases}$$

The definition of the parameters can refer to reference [1].

On the basis of system (1), Ding Deqiong from Harbin Institute of Technology of China made some improvements, he study the following dengue fever model [22]:

$$(2)\qquad \begin{cases} S_1' = \mu_1 N_1 - \beta_1 S_1 I_2 - \mu_1 S_1, \\ I_1' = \beta_1 S_1 I_2 - (\mu_1+\gamma_1) I_1, \\ R_1' = \gamma_1 I_1 - \mu_1 R_1, \\ S_2' = A - \beta_2 S_2 I_1 - \mu_2 S_2, \\ I_2' = \beta_2 S_2 I_1 - \mu_2 I_2 N_1 = S_1 + R_1 + I_1. \end{cases}$$

In this paper, on the basis of system (2), we make some improvements, we consider the effect of time delay to the dengue fever. The equations describing the model are given by:

$$(3)\qquad \begin{cases} S_1' = \mu_1 N_1(t) - \beta_1 S_1(t) I_2(t) - \mu_1 S_1(t), \\ I_1' = \beta_1 S_1(t) I_2(t-\tau_1) - (\mu_1+\gamma_1) I_1(t), \\ R_1' = \gamma_1 I_1(t) - \mu_1 R(t), \\ S_2' = A - \beta_2 S_2(t) I_1(t) - \mu_2 S_2(t), \\ I_2' = \beta_2 S_2(t) I_1(t-\tau_2) - \mu_2 I_2(t), \\ N_1(t) = S_1(t) + R(t) + I_1(t). \end{cases}$$

As we can see, in system (3), the variable R is not explicit in the first two equations and the last two equations, so system (3) can be simplified to be the following system

$$(4)\qquad \begin{cases} S_1' = \mu_1 N_1(t) - \beta_1 S_1(t) I_2(t) - \mu_1 S_1(t), \\ I_1' = \beta_1 S_1(t) I_2(t-\tau_1) - (\mu_1+\gamma_1) I_1(t), \\ S_2' = A - \beta_2 S_2(t) I_1(t) - \mu_2 S_2(t), \\ I_2' = \beta_2 S_2(t) I_1(t-\tau_2) - \mu_2 I_2(t), \\ N_1(t) = S_1(t) + I_1(t). \end{cases}$$

where μ_1 and μ_2 represents the death date of human and mosquito respectively, $S_1(t)$, $I_1(t)$, $S_2(t)$, $I_2(t)$ represents the number of susceptible patient, no-viruses-carrying mosquitoes, viruses-carrying mosquitoes at time t. $N_1(t)$ is the total population at time t, β_1 represents the infectious rate of the viruses-carrying mosquitoes to the susceptible, β_2 represents the infectious rate of the patient to the no-viruses-carrying mosquitoes, γ_1 represents the recovery rate of the patient, A represents the new rate of mosquitoes, τ_1 and τ_2 represent the rehabilitees immune period and incubation period to the diease respectively.

Define the basic reproductive ratio as follows:

$$R_0 = \frac{\beta_1\beta_2 N_1 A}{\mu_2^2(\mu_1+\gamma_1)}.$$

Define the closed region:

$$D = \left\{(S_1, I_1, S_2, I_2) \in R_+^4 : S_1 \le N_1, S_1 + I_1 \le N_1, S_2 \le \frac{A}{\mu_2}, S_2 + I_2 \le \frac{A}{\mu_2}\right\}.$$

As the same as reference [1], region D is a positive invariant set. This paper will study the dynamical behavior of system (4) on the closed region D. The dynamical behavior of R can be determined by the third equation of the system(3).

Obviously, for all non negative parameters, system (4) has no-disease equilibrium point $E_1^*(S_1, I_1, S_2, I_2) = \left(N_1, 0, \frac{A}{\mu_2}, 0\right)$. When the basic reproductive ratio $R_0 > 1$, the system has the only endemic equilibrium $E_1^*(\bar{S}_1, \bar{I}_1, \bar{S}_2, \bar{I}_2)$, where

$$\begin{cases} \bar{S}_1 = \dfrac{\mu_1 N_1}{\beta_1 \bar{I}_2 + \mu_1}, \\ \bar{I}_1 = \dfrac{\mu_1\mu_2^2(R_0 - 1)}{\beta_2(\beta_1 A + \mu_1\mu_2)}, \\ \bar{S}_2 = \dfrac{A}{\beta_2 \bar{I}_1 + \mu_2}, \\ \bar{S}_1 = \dfrac{\beta_2 \bar{S}_2 \bar{I}_1}{\mu_2}. \end{cases}$$

Do the transform as follow:

$$\begin{cases} U_1(t) = S_1(t) - \bar{S}_1, \\ U_2(t) = I_1(t) - \bar{I}_1, \\ V_1(t) = S_2(t) - \bar{S}_2, \\ V_2(t) = I_2(t) - \bar{I}_2. \end{cases} \tag{5}$$

then system (4) comes to be:

$$\begin{cases} U_1' = -(\beta_1\bar{I}_2 + \mu_1)U_1(t) - \beta_1\bar{S}_1 V_2(t) - \beta_1 U_1(t)V_2(t), \\ V_1' = -\beta_2\bar{S}_2 U_2(t) - (\beta_2\bar{I}_1 + \mu_2)V_1(t) - \beta_2 V_1(t)U_2(t), \\ U_2' = \beta_1\bar{S}_1 V_2(t-\tau_1) + \beta_1 U_1(t)V_2(t-\tau_1) - \beta_1\bar{I}_2 U_1(t) - (\mu_1+\gamma_1)U_2(t), \\ V_2' = \beta_2\bar{S}_2 U_2(t-\tau_2) + \beta_2 V_1(t)U_2(t-\tau_2) + \beta_2\bar{I}_1 V_1(t) - \mu_2 V_2(t). \end{cases} \tag{6}$$

Obviously, the origin point $(0,0,0,0)$ is the equilibrium point of system (6), linearizing the system, we gain the following system:

$$
\begin{cases}
U_1' = -(\beta_1\bar{I}_2+\mu_1)U_1(t) - \beta_1\bar{S}_1V_2(t),\\
V_1' = -\beta_2\bar{S}_2U_2(t) - (\beta_2\bar{I}_1+\mu_2)V_1(t),\\
U_2' = \beta_1\bar{S}_1V_2(t-\tau_1) - \beta_1\bar{I}_2U_1(t) - (\mu_1+\gamma_1)U_2(t),\\
V_2' = \beta_2\bar{S}_2U_2(t-\tau_2) + \beta_2\bar{I}_1V_1(t) - \mu_2V_2(t).
\end{cases}
\tag{7}
$$

Let

$$
\begin{aligned}
\mathbf{X(t)} &= (U_1(t), U_2(t), V_1(t), V_2(t))^T,\\
\mathbf{X(t-\tau_1)} &= (U_1(t-\tau_1), U_2(t-\tau_1), V_1(t-\tau_1), V_2(t-\tau_1))^T,\\
\mathbf{X(t-\tau_2)} &= (U_1(t-\tau_2), U_2(t-\tau_2), V_1(t-\tau_2), V_2(t-\tau_2))^T,
\end{aligned}
$$

and let

$$
\dot{\mathbf{X}}(\mathbf{t}) = \left(\ U_1', V_1', U_2', V_2'\ \right)^T.
$$

The coefficient matrix of linear system is

$$
\mathbf{A_0} = \begin{pmatrix}
-(\beta_1\bar{I}_2+\mu_1) & 0 & 0 & -\beta_1\bar{S}_1\\
-\beta_1\bar{I}_2 & -(\mu_1+\gamma_1) & 0 & 0\\
0 & -\beta_2\bar{S}_2 & -(\beta_2\bar{I}_1+\mu_2) & 0\\
0 & 0 & \beta_2\bar{I}_1 & -\mu_2
\end{pmatrix},
$$

$$
\mathbf{A_1} = \begin{pmatrix}
0 & 0 & 0 & 0\\
0 & 0 & 0 & \beta_1\bar{S}_1\\
0 & 0 & 0 & 0\\
0 & 0 & 0 & 0
\end{pmatrix},
$$

$$
\mathbf{A_2} = \begin{pmatrix}
0 & 0 & 0 & 0\\
0 & 0 & 0 & 0\\
0 & 0 & 0 & 0\\
0 & \beta_2\bar{S}_2 & 0 & 0
\end{pmatrix}.
$$

Then system (7) becomes

$$
\dot{X}(t) = A_0X(t) + A_1X(t-\tau_1) + A_2X(t-\tau_2). \tag{8}
$$

The characteristic equation corresponding to equation (8) is:

$$
\begin{aligned}
&\mathbf{P}(\lambda, \boldsymbol{\tau_1}, \boldsymbol{\tau_2})\\
&= \begin{vmatrix}
-\lambda-(\beta_1\bar{I}_2+\mu_1) & 0 & 0 & -\beta_1\bar{S}_1\\
-\beta_2\bar{I}_1 & -\lambda-(\mu_1+\gamma_1) & 0 & \beta_1\bar{S}_1e^{-\tau_1\lambda}\\
0 & -\beta_2\bar{S}_2 & -\lambda-(\beta_2\bar{I}_1+\mu_2) & 0\\
0 & \beta_2\bar{S}_2e^{-\tau_2\lambda} & \beta_2\bar{I}_1 & -\lambda-\mu_2
\end{vmatrix} = 0
\end{aligned}
\tag{9}
$$

Expanding equation (9) in accordance to the first column, we have:

$$
P(\lambda, \tau_1, \tau_2) = -\lambda^4 + L\lambda^3 + M\lambda^2 + N\lambda - Re^{-\tau_2\lambda} - Se^{-(\tau_1+\tau_2)\lambda} + T = 0, \tag{10}
$$

where L, M, N, R, S, T, A, B, C, U, V are defined in the Appendix 1.

3. The stability of the model

In this section, we discuss the time delay effect on the stability and the Hopf bifurcation of the dengue fever model in three cases.

3.1. $\tau_1 = \tau_2 = 0$:

When $\tau_1 = \tau_2 = 0$, equation (8) becomes

$$P(\lambda) = -\lambda^4 + L\lambda^3 + M_1\lambda^2 + N_1\lambda - R - S + T = 0. \tag{9}$$

where

$$\begin{aligned} M_1 &= \mu_1 A + B - \beta_1\beta_2\bar{S}_1\bar{S}_2, \\ N_1 &= (\mu_1 + \beta_1\bar{I}_2)B + C - \beta_1\beta_2^2\bar{S}_1\bar{S}_2\bar{I}_2 - U\beta_1\beta_2\bar{S}_1\bar{S}_2. \end{aligned}$$

L, R, S, T, A, B, C, U, V are defined in the Appendix 1.

When $A = \beta_1\bar{I}_2 + M_1$ and $(\mu_1 + \beta_1\bar{I}_2)B + C = \beta_1\beta_2^2\bar{S}_1\bar{S}_2\bar{I}_2 + U\beta_1\beta_2\bar{S}_1\bar{S}_2$, then equation (9) becomes

$$P(\lambda) = -\lambda^4 + M_1\lambda^2 - R - S + T = 0. \tag{10}$$

Let $x = \lambda^2$, then we have:

$$x^2 - M_1 x + R + S - T = 0. \tag{11}$$

then $\Delta = \sqrt{M_1^2 - 4(R + X - T)}$, then we have the following conclusions:

(1) when $\Delta > 0$, that is $M_1^2 > 4(R + S - T)$, equation (11) has two unequal real roots $x_{1,2} = \dfrac{M_1 \pm \sqrt{\Delta}}{2}$, then equation (10) has four unequal real roots $\lambda_{1,2} = \pm\sqrt{x_1}$, $\lambda_{3,4} = \pm\sqrt{x_2}$;

(2) when $\Delta = 0$, that is $M_1^2 = 4(R + S - T)$, equation (11) has two equal real roots $x_{1,2} = \dfrac{M_1}{2}$, then equation (10) has four equal real roots $\lambda_{1,2,3,4} = \pm\sqrt{x_1}$;

(3) when $\Delta < 0$, that is $M_1^2 < 4(R+S-T)$, equation (11) has two unequal and conjugate imaginary roots $x_{1,2} = \dfrac{M_1 \pm i\sqrt{\Delta}}{2}$, then equation (10) has four unequal and conjugate imaginary roots $\lambda_{1,2} = \pm\sqrt{x_1}$, $\lambda_{3,4} = \pm\sqrt{x_2}$.

In summary, we have the following theorem:

Theorem 1. *When $M_1^2 \geq 4(R + S - T)$, equation* (10) *has four real roots, the no-disease equilibrium E_1^* and endemic equilibrium E_2^* of system* (2) *is unstable. When $M_1^2 < 4(R+S-T)$, equation* (10) *has four unequal and conjugate imaginary roots, the no-disease equilibrium E_1^* and endemic equilibrium E_2^* of system* (2) *is stable.*

3.2. $\tau_1 = \tau, \tau_2 = 0$**:**

When $\tau_1 = \tau, \tau_2 = 0$, equation(8) becomes to the following equation:

$$P(\lambda, \tau) = -\lambda^4 + L\lambda^3 + M_2\lambda^2 + N_2\lambda - R - Se^{-\tau_1\lambda} + T = 0. \tag{12}$$

where

$$\begin{aligned} M_2 &= \mu_1 A + B - \beta_1\beta_2\bar{S}_1\bar{S}_2 e^{-\tau\lambda}, \\ N_2 &= (\mu_1 + \beta_1\bar{I}_2)B + C - \beta_1\beta_2\bar{S}_1\bar{S}_2\bar{I}_2 - U\beta_1\beta_2\bar{S}_1\bar{S}_2 e^{-\tau\lambda}. \end{aligned}$$

L, R, S, T, A, B, C, U, V are defined in the Appendix 1.

Let $\lambda_1 = i\omega$ is a root of equation(12), then we have

$$P(\lambda_1^*, \tau) = -(i\omega)^4 + L(i\omega)^3 + M_2(i\omega)^2 + N_2(i\omega) - R - S^{-\tau(i\omega)} + T = 0. \tag{13}$$

Separating the real and imaginary parts of equation(13), we have the following equations

$$-\omega^4 - M_2\omega^2 - R + T - S\cos\tau\omega = 0, \tag{14}$$

$$-L\omega^3 + N_2\omega + S\sin\tau\omega = 0. \tag{15}$$

From equations (14) and (15), we gain the following equation

$$\begin{aligned} \omega^8 + (2M_2 + L^2)\omega^6 + [M_2^2 + 2(R - T - LN_2)]\omega^4 + [2(R - T) + N_2^2]\omega^2 \\ +(R - T)^2 - S^2 = 0. \end{aligned} \tag{16}$$

When $2M_2 + L^2 = 0$ and $2(R - T) + N_2^2 = 0$, let $y = \omega^4$, then equation (16) becomes

$$y^2 + [M_2^2 + 2(R - T - LN_2)]y + (R - T)^2 - S^2. \tag{17}$$

Then $\Delta = [M_2^2 + 2(R - T - LN_2)]^2 - 4[(R - T)^2 - S^2]$, then we have the following conclusions:

(1) when $\Delta > 0$, that is $[M_2^2 + 2(R - T - LN_2)]^2 > 4[(R - T)^2 - S^2]$, equation (17) has two unequal real roots $y_{1,2} = \dfrac{M_2^2 + 2(R - T - LN_2) \pm \sqrt{\Delta}}{2}$, then equation (16) has eight unequal real roots $\lambda_{1,2,3,4} = \pm\sqrt{y_1}$, $\lambda_{5,6,7,8} = \pm\sqrt{y_2}$;

(2) when $\Delta = 0$, that is $[M_2^2 + 2(R - T - LN_2)]^2 = 4[(R - T)^2 - S^2]$, equation (17) has two equal real roots $y_{1,2} = \dfrac{M_2^2 + 2(R - T - LN_2)}{2}$, then equation (16) has eight unequal real roots $\lambda_{1,2,3,4} = \pm\sqrt{y_1}$, $\lambda_{5,6,7,8} = \pm\sqrt{y_2}$;

(3) when $\Delta<0$, that is $[M_2^2+2(R-T-LN_2)]^2<4[(R-T)^2-S^2]$, equation (17) has two unequal and conjugate imaginary roots $y_{1,2}=\dfrac{M_2^2+2(R-T-LN_2)\pm i\sqrt{\Delta}}{2}$, then equation(16) has eight unequal and conjugate imaginary roots $\lambda_{1,2,3,4} = \pm\sqrt{y_1}$, $\lambda_{5,6,7,8} = \pm\sqrt{y_2}$.

In summary, we have the following theorem:

Theorem 2. *When* $\tau_1 = \tau, \tau_2 = 0$ *and* $[M_2^2+2(R-T-LN_2)]^2 \geq 4[(R-T)^2-S^2]$, *equation* (12) *has four real roots, the no-disease equilibrium* E_1^* *and endemic equilibrium* E_2^* *of system* (2) *is unstable. When* $[M_2^2 + 2(R - T - LN_2)]^2 < 4[(R-T)^2 - S^2]$, *equation* (12) *has four unequal and conjugate imaginary roots, the no-disease equilibrium* E_1^* *and endemic equilibrium* E_2^* *of system* (2) *is stable.*

If ω_0 is a positive real root of equation (14), subs ω_0 to equation (14), we have

$$-\omega_0^4 - M_2\omega_0^2 - R + T - S\cos\tau_0\omega_0 = 0. \tag{18}$$

From equation (18), we gain the real positive root $\omega_0 = 1.3594$, then we can get the calculation formula of τ_0 as follows

$$\tau_0 = \frac{1}{\omega_0}\arccos\frac{T - R - M_2\omega_0^2 - \omega_0^4}{S}. \tag{19}$$

3.3. $\tau_1 = 0, \tau_2 = \tau$**:**

When $\tau_1 = 0, \tau_2 = \tau$, equation(8) becomes to the following equation:

$$P(\lambda,\tau) = -\lambda^4 + L\lambda^3 + M_3\lambda^2 + N_3\lambda - (R+S)e^{-\tau\lambda} + T \tag{20}$$

where

$$\begin{aligned} M_3 &= \mu_1 A + B - \beta_1\beta_2\bar{S}_1\bar{S}_2e^{-\tau\lambda}, \\ N_3 &= (\mu_1 + \beta_1\bar{I}_2)B + C - \beta_1\beta_2^2\bar{S}_1\bar{S}_2\bar{I}_2e^{-\tau\lambda} - U\beta_1\beta_2\bar{S}_1\bar{S}_2e^{-\tau\lambda}. \end{aligned}$$

L, R, S, T, A, B, C, U, V are defined in the Appendix 1.

Let $\lambda_2 = i\omega$ is a root of equation (20), separating the real and imaginary parts of equation (20), we have the following equations

$$\omega^4 + M_3\omega^2 + (R+S)\cos\tau\omega = 0, \tag{21}$$

$$L\omega^3 - N_3\omega - (R+S)\sin\tau\omega = 0. \tag{22}$$

From equations (21) and (22), we have the following equation

$$\omega^8 + (2M_3 + L^2)\omega^6 + (M_3^2 - 2LM_3)\omega^4 + N_3\omega^2 - (R+S)^2 = 0. \tag{23}$$

When $2M_3 + L^2 = 0$ and $\tau\lambda = ln\dfrac{\beta_1\beta_2\bar{S}_1\bar{S}_2(\beta_2\bar{I}_2 - U)}{(\mu_1 + \beta_1\bar{I}_2)B + C}$, let $z = \omega^4$, then equation (23) becomes

$$z^2 + (M_3^2 - 2LM_3)z - (R+S)^2 = 0. \tag{24}$$

Then $\Delta = (M_3^2 - 2LM_3)^2 - 4(R+S)^2$, then we have the following conclusions:

(1) when $\Delta > 0$, that is $(M_3^2 - 2LM_3)^2 > 4(R+S)^2$, equation (24) has two unequal real roots $z_{1,2} = \frac{(M_3^2 - 2LM_3) \pm \sqrt{\Delta}}{2}$, then equation (23) has eight unequal real roots $\lambda_{1,2,3,4} = \pm\sqrt{z_1}$, $\lambda_{5,6,7,8} = \pm\sqrt{z_2}$;

(2) when $\Delta = 0$, that is $(M_3^2 - 2LM_3)^2 = 4(R+S)^2$, equation(17) has two equal real roots $z_{1,2} = \frac{(M_3^2 - 2LM_3)}{2}$, then equation (16) has eight unequal real roots $\lambda_{1,2,3,4} = \pm\sqrt{z_1}$, $\lambda_{5,6,7,8} = \pm\sqrt{z_2}$;

(3) when $\Delta < 0$, that is $(M_3^2 - 2LM_3)^2 < 4(R+S)^2$, equation (17) has two unequal and conjugate imaginary roots $z_{1,2} = \frac{(M_3^2 - 2LM_3) \pm i\sqrt{\Delta}}{2}$, then equation (16) has eight unequal and conjugate imaginary roots $\lambda_{1,2,3,4} = \pm\sqrt{z_1}$, $\lambda_{5,6,7,8} = \pm\sqrt{z_2}$.

In summary, we have the following theorem:

Theorem 3. *When $\tau_1 = 0, \tau_2 = \tau$ and $(M_3^2 - 2LM_3)^2 \geq 4(R+S)^2$, equation (20) has four real roots, the no-disease equilibrium E_1^* and endemic equilibrium E_2^* of system (2) is unstable. When $(M_3^2 - 2LM_3)^2 < 4(R+S)^2$, equation (20) has four unequal and conjugate imaginary roots, the no-disease equilibrium E_1^* and endemic equilibrium E_2^* of system (2) is stable.*

If ω_0 is a positive real root of equation (21), subs ω_0 to the equation (21) and solve it, we gain the real positive root $\omega_0 = 0.7597$. Then we can get the calculation formula of τ_0^* as follows:

$$\tau_0^* = \frac{1}{\omega_0} \arccos \frac{\omega_0^4 + M_3\omega_0^2}{R+S} \tag{25}$$

.

3.3. $\tau_1 = \tau_2 = \tau$**:**

When $\tau_1 = \tau, \tau_2 = \tau$, equation(8) becomes to the following equation:

$$P(\lambda, \tau) = -\lambda^4 + L\lambda^3 + M_4\lambda^2 + N_4\lambda - Re^{-\tau\lambda} - \nu\beta_1\beta_2 S_1 S_2 e^{-2\tau\lambda}. \tag{26}$$

where

$$\begin{aligned} M_4 &= \mu_1 A + B - \beta_1\beta_2\bar{S}_1\bar{S}_2 e^{-\tau\lambda}, \\ N_4 &= (\mu_1 + \beta_1\bar{I}_2 B + C) - \beta_1\beta_2\bar{S}_1\bar{S}_2\bar{I}_2 e^{-\tau\lambda} - v\beta_1\beta_2\bar{S}_1\bar{S}_2 e^{-2\tau\lambda}. \end{aligned}$$

L, R, S, T, A, B, C, U, V are defined in the Appendix 1.

Let $\lambda_3 = i\omega$ is a root of equation (26), separating the real and imaginary parts of equation (26), we have the following equations:

$$\omega^4 + M_4\omega^2 + R\cos\tau\omega + S\cos 2\tau\omega - T = 0, \tag{27}$$

$$(N_4 - L)\omega + R\sin\tau\omega + S\sin 2\tau\omega = 0. \tag{28}$$

From equations (27) and (28), we have:

$$\cos\tau\omega = \frac{(T-\omega^4-M_4\omega^2)^2+(L-N_4)^2\omega^2-R^2-S^2}{2RS}. \tag{29}$$

If ω_0 is a positive real root of equation (27), subs ω_0 to equation (27). Then we can get the calculation formula of τ_0^{**} as follows:

$$\tau_0^{**} = \frac{1}{\omega_0}\arccos\frac{(T-\omega_0^4-M_4\omega_0^2)^2+(L-N_4)^2\omega_0^2-R^2-S^2}{2RS}. \tag{30}$$

4. Numerical simulation

For the analytical solution of equation (9) is too complex, so we solve its numerical solutions. When $\tau_1 = \tau_2 = 0$, set the value of the parameters are $\mu_1 = 0.6$, $\mu_1 = 0.6$, $\beta_1 = 0.01$, $\beta_2 = 0.04$, $\gamma_1 = 0.95$, $N_1 = 33$, $A = 38$, we solve equation (9) and gain the characteristic root are:

$$\begin{aligned} \lambda_1 &= 37.5675, \\ \lambda_2 &= 0.2279, \\ \lambda_3 &= -0.0828+0.1559i, \\ \lambda_4 &= -0.0828-0.1559i. \end{aligned}$$

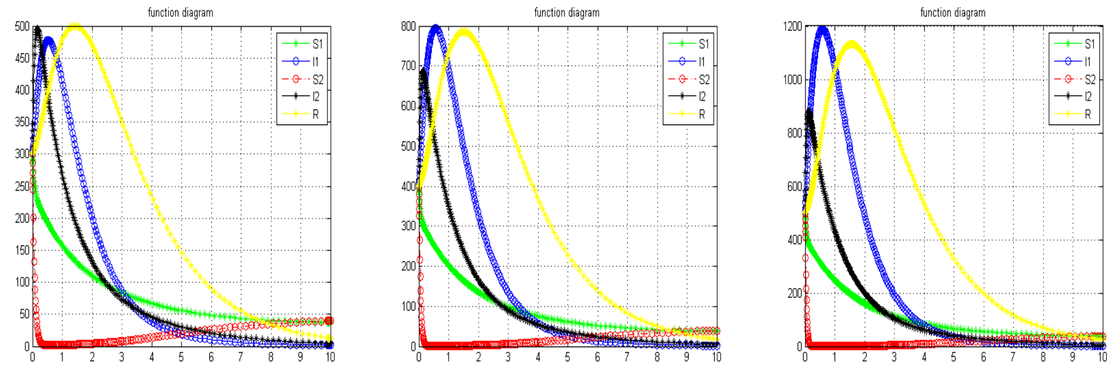
At this time, the basic reproductive ratio $R_0 = 0.3995 < 1$. The non-disease equilibrium point is $E_1^*(S_1, I_1, S_2, I_2) = (33, 0, 42.2222, 0)$, The endemic equilibrium point is $E_2^*(S_1, I_1, S_2, I_2) = (53.4864, 7.9302, 65.2035, 22.9813)$. Therefore, we have:

Theorem 4. *When $\tau_1 = \tau_2 = 0$, the non-disease equilibrium point is E_1^* and the endemic equilibrium point E_2^* are stable.*

We discuss the time delays affect on the number of susceptible and patients in three cases:(1)$\tau_1 = \tau, \tau_2 = 0$, (2)$\tau_1 = 0, \tau_2 = \tau$, (3)$\tau_1 = \tau_2 = \tau$.

4.1. $\tau_1 = \tau, \tau_2 = 0$

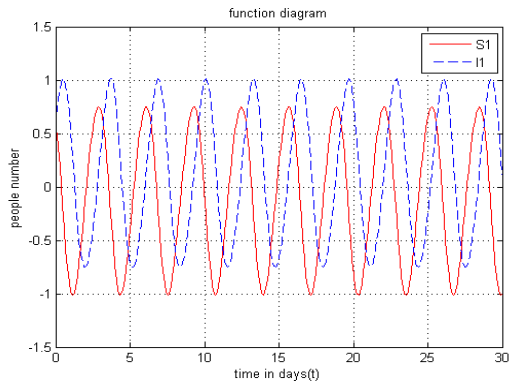
For the analytical solution of equation(14) is too complex, so we solve its numerical solutions. When $\tau_1 = 3, \tau_2 = 0$, set the value of the parameters are $\mu_1 = 0.6$, $\mu_1 = 0.6$, $\beta_1 = 0.01$, $\beta_2 = 0.04$, $\gamma_1 = 0.95, N_1 = 33$, $A = 38$, we solve equation (14) and gain the characteristic root is: $\omega = -0.2423 - 0.1502i$. At this time, the characteristic root is $\lambda_1 = i\omega = 0.1502 - 0.2423i$. The real positive root of equation (12) is $\omega_0 = 1.3594$, and $\tau_0^* = 0.6155$.



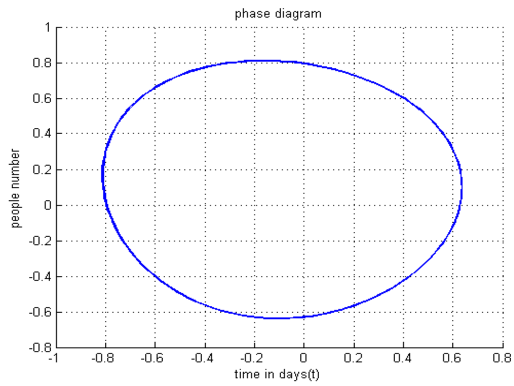
(a) (b) (c)

Figure 1: The function diagram of different N_1, where (a) is $N1 = 300$, (b) is $N1 = 400$, (c) is $N1 = 500$, the initial value $A = 60$, $(S_1, I_1, S_2, I_2, R) = (40, 40, 40, 40, 40)$.

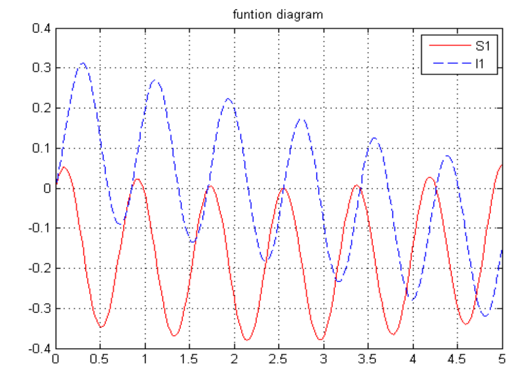
As the first case $\tau_1 = \tau, \tau_2 = 0$, we choose different iteration step (IS), to simulate the phase diagram of the system, when the time delay $\tau_2 > \tau_0$, the system appears periodic solution and limit cycle, the result are showed as the following figures.



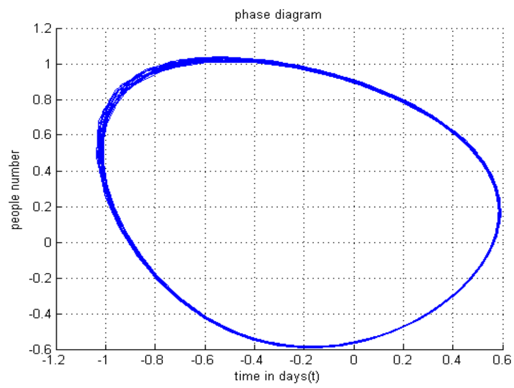
(a)The function diagram



(b)The phase diagram



(c)The function diagram



(d)The phase diagram

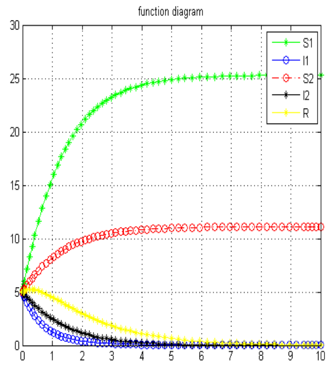
Figure 2: The function diagram and phase diagram of different iteration step, where (a) and (b) is the function diagram and phase diagram of IS=0.05, $\tau_1 \geq \tau_0$, τ_2=0, (c) and (d) is the function diagram and phase diagram of IS=0.5, $\tau_1 \geq \tau_0$, τ_2=0.

In summary, we have the following theorem:

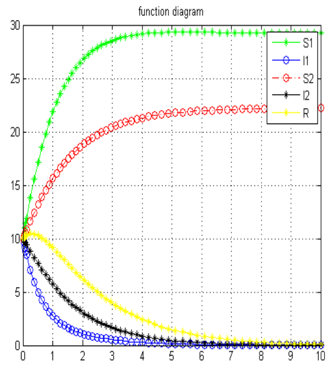
Theorem 5. *When $\tau < \tau_0$, the non-disease equilibrium point is E_1^* and the endemic equilibrium point E_2^* are asymptotically stable; when $\tau > \tau_0$, the endemic equilibrium point E_2^* is unstable, when τ is increasing and through τ_0, the endemic equilibrium point E_2^* branch into periodic small amplitude solutions.*

4.2. $\tau_1 = 0, \tau_2 = \tau$

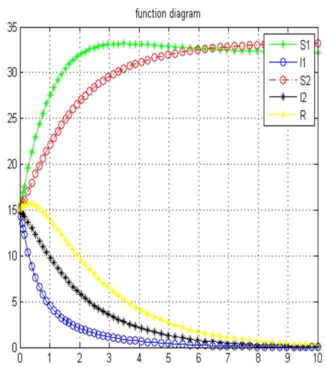
For the analytical solution of equation (20) is too complex, so we solve its numerical solutions.When $\tau_1 = 0, \tau_2 = 3$, set the value of the parameters are $\mu_1 = 0.6$, $\mu_1 = 0.6$, $\beta_1 = 0.01$, $\beta_2 = 0.04$, $\gamma_1 = 0.95$, $N_1 = 33$, $A = 38$, we solve equation (14) and gain the characteristic root is: $\omega = -0.0269 + 0.1885i$. At this time, the characteristic root is $\lambda_2 = i\omega = -0.1885 - 0.0269i$. The real positive root of equation(18) is $\omega_0 = 0.7597$, and $\tau_0^* = 0.0490$.



(a)

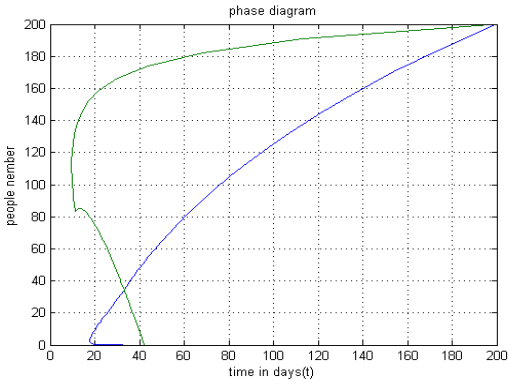


(b)

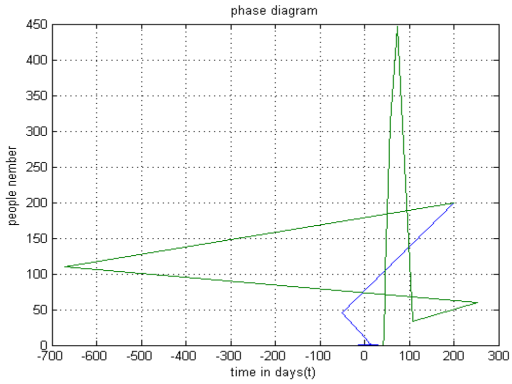


(c)

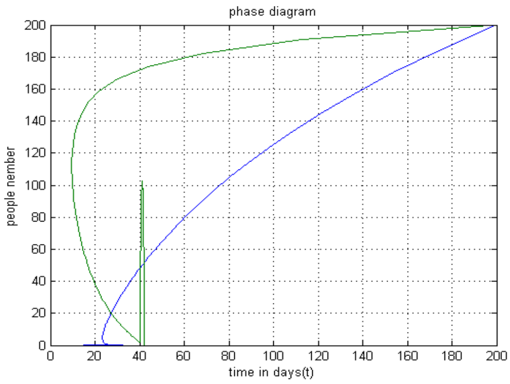
Figure 3: The function diagram of different N_1 and A, and of different initial value S_1, I_1, S_2, I_2, where (a) is $N1 = 30, A = 10, IV = 5$, $(S_1, I_1, S_2, I_2, R) = (5, 5, 5, 5, 5)$, (b) is $N1 = 40, A = 20, IV = 10$,$(S_1, I_1, S_2, I_2, R) = (10, 10, 10, 10, 10)$, (c) is $N1 = 50, A = 30, IV = 15$, $(S_1, I_1, S_2, I_2, R) = (15, 15, 15, 15, 15)$.



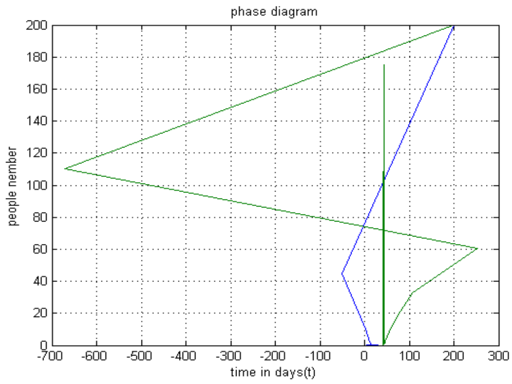
(a)



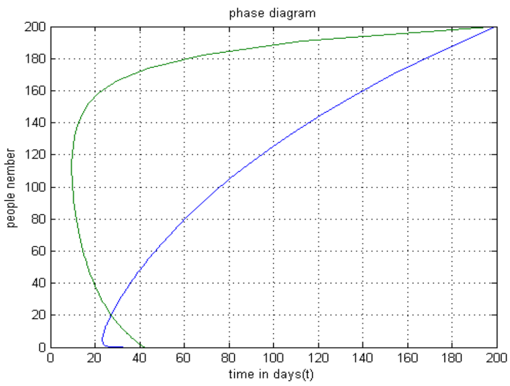
(b)



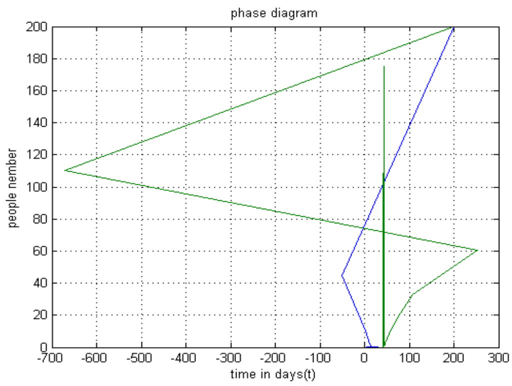
(c)



(d)



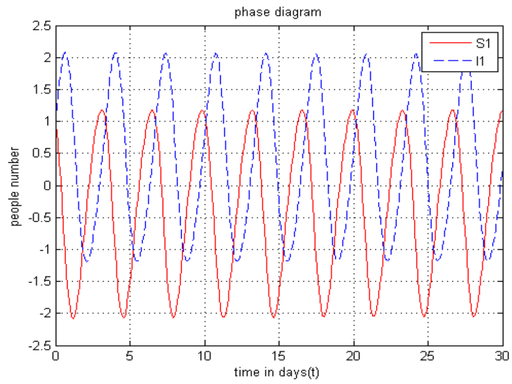
(e)



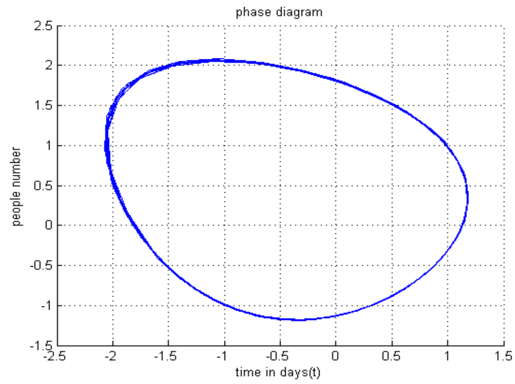
(f)

Figure 4: The phase diagram of different iteration step t, where (a) is $IS = 0.05, \tau_1 = 0, \tau_2 = 1$, (b) is $IS = 0.5, \tau_1 = 0, \tau_2 = 1$, (c) is $IS = 0.05, \tau_1 = 0, \tau_2 = 5$, (d) is $IS = 0.5, \tau_1 = 0, \tau_2 = 5$, (e) is $IS = 0.05, \tau_1 = 0, \tau_2 = 50$, (f) is $IS = 0.5, \tau_1 = 0, \tau_2 = 50$.

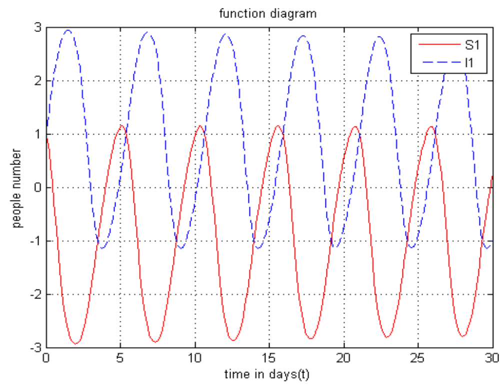
When the time delay $\tau_2 > \tau_0$,the system appear periodic solution and limit cycle, it is showed as the following figure.



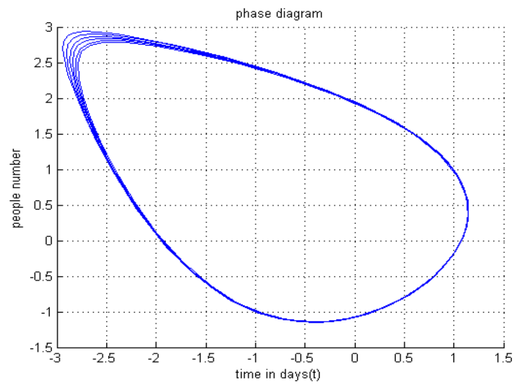
(a)The function diagram



(b)The phase diagram



(c)The function diagram



(d)The phase diagram

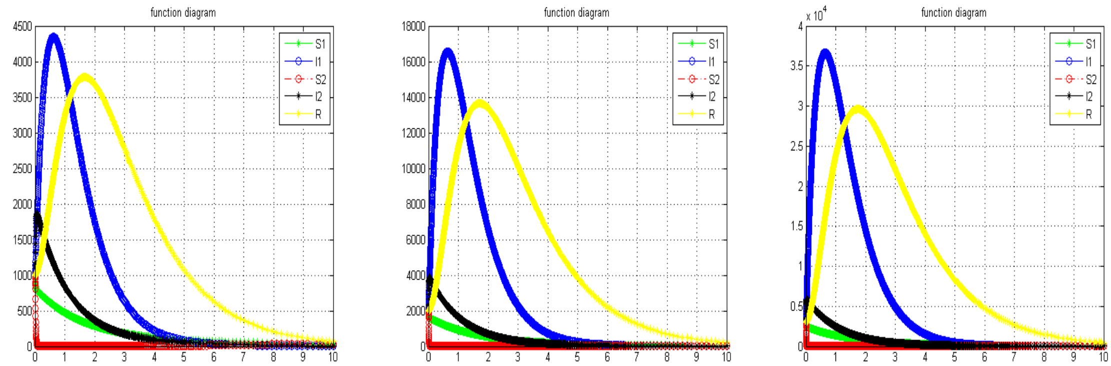
Figure 5: The function diagram and phase diagram of different iteration step, where (a) and (b) is the function diagram and phase diagram of $IS = 0.05, \tau_1 = 0, \tau_2 \geq \tau_0$, (c) and (d) is the function diagram and phase diagram of $IS = 0.5, \tau_1 = 0, \tau_2 \geq \tau_0$.

In summary, we have the following theorem:

Theorem 6. *When $\tau < \tau_0^*$, the non-disease equilibrium point is E_1^* and the endemic equilibrium point E_2^* are asymptotically stable; when $\tau > \tau_0^*$, the endemic equilibrium point E_2^* is unstable, when τ is increasing and through τ_0^*, the endemic equilibrium point E_2^* branch into periodic small amplitude solutions.*

4.3. $\tau_1 = \tau_2 = \tau$

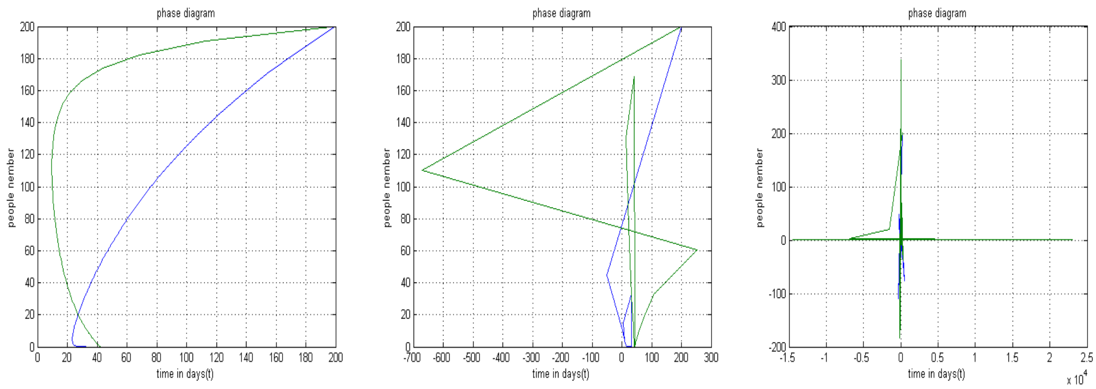
Solving equation (25), we gain $\omega = -0.0147i$. At this time, the characteristic root is $\lambda_3 = i\omega = 0.0147$. When $\tau_1 = \tau_2 \geq 60$, equation (23) has real positive root. By calculating, we gain $\omega_0 = 0.8861$, $\tau_0^{**} = 3.5454$.



(d) (e) (f)

Figure 6: The function diagram of different N_1, where (a) is $N1 = 1000$, (b) is $N1 = 2000$, (c) is $N1 = 3000$, the initial value $A = 60$, $(S_1, I_1, S_2, I_2, R) = (40, 40, 40, 40, 40)$.

In order to simulate the phase of the system when $\tau_1 = \tau_2 = \tau$, we choose the different iteration step (IS) t, the result are showed as the following figures.



(a)$\tau_1 = \tau_2 = 10$ (b)$\tau_1 = \tau_2 = 20$ (c)$\tau_1 = \tau_2 = 50$

Figure 7: The phase diagram of the system when $\tau_1 = \tau_2 = \tau$, where (a) is the $IS = 0.05$, (b) is the $IS = 0.5$, (c) is $IS = 1$

In summary, we have the following theorem:

Theorem 7. *When* $(T - \omega_0^4 - M_4\omega_0^2)^2 + (L - N)^2\omega_0^2 \geq R^2 + S^2$ *and for any* τ_0^{**} *the non-disease equilibrium point is* E_1^* *and the endemic equilibrium point* E_2^* *are unstable.*

5. Conclusion

Recently, dengue fever which is considered to be a tropical disease forms a major concern due to its severity and complexity in the world. It brings a serious threat to human health and life. Therefore, it is of great importance and meaning to study the infections of the dengue fever. There are many factors that can influence the spread of dengue fever, here we only consider the time delay.

In this paper, on the basis of the dengue model which was proposed by Tewa et.al. [21] and Dr. Ding Deqiong [22], we improve the model, present a model based on several features of dengue infection with time delay. We consider two time delays τ_1 and τ_2, where τ_1 and τ_2 represent the rehabilitees immune period and incubation period to the disease respectively. This paper analyzed the stability of the dengue model with time delay and gained the specific value of time delay threshold τ_0 at which the dengue model appears periodic solutions and Hopf bifurcation.

In this paper, we discuss the effect of the time delay in three cases, that are $(1)\tau_1 = \tau, \tau_2 = 0$, $(2)\tau_1 = 0, \tau_2 = \tau$, $(3)\tau_1 = \tau_2 = \tau$. On each case, by calculating the model, we gained the time delay threshold τ_0 at which the dengue model appears periodic solutions and Hopf bifurcation. On the first case, $\tau_1 = \tau, \tau_2 = 0$, $\tau_0 = 0.6155$; on the second case $\tau_1 = 0, \tau_2 = \tau$, $\tau_0 = 0.0490$; on the third case $\tau_1 = \tau_2 = \tau$, $\tau_0 = 3.5454$. From this, we can see that, all of the time delay τ_1 and τ_2 play importance role in the spreading of the dengue fever epidemic.

Our numerical results imply that in general the dengue fever epidemic latent in 7-14 days which is in agreement with clinical literature.

Appendix 1

$$
\begin{aligned}
L &= A - \beta_1 \bar{I}_2 - M_1 \\
M &= \mu_1 A + B - \beta_1\beta_2 \bar{S}_1 \bar{S}_2 e^{-(\tau_1+\tau_2)\lambda} \\
N &= (\mu_1 + \beta_1 \bar{I}_1)B + C - \beta_1\beta_2^2 \bar{S}_1 \bar{S}_2 \bar{I}_1 e^{-\tau\lambda} - U\beta_1\beta_2 \bar{S}_1 \bar{S}_2 e^{-(\tau_1+\tau_2)\lambda} \\
R &= (\mu_2 + \beta_2 \bar{I}_1)\beta_1\beta_2^2 \bar{S}_1 \bar{S}_2 \bar{I}_1 \\
S &= \beta_1\beta_2 \bar{S}_1 \bar{S}_2 V \\
T &= (\mu_1 + \beta_1 \bar{I}_2)C + \beta_1\beta_2^2 \bar{S}_1 \bar{S}_2 \bar{I}_1 \bar{I}_2 \\
A &= \mu_1 + 2\mu_2 + \beta_2 \bar{I}_1 + \gamma_1 \\
B &= (\mu_1 + \mu_2 + \gamma_1)\beta_2 \bar{I}_1 + 2(\mu_1 + \gamma_1)\mu_2 + \mu_2^2 \\
C &= (\mu_1 + \mu_2)\mu_2\beta_2 \bar{I}_1 + (\mu_1 + \gamma_1)\mu_2^2 \\
U &= \beta_1 \bar{I}_2 + \beta_2 \bar{I}_1 + \mu_1 + \mu_2 \\
V &= \beta_1\beta_2 \bar{I}_1 \bar{I}_2 + \mu_1\beta_2 \bar{I}_1 + \mu_2\beta_1 \bar{I}_2 + \mu_1\mu_2
\end{aligned}
$$

References

[1] World Health Organization. *Dengue and severe dengue*, 2013. http://www.who.int/mediacentre/factsheets/fs117/en/index.html. [accessed 22.09.13].

[2] World Health Organization. *Health topics* (dengue), 2013. http://www.who.int/topics/dengue/en/. [accessed 22.09.13]

[3] Hethcote, H.W., *The Mathematics of Infectious Disease* [J], SIAM Review, 42 (2000), 599-653.

[4] Freedman, H.I., Tang, M.X., Ruan, S.G., *Uniform Persistence and Flows near a Closed Positively Invariant Set* [J], Journal of Dynamics and Differential Equations, 6 (1994), 583-600.

[5] Li, M.Y., Graef, J.R., Wang, L., et al., *Global Dynamics of a SEIR Model with Varying Total Population Size* [J], Mathematical Biosciences, 160 (1999), 191-213.

[6] Feng, Z., Vealsco-Hernandez, V., *Competitive Exclusion in a Vector-host Model for the Dengue Fever* [J], Journal of Mathematical Biology, 1997, 35:523-544.

[7] Naresh, R., Tripathi, A., Sharma, D., *A Nonlinear AIDS Epidemic Model with Screening and Time Delay* [J], Journal of the Mathematical Society of Japan, 2011,217:4416-4426.

[8] McCluskey, C.C., *Global Stability for an SIR Epidemic Model with Delay and Nonlinear Incidence* [J], Nonlinear Analysis: Real World Applications, 2010, 11:3106-3109.

[9] Marsden, J.E., Sirovich, L., John, F., *Nonlinear Oscillations Dynamical Systems, and Bifurcations of Vector Fields* [M], Springer-Verlag, New York, 1997, 4-5.

[10] Kermack, W.O., McKendrick, A.G., *Contributions to the Mathematical Theory of Epidemics* [J], Proceedings of the Royal Society A, A115 (1927), 700-720.

[11] Kermack, W.O., McKendrick, A.G., *Contributions to the Mathematical Theory of Epidemics* [J], Proceedings of the Royal Society A, A138 (1932), 55-83.

[12] Hethcote, H.W., Li, Y., Jing, Z., *Hopf Bifurcation in Models for Pertussis Epidemiology* [J], Mathematical and Computer Modeling, 30 (1999), 29-45.

[13] *Babar, A., Jamil, A., Amjad, A., Rehan, Z.P., Samar, H.K.T., Umar, N., Tariq, S.*, *On the modeling and analysis of the regulatory network of dengue virus pathogenesis and clearance* [J], Computational Biology and Chemistry, 53 (2014), 277-291.

[14] Lung-Chang Chien, Hwa-Lung Yu, *Impact of meteorological factors on the spatiotemporal patterns of dengue fever incidence*, [J], Environment International, 73 (2014), 46-56.

[15] Tanvi, P., Gujarati, G.A., *Virus antibody dynamics in primary and secondary dengue infections* [J], Mathematical Biology, 69 (2014), 1773-1800.

[16] Garcia-Garaluz, E., Atencia, M., Joya, G., Garcia-Lagos, F., Sandoval, F., *Hopfield networks for identification of delay differential equaltions with an application to dengue fever epidemics in Cuba* [J], Neurocomputing, 74 (2011), 2691-2697.

[17] Newton, E.A., Reiter, A., *A model of the Transmission of Dengue Fever with an Evaluation of the Impact of Ultra-low Volume (ULV) Insecticide Applications on Dengue Epidemics* [J], American Journal of Tropical Medicine and Hygiene, 47 (1992), 709-720.

[18] Esteva, L., Vargas, C., *Analysis of a Dengue Disease Transmission Model* [J], Mathematical Biosciences, 150 (1998), 131-151.

[19] Esteva, L., Vargas, C., *A Model for Dengue Disease with Variable Human Population* [J], Journal of Mathematical Biology, 46 (1999), 220-240.

[20] Esteva, L., Vargas, C., *Influence of Vertical and Mechanical Transmission on The Dynamics of Dengue Disease* [J], Mathematical Biosciences, 167 (2000), 51-64.

[21] Tewa, J.J., Dimi, J.L., Bowong, S., *Lyapunov Functions for a Dengue Disease Transmission Model* [J], Chaos Solutions and Fractals, 39 (2009), 936-941.

[22] Ding, Deqiong, *The numerical solution and global stability of two classes of infectious disease* [D], Harbin Institute of Technology, 2011, 6.

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ENTROPY OF QUANTUM DYNAMICAL SYSTEMS WITH INFINITE PARTITIONS

Abolfazl Ebrahimzadeh[1]

Young Researchers and Elite Club
Zahedan Branch
Islamic Azad University
Zahedan
Iran
e-mail: Abolfazl35@yahoo.com

Zahra Eslami Giski

Department of Mathematics
Islamic Azad University
Sirjan Branch
Sirjan
Iran
e-mail: Eslamig_zahra@yahoo.com

Abstract. In this paper, the concepts of the entropy and relative entropy on quantum logic with countable partitions are defined and some ergodic properties of quantum dynamical systems are investigated. Finally, we show that the entropy is invariant under isomorphic relation.

Keywords: countable partition, entropy, dynamical system, quantum logic.

2010 MSC: 03-XX 05A18 28D20.

1. Introduction and countable partitions

Entropy is a tool to measure the amount of uncertainty in random event. Entropy has been applied in a variety of problem areas including physics, computer science, general systems theory, information theory, statistics, biology, chemistry, sociology and many other fields. The quantum logic approach was introduced by Birkhoff and Von Neumann [1]. Later Riecan and Dvurecenskij proposed a new model for quantum mechanics [3]. Yuan, Khare, Roy and Ebrahimzadeh using the notion of state of quantum logic, were introduced the entropy and logical entropy of finite partitions on quantum logic [2], [4], [5], [7]. In this paper, the entropy with countable partitions is defined and then by using this concept, entropy of

[1]Corresponding author.

dynamical systems is defined and some of its properties are investigated. Also, the relative entropy of countable partitions is defined and studied under the relation s-refinement.

At first, some basic definitions are presented that will be useful in further considerations.

Definition 1.1. [7] A quantum logic QL is a σ-orthomodular lattice, i.e., a lattice L $(L, \leq, \vee, \wedge, 0, 1)$ with the smallest element 0 and the greatest element 1, an operation $' : L \to L$ such that the following properties are hold for all $a, b \in L$:

(i) $a^{''} = a$, $a \leq b \Rightarrow b^{'} \leq a^{'}$, $a \vee a^{'} = 1$, $a \wedge a^{'} = 0$;

(ii) Given any countable sequence $(a_i)_{i \in I}$, $a_i \leq a_j^{'}$, $i \neq j$, the join $\vee_{i \in \mathbb{N}} a_i$ exists in L;

(iii) L is orthomodular: $a \leq b \Rightarrow b = a \vee (b \wedge a^{'})$.

Two elements $a, b \in QL$ are called orthogonal if $a \leq b^{'}$ and denoted by $a \perp b$. A sequence $(a_i)_{i \in I}$ is said orthogonal if $a_i \perp a_j$, $\forall i \neq j$.

Definition 1.2. [7] Let L be a QL. A map $s : L \to [0, 1]$ is a state iff $s(1) = 1$ and for any orthogonal sequence $(a_i)_{i \in I}$, $s(\vee_{i \in I} a_i) = \sum\limits_{i \in I} s(a_i)$.

Definition 1.3. Let $P = \{a_i : i \in \mathbb{N}\}$ be a countable system of elements of the QL, L. P is called to be a $\vee$-orthogonal system iff $\vee_{i=1}^{k} a_i \perp a_{k+1}$, $\forall k$.

Definition 1.4. We say a system $P = \{a_i : i \in \mathbb{N}\} \subset L$ is the partition of L corresponding to the state s iff:

(i) P is a $\vee$-orthogonal system;

(ii) $s(\vee_{i \in \mathbb{N}} a_i) = \sum\limits_{i=1}^{\infty} s(a_i) = 1$.

2. Entropy and relative entropy of countable partitions

Definition 2.1. Let $P = \{a_i : i \in \mathbb{N}\}$ and $R = \{b_j : j \in \mathbb{N}\}$ be two countable partitions of L. We say R is a s-refinement of P, denoted by $P \leq_s R$, if for each $b_j \in R$ there exists $a_i \in P$ with $s(b_j \wedge a_i) = s(b_j)$.

Let $P = \{a_i : i \in \mathbb{N}\}$ and $R = \{b_j : j \in \mathbb{N}\}$ be two countable partitions of L corresponding to a state s and $s(\vee_{i \in \mathbb{N}}(a_i \wedge b)) = s(b), \forall b \in L$. Then by Definition 1.2, we get $\sum_{i=1}^{\infty} s(a_i \wedge b) = s(b)$. In the remaining of the present paper, s has this property. Then the common refinement of these partitions is the partition

$$P \vee R = \{a_i \wedge b_j : a_i \in P, b_j \in R, i, j \in \mathbb{N}\}.$$

Definition 2.2. Let $P = \{a_i : i \in \mathbb{N}\} \subset L$ be a partition of the QL, L corresponding to a state s. The entropy of P with respect to state s is defined by

$$H_s(P) = -\sum_{i=1}^{\infty} s(a_i) \log s(a_i)$$

such that $0 \log 0 = 0$.

Definition 2.3. Let $P = \{a_i : i \in \mathbb{N}\}$ and $R = \{b_j : j \in \mathbb{N}\}$ be two countable partitions of L corresponding to a state s. The relative entropy of P with respect to R is defined as following:

$$H_s(P||R) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} s(a_i) \log \frac{s(a_i)}{s(b_j)} \tag{2.1}$$

whenever $s(b_j) \neq 0$.

In the next proposition, it is proved subadditivity of entropy of countable partitions on a QL.

Proposition 2.4. *Let P and R be countable partitions of L corresponding to a state s. Then*

(i) $H_s(P) \geq 0$;

(ii) $P \leq_s R$ *implies that* $H_s(P) \leq H_s(R)$;

(iii) $H_s(P \vee R) \leq H_s(P) + H_s(R)$.

Proof. (ii) For each $b_j \in R$, there exists $a_i \in P$ such that $s(b_j) = s(b_j \wedge a_i)$. By definition of state,

$$s(b_j) \leq \sum_{j=1}^{\infty} s(a_i \wedge b_j) = s(a_i).$$

(iii) Let $P = \{a_i : i \in \mathbb{N}\}$ and $Q = \{b_j : j \in \mathbb{N}\}$ be two countable partitions corresponding to a state s, $f(x) = x \log x$, for $x > 0$ and $f(x) = 0$, for $x = 0$. From [8], we have

$$f\left(\sum_{j=1}^{\infty} \alpha_j x_j\right) \leq \sum_{j=1}^{\infty} \alpha_j f(x_j),$$

where $\sum\limits_{j=1}^{\infty} \alpha_j = 1$ and $\alpha_j, x_j \in [0,1]$. Let $\alpha_j = s(b_j)$ and $x_j = \frac{s(a_i \wedge b_j)}{s(b_j)}$, $s(b_j) \neq 0$, $j \in \mathbb{N}$. We have $\alpha_j, x_j \in [0,1]$, $\sum\limits_{j=1}^{\infty} \alpha_j = \sum\limits_{j=1}^{\infty} s(b_j) = 1$. By definition of state, we can write, for each $i \in \mathbb{N}$,

$$\sum_{j=1}^{\infty} \alpha_j x_j = \sum_{j=1}^{\infty} s(b_j)\ \frac{s(a_i \wedge b_j)}{s(b_j)} = \sum_{j=1}^{\infty} s(a_i \wedge b_j) = s(a_i)$$

and

$$f\left(\sum_{j=1}^{\infty}\alpha_j x_j\right) = f(s(a_i)).$$

Similarly, we have

$$\begin{aligned}\sum_{j=1}^{\infty}\alpha_j f(x_j) &= \sum_{j=1}^{\infty} s(b_j)\ f\left(\frac{s(a_i\wedge b_j)}{s(b_j)}\right)\\ &= \sum_{j=1}^{\infty} s(a_i\wedge b_j)\ (\log(s(a_i\wedge b_j) - \log s(b_j))\\ &= \sum_{j=1}^{\infty} s(a_i\wedge b_j)\ \log s(a_i\wedge b_j) - \sum_{j=1}^{\infty} s(a_i\wedge b_j)\ \log s(b_j).\end{aligned}$$

So

$$\begin{aligned}f\left(\sum_{j=1}^{\infty}\alpha_j x_j\right) &= f(s(a_i)) \le \sum_{j=1}^{\infty}\alpha_j f(x_j)\\ &= \sum_{j=1}^{\infty} f(s(a_i\wedge b_j)) - \sum_{j=1}^{\infty} s(a_i\wedge b_j)\ \log s(b_j).\end{aligned}$$

Summarizing, we obtain

$$\sum_{i=1}^{\infty} f(s(a_i) \le \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} f(s(a_i\wedge b_j)) - \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} s(a_i\wedge b_j)\ \log s(b_j).$$

From definition of state, we have $\sum_{i=1}^{\infty} s(a_i\wedge b_j) = s(b_j), j\in\mathbb{N}$, therefore

$$\sum_{i=1}^{\infty} f(s(a_i) \le \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} f(s(a_i\wedge b_j)) - \sum_{j=1}^{\infty} f(s(b_j)).$$

This implies that $H_s(P\vee R)\le H_s(P)+H_s(R)$. ∎

Now, the relative entropy of infinite partitions under the relation s-refinement, will be studied.

Proposition 2.5. *Let P, R and M be countable partitions of L. Then*

(i) $H_s(P||P^0) = H_s(P)$ *where* $P^0 = \{1\}$*;*

(ii) $P\le_s M$ *implies that* $H_s(M||R)\le H_s(P||R)$*;*

(iii) $P\le_s M$ *implies that* $H_s(R||M)\ge H_s(R||P)$.

Proof. (i) $s(1) = 1$.

(ii) Let $P = \{p_i : i \in \mathbb{N}\}$, $R = \{r_j : j \in \mathbb{N}\}$ and $M = \{m_k : k \in \mathbb{N}\}$. Since $P \leq_s M$, for each m_k there exists p_i such that $s(m_k \wedge p_i) = s(m_k)$. Thus

$$s(m_k) = s(m_k \wedge p_i) \leq \sum_{k=1}^{\infty} s(m_k \wedge p_i) = s(p_i),$$

and this implies that

$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} s(m_k) \log \frac{s(m_k)}{s(r_j)} \leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} s(p_i) \log \frac{s(p_i)}{s(r_j)}.$$

So $H_s(M||R) \leq H_s(P||R)$.

(iii) As we proved in part ii), $s(m_k) \leq s(p_i)$. Therefore,

$$\frac{s(m_k)}{s(r_j)} \leq \frac{s(p_i)}{s(r_j)},$$

and this finished the proof. ∎

3. Entropy of dynamical systems

Definition 3.1. Let L be a QL and $T : L \to L$ be a map with the following properties:

(i) $T\left(\bigvee_{i=1}^{k} a_i\right) = \bigvee_{i=1}^{k} T(a_i)$, $T\left(\bigvee_{i=1}^{\infty} a_i\right) = \bigvee_{i=1}^{\infty} T(a_i)$, $\forall k \in \mathbb{N}, \forall a_i \in L$;

(ii) $T(a \wedge b) = T(a) \wedge T(b)$, $\forall a, b \in L$;

(iii) $T(a') = (T(a))'$, $\forall a \in L$.

$T : L \to L$ with respect to a state s is called state preserving if $s(T(a)) = s(a)$ for every $a \in L$. Then the triple (L, s, T) is said a quantum dynamical system.

Proposition 3.2. *Let (L, s, T) be a quantum dynamical system and $P = \{a_i : i \in \mathbb{N}\}$ be a partition of (L, s), then*

(i) *If $a_i \leq a_j$, then $T(a_i) \leq T(a_j)$;*

(ii) *$T(P) = \{T(a_i) : i \in \mathbb{N}\}$ is a partition;*

(iii) *$H_s(T(P)) = H_s(P)$.*

Proof. (i) $a_i \leq a_j$ so $a_i \wedge a_j = a_i$. By Definition 3.1, $T(a_i \wedge a_j) = T(a_i) \wedge T(a_j) = T(a_i)$. Thus $T(a_i) \leq T(a_j)$.
(ii) By Definition 3.1 and part i), the proof is obvious.
(iii) $s(T(a_i)) = s(a_i)$. ∎

Theorem 3.3. [6] *Let* $\{(a_i)\}_{i=1}^{\infty}$ *be sequence of nonnegative numbers such that* $a_{r+s} \leq a_r + a_s$ *for each* $r, s \in \mathbb{N}$. *Then* $\lim\limits_{n\to\infty} \frac{1}{n} a_n$ *exists.*

Proposition 3.4. $\lim\limits_{n\to\infty} \dfrac{1}{n} H_s(\vee_{i=1}^{n} T^i P)$ *exists.*

Proof. Let $a_n = H_s(\vee_{i=1}^{n} T^i P) \geq 0$. Then

$$H_s(\vee_{i=1}^{n+p} T^i P) \leq H_s(\vee_{i=1}^{n} T^i P) + H_s(\vee_{i=n+1}^{n+p} T^i P) = a_n + a_p.$$

So $a_{n+p} \leq a_n + a_p$, $\forall n, p$. Hence $\lim\limits_{n\to\infty} \frac{a_n}{n}$ exists. ∎

Definition 3.5. Let (L, s, T) be a quantum dynamical system and P be a countable partition of (L, s). The entropy of T respect to P is defined by

$$h_s(T, P) = \lim_{n\to\infty} \frac{1}{n} H_s(\vee_{i=1}^{n} T^i P). \tag{3.1}$$

Definition 3.6. Let (L, s, T) be a quantum dynamical system. The entropy of T is defined by

$$h_s(T) = \sup_P h_s(T, P) \tag{3.2}$$

where the supremum is taken over all countable partitions of (L, s).

In the following propositions, some ergodic properties of $h_s(T, P)$ and $h_s(T)$ will be studied.

Proposition 3.7. *If* (L, s, T) *is a quantum dynamical system, then*

(i) $h_s(T, P) \geq 0$;

(ii) $P \leq_s R$ *implies that* $h_s(T, P) \leq h_s(T, R)$;

(iii) $h_s(T, P) \leq H_s(P)$;

(iv) $h_s(T, P \vee R) \leq h_s(T, P) + h_s(T, R)$;

(v) $h_s(T, P) = h_s\left(T, \vee_{i=1}^{k} T^i P\right)$, *for* $k \in \mathbb{N}$.

Proof. (i) Obvious.
(ii) We have for each $i \in \mathbb{N}$, $\vee_{i=1}^{n} T^i P \leq_s \vee_{i=1}^{n} T^i R$. So, by definition and Proposition 2.4, it holds.
(iii) From Propositions 2.4, 3.2 we have for each $n \in \mathbb{N}$,

$$\frac{1}{n} H_s(\vee_{i=1}^{n} T^i P) \leq \frac{1}{n} \sum_{i=1}^{n} H_s(T^i P) = \frac{1}{n} \sum_{i=1}^{n} H_s(P) = H_s(P).$$

(iv) By Definition 3.1, $T(a \wedge b) = T(a) \wedge T(b)$, $\forall a, b \in B$. So, by using Proposition 2.4, we can write

$$\begin{aligned} H_s(\vee_{i=1}^n T^i(P \vee R)) &= H_s((\vee_{i=1}^n T^i P) \vee (\vee_{i=1}^n T^i R)) \\ &\leq H_s(\vee_{i=1}^n T^i P) + H_s(\vee_{i=1}^n T^i R). \end{aligned}$$

(v)

$$\begin{aligned} h_s(T, \vee_{i=1}^k T^i P) &= \lim_{n\to\infty} \frac{1}{n} H_s(\vee_{j=1}^n T^j (\vee_{i=1}^k T^i P)) \\ &= \lim_{n\to\infty} \frac{1}{n} H_s(\vee_{i=1}^{k+n-1} T^i P) \\ &= \lim_{n\to\infty} \left(\frac{k+n-1}{n}\right) \frac{1}{k+n-1} H_s(\vee_{i=1}^{k+n-1} T^i P) \\ &= h_s(T, P). \end{aligned}$$

■

Proposition 3.8. *If (L, s, T) is a quantum dynamical system, then*

(i) $h_s(id) = 0$;

(ii) *For* $k \geq 1$, $h_s(T^k) = k h_s(T)$.

Proof. (i) By Definition 2.1, we have $\vee_{i=1}^n T^i P = P$, for all $n \in \mathbb{N}$. Therefore,

$$h_s(id, P) = \lim_{n\to\infty} \frac{1}{n} H_s(\vee_{i=1}^n T^i P) = 0.$$

(ii) Let P be an arbitrary countable partition of (L, s). We have

$$\begin{aligned} h_s(T^k, \vee_{i=1}^n T^i P) &= \lim_{n\to\infty} \frac{1}{n} H_s(\vee_{j=1}^n (T^k)^j (\vee_{i=1}^n T^i P)) \\ &= \lim_{n\to\infty} \frac{1}{n} H_s(\vee_{j=1}^n \vee_{i=1}^k T^{kj+i} P) \\ &= \lim_{n\to\infty} \frac{1}{n} H_s(\vee_{i=1}^{nk-1} T^i P) \\ &= \lim_{n\to\infty} \frac{nk}{n} \frac{1}{nk} H_s(\vee_{i=1}^{nk-1} T^i P) \\ &= k h_s(T, P) \end{aligned}$$

So, $\sup_P h_s(T, P) = \sup_P h_s(T^k, \vee_{i=1}^n T^i P) \leq \sup_P h_\eta(T^k, P) = h_s(T^k)$
Since $P \leq_s \vee_{i=1}^k T^i P$, we have $h_s(T^k, P) \leq h_s(T^k, \vee_{i=1}^n T^i P) = k h_s(T, P)$. ■

Corollary 3.9. *If (L, s, T) is a quantum dynamical system with $T^k = id$ for some $k \in \mathbb{N}$, then $h_s(T) = 0$.*

Proof. Since $T^k = id$, by Proposition 3.8, $h_s(T^k) = h_s(id) = 0$, and so, $h_s(T) = \frac{1}{k} h_s(T^k) = 0$. ■

Definition 3.10. Two quantum dynamical systems (L_1, s_1, T_1) and (L_2, s_2, T_2) are called to be isomorphic if there exists a bijective map $g : L_1 \to L_2$ satisfying the following properties. For every $a, b \in L_1$,

(i) $g(a \vee b) = g(a) \vee g(b)$;
(ii) $g(a^{'}) = (g(a))^{'}$;
(iii) $s_1(a) = s_2(g(a))$;
(iv) $g(T_1(a)) = T_2(g(a))$.

In the following proposition, we prove that the entropy of quantum dynamical systems is invariant under isomorphism.

Proposition 3.11. *If (L_1, s_1, T_1) and (L_2, s_2, T_2) are isomorphic quantum dynamical systems, then $h_s(T_1) = h_s(T_2)$.*

Proof. By Definition 3.10, we have

$$\begin{aligned} h_s(T_1, P) &= \lim_{n\to\infty} \frac{1}{n} H_s\left(\bigvee_{i=1}^{n} T_1^i P\right) = \lim_{n\to\infty} \frac{1}{n} H_s\left(g\left(\bigvee_{i=1}^{n} T_1^i P\right)\right) \\ &= \lim_{n\to\infty} \frac{1}{n} H_s\left(\bigvee_{i=1}^{n} g(T_1^i P)\right) = \lim_{n\to\infty} \frac{1}{n} H_s\left(\bigvee_{i=1}^{n} T_2(g^i P)\right) = h_s(T_2, g(P)). \end{aligned}$$

So, $h_s(T_1) = \sup_P h_s(T_1, P) = \sup_P h_s(T_2, g(P)) \leq \sup_P h_s(T_2, P) = h_s(T_2)$. Therefore, $h_s(T_1) \leq h_s(T_2)$. Similarly, we obtain $h_s(T_2) \leq h_s(T_1)$. ∎

4. Conclusion

This paper has defined entropy and relative entropy of infinite partitions on a quantum logic. Also entropy of a dynamical system with infinite partitions was studied and some of its properties were proved.

References

[1] Birkhoff, G., Von Neumann, J., *The logic of quantum mechanics,* Ann. Math., 37. (1936). 8-23.

[2] Ebrahimzadeh, A., *Logical entropy of quantum dynamical systems,* Open Physics, 14 (2016). 1-5.

[3] Jauch, J.M., *Foundation of quantum mechanics,* Addison Wesley. Reading Massachusetts, 1968.

[4] Khare, M. Roy, Sh., *Entropy of quantum dynamical systems and sufficient families in orthomodular lattices with Bayessian State,* China. J Theor. Phys., 50 (2008) 551-556.

[5] Srivastava, P., Khare, M., Srivastava, Y.K., *m-equivalence, entropy and F-dynamical systems,* Fuzzy Sets and Systems, 121 (2001), 275-283.

[6] Walters, P., *An introduction to ergodic theory,* Springer Verlag, 1982.

[7] Yuan, H., *Entropy of partitions on quantum logic,* China. MM Research Preprints, 22 (2003), 409-414.

[8] Zalinescu, C., *Convex analysis in general vector spaces,* World Scientific, 2002.

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FURTHER PROPERTIES OF THE GENERALIZATION OF PRIMAL SUPERIDEALS

Ameer Jaber[1]
Moh'D Yasein

Department of Mathematics
The Hashemite University
Zarqa 13115
Jordan
e-mails: ameerj@hu.edu.jo
myasein@hu.edu.jo

Abstract. In [7], the author studied several generalizations of primal superideals of a commutative super-ring R. This paper is devoted to study further properties of ϕ-primal superideals of R, where $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ is any function and $\mathfrak{I}(R)$ is the set of all proper superideals of R. In particular, if $J \in \mathfrak{I}(R)$ then there is a one-to-one correspondence between ϕ-P^I-primal superideals I of R with $J \subseteq \phi(I)$ and ϕ_J-P^I/J-primal superideals of R/J. Moreover, for a multiplicatively closed subset S of $h(R)$, if $I - \phi(I) = \rho^{-1}(I_S) - \rho^{-1}(\phi_S(I_S))$ for any $I \in \mathfrak{I}(R)$, then there is a one-to-one correspondence between ϕ-P^I-primal superideals I of R and ϕ_S-P^I_S-primal superideals I_S of R_S with $P^I \cap S = \emptyset$.

Keywords: primal superideal, ϕ-P-primal superideal, ϕ-prime superideal.
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1. Introduction

Let R be any ring with unity, then R is called a super-ring if R is a $\mathbb{Z}_2$-graded ring such that if $a, b \in \mathbb{Z}_2$ then $R_aR_b \subseteq R_{a+b}$ where the subscripts are taken modulo 2. Let $h(R) = R_0 \cup R_1$ then $h(R)$ is the set of homogeneous elements in R and $1 \in R_0$.

Throughout, R will be a commutative super-ring with unity. By a proper superideal of R we mean a superideal I of R such that $I \neq R$. We will denote the set of all proper superideals of R by $\mathfrak{I}(R)$. If I and J are superideals of R, then the superideal $\{r \in R : rJ \subseteq I\}$ is denoted by $(I : J)$. Let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be any function. We call a proper superideal I of R ϕ-*prime* (*prime*) if for $x, y \in h(R)$

[1]Corresponding author

with $xy \in I - \phi(I)$ $(xy \in I)$ then $x \in I$ or $y \in I$. Since $I - \phi(I) = I - (\phi(I) \cap I)$, there is no loss of generality to assume that $\phi(I) \subseteq I$ for every proper superideal I of R. Let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be any function, we say that an element $a \in h(R)$ is *ϕ-prime* to I (*prime* to I), if whenever $ra \in I - \phi(I)$ $(ra \in I)$, where $r \in h(R)$, then $r \in I$. That is $a \in h(R)$ is ϕ-prime to I (*prime* to I), if $h((I : a)) - h((\phi(I) : a)) \subseteq h(I)$ (if $h((I : a)) \subseteq h(I)$).

Let $\nu(I)$ be the set of all homogeneous elements in R that are not prime to I. We define I to be *primal* if the set

$$P = (\nu(I))_0 + (\nu(I))_1 \cup \{0\}$$

forms a superideal in R. In this case we say that I is a P-primal superideal of R, and P is the adjoint superideal of I.

Let $\nu_\phi(I)$ be the set of all homogeneous elements in R that are not ϕ-prime to I. We define I to be *ϕ-primal* if the set

$$P = \begin{cases} [(\nu_\phi(I))_0 + (\nu_\phi(I))_1 \cup \{0\}] + \phi(I) & : \quad \text{if } \phi \neq \phi_\emptyset \\ (\nu_\phi(I))_0 + (\nu_\phi(I))_1 & : \quad \text{if } \phi = \phi_\emptyset \end{cases}$$

forms a superideal in R. In this case we say that I is a ϕ-P-primal superideal of R, and P is the adjoint superideal of I.

In [7], the author gave a generalization of primal superideals of R and he studied some properties of that generalization, he also studied the relation between ϕ-primary and ϕ-primal superideals. Moreover, the author introduced some conditions under which ϕ-primal superideals are primal.

In the next example we give some famous functions $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ and their corresponding ϕ-primal superideals.

Example 1.1.

$\phi_\emptyset$	$\phi_\emptyset(I) = \emptyset \ \ \forall\, I \in \mathfrak{I}(R)$	primal superideal.
ϕ_0	$\phi_0(I) = \{0\} \ \ \forall\, I \in \mathfrak{I}(R)$	weakly primal superideal.
ϕ_2	$\phi_2(I) = I^2 \ \ \forall\, I \in \mathfrak{I}(R)$	almost primal superideal.
ϕ_n	$\phi_n(I) = I^n \ \ \forall\, I \in \mathfrak{I}(R)$	n-almost primal superideal.
ϕ_ω	$\phi_\omega(I) = \cap_{n=1}^{\infty} I^n \ \ \forall\, I \in \mathfrak{I}(R)$	ω-primal superideal.

Observe that $\phi_\emptyset \leq \phi_0 \leq \phi_\omega \leq \cdots \leq \phi_{n+1} \leq \phi_n \leq \cdots \leq \phi_2$.

In this paper, we prove that if J is a ϕ-prime superideal of R then then there is a one-to-one correspondence between ϕ-P^I-primal superideals I of R with $J \subseteq \phi(I)$ and ϕ_J-P^I/J-primal superideals of R/J. Also, we prove that for a multiplicatively closed subset S of $h(R)$ and under the condition that $I - \phi(I) = \rho^{-1}(I_S) - \rho^{-1}(\phi_S(I_S))$ for any $I \in \mathfrak{I}(R)$, where $\rho : R \to R_S$ is the canonical homomorphism, there is a one-to-one correspondence between ϕ-P^I-primal superideals I of R and ϕ_S-P^I_S-primal superideals I_S of R_S with $P^I \cap S = \emptyset$.

2. ϕ_J-P/J-primal superideals

We start this section by the following two examples to show that the concepts "primal superideals" and "ϕ-primal superideals" are different.

Example 2.1. Let $R = \mathbb{Z}_{24} + u\mathbb{Z}_{24}$, where $u^2 = 0$, be a commutative super-ring and assume that $\phi = \phi_0$. Let $I = 8\mathbb{Z}_{24} + u\mathbb{Z}_{24}$.

(1) Since $0 \neq \bar{2} \cdot \bar{4} \in I$ with $\bar{2}, \bar{4} \notin I$, then we get that $\bar{2}$ and $\bar{4}$ are not ϕ-prime to I. Easy computations imply that $\bar{2} + \bar{4} = \bar{6}$ is ϕ-prime to I. Thus we obtain that I is not a ϕ-primal superideal of R.

(2) Set $P = 2\mathbb{Z}_{24} + u\mathbb{Z}_{24}$. We show that I is a primal superideal of R. It is easy to check that every element of $h(P)$ is not prime to I. Conversely, assume that $\bar{a} \in h(R) - h(P)$, then $\bar{a} \in \mathbb{Z}_{24}$ with $gcd(a, 8) = 1$. If $\bar{a} \cdot \bar{n} \in I$ for some $\bar{n} \in \mathbb{Z}_{24}$, then 8 divides n; hence $\bar{n} \in I$. Therefore, $h(P)$ is exactly the set of elements in $h(R)$ which are not prime to I. Thus I is a primal superideal of R.

Example 2.2. Let $\phi = \phi_0$, and let $T(R)$ be the collection of all homogeneous zero divisors of R. If R is not a superdomain such that $Z(R) = T_0(R) + T_1(R)$ is not a superideal of R. Then the trivial superideal of R is a ϕ-primal superideal which is not primal.

According to Examples 2.1 and 2.2 a primal superideal of R need not to be ϕ-primal and a ϕ-primal superideal of R need not to be primal.

Let R be a commutative super-ring with unity and let J be a proper superideal of R. Let $\phi : \Im(R) \to \Im(R) \cup \{\emptyset\}$ be any function. As a generalization of [2], we define $\phi_J : \Im(R/J) \to \Im(R/J) \cup \{\emptyset\}$ by $\phi_J(I/J) = (\phi(I)+J)/J$ for every superideal $I \in \Im(R)$ with $J \subseteq I$ (and $\phi_J(I/J) = \emptyset$ if $\phi = \phi_\emptyset$).

We leave the trivial proof of the next lemma to the reader.

Lemma 2.3. *Let R be a commutative super-ring with unity and let J be a proper superideal of R. Let $\phi : \Im(R) \to \Im(R) \cup \{\emptyset\}$ be any function. If P is a ϕ-prime superideal of R containing J. Then P/J is a ϕ_J-prime superideal of R/J.* ∎

Lemma 2.4. *Let R be a commutative super-ring with unity and let J be a proper superideal of R, and let $\phi : \Im(R) \to \Im(R) \cup \{\emptyset\}$ be any function. Let P be a superideal of R containing J. If P/J is a ϕ_J-prime superideal of R/J with $J \subseteq \phi(P)$, then P is a ϕ-prime superideal of R.*

Proof. Let a, b be homogeneous elements in R with $ab \in P - \phi(P)$. Then $ab \in J + P$ and $ab \notin J + \phi(P) = \phi(P)$. Thus, $ab \in (J + P) - (J + \phi(P))$, so $\overline{ab} \in (J + P)/J - (J + \phi(P))/J$ which implies that $\overline{ab} \in P/J - \phi_J(P/J)$, that is $\bar{a} \in P/J$ or $\bar{b} \in P/J$ so $a \in P$ or $b \in P$. Therefore, P is a ϕ-prime superideal of R. ∎

In the next result and under the condition that $J \subseteq \phi(I)$ we prove that I is a ϕ-primal superideal of R if and only if I/J is a ϕ_J-primal superideal of R/J.

Theorem 2.5. *Let R be a commutative super-ring with unity and let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be any function. Let I be a proper superideal of R, and let J be a superideal of R with $J \subseteq \phi(I)$. Then I is a ϕ-P-primal superideal of R if and only if I/J is a ϕ_J-P/J-primal superideal of R/J.*

Proof. Suppose that I is ϕ-P-primal superideal of R with $J \subseteq I$. Then, by [7, Theorem 2.5], P is a ϕ-prime superideal of R containing J. Therefore, by Lemma 2.3, P/J is a ϕ_J-prime superideal of R/J. We show that I/J is ϕ_J-P/J-primal superideal of R/J. That is we have to prove that

$$P/J = \begin{cases} [(\nu_{\phi_J}(I/J))_0 + (\nu_{\phi_J}(I/J))_1 \cup \{0\}] + \phi_J(I/J) & : \quad \text{if } \phi_J \neq \phi_\emptyset \\ (\nu_{\phi_J}(I/J))_0 + (\nu_{\phi_J}(I/J))_1 & : \quad \text{if } \phi_J = \phi_\emptyset \end{cases}.$$

Let $\bar{a} \in h(P/J)$ then $a \in h(P)$ is not ϕ-prime to I. That is there exists r in $h(R) - h(I)$ with $ra \in I - \phi(I)$. If $a \in J + \phi(I) = \phi(I)$ then $\bar{a} \in \phi_J(I/J)$. So we may assume that $a \notin J + \phi(I) = \phi(I)$. Therefore, $\bar{r}\bar{a} \in I/J - (J + \phi(I))/J = I/J - \phi_J(I/J)$ and because $\bar{r} \notin I/J$ we get that $\bar{a} \in \nu_{\phi_J}(I/J)$.

Now, assume that $\bar{b}$ is a homogeneous element in R/J such that $\bar{b} \in \nu_{\phi_J}(I/J)$. Then, there exists a homogeneous element $\bar{r}$ in $R/J - I/J$ such that $\bar{r}\bar{b} \in I/J - \phi_J(I/J)$, so $rb \in I - \phi(I)$ with $r \notin I$. Thus, b is not ϕ-prime to I which implies that $b \in P$, and hence $\bar{b} \in P/J$. Therefore,

$$P/J = \begin{cases} [(\nu_{\phi_J}(I/J))_0 + (\nu_{\phi_J}(I/J))_1 \cup \{0\}] + \phi_J(I/J) & : \quad \text{if } \phi_J \neq \phi_\emptyset \\ (\nu_{\phi_J}(I/J))_0 + (\nu_{\phi_J}(I/J))_1 & : \quad \text{if } \phi_J = \phi_\emptyset \end{cases}$$

and so I/J is ϕ_J-P/J-primal superideal of R/J.

Conversely, suppose that I/J is ϕ_J-P/J-primal superideal of R/J with the adjoint superideal P/J. We show that I is a ϕ-P-primal superideal of R. Now, by [7, Theorem 2.5], P/J is a ϕ_J-prime superideal of R/J with $J \subseteq P$, so by Lemma 2.4, P is a ϕ-prime superideal of R. To finish the proof we need to show that

$$P = \begin{cases} [(\nu_\phi(I))_0 + (\nu_\phi(I))_1 \cup \{0\}] + \phi(I) & : \quad \text{if } \phi \neq \phi_\emptyset \\ (\nu_\phi(I))_0 + (\nu_\phi(I))_1 & : \quad \text{if } \phi = \phi_\emptyset \end{cases}.$$

Clearly, $\phi(I) \subseteq I \subseteq P$. Let $a \in \nu_\phi(I)$, then there exists a homogeneous element $r \in R - I$ with $ra \in I - \phi(I)$. Since $ra \notin \phi(I) = J + \phi(I)$ we get that $\bar{r}\bar{a} \in I/J - (J + \phi(I))/J = I/J - \phi_J(I/J)$ and $\bar{r} \notin I/J$. So, $\bar{a} \in P/J$ and hence $a \in P$. Now, let $a \in h(P)$. Suppose that $\bar{a} \in I/J$, then $a \in I$. If $a \in \phi(I)$, then done. If $a \notin \phi(I)$, then $a \in I - \phi(I)$ and so, a is not ϕ-prime to I, hence $a \in \nu_\phi(I)$. Thus,

$$a \in \begin{cases} [(\nu_\phi(I))_0 + (\nu_\phi(I))_1 \cup \{0\}] + \phi(I) & : \quad \text{if } \phi \neq \phi_\emptyset \\ (\nu_\phi(I))_0 + (\nu_\phi(I))_1 & : \quad \text{if } \phi = \phi_\emptyset \end{cases}.$$

Therefore, we may assume that $\bar{a} \notin I/J$, so there exists $\bar{r} \in h(R/J) - h(I/J)$ with $\bar{r}\bar{a} \in I/J - \phi_J(I/J)$, and so $\bar{r}\bar{a} \notin (J + \phi(I))/J$ that is $ra \notin J + \phi(I) = \phi(I)$. Therefore, $ra \in I - \phi(I)$ with $r \notin I$ that is $a \in \nu_\phi(I)$. ∎

By using Theorem 2.5, we have the following result.

Theorem 2.6. *Let R be a commutative super-ring with unity, and let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be any function. let J be a superideal of R. Then there is one-to-one correspondence between the ϕ-P^I-primal superideals I of R containing J with $J \subseteq \phi(I)$ and ϕ_J-P^I/J-primal superideals of R/J.* ∎

3. Multiplicatively closed subsets

Let R be a commutative super-ring with unity, and let S be a multiplicatively closed subset of $h(R)$. Consider the canonical homomorphism $\rho : R \to R_S$ which is defined by $r \mapsto \frac{r}{1}$ for all $r \in h(R)$. Then ρ is a homogenous superhomomorphism of degree 0.

Now let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be any function. We define $\phi_S : \mathfrak{I}(R_S) \to \mathfrak{I}(R_S) \cup \{\emptyset\}$ by $\phi_S(J) = (\phi(\rho^{-1}(J)))_S$ for every $J \in \mathfrak{I}(R_S)$. Note that $\phi_S(J) \subseteq J$, since for $J \in \mathfrak{I}(R_S)$, we get that $\phi(\rho^{-1}(J)) \subseteq \rho^{-1}(J)$ implies $\phi_S(J) \subseteq (\rho^{-1}(J))_S \subseteq J$.

Example 3.1. Let $R = \mathbb{Z}_6 + u\mathbb{Z}_6$ with $u^2 = 0$. Let $S = \{1, 2, 4\}$, $\phi = \phi_2$. Then S is a multiplicatively closed subset of $h(R)$. If $P = \{0\}$ then one can easily check that P is ϕ_2-P-primal superideal of R. Moreover, $P_S = (\phi(P))_S = \{0\}$, hence $\phi_S(P_S) = P_S$ since $\rho^{-1}(P_S) = \{0, 3, 3u\}$, where $\rho : R \to R_S$ is the canonical homomorphism. Therefore, P_S is ϕ_S-P_S-primal superideal in R_S.

We start by proving the following properties about ϕ-prime superideals of R, where $\rho : R \to R_S$ is the canonical homomorphism.

Theorem 3.2. *Let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be any function, and let I be a ϕ-prime superideal of R with $I \cap S = \emptyset$, then I_S is a ϕ_S-prime superideal of R_S.*

Proof. Let $\frac{x}{s}, \frac{y}{t}$ be homogeneous elements in R_S with $(\frac{x}{s})(\frac{y}{t}) \in I_S - \phi_S(I_S)$, then for some $u \in S$ $xyu \in I - \phi(I)$, so $x \in I$ or $yu \in I$ and thus $\frac{x}{s} \in I_S$ or $\frac{y}{t} \in I_S$, hence I_S is a ϕ_S-prime superideal of R_S. ∎

Theorem 3.3. *Let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be any function, and let P be a ϕ-prime superideal of R with $h(P) \cap S = \emptyset$. If $\rho^{-1}((\phi(P))_S) \subseteq P$, then $\rho^{-1}(P_S) = P$.*

Proof. It is easy to see that $P \subseteq \rho^{-1}(P_S)$.

Conversely, let x be a homogeneous element in $\rho^{-1}(P_S)$, then for some $s \in S$, $xs \in P$. If $xs \notin \phi(P)$ then $xs \in P - \phi(P)$ and $s \notin P$ so $x \in P$. Therefore we may assume that $xs \in \phi(P)$, so x is a homogeneous element in $\rho^{-1}((\phi(P))_S)$. Thus,

$$\rho^{-1}(P_S) \subseteq P \cup \rho^{-1}((\phi(P))_S),$$

and since $\rho^{-1}((\phi(P))_S) \subseteq P$ then we have that $\rho^{-1}(P_S) = P$. ∎

Lemma 3.4. *Let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be any function, and let I be a ϕ-P-primal superideal of R with $h(P) \cap S = \emptyset$. If $a/s \in h(I_S) - h(\phi_S(I_S))$ then $a \in I$.*

Proof. Let $a/s \in h(I_S) - h(\phi_S(I_S))$, so $a/s \notin \phi_S(I_S)$, and hence $a/1 \notin \phi_S(I_S)$, thus $a/1 \notin (\phi(I))_S$ since $(\phi(I))_S \subseteq \phi_S(I_S)$. Now, $\frac{a}{1} = \frac{r}{u}$ for some $r \in h(I)$ and $u \in S$, so, $r = au \in I$ and $au \notin \phi(I)$ because if $au \in \phi(I)$ then $a/1 \in (\phi(I))_S$, a contradiction. Thus $au \in I - \phi(I)$, if $a \notin I$ then u is not a ϕ-prime to I which mplies that $u \in h(P) \cap S$, a contradiction. Therefore, $a \in I$. ∎

Lemma 3.5. Let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be any function, and let I be a ϕ-P-primal superideal of R with $h(P) \cap S = \emptyset$. Then $h(\rho^{-1}(I_S)) - h(\rho^{-1}(\phi_S(I_S))) \subseteq h(I) - h(\phi(I))$.

Proof. Let a be a homogeneous element in $\rho^{-1}(I_S)$ such that $a \notin h(\rho^{-1}(\phi_S(I_S)))$, then $a/1 \in h(I_S) - h(\phi_S(I_S))$ and by Lemma 3.4, $a \in I$. If $a \in \phi(I)$ then $a/1 \in (\phi(I))_S \subseteq \phi_S(I_S)$ implies that $a \in h(\rho^{-1}(\phi_S(I_S)))$ a contradiction. Therefore, $a \in h(I) - h(\phi(I))$. ∎

Lemma 3.6. *Let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be any function, and let I be a ϕ-P-primal superideal of R with $h(P) \cap S = \emptyset$. Then $\nu_{\phi_S}(I_S) \subseteq (\nu_\phi(I))_S$.*

Proof. If $a/s \in \nu_{\phi_S}(I_S)$, then we show in the proof of Lemma 3.4 that $au \in I - \phi(I) \subseteq \nu_\phi(I)$ for some u in S, hence $ua/us = a/s \in (\nu_\phi(I))_S$. ∎

Corollary 3.7. *Let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be any function, and let I be a ϕ-P-primal superideal of R with $h(P) \cap S = \emptyset$. If $I - \phi(I) = \rho^{-1}(I_S) - \rho^{-1}(\phi_S(I_S))$ then $\nu_{\phi_S}(I_S) = (\nu_\phi(I))_S$.*

Proof. In Lemma 3.6, we proved that $\nu_{\phi_S}(I_S) \subseteq (\nu_\phi(I))_S$.

Conversely, let $\frac{x}{s}$ be a homogeneous element in $(\nu_\phi(I))_S$, then $\frac{x}{s} = \frac{y}{t}$, where $y \in \nu_\phi(I)$. If $y \in I$, then $\frac{y}{t} \in I_S - (\phi(I))_S$ and so

$$y \in \rho^{-1}(I_S) - \rho^{-1}((\phi(I))_S) \subseteq I - \phi(I) = \rho^{-1}(I_S) - \rho^{-1}(\phi_S(I_S)).$$

Hence $\frac{y}{t} \in I_S - \phi_S(I_S) \subseteq \nu_{\phi_S}(I_S)$. Therefore we may assume that $y \notin I$. Since $y \in \nu_\phi(I)$ there exists a homogeneous element u in $R - I$ with $uy \in I - \phi(I)$, but $I - \phi(I) = \rho^{-1}(I_S) - \rho^{-1}(\phi_S(I_S))$, so $(\frac{u}{1})(\frac{y}{t}) \in I_S - \phi_S(I_S)$ with $\frac{u}{1} \notin I_S$, thus $\frac{y}{t} \in \nu_{\phi_S}(I_S)$. ∎

We recall that if J is a superideal in R, then $J \subseteq \rho^{-1}(J_S)$ and therefore we may assume that $(\phi(J))_S \subseteq \phi_S(J_S)$. Under the condition that

$$I - \phi(I) = \rho^{-1}(I_S) - \rho^{-1}(\phi_S(I_S))$$

for all superideals I of R, we have the following proposition.

Proposition 3.8. *Let S be a multiplicatively closed subset of $h(R)$ with $1 \in S$, let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be any function, and let I be a ϕ-P-primal superideal of R with $h(P) \cap S = \emptyset$. Then I_S is a ϕ_S-P_S-primal superideal of R_S.*

Proof. By Theorem 3.2, P_S is a ϕ_S-prime superideal of R_S. To show that I_S is a ϕ_S-P_S-primal superideal of R_S, we must prove that

$$P_S = \begin{cases} [(\nu_{\phi_S}(I_S))_0 + (\nu_{\phi_S}(I_S))_1 \cup \{0\}] + \phi_S(I_S) & : \quad \text{if } \phi_S \neq \phi_\emptyset \\ (\nu_{\phi_S}(I_S))_0 + (\nu_{\phi_S}(I_S))_1 & : \quad \text{if } \phi_S = \phi_\emptyset \end{cases}.$$

Clearly, $\phi_S(I_S) \subseteq P_S$, let a/s be a homogenous element in $\nu_{\phi_S}(I_S)$, then there exists $r/u \in h(R_S) - h(I_S)$ with $(\frac{r}{u}) \cdot (\frac{a}{s}) \in I_S - \phi_S(I_S)$ so, by Lemma 3.5, $ra \in I - \phi(I)$ and $r \notin h(I)$, thus $a \in \nu_\phi(I) \subseteq P$ and hence $a/s \in P_S$.

Conversely, let $a/s \in h(P_S)$ such that $a/s \notin \phi_S(I_S)$. If $a/s \in I_S$, then $(1/1)(a/s) \in I_S - \phi_S(I_S)$, $(1/1) \notin I_S$, so a/s is not ϕ_S-prime to I_S, thus $a/s \in \nu_{\phi_S}(I_S)$. Therefore, we may assume that $a/s \notin I_S$, that is $ta \notin I$ for every $t \in S$. Since $a/s \in P_S$, then for some $t \in S$, $ta \in P - I$, and so $ta \in \nu_\phi(I)$ which implies that $a/s \in (\nu_\phi(I))_S$, and by Corollary 3.7, we have that $\nu_{\phi_S}(I_S) = (\nu_\phi(I))_S$ thus $a/s \in \nu_{\phi_S}(I_S)$. ∎

Let R be a commutative super-ring with unity, and let S be a multiplicatively closed subset of $h(R)$. Let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be any function, then under the condition that

$$I - \phi(I) = \rho^{-1}(I_S) - \rho^{-1}(\phi_S(I_S))$$

for all proper superideals I of R, we have the following proposition.

Proposition 3.9. *Let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be any function, and let J be a ϕ_S-Q-primal superideal of R_S, then $\rho^{-1}(J)$ is ϕ-primal superideal of R with the adjoint superideal $\rho^{-1}(Q)$. Moreover, $J = (\rho^{-1}(J))_S$.*

Proof. To show that $\rho^{-1}(J)$ is ϕ-primal superideal of R with the adjoint superideal $\rho^{-1}(Q)$ we must show that

$$\rho^{-1}(Q) = \begin{cases} [(\nu_\phi(\rho^{-1}(J)))_0 + (\nu_\phi(\rho^{-1}(J)))_1 \cup \{0\}] + \phi(\rho^{-1}(J)) & : \quad \text{if } \phi \neq \phi_\emptyset \\ (\nu_\phi(\rho^{-1}(J)))_0 + (\nu_\phi(\rho^{-1}(J)))_1 & : \quad \text{if } \phi = \phi_\emptyset \end{cases}.$$

But $\phi(\rho^{-1}(J)) \subseteq \rho^{-1}(J) \subseteq \rho^{-1}(Q)$. Now let a be a homogenous element in $\nu_\phi(\rho^{-1}(J))$, then $\frac{a}{1} \in (\nu_\phi(\rho^{-1}(J)))_S$, but by Corollary 3.7, $(\nu_\phi(\rho^{-1}(J)))_S = \nu_{\phi_S}(J)$, so $\frac{a}{1} \in \nu_{\phi_S}(J) \subseteq Q$ and hence $a \in \rho^{-1}(Q)$.

Conversely, let a be a homogeneous element in $\rho^{-1}(Q)$, then $a/1$ in Q. We may assume that $a \notin \phi(\rho^{-1}(J))$, so $a/1 \notin \phi_S(J)$. If $a/1 \in J$, then $(a/1) \in J - \phi_S(J)$ and since $\phi(\rho^{-1}(J)) \subseteq \rho^{-1}(\phi_S(J))$, we have that $a \in \rho^{-1}(J) - \rho^{-1}(\phi_S(J)) \subseteq \rho^{-1}(J) - \phi(\rho^{-1}(J))$, but $1 \notin \rho^{-1}(J)$, so $a \in \nu_\phi(\rho^{-1}(J))$. If $a/1 \notin J$, then $a/1 \in Q - J$ and so $a/1 \in \nu_{\phi_S}(J)$. Let $\frac{x}{s}$ be a homogeneous element in $R_S - J$ with $(\frac{a}{1})(\frac{x}{s}) \in J - \phi_S(J)$ then $ax \in \rho^{-1}(J) - \rho^{-1}(\phi_S(J)) \subseteq \rho^{-1}(J) - \phi(\rho^{-1}(J))$, since $\frac{ax}{1} \in J$ and $\frac{ax}{1} \notin \phi_S(J)$, because if $\frac{ax}{1} \in \phi_S(J)$, then $\frac{ax}{s} \in \phi_S(J)$, a contradiction. Thus we have that $ax \in \rho^{-1}(J) - \phi(\rho^{-1}(J))$ and $x \notin \rho^{-1}(J)$, since $\frac{x}{s} \notin J$. Therefore, $a \in \nu_\phi(\rho^{-1}(J))$ and hence $\rho^{-1}(J)$ is ϕ-primal superideal of R with the adjoint superideal $\rho^{-1}(Q)$. Moreover, by [7, Theorem 2.5], $\rho^{-1}(Q)$ is a ϕ-prime superideal of R.

Finally, we show that $J = (\rho^{-1}(J))_S$. Clearly, $J \subseteq (\rho^{-1}(J))_S$.

Conversely, let $\frac{x}{s}$ be a homogeneous element in $(\rho^{-1}(J))_S$, then $xt \in \rho^{-1}(J)$ for some $t \in S$. Thus, $\frac{xt}{1} \in \rho(\rho^{-1}(J)) = J$, and hence $(\frac{xt}{1})(\frac{1}{st}) = \frac{x}{s} \in J$. Therefore, $J = (\rho^{-1}(J))_S$. ∎

By using Propositions 3.8 and 3.9, we have the following result.

Theorem 3.10. *Let R be a commutative super-ring with unity. Let S be a multiplicatively closed subset of $h(R)$, and let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be any function with the condition that $I - \phi(I) = \rho^{-1}(I_S) - \rho^{-1}(\phi_S(I_S))$ for any proper superideal I of R. Then there is one-to-one correspondence between the ϕ-P^I-primal superideals I of R and ϕ_S-P^I_S-primal superideals I_S of R_S, where P^I is a ϕ-prime superideal of R with $P^I \cap S = \emptyset$.* ∎

References

[1] Anderson, D.F., Badawi, A., *On n-absorbing ideals of commutative rings*, Comm. Algebra, 39 (2011), 1646-1672.

[2] Anderson, D., Bataineh, M., *Generalizations of prime ideals*, Comm. in Algebra, 36 (2008), 686-696.

[3] Atani, E., Darani, Y., *On weakly primal ideals* (I), Demonstratio Math., 40 (2007), 23-32.

[4] Darani, Y., *Generalizations of primal ideals in commutative rings*, Matematiqki Vesnik, 64 (2012), 1, 2531.

[5] Ebrahimpour, M., Nekooei, R., *On generalizations of prime ideals*, Comm. Algebra, 40 (2012), 1268-1279.

[6] Fuchs, L., *On primal ideals*, Amer. Math. Soc., 1 (1950), 1-6.

[7] Jaber, A., *Generalization of primal superideals*, Algebra and Discrete Mathematics, 21 (2) (2016), 202-213.

[8] Jaber, A., *Weakly primal graded superideals*, Tamkang Journal of Mathematics, 43 (1) (2012), 123-135.

[9] Jabbar, A. K., Ahmed, C. A., *On almost primary ideals*, International Journal of Algebra, 5 (2011) 627-636.

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COUNT AND CRYPTOGRAPHIC PROPERTIES OF GENERALIZED SYMMETRIC BOOLEAN FUNCTIONS

Shashi Kant Pandey
Department of Mathematics
University of Delhi
Delhi – 110007
India
e-mail: shashikantshvet@gmail.com

P.R. Mishra
Scientific Analysis Group
Metcalfe House Complex, DRDO
Delhi – 110054
India
e-mail: prasanna.r.mishra@gmail.com

B.K. Dass
Department of Mathematics
University of Delhi
Delhi – 110007
India
e-mail: dassbk@rediffmail.com

Abstract. Boolean functions with symmetry have been the object of interest to the researchers. With their concise representation and ease of computation, they offer themselves as cut above the rest candidates for the filtering and exploration of optimal Boolean functions. Generalized Boolean functions have been explored as a number of trade-offs in usual Boolean hinder, the process of selecting good Boolean functions required for a specific application. Therefore, it would be interesting to investigate symmetry in Boolean functions in a generalized scenario. We look into three different symmetries in generalized Boolean function according to different parameter of symmetry and present enumeration formulae. We also present an exhaustive construction of bent and balanced symmetric generalized functions (in form of ANF) on smaller domains.

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1. Introduction

Boolean functions are amply used in cryptographic designs especially in stream cipher designs where these functions play roles for combiners and filters. Not every Boolean function is a suitable candidate to be used in a cryptographic designs. A cryptograhically suited Boolean function must possess some properties which make a design resistant to cryptanalytic attacks. The computational search for cryptographically suited Boolean functions becomes harder and harder as the number of variables increases. This is because the search space grows exponentially with the number of variables. That is why it is sometimes easier to search for good Boolean functions in a relatively smaller but wisely chosen subset of set of all Boolean functions. Examples of such subsets are the sets of symmetric and rotational symmetric Boolean function.

Another problem with Boolean functions is the presence of a number of trade-offs between various properties[1], [3], [7], [9], [11]. For example, Camion et al. [2] has shown that for a p variable Boolean function, there is a trade-off between the correlation immunity k and the degree d, i.e., $(d+k) < (p-1).k$. Such trade-offs do not let us maximize the desired properties to their maximum possible values.

Therefore, study of symmetries of Boolean functions in a generalized scenario would be an interesting direction to explore into. Not much has been reported so far in this direction. Though some work on generalization of partial symmetric Boolean functions can be found in [3]. We are presenting here generalization of three type of symmetric Boolean functions and their characterizations. The direct generalization of symmetric Boolean functions from binary field provide us a chance to generalized the result of Savicky [4] on symmetric bent function and we landed up to a conjecture which is given in the last section of this paper. In Section 2, we provide enumeration of all three generalized symmetric Boolean functions and, in Section 3, we proved some results in correspondence with bent properties and balancedness of generalized symmetric functions.

2. Generalized symmetric Boolean functions

Symmetries of Boolean functions are seen as behaviour of the function at orbits under the action of some permutation group. On similar lines, we can define symmetries for Generalized functions. Below we define a symmetric function on generalized domain.

Definition 1. Let $f : \mathbb{Z}_q^n \to \mathbb{Z}_q$ be an arbitrary function. f is said to be generalized symmetric Boolean function (GSBF) if for every $\mathbf{x} \in \mathbb{Z}_q^n$, f is constant on the orbit of $\mathbf{x}$ under the action of $\mathfrak{S}_n$ on $\mathbf{x} \in \mathbb{Z}_q^n$.

We can see that on the basis of the selection of the permutation group we can categorize the action of these group on the set of vectors. The next definition of a rotational symmetric function is just another choice of the permutations. To this end, we first define a shift operator ρ_n^k on $\mathbb{Z}_q^n$ as:

Given $\mathbf{x} = (x_1, x_2, ..., x_n) \in \mathbb{Z}_q^n$ $1 \le i \le n$ and $1 \le k \le n$ we define

$$\rho_n^k(\mathbf{x}) = (x_{t_1}, x_{t_2}, ..., x_{t_n})$$

where, $t_i = \begin{cases} t_{i+k}, & \text{if } i+k \le n, \\ t_{i+k-n}, & \text{otherwise,} \end{cases}$ for all $1 \le i \le n$.

Definition 2. Let $f : \mathbb{Z}_q^n \to \mathbb{Z}_q$ be an arbitrary function. f is said to be generalized rotational symmetric Boolean function (GRSBF) if for every $\mathbf{x} \in \mathbb{Z}_q^n$, f is constant on the orbit of $\mathbf{x}$ under the action of ρ_n^i on $\mathbf{x} \in \mathbb{Z}_q^n$. In other words, for any $(x_1, x_2, ..., x_n) \in \mathbb{Z}_q^n$, $f(\rho_n^k(x_1, x_2, ..., x_n)) = f(x_1, x_2, ..., x_n)$.

Now, we are showing the next generalization of symmetric Boolean function on the basis of direct extension of the definition of symmetric Boolean function for $q = 2$.

Definition 3. *Let $f : \mathbb{Z}_q^n \to \mathbb{Z}_q$ be an arbitrary function. f is said to be generalized super symmetric Boolean function (GSSBF) if for every $\mathbf{x} \in \mathbb{Z}_q^n$, f is constant on all $\mathbf{x}$ of same hamming weight, irrespective of its permutation.*

We can express this function as $f(\mathbf{x}) = v_{wt(\mathbf{x})} \in \mathbb{Z}_q$, where $wt(\mathbf{x})$ denotes Hamming weight of vector $\mathbf{x}$.

Definition 4. *A function $f : \mathbb{Z}_q^n \to \mathbb{Z}_q$ is said to be balanced if for $0 \leq k \leq q-1$, the cardinality of $S = \{x \in Z_q^n : f(x) = k\}$ is independent of k.*

3. Walsh transform of generalized symmetric and super symmetric functions

It is obvious that, for $\mathbf{x}, \mathbf{y} \in \mathbb{Z}_q^n \implies \mathfrak{S}_n.\mathbf{x} = \mathfrak{S}_n.\mathbf{y}$ if and only if $\{x_1, x_2, ..., x_n\} = \{y_1, y_2, ..., y_n\}$, where $\mathbf{x} = (x_1, x_2, ..., x_n), \mathbf{y} = (y_1, y_2, ..., y_n)$. We define the set $U = \{\mathbf{u} | \mathbf{u} = (u_1, u_2, ..., u_n), u_1 \leq u_2 \leq \cdots u_n\}$. Clearly, U is the maximal subset of $\mathbb{Z}_q^n$ with the property $\mathbf{u_1}, \mathbf{u_2} \in U_k, \mathbf{u_1} \neq \mathbf{u_2} \implies \mathfrak{S}_n.\mathbf{u}_1 \neq \mathfrak{S}_n.\mathbf{u}_2$. Hence $\mathbb{Z}_q^n$ can be written as disjoint union of orbits of elements in U.

$$\mathbb{Z}_q^n = \bigcup_{\mathbf{u} \in U} \mathfrak{S}_n.\mathbf{u} \tag{1}$$

As per definition, f is constant on orbits. Therefore, we define $f(x) = v_{\mathbf{u}}$, $\forall \mathbf{x} \in \mathfrak{S}_n.\mathbf{u}$. The Walsh transform of f at $\mathbf{w} \in \mathbb{Z}_q^n$ can be given as

$$W_f(\mathbf{w}) = \sum_{\mathbf{x} \in \mathbb{Z}_q^n} \zeta^{f(\mathbf{x}) + \mathbf{w}.\mathbf{x}} = \sum_{\mathbf{u} \in U} \sum_{\mathbf{x} \in \mathfrak{S}_n.\mathbf{u}} \zeta^{f(\mathbf{x}) + \mathbf{w}.\mathbf{x}} = \sum_{\mathbf{u} \in U} \zeta^{v_{\mathbf{u}}} \sum_{\mathbf{x} \in \mathfrak{S}_n.\mathbf{u}} \zeta^{\mathbf{w}.\mathbf{x}}. \tag{2}$$

We can see that the rightmost summation in (2) is independent of the choice of f and to calculate the internal sum we use Bernside's lemma [12]. So the order of U can be given by

$$|U| = |\mathbb{Z}_q^n / \mathfrak{S}_n| = \frac{1}{n!} \sum_{s \in \mathfrak{S}_n} \mathbb{Z}_q^{n^s}, \tag{3}$$

where $\mathbb{Z}_q^{n^s} = \{\mathbf{x} \in \mathbb{Z}_q^n | s.\mathbf{x} = \mathbf{x}\}$. Also, from the orbit stabilizer theorem,

$$|\mathfrak{S}_n.\mathbf{x}| = \frac{n!}{Stab(\mathbf{x})},$$

where $Stab(\mathbf{x}) = \{s \in \mathfrak{S}_n | s.\mathbf{x} = \mathbf{x}\}$. So to enumerate U we need $|Stab(\mathbf{x})|$ under the action of s.

The Walsh transformation of GSBF is defined as

$$W_f(w) = \sum_{\mathbf{x}\in\mathbb{Z}_q^n} \zeta^{f(\mathbf{x})+w.\mathbf{x}} = \sum_{k=0}^{n} \zeta^{v_k} \sum_{wt(\mathbf{x})=k} \zeta^{w.\mathbf{x}}$$

3.1. Enumeration of generalized symmetric and rotational symmetric Boolean function

Theorem 3.1. *Total number of GSBF $f : \mathbb{Z}_q^n \to \mathbb{Z}_q$ is $q^{\binom{n+q-1}{q-1}}$.*

Proof. By the definition of GSBF the number of GSBF is given as,

$$\#GSBF = q^{|U|} \tag{4}$$

We have shown in (3) that how $|U|$ can be calculated. We here proceed in a different manner to estimate $|U|$. From the definition of U, it is clear that U has one to one correspondence with the set of all combinations of n objects each can be repeated q times. This set has cardinality $\binom{n+q-1}{q-1}$. Therefore,

$$|U| = \binom{k+q-1}{q-1} \tag{5}$$

(4) and (5) together give

$$\#GSBF = q^{\binom{n+q-1}{q-1}}. \tag{6}$$

■

The next theorem deals with the enumeration of GRSBF.

Theorem 3.2. *The number of GRSBF on $\mathbb{Z}_q^n$ is q^{s_n} where $s_n = (1/n)\sum_{m|n} \phi(m)q^{n/m}$.*

Proof. The set of all rotations (denote it by S_n) is a subgroup of $\mathfrak{S}_n$. The orbit of $\mathbf{x} = (x_1, ..., x_n)$ under the action of S_n is given as

$$S_n \cdot \mathbf{x} = \{\rho_n^k(\mathbf{x}) | 1 \le k \le n\}.$$

Let s_n is the number of orbits formed by this action. Clearly, the number of rotational symmetric $f : \mathbb{Z}_q^n \to \mathbb{Z}_q$ is q^{s_n}.

We derive the expression for s_n using Burnside's lemma. The disjoint cyclic decomposition of ρ_n^i can be written as

$$\rho_n^i = C_1 C_2 \dots C_k,$$

where $k = \gcd(n, i)$ and each C_i is of length $n/\gcd(n, i)$.

To apply Burnside's lemma we need the number of fixed points of ρ_n^i, $i = 1, 2, ..., n$. It is easy to observe that any $x \in \mathbb{Z}_q^n$ is fixed by ρ_n^i for $i = 1, 2, ..., n$ if and only if it is fixed by each of $C_i, 1 \le i \le k$. Further, if a cycle C_i permutes $x_{j_1}, x_{j_2}, ..., x_{j_{n/k}}$, then it fixes $\mathbf{x}$ if and only if $x_{j_1} = x_{j_2} = \cdots = x_{j_{n/k}}$. Clearly, there are q choices for each $C_i, 1 \leq i \leq k$. Hence there are q^k number of fixed points of ρ_n^i. Now, the Burnside's lemma implies

$$\begin{aligned} s_n &= (1/n)\sum_{i=1}^{n} q^{\gcd(n,i)} = (1/n)\sum_{m|n}\sum_{\gcd(n,i)=m}^{n} q^m \\ &= (1/n)\sum_{m|n} q^m \sum_{j,\gcd(n/m,j)=1} 1 = (1/n)\sum_{m|n}\phi(m)q^{n/m}. \end{aligned}$$

■

The enumeration of GSSBF is based on the classification of $\mathbb{Z}_q^n$ under the constraints discussed earlier in definition 3. Since the number of class of $\mathbb{Z}_q^n$ depends on the number of the possible choices of the weight of the vectors in $\mathbb{Z}_q^n$ so in next theorem we are showing the count of GSSBF.

Theorem 3.3. *Total number of GSSBF $f : \mathbb{Z}_q^n \to \mathbb{Z}_q$ is q^{n+1}.*

Proof. We can do the classification of Z_q^n on the basis of hamming weight of all $x \in Z_q^n$. Let $S_k = \{x \in Z_q^n : wt(x) = k\}$, so we can write

$$Z_q^n = \bigcup_{k=0}^{n} S_k.$$

Therefore, $n+1$ total number of classes are there for Z_q^n. Now by definition of super symmetric function $f : Z_q^n \to Z_q$ maps on maximum possibly $n+1$ distinct values from Z_q for all $x \in Z_q^n$. Hence total number of GSSBF are q^{n+1}. ■

4. Bent functions

A function $f : \mathbb{Z}_q^n \to \mathbb{Z}_q$ is said to Bent function if function having $\{\pm q^{n/2}\}$ valued Walsh spectrum. This spectrum furnished it an optimal non-linearity among all boolean functions. From cryptographic point of view a Boolean functions having good non-linearity profile are useful in designing linear attack immune cryptosystems. A rich Boolean function required to satisfy another criteria simultaneously viz. high algebraic degree and balancedness. For $q = 2$ it is proved that a bent function never be a balanced function. In the next theorem, we are presenting the proof for any prime q.

Proposition 1. *$f : \mathbb{Z}_q^n \to \mathbb{Z}_q$ is balanced if and only if $\sum_{x\in Z_q^n, f(x)=k} 1 = q^{n-1}$ for all $0 \leq k < q$.*

Proof. The proof directly follows the definition of balancedness. ■

Theorem 4.4. *$f : \mathbb{Z}_q^n \to \mathbb{Z}_q$ is balanced if and only if $W_f(\boldsymbol{0}) = 0$ where $\boldsymbol{0} \in \mathbb{Z}_q^n$ and q is any prime number.*

Proof. Let f be a balanced function. The Walsh transform at $\mathbf{0}$ can be calculated by

$$W_f(\mathbf{0}) = \sum_{x \in Z_q^n} \zeta^{f(x)} = \sum_{k=0}^{p-1} \sum_{x \in z_q^n, f(x)=k} \zeta^k = \sum_{k=0}^{q-1} \zeta^k \sum_{x \in z_q^n, f(x)=k} 1$$

Since f is balanced so

$$W_f(\mathbf{0}) = q^{n-1} \sum_{k=0}^{q-1} \zeta^k = 0 \quad \text{as} \quad \sum_{k=0}^{q-1} \zeta^k = 0.$$

Now, conversely, if $W_f(\mathbf{0}) = 0$, then

$$\sum_{k=0}^{q-1} \zeta^k \sum_{x \in z_q^n, f(x)=k} 1 = 0. \tag{7}$$

Let $\sum_{x \in Z_q^n, f(x)=k} 1 = a_k$ for $0 \le k < q$. Then equation (7) implies

$$\sum_{k=0}^{q-1} a_k = q^n \tag{8}$$

$$\sum_{k=0}^{q-1} a_k \zeta^k = 0. \tag{9}$$

But $x^{q-1} + x^{q-2} + ... + x + 1$ is the minimal polynomial of ζ. Hence we have

$$\frac{a_0}{1} = \frac{a_1}{1} = ... = \frac{a_{q-1}}{1} = a(say).$$

From equation (8) we have,

$$qa = q^n \Rightarrow a = q^{n-1} \Longrightarrow \sum_{x \in Z_q^n, f(x)=k} 1 = q^{n-1} \text{ for all } 0 \le k < q.$$

Hence f is balanced. ∎

Now, the above theorem is obviously implying the proof of next corollary.

Corollary 1. *Bent functions $f : \mathbb{Z}_q^n \to \mathbb{Z}_q$ are not balanced.*

4.1. Generalized symmetric bent functions

The number of generalized symmetric, rotational symmetric and super symmetric boolean functions on $\mathbb{Z}_q^n$ are very less with respect to the total number of Boolean functions on $\mathbb{Z}_q^n$ i.e. q^{q^n}. Search of these all possible functions having bent as well as various symmetric properties can be achievable from some tricky approach. Table 1 shows a comparative results of enumeration of symmetric functions and total number of Boolean functions based on some chosen parameters.

Table 1: Distribution of GSBF, GRSBF and GSSBF

n	q	$\#SBF$	$\#SSBF$	$\#RSBF$	$\#BF$
1	3	27	9	27	27
2	3	729	27	729	19683
3	3	59049	81	177147	7625597484987

Table 2: Truth table of SBF on $\mathbb{Z}_3^2$

x_1	x_2	$f(x_1,x_2)$	$W_f(x)$
0	0	v_0	$\zeta^{v_0}+2\zeta^{v_1}+2\zeta^{v_2}+\zeta^{v_4}+2\zeta^{v_3}+\zeta^{v_5}$
0 1	1 0	v_1	$\zeta^{v_0}-\zeta^{2+v_1}-\zeta^{1+v_2}+\zeta^{1+v_4}-\zeta^{v_3}+\zeta^{2+v_5}$
0 2	2 0	v_2	$\zeta^{v_0}-\zeta^{1+v_1}-\zeta^{2+v_2}+\zeta^{2+v_4}-\zeta^{v_3}+\zeta^{1+v_5}$
1 2	2 1	v_3	$\zeta^{v_0}-\zeta^{v_1}-\zeta^{v_2}+\zeta^{v_4}-\zeta^{v_3}+\zeta^{v_5}$
1	1	v_4	$\zeta^{v_0}+2\zeta^{1+v_1}+2\zeta^{2+v_2}+\zeta^{2+v_4}+2\zeta^{v_3}+\zeta^{1+v_5}$
2	2	v_5	$\zeta^{v_0}+2\zeta^{2+v_1}+2\zeta^{1+v_2}+\zeta^{1+v_4}+2\zeta^{v_3}+\zeta^{2+v_5}$

Table 3: Truth table of SSBF on $\mathbb{Z}_3^2$

x_1	x_2	$f(x_1,x_2)$	$W_f(x)$
0	0	v_0	$\zeta^{v_0}+4\zeta^{v_1}+4\zeta^{v_2}$
0 1 0 2	1 0 2 0	v_1	$\zeta^{v_0}+\zeta^{v_1}-2\zeta^{v_2}$
1 1 2 2	1 2 1 2	v_2	$\zeta^{v_0}-2\zeta^{v_1}+\zeta^{v_2}$

Since the classification under the constraints of symmetric and rotational symmetric permutation of Z_q^n for $n=2$ are same, so Table 2 also represents the Walsh spectrum of rotational symmetric Boolean function for $q=3$. The next theorem provides us major clues about the existence and construction of both functions.

Theorem 4.5. *There are only* 6 *balanced symmetric or rotational symmetric function from* $\mathbb{Z}_3^2$ *to* $\mathbb{Z}_3$

Proof. We can classify all $x \in Z_3^2$ on the basis of its hamming weight$(0 \leq j \leq 2)$ and rotational symmetry in the following manner

$$A_j = \{x \in Z_3^2 : wt(x) = j, x_1 = x_2\}$$
$$B_j = \{x \in Z_3^2 : wt(x) = j, x_1 \neq x_2\}$$

We can see that $|A_0| = 0$, $|A_j| = 1$ and $|B_j| = 2$ for all $0 \leq j \leq 2$. Now by definition to the balancedness of 3−ary boolean functions the possibility of one combination of set of $x \in Z_3^2$ viz. A_j and B_j are

$$\{A_0, B_1\}, \{A_2, B_1\}, \{A_2, B_2\}$$

Similarly we can see that there are only 6 possible combination of these class under the balancedness condition. Hence we can say that there are only 6 balanced symmetric Boolean functions. ■

Theorem 4.6. *There are only* 8 *symmetric and rotational symmetric bent functions* f *from* $\mathbb{Z}_3^2$ *to* $\mathbb{Z}_3$ *and all of them are not balanced.*

Proof. By the definition of Walsh transformation and table 2. for all $w \in \mathbb{Z}_3^2$, $W_f(w) \in \{\zeta^{v_0} + 2\zeta^{v_1} + 2\zeta^{v_2} + \zeta^{v_4} + 2\zeta^{v_3} + \zeta^{v_5}, \zeta^{v_0} - \zeta^{2+v_1} - \zeta^{1+v_2} + \zeta^{1+v_4} - \zeta^{v_3} + \zeta^{2+v_5}, \zeta^{v_0} + 2\zeta^{1+v_1} + 2\zeta^{2+v_2} + \zeta^{2+v_4} + 2\zeta^{v_3} + \zeta^{1+v_5}, \zeta^{v_0} - \zeta^{v_1} - \zeta^{v_2} + \zeta^{v_4} - \zeta^{v_3} + \zeta^{v_5}, \zeta^{v_0} + 2\zeta^{1+v_1} + 2\zeta^{2+v_2} + \zeta^{2+v_4} + 2\zeta^{v_3} + \zeta^{1+v_5}, \zeta^{v_0} + 2\zeta^{2+v_1} + 2\zeta^{1+v_2} + \zeta^{1+v_4} + 2\zeta^{v_3} + \zeta^{2+v_5}\}$. Now, if f is bent, then $|W_f(w)|^2 = 9$ for all $w \in \mathbb{Z}_3^2$ which obviously imply the unbalanced property of all possible f by theorem 3.1, so we can get six different equations for all $W_f(w)$. Computationally we search the solution of these equations and found only eight set of solutions. In Table 4, we are showing these solutions.

Table 4: Bent GSBF and GRSBF on $\mathbb{Z}_3^2$

	f^1	f^2	f^3	f^4	f^5	f^6	f^7	f^8
v_0	0	0	0	0	1	1	2	2
v_1	0	1	1	2	1	2	0	2
v_2	2	0	1	2	1	0	1	2
v_3	2	1	2	1	2	0	0	1
v_4	0	2	2	1	0	1	0	0
v_5	1	0	2	1	0	0	2	0

In [5], Hou has shown the uniqueness of representation of generalized Boolean function as a multivariate polynomial. Corresponding to each solution in the above table, below we are giving multivariate representations.

Bent Symmetric and Rotational symmetric Boolean functions from $\mathbb{Z}_3^2$ to $\mathbb{Z}_3$

$$
\begin{aligned}
f^1(x_1, x_2) &= 2x_2 + 2x_1 + x_2^2 + x_1^2 \\
f^2(x_1, x_2) &= 2x_2 + 2x_1 + 2x_2^2 + 2x_1^2 \\
f^3(x_1, x_2) &= x_1^2 + x_2^2 \\
f^4(x_1, x_2) &= 2x_2^2 + 2x_1^2 \\
f^5(x_1, x_2) &= 1 + 2x_1x_2 \\
f^6(x_1, x_2) &= 1 + x_2 + x_1 + x_1x_2 \\
f^7(x_1, x_2) &= 2 + x_2 + x_1 + 2x_1x_2 \\
f^8(x_1, x_2) &= 2 + x_1x_2
\end{aligned}
$$

Hence the proof is completed. ∎

Theorem 4.7. *There is no any super symmetric bent function for $q = 3$ and $n = 2$.*

Proof. By the definition of SSBF and the bent property of a Boolean function, we know that $|W_f(w)|^2 = 9$. We can write three equations from the Walsh spectrum values given in the table 3.

$$
\begin{aligned}
|\zeta^{v_0} + 4\zeta^{v_1} + 4\zeta^{v_2}| &= 3 \\
|\zeta^{v_0} + \zeta^{v_1} - 2\zeta^{v_2}| &= 3 \\
|\zeta^{v_0} - 2\zeta^{v_1} + \zeta^{v_2}| &= 3
\end{aligned}
$$

Let $\zeta = \exp^{i\theta}$, where $\theta = 2n\pi/3$. Above relation can be reduced in the real coefficients as follows

$$\cos(v_0 - v_1)\theta + 4\cos(v_1 - v_2)\theta + \cos(v_1 - v_2)\theta = -7 \tag{10}$$

$$2\cos(v_0 - v_1)\theta - 4\cos(v_1 - v_2)\theta - 4\cos(v_2 - v_0)\theta = 3 \tag{11}$$

$$-4\cos(v_0 - v_1)\theta - 4\cos(v_1 - v_2)\theta + 2\cos(v_2 - v_0)\theta = 3 \tag{12}$$

Now, let $\cos(v_0 - v_1)\theta = x$, $y = \cos(v_1 - v_2)\theta$ and $\cos(v_2 - v_0)\theta = z$ then $7, 8$ and 9 can be written as

$$x + 4y + z = -7 \tag{13}$$

$$2x - 4y - 4z = 3 \tag{14}$$

$$-4x - 4y + 2z = 3 \tag{15}$$

Inconsistency of above three equations implies the non existence of super symmetric bent function on $\mathbb{Z}_3^2$. ∎

Based on our analysis we present two conjectures related to the generalized super symmetric bent functions and symmetric bent functions.

Conjecture 1. *For any prime q there is no any generalized super symmetric bent function from $\mathbb{Z}_q^n$ to $\mathbb{Z}_q$.*

Conjecture 2. *For any prime q there is no any generalized homogeneous symmetric bent function from $\mathbb{Z}_q^n$ to $\mathbb{Z}_q$ of degree greater then 2.*

References

[1] Canteaut, A., Videau, M., *Symmetric Boolean functions*, IEEE Transactions on Information Theory, 51 (8) (2005), 2791-2811.

[2] Camion, P., Canteaut, A., *Construction of t-resilient functions over a finite alphabet*, Advances in Cryptology-Eurocrypt 96, Springer, Berlin, 1996, 283-293.

[3] Kravets, V.N., *Generalized symmetries in Boolean functions*, Computer Aided Design, ICCAD-2000. IEEE/ACM International Conference, Nov. 2000, 526-532.

[4] Savicky, P., *On the bent Boolean functions that are symmetric*, Europian J. Combin., 15 (1994), 407-410.

[5] Hou, X.-D., *p-ary and q-ary versions of certain results about bent functions and resilient functions*, Finite Fields and Their Applications, 10 (2004), 566-582.

[6] Stanica, P., Subhamoy Maitra, S., *Rotation symmetric Boolean functions-Count and cryptographic properties*, Discrete Applied Mathematics, 156 (2008), 1567-1580.

[7] Maitra, S., Sarkar, P., *Maximum nonlinearity of symmetric Boolean functions on odd number of variables*, IEEE Trans. Inf. Theory, 48 (9) (2002), 2626-2630.

[8] Gopalakrishnan, K., Hoffman, D., Stinson, D., *A note on a conjecture concerning symmetric resilient functions*, Inform. Process. Lett., 47 (3) (1993), 139-143.

[9] Sarkar, P., Maitra, S., *Balancedness and correlation immunity of symmetric Boolean functions*, Proc. R.C. Bose Centenary Symp., 15(2003), 178-183.

[10] Cunkle, C.H., *Symmetric Boolean Functions*, The American Mathematical Monthly, 70 (8) (1963), 833-836.

[11] Paturi, R., *On the degree of polynomials that approximate symmetric Boolean functions*, STOC '92 Proceedings of the twenty-fourth annual ACM symposium on Theory of computing, 1992, 468-474.

[12] Lidl, R., Pilz, G., *Applied Abstract Algebra*, Second edition, Springer, 1998.

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RESULTS ON PRIME IDEALS IN PMV-ALGEBRAS AND MV-MODULES

S. Saidi Goraghani[1]

Department of Mathematics
Farhangian University
Iran
e-mail: SiminSaidi@yahoo.com

R.A. Borzooei

Department of Mathematics
Shahid Beheshti University
Tehran
Iran
e-mail: borzooei@sbu.ac.ir

Abstract. In this paper, by considering the notions of MV-modules and PMV-algebras, we study $\cdot$-prime ideals in PMV-algebras, prime A-ideals in MV-modules and investigate some properties on them. Also, we present the definitions of radical of a $\cdot$-ideal in PMV-algebras, radical of an A-ideal in MV-modules and verify some properties on them. Finally, we state a method to obtain the radical of a $\cdot$-ideal in PMV-algebras.

Keywords: (MV, PMV)-algebra, MV-module, $\cdot$-prime ideal, prime A-ideal, radical.

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1. Introduction

MV-algebras were defined by C.C. Chang [1], [2] as algebras corresponding to the Lukasiewicz infinite valued propositional calculus. These algebras have appeared in the literature under different names and polynomially equivalent presentation: CN-algebras, Wajsberg algebras, bounded commutative BCK-algebras and bricks. It is discovered that MV-algebras are naturally related to the Murray-von Neumann order of projections in operator algebras on Hilbert spaces and that they play an interesting role as invariants of approximately finite-dimensional C^*-algebras. They are also naturally related to Ulam's searching games with lies. MV-algebras admit a natural lattice reduct and hence a natural order structure. Many important properties can be derived from the fact, established by Chang

[1]Corresponding author.

that nontrivial MV-algebras are subdirect products of MV-chains, that is, totally ordered MV-algebras. To prove this fundamental result, Chang introduced the notion of prime ideal in an MV-algebra. The categorical equivalence between MV-algebras and lu-groups leads to the problem of defining a product operation on MV-algebras, in order to obtain structures corresponding to l-rings. A *product MV-algebra* (or PMV-algebra, for short) is an MV-algebra which has an associative binary operation ".". It satisfies an extra property which will be explained in Preliminaries. During the last years, PMV-algebras were considered and their equivalence with a certain class of l-rings with strong unit was proved. It seems quite natural to introduce modules over such algebras, generalizing the divisible MV-algebras and the MV-algebras obtained from Riesz spaces and to prove natural equivalence theorems. Hence, the notion of MV-modules was introduced as an action of a PMV-algebra over an MV-algebra by A. Di Nola [5]. In 2014, F. Forouzesh, E. Eslami and A. Borumand Saeid defined prime A-ideals in MV-modules and $\cdot$-prime ideals in PMV-algebras [7]. Also, they defined radical of A-ideals by maximal A-ideals. Since MV-modules are in their infancy, stating and opening of any subject in this field can be useful. Hence, in this paper, we investigate prime A-ideals in MV-modules and verify some properties on them. For example, we state some conditions for obtaining a prime A-ideal. Also, we present the definition of radical of an A-ideal by prime A-ideals in MV-modules, radical of a $\cdot$-ideal in PMV-algebras and verify some properties on them. Then we state a method to obtain a radical of a $\cdot$-ideal in PMV-algebras. In fact, we open new fields to anyone that is interested to studying and development of MV-modules.

2. Preliminaries

In this section, we review related lemmas and theorems that we use in the next sections.

Definition 2.1 [3] An *MV-algebra* is a structure $M = (M, \oplus, ', 0)$ of type $(2, 1, 0)$ such that:
$(MV1)$ $(M, \oplus, 0)$ is an Abelian monoid,
$(MV2)$ $(a')' = a$,
$(MV3)$ $0' \oplus a = 0'$,
$(MV4)$ $(a' \oplus b)' \oplus b = (b' \oplus a)' \oplus a$,
If we define the constant $1 = 0'$ and operations $\odot$ and $\ominus$ by $a \odot b = (a' \oplus b')'$, $a \ominus b = a \odot b'$, then
$(MV5)$ $(a \oplus b) = (a' \odot b')'$,
$(MV6)$ $x \oplus 1 = 1$,
$(MV7)$ $(a \ominus b) \oplus b = (b \ominus a) \oplus a$,
$(MV8)$ $a \oplus a' = 1$,
for every $a, b \in M$. It is clear that $(M, \odot, 1)$ is an Abelian monoid. Now, if we define auxiliary operations $\vee$ and $\wedge$ on M by $a \vee b = (a \odot b') \oplus b$ and $a \wedge b = a \odot (a' \oplus b)$, for every $a, b \in M$, then $(M, \vee, \wedge, 0)$ is a *bounded distributive lattice.*

An MV-algebra M is a *Boolean* algebra if and only if the operation " $\oplus$ " is idempotent, that is, $x \oplus x = x$, for every $x \in M$. In an MV-algebra M, the following conditions are equivalent: (i) $a' \oplus b = 1$, (ii) $a \odot b' = 0$, (iii) $b = a \oplus (b \ominus a)$, (iv) $\exists c \in M$ such that $a \oplus c = b$, for every $a, b, c \in M$. For any two elements a, b of the MV-algebra M, $a \leq b$ if and only if a, b satisfy the above equivalent conditions $(i) - (iv)$. An ideal of MV-algebra M is a subset I of M, satisfying the following conditions: $(I1)$: $0 \in I$, $(I2)$: $x \leq y$ and $y \in I$ imply $x \in I$, $(I3)$: $x \oplus y \in I$, for every $x, y \in I$. A proper ideal I of M is a prime ideal of M if and only if $x \ominus y \in I$ or $y \ominus x \in I$ (or $x \wedge y \in I$ implies that $x \in I$ or $y \in I$), for every $x, y \in M$. In an MV-algebra M, the *distance function* $d : M \times M \to M$ is defined by $d(x, y) = (x \ominus y) \oplus (y \ominus x)$ which satisfies (i): $d(x, y) = 0$ if and only if $x = y$, (ii): $d(x, y) = d(y, x)$, (iii): $d(x, z) \leq d(x, y) \oplus d(y, z)$, (iv): $d(x, y) = d(x', y')$, (v): $d(x \oplus z, y \oplus t) \leq d(x, y) \oplus d(z, t)$, for every $x, y, z, t \in M$. Let I be an ideal of an MV-algebra M. We denote $x \sim y$ ($x \equiv_I y$) if and only if $d(x, y) \in I$, for every $x, y \in M$. So $\sim$ is a congruence relation on M. Denote the equivalence class containing x by $\frac{x}{I}$ and $\frac{M}{I} = \{\frac{x}{I} : x \in X\}$. Then $(\frac{M}{I}, \oplus, ', \frac{0}{I})$ is an MV-algebra, where $(\frac{x}{I})' = \frac{x'}{I}$ and $\frac{x}{I} \oplus \frac{y}{I} = \frac{x \oplus y}{I}$, for all $x, y \in M$. Let M and K be two MV-algebras. A mapping $f : M \to K$ is called an MV-*homomorphism* if $(H1)$: $f(0) = 0$, $(H2)$: $f(x \oplus y) = f(x) \oplus f(y)$ and $(H3)$: $f(x') = (f(x))'$, for every $x, y \in M$. If f is one to one (onto), then f is called an MV-monomorphism (epimorphism) and if f is onto and one to one, then f is called an MV-isomorphism (see [4, 10])

Proposition 2.2 [3] *Let M be an MV-algebra and $z \in M$. Then the principal ideal generated by z is denoted by $\prec z \succ$ and*

$$\prec z \succ = \{x \in M : nz = \underbrace{z \oplus \cdots \oplus z}_{n\ times} \geq x,\ for\ some\ n \geq 0\}.$$

Proposition 2.3 [3] *Let I be an ideal of A. Then*

$$\prec I \cup \{z\} \succ = \{x \in A : x \leq nz \oplus a,\ for\ some\ n \in \mathbb{N}\ and\ a \in I\}.$$

Proposition 2.4 [3] *In every MV-algebra A, the natural order " $\leq$ " has the following properties:*
(i) $x \leq y$ if and only if $y' \leq x'$,
(ii) if $x \leq y$, then $x \oplus z \leq y \oplus z$ and $x \odot z \leq y \odot z$, for every $x, y, z \in A$.

Definition 2.5 [5], [6] (i) An l-group is an algebra $(G, +, -, 0, \vee, \wedge)$, where the following properties hold:
(a) $(G, +, -, 0)$ is a group,
(b) $(G, \vee, \wedge)$ is a lattice,
(c) $x \leq y$ implies that $x + a \leq y + a$, for any $x, y, a, b \in G$.
A strong unit $u > 0$ is a positive element with property that for any $g \in G$ there exits $n \in \omega$ such that $g \leq nu$. The Abelian l-groups with strong unit will be simply called lu-groups.

The category whose objects are MV-algebras and whose homomorphisms are MV-homomorphisms is denoted by MV. The category whose objects are pairs (G, u), where G is an Abelian l-group and u is a strong unit of G and whose homomorphisms are l-group homomorphisms is denoted by Ug. The functor that establishes the categorial equivalence between MV and Ug is

$$\Gamma : Ug \longrightarrow MV,$$

where $\Gamma(G, u) = [0, u]_G$, for every lu-group (G, u) and $\Gamma(h) = h|_{[0,u]}$, for every lu-group homomorphism h. The above results allows us to consider an MV-algebra, when necessary, as an interval in the positive cone of an l-group. Thus, many definitions and properties can be transferred from l-groups to MV-algebras. For example, the group addition becomes a partial operation when it is restricted to an interval, so we define a *partial addition* on an MV-algebra M as follows:
$x + y$ is defined if and only if $x \leq y'$ and in this case, $x + y = x \oplus y$, for every $x, y \in M$. Moreover, if $z + x \leq z + y$, then $x \leq y$.
(ii) An l-ring is a structure $(R, +, ., 0, \leq)$, where $(R, +, 0, \leq)$ is an L-group such that, for any $x, y \in R$,

$$x \geq 0 \textit{ and } y \geq 0 \textit{ implies } x.y \geq 0.$$

(iii) A *product* MV*-algebra* (or PMV-algebra, for short) is a structure $A = (A, \oplus, ., ', 0)$, where $(A, \oplus, ', 0)$ is an MV-algebra and "." is a binary associative operation on A such that the following property is satisfied: if $x+y$ is defined, then $x.z+y.z$ and $z.x+z.y$ are defined and $(x+y).z = x.z+y.z$, $z.(x+y) = z.x+z.y$, for every $x, y, z \in A$, where " + " is the partial addition on A. A unity for the product is an element $e \in A$ such that $e.x = x.e = x$, for every $x \in A$. If A has a unity for product, then $e = 1$. A PMV-homomorphism is an MV-homomorphism which also commutes with the product operation. A $\cdot$-ideal of A is an ideal I of A such that if $a \in I$ and $b \in A$ entail $a.b \in I$ and $b.a \in I$. The set of $\cdot$-ideals of A is denoted by $Id(A)$.

Lemma 2.6 [4] *Let A be a PMV-algebra. Then $a \leq b$ implies that $a.c \leq b.c$ and $c.a \leq c.b$ for every $a, b, c \in A$.*

Lemma 2.7 [5] *Let M be an MV-algebra . Then for every $x, y, z \in M$,*
(b) $x + 0 = x$,
(c) $x \vee y = x + (x' \odot y)$,
(d) *if $x + y$ and $(x + y) + z$ are defined, then $y + z$ and $x + (y + z)$ are defined and* $(x + y) + z = x + (y + z)$,
(f) *if $z + x \leq z + y$, then $x \leq y$,*
(h) *if $z + x = z + y$, then $x = y$,*
where $+$ is the partial addition on M.

Definition 2.8 [5] Let $A = (A, \oplus, ., ', 0)$ be a PMV-algebra, $M = (M, \oplus, ', 0)$ be an MV-algebra and the operation $\Phi : A \times M \longrightarrow M$ is defined by $\Phi(a, m) = am$, which satisfies the following axioms:

$(AM1)$ If $x+y$ is defined in M, then $ax+ay$ is defined in M and $a(x+y) = ax+ay$,
$(AM2)$ If $a+b$ is defined in A, then $ax+bx$ is defined in M and $(a+b)x = ax+bx$,
$(AM3)$ $(a.b)x = a(bx)$, for every $a, b \in A$ and $x, y \in M$.
Then M is called a (left) *MV-module* over A or briefly an A-module. We say that M is a *unitary* MV-module if A has a unity 1_A for the product that is
$(AM4)$ $1_A x = x$, for every $x \in M$.

Lemma 2.9 [5] *Let A be a PMV-algebra and M be an A-module. Then*
(a) $0x = 0$,
(b) $a0 = 0$,
(c) $ax' \leq (ax)'$,
(d) $a'x \leq (ax)'$,
(e) $(ax)' = a'x + (1x)'$,
(f) $x \leq y$ *implies* $ax \leq ay$,
(g) $a \leq b$ *implies* $ax \leq bx$,
(h) $a(x \oplus y) \leq ax \oplus ay$,
(i) $d(ax, ay) \leq ad(x, y)$,
(j) *if M is a unitary MV-module, then* $(ax)' = a'x + x'$,
(k) $(ax) \odot (ay)' \leq a(x \odot y)$, *for every $a, b \in A$ and $x, y \in M$.*

Definition 2.10 [5] Let A be a PMV-algebra, M_1 and M_2 be two A-modules. A map $f : M_1 \to M_2$ is called an *A-module* homomorphism or (*A-homomorphism*, for short) if f is an MV-homomorphism and
$(H4)$: $f(ax) = af(x)$, for every $x \in M_1$ and $a \in A$.

Definition 2.11 [5] Let A be a PMV-algebra and M be an A-module. Then an ideal $N \subseteq M$ is called an *A-ideal* of M if $(I4)$: $ax \in N$, for every $a \in A$ and $x \in N$.

Definition 2.12 [7] Let M be an A-module and N be a proper A-ideal of M. Then N is called a *prime* A-ideal of M, if $am \in N$ implies that $m \in N$ or $a \in (N : M)$, for any $a \in A$ and $m \in M$, where $(N : M) = \{a \in A : aM \subseteq N\}$. Moreover, the set of all prime A-ideals of M is showed by $Spec(M)$.

Definition 2.13 [8] Let M be an A-module. An A-ideal N of M is called a maximal A-ideal of M, if there exist no A-ideal K of M containing N such that $N \subsetneq K \subsetneq M$. The set of all maximal A-ideals of M is showed by $Max(M)$. Let I be a proper A-ideal in M. The intersection of all maximal A-ideals of M which contain I is called the radical of I and is denoted by $Rad(I)$.

Definition 2.14 [7] Let I be a proper $\cdot$-ideal of A. I is called a $\cdot$-prime ideal of A, if $x.y \in I$ implies that $x \in I$ or $y \in I$, for any $x, y \in A$.

Note. From now on, in this paper, we let A be a PMV-algebra, M be an MV-algebra, $\sum_{i=1}^{n} x_i$ means $x_1 \oplus x_2 \oplus \cdots \oplus x_n$ and in particular case, $\underbrace{x \oplus x \oplus \cdots \oplus x}_{n \text{ times}} = nx$.

3. Prime A-ideals in MV-modules and prime $\cdot$-ideals in PMV-algebras

In this section, we study prime A-ideals in MV-modules and $\cdot$-prime ideals in PMV-algebras. Then we state and prove some conditions to obtain them.

Remark. Let A be unital and I be an ideal of A. Then by Lemma 2.6, since $x \leq 1$, $x.y \leq x.1 = x \in I$ and $y.x \leq y.1 = y \in I$ and so $x.y, y.x \in I$, for every $x, y \in I$. It means that if A is unital, then every ideal of A is a $\cdot$-ideal of A.

Example 3.1 Let $A = \{0, 1, 2, 3\}$ and the operations " $\oplus$ " and "." on A be defined as follows:

$\oplus$	0	1	2	3
0	0	1	2	3
1	1	1	3	3
2	2	3	2	3
3	3	3	3	3

.	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

Consider $0' = 3$, $1' = 2$, $2' = 1$ and $3' = 0$. Then it is easy to show that $(A, \oplus, ', ., 0)$ is a PMV-algebra and $(A, \oplus, ', 0)$ is an MV-algebra. Now, let the operation $\bullet : A \times A \longrightarrow A$ be defined by $a \bullet b = a.b$, for every $a, b \in A$. It is easy to show that A is an MV-module on A, $I = \{0, 1\}$, $J = \{0, 2\}$ are prime A-ideals of A and $\{0\}$ is not a prime A-ideal of A. Also, $I = \{0, 1\}$ is a $\cdot$-prime ideal of A and $I = \{0\}$ is not a $\cdot$-prime ideal of A.

Proposition 3.2 *Let M be an A-module and N be an A-ideal of M. Then $(N : M) = \{a \in A : aM \subseteq N\}$ is an ideal of A.*

Proof. It is clear that $0 \in (N : M)$. Let $a, b \in (N : M)$. Then $am, bm \in N$, for every $m \in N$. Similar the proof of Lemma 2.9 (k), we have $am \odot (bm)' \leq (a \odot b')m$, for every $a, b \in A$ and $m \in M$. If we set $a \oplus b$ instead of a, then by Lemma 2.9 (g), we have $(a \oplus b)m \odot (bm)' \leq ((a \oplus b) \odot b')m = (a \wedge b')m \leq am$. Since

$$(a \oplus b)m = (a \oplus b)m \vee bm = (a \oplus b)m \odot (bm)' \oplus bm \leq am \oplus bm \in N,$$

$a \oplus b \in (N : M)$. Now, let $a \leq b$ and $b \in (N : M)$. Then by Lemma 2.9(g), $am \leq bm \in N$ and so $am \in N$, for every $m \in M$. It means that $a \in (N : M)$. ■

Remark. Similar the Proposition 3.2, $(N : m)$ is an ideal of A, for every $m \in M$.

Proposition 3.3 *Let M be a unitary A-module and N, L be A-ideals of M. Then*
(i) $(N : M)$ is a prime ideal of A or $\frac{A}{(N:M)}$ has at least two elements,
(ii) if N is a prime A-ideal of M and $m \notin N$, then $(N : m)$ is a $\cdot$-prime ideal of A,
(iii) N is a prime A-ideal of M if and only if $(N : m) = (N : M)$, where $m \notin N$,
(iv) $N \subseteq L$ implies that $(N : M) \subseteq (L : M)$,
(v) if N is a prime A-ideal of M, then $(N : M)$ is a $\cdot$-prime ideal of A.

Proof. (i) By Proposition 3.2, $(N : M)$ is an ideal of A. Let $(N : M)$ is not a prime ideal of A. Then there exist $x, y \in A$ such that $x \ominus y \notin (N : M)$ and $y \ominus x \notin (N : M)$ and so $d(x,y) = (x \ominus y) \oplus (y \ominus x) \notin (N : M)$. It means that $\frac{x}{(N:M)} \neq \frac{y}{(N:M)}$ and so $\frac{A}{(N:M)}$ has at least two elements. Now, let $\frac{A}{(N:M)}$ have only element $\frac{0}{(N:M)}$. Then $\frac{x}{(N:M)} = \frac{y}{(N:M)}$, for every $x, y \in A$ and so $x \ominus y \leq (x \ominus y) \oplus (y \ominus x) = d(x,y) \in (N : M)$. Hence, $x \ominus y \in (N : M)$ and so $(N : M)$ is a prime ideal of A.
$(ii), (iii), (iv)$ The proofs are easy.
(v) By Proposition 3.2, $(N : M)$ is an ideal of A. If $A = (N : M)$, then $1 \in (N : M)$ and so $M = N$, which is a contradiction. Hence, $(N : M)$ is a proper ideal of A. Since A is unital, $(N : M)$ is a $\cdot$-ideal of A. Let $x.y \in (N : M)$, for any $x, y \in A$. Then $x(ym) = (x.y)m \in N$, for every $m \in M$ and so $ym \in N$ or $x \in (N : M)$. Let $x \notin (N : M)$. Then $ym \in N$ and so $y \in (N : M)$ or $m \in N$. If $m \in N$, then $N = M$, which is a contradiction. Hence, $y \in (N : M)$ and so $(N : M)$ is a $\cdot$-prime ideal of A. ∎

Lemma 3.4 *Let M be a unitary A-module and $m \in M$. Then*

$$I_m = \left\{ \sum_{i=1}^{k} t_i m : \sum_{i=1}^{k} t_i m \leq nm, \ \textit{for some } n, k \in \mathbb{N} \cup \{0\}, \right.$$
$$\left. \textit{where } t_i \in A \textit{ and } t_1 m + \cdots + t_k m \textit{ is defined} \right\}$$

is an A-ideal of M.

Proof. (I_1) It is clear that $0 \in I_m$.
(I_2) Let $x \leq \sum_{i=1}^{k} t_i m \in I_m$, for some $x \in M$. Then $x = 1x \leq \sum\limits_{i=1}^{k} t_i m \leq nm \in I_m$, for some $n \geq 0$ and so $x \in I_m$.
(I_3) Let $\sum\limits_{i=1}^{k} t_i m, \sum\limits_{i=1}^{w} s_i m \in I_m$. Then there exist $n_1, n_2 \geq 0$ such that $\sum\limits_{i=1}^{k} t_i m \leq n_1 m$ and $\sum\limits_{i=1}^{w} s_i m \leq n_2 m$ and so

$$\begin{aligned} \sum_{i=1}^{k+w} c_i m &\leq \sum_{i=1}^{k} t_i m \oplus \sum_{i=1}^{w} s_i m \\ &\leq n_1 m \oplus n_2 m = \underbrace{m \oplus \cdots \oplus m}_{n_1 \ times} \oplus \underbrace{m \oplus \cdots \oplus m}_{n_2 \ times} = (n_1 + n_2) m, \end{aligned}$$

where $c_i \in A$, for $1 \leq i \leq k + w$. It means that $\sum\limits_{i=1}^{k} t_i m \oplus \sum\limits_{i=1}^{w} s_i m \in I_m$.
(I_4) Let $a \in A$ and $\sum\limits_{i=1}^{k} t_i m \in I_m$. Then there exists $n \geq 0$ such that $\sum\limits_{i=1}^{k} tm \leq nm$.

Since $\sum_{i=1}^{k} t_i m \leq nm$, by Proposition 2.9(f) and (h),

$$a\left(\sum_{i=1}^{k} t_i m\right) \leq a(m \oplus \cdots \oplus m) \leq \underbrace{am \oplus \cdots \oplus am}_{n \ times}.$$

By Proposition 2.9(j), since $(am)' \oplus m = a'm \oplus m' \oplus m = 1$, $am \leq m$ and so $a\left(\sum_{i=1}^{k} t_i m\right) \leq \underbrace{m \oplus \cdots \oplus m}_{n \ times} = nm$. It results that $\sum_{i=1}^{k}(a.t_i)m = \sum_{i=1}^{k} a(t_i m) \in I_m$. ■

Definition 3.5 A PMV-algebra A is called *commutative*, if $x.y = y.x$, for every $x, y \in A$.

Example 3.6 In Example 3.1, A is a commutative PMV-algebra.

Notation: For A-module M, $I \subseteq A$ and A-ideal N of M, we let

$$IN = \{xm : x \in I, m \in N\}.$$

Theorem 3.7 *Let A be commutative, M be an A-module, N be a proper A-ideal of M and $x \oplus x = x$, for every $x \in A$. Then N is a prime A-ideal of M if and only if for every ideal I of A and A-ideal D of M, $ID \subseteq N$ implies that $I \subseteq (N : M)$ or $D \subseteq N$.*

Proof. ($\Rightarrow$) Let N be a prime A-ideal of M, I be an ideal of A and D be an A-ideal of M such that $ID \subseteq N$. We show that $I \subseteq (N : M)$ or $D \subseteq N$. Let $I \nsubseteq (N : M)$ and $D \nsubseteq N$. Then there exist $x \in I$ and $d \in D$ such that $xM \nsubseteq N$ and $d \notin N$. On the other hand, $ID \subseteq N$ implies that $xd \in N$. Since N is a prime A-ideal of M and $d \notin N$, $xM \subseteq N$, which is a contradiction.
($\Leftarrow$) Let for every ideal I of A and A-ideal D of M, $ID \subseteq N$ implies that $I \subseteq (N : M)$ or $D \subseteq N$. Let there exist $x \in A$ and $m \in M$ such that $xm \in N$ and $m \notin N$. By Proposition 2.2 and Lemma 3.4, let $I = \prec x \succ$ and $D = I_m$. Then for $y \in I$, by Proposition 2.2, there exists $n \geq 0$ such that $y \leq nx$ and so $y \ominus nx = 0$. Hence, $ym = (y \ominus 0)m = (y \ominus (y \ominus nx))m = (y \odot (y \odot (nx)')')m = (y \odot (y' \oplus nx))m = (y \wedge nx)m$. By Proposition 2.9(g), since $y \wedge nx \leq nx$, $ym = (y \wedge nx)m \leq (nx)m = xm \in N$. Hence, $ym \in N$ and so $ID = \{y(\sum_{i=1}^{k} t_i m) : y, t \in A\} = \{\sum_{i=1}^{k} t_i(ym) : y, t \in A\} \subseteq N$ and so $I \subseteq (N : M)$ or $D \subseteq N$. Since $m \notin N$, $I \subseteq (N : M)$ and so $xM \subseteq N$. Therefore, N is a prime A-ideal of M. ■

Definition 3.8 Let M be an A-module. M is called *torsion free* if $xm = 0$ implies that $x = 0$ or $m = 0$, for any $x \in A$ and $m \in M$.

Example 3.9 (i) Consider $L_2 = \{0, 1\}$, $L_4 = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$, $a \oplus b = min\{1, a + b\}$, $a' = 1 - a$ and $+, -, .$ are ordinary operations in $\mathbb{R}$. Then it is routine to show that $(L_2, \oplus, ', ., 0)$ is a PMV-algebra and $(L_4, \oplus, ', 0)$ is an MV-algebra. Let operation $\bullet : L_2 \times L_4 \longrightarrow L_4$ is defined by $a \bullet b = a.b$, for any $a \in L_2$ and $b \in L_4$. Then it is easy to show that L_4 is a torsion free L_2-module.
(ii) In Example 3.1, A is not a torsion free A-module.

Lemma 3.10 *Let M be an A-module. Then $d(\alpha m, \beta m) \leq d(\alpha, \beta)m$, for every $\alpha, \beta \in A$ and $m \in M$.*

Proof. The proof is similar to the proof of Lemma 2.9 (i) in [5]. ∎

Theorem 3.11 *Let M be a unitary A-module and K be an A-ideal of M. Then K is a prime A-ideal of M if and only if $P = (K : M)$ is a $\cdot$-prime ideal of A and $\frac{M}{K}$ is a torsion free $\frac{A}{P}$-module.*

Proof. $(\Rightarrow)$ Let K be a prime A-ideal of M. By Proposition 3.3 (v), $P = (K : M)$ is a $\cdot$-prime ideal of A. Now, we show that $\frac{M}{K}$ is a torsion free $\frac{A}{P}$-module. Let operation $\bullet : \frac{A}{P} \times \frac{M}{K} \longrightarrow \frac{M}{K}$ is defined by $\frac{x}{P}\frac{m}{K} = \frac{x}{P} \bullet \frac{m}{K} = \frac{xm}{K}$, for every $x \in A$ and $m \in M$. If $(\frac{x_1}{P}, \frac{m_1}{K}) = (\frac{x_2}{P}, \frac{m_2}{K})$, then $d(x_1, x_2) \in P$ and $d(m_1, m_2) \in K$, for any $x_1, x_2 \in A$ and $m_1, m_2 \in M$. By Proposition 2.9(i), $d(x_1m_1, x_1m_2) \leq x_1 d(m_1, m_2) \in K$ and so $d(x_1m_1, x_1m_2) \in K$. Also, by Lemma 3.10, $d(x_1m_2, x_2m_2) \leq d(x_1, x_2)m_2 \in PM \subseteq K$. Since $d(x_1m_1, x_2m_2) \leq d(x_1m_1, x_1m_2) \oplus d(x_1m_2, x_2m_2) \in K$, $d(x_1m_1, x_2m_2) \in K$ and so $\frac{x_1m_1}{K} = \frac{x_2m_2}{K}$. Hence, $\bullet$ is well defined. Let $x_1, x_2, x \in A$ and $m_1, m_2, m \in M$.

$(\frac{A}{P}\frac{M}{K}1)$: If $\frac{m_1}{K} + \frac{m_2}{K}$ is defined in $\frac{M}{K}$, then $\frac{m_1}{K} \leq \frac{m_2'}{K}$ and so by Proposition 2.9 (f, c), $\frac{xm_1}{K} \leq \frac{xm_2'}{K} \leq \frac{(xm_2)'}{K}$. It results that $\frac{xm_1}{K} + \frac{xm_2}{K}$ is defined and so $\frac{x}{P}(\frac{m_1}{K} + \frac{m_2}{K}) = \frac{xm_1}{K} + \frac{xm_2}{K} = \frac{x}{P}\frac{m_1}{K} + \frac{x}{P}\frac{m_2}{K}$.
$(\frac{A}{P}\frac{M}{K}2)$: if $\frac{x1}{P} + \frac{x_2}{P}$ is defined in $\frac{A}{P}$, then $\frac{x1}{P} \leq \frac{x_2'}{P}$ and so $\frac{x_1' \oplus x_2'}{P} = \frac{x_1'}{P} \oplus \frac{x_2'}{P} = \frac{1}{P}$. It means that $x_1 \ominus x_2' = d(x_1' \oplus x_2', 1) \in P = (K : M)$ and so $(x_1 \ominus x_2')m \in K$, for every $m \in M$. Since $x_1 \ominus x_2' \leq x_1$, $(x_1 \ominus x_2')m \leq x_1m$, for every $m \in M$. By Propositions 2.4 (ii) and 2.9 (d),

$$\begin{aligned}
(x_1 \ominus x_2')m &= (x_1 \ominus x_2')m \wedge x_1m \\
&= ((x_1 \ominus x_2')m \oplus (x_1m)') \odot (x_1m) = ((x_1 \ominus x_2')m + (x_1m)') \odot (x_1m) \\
&\geq ((x_1 \ominus x_2')m \oplus x_1'm) \odot (x_1m) = ((x_1 \ominus x_2') + x_1')m \odot (x_1m) \\
&= ((x_1 \odot x_2) + x_1')m \odot (x_1m)l = (x_1' \vee x_2)m \odot x_1m \\
&\geq x_2m \odot x_1m = x_2m \ominus (x_1m)'.
\end{aligned}$$

Then $x_2m \ominus (x_1m)' \in K$ and so $d((x_2m)' \oplus (x_1m)', 1) = x_2m \ominus (x_1m)' \in K$. It results that $\frac{(x_1m)'}{K} \oplus \frac{(x_2m)'}{K} = \frac{1}{K}$ and so $\frac{x1m}{K} \leq \frac{(x_2m)'}{K}$. Hence, $\frac{x_1m}{K} + \frac{x_2m}{K}$ is defined and so $(\frac{x1}{P} + \frac{x_2}{P})\frac{m}{K} = \frac{x1}{P}\frac{m}{K} + \frac{x_2}{P}\frac{m}{K}$.
$(\frac{A}{P}\frac{M}{K}3)$: The proof is routine.

Now, let $\frac{x}{P}\frac{m}{K} = \frac{0}{K}$, for any $x \in A$ and $m \in M$. Then $xm = d(xm, 0) \in K$ and so $m \in K$ or $x \in (K : M) = P$. Hence, $\frac{m}{K} = \frac{0}{K}$ or $\frac{x}{P} = \frac{0}{P}$ and so $\frac{M}{K}$ is a torsion free $\frac{A}{P}$-module.

$(\Leftarrow)$ Let P be a prime A-ideal of M and $\frac{M}{K}$ be a torsion free $\frac{A}{P}$-module. If $K = M$, then $P = (K : M) = (M : M) = A$, which is a contradiction. Now, let $xm \in K$, for any $x \in A$ and $m \in M$. Then $\frac{x}{P}\frac{m}{K} = \frac{xm}{K} = \frac{0}{K}$. Since $\frac{M}{K}$ is torsion free, $\frac{x}{P} = \frac{0}{P}$ or $\frac{m}{K} = \frac{0}{K}$ and so $x \in P = (K : M)$ or $m \in K$. Therefore, K is a prime A-ideal of M. ∎

4. On radical of A-ideals and $\cdot$-ideals

In this section, we present the definition of radical of an A-ideal ($\cdot$-ideal) in MV-modules (MV-algebras) and obtain some properties on it. Also, we characterize radical of a $\cdot$-ideal via elements of A.

Definition 4.1 Let M be an A-module and N be an A-ideal of M. The intersection of all prime A-ideals of M, including N, is called *radical* of N and it is shown by $rad_M(N)$ or $rad(N)$. If N is a prime A-ideal of M, then it is clear that $rad(N) = N$. If there exist no prime A-ideal of M including N, then we let $rad_M(N) = M$.

Example 4.2 In Example 3.1, $rad(I) = \{0, 1\}$ and $rad(\{0\}) = \{0\}$.

Lemma 4.3 *Let M be an A-module and N be an A-ideal of M. Then $x \in (N : M)$ if and only if $x1 \in N$, for every $x \in A$.*

Proof. Let $x \in (N : M)$. Then it is clear that $x1 \in N$. Now, let $x1 \in N$ and $m \in M$. Since $m \leq 1$, by Proposition 2.9(f), $xm \leq x1 \in N$ and so $xm \in N$, for every $m \in M$. Hence, $x \in (N : M)$. ∎

Lemma 4.4 *Let M be a unitary A-module, K be an A-ideal of M and $m \in M$. Then*
(i) $\prec K \cup \{m\} \succ = \{x \in M : x \leq nm \oplus s,\ for\ some\ n \in \mathbb{N}\ and\ s \in K\}$ is an A-ideal of M.
Moreover, if K is a maximal A-ideal of M. Then
(ii) K is a prime A-ideal of M,
(iii) $(K : M)$ is a maximal ideal of A.

Proof. (i) By Proposition 2.3, $\prec K \cup \{m\} \succ$ is an ideal of M. Now, let $a \in A$ and $t \in \prec K \cup \{m\} \succ$. Then $t \leq nm \oplus s$, for some $n \in \mathbb{N}$ and $s \in K$. Since $t \leq nm \oplus s$ and $am \leq m$, by Proposition 2.9(f) and (h),

$$at \leq a(nm \oplus s) \leq \underbrace{am \oplus \cdots \oplus am}_{n\ times} \oplus as \leq \underbrace{m \oplus \cdots \oplus m}_{n\ times} \oplus as = nm \oplus as$$

and so $at \in \prec K \cup \{m\} \succ$. Therefore, $\prec K \cup \{m\} \succ$ is an A-ideal of M.

(ii) Let $xm \in K$, where $x \in A$ and $m \in M$. If $x \notin (K : M)$, then by Lemma 4.3, $x1 \notin K$. Let $m \notin K$. Then we consider $\prec K \cup \{m\} \succ$. By (i), $\prec K \cup \{m\} \succ$ is an A-ideal of M. Since K is maximal, $\prec K \cup \{m\} \succ = M$ and so $1 \in \prec K \cup \{m\} \succ$. Hence, $1 \leq nm \oplus t$, for some $n \in \mathbb{N}$ and $t \in K$. By Proposition 2.9(f) and (h), $x1 \leq x(nm \oplus t) \leq \underbrace{xm \oplus \cdots \oplus xm}_{n\ times} \oplus xt \in K$ and so $x1 \in K$, which is a contradiction. It results that $m \in M$. Therefore, K is a prime A-ideal of A.

(iii) Let I be an ideal of A such that $(K : M) \subsetneq I \subsetneq A$. Let $a \in A$ such that $a \notin (K : M)$. Then there exists $m \in M$ such that $am \notin K$. Now, similar to the proof of (ii), the result will be obtain. ∎

Theorem 4.5 *Let $J(A)$ be the intersection of all maximal ideals of A, $N(A)$ be the intersection of all prime $\cdot$-ideals of A and M be an A-module. Then*
(i) $J(A)M \subseteq Rad(M)$,
(ii) $rad_M(0) \subseteq Rad(M)$
(iii) $N(A)M \subseteq rad_M(0)$.

Proof. (i) Let N be a maximal A-ideal of M. Then by Lemma 4.4 (ii, iii), N is a prime A-ideal of M and $(N : M)$ is a maximal ideal of A. Hence,

$$(Rad(M) : M) = (\bigcap_{N \in Max(M)} N : M) = \bigcap_{N \in Max(M)} (N : M) \supseteq J(A)$$

and so $J(A)M \subseteq Rad(M)$.
(ii) By Lemma 4.4(ii), it is clear that $rad_M(0) \subseteq Rad(M)$.
(iii) If $rad_M(0) = M$, then $N(A)M \subseteq rad_M(0)$. Let $rad_M(0) \neq M$ and N be a prime A-ideal of M. If $a \in N(A)$, then by Proposition 3.3 (v), $a \in (P : M)$ and so $aM \subseteq P$, for every prime A-ideal P of M. Hence, $N(A)M \subseteq P$, for every prime A-ideal P of M. Therefore, $N(A)M \subseteq rad_M(0)$. ∎

Theorem 4.6 *Let M be an A-module and L, N be A-ideals of M. Then*
(i) $N \subseteq rad(N)$,
(ii) if $L \subseteq N$, then $rad(L) \subseteq rad(N)$,
(ii) $rad(rad(N) = rad(N)$,
(iv) $rad(N \cap L) \subseteq rad(N) \cap rad(L)$.

Proof. The proof is routine. ∎

Definition 4.7 Let I be an ideal of A. The intersection of all $\cdot$-prime ideals of A including I is denoted by $r_A(I)$ or $r(I)$. If there exists no $\cdot$-prime ideal of A including I, then we let $r_A(I) = A$.

Example 4.8 In Example 3.1, $r(I) = \{0, 1\}$ and $r(\{0\}) = \{0\}$.
(ii) Let $M_2(\mathbb{R})$ be the ring of square matrixes of order 2 with real elements and let 0 be the matrix with all elements 0. If we define the order relation on components

$$A = (a_{ij})_{i,j=1,2} \geq 0 \; if \; and \; only \; if \; a_{ij} \geq 0 \; for \; any \; i, j,$$

then $M_2(\mathbb{R})$ is an l-ring. If $v = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$, then $(M_2(\mathbb{R}), v)$ is an lu-ring and so $A = \Gamma(M_2(\mathbb{R}), v)$ is a PMV-algebra. It is easy to see that $Id(A) = \{\{0\}, A\}$ and $\{0\}$ is not a $\cdot$-prime ideal of A. Then $r(\{0\}) = A$.

Lemma 4.9 *$(\alpha \oplus \beta)a \leq \alpha a \oplus \beta a$, for every $\alpha, \beta, a \in A$.*

Proof. Since $\beta a \leq (\alpha a)' \oplus \beta a$, by Proposition 2.4(i), $(\alpha a) \odot (\beta a)' = ((\alpha a)' \oplus \beta a)' \leq (\beta a)'$ and so $(\alpha a) \odot (\beta a)' + \beta a$ is defined, where " $+$ " is the partial addition on A. Similarly, $\alpha \odot \beta' + \beta$ is defined, too. Consider A as A-module, where $ab = a.b$, for every $a, b \in A$. Then by Lemma 2.9(d) and (g), since $\alpha \odot \beta' \leq \beta'$,

$(\alpha\odot\beta')a\leq\beta' a\leq(\beta a)'$ and so $(\alpha\odot\beta')a+\beta a$ is defined. Now, $\alpha\leq\alpha\vee\beta$ implies that $\alpha a\leq(\alpha\vee\beta)a$. Then $\alpha a\vee\beta a\leq(\alpha\vee\beta)a$ and so by Lemma 2.7(c),

$$(\alpha a)\odot(\beta a)'+\beta a=\alpha a\vee\beta a\leq(\alpha\vee\beta)a=(\alpha\odot\beta'\oplus\beta)a=(\alpha\odot\beta'+\beta)a=(\alpha\odot\beta')a+\beta a.$$

By Lemma 2.7(f), $\alpha a\odot(\beta a)'\leq(\alpha\odot\beta')a$. If we set $\alpha\oplus\beta$ instead of α, then by Lemma 2.9 (g), we have $(\alpha\oplus\beta)a\odot(\beta a)'\leq((\alpha\oplus\beta)\odot\beta')a=(\alpha\wedge\beta')a\leq\alpha a$ and so

$$(\alpha\oplus\beta)a=(\alpha\oplus\beta)a\vee\beta a=(\alpha\oplus\beta)a\odot(\beta a)'\oplus\beta a\leq\alpha a\oplus\beta a. \qquad \blacksquare$$

.

Definition 4.10 $S\subseteq A$ is called a $\cdot$-closed subset of A, if $x.y\in S$, for every $x,y\in S$.

Example 4.11 In Example 3.1, $S=\{1,3\}$ is a $\cdot$-closed subset of A.

Theorem 4.12 *Let I be an ideal of A and S be a $\cdot$-closed subset of A such that $I\cap S=\emptyset$. Then there exists a $\cdot$-prime ideal P of A such that $I\subseteq P$ and $P\cap S=\emptyset$.*

Proof. Let $T=\{J: J\ is\ an\ ideal\ of\ A, I\subseteq J\ and\ J\cap S=\emptyset\}$. Since $I\in T$, $T\neq\emptyset$. By *Zorn's* Lemma, T has a maximal element P. We show that P is a $\cdot$-prime ideal of A. Let $x.y\in P$ and $x,y\notin P$. Consider $\prec P\cup\{x\}\succ$ and $\prec P\cup\{y\}\succ$. By maximality P, $\prec P\cup\{x\}\succ\cap S\neq\emptyset$ and $\prec P\cup\{y\}\succ\cap S\neq\emptyset$ and so there exist $\alpha\in\prec P\cup\{x\}\succ\cap S$ and $\beta\in\prec P\cup\{y\}\succ\cap S$. Then by Proposition 2.3, there exist $a,b\in P$ and $n,m\in\mathbb{N}\cup\{0\}$ such that $\alpha\leq nx\oplus a$ and $\beta\leq my\oplus b$. By Proposition 2.9 (f,g), $\alpha.\beta\leq(nx\oplus a).(my\oplus b)$. If we consider A as A-module, where $xy=x.y$, for every $x,y\in A$, then the same as Lemma 4.9 for PMV-algebras and by Proposition 2.9 (h),

$$\begin{aligned}\alpha.\beta &\leq (nx\oplus a).my\oplus(nx\oplus a).b\leq nx.my\oplus a.my)\oplus nx.b\oplus a.b\\ &\leq \underbrace{x.y\oplus\cdots\oplus x.y}_{mn\ times}\oplus\underbrace{a.y\oplus\cdots\oplus a.y}_{m\ times}\oplus\underbrace{x.b\oplus\cdots\oplus}_{n\ times}\oplus a.b\in P\end{aligned}$$

and so $\alpha.\beta\in P$. Since S is a $\cdot$-closed subset of A, $\alpha.\beta\in P\cap S$, which is a contradiction. Therefore, P is a $\cdot$-prime ideal of A. $\blacksquare$

Proposition 4.13 *Let I be an ideal of A and $c\in I$. Then $a.c\in I$, for every $a\in A$.*

Proof. The proof is easy. $\blacksquare$

Theorem 4.14 *Let I be an ideal of A. Then*

$$r(I)=\{x\in A: x^n=\underbrace{x.x.\cdots.x}_{n\ times}\in I, for\ some\ n\in\mathbb{N}\}.$$

Proof. Let $T=\{x\in A: x^n=\underbrace{x.x.\cdots.x}_{n\ times}\in I, for\ some\ n\in\mathbb{N}\}$. It is easy to show that $T\subseteq r(I)$. Let $x\in r(I)$. If $x\notin T$, then $x^n\notin I$, for every $n\in\mathbb{N}$. Consider $S=\{x^n\oplus a: n\in\mathbb{N}\cup\{0\}, a\in I\ and\ x^n\leq a'\}$. Let $x^n\oplus a,\ x^m\oplus b\in S$, for $a,b\in I$ and $n,m\in\mathbb{N}$. Since $x^n\leq a'$ and $x^m\leq a'$, x^n+a and x^m+a are defined in A. Then

$$\begin{aligned}(x^n \oplus a).(x^m \oplus b) &= (x^n + a).(x^m + b)\\ &= x^{m+n} + a.x^m + x^n.b + a.b\\ &= x^{n+m} \oplus t \in S,\end{aligned}$$

where by Proposition 4.13, $t \in I$. It results that S is a $\cdot$-closed subset of A. It is easy to see that $S \cap I = \emptyset$. Then by Theorem 4.12, there is a $\cdot$-prime ideal of A such that $I \subseteq P$ and $S \cap P = \emptyset$. Now, since $x \in r(I)$ and $x = x^1 \oplus 0 \in S$, $x \in P \cap S$, which is a contradiction. Therefore, $x \in T$ and therefore $T = r(I)$. ∎

Theorem 4.15 *Let A be unital and $I, I_1, \cdots, I_n$ be ideals of A. Then*
(i) $r(r(I)) = r(I)$,
(ii) $r\left(\bigcap\limits_{k=1}^{n} I_k\right) = \bigcap\limits_{k=1}^{n} r(I_k)$.
(iii) $r(N : M) \subseteq (rad(N) : M)$.

Proof. *(i)* The proof is routine.
(ii) By Theorem 4.14, it is easy to show that $r\left(\bigcap\limits_{k=1}^{n} I_k\right) \subseteq \bigcap\limits_{k=1}^{n} r(I_k)$. Let $x \in \bigcap\limits_{k=1}^{n} r(I_k)$. Then $x \in r(I_k)$, for every $1 \leq i \leq n$ and so there exists $m_k \in \mathbb{N}$ such that $x^{m_k} \in I_k$. Let $m = \max\{m_1, ..., m_n\}$. If we consider A as A-module, where $xy = x.y$, for every $x, y \in A$, then by Proposition 2.9 (j), $x^{m-m_k}.x^{m_k} \leq x^{m_k}$. Since $x^m = x^{m-m_k}.x^{m_k} \leq x^{m_k} \in I_k$, $x^m \in I_k$, for every $1 \leq k \leq n$ and so $x^m \in \bigcap\limits_{k=1}^{n} I_k$. Hence, by Theorem 4.14, $x \in r\left(\bigcap\limits_{k=1}^{n} I_k\right)$. Therefore, $r\left(\bigcap\limits_{k=1}^{n} I_k\right) = \bigcap\limits_{k=1}^{n} r(I_k)$.
(iii) Let P be an arbitrary prime A-ideal of M containing N. Since $N \subseteq P$, by Proposition 3.3 (iv), $(N : M) \subseteq (P : M)$. Hence, $r(N : M) \subseteq (P : M)$ and so

$$r(N : M) \subseteq \bigcap_{N \subseteq P \in Spec(M)} (P : M) = \left(\bigcap_{N \subseteq P \in Spec(M)} P : M\right) = (rad(N) : M). \quad ∎$$

5. Conclusion

The categorical equivalence between MV-algebras and lu-groups leads to the problem of defining a product operation on MV-algebras, in order to obtain structures corresponding to l-rings. In fact, by defining MV-modules, MV-algebras were extended. PMV-algebras are MV-algebras whose product operation "." is defined on the whole MV-algebra. we studied $\cdot$-prime ideals in PMV-algebras, prime A-ideals in MV-modules and presented the definition of radical of a $\cdot$-ideal in PMV-algebra A and characterize it via elements of A. Also, we presented definition of the radical of an A-ideal in MV-modules by prime A-ideals that in [8], was defined by maximal A-ideals. The obtained results in the last sections encouraged us to continue this way in order to introduce the notion of primary decomposition of an A-ideal in MV-modules by prime A-ideals, primary decomposition of a $\cdot$-ideal in PMV-algebras and other results.

References

[1] Chang, C.C., *Algebraic analysis of many-valued logic*, Transactions of the American Mathematical Society, 88 (1958), 467-490.

[2] Chang, C.C., *A new proof of the completeness of the Lukasiewicz axioms*, Transactions of the American Mathematical Society, 93 (1959), 74-80.

[3] Cignoli, R., D'Ottaviano, I.M.L., Mundici, D., *Algebraic Foundations of Many-valued Reasoning*, Kluwer Academic, Dordrecht, 2000.

[4] Di Nola, A., Dvurečenskij, A., *Product MV-algebras*, Multiple-Valued Logics, 6 (2001), 193-215.

[5] Di Nola, A., Flondor, P., Leustean, I., *MV-modules*, Journal of Algebra, 267 (2003), 21-40.

[6] Dvurečenskij, A., *On Partial addition in Pseudo MV-algebras*, Proceedings of the Fourth International Symposium on Economic Informatics, (1999), 952-960.

[7] Forouzesh,F., Eslami, E., Borumand Saeid, A., *On Prime A-ideals in MV-modules*, University Politechnica of Bucharest Scientific Bulletine, 76 (2014), 181-198.

[8] Forouzesh, F., Eslami, E., Borumand Saeid, A., *Radical of A-ideals in MV-modules*, Annals of the Alexandru Ioan Cuza University, Math., 10.2478 (2014), 1-24.

[9] Kroupa, T., *Conditional probability on MV-algebras*, Fuzzy Sets and Systems, 149 (2005), 369-381.

[10] Teheux, B., *Lattice of subalgebras in the finitely generated varieties of MV-algebras*, Discrete Mathemathics, 307 (2007), 2261-2275.

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EOQ MODELS FOR NON-INSTANTANEOUS/INSTANTANEOUS DETERIORATING ITEMS WITH CUBIC DEMAND RATE UNDER INFLATION AND PERMISSIBLE DELAY IN PAYMENTS

K. Rangarajan

Research Scholar
Department of Mathematics, SAS
VIT University
Vellore
India

K. Karthikeyan

Associate Professor
Department of Mathematics, SAS
VIT University
Vellore
India

Abstract. In this article, an attempt is made to develop two inventory models for non-instantaneous deteriorating items and two inventory models for instantaneous deteriorating items with linear deterioration rate and cubic demand rate. That is, the demand rate is a piecewise cubic function of time under inflation and permissible delay in payments. This model supports in minimizing the total inventory cost by finding an optimal replenishment policy; where shortages are allowed, partially backlogged and completely backlogged cases are considered. The backlogging rate is variable and dependent on the waiting time for the next replenishment. Numerical examples are given to establish the analytical results. Sensitivity analysis of the optimal solution with respect to major parameters is carried out and the effects are discussed in detail.

Keywords: inventory, non-instantaneous deterioration, cubic demand, inflation, permissible delay in payments, allowable shortages.

AMS Mathematics Subject Classification: 90B05.

1. Introduction

Inventory system is one of the main streams of the Operation Research, which is essential in business enterprises and industries. Inventory may be considered as accumulation of a product that would be used to satisfy future demands for that product. It needs scientific way of exercising inventory model. Generally, deterioration is defined as the damage, spoilage, dryness, vaporization, etc., that results in the decrease of usefulness of the commodity. Deterioration of goods is a common phenomenon and unavoidable in daily life. Therefore, to control and maintain the inventory of deteriorating items becomes an important factor for decision makers. In all the inventory models for deteriorating items,it is assumed that deterioration starts as soon as the retailer receives the inventory. But most of the goods have a span of maintaining quality or the original condition in real situation. During that period, there was no occurrence of deterioration (e.g., vegetables, fruits, meat, fish and so on). This phenomenon is termed as *non-instantaneous deterioration*. It has been observed that only few researchers considered inventory models for non-instantaneous deteriorating items with inflation and permissible delay in payments simultaneously. They play important role in the optimal order policy and influences the demand of certain products. The inventory model for fashion goods deteriorating at the end of prescribed period was first studied by Whitin [1957]. Then Ghare and Schrader [1963] developed an EOQ model with constant rate of deterioration. Covert and Philip [1973] extended Ghare and Schrader's [1963] model by considering variable rate of deterioration. Further, Shah [1997] extended Covert and Philip's [1973] model by considering shortages.Ouyang et al. [2005] considered an optimal replenishment policy for non-instantaneous deteriorating items with stock dependent and partial backlogging. Again, Ouyang et al. [2006] considered an appropriate inventory model for non- instantaneous deteriorating items with permissible delay in payments. Chung [2008] completed the incomplete proof of Ouyang et al. [2006] model. Geetha and Uthayakumar [2009] proposed an EOQ based model for non-instantaneous deteriorating items with permissible delay in payments. Goyal et al. [2010] proposed an optimal replenishment policies for non-instantaneous deteriorating items stock dependent demand. Soni [2013] extended Goyal et al.[2010] model from two aspects (i) demand rate as multivariate function of price and level of inventory, and (ii) delay in payment permissible. Further Ouyang et al. [2013] extended Soni [2013] model by considering selling those inventories as salvages and all possible replenishment cycle, which may be shorter than the period of non-deterioration. Retailer promotional activity has become prevalent in the business world. Promotional efforts impact the replenishment policy and the sale price of goods. Reza Maihami and Behrooz Karimi [2014] considered the problem of replenishment policy and pricing for non-instantaneous deteriorating items subject to promotional effort and they adopted a price dependent stochastic demand function in which shortages are allowed and partially backlogged. Priyan and Uthayakumar [2015] considered a distributor and a warehouse consisting of a serviceable part and a recoverable part supply chain problem.Chang and Dye[1999] were the first to give a definition for time dependent partial backlogging rate. They considered an EOQ model for deteriorating items with time varying demand and partial backlogging Goyal [1985] was the first to consider the economic order quantity model under conditions of permissible delay in payments. Goyal's[1985] model was extended by Aggarwal and Jaggi [1995] for deteriorating items. Jamal et al. [1997] further extended Aggarwal and Jaggi's [1995] model to consider shortages. Goyal et al.[2005] developed the optimal inventory policies under permissible delay in payments depredating on the ordering quantity. Chang et al. [2015] proposed an inventory system with non-instantaneously deteriorating items in circumstances where the supplier provides the retailer with various trade credits linked to order quantity. Mohsen Lashgari1 et al. [2016] developed an EOQ model with down-stream partial delayed payment and up-stream partial prepayment under three different scenarios: without shortage, with full backordering and with partial backordering. Buzacott [1975] was the first to develop economic order quantity model by considering the effect of inflation. Datta and Pal [1991] studied the effects of inflation and time value of money with linear time dependent demand rate and shortages. Hariga and Ben-Daya [1996] considered optimal time varying lot sizing models underinflationary conditions. Liao et al. [2000] developed an inventory model with deteriorating items under inflation when a delay in payment is permissible. The EOQ model for ameliorating / deteriorating items with time varying demand pattern over a finite planning horizon taking into account the effect of inflation and time value of money was considered by Moon et al. [2005]. Yang et al.[2010] developed an inventory model under inflation for deteriorating items with stock dependent consumption rate and partial backlogging shortages. Singh [2011] considered an EOQ model for items having linear demand under inflation and permissible delay in payments. Yashveer Singh et al. [2014] developed an inflation induced stock dependent demand inventory model with permissible delay in payments. An appropriate inventory model for non-instantaneous deteriorating items with cubic demand rate under inflation and permissible delay in payments is proposed in this article. In this model shortages are allowed and partially backlogged. Two inventory models for non- instantaneous deteriorating items and two inventory models for instantaneous deteriorating items with linear deterioration rate and cubic demand rate, that is, the demand rate is a piecewise cubic function of time under inflation and permissible delay in payments are developed. This models supports in minimizing the total inventory cost by finding an optimal replenishment policy. Partially backlogged and completely backlogged cases are considered for all models. The backlogging rate is variable and dependent on the waiting time for the next replenishment. Numerical examples are given to establish the analytical results. Sensitivity analysis of the optimal solution with respect to major parameters is carried out and the effects are discussed in detail.The rest of the article is organized as follows: In section II, the assumptions and notations, which are used throughout this article, are described. In section III, the mathematical formulation and solution of the model to minimize the total inventory cost is established. Numerical examples for all models are provided in section IV. Sensitivity analysis and their observations are discussed in section V. This is followed by conclusion and future research.

2. Assumptions and notations

The following assumptions are made in developing the model:

1. The demand of the product is declining as a cubic function of time.
2. Replenishment rate is infinite and instantaneous.

3. Lead time is zero.
4. Shortages are allowed and are partially backlogged.
5. The deteriorated units can neither be repaired nor replaced during the cycle time.
6. During the time, the account is not settled; generated sales revenue is deposited in an interest bearing account.At the end of the credit period, the account is settled as well as the buyer pays off all units sold and start paying for the interest charges on the items in stocks.

The following notations have been used in developing the model:

1. $D(t)$: $D\left(t\right) = a + bt{+}ct^2 + dt^3$ is the demand rate, it is a cubic function of time, where a,b,c and d are the positive constants.
2. $I(t)$: Inventory level at any time t, $0 \leq t \leq T$.
3. Q: Order quantity.
4. Q_1: Inventory level at time $t = 0$.
5. Q_2: Shortage of inventory.
6. C_p: Unit purchase cost of an item.
7. p: Unit selling price of an item.
8. C_2: Shortage cost per unit item.
9. C_3: Cost of lost sales per unit item.
10. δ: Lost sales.
11. I_e: Interest earned per year.
12. I_p: Interest paid in stocks per year.
13. R: Inflation Rate.
14. M: Permissible period of delay in setting the accounts with the supplier.
15. T: The time interval between two successive orders.
16. $\theta\left(t\right)$: $\theta\left(t\right) = \theta_1{+}\theta_2 t$ is the deterioration rate of an item, where $\theta_1 > 0$ and $0 < \theta_1 < 1$.
17. HC: Holding Cost per unit (excluding interest charges) is linear function of time $H(t) = \alpha + \beta t$, $\alpha > 0$, $\beta > 0$.
18. TC: Total Cost per unit time.
19. A: Ordering Cost per unit order is known and constant.

3. Mathematical formulation and solution of the model

The instantaneous inventory level $I_1(t)$ at any time t during the cycle time $(0, t_1)$ is governed by the following differential equation

$$\frac{dI_1(t)}{dt} + \theta I_1(t) = -\left(a + bt + ct^2 + dt^3\right), \quad 0 \le t \le t_1 \tag{1}$$

The solution of above equation with boundary condition $I_1(0) = Q_1$ is

$$I_1(t) = Q_1 - (at + \frac{bt^2}{2} + \frac{ct^3}{3} + \frac{dt^4}{4}) \tag{2}$$

The instantaneous inventory level $I_2(t)$ at any time t during the cycle time (t_1,t_2) is governed by the following differential equation

$$\frac{dI_2(t)}{dt} + (\theta_1 + \theta_2 t) I_2(t) = -\left(a + bt + ct^2 + dt^3\right), \quad t_1 \le t \le t_2 \tag{3}$$

The solution of above equation with boundary condition $I_2(t_1) = 0$ is

$$I_2(t) = \begin{bmatrix} a\{(t_2 - t) + \frac{\theta_1}{2}(t_2^2 - 2tt_2 + t^2) + \frac{\theta_2}{6}(t_2^3 - 3t^2 t_2 + 2t^3)\} \\ + b\{\frac{1}{2}(t_2^2 - t^2) + \frac{\theta_1}{6}(2t_2^3 - 3tt_2^2 + t^3) + \frac{\theta_2}{8}(t_2^4 - 2t^2 t_2^2 + t^4)\} \\ + c\{\frac{1}{3}(t_3^2 - t^3) + \frac{\theta_1}{12}(3t_2^4 - 4tt_3^2 + t^4) + \frac{\theta_2}{30}(3t_2^5 - 5t^2 t_2^3 + 2t^5)\} \\ + d\{\frac{1}{4}(t_2^4 - t^4) + \frac{\theta_1}{20}(4t_2^5 - 5tt_2^4 + t^5) + \frac{\theta_2}{24}(2t_2^6 - 3t^2 t_2^4 + t^6)\} \end{bmatrix} \tag{4}$$

Due to continuity of $I(t)$ at $t = t_1$, it follows from equation (2) and (4), which implies that $I_1(t_1) = I_1(t_2)$, we get

$$Q_1 = \begin{bmatrix} a\{(t_2) + \frac{\theta_1}{2}(t_2^2 - 2t_1 t_2 + t_1^2) + \frac{\theta_2}{6}(t_2^3 - 3t_1^2 t_2 + 2t_1^3)\} \\ + b\{\frac{1}{2}(t_2^2) + \frac{\theta_1}{6}(2t_2^3 - 3t_1 t_2^2 + t_1^3) + \frac{\theta_2}{8}(t_2^4 - 2t_1^2 t_2^2 + t_1^4)\} \\ + c\{\frac{1}{3}(t_3^2) + \frac{\theta_1}{12}(3t_2^4 - 4t_1 t_3^2 + t_1^4) + \frac{\theta_2}{30}(3t_2^5 - 5t_1^2 t_2^3 + 2t_1^5)\} \\ + d\{\frac{1}{4}(t_2^4) + \frac{\theta_1}{20}(4t_2^5 - 5t_1 t_2^4 + t_1^5) + \frac{\theta_2}{24}(2t_2^6 - 3t_1^2 t_2^4 + t_1^6)\} \end{bmatrix} \tag{5}$$

$$I_1(t) = \begin{bmatrix} a\{(t_2 - t) + \frac{\theta_1}{2}(t_2^2 - 2t_1 t_2 + t_1^2) + \frac{\theta_2}{6}(t_2^3 - 3t_1^2 t_2 + 2t_1^3)\} \\ + b\{\frac{1}{2}(t_2^2 - t^2) + \frac{\theta_1}{6}(2t_2^3 - 3t_1 t_2^2 + t_1^3) + \frac{\theta_2}{8}(t_2^4 - 2t_1^2 t_2^2 + t_1^4)\} \\ + c\{\frac{1}{3}(t_3^2 - t^3) + \frac{\theta_1}{12}(3t_2^4 - 4t_1 t_3^2 + t_1^4) + \frac{\theta_2}{30}(3t_2^5 - 5t_1^2 t_2^3 + 2t_1^5)\} \\ + d\{\frac{1}{4}(t_2^4 - t^4) + \frac{\theta_1}{20}(4t_2^5 - 5t_1 t_2^4 + t_1^5) + \frac{\theta_2}{24}(2t_2^6 - 3t_1^2 t_2^4 + t_1^6)\} \end{bmatrix} \tag{6}$$

During the shortage period (t_2, T), the demand rate at time "t" is partially backlogged at rate of $e^{-\delta(T-t)}(a + bt + ct^2 + dt^3)$.

The instantaneous inventory level $I_3(t)$ at any time t during the cycle time (t_2,T) is governed by the following differential equation

$$\frac{dI_3(t)}{dt} = -e^{-\delta(T-t)}\left(a + bt + ct^2 + dt^3\right), \quad t_2 \le t \le T. \tag{7}$$

The solution of the above equation with the boundary conditions $I_3(t_2) = 0$ and $I_3(T) = -Q_2$ is

$$I_3(t) = \begin{bmatrix} a\{(t_2 - t)(1 - \delta T) + \frac{\delta}{2}(t_2^2 - t^2)\} \\ + b\{\frac{1}{2}(t_2^2 - t^2)(1 - \delta T) + \frac{\delta}{3}(t_2^3 - t^3)\} \\ + c\{\frac{1}{3}(t_3^2 - t^3)(1 - \delta T) + \frac{\delta}{4}(t_2^4 - t^4)\} \\ + d\{\frac{1}{4}(t_2^4 - t^4)(1 - \delta T) + \frac{\delta}{5}(t_2^5 - t^5)\} \end{bmatrix}, \quad t_2 \le t \le T \tag{8}$$

$$Q_2 = \begin{bmatrix} a\{(T - t_2)(1 - \delta T) + \frac{\delta}{2}(T^2 - t_2^2)\} \\ + b\{\frac{1}{2}(T^2 - t_2^2)(1 - \delta T) + \frac{\delta}{3}(T^3 - t_2^3)\} \\ + c\{\frac{1}{3}(T^3 - t_2^3)(1 - \delta T) + \frac{\delta}{4}(T^4 - t_2^4)\} \\ + d\{\frac{1}{4}(T^4 - t_2^4)(1 - \delta T) + \frac{\delta}{5}(T^5 - t_2^5)\} \end{bmatrix}, \quad t_2 \le t \le T. \tag{9}$$

The optimum order quantity is given by

$$I(0) = Q = Q_1 + Q_2. \tag{10}$$

The Total Cost (TC) per unit time consists of the following costs:

1. *Ordering Cost:*

$$OC = \frac{A}{T} \tag{11}$$

2. *Holding Cost*:

$$HC = \frac{1}{T}\int_0^{t_1} (\alpha + \beta t) I(t)\, dt \text{ (see appendix 1)} \tag{12}$$

3. *Deterioration Cost*: $DC = \frac{C_p}{T}\int_0^{t_2} D(t)e^{-RT} dt$

$$DC = \frac{C_p}{T}\begin{bmatrix} a\left\{ \begin{array}{c} t_1 + \frac{\theta_1}{2}(t_2^2 - 2t_1t_2 + t_1^2) \\ + \frac{\theta_2}{6}(t_2^3 - 3t_1^2t_2 + 2t_1^3) + \frac{R}{2}(t_2^2 - t_1^2) \end{array} \right\} \\ + b\left\{ \begin{array}{c} \frac{t_2^2}{2} + \frac{\theta_1}{6}(2t_2^3 - 3t_1t_2^2 + t_1^3) \\ + \frac{\theta_2}{8}(t_2^4 - 2t_1^2t_2^2 + t_1^4) + \frac{R}{2}(t_2^3 - t_1^3) \end{array} \right\} \\ + c\left\{ \begin{array}{c} \frac{t_3^2}{3} + \frac{\theta_1}{12}(3t_2^4 - 4t_1t_3^2 + t_1^4) \\ + \frac{\theta_2}{30}(3t_2^5 - 5t_1^2t_2^3 + 2t_1^5) + \frac{R}{2}(t_2^4 - t_1^4) \end{array} \right\} \\ + d\left\{ \begin{array}{c} \frac{t_2^4}{4} + \frac{\theta_1}{20}(4t_2^5 - 5t_1t_2^4 + t_1^5) \\ + \frac{\theta_2}{24}(2t_2^6 - 3t_1^2t_2^4 + t_1^6) + \frac{R}{2}(t_2^5 - t_1^5) \end{array} \right\} \end{bmatrix} \tag{13}$$

4. *Shortage Cost*:

$$SC = \frac{-C_2}{T}\int_{t_2}^{T} I_3(t)e^{-RT} dt \quad \text{(see appendix 2)} \tag{14}$$

5. *Cost due to lost sales*: $CLS = \frac{C_3}{T}\int_{t_2}^{T} D(t)\left(1 - e^{-\delta(T-t)}\right)e^{-RT}dt$

$$CLS = \frac{C_3}{T}\begin{bmatrix} a\left(T^2 - Tt_2\right) \\ +\frac{1}{2}\begin{Bmatrix} bT^3 - bTt_2^2 - aT^2 \\ +at_2^2 - aRT^3 + aRt_2^2T \end{Bmatrix} \\ +\frac{1}{3}\begin{Bmatrix} cT^4 - cTt_2^3 - bT^3 + bt_2^3 \\ -bRT^4 + bRTt_2^3 + aRT^3 - aRt_2^3 \end{Bmatrix} \\ +\frac{1}{4}\begin{Bmatrix} dT^5 - dTt_2^4 - cT^4 + ct_2^4 \\ -cRT^5 + cRTt_2^4 + bRT^4 - bRt_2^4 \end{Bmatrix} \\ +\frac{1}{5}\left(dRT^6 - dRt_2^6\right) \end{bmatrix} \tag{15}$$

To determine the interest earned and interest payable, there will be three cases as follows:

Case I: $0 \leq M < t_1$

In this case, the retailer can earn interest on revenue generated from the sales up to M. Although, he has to settle the accounts at M, for that he has to arrange money at some specified rate of interest in order to get his remaining stocks financed for the period M to t_1.

1. *Interest earned per cycle:* $InE_1 = \frac{pI_e}{T}\int_0^M D(t)e^{-RT}dt$

$$InE_1 = \frac{pI_e}{T}\left[\frac{aM^2}{2} + \frac{bM^3}{3} + \frac{cM^4}{4} + \frac{dM^5}{5} - R\left(\frac{aM^3}{3} + \frac{bM^4}{4} + \frac{cM^5}{5} + \frac{dM^6}{6}\right)\right] \tag{16}$$

2. *Interest payable per cycle for the inventory not sold after the due period M is*

$$InP_1 = \frac{C_pI_p}{T}\int_M^{t_2} I(t)e^{-RT}dt \quad \text{(see appendix 3)} \tag{17}$$

The Total Cost per unit time is given by

$$TC_1 = OC + HC + DC + SC + CLS + InP_1 - InE_1.$$

Our objective is to minimize the total cost.

The necessary condition for total cost to be minimized are

(i) $\dfrac{\partial(TC_1)}{\partial t_2} = 0$ and

(ii) $\dfrac{\partial^2(TC_1)}{\partial t_2^2} > 0.$

The optimal value of t_2 can be obtained by using the condition (i). Condition (ii) is also satisfied for the value of t_2 obtained from condition (i). The value of t_2 is used to find the optimal values Q and TC_1. Since equation (i) is nonlinear, it is solved by using MATLAB software.

Case II: $t_1 \leq M < t_2$

In this case, the retailer can earn interest on revenue generated from the sales up to M. Although, he has to settle the accounts at M, for that he has to arrange money at some specified rate of interest in order to get his remaining stocks financed for the period M to t_2.

1. *Interest earned per cycle*: $InE_2 = \frac{pI_e}{T}\int_0^M D(t)e^{-RT}dt$

$$InE_2 = \frac{pI_e}{T}\left[\frac{aM^2}{2}+\frac{bM^3}{3}+\frac{cM^4}{4}+\frac{dM^5}{5}-R\left(\frac{aM^3}{3}+\frac{bM^4}{4}+\frac{cM^5}{5}+\frac{dM^6}{6}\right)\right] \tag{18}$$

2. *Interest payable per cycle for the inventory not sold after the due period M is*

$$InP_2 = \frac{C_pI_p}{T}\int_M^{t_2} I(t)e^{-RT}dt \quad \text{(see appendix 4)} \tag{19}$$

The Total Cost per unit time is given by

$$TC_2 = OC + HC + DC + SC + CLS + InP_2 - InE_2.$$

Our objective is to minimize the total cost.

The necessary condition for total cost to be minimized are

(i) $\dfrac{\partial(TC_2)}{\partial t_2} = 0$ and

(ii) $\dfrac{\partial^2(TC_2)}{\partial t_2^2} > 0.$

The optimal value of t_2 can be obtained by using condition (i). Condition (ii) is also satisfied for the value of t_2 obtained from condition (i). The value of t_2 is used to find the optimal values Q and TC_2. Since equation (i) is nonlinear, it is solved by using MATLAB software.

Case III: $t_2 \leq M < T$

In this case, the retailer earns interest on the sales revenue up to the permissible delay period and no interest is payable during this period.

1. *Interest earned per cycle*: $InE_3 = \frac{pI_e}{T}\int_0^M D(t)e^{-RT}dt$

$$InE_3 = \frac{pI_e}{T}\left[\begin{array}{l}\left\{\begin{array}{l}\dfrac{at_2^2}{2}+\dfrac{bt_2^3}{3}+\dfrac{ct_2^4}{4}+\dfrac{dt_2^5}{5}\\ -R\left(\dfrac{at_2^3}{3}+\dfrac{bt_2^4}{4}+\dfrac{ct_2^5}{5}+\dfrac{dt_2^6}{6}\right)\end{array}\right\}\\ \\ +(M-t_2)\left\{\begin{array}{l}\dfrac{at_2^2}{2}+\dfrac{bt_2^3}{3}+\dfrac{ct_2^4}{4}+\dfrac{dt_2^5}{5}\\ -R\left(\dfrac{at_2^3}{3}+\dfrac{bt_2^4}{4}+\dfrac{ct_2^5}{5}+\dfrac{dt_2^6}{6}\right)\end{array}\right\}\end{array}\right] \tag{20}$$

2. *Interest payable per cycle for the inventory not sold after the due period M is*

$$InP_3 = 0 \tag{21}$$

The Total Cost per unit time is given by

$$TC_3 = OC + HC + DC + SC + CLS + InP_3 - InE_3$$

Our objective is to minimize the total cost.

The necessary condition for total cost to be minimized are

$$\text{(i)} \quad \frac{\partial(TC_3)}{\partial t_2} = 0 \text{ and } \quad \text{(ii)} \quad \frac{\partial^2(TC_3)}{\partial t_2^2} > 0.$$

The optimal value of t_2 can be obtained by using condition (i). Condition (ii) is also satisfied for the value of t_2 obtained from condition (i). The value of t_2 is used to find the optimal values Q and TC_3. Since equation (i) is non-linear, it is solved by using MATLAB software.

4. Numerical analysis

MODEL I: Inventory Model for Non Instantaneous Deteriorating Items with partial backlogging

Case I: *Considering* $A = Rs.1000$, $C_p = Rs.25$, $p = Rs.0.15$, $I_p = Rs.0.15$, $I_e = Rs.0.12$, $M = 0.5$ *years,* $\theta_1 = 0.04$, $\theta_2 = 0.04$, $a = 1000$, $b = 500$, $c = 250$, $d = 125$, $\alpha = 5$, $\beta = 0.05$, $C_2 = Rs.8$, $C_3 = Rs.2$, $R = 0.1$, $\delta = 0.8$, $t_1 = 1$ *year,* $T = 3$ *years in appropriate units. Then the optimal value of* $t_2 = 1.8240$, *the optimal total cost* $TC_1 = Rs.24,049$ *and the optimum order quantity* $Q = 7475.9$

Case II: *Considering* $A = Rs.1000$, $C_p = Rs.25$, $p = Rs.0.15$, $I_p = Rs.0.15$, $I_e = Rs.0.12$, $M = 1.5$ *years,* $\theta_1 = 0.04$, $\theta_2 = 0.04$, $a = 1000$, $b = 500$, $c = 250$, $d = 125$, $\alpha = 5$, $\beta = 0.05, C_2 = Rs.8$, $C_3 = Rs.2$, $R = 0.1$, $\delta = 0.8$, $t_1 = 1$ *year,* $T = 3$ *years in appropriate units. Then the optimal value of* $t_2 = 1.7601$, *the optimal total cost* $TC_2 = Rs.18,818$ *and the optimum order quantity* $Q = 7248.3$

Case III: *Considering* $A = Rs.1000$, $C_p = Rs.25$, $p = Rs.0.15$, $I_p = Rs.0.15$, $I_e = Rs.0.12$, $M = 2.5$ *years,* $\theta_1 = 0.04$, $\theta_2 = 0.04$, $a = 1000$, $b = 500$, $c = 250$, $d = 125$, $\alpha = 5$, $\beta = 0.05, C_2 = Rs.8$, $C_3 = Rs.2$, $R = 0.1$, $\delta = 0.8$, $t_1 = 1$ *year,* $T = 3$ *years in appropriate units. Then the optimal value of* $t_2 = 1.5756$, *the optimal total cost* $TC_1 = Rs.14,054$ *and the optimum order quantity* $Q = 6601.5$

MODEL II: Inventory Model for Instantaneous Deteriorating Items with partial backlogging

Case I: *Considering* $A = Rs.1000$, $C_p = Rs.25$, $p = Rs.0.15$, $I_p = Rs.0.15$, $I_e = Rs.0.12$, $M = 0.5$ *years,* $\theta_1 = 0.04$, $\theta_2 = 0.04$, $a = 1000$, $b = 500$, $c = 250$, $d = 125$, $\alpha = 5$, $\beta = 0.05, C_2 = Rs.8$, $C_3 = Rs.2$, $R = 0.1$, $\delta = 0.8$, $t_1 = 0$, $T = 3$ *years in appropriate units. Then the optimal value of* $t_2 = 1.8484$, *the optimal total cost* $TC_1 = Rs.14,868$ *and the optimum order quantity* $Q = 7737.7$

Case II: *Considering* $A = Rs.1000$, $C_p = Rs.25$, $p = Rs.0.15$, $I_p = Rs.0.15$, $I_e = Rs.0.12$, $M = 1.5$ *years,* $\theta_1 = 0.04$, $\theta_2 = 0.04$, $a = 1000$, $b = 500$, $c = 250$, $d = 125$, $\alpha = 5$, $\beta = 0.05, C_2 = Rs.8$, $C_3 = Rs.2$, $R = 0.1$, $\delta = 0.8$, $t_1 = 0$, $T = 3$ *years in appropriate units. Then the optimal value of* $t_2 = 1.7913$, *the optimal total cost* $TC_2 = Rs.9,535.5$ *and the optimum order quantity* $Q = 7521.8$

Case III: *Considering* $A = Rs.1000$, $C_p = Rs.25$, $p = Rs.0.15$, $I_p = Rs.0.15$, $I_e = Rs.0.12$, $M = 2.5$ *years,* $\theta_1 = 0.04$, $\theta_2 = 0.04$, $a = 1000$, $b = 500$, $c = 250$, $d = 125$, $\alpha = 5$, $\beta = 0.05, C_2 = Rs.8$, $C_3 = Rs.2$, $R = 0.1$, $\delta = 0.8$, $t_1 = 0$, $T = 3$ *years in appropriate units. Then the optimal value of* $t_2 = 1.6393$, *the optimal total cost* $TC_3 = Rs.4,478.7$ *and the optimum order quantity* $Q = 6955.8$

MODEL III: Inventory Model for Non Instantaneous Deteriorating Items with complete backlogging

Case I: *Considering* $A = Rs.1000$, $C_p = Rs.25$, $p = Rs.0.15$, $I_p = Rs.0.15$, $I_e = Rs.0.12$, $M = 0.5$ *years,* $\theta_1 = 0.04$, $\theta_2 = 0.04$, $a = 1000$, $b = 500$, $c = 250$, $d = 125$, $\alpha = 5$, $\beta = 0.05, C_2 = Rs.8$, $C_3 = Rs.2$, $R = 0.1$, $\delta = 1$, $t_1 = 1$ *year,* $T = 3$ *years in appropriate units. Then the optimal value of* $t_2 = 1.8480$, *the optimal total cost* $TC_1 = Rs.23,124$, *and the optimum order quantity* $Q = 6920.4$

Case II: *Considering* $A = Rs.1000$, $C_p = Rs.25$, $p = Rs.0.15$, $I_p = Rs.0.15$, $I_e = Rs.0.12$, $M = 1.5$ *years,* $\theta_1 = 0.04$, $\theta_2 = 0.04$, $a = 1000$, $b = 500$, $c = 250$, $d = 125$, $\alpha = 5$, $\beta = 0.05, C_2 = Rs.8$, $C_3 = Rs.2$, $R = 0.1$, $\delta = 1$, $t_1 = 1year$, $T = 3$ *years in appropriate units. Then the optimal value of* $t_2 = 1.7933$, *the optimal total cost* $TC_2 = Rs.17,775$ *and the optimum order quantity* $Q = 6679.9$

Case III: *Considering* $A = Rs.1000$, $C_p = Rs.25$, $p = Rs.0.15$, $I_p = Rs.0.15$, $I_e = Rs.0.12$, $M = 2.5$ *years,* $\theta_1 = 0.04$, $\theta_2 = 0.04$, $a = 1000$, $b = 500$, $c = 250$, $d = 125$, $\alpha = 5$, $\beta = 0.05, C_2 = Rs.8$, $C_3 = Rs.2$, $R = 0.1$, $\delta = 1$, $t_1 = 1$ *year,* $T = 3$ *years in appropriate units. Then the optimal value of* $t_2 = 1.6568$, *the optimal total cost* $TC_3 = Rs.12,666$, *and the optimum order quantity* $Q = 6084.5$

MODEL IV: Inventory Model for Instantaneous Deteriorating Items with complete backlogging

Case I: *Considering* $A = Rs.1000$, $C_p = Rs.25$, $p = Rs.0.15$, $I_p = Rs.0.15$, $I_e = Rs.0.12$, $M = 0.5$ *years,* $\theta_1 = 0.04$, $\theta_2 = 0.04$, $a = 1000$, $b = 500$, $c = 250$, $d = 125$, $\alpha = 5$, $\beta = 0.05$, $C_2 = Rs.8$, $C_3 = Rs.2$, $R = 0.1$, $\delta = 1$, $t_1 = 0$, $T = 3$ *years in appropriate units. Then the optimal value of* $t_2 = 1.8695$, *the optimal total cost* $TC_1 = Rs.13,986$ *and the optimum order quantity* $Q = 7194.1$

Case II: *Considering* $A = Rs.1000$, $C_p = Rs.25$, $p = Rs.0.15$, $I_p = Rs.0.15$, $I_e = Rs.0.12$, $M = 1.5$ *years,* $\theta_1 = 0.04$, $\theta_2 = 0.04$, $a = 1000$, $b = 500$, $c = 250$, $d = 125$, $\alpha = 5$, $\beta = 0.05, C_2 = Rs.8$, $C_3 = Rs.2$, $R = 0.1$, $\delta = 1$, $t_1 = 0$, $T = 3$ *years in appropriate units. Then the optimal value of* $t_2 = 1.8195$, *the optimal total cost* $TC_2 = Rs.8,550.4$ *and the optimum order quantity* $Q = 6963.4$

Case III: *Considering* $A = Rs.1000$, $C_p = Rs.25$, $p = Rs.0.15$, $I_p = Rs.0.15$, $I_e = Rs.0.12$, $M = 2.5$ *years,* $\theta_1 = 0.04$, $\theta_2 = 0.04$, $a = 1000$, $b = 500$, $c = 250$, $d = 125$, $\alpha = 5$, $\beta = 0.05, C_2 = Rs.8$, $C_3 = Rs.2$, $R = 0.1$, $\delta = 1$, $t_1 = 0$, $T = 3$ *years in appropriate units. Then the optimal value of* $t_2 = 1.6756$, *the optimal total cost* $TC_3 = Rs.3,207.7$ *and the optimum order quantity* $Q = 6411.9$

5. Sensitivity analysis

On the basis of the data given in above examples, it is studied the sensitivity analysis by changing the parameters one at a time by and keeping the rest fixed. From Table 1, the following points are observed:

1. with increase in parameters a, β, θ_1 and θ_2 there is corresponding increase in t_2, total cost and total quantity for all cases;

2. with decrease in parameters a, β, θ_1 and θ_2 there is corresponding decrease in t_2, total cost and total quantity for all cases;

3. with increase in parameter α, there is corresponding decrease in t_2, and total quantity for cases I and II and increase in t_2, and total quantity for case III. Also there is corresponding increase in total cost for all cases;

4. with decrease in parameter α, there is corresponding increase in t_2, and total quantity for cases I and II and decrease in t_2, and total quantity for case III. Also there is corresponding decrease in total cost for all cases;

5. with increase in parameters M and R there is corresponding decrease in t_2, total cost and total quantity for all cases;

6. with decrease in parameters M and R there is corresponding increase in t_2, total cost and total quantity for all cases;
7. with increase in parameter δ, there is corresponding increase in t_2 for all cases and decrease in total cost and total quantity for all cases;
8. with decrease in parameter δ, there is corresponding decrease in t_2 for all cases and increase in total cost and total quantity for all cases.

From Tables 2, 3 and 4, the following points are observed:

1. with increase in parameters a,β, θ_1, θ_2 and δ there is corresponding increase in t_2 and total quantity for all cases and decrease in total cost for all cases;
2. with decrease in parameters a,β, θ_1, θ_2 and δ there is corresponding decrease in t_2 and total quantity for all cases and decrease in total cost for all cases;
3. with increase in parameter α, there is corresponding decrease in t_2, and total quantity for all cases and decrease in total cost for all cases;
4. with decrease in parameter α, there is corresponding increase in t_2, and total quantity for all cases and decrease in total cost for all cases;
5. with increase in parameters M and R there is corresponding decrease in t_2, and total quantity for all cases and decrease in total cost for all cases;
6. with decrease in parameters M and R there is corresponding increase in t_2, and total quantity for all cases and decrease in total cost for all cases.

From the solutions of numerical examples for all models and their cases, the following points are observed:

1. Total cost of Inventory Model for Instantaneous Deteriorating Items with partial backlogging 38.17% for case I, 49.32% for case II and 68.13% for case III are less than the total cost of Inventory Model for Non-Instantaneous Deteriorating Items with partial backlogging.
2. Total cost of Inventory Model for Instantaneous Deteriorating Items with complete backlogging 39.51% for case I, 51.89% for case II and 74.67% for case III are less than the total cost of Inventory Model for Non-Instantaneous Deteriorating Items with complete backlogging.
3. Total cost of Inventory Model for Non-Instantaneous Deteriorating Items with partial backlogging 3.84% for case I, 5.54% for case II and 9.87% for case III are more than the total cost of Inventory Model for Non-Instantaneous Deteriorating Items with complete backlogging.
4. Total cost of Inventory Model for Instantaneous Deteriorating Items with partial backlogging 5.93% for case I, 10.32% for case II and 28.37% for case III are more than the total cost of Inventory Model for Instantaneous Deteriorating Items with complete backlogging.

5. The value of Q for Inventory Model for Instantaneous Deteriorating Items with partial backlogging 3.38% for case I, 3.63% for case II and 5.09% for case III are more than the value of Q for Inventory Model for Non-Instantaneous Deteriorating Items with partial backlogging.
6. The value of Q for Inventory Model for Instantaneous Deteriorating Items with complete backlogging 3.8% for case I, 4.06% for case II and 5.10% for case III are more than the value of Q for Inventory Model for Non-Instantaneous Deteriorating Items with complete backlogging.
7. The value of Q for Inventory Model for Non-Instantaneous Deteriorating Items with partial backlogging 7.43% for case I, 7.84% for case II and 7.83% for case III are more than the value of Q for Inventory Model for Non-Instantaneous Deteriorating Items with complete backlogging.
8. The value of Q for Inventory Model for Instantaneous Deteriorating Items with partial backlogging 7.02% for case I, 7.42% for case II and 7.81% for case III are more than the value of Q for Inventory Model for Instantaneous Deteriorating Items with complete backlogging.
9. The value of for Inventory Model for Instantaneous Deteriorating Items with partial backlogging 1.32% for case I, 1.74% for case II and 3.88% for case III are more than the value of for Inventory Model for Non-Instantaneous Deteriorating Items with partial backlogging.
10. The value of for Inventory Model for Instantaneous Deteriorating Items with complete backlogging 1.15% for case I, 1.43% for case II and 1.12% for case III are more than the value of for Inventory Model for Non-Instantaneous Deteriorating Items with complete backlogging.
11. The value of for Inventory Model for Non-Instantaneous Deteriorating Items with partial backlogging 1.29% for case I, 1.85% for case II and 4.9% for case III are less than the value of for Inventory Model for Non-Instantaneous Deteriorating Items with complete backlogging.
12. The value of for Inventory Model for Instantaneous Deteriorating Items with partial backlogging 1.12% for case I, 1.54% for case II and 2.16% for case III are more than the value of for Inventory Model for Instantaneous Deteriorating Items with complete backlogging.

6. Conclusion

An inventory model for non-instantaneous deteriorating items with cubic demand rate under inflation and permissible delay in payments is proposed in this article. In this model shortages are allowed and partially backlogged. Two inventory models for non- instantaneous deteriorating items and two inventory models for instantaneous deteriorating items with linear deterioration rate and cubic demand rate under inflation and permissible delay in payments are developed. This model supports in minimizing the total inventory cost by finding an optimal replenishment policy. Partially backlogged and completely backlogged cases are considered for all models. From the numerical examples and sensitivity analysis, the following conclusions are obtained for all cases:

(i) *Total cost of Inventory Model for Instantaneous Deteriorating Items with partial/complete backlogging is less than the total cost of Inventory Model for Non- Instantaneous Deteriorating Items with partial /complete backlogging.*

(ii) *Total cost of Inventory Model for Non-instantaneous/Instantaneous Deteriorating Items with partial backlogging is more than the total cost of Inventory Model for Non-instantaneous/Instantaneous Deteriorating Items with complete backlogging.*

(iii) *The value of Q for Inventory Model for Instantaneous Deteriorating Items with partial /complete backlogging is more than the value of Q for Inventory Model for Non- Instantaneous Deteriorating Items with partial /complete backlogging.*

(iv) *The value of Q for Inventory Model for Non-instantaneous/Instantaneous Deteriorating Items with partial backlogging is more than the value of Q for Inventory Model for Non-instantaneous/ Instantaneous Deteriorating Items with complete backlogging.*

(v) *The value of occurrence of shortage period for Inventory Model for Instantaneous Deteriorating Items with partial/complete backlogging is more than the value of occurrence of shortage period for Inventory Model for Non-Instantaneous Deteriorating Items with partial/complete backlogging.*

(vi) *The value of occurrence of shortage period for Inventory Model for Non-instantaneous/Instantaneous Deteriorating Items with partial backlogging is more than the value of occurrence of shortage period for Inventory Model for Noninstantaneous/Instantaneous Deteriorating Items with complete backlogging.*

In future, the proposed model can be extended in several ways. For instance, this inventory model may be extended incorporating with various considerations like fuzzy environment, probabilistic demand rates, probabilistic deterioration rate including shortages, price discount, quantity discount and others.

References

[1] AGGARWAL, S.P., JAGGI, C.K., *Ordering policies for deteriorating items under permissible delay in payments*, J. Oper. Res. Soc., 46 (1995), 658-662.

[2] BUZACOTT, J.A., *Economic order quantities with inflation* , Operation Research., 26 (1975), 553-558.

[3] CHANG, H.J., DYE, C.Y., *An EOQ model for deteriorating items with time varying demand and partial backlogging* , Journal of the Operational Research Society, 50 (1999), 1175-1182.

[4] CHUN-TAO CHANG., MEI-CHUAN CHENG., LIANG-YUH OUYANG., *Optimal pricing and ordering policies for non-instantaneously deteriorating items under order-size-dependent delay in payments*, Applied Mathematical Modelling, 39 (2015), 747-763.

[5] CHERN, M.S., YANG, H.L., TENG J.T., PAPACHRISTOS, S., *Partial backlogging inventory lot size models for deteriorating items with fluctuating demand under inflation*, Euro. J. Oper. Res., 191 (2008), 127-141.

[6] CHUNG, K.J., GOYAL, S.K., HUANG, Y.F.,*The optimal inventory policies under permissible delay in payments depredating on the ordering quantity*, International Journal of production economics, 95 (2005), 203-213.

[7] CHUNG, K.J., *A complete proof on the solution procedure for non-instantaneous deteriorating items with permissible delay in payment*, Computers and Industrial Engineering, 56 (2008), 267-273.

[8] COVERT, R.P., PHILIP, G.C., *An EOQ model for items with Weibull distribution deterioration*, American Institute of Industrial Engineering Transactions, 5 (1973), 323-328.

[9] DATTA, T.K., PAL, A.K., *Effects of inflation and time value of money with linear time dependent demand rate and shortages*, Euro. J. Oper. Res, 52 (1991), 1-8.

[10] GEETHA, K.V., UTHAYAKUMAR,R., *Economic design of an inventory policy for non-instantaneous deteriorating items under permissible delay in payments* , Journal of computational and applied mathematics, 233 (2009), 2492-2505.

[11] GHARE, P.M., SCHRADER, G.F., *A model for exponentially decaying inventories*, J. Indus. Engg, 14 (1963), 238-243.

[12] GOYAL, S.K.,*Economic order quantity under conditions of permissible delay in payments*, J. O.R. Soc, 36 (1985), 335-338.

[13] HARDIK N SONI., *Optimal replenishment policies for non instantaneous deteriorating items with price and stock sensitive demand under permissible delay in payments*, Int. J. Production Economics, 146 (2013), 259-268.

[14] HARIGA, M.A., and BEN-DAYA, M., *Optimal time varying lot sizing models under inflationary conditions* , Euro. J. of Oper. Res, 89 (1996), 313-325.

[15] JAMAL, A.M.M., SARKER, B.R., WANG, S., *An ordering policy for deteriorating items with allowable shortages and permissible delay in payment*, Oper. Res. Soc, 48 (1997), 826-833.

[16] LIAO, H.C., TSAI, C.H., SU, T.C., *An inventory model with deteriorating items under inflation when a delay in payment is permissible*, Int. J. Prod. Eco, 63 (2000), 207-214.

[17] MOHSEN LASHGARI., ATA ALLAH TALEIZADEH., ABBAS AHMADI., *Partial up-stream advanced payment and partial down-stream delayed payment in a three-level supply chain*, Annals of Operations Research, 238 (2016), 329-354.

[18] MOON, I., GIRI, B.C., KO, B., *Economic order quantity model for ameliorating deteriorating items under inflation and time discounting*, Euro. J. Oper. Res, 162 (2005), 773-785.

[19] PRIYAN.S., UTHAYAKUMAR.R., *Two-echelon multi-product multi-constraint product returns inventory model with permissible delay in payments and variable lead time*, Journal of Manufacturing Systems, 36 (2015), 244262.

[20] REZAI MAIHAMI., BEHROOZ KARIMI., *Optimizing the pricing and replenishment policy for non-instantaneous deteriorating items with stochastic demand and promotional efforts*, Computers and Operations Research, 51 (2014), 302312.

[21] SHAK, Y.K., *An order level lot size inventory for deteriorating items* American Institute of Industrial Engineering Transactions, 9 (1997), 108-112.

[22] SINGH, S., *An economic order quantity model for items having linear demand under inflation and permissible delay in payments* , International J. of Computer Applications, 33 (2011), 48- 55.

[23] YANG, H.L., TENG, J.T., CHERN, M.S., *An inventory model under inflation for deteriorating items with stock dependent consumption rate and partial backlogging shortages* , International J. of Production Economics, 123 (2010), 8-19.

[24] YASHVEER SINGH., MALIK., SATHISH KUMAR., *An inflation induced stock-dependent demand inventory model with permissible delay in payments*, International journal of computer applications, 96 (2014), 14-18.

[25] WHITIN, T.M., *Theory of inventory management* , Princeton University Press, Princeton, NJ, (1957), 62-72.

[26] WU, K.S., OUYANG, L. Y., YANG, C.T., *An optimal replenishment policy for non-instantaneous deteriorating items with stock dependent demand and partial backlogging*, International J. of Production Economics, 101 (2005), 369-384.

[27] WU, K.S., OUYANG, L. Y., YANG, C.T., *A study on an non-instantaneous deteriorating items with permissible delay in payments* , Computers and industrial engineering, 51 (2006), 637-651.

[28] WU, J., SKOURI, K., OUYANG, L. Y., TENG, J.T., *A note on optimal replenishment policies for non instantaneous deteriorating items with price and stock sensitive demand under permissible delay in payments*, Int. J. Production Economics, 155 (2013), 324-329.

Appendix 1

$$HC = \frac{1}{T}\left[\begin{array}{l}
\alpha\left[\begin{array}{l}
a\left\{\begin{array}{c}\frac{t_2^2}{2}+\frac{\theta_1}{6}\left(t_2^3-3t_1^2t_2+2t_1^3\right)\\ +\frac{\theta_2}{12}\left(t_2^4-4t_1^3t_2+3t_1^4\right)\end{array}\right\}\\
b\left\{\begin{array}{c}\frac{t_2^3}{3}+\frac{\theta_1}{8}\left(t_2^4-2t_1^2t_2^2+t_1^4\right)\\ +\frac{\theta_2}{30}\left(2t_2^5-5t_1^3t_2^2+3t_1^5\right)\end{array}\right\}\\
c\left\{\begin{array}{c}\frac{t_2^4}{4}+\frac{\theta_1}{30}\left(3t_2^5-5t_1^2t_2^3+2t_1^5\right)\\ +\frac{\theta_2}{18}\left(t_2^6-2t_1^3t_2^3+t_1^6\right)\end{array}\right\}\\
d\left\{\begin{array}{c}\frac{t_2^5}{5}+\frac{\theta_1}{24}\left(2t_2^6-3t_1^2t_2^4+t_1^6\right)\\ +\frac{\theta_2}{84}\left(4t_2^7-7t_1^3t_2^4+3t_1^7\right)\end{array}\right\}
\end{array}\right]\\
+(\beta-\alpha R)\left[\begin{array}{l}
a\left\{\begin{array}{c}\frac{t_2^3}{6}+\frac{\theta_1}{24}\left(t_2^4-4t_1^3t_2+3t_1^4\right)\\ +\frac{\theta_2}{40}\left(t_2^5-5t_1^4t_2+4t_1^5\right)\end{array}\right\}\\
b\left\{\begin{array}{c}\frac{t_2^4}{8}+\frac{\theta_1}{60}\left(2t_2^5-5t_1^3t_2^2+3t_1^5\right)\\ +\frac{\theta_2}{48}\left(t_2^6-3t_1^4t_2^2+2t_1^6\right)\end{array}\right\}\\
c\left\{\begin{array}{c}\frac{t_2^5}{10}+\frac{\theta_1}{36}\left(t_2^6-2t_1^3t_2^3+t_1^6\right)\\ +\frac{\theta_2}{168}\left(3t_2^7-7t_1^5t_2^2+5t_1^7\right)\end{array}\right\}\\
d\left\{\begin{array}{c}\frac{t_2^6}{12}+\frac{\theta_1}{168}\left(4t_2^7-7t_1^3t_2^4+3t_1^7\right)\\ +\frac{\theta_2}{64}\left(t_2^8-2t_1^4t_2^4+t_1^8\right)\end{array}\right\}
\end{array}\right]\\
-\beta R\left[\begin{array}{l}
a\left\{\begin{array}{c}\frac{t_2^4}{12}+\frac{\theta_1}{60}\left(t_2^5-5t_1^4t_2+4t_1^5\right)\\ +\frac{\theta_2}{90}\left(t_2^6-6t_1^5t_2+5t_1^6\right)\end{array}\right\}\\
b\left\{\begin{array}{c}\frac{t_2^5}{15}+\frac{\theta_1}{72}\left(t_2^6-3t_1^4t_2^2+2t_1^6\right)\\ +\frac{\theta_2}{210}\left(2t_2^7-7t_1^5t_2^2+5t_1^7\right)\end{array}\right\}\\
c\left\{\begin{array}{c}\frac{t_2^6}{18}+\frac{\theta_1}{252}\left(3t_2^7-7t_1^4t_2^3+4t_1^7\right)\\ +\frac{\theta_2}{360}\left(3t_2^8-8t_1^5t_2^3+5t_1^8\right)\end{array}\right\}\\
d\left\{\begin{array}{c}\frac{t_2^7}{21}+\frac{\theta_1}{96}\left(t_2^8-2t_1^4t_2^4+t_1^7\right)\\ +\frac{\theta_2}{540}\left(4t_2^9-9t_1^5t_2^4+5t_1^9\right)\end{array}\right\}
\end{array}\right]
\end{array}\right]$$

Appendix 2

$$SC = \frac{C_2}{T}\left[\begin{array}{l}
-\left[\begin{array}{l}
\frac{a}{2}\left\{\begin{array}{c}\left(2Tt_2-t_2^2-T^2\right)\\ -\delta\left(2T^2t_2-2T^2t_2-T^3\right)\end{array}\right\}\\
\frac{1}{6}\left\{\begin{array}{c}(a\delta+b)\left(3Tt_2^2-2t_2^3-T^3\right)\\ -b\delta\left(3T^2t_2^2-2Tt_2^3-T^4\right)\end{array}\right\}\\
\frac{1}{12}\left\{\begin{array}{c}(b\delta+c)\left(4Tt_2^3-3t_2^4-T^4\right)\\ -c\delta\left(4T^2t_2^3-3Tt_2^4-T^5\right)\end{array}\right\}\\
\frac{1}{20}\left\{\begin{array}{c}(c\delta+d)\left(3Tt_2^2-2t_2^6-T^6\right)\\ -d\delta\left(3T^3t_2^4-2Tt_2^6-T^7\right)\end{array}\right\}\\
\qquad +\frac{d\delta}{70}\left(7T^2t_2^5-5t_2^7-2T^7\right)
\end{array}\right]\\
+R\left[\begin{array}{l}
\frac{a}{6}\left\{\begin{array}{c}\left(3T^2t_2-t_2^3-2T^3\right)\\ -\delta\left(3T^3t_2-T^3t_2-2T^4\right)\end{array}\right\}\\
\frac{1}{8}\left\{\begin{array}{c}(a\delta+b)\left(2T^2t_2^2-t_2^4-T^4\right)\\ -b\delta\left(2T^3t_2^2-Tt_2^4-T^5\right)\end{array}\right\}\\
\frac{1}{30}\left\{\begin{array}{c}(b\delta+c)\left(5T^2t_2^3-3t_2^5-2T^5\right)\\ -c\delta\left(5T^3t_2^3-3Tt_2^5-2T^6\right)\end{array}\right\}\\
\frac{1}{24}\left\{\begin{array}{c}(c\delta+d)\left(3T^2t_2^4-2t_2^6-T^6\right)\\ -d\delta\left(3T^3t_2^4-2Tt_2^6-T^7\right)\end{array}\right\}\\
\qquad +\frac{d\delta}{70}\left(7T^2t_2^5-5t_2^7-2T^7\right)
\end{array}\right]
\end{array}\right]$$

Appendix 3

$$InP_1 = \frac{C_pI_p}{T}\left[\begin{array}{c} \left[\begin{array}{l} a\left\{\begin{array}{c} \frac{1}{2}\left(-2Mt_2+M^2+t_2^2\right) \\ +\frac{\theta_1}{6}\left(t_2^3-3t_1^2t_2+2t_1^3-3M\left(t_2^2-2t_1t_2+t_1^2\right)\right) \\ +\frac{\theta_2}{12}\left(t_2^4-4t_1^3t_2+3t_1^4-2M\left(t_2^3-3t_1^2t_2+2t_1^3\right)\right) \end{array}\right\} \\ +b\left\{\begin{array}{c} \frac{1}{6}\left(-3Mt_2^2+M^3+2t_2^3\right) \\ +\frac{\theta_1}{24}\left(3t_2^4-6t_1^2t_2^2+3t_1^4-4M\left(2t_2^3-3t_1t_2^2+t_1^3\right)\right) \\ +\frac{\theta_2}{120}\left(8t_2^5-20t_1^3t_2^2+12t_1^5-15M\left(t_2^4-2t_1^2t_2+t_1^4\right)\right) \end{array}\right\} \\ +c\left\{\begin{array}{c} \frac{1}{12}\left(-4Mt_2^3+M^4+3t_2^4\right) \\ +\frac{\theta_1}{60}\left(6t_2^5-10t_1^2t_2^3+4t_1^4-5M\left(3t_2^4-4t_1t_2^3+t_1^4\right)\right) \\ +\frac{\theta_2}{90}\left(5t_2^6-10t_1^3t_2^3+5t_1^6-3M\left(3t_2^5-5t_1^2t_2^2+2t_1^5\right)\right) \end{array}\right\} \\ +d\left\{\begin{array}{c} \frac{1}{20}\left(-5Mt_2^4+M^5+4t_2^5\right) \\ +\frac{\theta_1}{120}\left(10t_2^6-15t_1^2t_2^4+5t_1^4-6M\left(4t_2^5-5t_1t_2^4+t_1^5\right)\right) \\ +\frac{\theta_2}{168}\left(8t_2^7-14t_1^3t_2^4+6t_1^7-7M\left(2t_2^6-3t_1^2t_2^4+t_1^6\right)\right) \end{array}\right\} \end{array}\right] \\ -R\left[\begin{array}{l} a\left\{\begin{array}{c} \frac{1}{6}\left(-3M^2t_2+2M^3+t_2^3\right) \\ +\frac{\theta_1}{24}\left(t_2^4-4t_1^3t_2+3t_1^4-6M^2\left(t_2^2-2t_1t_2+t_1^2\right)\right) \\ +\frac{\theta_2}{120}\left(3t_2^5-15t_1^4t_2+12t_1^5-10M^2\left(t_2^3-3t_1^2t_2+2t_1^3\right)\right) \end{array}\right\} \\ +b\left\{\begin{array}{c} \frac{1}{8}\left(-2M^2t_2^2+M^4+t_2^4\right) \\ +\frac{\theta_1}{120}\left(4t_2^5-10t_1^3t_2^2+6t_1^5-10M^2\left(2t_2^3-3t_1t_2^2+t_1^3\right)\right) \\ +\frac{\theta_2}{48}\left(t_2^6-3t_1^4t_2^2+2t_1^6-3M^2\left(t_2^4-2t_1^2t_2^2+t_1^4\right)\right) \end{array}\right\} \\ +c\left\{\begin{array}{c} \frac{1}{30}\left(-5M^2t_2^3+2M^5+3t_2^5\right) \\ +\frac{\theta_1}{216}\left(6t_2^6-12t_1^3t_2^3+6t_1^6-9M^2\left(3t_2^4-4t_1t_2^3+t_1^4\right)\right) \\ +\frac{\theta_2}{840}\left(15t_2^7-35t_1^4t_2^3+20t_1^6-14M^2\left(3t_2^5-5t_1^2t_2^3+2t_1^5\right)\right) \end{array}\right\} \\ +d\left\{\begin{array}{c} \frac{1}{24}\left(-3M^2t_2^4+M^6+2t_2^6\right) \\ +\frac{\theta_1}{840}\left(20t_2^7-35t_1^3t_2^4+15t_1^7-21M^2\left(4t_2^5-5t_1t_2^4+t_1^5\right)\right) \\ +\frac{\theta_2}{192}\left(3t_2^8-6t_1^4t_2^4+3t_1^8-4M^2\left(2t_2^6-3t_1^2t_2^4+t_1^6\right)\right) \end{array}\right\} \end{array}\right] \end{array}\right]$$

Appendix 4

$$InP_2 = \frac{C_pI_p}{T}\left[\begin{array}{c} \left[\begin{array}{l} a\left\{\begin{array}{c} \frac{1}{2}\left(-2Mt_2+M^2+t_2^2\right) \\ +\frac{\theta_1}{6}\left(t_2^3-3Mt_2^2+3M^2t_2-M^3\right) \\ +\frac{\theta_2}{12}\left(t_2^4-2Mt_2^3+2M^3t_2-M^4\right) \end{array}\right\} \\ +b\left\{\begin{array}{c} \frac{1}{6}\left(-3Mt_2^2+M^3+2t_2^3\right) \\ +\frac{\theta_1}{24}\left(3t_2^4-8Mt_2^3+6M^2t_2^2-M^4\right) \\ +\frac{\theta_2}{120}\left(8t_2^5-15Mt_2^4+10M^3t_2^2-3M^5\right) \end{array}\right\} \\ +c\left\{\begin{array}{c} \frac{1}{12}\left(-4Mt_2^3+M^4+3t_2^4\right) \\ +\frac{\theta_1}{60}\left(6t_2^5-15Mt_2^4+10M^2t_2^3-M^5\right) \\ +\frac{\theta_2}{90}\left(5t_2^6-9Mt_2^5+5M^3t_2^3-M^6\right) \end{array}\right\} \\ +d\left\{\begin{array}{c} \frac{1}{20}\left(-5Mt_2^2+M^5+4t_2^5\right) \\ +\frac{\theta_1}{120}\left(10t_2^6-24Mt_2^5+15M^2t_4-M^6\right) \\ +\frac{\theta_2}{168}\left(8t_2^7-14Mt_2^6+7M^3t_2^4-M^7\right) \end{array}\right\} \end{array}\right] \\ -R\left[\begin{array}{l} a\left\{\begin{array}{c} \frac{1}{6}\left(-3M^2t_2+2M^3+t_2^3\right) \\ +\frac{\theta_1}{24}\left(t_2^4-6M^2t_2^2+8M^3t_2-3M^4\right) \\ +\frac{\theta_2}{120}\left(3t_2^5-10M^2t_2^3+15M^4t_2-8M^5\right) \end{array}\right\} \\ +b\left\{\begin{array}{c} \frac{1}{8}\left(-2M^2t_2^2+M^4+t_2^4\right) \\ +\frac{\theta_1}{30}\left(t_2^5-5M^2t_2^3+5M^3t_2^2-M^5\right) \\ +\frac{\theta_2}{48}\left(t_2^6-3M^2t_2^4+3M^4t_2^2-M^6\right) \end{array}\right\} \\ +c\left\{\begin{array}{c} \frac{1}{30}\left(-5M^2t_2^3+2M^5+3t_2^5\right) \\ +\frac{\theta_1}{216}\left(6t_2^6-27M^2t_2^4+24M^3t_2^3-M^6\right) \\ +\frac{\theta_2}{840}\left(15t_2^7-42M^2t_2^5+35M^4t_2^3-8M^7\right) \end{array}\right\} \\ +d\left\{\begin{array}{c} \frac{1}{24}\left(-3M^2t_2^4+M^6+2t_2^6\right) \\ +\frac{\theta_1}{420}\left(10t_2^7-42M^2t_2^5+35M^3t_2^4-3M^7\right) \\ +\frac{\theta_2}{192}\left(3t_2^8-8M^2t_2^6+6M^4t_2^4-M^8\right) \end{array}\right\} \end{array}\right] \end{array}\right]$$

Table 1: MODEL I

		Case I			**Case II**			**Case III**		
Parameter	%	t_2	TC	Q	t_2	TC	Q	t_2	TC	Q
a	+50%	1.8606	31,071	8864.2	1.7976	24,484	8607.0	1.6342	18,177	7940.7
	+25%	1.8430	27,546	8168.1	1.7796	21,640	7925.5	1.6067	16,105	7269.1
	-25%	1.8034	20,581	6788.3	1.7388	16,018	6575.7	1.5397	12,027	5938.0
	-50%	1.7807	17,145	6105.1	1.7153	13,243	5908.3	1.4970	10,027	5278.4
α	+50%	1.7646	25,249	7264.3	1.7156	20,230	7090.7	1.5925	15,605	6659.9
	+25%	1.7891	24,613	7351.4	1.7336	19,509	7154.4	1.5862	14,828	6638.1
	-25%	1.8788	23,631	7672.2	1.8030	18,190	7401.0	1.5544	13,285	6528.4
	-50%	1.9803	23,601	8037.1	1.8866	17,740	7700.2	1.4892	12,533	6305.5
β	+50%	1.8406	24,287	7535.3	1.7764	18,987	7306.2	1.5916	14,115	6656.8
	+25%	1.8322	24,166	7505.3	1.7681	18,901	7276.7	1.5835	14,084	6628.8
	-25%	1.8162	23,937	7448.1	1.7523	18,738	7220.6	1.5680	14,026	6575.2
	-50%	1.8086	23,830	7420.9	1.7448	18,662	7194.0	1.5621	13,998	6554.9
θ_1	+50%	1.8332	24,281	7528.9	1.7697	18,986	7298.9	1.5915	14,143	6665.4
	+25%	1.8286	24,164	7502.3	1.7649	18,901	7273.5	1.5836	14,098	6633.4
	-25%	1.8194	23,936	7449.9	1.7552	18,735	7223.0	1.5675	14,012	6569.5
	-50%	1.8148	23,824	7424.1	1.7503	18,654	7198.0	1.5593	13,971	6537.5
θ_2	+50%	1.8367	24,339	7547.5	1.7734	19,033	7316.8	1.5962	14,161	6683.7
	+25%	1.8304	24,193	7511.6	1.7667	18,924	7282.1	1.5859	14,107	6642.3
	-25%	1.8177	23,908	7441.1	1.7533	18,714	7214.3	1.5652	14,004	6560.8
	-50%	1.8113	23,771	7406.5	1.7466	18,612	7181.1	1.5548	13,956	6520.6
M	+50%	1.8091	22,891	7422.7	1.7101	12,392	7071.3	0.3743	10,645	3016.3
	+25%	1.8166	23,475	7449.5	1.7352	16,070	7160.0	1.3963	11,767	5993.0
	-25%	1.8314	24,618	7502.4	1.7845	20,995	7335.0	1.6654	16,637	6914.1
	-50%	1.8386	25,184	7528.1	1.8084	22,873	7420.2	1.7282	19,412	7135.2
δ	+50%	1.8683	22,241	6385.1	1.8200	16,792	6132.8	1.7078	11,423	5548.8
	+25%	1.8480	23,124	6920.4	1.7933	17,775	6679.9	1.6568	12,666	6084.5
	-25%	1.7948	25,026	8056.9	1.7156	19,940	7841.4	-0.7380	17,693	2884.9
	-50%	1.7569	26,070	8670.0	1.6450	21,183	8461.1	-1.4823	30,308	4632.2
R	+50%	1.7320	24,435	8623.0	1.6795	17,948	6963.6	1.5539	13,822	6526.7
	+25%	1.7711	25,148	8697.0	1.7134	18,336	7082.0	1.5625	13,937	6556.3
	-25%	1.9123	26,897	8970.0	1.8304	19,484	7498.8	1.5980	14,178	6679.0
	-50%	2.0372	28,517	9218.5	1.9552	20,640	7946.8	1.6464	14,321	6847.5

Table 2: MODEL II

		Case I			**Case II**			**Case III**		
Parameter	%	t_2	TC	Q	t_2	TC	Q	t_2	TC	Q
a	+50%	1.8846	14,809	7875.3	1.8283	9,488	7661.5	1.6920	4,431.2	7150.4
	+25%	1.8672	14,852	7809.1	1.8105	9,523.0	7594.2	1.6671	4,466.2	7058.3
	-25%	1.8280	14,851	7604.1	1.7704	9,521.7	7443.2	1.6076	4,464.4	6839.7
	-50%	1.8055	14,796	7575.3	1.7473	9,477.7	7356.5	1.5708	4,416.3	6705.8
α	+50%	1.7849	14,717	7497.7	1.7404	9,459.2	7330.7	1.6332	4,478.2	6933.4
	+25%	1.8111	14,813	7596.5	1.7612	9,507.6	7408.6	1.6356	4,478.5	6942.3
	-25%	1.9062	14,712	7957.7	1.8394	9,453.8	7703.6	1.6460	4,478.0	6980.5
	-50%	2.0119	13,357	8362.6	1.9307	8,720.7	8051.3	1.6619	4,470.5	7039.1
β	+50%	1.8650	14,856	7800.8	1.8077	9,526.4	7583.6	1.6551	4,474.8	7014.0
	+25%	1.8566	14,865	7768.8	1.7994	9,533.1	7552.3	1.6471	4,477.8	6984.5
	-25%	1.8405	14,865	7707.7	1.7835	9,533.3	7492.4	1.6317	4,477.9	6927.9
	-50%	1.8329	14,858	7678.9	1.7760	9,527.9	7464.2	1.6244	4,475.5	6901.2
θ_1	+50%	1.8642	14,857	7797.7	1.8094	9,524.4	7590.1	1.6692	4,464.1	7066.0
	+25%	1.8564	14,865	7768.1	1.8005	9,532.5	7556.5	1.6546	4,475.0	7012.1
	-25%	1.8402	14,865	7706.6	1.7819	9,532.5	7486.4	1.6232	4,474.9	6896.8
	-50%	1.8319	14,857	7675.2	1.7723	9,524.0	7450.3	1.6062	4,463.1	6834.6
θ_2	+50%	1.8643	14,857	7798.1	1.8085	9,525.5	7586.7	1.6654	4,467.7	7052.0
	+25%	1.8564	14,865	7768.1	1.8000	9,532.8	7554.6	1.6525	4,476.0	7004.4
	-25%	1.8403	14,865	7707.0	1.7825	9,532.8	7488.7	1.6257	4,476.0	6905.9
	-50%	1.8321	14,857	7675.9	1.7737	9,525.6	7455.6	1.6117	4,467.7	6854.7
M	+50%	1.8338	14,859	7682.4	1.7479	9,479.2	7358.8	0.9880	2,431.0	4741.4
	+25%	1.8411	14,865	7710.0	1.7696	9,520.7	7440.2	1.5316	4,335.8	6564.3
	-25%	1.8556	14,865	7765.0	1.8130	9,519.5	7603.7	1.7093	4,391.9	7214.7
	-50%	1.8626	14,859	7791.6	1.8345	9,470.1	7685.0	1.7624	4,178.7	7413.1
δ	+50%	1.8877	14,798	7887.1	1.8430	9,440.4	7717.2	1.7407	4,284.2	7331.8
	+25%	1.8695	14,848	7817.9	1.8195	9,508.3	7628.2	1.6990	4,416.9	7176.4
	-25%	1.8234	14,843	7643.0	1.7554	9,496.4	7386.9	1.5222	4,312.9	6530.6
	-50%	1.7923	14,748	7525.6	1.7050	9,330.0	7198.7	-1.4720	-8,289.8	-975.533
R	+50%	1.7537	14,550	7380.5	1.7066	9,337.3	7204.7	1.5998	4,456.7	6811.3
	+25%	1.7940	14,755	7532.0	1.7423	9,464.5	7337.8	1.6158	4,470.7	6869.7
	-25%	1.9283	14,558	8042.1	1.8645	9,337.3	7798.9	1.6774	4,454.6	7096.3
	-50%	2.0655	11,925	8568.7	1.9929	7,615.0	8289.7.0	1.7528	4,228.8	7377.1

Table 3: MODEL III

		Case I			Case II			Case III		
Parameter	%	t_2	TC	Q	t_2	TC	Q	t_2	TC	Q
a	+50%	1.8827	23,068	7073.2	1.8283	17,730	6833.7	1.7038	12,621	6288.6
	+25%	1.8660	23,109	6999.7	1.8115	17,763	6759.9	1.6815	12,654	6191.6
	-25%	1.8284	23,108	6834.2	1.7734	17,762	6592.7	1.6292	12,652	5965.2
	-50%	1.8069	23,055	6739.7	1.7516	17,720	6497.2	1.5977	12,607	5829.6
α	+50%	1.7857	22,972	6646.6	1.7430	17,696	6459.6	1.6440	12,663	6029.1
	+25%	1.8116	23,070	6760.3	1.7637	17,746	6550.2	1.6491	12,665	6051.2
	-25%	1.9038	22,974	7166.2	1.8396	17,695	6883.4	1.6698	12,663	6140.8
	-50%	2.0048	21,710	7611.2	1.9258	17,010	7263.2	1.6963	12,635	6255.9
β	+50%	1.8639	23,113	6990.4	1.8088	17,766	6748.0	1.6712	12,662	6146.9
	+25%	1.8558	23,121	6954.7	1.8009	17,773	6713.3	1.6639	12,665	6115.2
	-25%	1.8404	23,121	6887.0	1.7859	17,773	6647.5	1.6500	12,665	6055.1
	-50%	1.8331	23,115	6854.8	1.7787	17,768	6615.9	1.6435	12,663	6026.1
θ_1	+50%	1.8562	23,121	6956.5	1.8016	17,772	6716.4	1.6682	12,663	6133.9
	+25%	1.8521	23,123	6938.5	1.7974	17,774	6697.9	1.6625	12,665	6109.2
	-25%	1.8438	23,123	6901.9	1.7891	17,774	6661.5	1.6511	12,665	6059.8
	-50%	1.8396	23,121	6883.4	1.7848	17,772	6642.6	1.6454	12,663	6035.2
θ_2	+50%	1.8595	23,118	6971.0	1.8050	17,770	6731.3	1.6719	12,661	6149.9
	+25%	1.8537	23,122	6945.5	1.7991	17,774	6705.4	1.6644	12,665	6117.4
	-25%	1.8422	23,122	6894.9	1.7874	17,774	6654.0	1.6493	12,663	6052.0
	-50%	1.8364	23,118	6869.4	1.7815	17,770	6628.2	1.6417	12,662	6019.2
M	+50%	1.8349	23,117	6862.8	1.7528	17,723	6502.5	1.4443	12,088	5179.3
	+25%	1.8415	23,122	6891.8	1.7729	17,761	6590.5	1.5747	12,558	5731.0
	-25%	1.8544	23,122	6948.6	1.8138	17,760	9770.0	1.7176	12,589	6348.7
	-50%	1.8608	23,117	6976.8	1.8343	17,713	6860.1	1.7661	12,393	6560.7
δ	+50%	1.8942	23,023	7123.9	1.8525	17,641	6940.2	1.7613	12,419	6539.7
	+25%	1.8730	23,095	7030.5	1.8260	17,736	6823.6	1.7181	12,588	6350.9
	-25%	1.8173	23,085	6785.4	1.7503	17,716	6491.5	1.5438	12,472	5599.1
	-50%	1.7773	22,931	6609.8	1.6856	17,445	6209.4	-1.3511	-8.6484	-3.3961
R	+50%	1.7532	22,791	6504.2	1.7076	17,558	6305.1	1.6088	12,627	5877.3
	+25%	1.7939	23,008	6682.6	1.7442	17,699	6464.8	1.6289	12,652	5963.9
	-25%	1.9263	22,817	7265.4	1.8650	17,574	6995.3	1.6991	12,630	6268.1
	-50%	2.0588	20,307	7848.4	1.9880	15,922	7537.3	1.7748	12,342	6598.8

Table 4: MODEL IV

		Case I			Case II			Case III		
Parameter	%	t_2	TC	Q	t_2	TC	Q	t_2	TC	Q
a	+50%	1.9041	13,925	7354.2	1.8544	8,500.5	7124.4	1.7441	3,157.5	6617.4
	+25%	1.8874	13,970	7276.9	1.8376	8,537.4	7046.8	1.7226	3,194.5	6519.3
	-25%	1.8500	13,969	7104.0	1.7999	8,536.0	6873.2	1.6727	3,192.7	6292.7
	-50%	1.8287	13,912	7005.8	1.7783	8,489.2	6774.0	1.6427	3,142.6	6157.2
α	+50%	1.8038	13,801	6891.1	1.7644	8,443.7	6710.3	1.6748	3,195.0	6302.2
	+25%	1.8312	13,921	7017.3	1.7872	8,512.2	6814.9	1.6846	3,203.1	6346.6
	-25%	1.9279	13,805	7464.5	1.8698	8,443.8	7195.5	1.7226	3,194.5	6519.3
	-50%	2.0327	12,288	7950.7	1.9620	7,540.4	7622.7	1.7693	3,079.9	6732.8
β	+50%	1.8855	13,974	7268.1	1.8353	8,540.5	7036.2	1.7136	3,202.7	6478.3
	+25%	1.8774	13,983	7230.7	1.8273	8,548.0	6999.3	1.7062	3,206.5	6444.7
	-25%	1.8618	13,983	7158.5	1.8121	8,548.2	6929.3	1.6921	3,206.6	6380.6
	-50%	1.8544	13,976	7124.4	1.8049	8,542.3	6896.2	1.6853	3,203.5	6349.8
θ_1	+50%	1.8838	13,976	7260.3	1.8354	8,540.4	7036.7	1.7214	3,195.8	6513.8
	+25%	1.8767	13,984	7227.4	1.8275	8,547.9	7000.3	1.7104	3,204.7	6463.8
	-25%	1.8621	13,984	7159.9	1.8114	8,547.8	6926.1	1.6873	3,204.6	6358.8
	-50%	1.8546	13,976	7125.3	1.8031	8,540.2	6887.9	1.6752	3,195.4	6304.0
θ_2	+50%	1.8840	13,976	7261.2	1.8348	8,541.1	7033.9	1.7192	3,198.1	6503.8
	+25%	1.8768	13,984	7227.9	1.8272	8,548.1	6998.9	1.7092	3,205.3	6458.3
	-25%	1.8621	13,984	7159.9	1.8118	8,548.1	6927.9	1.6887	3,205.3	6365.2
	-50%	1.8546	13,976	7125.3	1.8040	8,541.3	6892.1	1.6782	3,198.2	6317.6
M	+50%	1.8630	13,978	7134.1	1.7831	8,502.2	6796.0	1.5376	2,762.4	5688.0
	+25%	1.8630	13,984	7164.1	1.8011	8,537.6	6878.7	1.6322	3,117.8	6110.0
	-25%	1.8759	13,984	7223.7	1.8383	8,536.3	7050.0	1.7512	3,139.6	6649.9
	-50%	1.8822	13,978	7252.9	1.8571	8,492.2	7136.8	1.7942	2,962.2	6847.0
δ	+50%	1.9113	13,896	7387.6	1.8721	8,433.4	7206.1	1.7869	3,001.3	6813.5
	+25%	1.8919	13,961	7297.7	1.8483	8,516.8	7096.2	1.7494	3,144.4	6641.7
	-25%	1.8426	13,953	7069.9	1.7832	8,502.5	6796.5	1.6189	3,081.5	6050.2
	-50%	1.8088	13,827	6914.1	1.7328	8,300.2	6565.8	-1.3243	-2,003.5	-3441.5
R	+50%	1.7726	13,605	6747.9	1.7308	8,289.4	6556.7	1.6428	3,142.9	6157.7
	+25%	1.8142	13,853	6939.0	1.7688	8,459.3	6730.5	1.6664	3,184.9	6264.2
	-25%	1.9493	13,636	7563.8	1.8936	8,309.0	7305.6	1.7479	3,148.3	6634.8
	-50%	2.0841	10,768	8188.8	2.0200	6,327.1	7891.8	1.8348	2,668.3	7033.9

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ON THE LINEAR QUADRATIC DYNAMIC OPTIMIZATION PROBLEMS WITH FIXED-LEVELS CONTROL FUNCTIONS

Vadim Azhmyakov

Department of Basic Sciences
Universidad de Medellin
Medellin
Colombia
e-mail: vazhmyakov@udem.edu.co

Luz Adriana Guzman Trujillo

School of Ingeniering
Universidad de Medellin
Medellin
Colombia
e-mail: lguzman@udem.edu.co

Abstract. This paper deals with a constrained LQ-type optimal control problem (OCP) in the presence of fixed levels input restrictions. We consider control processes governed by linear differential equations with a priori known control switching structure. The set of admissible inputs reflects some important natural engineering applications and moreover, can also be interpreted as a result of a quantization procedure applied to the original dynamic system. We propose a novel implementable algorithm that makes it possible to calculate a (numerically consistent) approximative solution to the constrained LQ-type OCPs under consideration. Our contribution mainly discusses theoretic aspects of the proposed solution scheme and contains an illustrative numerical example.

Keywords: optimal control, systems theory, convex optimization, numerical methods.

1. Introduction

Optimal control methodology is nowadays a mature powerful approach to the practical synthesis of several types of modern switched-type and interconnected dynamic systems (see e.g., [3], [4], [8], [14], [16], [18], [19], [20], [22], [27], [29], [30]). In this context let us also refer to [7], [9], [10], [29], [30], [32], [34], [39], [45] for some examples of specific optimization techniques and concrete real-world appplications. Recently, the problem of effective numerical methods for the constrained LQ based systems optimization has attracted a lot of attention, thus both theoretical results and applications were developed (see, e.g., [5], [6], [25], [22], [26], [27], [28], [31], [32],

[34], [36], [39], [40], [45] and the references therein). Note that handling various types of constraints in practical system design is an important issue in most, if not all, real world applications. It is readily appreciated that the implementable dynamic systems have a corresponding set of constraints; for example, inputs always have maximum and minimum values and states are usually required to lie within certain ranges. Moreover, it is generally true that optimal levels of performance are associated with operating on, or near, constraint boundaries (see [14], [21], [42]).Thus, a control engineer really can not ignore constraints without incurring a performance penalty.

The aim of our contribution is to elaborate a consistent computational algorithm for an LQ-type OCP in systems with piecewise constant control inputs. The given restrictive structure of the admissible control function under consideration is motivated by some important engineering applications (see [5], [6], [12], [13], [22], [26], [25], [27], [28], [30], [36], [40], [42]) as well as by application of common quantization procedures to the original dynamics (see e.g., [17], [31], [34]). Note that quadratic optimal control of piecewise linear systems was addressed earlier in [28], [36]. The treatment there was based on the backward solutions of Riccati differential equations, and the optimum had to be recomputed for each new final state. Computation of non-linear gain using the Hamilton-Jacobi-Bellman equation and the convex optimization techniques has also been done in [36]. Let us also refer to a sophisticated solution techniques for non-linear OCPs proposed in [11]. This approach is based on a newly developed relaxation procedure. On the other hand, the common optimization approaches to linear constrained and switched-type systems are not sufficiently advanced to LQ-type problems for linear systems with fixed levels controls. In our paper we propose a new numerical method that includes a specific relaxation scheme in combination with the classic projection approach. Moreover, it should be noted already at this point that the optimization algorithm we propose can be effectively used as a part of a concrete control design procedure for some classes of dynamic systems with switched nature.

Recall that the general switched (and hybrid) systems constitute formal framework of systems where two types of dynamics are present, continuous and discrete event behaviour (see, e.g., [16], [31]). Evidently, a dynamic model with fixed levels control inputs constitutes a simple example of a switched system. In order to understand how these systems can be operated efficiently, both aspects of the dynamics/controls have to be taken into account during the system optimization phase. The non-stationary linear systems we study in this paper include a particular family of switched systems with the time-driven location transitions. We refer to [1], [4], [15], [18], [19], [20], [23], [32], [38], [41], [43], [45] for some relevant examples and abstract concepts.

The remainder of our paper is organized as follows: Section 2 contains a problem statement, the necessary preliminary facts and basic concepts. Section 3 deals with a simple relaxation scheme of the initial constrained LQ-type OCP. We propose a projected gradient method for the concrete numerical treatment of the OCPs under consideration (see e.g. [4], [24], [33]). In Section 4 we discuss a specific controllability condition for the concrete dynamic system under consideration. Section 5 is devoted to the numerical aspects of the proposed algorithm and contains an illustrative example. Section 6 summarizes the paper.

2. Problem formulation and some basic facts

Consider the following linear non-stationary system with a switched control structure

$$\dot{x}(t) = A(t)x(t) + B(t)u(t),\ t \in [t_0, t_f],\ x(t_0) = x_0, \tag{1.1}$$

where $A(\cdot) \in \mathbb{L}^\infty[t_0, t_f; \mathbb{R}^{n\times n}]$, $B(\cdot) \in \mathbb{L}^\infty[t_0, t_f; \mathbb{R}^{n\times m}]$. Here $\mathbb{L}^\infty[t_0, t_f; R^{n\times n}]$ and $\mathbb{L}^\infty[t_0, t_f; \mathbb{R}^{n\times m}]$ are the standard Lebesgue spaces of the essentially bounded matrix-functions defined on a bounded time interval $[t_0, t_f]$. Similarly to the classic LQR (the Linear Quadratic Regulator) theory it is desired to minimize the following quadratic cost functional associated with (1.1)

$$J(u(\cdot)) = \frac{1}{2}\int_{t_0}^{t_f} (\langle Q(t)x(t), x(t)\rangle + \langle R(t)u(t), u(t)\rangle)dt + \frac{1}{2}\langle Gx(t_f), x(t_f)\rangle, \tag{1.2}$$

where $G \in \mathbb{R}^{n\times n}$ and the matrix-functions $Q(\cdot)$ and $R(\cdot)$ are assumed to be integrable. Following the conventional LQR theory we next introduce the standard regularity/positivity hypothesis: $G \geq 0$, $Q(t) \geq 0$, $R(t) \geq \delta I$, $\delta > 0$ $\forall t \in [t_0, t_f]$. It is well known that the classic LQ optimal control strategy $u^{opt}(\cdot)$ does not incorporate any additional (state or control) restrictions into the resulting design procedure. Let us recall here the explicit formula for $u^{opt}(\cdot)$ (see e.g., [21], [27])

$$u^{opt}(t) = -R^{-1}(t)\left[B^T(t)P(t)\right]x^{opt}(t), \tag{1.3}$$

where $P(\cdot)$ is the matrix-function, namely, the solution to the classic differential matrix Riccati equation associated with the LQ problem (1.1)-(1.2). In the above-mentioned conventional case (1.1)-(1.2) the optimization problem is formally studied in the full space $\mathbb{L}^2[t_0, t_f; \mathbb{R}^m]$ of square integrable control functions. In contrast to the classic case, we consider system (1.1) in combination with the specific piecewise constant admissible inputs $u(\cdot)$ of the following type (see Fig. 1).

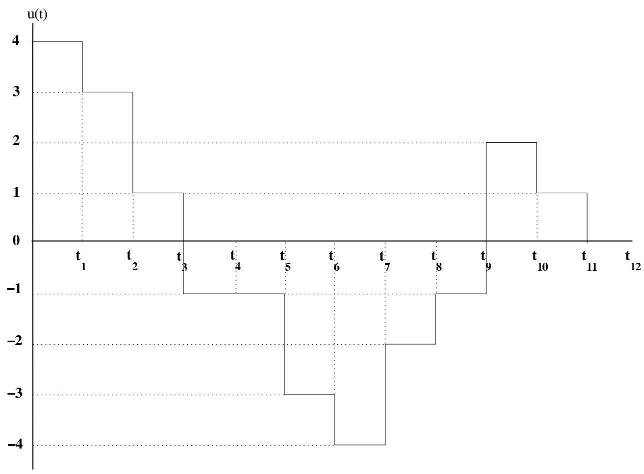

Figure 1: The admissible switched-type control inputs $u(\cdot)$

Resulting from the admissibility assumption the main minimization problem for the linear system (1.1) can be interpreted as a restricted LQ optimization problem. For example, the control signal $u(\cdot)$ showed in Fig. 1 can take a value (level) within the finite set $\mathbb{Q} = \{-4,-3,-2,-1,0,1,2,3,4\}$ during the time interval $[t_{i-1},t_i)$, $i = 1,\dots,12$. In addition the control signal here is only allowed to change its value at the times $t_0,t_1,\dots,t_f$ being fixed between these times.

Let us now specify formally the set of admissible piecewise constant control functions for system (1.1) in a general case. For each component $u^{(k)}(\cdot)$ of the feasible control input $u(\cdot) = [u^{(1)}(\cdot),\dots,u^{(m)}(\cdot)]^T$ we introduce the following finite set of feasible value levels: $\mathbb{Q}^k := \{q_j^{(k)} \in \mathbb{R},\ j = 1,\dots,M_k\},\ M_k \in \mathbb{N},\quad k = 1,\dots,m$. In general, all the sets $\mathbb{Q}^k$ are different (contains different levels) and have various numbers of elements. In addition, each $\mathbb{Q}^k$ possesses a strict order property

$$q_1^{(k)} < q_2^{(k)} < \dots < q_{M_k}^{(k)}.$$

We now introduce the set of switching times associated with an admissible control function $\mathbb{T}^k := \{t_i^{(k)} \in \mathbb{R}_+, i = 1,\dots,N_k\},\quad N_k \in \mathbb{N},\quad k = 1,\dots,m$. The sets $\mathbb{T}^k$ are assumed to be defined for each control component $u^{(k)}(\cdot)$, $k = 1,\dots,m$, where $\mathbb{R}_+$ denotes a non-negative semi-axis. Let us consider an ordered sequence of time instants: $t_0 < t_1^{(k)} < \dots < t_{N_k}^{(k)}$. For the final time instants associated with each set $\mathbb{T}^k$ we put $t_{N_1}^{(1)} = \dots = t_{N_m}^{(m)} = t_f$. Using the notation of the level sets $\mathbb{Q}^k$ and the fixed switching times $\mathbb{T}^k$ introduced above, the set of admissible controls $\mathscr{S}$ can now be easily specified by the Cartesian product

$$\mathscr{S} := \mathscr{S}_1 \times \dots \times \mathscr{S}_m, \tag{1.4}$$

where each set $\mathscr{S}_k$, $k = 1,...,m$ is defined as follows

$$\mathscr{S}_k := \{v(\cdot) \mid v(t) = \sum_{i=1}^{N_k} I_{[t_{i-1}^{(k)},t_i^{(k)})}(t) q_{j_i}^{(k)}; q_{j_i}^{(k)} \in \mathbb{Q}^k; j_i \in \mathbb{Z}[1,M_k];\ t_i^{(k)} \in \mathbb{T}^k\}.$$

By $\mathbb{Z}[1,M_k]$ we denote here the set of all integers into the interval $[1,M_k]$ and $I_{[t_{i-1}^k,t_i^k)}(t)$ is the characteristic function of the interval $[t_{i-1}^k,t_i^k)$. Evidently, the set of admissible control inputs $\mathscr{S}$ can be qualitatively interpreted as the set of all the possible functions $u : [t_0,t_f] \to \mathbb{R}^m$, such that each component $u^{(k)}(\cdot)$ of $u(\cdot)$ attains a constant level value $q_{j_i}^{(k)} \in \mathbb{Q}^k$ for $t \in [t_{i-1}^k,t_i^k)$. Moreover, the component level changes occur only at the prescribed times $t_i^k \in \mathbb{T}^k$, $i = 1,...,N_k-1$. The clear combinatorial character of the examined control functions associated with the initial system (1.1) can be illustrated by a simple example.

Example 1.1 Suppose $u(t) \in \mathbb{R}^2$ and $\mathbb{Q}^1 = \{0,1,2\}$, $\mathbb{Q}^2 = \{0,-1\}$. Furthermore, the set of switching times for each control component is assumed to be given by $\mathbb{T}^1 = \{0,0.5,1\}$, $\mathbb{T}^2 = \{0,0.33,0.66,1\}$. Resulting from the above definitions, the set $\mathscr{S}$ in (1.4) can be written as $\mathscr{S} = \mathscr{S}_1 \times \mathscr{S}_2$, where:

$$\mathscr{S}_1 = \{v : [t_0, t_f] \to \mathbb{R} \mid v(t) = I_{[0,0.5)}(t) q^{(1)}_{j_1} + I_{[0.5,1)}(t) q^{(1)}_{j_2},\ q^{(1)}_{j_i} \in \mathbb{Q}^1\};$$
$$\mathscr{S}_2 = \{w : [t_0, t_f] \to \mathbb{R} \mid w(t) = I_{[0,0.33)}(t) q^{(2)}_{j_1} +$$
$$I_{[0.33,0.66)}(t) q^{(2)}_{j_2} + I_{[0.66,1)}(t) q^{(2)}_{j_3}, q^{(2)}_{j_i} \in \mathbb{Q}^2\}$$

In that concrete case we evidently have: $M_1 = 3$, $M_2 = 2$, $N_1 = 2$ and $N_2 = 3$.

The cardinality of the control set $\mathscr{S}$ is given as follows: $|\mathscr{S}| = 3^2 \cdot 2^3 = 72$. In other words, we have 72 admissible control inputs, among which we must find the one that minimizes the quadratic performance criterion.

In general, the cardinality of the set $\mathscr{S}$ of admissible controls $u(\cdot)$ with $u(t) \in \mathbb{R}^m$ can be expressed as follows

$$|\mathscr{S}| = \prod_{l=1}^{m} M_l^{N_l}. \tag{1.5}$$

Motivating from various engineering applications, we now can formulate the following specific constrained LQ-type OCP

$$\begin{aligned} &\text{minimize } J(u(\cdot)) \\ &\text{subject to } u(\cdot) \in \mathscr{S}, \end{aligned} \tag{1.6}$$

where $J(\cdot)$ is the costs functional defined in (1.2). Note that $\mathscr{S}$ constitutes a nonempty subset of the space $\mathbb{L}^2[t_0, t_f; \mathbb{R}^m]$. However, the classically LQ-optimal control input $u^{opt}(\cdot)$ in (1.3) does not belong to the introduced specific set $\mathscr{S}$. Due to the highly restrictive condition $u(\cdot) \in \mathscr{S}$, the main optimization problem (1.6) can not be generally solved by a direct application of the classic Pontryagin Maximum Principle. A possible application of a suitable hybrid version of the conventional Maximum principle from [4], [16], [23], [38], [41], [45] is also complicated by a non-standard structure of the simple control inputs under consideration. Let us additionally note that the value of an exponentially growing cardinality $|\mathscr{S}|$ in (1.5) exacerbates crucially a possible application of some combinatorial and various state/control discretization based numerical algorithms for OCPs (see e.g., [7], [8], [13], [24], [33], [34], [35], [38], [42], [45] and the references therein).

The aim of this contribution is to propose a relative simple implementable computational procedure for a consistent numerical treatment of the constrained OCP (1.6). We use a simple relaxation technique associated with the main OCP (1.6) in combination with a gradient based algorithm for this purpose. We first obtain an optimal solution of a convex relaxed OCP. Next we use it in a constructive solution procedure for the original problem (1.6).

3. The gradient-based approach to the relaxed optimal control problem

In this section we propose a constructive computational scheme for the constrained LQ-type OCP (1.6) formulated above. The proposed approach incorporates a simply relaxed OCPs associated with the initial problem (1.6). Let us first recall a necessary auxiliary result from the classic convex analysis (see [3], [33], [37]): it is a well known fact that a composition of two convex functionals is not necessarily convex. In the following we will need a basic result providing conditions that ensure convexity of the composition (see e.g., [3], [37]).

Lemma 3.1 *Let $g^1 : \mathscr{W} \to \mathbb{R}$ be a convex functional determined on a convex set $\mathscr{W} \subseteq \mathbb{R}^p$ and $g^2 : \mathscr{V} \to \mathscr{W}$ be an affine mapping defined on a convex subset $\mathscr{V}$ of a real* Hilbert *space H. Then the composed functional $g : \mathscr{V} \to \mathbb{R}$, $g(\cdot) := g^1(g^2(\cdot))$ is convex.*

Let now $x^u(\cdot)$ be a solution to the initial value problem (1.1) generated by an admissible control $u(\cdot) \in \mathscr{S}$. Evidently, every component of $x^u(\cdot)$ is an affine function (functional) of $u(\cdot)$

$$x(t,u) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)d\tau. \tag{3.1}$$

Here $\Phi(\cdot,\tau)$ is the fundamental solution matrix associated with (1.1). Let us note that set of admissible controls $\mathscr{S}$ constitutes a non-convex set. This fact is due to the originally combinatorial structure of $\mathscr{S}$ determined in (1.4).

Example 3.2 Under assumptions of Example 1.1 we have $\mathscr{S} := \mathscr{S}_1 \times \mathscr{S}_2$ and moreover,

$$\begin{aligned}
&\mathscr{S}_1 = \{(0 \times I_{[0,0.5)}(t) + 1 \times I_{[0.5,1)}(t));\ (1 \times I_{[0,0.5)}(t) + 0 \times I_{[0.5,1)}(t));\\
&(0 \times I_{[0,0.5)}(t) + 2 \times I_{[0.5,1)}(t));\ (2 \times I_{[0,0.5)}(t) + 0 \times I_{[0.5,1)}(t));\\
&(1 \times I_{[0,0.5)}(t) + 2 \times I_{[0.5,1)}(t));\ (2 \times I_{[0,0.5)}(t) + 1 \times I_{[0.5,1)}(t));\ (0);\ (1);\ (2)\}\\
&\mathscr{S}_2 = \{(0 \times I_{[0,0.33)}(t) + (-1) \times I_{[0.33,0.66)}(t) + (-1) \times I_{[0.66,1)}(t));\\
&(0 \times I_{[0,0.33)}(t) + (-1) \times I_{[0.33,0.66)}(t) + 0 \times I_{[0.66,1)}(t));\\
&(0 \times I_{[0,0.33)}(t) + 0 \times I_{[0.33,0.66)}(t) + (-1) \times I_{[0.66,1)}(t));\\
&((-1) \times I_{[0,0.33)}(t) + (-1) \times I_{[0.33,0.66)}(t) + 0 \times I_{[0.66,1)}(t));\\
&((-1) \times I_{[0,0.33)}(t) + 0 \times I_{[0.33,0.66)}(t) + 0 \times I_{[0.66,1)}(t));\\
&((-1) \times I_{[0,0.33)}(t) + 0 \times I_{[0.33,0.66)}(t) + (-1) \times I_{[0.66,1)}(t));\ (0);\ (-1)\}
\end{aligned}$$

The combinatorial structure of $\mathscr{S}$ is evident. Recall that a combinatorial set is a non-convex set. The convex hull $\mathrm{conv}(\mathscr{S})$ of the original set $\mathscr{S}$ has a simple expression:

$$\begin{aligned}
&\mathrm{conv}(\mathscr{S}) = \{(C_1 \times I_{[0,0.5)}(t), C_2 \times I_{[0.5,1)}(t))\} \times\\
&\{(D_1 \times I_{[0,0.33)}(t) + D_2 \times I_{[0.33,0.66)}(t) + D_3 \times I_{[0.66,1)}(t))\}
\end{aligned}$$

where $C_1, C_2 \in [0,2]$ and $D_1, D_2, D_3 \in [0,-1]$.

Motivated from the above facts let us consider the convex hull $\mathrm{conv}(\mathscr{S})$ associated with $\mathscr{S}$

$$\mathrm{conv}(\mathscr{S}) := \{v(\cdot) \mid v(t) = \sum_{s=1}^{|\mathscr{S}|} \lambda_s u_s(t),\ \sum_{s=1}^{|\mathscr{S}|} \lambda_s = 1,$$

where $\lambda_s \geq 0,\ u_s(\cdot) \in \mathscr{S},\ s = 1,\dots,|\mathscr{S}|\}$. From the definition of $\mathscr{S}$ we conclude that the convex set conv$(\mathscr{S})$ is closed and bounded. Using (1.4), we also can give the alternative characterization: $\mathrm{conv}(\mathscr{S}) = \mathrm{conv}(\mathscr{S}_1) \times \ldots \times \mathrm{conv}(\mathscr{S}_m)$, where conv$(\mathscr{S}_k)$ is a convex hull of $\mathscr{S}_k$ $k = 1,\dots,m$. Since $\mathrm{conv}(\mathbb{Q}^k) \equiv [q_1^{(k)}, q_{M_k}^{(k)}]$, we have

$$\mathrm{conv}(\mathscr{S}_k) := \{v(\cdot) \mid v(t) = \sum_{i=1}^{N_k} I_{[t_{i-1}^{(k)}, t_i^{(k)})}(t) q_{j_i}^{(k)};\ q_{j_i}^{(k)} \in [q_1^{(k)}, q_{M_k}^{(k)}];\ j_i \in \mathbb{Z}[1, M_k];\ t_i^{(k)} \in \mathbb{T}^k\}.$$

Roughly speaking conv$(\mathscr{S})$ contains all the piecewise constant functions $u(\cdot)$ such that the constant value $u^{(k)}(t)$ belongs to the interval $[q_1^{(k)}, q_{M_k}^{(k)}]$ for all $t \in [t_{i-1}^{(k)}, t_i^{(k)})$. Let us note that in contrast to the initial set $\mathscr{S}$, the corresponding convex hull conv$(\mathscr{S})$ is an infinite dimensional space. Using the above convex construction, we can formulate the following auxiliary OCP

$$\begin{aligned} &\text{minimize } J(u(\cdot)) \\ &\text{subject to } u(\cdot) \in \mathrm{conv}(\mathscr{S}). \end{aligned} \tag{3.2}$$

The problem (3.2) formulated above is in fact a simple convex relaxation of the initial OCP (1.6). We will study this problem and use it for a constructive numerical treatment of (1.6). Let us firstly formulate the following key property of the auxiliary OCP (3.2).

Theorem 3.3 *The cost functional* $J : \mathrm{conv}(\mathscr{S}) \to \mathbb{R}$

$$J(u(\cdot)) = \frac{1}{2}\int_{t_0}^{t_f} [\langle Q(t)x^u(t), x^u(t)\rangle + \langle R(t)u(t), u(t)\rangle]dt + \frac{1}{2}\langle Gx^u(t_f), x(t_f)\rangle$$

is convex and the auxiliary OCP (3.2) *constitutes a convex optimization problem in the* Hilbert *space* $\mathbb{L}^2[t_0, t_f; \mathbb{R}^m]$.

Proof. Evidently, conv$(\mathscr{S})$ is a bounded closed and convex subset of $\mathbb{L}^2[t_0, t_f; \mathbb{R}^m]$. The cost functional $J(\cdot)$ is in fact a sum of two functionals:

$$J(u(\cdot)) = J_1(u(\cdot)) + J_2(u(\cdot)),\ J_1(u(\cdot)) := \frac{1}{2}\int_{t_0}^{t_f} [\langle R(t)u(t), u(t)\rangle]dt,$$
$$J_2(u(\cdot)) := \frac{1}{2}\int_{t_0}^{t_f} [\langle Q(t)x^u(t), x^u(t)\rangle]dt + \frac{1}{2}\langle Gx^u(t_f), x^u(t_f)\rangle.$$

The first one, namely, the functional $J_1(\cdot)$ is convex (recall that its Hessian is positive definite matrix). Moreover, $J_2(\cdot)$ is a composition of a convex (quadratic) functional of $x^u(\cdot)$, where $x^u(\cdot)$ is an affine mapping with respect to $u(\cdot)$ (see (3.1)). Applying Lemma 3.1, we now easily deduce the convexity of $J_2(\cdot)$. Since the sum of two convex functions is convex, we obtain the desired convexity result for $J(\cdot)$. The proof is completed. ∎

As we can see, (3.2) is a convex relaxation of the initial OCP (1.6). The proved convexity of OCP (3.2) makes it possible to apply the powerful numerical convex programming approaches to this auxiliary optimization problem. In this paper, we use a variant of the projected gradient method for a concrete numerical treatment of (3.2). Note that under the basic assumptions introduced in Section the following mapping $x^u(t) : \mathbb{L}^2[t_0,t_f;\mathbb{R}^m] \to \mathbb{R}^n$ is Fréchet differentiable for every $t \in [t_0,t_f]$ (see [17, 23]). Therefore, the quadratic costs functional $J(\cdot)$ in (3.2) is also Fréchet differentiable. We refer to [23, 29] for the corresponding differentiability concept. Assume $u^*(\cdot) \in \text{conv}(\mathscr{S})$ is an optimal solution of (3.2). The existence of an optimal input $u^*(\cdot)$ is guaranteed in the convex problem (3.2) (see e.g., [33]). By $x^*(\cdot)$ we next denote the corresponding optimal trajectory (solution) of (1.1) generated by $u^*(\cdot)$. The projected gradient method for problem (3.2) can now be expressed as follows:

$$u_{l+1}(\cdot) = \mathscr{P}_{\text{conv}(\mathscr{S})}\left[u_l(\cdot) - \alpha_l \nabla J(u_l(\cdot))\right] \tag{3.3}$$

where $\mathscr{P}_{\text{conv}(\mathscr{S})}$ is the operator of projection on to convex set $\text{conv}(\mathscr{S})$ and $\{\alpha_l\}$ is a sequence of step sizes. The conventional projection operator $\mathscr{P}_{\text{conv}(\mathscr{S})}$ is defined as usual:

$$\mathscr{P}_{\text{conv}(\mathscr{S})}(u(\cdot)) := \text{Argmin}_{v(\cdot)\in\text{conv}(\mathscr{S})}\left(||v(\cdot) - u(\cdot)||_{\mathbb{L}^2[t_0,t_f;\mathbb{R}^m]}\right)$$

Recall that the projected gradient iterations (3.3) generate a minimizing sequence for the convex optimization problem (3.2). Some useful mathematically exact convergence theorems for iterations (3.3) can be found in [11], [24], [33], [37]. We also refer to [9], [10], [19] for the related results. In the context of OCP (3.2) and method (3.3) the basic convergence result from [33], [37] can be reformulated as follows.

Theorem 3.4 *Assume that all the hypotheses from* Section *are satisfied. Consider a sequence of control functions generated by* (3.3)*. Then there exists an admissible initial data* $(u^0(\cdot), x^0(\cdot))$ *and a sequence of the step-sizes* $\{\alpha_l\}$ *such that* $\{u_l(\cdot)\}$ *is a minimizing sequence for* (3.2)*, i.e.,* $\lim_{l\to\infty} J(u_l(\cdot) = J(u^*(\cdot))$.

The proposed gradient-type method (3.3) provides a basis for the computational approach to (3.2). Using an optimal solution $u^*(\cdot) \in \text{conv}(\mathscr{S})$ we next need to determine a suitable approximation for a solution to the original OCP (1.1). In the next sections we propose a constructive numerical procedure for this purpose.

4. On the controllability condition for the linear system with a switched control structure

The study of OCPs with piecewise constant controls also involves a question of the general interest. Consider the initial dynamic system (1.1) determined on the given set of admissible controls $\mathscr{S}$ and reformulate the classical controllability question associated with the specific control set of piecewise constant inputs: system (1.1) on $\mathscr{S}$ is said to be controllable if for any initial state $x(t_0)$ and any final state $x(t_f)$, there exist an admissible function $u(\cdot) \in \mathscr{S}$ that transfers $x(t_0)$ to $x(t_f)$ in finite time. It is necessary to stress, that there are some (expectable) examples of non-controllable linear system involving the piecewise constant controls. In connection with this observation we formulate here a simple controllability criterion for the specific case of constant system/control matrices $A(t) \equiv A$, $B(t) \equiv B$ and unified switching times $N_k \equiv N$ $\mathbb{T}^k \equiv \mathbb{T}$ for all $k = 1,\ldots,m$.

Theorem 4.1 *Consider the stationary variant of the linear system* (1.1) *for* $u(\cdot) \in \mathscr{S}$ *and assume* $N_k \equiv N,\ \ \mathbb{T}^k \equiv \mathbb{T},\ \ k = 1,\ldots,m$. *Let*

$$W(N) := \sum_{i=1}^{N} \left[\int_{t_{i-1}}^{t_i} e^{-A\tau} d\tau BB^T \int_{t_{i-1}}^{t_i} e^{-A^T\tau} d\tau \right],\ t_i \in \mathbb{T}.$$

and assume that

$$-B^T \int_{t_{i-1}}^{t_i} e^{-A^T\tau} d\tau W(N)^{-1} \left(x(t_0) - e^{-At_f} x(t_f) \right) \in \mathbb{Q}, \tag{4.1}$$

where $\mathbb{Q} := \mathbb{Q}^1 \times \ldots \times \mathbb{Q}^m$. *Then system* (1.1) *is controllable if and only if matrix* $W(N)$ *is non-singular.*

Proof. Let $W(N)$ be non-singular. Then

$$x(t_f) = e^{At_f} x(t_0) + \int_{t_0}^{t_f} e^{A(t_f-\tau)} Bu(\tau) d\tau,$$

or equivalently

$$x(t_f) = e^{At_f} \left[x(t_0) + \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} e^{-A\tau} d\tau Bu^i \right], \tag{4.2}$$

where $u^i \in R^m$ is a constant vector associated with the interval $[t_{i-1}, t_i)$. The resulting input value u^i such that $u(t) = u^i$ for $t \in [t_{i-1}, t_i)$ and $x(t_0)$, $x(t_f)$ belongs to the corresponding trajectory of (1.1) generated by $u(\cdot)$ is given by

$$u^i = -B^T \int_{t_{i-1}}^{t_i} e^{-A^T\tau} d\tau W(N)^{-1} \left(x(t_0) - e^{-At_f} x(t_f) \right).$$

From (4.1) it follows $u^i \in Q$. Substituting the obtained expression in (4.2), we next obtain

$$x(t_f) = e^{At_f} [x(t_0) - W(N)W(N)^{-1}(x(t_0) - e^{-At_f} x(t_f))] = x(t_f).$$

We conclude that the given system is controllable under piecewise constant inputs.

Let now the initial system (1.1) be controllable by piecewise constant controls from $\mathscr{S}$. Assume that the symmetric matrix $W(N)$ is not a (strictly) positive definite matrix. This hypothesis implies the existence of a non-trivial vector $v \in \mathbb{R}^n$ such that $v^T W(N) v = 0$, or equivalently:

$$0 = v^T \sum_{i=1}^{N} \left[\int_{t_{i-1}}^{t_i} e^{-A\tau} d\tau BB^T \int_{t_{i-1}}^{t_i} e^{-A^T\tau} d\tau \right] v = \sum_{i=1}^{N} ||v^T \int_{t_{i-1}}^{t_i} e^{-A\tau} d\tau B||^2.$$

The last fact evidently implies the following $v^T \int_{t_{i-1}}^{t_i} e^{-A\tau} d\tau B = 0 \; \forall i = 1, \dots, N$. Since the controllability of the system (1.1) for $u(\cdot) \in \mathscr{S}$ is assumed, there exist a sequence of values $\{u^i\}$ such that the state $x(t_0) \equiv v$ can be transferred into $x(t_f) \equiv 0$. Therefore, we deduce the next consequence

$$0 = e^{At_f} \left[v + \sum_{i=1}^{N} \left(\int_{t_{i-1}}^{t_i} e^{-A\tau} d\tau \right) Bu^i \right]. \tag{4.3}$$

Evidently, (4.3) holds if and only if

$$0 = v + \sum_{i=1}^{N} \left(\int_{t_{i-1}}^{t_i} e^{-A\tau} d\tau \right) Bu^i. \tag{4.4}$$

We now multiply (4.4) by v^T

$$0 = v^T v + \sum_{i=1}^{N} v^T \left(\int_{t_{i-1}}^{t_i} e^{-A\tau} d\tau \right) Bu^i = v^T v$$

and obtain the contradiction with the non-triviality hypothesis $v \neq 0$. Therefore, $W(N)$ is a positive definite symmetric matrix and the existence of the inverse $W(N)^{-1}$ follows immediately. The proof is completed. ∎

Note that Theorem 4.1 makes it possible to establish the existence of an optimal solution to the restricted OCP of the type (1.6) with additional terminal constraint $x(t_f) = x_f$, where $x_f \in \mathbb{R}^n$ is a prescribed final state. We refer to [21], [44] for the classic result and for the corresponding regularity conditions in some classes of constrained OCPs.

5. A relaxation based numerical method for the initial optimal control problem

Theorem 3.4 and the classic gradient-type iterations (3.3) provide an analytic basis for a consistent computational approach to the initial OCP (1.6). Recall that in contrast to the relaxed optimization problem (3.2) the original OCP (1.6) does not possesses any convexity property. However, the simple relaxed OCP (3.2) can be effectively used for an approximative numerical treatment of the original problem (1.6). Let us introduce the formal Hamiltonian associated with problems (1.6) and (3.2)

$$H(t,x,u,p) = \langle p, A(t)x + B(t)u \rangle - \frac{1}{2} \left(\langle Q(t)xt, x \rangle + \langle R(t)u, u \rangle \right).$$

where $p \in \mathbb{R}^n$ is the adjoint variable. By $\hat{u}(\cdot) \in \mathscr{S}$ we now denote an optimal solution to the initial OCP (1.6). Using the explicit representation of the gradient $\nabla J(u_l(\cdot))$ in OCPs with ordinary differential equations (see e.g., [3], [4], [20], [33], [38], [42]), we can propose a conceptual computational scheme for (1.6).

Conceptual Algorithm 1

(0) *Set the initial condition for the iterative scheme* $u_{(0)}(\cdot) := \mathscr{P}_{\mathrm{conv}(\mathscr{S})}(u^{opt}(\cdot))$, *where* $u^{opt}(\cdot)$ *is the optimal control input* (1.3) *from the classic* LQ *problem. Calculate the corresponding trajectory* $x_{(l)}(\cdot)$ *of* (1.1) *and put the iterations index* $l := 0$.

(1) *Calculate* $\nabla J(u_{(l)})(\cdot)$ *as* (see [3], [4], [7], [33], [42])

$$\nabla J(u_{(l)})(t) = -\frac{\partial H(t, x_{(l)}(t), u_{(l)}(t), p(t))}{\partial u} = -B^T(t)p(t) + R(t)u_{(l)},$$

where the adjoint variable $p(\cdot)$ *is a solution to the usual boundary value problem*

$$\dot{p}(t) = -\frac{\partial H(t, x_{(l)}(t), u_{(l)}(t), p)}{\partial x} = -A^T(t)p(t) + Q(t)x_{(l)}(t),$$
$$p(t_f) = -Gx_{(l)}(t_f).$$

(2) *Calculate the projection of* $u_{(l)}(\cdot) - \alpha_{(l)}\nabla J(u_{(l)}(\cdot))$ *on the convex* (*relaxed*) *restriction set* $\mathrm{conv}(\mathscr{S})$ *and determine* $\bar{u}_{(l+1)}(\cdot) := \mathscr{P}_{\mathrm{conv}(\mathscr{S})}(\bar{u}_{(l)}(\cdot))$.

(3) *Evaluate the* $(l+1)$ *iteration of the control function given by components*

$$u^{(k)}_{(l+1)}(t) = \sum_{i=1}^{N_k} I_{[t^{(k)}_{i-1}, t^{(k)}_i)}(t)\bar{q}^{(k)}_{i,n} \quad \forall k = 1, \dots, m,$$

where:

$$\bar{q}^{(k)}_{i,l} := \begin{cases} q^{(k)}_1, & \bar{\bar{q}}^{(k)}_{i,l} < q^{(k)}_1 \\ \bar{\bar{q}}^{(k)}_{i,l}, & q^{(k)}_1 \leq \bar{\bar{q}}^{(k)}_{i,l} \leq q^{(k)}_{M_k} \\ q^{(k)}_{M_k}, & q^{(k)}_{M_k} \leq \bar{\bar{q}}^{(k)}_{i,l} \end{cases} , i = 1, \dots, N_k.$$

and $q^{(k)}_j \in \mathbb{Q}^k$, $\forall j = 1, \dots, M_k$, $\bar{\bar{q}}^{(k)}_{i,l} := \frac{1}{\Delta_i}\int_{t_{i-1}}^{t_i} \bar{u}^{(k)}_{(l)}(t)dt$, $\Delta_i := t_i - t_{i-1}$.

(4) *Calculate the difference* $|J(u_{(l+1)}(\cdot)) - J(u_{(l)}(\cdot))|$. *If it is less than a prescribed accuracy* $\varepsilon > 0$, *then we put* $u^*(\cdot) \equiv u_{(l+1)}(\cdot)$ *(an approximating optimal solution to* (3.2)) *and* Stop. *Else, update the iteration register and go to* Step (1).

(5) *Using the evaluated function* $u^*(\cdot)$ *the approximating optimal control* $\hat{u}(\cdot) \in \mathscr{S}$ *can finally be calculated by components*

$$\hat{u}^{(k)}(\cdot) = \sum_{i=1}^{N_k} I_{[t^{(k)}_{i-1}, t^{(k)}_i)}(t)\hat{q}^{(k)}_i \quad \forall k = 1, \dots, m. \tag{5.1}$$

where $\hat{q}^k_i := \mathrm{Arg}\min_{v \in \mathbb{Q}^k} |v - \bar{q}^{(k)}_{i,l+1}|$. *Solve* (1.1) *with the obtained control input* $\hat{u}(\cdot) \in \mathscr{S}$ *and obtain the approximating optimal trajectory* $\hat{x}(\cdot)$. Stop.

Using Theorem 3.4 and the continuity property of the objective functional one can establish the convergence of the proposed Conceptual Algorithm 1. Note that this type of convergence is determined as a "convergence in functional". To put it another way, the control sequence $\{u_l(\cdot)\}$ generated by Steps (0)-(4) of the above Algorithm is a minimizing sequence (see Theorem 3.4). By implementation and taking into consideration the continuity of the objective functional, we finally can establish the convergence property ("in functional") of the resulting sequences $\{\hat{u}^{(k)}(\cdot)\}$, $k = 1,\dots,m$ obtained in Step (5) of Algorithm.

We now illustrate the effectiveness of the proposed Conceptual Algorithm 1 and consider two simple examples.

Example 5.1 Consider the following linear system

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -x_1(t) + u(t) \end{bmatrix},\ t \in [0,5],\ x(0) = (1,-1)^T$$

associated with the quadratic cost functional

$$J(u(\cdot)) = \frac{1}{2}\int_0^5 \left(x_1^2(t) + 10x_2^2(t) + u^2(t)\right) dt,$$

Let $\mathbb{Q} = \{0, 0.25, 0.5, 0.75, 1, 1.25, 1.5, \dots, 5\}$ be the given finite set of constant control values. Assume $N_k = 10$, $k = 1$ and the set $\mathbb{T}$ is not given a priory. The classic LQ optimal control $u^{opt}(\cdot)$ can be here easily calculated. Applying the proposed Conceptual Algorithm 1, we now compute the approximating optimal control $\hat{u}(\cdot)$ (see Fig. 2).

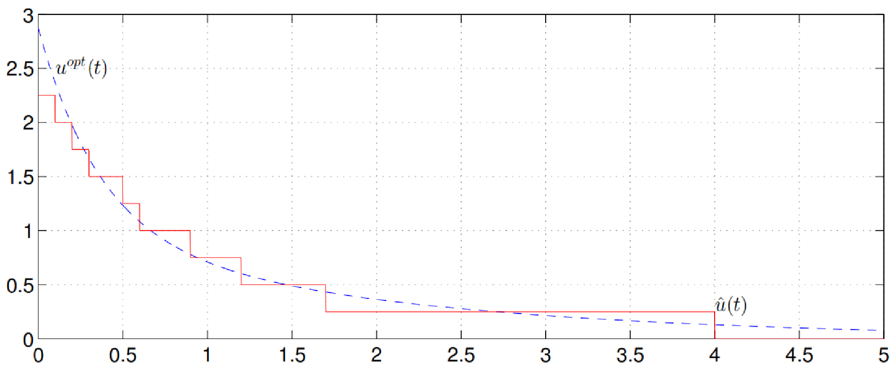

Figure 2: Control inputs $u^{opt}(t)$ and $\hat{u}(t)$

The associated trajectory $\hat{x}(\cdot)$ is indicated on Fig. 3. The calculated cost in problem (1.6) associated with our example was evaluated as follows: $J(\hat{u}(\cdot)) = 7.5362$. Evidently, this value is higher in comparison with the optimal cost in the conventional (non-restricted) LQ problem.

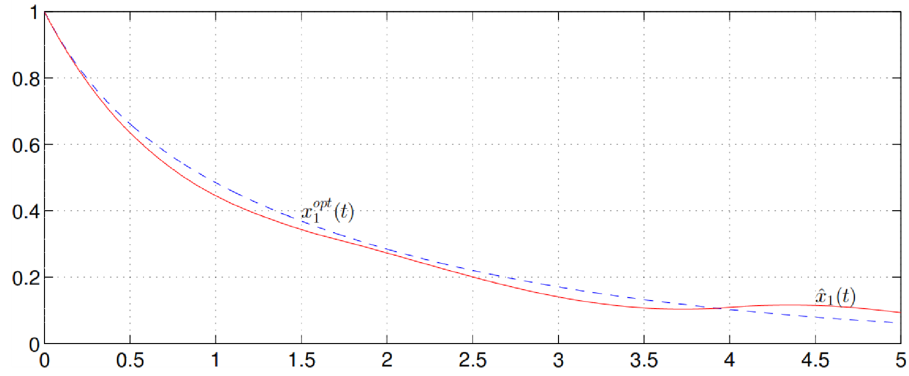

Figure 3: First components of the optimal trajectories $x_1^{opt}(t)$ and $\hat{x}_1(t)$

Example 5.2 We now consider (1.1) for $n = 3$

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -0.875x_2(t) - 20x_3(t) \\ -50x_3(t) + 50u(t) \end{bmatrix},$$
$$x(0) = (1, 0, -1)^T,$$

where $t \in [0, 1]$. The quadratic cost functional in problem (1.6) associated with our example has been given in the following concrete form

$$J(u(\cdot)) = \frac{1}{2} \int_0^1 \left(3x_1^2(t) + x_2^2(t) + 2x_3^2(t) + u^2(t)\right) dt,$$

We next assume

$$\mathbb{Q} = \{-5, -4.5, -4, -3.5, \ldots, 3.5, 4, 4.5, 5\},\ N_k = 3,\ k = 1.$$

The set $\mathbb{T}$ is not given a priory. Application of the proposed numerical solution procedure, namely, of Conceptual Algorithm 1 leads to the computational results for the quasi-optimal control $\hat{u}(\cdot)$ and the corresponding trajectory $\hat{x}(\cdot)$). These numerically optimal functions are indicated on Fig. 4 and Fig. 5, respectively.

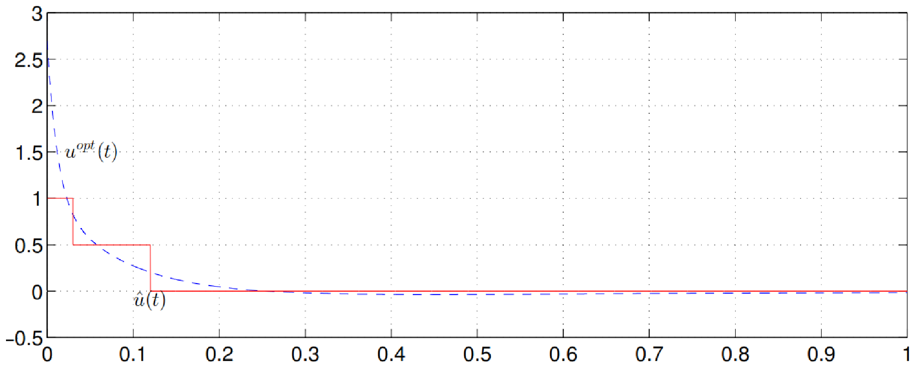

Figure 4: Control inputs $u^{opt}(t)$ and $\hat{u}(t)$

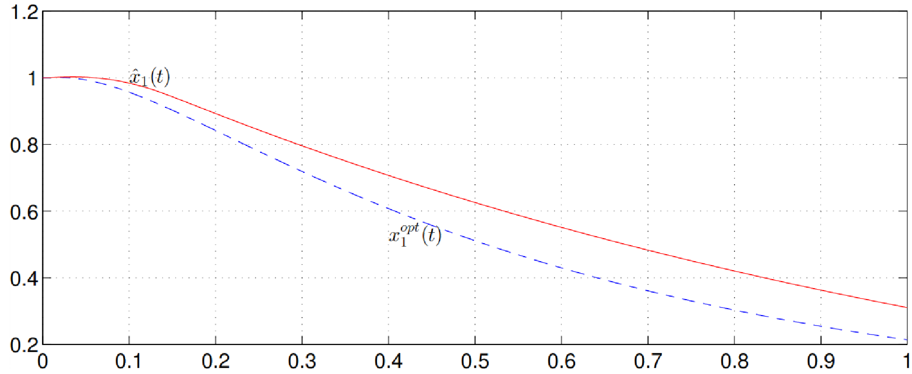

Figure 5: First components of the trajectories $x_1^{opt}(t)$ and $\hat{x}_1(t)$

The calculated cost associated with the initial OCP (1.6) in this example was evaluated as follows: $J(\hat{u}(\cdot)) = 2.0237$. As mentioned above, the calculated optimal cost here has a higher value in comparison with the optimal cost in the conventional LQ problem. This fact is a simple consequence of the evident inclusion $\mathscr{S} \subset \mathbb{L}^2[t_0,t_f;\mathbb{R}^m]$ that constitutes the admissible control set restrictions in the constrained LQ problem under consideration.

Finally, note that implementations of the Conceptual Algorithm 1 presented in Examples 4.1 and 4.2, was carried out, using the standard MATLAB packages and the Authors programs.

6. Conclusion

In this contribution, we have developed a new implementable numerical approach to a constrained LQ-type OCP. This computational method is based on a simple convex relaxation procedure applied to the initial problem in combination with the conventional gradient-based numerical technique. We firstly rewrite the original (non-convex) OCP in a relaxed form and establish the convexity properties. We next use the obtained convex relaxation as an auxiliary tool in a concrete solution scheme for the initial OCP. The convex structure of the auxiliary OCP makes it possible to take into consideration diverse powerful computational algorithms from the classic convex programming. Let us note that various variants of the basic gradient method, namely, Armijo-type gradient schemes can be applied to the obtained relaxed OCP (see [1], [2], [9], [10], [11], [19], [20], [42]). In the presented paper we also discussed the general controllability question associated with the stationary variant of the constrained linear dynamic systems under consideration. The general controllability concept for linear systems of the type (1.1) involves a full theoretic justification of the OCP under consideration and finally, makes it possible to establish the applicability of the gradient method in the presented form (see [10], [11] for details).

It is common knowledge that modern numerical algorithms mainly use specific non-equidistant discretizations with the aim to increase the effectiveness of the resulting algorithm. The specific type of the control functions discussed in our contribution is motivated by the initially given physical nature of the class of controlled processes under consideration. Note that there are various formal models that involve a non-equally spaced inputs grid. The necessary investigation of these types of models and the corresponding engineering applications (mainly from the modern communication science) constitutes an interesting subject of a next contribution. Our paper focuses on equally-spaced controlled models since they are simpler to analyse while capturing the salient features of the newly elaborated control method we propose. Let us also note that the presented Conceptual Algorithm 1 need to be analysed in comparison with some powerful numerical schemes. For example, one needs to compare it with the various implementations of the direct search (see e.g., [27], [42]).

Finally, note that the theoretical and computational approaches presented in this paper can be applied to some alternative classes of constrained OCPs. Let us also note that the proposed numerical algorithm can also constitute a constructive tool of some general numerical techniques based on discretizations and linear approximations associated with the common types of non-linear OCPs.

References

[1] ALI, U., CAI, H., MOSTOFI, Y., WARDI, Y., *Motion and communication co-optimization with path planning and online channel estimation*, 2016 American Control Conference, submitted.

[2] ARMIJO, L., *Minimization of functions having Lipschitz continuous first partial derivatives*, Pacific Journal of Mathematics, 16 (1966), 1-3.

[3] AZHMYAKOV, V., RAISCH, J., *Convex control systems and convex optimal control problems with constraints*, IEEE Transactions on Automatic Control, 53 (2008), 993 – 998.

[4] AZHMYAKOV, V., BOLTYANSKI, V., POZNYAK, A., *Optimal control of impulsive hybrid systems*, Nonlinear Analysis: Hybrid Systems, 2 (2008), 1089-1097.

[5] AZHMYAKOV, V., GALVAN-GUERRA, R., EGERSTEDT, M., *Hybrid LQ-optimization using Dynamic Programming*, in: ”Proceedings of the 2009 American Control Conference”, St. Louis, USA, 2009, 3617-3623.

[6] AZHMYAKOV, V., GALVAN-GUERRA, R., EGERSTEDT, M., *On the LQ-based optimization technique for impulsive hybrid control systems*, in: ”Proceedings of the 2010 American Control Conference”, Baltimore, USA, 2010, 129-135.

[7] AZHMYAKOV, V., BASIN, M.V., GIL GARCIA, A.E., *Optimal control processes associated with a class of discontinuous control systems: applications to sliding mode dynamics*, Kybernetika, 50 (1) (2014), 5-18.

[8] AZHMYAKOV, V., SERREZUELA, R.R., TRUJILLO, L.A.G., *Approximations based optimal control design for a class of switched dynamic systems*, in: ”Proceedings of the 40th Annual Conference of the IEEE Industrial Electronics Society”, Dallas, USA, 2014, 90-95.

[9] AZHMYAKOV, V., JUAREZ, R., *On the projected gradient methods for switched-mode systems optimization*, in: ”Proceedings of the 5th IFAC Conference on Analysis and Design of Hybrid Systems”, Atlanta, USA, 2015, 181-186.

[10] AZHMYAKOV, V., JUAREZ, R., PICKL, S., *On the local convexity of singular optimal control problems associated with the switched-mode dynamics*, in: ”Proceedings of the 16th IFAC Workshop on Control Applications of Optimization”, Garmisch-Partenkirchen, Germany, 2015, 271-276.

[11] AZHMYAKOV, V., MARTINEZ, J.C., POZNYAK, A., *Optimal fixed-levels control for non-linear systems with quadratic cost-functionals*, Optimal Control: Applications and Methods, to appear in 2016.

[12] BANKS, S.P., KHATHUR, S.A., *Structure and control of piecewise-linear systems*, International Journal of Control, 50 (1989), 667-686.

[13] BETTS, J., *Practical Methods for Optimal Control Problems Using Nonlinear Programming*, SIAM, Philadelphia, USA, 2001.

[14] BOLTYANSKI, V., POZNYAK, A., *The Robust Maximum Principle*, Birkhaeuser, New York, USA, 2012.

[15] BONILLA, M., MALABRE, M., AZHMYAKOV, V., *An implicit systems characterization of a class of impulsive linear switched control processes*. Part 1: *Modeling*, Nonlinear Analysis: Hybrid Systems, 15 (2015), 157-170.

[16] BRANICKY, M.S., BORKAR, V.S., MITTER, S.K., *A unifed framework for hybrid control: model and optimal control theory*, IEEE Transactions on Automatic Control, 43 (1998), 31-45.

[17] BROCKETT, R.W., LIBERZON, D., *Quantized feedback stabilization of linear systems*, IEEE Transactions on Automatic Control, 45 (2000), 1279-1289.

[18] CASSANDRAS, C., PEPYNE, D.L., WARDI, Y., *Optimal control of class of hybrid systems*, IEEE Transactions on Automatic Control, 46 (2001), 398-415.

[19] DING, X.C., WARDI, Y., EGERSTEDT, M., *On-line optimization of switched-mode dynamical systems*, IEEE Transactions on Automatic Control, 54 (2009), 2266-2271.

[20] EGERSTEDT, M., WARDI, Y., AXELSSON, H., *Transition-time optimization for switched-mode dynamical systems*, IEEE Transactions on Automatic Control, 51 (2006), 110-115.

[21] FATTORINI, H.O., *Infinite-Dimensional Optimization and Control Theory*, Cambridge University Press, Cambridge, UK, 1999.

[22] GALVAN-GUERRA, R., AZHMYAKOV, V., EGERSTEDT, M., *Optimization of the multiagent systems with increasing state dimensions: a hybrid LQ approach*, in: "Proceedings of the 2011 American Control Conference", San Francisco, USA, 2011, 881-887.

[23] GARAVELLO, M., PICCOLI, B., *Hybrid necessary principle*, SIAM Journal on Control and Optimization, 43 (2005), 1867-1887.

[24] GILL, P.E., MURRAY, W., WRIGHT, M.H., *Practical Optimization*, Academic Press, New York, USA, 1981.

[25] GOEBEL, R., SUBBOTIN, M., *Continuous time linear quadratic regulator with control constraints via convex duality*, IEEE Transactions on Automatic Control, AC-52, 2007, 886-892.

[26] GOODWIN, G.C., SERON, M.M., DONA, J.A., *Constrained Control and Estimation*, Springer, London, UK, 2005.

[27] KIRK, D., *Optimal Control Theory*, Dover, New York, USA, 1998.

[28] KOJIMAA, A., MORARI, M., *LQ control for constrained continuous-time systems*, Automatica, 40 (2004), 1143-1155.

[29] LOZOVANU, D., PICKL, S., *Determining the optimal strategies for discrete control problems on stochastic networks with discounted costs*, Discrete Applied Mathematics, 182 (2015), 169-180.

[30] LOZOVANU, D., PICKL, ST., *Optimization of Stochastic Discrete Systems and Control on Complex Networks: Computational Networks*, Springer, Basel, Switzerland, 2014.

[31] LYGEROS, J., *Lecture Notes on Hyrid Systems*, Cambridge University Press, Cambridge, UK, 2003.

[32] MARTINEZ, J.C., AZHMYAKOV, V., *Optimal switched-type control design for a class of nonlinear systems*, in: "Proceedings of the IEEE International Conference on Automation Science and Engineering", Madison, USA, 2013, pp. 1069-1074.

[33] POLAK, E., *Optimization*, Springer-Verlag, New York, USA, 1997.

[34] POZNYAK, A., POLYAKOV, A., AZHMYAKOV, V., *Attractive Ellipsoids in Robust Control*, Birkhäuser, Basel, Switzerland, 2014.

[35] PYTLAK, R., *Numerical Methods for Optimal Control Problems with State Constraints*, Springer-Verlag, Berlin, Germany, 1999.

[36] RANTZER, A., JOHANSSON, M., *Piecewise linear quadratic optimal control*, IEEE Transactions on Automatic Control, 45 (2000), 629-637.

[37] ROCKAFELLAR, R.T., *Convex Analysis*, Princeton University Press, Princeton, USA, 1970.

[38] SHAIKH, M.S., CAINES, P.E., *On the hybrid optimal control problem: theory and algorithms*, IEEE Transactions on Automatic Control, 52 (2007), 1587-1603.

[39] STANTON, S.A., MARCHAND, B.G., *Finite set control transcription for optimal control applications*, Journal of Spacecraft and Rockets, 47 (2010), 457-471.

[40] STOER, J., BULIRSCH, R., *Introduction to Numerical Analysis*, Springer, New York, USA, 2002.

[41] TARINGOO, F., CAINES, P., *The sensitivity of hybrid systems optimal cost functions with respect to switching manifold parameters*, in: Lecture Notes in Computer Science, vol. 5469, Springer, Berlin, Germany, 2009, 475-479.

[42] TEO, K.L., GOH, C.J., WONG, K.H., *A Unifed Computational Approach to Optimal Control Problems*, Wiley, New York, USA, 1991.

[43] VERRIEST, E.I., *Multi-mode multi-dimensional systems with Poissonian sequencing*, Communications in Information and Systems, 9 (1) (2009), 77-102.

[44] WONHAM, W.M., *Linear Multivariable Control: A Geometric Approach*, Springer, New York, USA, 1985.

[45] XU, X., ANTSAKLIS, P.J., *Optimal control of hybrid autonomous systems with state jumps*, in: "Proceedings of the 2003 American Control Conference", Denver, 2003, 5191-5196.

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FALLING FUZZY GÖDEL IDEALS OF BL-ALGEBRAS

Biao Long Meng[1]

Department of Mathematics
Northwest University
Xi'an, 710127
P.R. China
College of Science
Xi'an University of Science and Technology
Xi'an, 710054
P.R. China

Xiao Long Xin

Department of Mathematics
Northwest University
Xi'an, 710127
P.R. China

Abstract. The notions of falling fuzzy (resp. Gödel, Boolean, implicative) ideals of a BL-algebra are introduced based on the theory of falling shadows and fuzzy sets. Several characterizations and relations of these notions are studied. Finally we apply the concept of falling fuzzy inference relations to ideal theory of BL-algebras and obtain some related results.

Keywords: BL-algebra, falling shadow, ideal, fuzzy ideal, falling fuzzy ideal, falling fuzzy Gödel(resp. Boolean, implicative) ideal, falling fuzzy relation.

1. Introduction

Goodman [2] pointed out the equivalence of a fuzzy set and a class of random sets by means of combining probability and fuzzy set theory. Also, Wang and Sanchez [18] introduced the theory of falling shadows which directly relates probability concepts with the membership function of fuzzy sets. Falling shadow representation theory shows us the way of selection related on the joint degrees distributions. It is a reasonable and convenient approach for the theoretical development and the practical applications of fuzzy sets and fuzzy logics. The mathematical structure of the theory of falling shadows is formulated. In particular, Tan et al. ([12], [13]) established a theoretic approach to define a fuzzy inference

[1]Corresponding author. E-mail address: mengbl_100@139.com

relation and fuzzy set operation based on the theory of falling shadows. Yuan et al. [19] considered a fuzzy subgroup (resp. subring, ideal) as the falling shadow of a cloud of the subgroups (resp. subring, ideal). Recently, Jun et al. ([5], [6]) considered the notions of falling fuzzy (resp. positive implicative, commutative, implicative) ideals of BCK-algebras. Zhan et al. [21] applied the falling shadow theory to filter theory of BL-algebras.

The notion of BL-algebra was initiated by Hájek [3] in order to provide an algebraic proof of the completeness theorem of Basic Logic (BL, in short). Soon after Cignoli et al. [1] proved that Hájek's logic really is the logic of continuous t-norms as conjectured by Hájek. One important aspect of BL-algebras is its filter theory. Researchers started a systematic study of BL-algebras with filter theory ([4], [10], [14], [15]). Another important aspect of BL-algebras is ideal theory, which was introduced by Hájek [3]. Some properties of ideals were investigated by ([8],[11]). Fuzzy ideal theory in BL-algebras were studied by Zhang et al. [22] and Meng et al. [9]. The notions of fuzzy prime ideals, fuzzy irreducible ideals, fuzzy Gödel ideals and fuzzy Boolean ideals were introduced. Several relations among these ideals were discussed. In the present paper we introduce the notions of falling fuzzy (resp. Gödel, Boolean, implicative) ideals of a BL-algebra based on the theory of falling shadows and fuzzy sets, and study several characterizations and relations of these notions. Finally we apply the concept of falling fuzzy inference relations to BL-algebras and obtain some related results.

2. Preliminaries

Let us recall some definitions and results on BL-algebras.

Definition 2.1. [3] An algebra $(A;\ \wedge, \vee, *, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ is called a BL-algebra if it satisfies the following conditions:

(BL1) $(A;\ \wedge, \vee, 0, 1)$ is a bounded lattice,
(BL2) $(A;\ *, 1)$ is a commutative monoid,
(BL3) $x * z \leq y$ if and only if $z \leq x \rightarrow y$ (residuation),
(BL4) $x \wedge y = x * (x \rightarrow y)$, thus $x * (x \rightarrow y) = y * (y \rightarrow x)$ (divisibility),
(BL5) $(x \rightarrow y) \vee (y \rightarrow x) = 1$ (prelinearity).

Throughout this paper, A means a BL-algebra without mentioned otherwise. In the sequel, we agree the operations $*$, $\vee$, $\wedge$ have priority towards the operation $\rightarrow$.

Proposition 2.2. ([3],[8],[10],[14]) *For all $x, y, z \in A$, the following are valid:*

(1) $x * (x \rightarrow y) \leq y$,
(2) $x \leq y \rightarrow (x * y)$,
(3) $x \leq y$ if and only if $x \rightarrow y = 1$,
(4) $x \rightarrow (y \rightarrow z) = (x * y) \rightarrow z = y \rightarrow (x \rightarrow z)$,
(5) $x \leq y$ implies $z \rightarrow x \leq z \rightarrow y$, $y \rightarrow z \leq x \rightarrow z$,

(6) $y \leq (y \to x) \to x$,
(7) $(x \to y) * (y \to z) \leq x \to z$,
(8) $y \to x \leq (z \to y) \to (z \to x)$,
(9) $x \to y \leq (y \to z) \to (x \to z)$,
(10) $x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x)$,
(11) $x \leq y$ implies $y^- \leq x^-$,
(12) $1 \to x = x,\ x \to x = 1,\ x \to 1 = 1$,
(13) $x \leq y \to x$, *or equivalently,* $x \to (y \to x) = 1$,
(14) $((x \to y) \to y) \to y = x \to y$,
(15) $1^- = 0,\ 0^- = 1, 1^{--} = 1, 0^{--} = 0$,
(16) $x \to (y \vee z) = (x \to y) \vee (x \to z), x \to (y \wedge z) = (x \to y) \wedge (x \to z)$,
(17) $(x \vee y) \to z = (x \to z) \wedge (y \to z), (x \wedge y) \to z = (x \to z) \vee (y \to z)$,
(18) $(x \vee y)^- = x^- \wedge y^-,\ \ (x \wedge y)^- = x^- \vee y^-$,
(19) $x \to y \leq x * z \to y * z$,
(20) $x \to y \leq x \wedge z \to y \wedge z, x \to y \leq x \vee z \to y \vee z$,
(21) $(x \to y^-)^{--} = x \to y^-$,
(22) $x * x^- = 0$,
(23) $x * (y \vee z) = (x * y) \vee (x * z), x * (y \wedge z) = (x * y) \wedge (x * z)$.

where $x^- = x \to 0$.

Definition 2.3.[3] A nonempty subset I of A is said to be an ideal of A if it satisfies:

(I1) $0 \in I$,
(I2) $x \in I$ and $(x^- \to y^-)^- \in I$ implies $y \in I$ for all $x, y \in A$.

Proposition 2.4. [8] *Let I be an ideal of A. Then for any $x \in A$, $x \in I$ if and only if $x^{--} \in I$.*

Definition 2.5. [20] A fuzzy set in X is a mapping $\mu : X \longrightarrow [0, 1]$.

Let μ be a fuzzy set in X, $t \in [0, 1]$, the set $\mu_t = \{x \in X \mid \mu(x) \geq t\}$ is called a level subset of μ.

By $\mu \leq \nu$ we mean that $\mu(x) \leq \nu(x)$ for all $x \in X$.

Definition 2.6. [22] A fuzzy set μ in A is called a fuzzy ideal of A, if for all $x, y \in A$,

(FI1) $\mu(0) \geq \mu(x)$,
(FI2) $\mu(y) \geq \mu(x) \wedge \mu((x^- \to y^-)^-)$.

Proposition 2.7. [22] *Let μ be a fuzzy set in A. Then μ is a fuzzy ideal of A if and only if, for any $t \in [0, 1]$, μ_t is either empty or an ideal of A.*

In what follows we display the basic theory of falling shadows([12],[13], [17]).

Given a universe of discourse U and $\mathscr{P}(U)$ be the power set of U. For each $u \in U$ and $E \in \mathscr{P}(U)$, denote

(FS1) $\dot{u} := \{E \in \mathscr{P}(U) \mid u \in E\}$;
(FS2) $\dot{E} := \{\dot{u} \mid u \in E\}$.

An ordered pair $(\mathscr{P}(U), \mathscr{B})$ is called a hyper measurable structure if $\mathscr{B}$ is a σ-field in $\mathscr{P}(U)$ and $\dot{U} \subseteq \mathscr{B}$.

Let $(\Omega, \mathscr{A}, P)$ be a probability space and $(\mathscr{P}(U), \mathscr{B})$ be a hyper measurable structure. Suppose that a mapping $\xi : \Omega \to \mathscr{P}(U)$ satisfies, for any $C \in \mathscr{B}$, the set $\xi^{-1}(C) = \{\omega \in \Omega \mid \xi(\omega) \in C\} \in \mathscr{A}$. Then ξ is called a random set on U.

Suppose that ξ is a random set on U. Let

$$\tilde{H}(u) := P(\{\omega \in \Omega \mid u \in \xi(\omega)\}) \text{ for each } \ u \in U.$$

Then $\tilde{H}$ is a kind of fuzzy set in U. We call $\tilde{H}$ a falling shadow of the random set ξ, and ξ is a cloud of $\tilde{H}$.

For example, $(\Omega, \mathscr{A}, P) = ([0,1], \mathscr{A}, m)$, where $\mathscr{A}$ is a Borel field on $[0,1]$ and m the usual Lebesgue measure. Let $\tilde{H}$ be a fuzzy set in U and $\tilde{H}_t := \{u \in U \mid \tilde{H}(u) \geq t\}$ be a t-cut of $\tilde{H}$. Then

$$\xi : [0,1] \to \mathscr{P}(U), t \mapsto \tilde{H}_t.$$

is a random set and ξ is a cloud of $\tilde{H}$. We shall call ξ defined above as the cut-cloud of $\tilde{H}$([2]).

3. Falling fuzzy ideals

Definition 3.1. Let $(\Omega, \mathscr{A}, P)$ be a probability space and $\xi : \Omega \to \mathscr{P}(A)$ be a random set. If $\xi(\omega)$ is an ideal of A for any $\omega \in \Omega$, then the falling shadow $\tilde{H}$ of the random set ξ where

$$\tilde{H}(x) = P(\{\omega \in \Omega \mid x \in \xi(\omega)\}) \text{ for each } x \in A.$$

is called a falling fuzzy ideal of A.

Let $(\Omega, \mathscr{A}, P)$ be a probability space and $F(A) = \{f \mid f : \Omega \to A\}$. Define two binary operations $\otimes$ and $\rightharpoonup$ on $F(A)$ by $(f \otimes g)(\omega) = f(\omega) * g(\omega)$ and $(f \rightharpoonup g)(\omega) = f(\omega) \to g(\omega)$ for all $\omega \in \Omega$ and for all $f, g \in F(A)$.

Let $e, \theta \in F(A)$ be defined by $e(\omega) = 1, \theta(\omega) = 0$ for all $\omega \in \Omega$. For short, denotes $f^- := f \rightharpoonup \theta$. Then it is easy to check that $(F(A), \otimes, \rightharpoonup, \theta, e)$ is a BL-algebra.

For any subset B of A and $f \in F(A)$, let $B_f := \{\omega \in \Omega \mid f(\omega) \in B\}$,

$$\begin{aligned} \xi :\ & \Omega \to \mathscr{P}(F(A)), \\ & \omega \mapsto \{f \in F(A) \mid f(\omega) \in B\}. \end{aligned}$$

then $B_f \in \mathscr{A}$([21]).

Theorem 3.2. *If B is an ideal of A, then $\xi(\omega) = \{f \in F(A) \mid f(\omega) \in B\}$ is an ideal of $F(A)$.*

Proof. Let B be an ideal of A and $\omega \in \Omega$. By $\theta(\omega) = 0 \in B$ it follows that $\theta \in \xi(\omega)$. If $(f^- \rightharpoonup g^-)^- \in \xi(\omega)$ and $f \in \xi(\omega)$ for any $f, g \in F(A)$, then $((f(\omega))^- \to (g(\omega))^-)^- = (f^- \rightharpoonup g^-)^-(\omega) \in B$ and $f(\omega) \in B$. Since B is an ideal of A, we have $g(\omega) \in B$, and so $g \in \xi(\omega)$. Therefore $\xi(\omega)$ is an ideal of A. ∎

Example 3.3. [8] Let $A = \{0, a, b, 1\}$. Define $*$, $\to$, $\vee$ and $\wedge$ as follows:

$*$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	1	1	1

$\vee$	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

$\wedge$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

Then $(A; \vee, \wedge, *, \to, 0, 1)$ is a BL-algebra. It is easy to check that $\{0\}, \{0, a\}, \{0, b\}, A$ are ideals of A. But $\{0, a, b\}$ is not an ideal of A, because $(a^- \to 1^-)^- = (b \to 0)^- = a^- = b \in \{0, a, b\}$ and $1 \notin \{0, a, b\}$. Let $(\Omega, \mathscr{A}, P) = ([0, 1], \mathscr{A}, m)$, where $\mathscr{A}$ is a Borel field on $[0, 1]$ and m the usual Lebesgue measure. Let $\xi : [0, 1] \to \mathscr{P}(A)$ be defined by

$$\xi(t) := \begin{cases} \{0, a\} & \text{if } t \in [0, 0.4), \\ \{0, b\} & \text{if } t \in [0.4, 0.9), \\ \{0, a, b, 1\} & \text{if } t \in [0.9, 1]. \end{cases}$$

Thus $\xi(t)$ is an ideal of A for all $t \in [0, 1]$. Hence $\tilde{H}$ is a falling fuzzy ideal of A where $\tilde{H}(x) = P(\{t \in [0, 1] \mid x \in \xi(t)\})$ and

$$\tilde{H}(x) := \begin{cases} 1 & \text{if } x = 0, \\ 0.5 & \text{if } x = a, \\ 0.6 & \text{if } x = b, \\ 0.1 & \text{if } x = 1. \end{cases}$$

Note 3.4. Observe that in the above example, $\tilde{H}_{0.5} = \{0, a, b\}$ is not an ideal of A, hence $\tilde{H}$ is not a fuzzy ideal in A.

Example 3.5. [22] Let $A = \{0, a, b, c, d, 1\}$. Define $\vee$, $\wedge$, $*$ and $\to$ as follows:

$*$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	d	c	0	d	a
b	0	c	b	c	0	b
c	0	0	c	0	0	c
d	0	d	0	0	d	d
1	0	a	b	c	d	1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	b	b	a	1
b	d	a	1	a	d	1
c	a	1	1	1	a	1
d	b	1	b	b	1	1
1	0	a	b	c	d	1

$\vee$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	a	1	a	a	1
b	b	1	b	b	1	1
c	c	a	b	c	a	1
d	d	a	1	a	d	1
1	1	1	1	1	1	1

$\wedge$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	c	c	d	a
b	0	c	b	c	0	b
c	0	c	c	c	0	c
d	0	d	0	0	d	d
1	0	a	b	c	d	1

Then $(A; \vee, \wedge, *, \to, 0, 1)$ is a BL-algebra. It can check that $\{0\}, \{0, b, c\}$ and A are ideals of A. Let $\xi : [0,1] \to \mathscr{P}(A)$ be defined by

$$\xi(t) := \begin{cases} \{0, b, c\} & \text{if} \quad t \in [0, 0.5), \\ A & \text{if} \quad t \in [0.5, 1]. \end{cases}$$

Thus $\xi(t)$ is an ideal of A for all $t \in [0,1]$. Hence $\tilde{H}$ is a falling fuzzy ideal of A where $\tilde{H}(x) = P(\{t \in [0,1] \mid x \in \xi(t)\})$ and

$$\tilde{H}(x) := \begin{cases} 1 & \text{if} \quad x = 0, b, c \\ 0.5 & \text{if} \quad x = a, d, 1. \end{cases}$$

Note 3.6. Observe that in the Example 3.5, $\tilde{H}$ is also a fuzzy ideal in A.

Theorem 3.7. *Each fuzzy ideal μ in A is a falling fuzzy ideal.*

Proof. Suppose that $\mu : A \to [0,1]$ is a fuzzy ideal in A, then for all $t \in [0,1]$, μ_t is an ideal of A where $\mu_t \neq \emptyset$. Let $\xi : [0,1] \to \mathscr{P}(A)$ is a random set such that $\xi(t) = \mu_t$, then μ is a falling fuzzy ideal of A. ∎

Let $(\Omega, \mathscr{A}, P)$ be a probability space and $\tilde{H}$ be a falling shadow of a random set $\xi : \Omega \to \mathscr{P}(A)$. Denote

$$\Omega(x; \xi) := \{\omega \in \Omega \mid x \in \xi(\omega)\}.$$

Then $\Omega(x; \xi) \in \mathscr{A}$. Hence $\tilde{H}(x) = P(\Omega(x; \xi))$.

Proposition 3.8. *Let $\tilde{H}$ be the falling shadow of a random set $\xi : \Omega \to \mathscr{P}(A)$. If $\tilde{H}$ is a falling fuzzy ideal of A, then $x^- \leq y^- \Rightarrow \Omega(x; \xi) \subseteq \Omega(y; \xi)$ for any $x, y \in A$.*

Proof. If $x^- \leq y^-$ then $(x^- \to y^-)^- = 0$. Let $\omega \in \Omega(x;\xi)$, hence $x \in \xi(\omega)$ and $(x^- \to y^-)^- \in \xi(\omega)$. Since $\xi(\omega)$ is an ideal of A, we have $y \in \xi(\omega)$, and so $\omega \in \Omega(y;\xi)$. Thus $\Omega(x;\xi) \subseteq \Omega(y;\xi)$. ∎

Corollary 3.9. *Let $\tilde{H}$ be the falling shadow of a random set $\xi : \Omega \to \mathscr{P}(A)$. If $\tilde{H}$ is a falling fuzzy ideal of A, then $x \leq y \Rightarrow \Omega(y;\xi) \subseteq \Omega(x;\xi)$ for any $x, y \in A$.*

Proof. Trivial. ∎

Theorem 3.10. *Let $\tilde{H}$ be the falling shadow of a random set $\xi : \Omega \to \mathscr{P}(A)$. Then $\tilde{H}$ is a falling fuzzy ideal of A if and only if the following hold:*

(i) $\Omega(x;\xi) \subseteq \Omega(0;\xi)$ *for any* $x \in A$,

(ii) $\Omega((x^- \to y^-)^-;\xi) \cap \Omega(x;\xi) \subseteq \Omega(y;\xi)$ *for any* $x, y \in A$.

Proof. Suppose that the falling shadow $\tilde{H}$ of a random set $\xi : \Omega \to \mathscr{P}(A)$ is a falling fuzzy ideal of A. Since $0 \leq x$ for all $x \in A$, by Corollary 3.9 we have $\Omega(x;\xi) \subseteq \Omega(0;\xi)$ for any $x \in A$. (i) holds. Now let $\omega \in \Omega((x^- \to y^-)^-;\xi) \cap \Omega(x;\xi)$, then $\omega \in \Omega((x^- \to y^-)^-;\xi)$ and $\omega \in \Omega(x;\xi)$, i.e., $(x^- \to y^-)^- \in \xi(\omega)$ and $x \in \xi(\omega)$. Thus $y \in \xi(\omega)$ since $\xi(\omega)$ is an ideal of A. This shows that $\omega \in \Omega(y;\xi)$. (ii) holds.

Conversely, suppose that $\tilde{H}$ satisfies (i) and (ii). For any $\omega \in \Omega$ and $x \in \xi(\omega)$, i.e., $\omega \in \Omega(x;\xi)$, then it follows from (i) that $\omega \in \Omega(0;\xi)$, hence $0 \in \xi(\omega)$, thus $\xi(\omega)$ satisfies (I1). Now let $x \in \xi(\omega)$ and $(x^- \to y^-)^- \in \xi(\omega)$, then $\omega \in \Omega(x;\xi)$ and $\omega \in \Omega((x^- \to y^-)^-;\xi)$. By (ii) we have

$$\omega \in \Omega((x^- \to y^-)^-;\xi) \cap \Omega(x;\xi) \subseteq \Omega(y;\xi).$$

And so $y \in \xi(\omega)$. Therefore $\xi(\omega)$ satisfies (I2). This proves that $\tilde{H}$ is a falling fuzzy ideal of A. ∎

Theorem 3.11. *Let $\tilde{H}$ be the falling shadow of a random set $\xi : \Omega \to \mathscr{P}(A)$. Then $\tilde{H}$ is a falling fuzzy ideal of A if and only if , for any $x, y, z \in A$,*

$$(*) \quad z^- \to (y^- \to x^-) = 1 \Rightarrow \Omega(z;\xi) \cap \Omega(y;\xi) \subseteq \Omega(x;\xi)).$$

Proof. Suppose that $\tilde{H}$ is a falling fuzzy ideal of A. Let $z^- \to (y^- \to x^-) = 1$. If $\omega \in \Omega(z;\xi) \cap \Omega(y;\xi)$, then $z, y \in \xi(\omega)$. Since $\xi(\omega)$ is an ideal of A, it follows from $z^- \to (y^- \to x^-) = 1$ that $x \in \xi(\omega)$, i.e., $\omega \in \Omega(x;\xi)$, $(*)$ holds.

Conversely, suppose that $(*)$ is true. For any $\omega \in \Omega$, let $z^- \to (y^- \to x^-) = 1$. If $z, y \in \xi(\omega)$, then $\omega \in \Omega(z;\xi) \cap \Omega(y;\xi)$. By $(*)$ we have $\omega \in \Omega(x;\xi)$, thus $x \in \xi(\omega)$. This shows that $\xi(\omega)$ is an ideal of A. Therefore $\tilde{H}$ is a falling fuzzy ideal of A. ∎

By induction we have the following conclusion.

Corollary 3.12. *Let $\tilde{H}$ be the falling shadow of a random set $\xi : \Omega \to \mathscr{P}(A)$. Then $\tilde{H}$ is a falling fuzzy ideal of A if and only if , for any $x_1, \cdots, x_n, z \in A$, $(x_n^- \to (\cdots \to (x_1^- \to z^-) \cdots) = 1$ implies $\Omega(x_1;\xi) \cap \cdots \cap \Omega(x_n;\xi) \subseteq \Omega(z;\xi)$.*

Theorem 3.13. *If $\tilde{H}$ is a falling fuzzy ideal of A, then for any $x, y \in A$,*

(i) $\tilde{H}(0) \geq \tilde{H}(x)$,
(ii) $\tilde{H}(y) \geq T(\tilde{H}((x^- \to y^-)^-), \tilde{H}(x))$,

where $T(s,t) = \max\{s+t-1, 0\}$ *for any* $s, t \in [0,1]$.

Proof. Since $\xi(\omega)$ is an ideal of A for any $\omega \in \Omega$, by Theorem 3.10(i) it follows that for any $x \in A$, $\tilde{H}(x) = P(\Omega(x;\xi)) \leq P(\Omega(0;\xi)) = \tilde{H}(0)$, (i) holds. Furthermore, by Theorem 3.10(ii) we have for any $x, y \in A$,

$$\begin{array}{rcl}\tilde{H}(y) & = & P(\Omega(y;\xi)) \\ & \geq & P(\Omega((x^- \to y^-)^-;\xi) \cap \Omega(x;\xi)) \\ & = & P(\Omega((x^- \to y^-)^-;\xi)) + P(\Omega(x;\xi)) - P(\Omega((x^- \to y^-)^-;\xi) \cup \Omega(x;\xi)) \\ & \geq & \tilde{H}((x^- \to y^-)^-) + \tilde{H}(x) - 1.\end{array}$$

Therefore

$$\tilde{H}(y) \geq \max\{\tilde{H}((x^- \to y^-)^-) + \tilde{H}(x) - 1, 0\} = T(\tilde{H}((x^- \to y^-)^-), \tilde{H}(x)).$$

(ii) holds, ending the proof. ∎

Note 3.14. Theorem 3.13 shows that every falling fuzzy ideal of A is a T-fuzzy ideal of A.

4. Falling fuzzy Gödel ideals

In this section, we introduce the notion of falling fuzzy Gödel ideals of BL-algebras and investigate some of its basic properties.

Definition 4.1.[8] Let I be an ideal of A. I is said to be a Gödel ideal if it satisfies: $(x^- \to (x^-)^2)^- \in I$ for any $x \in A$.

Definition 4.2. [9] Let μ be a fuzzy ideal of A. Then μ is said to be a fuzzy Gödel ideal if it satisfies: $\mu((x^- \to (x^-)^2)^-) = \mu(0)$ for any $x \in A$.

Lemma 4.3. [9] *Let μ be a fuzzy set in A. Then μ is a fuzzy Gödel ideal if and only if , for each $t \in [0,1]$, μ_t is a Gödel ideal of A where $\mu_t \neq \emptyset$.*

Definition 4.4. Let $(\Omega, \mathscr{A}, P)$ be a probability space and $\xi : \Omega \to \mathscr{P}(A)$ be a random set. If $\xi(\omega)$ is a Gödel ideal of A for any $\omega \in \Omega$, then the falling shadow $\tilde{H}$ of the random set ξ is called a falling fuzzy Gödel ideal of A.

It is obvious that if A is a Gödel algebra then every falling fuzzy ideal $\tilde{H}$ of A is a falling fuzzy Gödel ideal of A.

Example 4.5. [22] Let A be the BL-algebra in Example 3.5. It can check that $\{0, b, c\}$ and A are Gödel ideals of A. Let ξ and $\tilde{H}$ be defined as in Example 3.5.

Then $\xi(t)$ is a Gödel ideal of A for all $t \in [0,1]$. Hence $\tilde{H}$ is a falling fuzzy Gödel ideal of A.

Example 4.6. In the above example, let $\xi : [0,1] \to \mathscr{P}(A)$ be defined by

$$\xi(t) := \begin{cases} \{0\} & \text{if} \quad t = [0, 0.2) \\ \{0, d\} & \text{if} \quad t \in [0.2, 0.5), \\ A & \text{if} \quad t \in [0.5, 1]. \end{cases}$$

Thus $\xi(t)$ is an ideal of A for all $t \in [0,1]$. Hence $\tilde{H}$ is a falling fuzzy ideal of A where $\tilde{H}(x) = P(\{t \in [0,1] \mid x \in \xi(t)\})$ and

$$\tilde{H}(x) := \begin{cases} 1 & \text{if} \quad x = 0, \\ 0.8 & \text{if} \quad x = d, \\ 0.5 & \text{if} \quad x = a, b, c, 1. \end{cases}$$

Because $(a^- \to (a^-)^2)^- = c \notin \{0\}$, hence $\xi(0) = \{0\}$ is an ideal of A but it is not a Gödel ideal of A. Thus $\tilde{H}$ is not a falling fuzzy Gödel ideal of A.

Theorem 4.7. *Each fuzzy Gödel ideal in A is a falling fuzzy Gödel ideal of A.*

Proof. Suppose that $\mu : A \to [0,1]$ is a fuzzy Gödel ideal in A, then by Lemma 4.3, for any $t \in [0,1]$, μ_t is a Gödel ideal of A where $\mu_t \neq \emptyset$. Let $\xi : [0,1] \to \mathscr{P}(A)$ is a random set such that $\xi(t) = \mu_t$, then μ is a falling fuzzy Gödel ideal of A. ■

Theorem 4.8. *Let A be a Gödel algebra. Then every falling fuzzy ideal of A is a falling fuzzy Gödel ideal of A.*

Proof. Trivial. ■

Theorem 4.9. *Let $(\Omega, \mathscr{A}, P)$ be a probability space and $\xi : \Omega \to \mathscr{P}(A)$ be a random set. If $\tilde{H}$ is the falling fuzzy ideal of A, then the following are equivalent: for for any $x, y, z \in A$,*

(i) *$\tilde{H}$ is a falling fuzzy Gödel ideal of A;*
(ii) $\Omega(((x^-)^2 \to y^-)^-; \xi) \subseteq \Omega((x^- \to y^-)^-; \xi)$;
(iii) $\Omega(((x^- * y^-) \to z^-)^-; \xi) \subseteq \Omega(((x^- \to y^-) \to (x^- \to z^-))^-; \xi)$.

Proof. (i)⇒(ii) Suppose $\tilde{H}$ is a falling fuzzy Gödel ideal of A.

For any $\omega \in \Omega$, then $\xi(\omega)$ is a Gödel ideal of A, and so $(x^- \to (x^-)^2)^- \in \xi(\omega)$. If $\omega \in \Omega(((x^-)^2 \to y^-)^-; \xi)$, then $((x^-)^2 \to y^-)^- \in \xi(\omega)$. Since

$$(x^- \to (x^-)^2) * ((x^-)^2 \to y^-) \leq x^- \to y^-,$$

it follows that

$$(x^- \to (x^-)^2) \leq ((x^-)^2 \to y^-) \to (x^- \to y^-).$$

Hence

$$\begin{array}{rcl}(x^- \to (x^-)^2)^{--} & \leq & (((x^-)^2 \to y^-) \to (x^- \to y^-))^{--} \\ & = & ((x^-)^2 \to y^-) \to (x^- \to y^-) \\ & = & ((x^-)^2 \to y^-)^{--} \to (x^- \to y^-)^{--},\end{array}$$

So we have

$$((x^- \to (x^-)^2)^{--} \to (((x^-)^2 \to y^-)^{--} \to (x^- \to y^-)^{--})^{--})^- = 0 \in \xi(\omega).$$

By $(x^- \to (x^-)^2)^- \in \xi(\omega)$ and (I2) we get

$$((x^-)^2 \to y^-)^{--} \to (x^- \to y^-)^{--})^- \in \xi(\omega).$$

By $((x^-)^2 \to y^-)^- \in \xi(\omega)$ and (I2) we obtain $(x^- \to y^-)^- \in \xi(\omega)$. Thus, $\omega \in \Omega((x^- \to y^-)^-; \xi)$. This proves that

$$\Omega(((x^-)^2 \to y^-)^-; \xi) \subseteq \Omega((x^- \to y^-)^-; \xi),$$

(ii) holds.

(ii)⇒(iii) Suppose that (ii) is true.

For any $\omega \in \Omega$, if $\omega \in \Omega(((x^- * y^-) \to z^-)^-; \xi)$, then $((x^- * y^-) \to z^-)^- \in \xi(\omega)$. Since $y^- \to z^- \leq (x^- \to y^-) \to (x^- \to z^-)$, then

$$\begin{array}{rcl}(x^- * y^-) \to z^- & = & x^- \to (y^- \to z^-) \\ & \leq & x^- \to ((x^- \to y^-) \to (x^- \to z^-)) \\ & = & x^- \to (x^- \to ((x^- \to y^-) \to z^-)) \\ & = & (x^-)^2 \to ((x^- \to y^-) \to z^-),\end{array}$$

we have $((x^-)^2 \to ((x^- \to y^-) \to z^-))^- \leq ((x^- * y^-) \to z^-)^-$, and so

$$((x^-)^2 \to ((x^- \to y^-) \to z^-))^- \in \xi(\omega).$$

Thus $\omega \in \Omega(((x^-)^2 \to ((x^- \to y^-) \to z^-))^-; \xi)$. By (ii) it follows that

$$\omega \in \Omega((x^- \to ((x^- \to y^-) \to z^-))^-; \xi) = \Omega(((x^- \to y^-) \to (x^- \to z^-))^-; \xi).$$

This shows that (iii) is true.

(iii)⇒(i) Suppose that (iii) is true. For any $\omega \in \Omega$ and $x \in A$, then

$$(x^- \to (x^- \to (x^-)^2))^- = (x^- * x^- \to (x^-)^2))^- = 0 \in \xi(\omega),$$

i.e., $\omega \in \Omega((x^- * x^- \to (x^-)^2))^-; \xi)$. It follows from (iii) that

$$\omega \in \Omega(((x^- \to x^-) \to (x^- \to (x^-)^2)^-); \xi).$$

Since $(x^- \to x^-) \to (x^- \to (x^-)^2)^- = (x^- \to (x^-)^2)^-$, then we have

$$\omega \in \Omega((x^- \to (x^-)^2)^-; \xi),$$

thus $(x^- \to (x^-)^2)^- \in \xi(\omega)$. This shows that for each $\omega \in \Omega$, $\xi(\omega)$ is a Gödel ideal of A. Therefore $\tilde{H}$ is a falling fuzzy Gödel ideal of A. (i) holds. ∎

Theorem 4.10. *Let $(\Omega, \mathscr{A}, P)$ be a probability space and $\xi : \Omega \to \mathscr{P}(A)$ be a random set. If $\tilde{H}$ is a falling fuzzy ideal of A, then $\tilde{H}$ is a falling fuzzy Gödel ideal of A if and only if for any $x, y, z \in A$,*

$$(**) \qquad \Omega((x^- \to (y^- \to z^-))^-; \xi) \cap \Omega((x^- \to y^-)^-; \xi) \subseteq \Omega((x^- \to z^-)^-; \xi).$$

Proof. Suppose that $\tilde{H}$ is a falling fuzzy Gödel ideal of A. For any $\omega \in \Omega$, if

$$\omega \in \Omega((x^- \to (y^- \to z^-))^-; \xi) \cap \Omega((x^- \to y^-)^-; \xi),$$

then, by Theorem 4.9(iii), we have

$$\begin{aligned} \omega &\in \Omega(((x^- \to y^-) \to (x^- \to z^-))^-; \xi) \cap \Omega((x^- \to y^-)^-; \xi) \\ &= \Omega(((x^- \to y^-)^{--} \to (x^- \to z^-)^{--})^-; \xi) \cap \Omega((x^- \to y^-)^-; \xi). \end{aligned}$$

Since $\tilde{H}$ is a falling fuzzy ideal of A, by Theorem 3.10(ii) we have

$$\omega \in \Omega((x^- \to z^-)^-; \xi).$$

Thus $(**)$ holds.

Conversely, suppose that the falling fuzzy ideal $\tilde{H}$ of A satisfies $(**)$. For any $\omega \in \Omega$ and $x \in A$, since

$$(x^- \to (x^- \to (x^-)^2))^- = ((x^-)^2 \to (x^-)^2)^- = 1^- = 0 \in \xi(\omega),$$

and

$$(x^- \to x^-)^- = 0 \in \xi(\omega),$$

we have $\omega \in \Omega\{(x^- \to (x^- \to (x^-)^2))^-; \xi\}$ and $\omega \in \{(x^- \to x^-)^-; \xi\}$, and so

$$\omega \in \Omega\{(x^- \to (x^- \to (x^-)^2))^-; \xi\} \cap \Omega\{(x^- \to x^-)^-; \xi\}.$$

It follows from $(**)$ that $\omega \in \Omega\{(x^- \to (x^-)^2)^-; \xi\}$. Hence $(x^- \to (x^-)^2)^- \in \xi(\omega)$, which shows that $\xi(\omega)$ is a Gödel ideal of A. Therefore $\tilde{H}$ is a falling fuzzy Gödel ideal of A. ∎

Theorem 4.11. *Let $(\Omega, \mathscr{A}, P)$ be a probability space and $\xi : \Omega \to \mathscr{P}(A)$ be a random set. If $\tilde{H}$ is a falling fuzzy ideal of A, then $\tilde{H}$ is a falling fuzzy Gödel ideal of A if and only if for any $x, y, z \in A$,*

$$(***) \qquad \Omega((x^- \to ((y^-)^2 \to z^-))^-; \xi) \cap \Omega(x; \xi) \subseteq \Omega((y^- \to z^-)^-; \xi).$$

Proof. Suppose that $\tilde{H}$ is a falling fuzzy Gödel ideal of A. If

$$\omega \in \Omega((x^- \to ((y^-)^2 \to z^-))^-; \xi) \cap \Omega(x; \xi),$$

then $\omega \in \Omega((x^- \to ((y^-)^2 \to z^-))^-; \xi)$ and $\omega \in \Omega(x; \xi)$. Thus $(x^- \to ((y^-)^2 \to z^-))^- \in \xi(\omega)$ and $x \in \xi(\omega)$, i.e., $(x^- \to ((y^-)^2 \to z^-)^{--})^- \in \xi(\omega)$ and $x \in \xi(\omega)$. Since $\xi(\omega)$ is an ideal of A, it follows that $((y^-)^2 \to z^-)^- \in \xi(\omega)$. Hence $\omega \in \Omega\{((y^-)^2 \to z^-)^-; \xi\}$, and so

$$\Omega((x^- \to ((y^-)^2 \to z^-))^-; \xi) \cap \Omega(x; \xi) \subseteq \Omega\{((y^-)^2 \to z^-)^-; \xi\}.$$

By Theorem 4.9(ii) we have

$$\Omega((x^- \to ((y^-)^2 \to z^-))^-; \xi) \cap \Omega(x; \xi) \subseteq \Omega\{(y^- \to z^-)^-; \xi\}.$$

$(***)$ holds.

Conversely, suppose that $(***)$ is true and $\omega \in \Omega((x^- \to (y^- \to z^-))^-; \xi) \cap \Omega((x^- \to y^-)^-; \xi)$. Since

$$(x^-)^2 * (x^- \to (y^- \to z^-)) * (x^- \to y^-) \leq z^-,$$

we have

$$\begin{aligned} &(x^- \to (y^- \to z^-)) * (x^- \to y^-) \leq (x^-)^2 \to z^-, \\ &x^- \to (y^- \to z^-) \leq (x^- \to y^-) \to ((x^-)^2 \to z^-), \\ &((x^- \to y^-) \to ((x^-)^2 \to z^-))^- \leq (x^- \to (y^- \to z^-))^-. \end{aligned}$$

By Corollary 3.9, we get

$$\Omega((x^- \to (y^- \to z^-))^-; \xi) \subseteq \Omega(((x^- \to y^-) \to ((x^-)^2 \to z^-))^-; \xi).$$

Therefore,

$$\begin{aligned} & \Omega((x^- \to (y^- \to z^-))^-; \xi) \cap \Omega((x^- \to y^-)^-; \xi) \\ \subseteq\ & \Omega(((x^- \to y^-) \to ((x^-)^2 \to z^-))^-; \xi) \cap \Omega((x^- \to y^-)^-; \xi) \\ =\ & \Omega(((x^- \to y^-)^{--} \to ((x^-)^2 \to z^-))^-; \xi) \cap \Omega((x^- \to y^-)^-; \xi) \\ \subseteq\ & \Omega((x^- \to z^-)^-). \end{aligned}$$

By Theorem 4.10, $\tilde{H}$ is a falling fuzzy ideal of A. ■

Theorem 4.12. *Let $\tilde{H}$ be the falling shadow of a random set $\xi : \Omega \to \mathscr{P}(A)$. Then $\tilde{H}$ is a falling fuzzy Gödel ideal of A if and only if the following hold:*

(i) $\Omega(x; \xi) \subseteq \Omega(0; \xi)$ *for any* $x \in A$,

(ii) $\Omega(x; \xi) = \Omega(x^{--}; \xi)$ *for any* $x \in A$,

(iii) $\Omega((x^- \to (y^- \to z^-))^-; \xi) \cap \Omega((x^- \to y^-)^-; \xi) \subseteq \Omega((x^- \to z^-)^-; \xi)$ *for any* $x, y, z \in A$.

Proof. Suppose that $\tilde{H}$ is a falling fuzzy Gödel ideal of A. By Theorem 3.10, $\tilde{H}$ satisfies (i). Since for any $x \in A$ we have

$$\omega \in \Omega(x; \xi) \Leftrightarrow x \in \xi(\omega) \Leftrightarrow x^{--} \in \xi(\omega) \Leftrightarrow \omega \in \Omega(x^{--}; \xi),$$

it follows that $\Omega(x; \xi) = \Omega(x^{--}; \xi)$, (ii) holds. By Theorem 4.10, (iii) holds.

Conversely, suppose that (i)-(iii) hold. For any $\omega \in \Omega$ and $x \in \xi(\omega)$, then we have $\omega \in \Omega(x;\xi)$. It follows from (i) that $\omega \in \Omega(0;\xi)$, hence $0 \in \xi(\omega)$, thus $\xi(\omega)$ satisfies (I1). Let $y \in \xi(\omega)$ and $(y^- \to x^-)^- \in \xi(\omega)$. Then $\omega \in \Omega(y;\xi)$ and $\omega \in \Omega((y^- \to x^-)^-;\xi)$. By (ii) we have $\omega \in \Omega(y^{--};\xi)$. Since $\Omega(y^{--};\xi) = \Omega((0^- \to y^-)^-;\xi)$ and $\Omega((y^- \to x^-)^-;\xi) = \Omega((0^- \to (y^- \to x^-))^-;\xi)$. It follows that $\omega \in \Omega((0^- \to y^-)^-;\xi)$ and $\omega \in \Omega((0^- \to (y^- \to x^-))^-;\xi)$. By (iii) and (ii) we have

$$\begin{array}{rcl} \omega & \in & \Omega((0^- \to y^-)^-) \cap \Omega((0^- \to (y^- \to x^-))^-;\xi) \\ & \subseteq & \Omega((0^- \to x^-)^-;\xi) \\ & = & \Omega(x^{--};\xi) \\ & = & \Omega(x;\xi), \end{array}$$

thus $x \in \xi(\omega)$. This shows that $\xi(\omega)$ is an ideal of A for any $\omega \in \Omega$. Hence $\tilde{H}$ is a falling fuzzy ideal of A. By (iii) and Theorem 4.10, $\tilde{H}$ is a falling fuzzy Gödel ideal of A. ∎

Theorem 4.13. *Let $\tilde{H}$ be the falling shadow of a random set $\xi : \Omega \to \mathscr{P}(A)$. Then $\tilde{H}$ is a falling fuzzy Gödel ideal of A if and only if the following hold:*

(i) $\Omega(x;\xi) \subseteq \Omega(0;\xi)$ *for any* $x \in A$,

(ii) $\Omega(x;\xi) = \Omega(x^{--};\xi)$ *for any* $x \in A$,

(iii) $\Omega((z^- \to ((y^-)^2 \to x^-))^-;\xi) \cap \Omega(z;\xi) \subseteq \Omega((y^- \to x^-)^-;\xi)$ *for any* $x, y, z \in A$.

Proof. Suppose that $\tilde{H}$ is a falling fuzzy Gödel ideal of A. By the "only if" part of Theorem 4.12 we know (i) and (ii) hold. Then by Theorem 4.11, (iii) holds.

Conversely, suppose that (i)-(iii) are true. In (iii), let $y = 0$, we get $\Omega((z^- \to x^-)^-;\xi) \cap \Omega(z;\xi) \subseteq \Omega(x^{--};\xi)$ for any $x, y \in A$. From (ii) it follows that

$$\Omega((z^- \to x^-)^-;\xi) \cap \Omega(z;\xi) \subseteq \Omega(x;\xi).$$

By (i) and Theorem 3.10 we know that $\tilde{H}$ is a falling fuzzy ideal. Furthermore, we get that $\tilde{H}$ is a falling fuzzy Gödel ideal of A by (iii) and Theorem 4.11. ∎

Theorem 4.14. *If $\tilde{H}$ is a falling fuzzy Gödel ideal of A, then for any $x, y, z \in A$,*

(i) $\tilde{H}(0) \geq \tilde{H}(x)$,

(ii) $\tilde{H}(x) = \tilde{H}(x^{--})$,

(iii) $\tilde{H}((x^- \to z^-)^-) \geq T(\tilde{H}((x^- \to (y^- \to z^-))^-), \tilde{H}((x^- \to y^-)^-))$,

where $T(s,t) = \max\{s + t - 1, 0\}$ for any $s, t \in [0,1]$.

Proof. It is easy to see that (i) and (ii) are true. We just prove (iii). By Theorem 4.12(iii), then for any $x, y, z \in A$ we have

$$\begin{array}{rl} & \tilde{H}((x^- \to z^-)^-) \\ = & P(\Omega((x^- \to z^-)^-;\xi)) \\ \geq & P(\Omega((x^- \to (y^- \to z^-))^-)^-;\xi) \cap \Omega((x^- \to y^-)^-;\xi)) \\ = & P(\Omega((x^- \to (y^- \to z^-))^-)^-;\xi)) + P(\Omega((x^- \to y^-)^-;\xi)) \\ & -P(\Omega((x^- \to (y^- \to z^-))^-;\xi) \cup \Omega((x^- \to y^-)^-;\xi)) \\ \geq & \tilde{H}((x^- \to (y^- \to z^-)^-) + \tilde{H}((x^- \to y^-)^-) - 1. \end{array}$$

Therefore

$$\begin{aligned}\tilde{H}(x) &\geq \max\{\tilde{H}((x^- \to (y^- \to z^-)^-) + \tilde{H}((x^- \to y^-)^-) - 1, 0\} \\ &= T(\tilde{H}((x^- \to (y^- \to z^-)^-), \tilde{H}((x^- \to y^-)^-)),\end{aligned}$$

and so (iii) holds. ∎

Note 4.15. Theorem 4.14 shows that every falling fuzzy Gödel ideal of A is a T-fuzzy Gödel ideal of A.

By similar argument we can prove the following conclusion and the details are omitted.

Theorem 4.16. *If $\tilde{H}$ is a falling fuzzy Gödel ideal of A, then for any $x, y, z \in A$*

(i) $\tilde{H}(0) \geq \tilde{H}(x)$,
(ii) $\tilde{H}(x) = \tilde{H}(x^{--})$,
(iii) $\tilde{H}((y^- \to z^-)^-) \geq T(\tilde{H}((x^- \to ((y^-)^2 \to z^-))^-), \tilde{H}(x))$,

where $T(s,t) = \max\{s + t - 1, 0\}$ for any $s, t \in [0,1]$.

5. Falling fuzzy Boolean ideals

In this section, we introduce the notion of falling fuzzy Boolean ideals of BL-algebras and investigate some of its basic properties. We also discuss relation between falling fuzzy Boolean ideals and falling fuzzy Gödel ideals.

Definition 5.1. An ideal I of A is said to be a Boolean ideal if $x \wedge x^- \in I$ for all $x \in A$.

Lemma 5.2. *Let I be a Boolean ideal of A. Then for any $x \in A$, $(x \to x^-)^- \in I$ implies $x \in I$.*

Proof. Since

$$\begin{aligned} &((x \to x^-)^{--} \to x^-)^- \\ =\ & ((x \to x^-) \to x^-)^- \\ =\ & ((x \to x^-) \to (x \to 0))^- \\ =\ & (((x \to x^-) * x) \to 0)^- \\ =\ & ((x \wedge x^-) \to 0)^- \\ =\ & (x^- \vee x^{--})^- \\ =\ & x^{--} \wedge x^- \in I,\end{aligned}$$

by $(x \to x^-)^- \in I$ and (I2) we get $x \in I$. ∎

Definition 5.3. [22] A fuzzy ideal μ in A is said to be a fuzzy Boolean ideal if $\mu(x \wedge x^-) = \mu(0)$ for all $x \in A$. In this case we also say that the fuzzy ideal μ in A is Boolean.

Proposition 5.4. *A fuzzy ideal μ in A is Boolean if, for any $t \in [0,1]$, μ_t is a Boolean ideal of A where $\mu_t \neq \emptyset$.*

Definition 5.5. Let $(\Omega, \mathscr{A}, P)$ be a probability space and $\xi : \Omega \to \mathscr{P}(A)$ be a random set. If $\xi(\omega)$ is a Boolean ideal of A for any $\omega \in \Omega$, then the falling shadow $\tilde{H}$ of the random set ξ is called a falling fuzzy Boolean ideal of A.

Example 5.6. Let A be the BL-algebra in Example 3.3. It is easy to see that $\{0\}, \{0, a\}, \{0, b\}, A$ are ideals of A. Since

$$\begin{array}{r|l} 1^- = 0 & 0 \wedge 0^- = 0 \\ 0^- = 1 & 1 \wedge 1^- = 0 \\ a^- = b & a \wedge a^- = a \wedge b = 0 \\ b^- = a & b \wedge b^- = b \wedge a = 0, \end{array}$$

It follows that $\{0\}, \{0, a\}, \{0, b\}, A$ are Boolean ideals of A. Let $(\Omega, \mathscr{A}, P) = ([0, 1], \mathscr{A}, m)$, where $\mathscr{A}$ is a Borel field on $[0, 1]$ and m the usual Lebesgue measure. Let $\xi : [0, 1] \to \mathscr{P}(A)$ be defined by

$$\xi(t) := \begin{cases} \{0\} & \text{if } \ t \in [0, 0.2), \\ \{0, a\} & \text{if } \ t \in [0.2, 0.4), \\ \{0, b\} & \text{if } \ t \in [0.4, 0.9), \\ \{0, a, b, 1\} & \text{if } \ t \in [0.9, 1]. \end{cases}$$

Thus $\xi(t)$ is a Boolean ideal of A for all $t \in [0, 1]$. Hence $\tilde{H}(x)$ is a falling fuzzy Boolean ideal of A where $\tilde{H}(x) = P(\{t \in [0, 1] \mid x \in \xi(t)\})$ and

$$\tilde{H}(x) := \begin{cases} 1 & \text{if } \ x = 0, \\ 0.3 & \text{if } \ x = a, \\ 0.6 & \text{if } \ x = b, \\ 0.1 & \text{if } \ x = 1. \end{cases}$$

If $\xi : [0, 1] \to \mathscr{P}(A)$ be defined by

$$\xi(t) := \begin{cases} \{0\} & \text{if } \ t \in [0, 0.2), \\ \{0, a, b\} & \text{if } \ t \in [0.4, 0.9), \\ \{0, a, b, 1\} & \text{if } \ t \in [0.9, 1]. \end{cases}$$

then $\tilde{H}$ is not a falling fuzzy Boolean ideal of A, because $\{0, a, b\}$ is not an ideal of A.

Theorem 5.7. *Each fuzzy Boolean ideal in A is a falling fuzzy Boolean ideal of A.*

Proof. It is similar to Theorem 4.7. ∎

Theorem 5.8. *Let $(\Omega, \mathscr{A}, P)$ be a probability space and $\xi : \Omega \to \mathscr{P}(A)$ be a random set. If $\tilde{H}$ is the falling fuzzy Boolean ideal of A, then $\tilde{H}$ is a falling fuzzy Gödel ideal of A.*

Proof. Let $\tilde{H}$ is the falling fuzzy Boolean ideal of A. For any $\omega \in \Omega$, suppose $\omega \in \Omega(((x^-)^2 \to y^-)^-; \xi)$, then $((x^-)^2 \to y^-)^- \in \xi(\omega)$. Since

$$\begin{aligned} & ((x \wedge x^-)^- \to (x^- \to y^-)^{--})^- \\ = & ((x \wedge x^-)^- \to (x^- \to y^-))^- \\ = & ((x^- \vee x^{--}) \to (x^- \to y^-))^- \\ = & (((x^- \vee x^{--}) * x^-) \to y^-)^- \\ = & (((x^-)^2 \vee (x^- * x^{--})) \to y^-)^- \\ = & (((x^-)^2 \vee 0) \to y^-)^- \\ = & (((x^-)^2 \to y^-)^- \\ \in & \xi(\omega), \end{aligned}$$

from $x \wedge x^- \in \xi(\omega)$ it follows that $(x^- \to y^-)^- \in \xi(\omega)$, hence $\omega \in \Omega((x^- \to y^-)^-; \xi)$. This shows that $\Omega(((x^-)^2 \to y^-)^-; \xi) \subseteq \Omega((x^- \to y^-)^-; \xi)$. By Theorem 4.9, $\tilde{H}$ is a falling fuzzy Gödel ideal of A. ■

Note 5.9. From above we know that a falling fuzzy Boolean ideal is a falling fuzzy Gödel ideal, but the converse is whether or not true?

Theorem 5.10. *Let $(\Omega, \mathscr{A}, P)$ be a probability space and $\xi : \Omega \to \mathscr{P}(A)$ be a random set. If $\tilde{H}$ is a falling fuzzy ideal of A and satisfies: for any $x, y \in A$,*

$$(*_1) \quad \Omega((x \to (y \to x)^-)^-; \xi) \subseteq \Omega(x; \xi),$$

then $\tilde{H}$ is a falling fuzzy Boolean ideal of A.

Proof. For any $\omega \in \Omega$ and $(x \to (y \to x)^-)^- \in \xi(\omega)$, then $\omega \in \Omega((x \to (y \to x)^-)^-; \xi)$. By $(*_1)$ we get $\omega \in \Omega(x; \xi)$, that is, $x \in \xi(\omega)$. Thus, we have proved that

$$(*') \qquad (x \to (y \to x)^-)^- \in \xi(\omega) \text{ implies } x \in \xi(\omega) \text{ for any } x, y \in A.$$

Since, for any $x \in A$, we have

$$\begin{aligned} ((x \wedge x^-) \to (1 \to (x \wedge x^-))^-)^- &= ((x \wedge x^-) \to (x \wedge x^-)^-)^- \\ &= ((x \wedge x^-) \to (x^- \vee x^{--}))^- \\ &= 1^- = 0 \in \xi(\omega), \end{aligned}$$

it follows from $(*')$ that $x \wedge x^- \in \xi(\omega)$. Hence $\xi(\omega)$ is a Boolean ideal of A, and so $\tilde{H}$ is a falling fuzzy Boolean ideal of A. ■

Definition 5.11. An ideal I of A is said to be an *implicative ideal* if it satisfies: for any $x, y \in A$,

$$(*_2) \qquad (x \to (y \to x)^-)^- \in I \text{ implies } x \in I.$$

Definition 5.12. [22] A fuzzy ideal μ of A is said to be an *fuzzy implicative ideal* if it satisfies: for any $x, y \in A$,

$$(*_3) \qquad \mu((x \to (y \to x)^-)^-) = \mu(0) \text{ implies } \mu(x) = \mu(0).$$

Definition 5.13. Let $(\Omega, \mathscr{A}, P)$ be a probability space and $\xi : \Omega \to \mathscr{P}(A)$ be a random set. If $\xi(\omega)$ is an implicative ideal of A for any $\omega \in \Omega$, then the falling shadow $\tilde{H}$ of the random set ξ is called a falling fuzzy implicative ideal of A.

Theorem 5.14. *Let $(\Omega, \mathscr{A}, P)$ be a probability space and $\xi : \Omega \to \mathscr{P}(A)$ be a random set. If $\tilde{H}$ is the falling fuzzy ideal of A, then $\tilde{H}$ is a falling fuzzy implicative ideal of A if and only if it satisfies: for any $x, y \in A$,*

$$(*_4) \qquad \Omega((x \to (y \to x)^-)^-; \xi) \subseteq \Omega(x; \xi).$$

Proof. It is easy and omitted. ∎

Theorem 5.15. *Let $(\Omega, \mathscr{A}, P)$ be a probability space and $\xi : \Omega \to \mathscr{P}(A)$ be a random set. If $\tilde{H}$ is a falling fuzzy ideal of A, then $\tilde{H}$ is a falling implicative ideal of A if and only if $\tilde{H}$ is a falling fuzzy Boolean ideal of A.*

Proof. $(\Rightarrow)$ It is immediate by Theorem 5.9.

$(\Leftarrow)$ Suppose that $\tilde{H}$ is a falling fuzzy Boolean ideal of A and $\omega \in \Omega((x \to (y \to x)^-)^-; \xi)$. Hence $(x \to (y \to x)^-)^- \in \xi(\omega)$. Since

$$\begin{aligned} & ((x \to (y \to x)^-)^{--} \to (x \to x^-)^{--})^- \\ = \; & ((x \to x^-)^- \to (x \to (y \to x)^-)^-))^- \\ = \; & ((x \to (y \to x)^-) \to (x \to x^-)^{--})^- \\ = \; & ((x \to (y \to x)^-) \to (x \to x^-))^- \\ \leq \; & ((y \to x)^- \to x^-)^- \\ \leq \; & (x \to (y \to x))^- = 1^- = 0 \in \xi(\omega), \end{aligned}$$

we have $((x \to (y \to x)^-)^{--} \to (x \to x^-)^{--})^- \in \xi(\omega)$.

From $(x \to (y \to x)^-)^- \in \xi(\omega)$ it follows that $(x \to x^-)^- \in \xi(\omega)$. Since $\xi(\omega)$ is a Boolean ideal of A, by lemma 5.2 we get $x \in \xi(\omega)$, i.e., $\omega \in \Omega(x; \xi)$. This shows that $\Omega((x \to (y \to x)^-)^-; \xi) \subseteq \Omega(x; \xi)$. Therefore $\tilde{H}$ is a falling implicative ideal of A. ∎

6. Falling fuzzy inference relations

Based on the theory of falling shadows, Tan et al.[12] establish a theoretical approach to define a fuzzy inference relation. Let B and C be fuzzy sets in the universes U and V, respectively, ξ and η be cut-clouds of B and C, respectively. Then fuzzy inference relation $T_{B \to C}$ of the implication $B \to C$ be defined to be

$$\begin{aligned} I_{B \to C}(u, v) = \; & P(\{(s, t) \mid (u, v) \in I_{B_s \to C_t}\}) \\ = \; & P(\{(s, t) \mid (u, v) \in (B_s \times C_t) \cup (B_s^c \times V)\}) \end{aligned}$$

where P is a joint probability on $[0, 1]^2$. So different probability distribution P will generate different formulas for the fuzzy inference relations. The following three basic cases are considered.

Theorem 6.1. [12]

(1) *If the whole probability P of (s,t) on $[0,1]^2$ is concentrated and uniformly distributed on the main diagonal $\{(s,s) \mid s \in [0,1]\}$ of the unit square $[0,1]^2$, then P is the diagonal distribution and $I_{B\to C}(s,t) = \min\{1 - B(s) + C(t), 1\}$.*

(2) *If the whole probability P of (s,t) on $[0,1]^2$ is concentrated and uniformly distributed on the anti-diagonal $\{(s, 1-s) \mid s \in [0,1]\}$ of the unit square $[0,1]^2$, then P is the anti-diagonal distribution and $I_{B\to C}(s,t) = \max\{1 - B(s), C(t)\}$.*

(3) *If the whole probability P of (s,t) on $[0,1]^2$ is uniformly distributed on the unit square $[0,1]^2$, then P is the independence distribution and $I_{B\to C}(s,t) = 1 - B(s) + B(s)C(t)$.*

We call the three fuzzy inference relations falling implication operators on $[0,1]$. In what follows we consider the concept of I-fuzzy ideals of BL-algebras.

Definition 6.2. Let μ be a fuzzy set of A, I be a falling implication operator over $[0,1]$ and $t \in (0,1]$. Then μ is called an I-fuzzy ideal of A if, for all $x, y \in A$, the following conditions are satisfied:

(FFI1) $I(\mu(x), \mu(0)) \geq t$;
(FFI2) $I(\min\{\mu((x^- \to y^-)^-), \mu(x)\}, \mu(y)) \geq t$.

Obviously, if $t = 1$ and P is the diagonal distribution, then Definition 6.2 is equivalent to Definition 2.6.

Theorem 6.3. *Let μ be a fuzzy set of A and $t = 0.5$. Then*

(1) *if P is the diagonal distribution then μ is an I-fuzzy ideal of A if and only if it satisfies* (a1) *and* (b1), *where*

(a1) $\mu(x) \leq \mu(0)$ *or* $0 < \mu(x) - \mu(0) \leq 0.5$ *for all* $x, y \in A$,

(b1) $\min\{\mu((x^- \to y^-)^-), \mu(x)\} \leq \mu(y)$ *or* $0 < \min\{\mu((x^- \to y^-)^-), \mu(x)\} - \mu(y) \leq 0.5$ *for all* $x, y \in A$.

(2) *if P is the anti-diagonal distribution then μ is an I-fuzzy ideal of A if and only if it satisfies* (a2) *and* (b2), *where*

(a2) $\mu(x) \leq \max\{\mu(0), 0.5\}$ *or* $\min\{\mu(x), 0.5\} \leq \mu(0)$ *for all* $x, y \in A$,

(b2) $\min\{\mu((x^- \to y^-)^-), \mu(x)\} \leq \max\{\mu(y), 0.5\}$ *or* $\min\{\mu((x^- \to y^-)^-), \mu(x), 0.5\} \leq \mu(y)$ *for all* $x, y \in A$.

(3) *if P is the independent distribution then μ is an I-fuzzy ideal of A if and only if it satisfies* (a3) *and* (b3), *where*

(a3) $\mu(x)(1 - \mu(0)) \leq 0.5$ *for all* $x, y \in A$,

(b3) $\min\{\mu((x^- \to y^-)^-), \mu(x)\}(1 - \mu(y)) \leq 0.5$ *for all* $x, y \in A$.

Proof. (1) Suppose P is the diagonal distribution. Then

$$I(\mu(x), \mu(0)) = \min\{1 - \mu(x) + \mu(0), 1\}.$$

If μ is an I-fuzzy ideal of A, then $\min\{1 - \mu(x) + \mu(0), 1\} \geq 0.5$ by (FFI1). When $\mu(x) > \mu(0)$, we have $\mu(x) - \mu(0) > 0$, and so $1 - \mu(x) + \mu(0) < 1$. Thus $\min\{1 - \mu(x) + \mu(0), 1\} = 1 - \mu(x) + \mu(0) \geq 0.5$, and $0 < \mu(x) - \mu(0) \leq 0.5$. Hence (a1) holds. By (FFI2) we have $I(\min\{\mu((x^- \to y^-)^-), \mu(x)\}, \mu(y)) \geq 0.5$, that is,

$$\min\{\min\{\mu((x^- \to y^-)^-), \mu(x)\} - \mu(y), 1\} > 0.5.$$

If $\min\{\mu((x^- \to y^-)^-), \mu(x)\} > \mu(y)$, then $0 < \min\{\mu((x^- \to y^-)^-), \mu(x)\} - \mu(y) \leq 0.5$. Thus (b1) holds.

Conversely, suppose that (a1) and (b1) hold. Let P be the diagonal distribution. By Theorem 6.1 we have $I(\mu(x), \mu(0)) = \min\{1 - \mu(x) + \mu(0), 1\}$. When $\mu(x) \leq \mu(0)$, we have $1 - \mu(x) + \mu(0) = 1 + [\mu(0) - \mu(x)] \geq 1$, and so $I(\min\{1 - \mu(x) + \mu(0), 1\}) = 1 > 0.5$. When $0 < \mu(x) - \mu(0) \leq 0.5$, we have

$$\begin{aligned} I(\mu(x), \mu(0)) &= \min\{1 - \mu(x) + \mu(0), 1\} \\ &= 1 - [\mu(x) - \mu(0)] \\ &\geq 0.5. \end{aligned}$$

Hence (FFI1) is true. Also by Theorem 6.1,

$$\begin{aligned} & I(\min\{\mu((x^- \to y^-)^-), \mu(x)\}, \mu(y)) \\ = \; & \min\{1 - \min\{\mu((x^- \to y^-)^-), \mu(x)\} + \mu(y), 1\}. \end{aligned}$$

When $\min\{\mu((x^- \to y^-)^-), \mu(x)\} \leq \mu(y)$, we have

$$\begin{aligned} & 1 - \min\{\mu((x^- \to y^-)^-), \mu(x)\} + \mu(y) \\ = \; & 1 + [\mu(y) - \min\{\mu((x^- \to y^-)^-), \mu(x)\}] \\ \geq \; & 1, \end{aligned}$$

and so $I(\min\{\mu((x^- \to y^-)^-), \mu(x)\}, \mu(y)) = 1 > 0.5$.

When $0 < \min\{\mu((x^- \to y^-)^-), \mu(x)\} - \mu(y) \leq 0.5$, we have

$$\begin{aligned} & I(\min\{\mu((x^- \to y^-)^-), \mu(x)\}, \mu(y)) \\ = \; & \min\{1 - \min\{\mu((x^- \to y^-)^-), \mu(x)\} + \mu(y), 1\} \\ = \; & \min\{1 - [\min\{\mu((x^- \to y^-)^-), \mu(x)\} - \mu(y)], 1\} \\ \geq \; & 0.5, \end{aligned}$$

i.e., $I(\min\{\mu((x^- \to y^-)^-), \mu(x)\}, \mu(y)) \geq 0.5$. (FFI2) holds. This proves (1).

(2) Suppose that P is the anti-diagonal distribution and μ is an I-fuzzy ideal of A. By (FFI1), $I(\mu(x), \mu(0)) = \max\{1 - \mu(x), \mu(0)\} \geq 0.5$. If $\mu(x) > \max\{\mu(0), 0.5\}$, then $\mu(x) > 0.5$, and $1 - \mu(x) < 0.5$. Hence we have $\mu(0) = \max\{1 - \mu(x), \mu(0)\} \geq 0.5$, and so $\min\{\mu(x), 0.5\} \leq \mu(0)$. Thus (a2) holds. By the same argument we can prove that (b2) holds.

Conversely, suppose that (a2) and (b2) hold. Let P be the anti-diagonal distribution. By Theorem 6.1 we have $I(\mu(x), \mu(0)) = \max\{1 - \mu(x), \mu(0)\}$. Let $\mu(x) \leq \max\{\mu(0), 0.5\}$. If $\mu(0) \geq 0.5$, then $I(\mu(x), \mu(0)) \geq 0.5$. If $\mu(0) < 0.5$, then $\mu(x) \leq 0.5$. Thus $1 - \mu(x) \geq 0.5$, and $I(\mu(x), \mu(0)) \geq 0.5$. Let $\min\{\mu(x), 0.5\} \leq \mu(0)$. If $0.5 \leq \mu(x)$ then $0.5 \leq \mu(0)$. Thus $I(\mu(x), \mu(0)) = \max\{1 - \mu(x), \mu(0)\} \geq 0.5$. If $0.5 \geq \mu(x)$ then $1-\mu(x) \geq 0.5$. Thus $I(\mu(x), \mu(0)) = \max\{1-\mu(x), \mu(0)\} \geq 0.5$. Therefore (FFI1) holds. This proves (2).

(3) Suppose that P is the independent distribution and μ is an I-fuzzy ideal of A. By (FFI1), $I(\mu(x), \mu(0)) = 1 - \mu(x) + \mu(x)\mu(0) \geq 0.5$. Hence $0.5 \leq \mu(x) - \mu(x)\mu(0) = \mu(x)(1-\mu(0))$, (a3) holds. By the same argument we can prove that (b3) holds.

Conversely, suppose that (a3) and (b3) hold. Let P be the independent distribution. By Theorem 6.1, $I(\mu(x), \mu(0)) = 1-\mu(x)+\mu(x)\mu(0)$. By (a3) we have $0.5 \geq \mu(x)(1-\mu(0))$. Thus $1-\mu(x)+\mu(x)\mu(0) = 1-\mu(x)(1-\mu(0)) \geq 1-0.5 = 0.5$ (FFI1) is true. By the same argument we can prove that (FFI2) holds. This proves (3). ■

Definition 6.4. Let μ be a fuzzy set of A, I be a falling implication operator over $[0, 1]$ and $t \in (0, 1]$. Then μ is called an I-fuzzy Gödel ideal of A, if it is satisfies (FFI1) and (FFI3), where

(FFI3) $I(\min\{\mu((x^- \to (y^- \to z^-))^-), \mu((x^- \to y^-)^-)\}, \mu((x^- \to z^-)^-)) \geq t$ for any $x, y, z \in A$.

If $t = 1$ and P is the diagonal distribution, then we easily prove that Definition 6.4 is equivalent to Definition 4.2.

By Theorem 6.3 and Definition 6.4 we can prove the following

Theorem 6.5. *Let μ be a fuzzy set of A and $t = 0.5$, then for all $x, y \in A$*

(1) if P is the diagonal distribution then μ is an I-fuzzy Gödel ideal of A if and only if it satisfies (a1) and (b4), where

 (b4) $\min\{\mu((x^- \to (y^- \to z^-))^-), \mu((x^- \to y^-)^-)\} \leq \mu((x^- \to z^-)^-)$ *or* $0 < \min\{\mu((x^- \to (y^- \to z^-))^-), \mu((x^- \to y^-)^-)\} - \mu((x^- \to z^-)^-) \leq 0.5$.

(2) *if P is the anti-diagonal distribution then μ is an I-fuzzy Gödel ideal of A if and only if it satisfies* (a2) *and* (b5), *where*

 (b5) $\min\{\mu(\mu((x^- \to (y^- \to z^-))^-), \mu((x^- \to y^-)^-)\} \leq \max\{\mu((x^- \to z^-)^-), 0.5\}$ *or* $\min\{\mu(\mu((x^- \to (y^- \to z^-))^-), \mu((x^- \to y^-)^-), 0.5\} \leq \mu((x^- \to z^-)^-)$.

(3) *if P is the independent distribution then μ is an I-fuzzy Gödel ideal of A if and only if it satisfies* (a3) *and* (b6), *where*

 (b6) $\min\{\mu((x^-\to (y^- \to z^-))^-), \mu((x^-\to y^-)^-)\}(1-\mu((x^-\to z^-)^-)) \leq 0.5$.

7. Conclusion

The theory of falling shadows relates to probability concepts with the membership functions of fuzzy sets. Falling shadow representation theory is a reasonable and convenient approach for the theoretical development and the practical applications of fuzzy sets and fuzzy logics. In this paper we apply falling theory to ideal theory of BL-algebras, and obtain some results. Also we consider falling fuzzy inference relations to BL-algebras. As the continuation of these results, we will further apply falling shadow theory and falling fuzzy inference relations to information systems and computer.

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References

[1] Cignoli, R., Esteva, F., Godo,, L., Torrens, A., *Basic fuzzy logic is the logic of continuous T-norm and their residua*, Soft Comput., **12**(2000), 106-112.

[2] Goodman, I.R., *Fuzzy sets as equivalence classes of random sets*, in: R.Yager(ed.), Recent developments in fuzzy sets and probability theory, Pergamon, New York, NY, USA, 1982.

[3] Hájek, P., *Metamathematics of fuzzy logic*, Kluwer Academic Publishers, Dordrecht, 1998.

[4] Haveshki, M., Saeid, A.B., Eslami, E., *Some types of filters in BL-algebras*, Soft Computing, 10 (2006), 657-664.

[5] Jun, Y.B., Kang, M.S., *Fuzzy positive implicative ideals of BCK-algebras based on the theory of falling shadows*, Comput. Math. Apll., 61 (2011), 62-67.

[6] Jun, Y.B., Kang, M.S., Park, C.H., *Implicative ideals of BCK-algebras based on the fuzzy sets and the theory of falling shadows*, Int. J. Math Math Sci., Article ID 819463 (2011), 11 pages.

[7] Liu, L.Z., Li, K.T., *Fuzzy filters in BL-algebras*, Inform Sci., 173 (2005), 141-154.

[8] Meng, B.L., Xin, X.L., *Prime ideals and Gödel ideals of BL-algebras*, Journal of Advances in Mathematics, 9 (2015), 2989-3005.

[9] Meng, B.L., Xin, X.L., *On fuzzy ideal theory of BL-algebras*, Scientific World J., vol. 2014, Article ID 757382, 12 pp.

[10] Saeid, A.B., Motamed, S., *Normal filters in BL-Algebras*, World Applied Sciences Journal, 7 (2009), 70-76.

[11] Saeid, A.B., Ahadpanah, A., Torkzadeh, L., *Smarandache BL-Algebras*, Journal of Applied Logic, 8 (2010), 253-261.

[12] Tan, S.K., Wang, P.Z., Zhang, X.Z., *Fuzzy inference relation based on the theory of falling shadows*, Fuzzy Sets and Systems, 53 (1993), 179-188.

[13] Tan, S.K., Wang, P.Z., Zhang, X.Z., *Fuzzy set operations based on the theory of falling shadows*, J. Math. Anal. Appl., 174 (1993), 242-255.

[14] Turunen, E., *BL-algebras of basic fuzzy logic,* Mathware & Soft Comp., 6 (1999), 49-61.

[15] Turunen, E., *Boolean deductive systems of BL-algebras,* Arch. Math. Logic, 40 (2001), 467-473.

[16] Wang, G.J., *On the logic foundation of fuzzy reasoning*, Inform. Sci., 117 (1999), 47-88.

[17] Wang, P.Z., *Fuzzy sets and falling shadows of random sets*, Beijing Normal Univ. Press, People's Republic of China (in Chinese), 1985.

[18] Wang, P.Z., Sanchez, E., *Treating a fuzzy subsets as a projectable random set in Fuzzy Information and Decision*, M.M. Gupta and E. Sanchez, Eds., Pergamon, New York, NY, USA, 1982, 212-219.

[19] Yuan, X.H., Lee, E.S., *A fuzzy algebraic system based on the theory of falling shadows*, J. Math. Anal. Appl., 208 (1997), 243-251.

[20] Zadeh, L.A., *Fuzzy sets*, Inform. Control, 8 (1965), 338-353.

[21] Zhan, J.M., Jun, Y.B., Kim, H.S., *Some types of falling fuzzy filters in BL-algebras and its applications*, J. Intelligent & Fuzzy systems, 26 (2014), 1675-1685.

[22] Zhang, X.H., Jun, Y.B., Doh, M.I., *On fuzzy filters and fuzzy ideals of BL-algebras*, Fuzzy Systems and Mathematics, 20 (2006), 8-20.

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FUZZY p-IDEALS IN MV-ALGEBRAS

F. Forouzesh

Department of Mathematics
Higher Education Complex of Bam
Kerman
Iran
e-mail: frouzesh@bam.ac.ir

Abstract. In this paper, we introduce the notion of p-ideals of MV-algebras and characterization of p-ideals is given. Also, we show that p-ideals equivalent to Boolean ideals in MV-algebras. In addition, we introduce the notion of fuzzy p-ideals of an MV-algebra and show that in any MV-algebra, the concept of fuzzy p-ideals is equivalent to fuzzy Boolean ideals and fuzzy implicative ideals. Also, several characterizations of these fuzzy ideals are given and prove that extension theorem of fuzzy p-ideals. Furthermore, we describe the transfer principle for fuzzy p-ideals in terms of level subsets. Finally, by using the notions of maximal and normal fuzzy p-ideals, we show that under certain conditions a fuzzy p-ideal is two valued and takes the values 0 and 1.

Keywords: p-ideal, fuzzy p-ideal, fuzzy Boolean ideal, normal.
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1. Introduction and preliminaries

C. Chang introduced the notion of MV-algebras to provide a proof for the completeness of the Łukasiewicz axioms for infinite valued propositional logic [1]. In fact MV-algebras are now algebraic counterparts of Łukasiewicz many valued logics. Also, D. Mundici [10] extended such a correspondence to a functor Γ from lattice ordered abelian groups with strong unit to MV-algebras.

In [8], Iseki proposed the notion of implicative ideals in BCK-algebras and obtained some results. Subsequently, Hoo and Sessa [6] proposed the notion of Boolean ideals in MV-algebras and proved that implicative ideals and Boolean ideals are equivalent in MV-algebras.

The concept of fuzzy set was introduced by Zadeh (1965) [13]. This idea has been applied to other algebraic structures such as semi-group, group, ideals, modules and topologies.

In 1991, Xi [12] applied the concept of fuzzy sets to BCK-algebras and proposed the notion of fuzzy implicative ideals. Afterwards, Hoo [5] proved that fuzzy implicative and fuzzy Boolean ideals are equivalent in MV-algebras.

In this paper, we introduce the notion of fuzzy p-ideals. We obtain some equivalent definitions of fuzzy p-ideals, and establish the extension theorem of fuzzy p-ideals in MV-algebras.

Furthermore, we prove that fuzzy Boolean ideals and fuzzy p-ideals are equivalent and using a level set of fuzzy set in an MV-algebra, we give a characterization of fuzzy p-ideals.

We define the concept of p-ideals and we give characterization of them. We prove that p-ideals are equivalent to Boolean ideals in MV-algebras.

Finally, we introduce normal fuzzy p-ideals and we show that the maximal fuzzy p-ideals of an MV-algebra A are normal and take only the values 0 and 1.

We recollect some definitions and results which will be used in the following:

Definition 1.1. [1], [2], [11] An MV-algebra is a structure $(A, \oplus, {}^*, 0)$ where $\oplus$ is a binary operation, * is a unary operation, and 0 is a constant such that the following axioms are satisfied for any $a, b \in A$:

$(MV1)$ $(A, \oplus, 0)$ is an abelian monoid,

$(MV2)$ $(a^*)^* = a$,

$(MV3)$ $0^* \oplus a = 0^*$,

$(MV4)$ $(a^* \oplus b)^* \oplus b = (b^* \oplus a)^* \oplus a$.

Note that $1 = 0^*$ and the auxiliary operation $\odot$ as follows: $x \odot y = (x^* \oplus y^*)^*$.

We recall that the natural order determines a bounded distributive lattice structure such that

$$x \vee y = x \oplus (x^* \odot y) = y \oplus (x \odot y^*) \quad \text{and} \quad x \wedge y = x \odot (x^* \oplus y) = y \odot (y^* \oplus x).$$

Lemma 1.2. [2], [11] *In each MV-algebra, the following relations hold for all $x, y, z \in A$:*

(1) $x \leq y$ *if and only if* $y^* \leq x^*$,

(2) *If* $x \leq y$, *then* $x \oplus z \leq y \oplus z$ *and* $x \odot z \leq y \odot z$,

(3) $x \leq y$ *if and only if* $x^* \oplus y = 1$ *if and if* $x \odot y^* = 0$,

(4) $x, y \leq x \oplus y$ *and* $x \odot y \leq x, y$, $x \leq nx = x \oplus x \oplus \cdots \oplus x$ *and* $x^n = x \odot x \odot \cdots \odot x \leq x$,

(5) $x \oplus x^* = 1$ *and* $x \odot x^* = 0$,

(6) *If* $x \leq y$ *and* $z \leq t$, *then* $x \oplus z \leq y \oplus t$,

(7) $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$.

Definition 1.3. [1], [11] An ideal of an MV-algebra A is a nonempty subset I of A satisfying the following conditions:

$(I1)$ If $x \in I$, $y \in A$ and $y \leq x$ then $y \in I$,

$(I2)$ If $x, y \in I$, then $x \oplus y \in I$.

We denote by $Id(A)$ the set of ideals of an MV-algebra A.

Definition 1.4. [2] Let I be an ideal of an MV-algebra A. Then I is a proper ideal of A if $I \neq A$.

- [6] An ideal I of an MV-algebra A is called Boolean ideal if $x \wedge x^* \in I$, for all $x \in A$.
- [6] An ideal I of an MV-algebra A is called an implicative ideal of A if for any $x, y, z \in A$ such that $z \odot (y^* \odot x^*) \in I$ and $y \odot x^* \in I$, then $z \odot x^* \in I$.

Definition 1.5. Let X and Y be two MV-algebras. A function $f : X \to Y$ is called homomorphism of MV-algebras if and only if

(1) $f(0) = 0$,

(2) $f(x \oplus y) = f(x) \oplus f(y)$,

(3) $f(x^*) = (f(x))^*$.

Remark 1.6. [2] In an MV-algebra A, the distance function is

$$d : \quad A \times A \longrightarrow A, \qquad d(x, y) := (x \odot y^*) \oplus (y \odot x^*).$$

Suppose that I is an ideal of an MV-algebra A. Define $x \sim_I y$ if and only if $d(x, y) \in I$ if and only if $x \odot y^* \in I$ and $y \odot x^* \in I$. Then $\sim_I$ is a congruence relation on A. The set of all congruence classes is denoted by A/I then $A/I = \{[x] : x \in A\}$, where $[x] = \{y \in A : x \sim_I y\}$. We can easily to see that $x \in I$ if and only if $x/I = 0/I$. The MV-algebra operations on A/I given by $x/I \oplus y/I = (x \oplus y)/I$ and $(x/I)^* = x^*/I$, are well defined. Hence $(A/I, \oplus, *, [0])$ becomes an MV-algebra [2], [11].

Definition 1.7. [4] A fuzzy set in A is a mapping $\mu : A \to [0, 1]$. Let A be an MV-algebra and μ be a fuzzy set on A. Then μ is a fuzzy ideal of A, if it satisfies

$(MV1)$ $\mu(0) \geq \mu(x)$, for all $x \in A$,

$(MV2)$ $\mu(y) \geq \mu(x) \wedge \mu(y \odot x^*)$, for all $x, y \in A$.

Lemma 1.8. [4] *Let A be an MV-algebra and μ be a fuzzy set on A. Then μ is called a fuzzy ideal on A, if and only if*

(1) *$\mu(x) \leq \mu(0)$, for all $x \in A$ and*

(2) *$\mu(x \oplus y) \geq \mu(x) \wedge \mu(y)$, for all $x, y \in A$,*

(3) *If $x \leq y$, then $\mu(x) \geq \mu(y)$.*

Definition 1.9. [13] Let X, Y be two MV-algebras. μ is a fuzzy subset of X, μ' is a fuzzy subset of Y and $f : X \to Y$ is a homomorphism. The image of μ under f denoted by $f(\mu)$ is a fuzzy set of Y defined by:

For all $y \in Y$,

$$\begin{aligned} f(\mu)(y) &= \sup_{x \in f^{-1}(y)} \mu(x), \quad \text{if } f^{-1}(y) \neq \emptyset, \text{ and} \\ f(\mu)(y) &= 0 \qquad\qquad\qquad \text{if } f^{-1}(y) = \emptyset. \end{aligned}$$

The preimage of μ' under f denoted by $f^{-1}(\mu')$ is a fuzzy set of X defined by:

For all $x \in X$,

$$f^{-1}(\mu')(x) = \mu'(f(x)).$$

Definition 1.10. [13] A fuzzy subset μ of X has sup-property if for any nonempty subset Y of X, there exists $y_0 \in Y$ such that $\mu(y_0) = \sup_{y \in Y} \mu(y)$.

Definition 1.11. [13] Let μ be a fuzzy set in A, $t \in [0, 1]$. The set $\mu_t = \{x \in A : \mu(x) \geq t\}$ is called a level subset of μ. For any fuzzy sets μ, ν in A, we define

$$\mu \subseteq \nu \text{ if and only if } \mu(x) \leq \nu(x) \text{ for all } x \in A.$$

Theorem 1.12. [4] *Let μ be a fuzzy ideal in A. For any $x, y, z \in A$, the following hold:*

(1) $\mu(x \oplus y) = \mu(x) \wedge \mu(y)$,

(2) $\mu(x \odot y) \geq \mu(x) \wedge \mu(y)$,

(3) $\mu(x \wedge y) \geq \mu(x) \wedge \mu(y)$,

(4) $\mu(nx) = \mu(x)$,

(5) $\mu(x \vee y) = \mu(x) \wedge \mu(y)$.

Theorem 1.13. [3] *Let μ be a fuzzy set in A. μ is a fuzzy ideal if and only if for all $t \in [0, 1]$, μ_t is either empty or an ideal of A.*

Corollary 1.14. *Let I be a nonempty subset of A. I is an ideal if and only if χ_I is a fuzzy ideal of A, where χ_I is characteristic function of I.*

2. P-ideals of MV-algebras

Definition 2.1. I is a p-ideal if it satisfies the following conditions:

(i) $0 \in I$,

(ii) For all $x, y, z \in A$, if $y \odot (z^* \oplus y) \odot x^* \in I$ and $x \in I$, then $y \in I$.

Example 2.2. Let $A = \{0, a, b, 1\}$, where $0 < a, b < 1$. Define $\odot$, $\oplus$ and $*$ as follows:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\oplus$	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

$*$	0	a	b	1
	1	b	a	0

Then $(A, \oplus, \odot, *, 0, 1)$ is an MV-algebra [7]. Simple computations prove that $I_1 = \{0, a\}$ and $I_2 = \{0, b\}$ are p-ideals of A.

The following example shows that an ideal may not be a p-ideal.

Example 2.3. Let $A = \{0, a, b, c, d, 1\}$, where $0 < a, c < d < 1$ and $0 < a < b < 1$. Define $\odot$, $\oplus$ and $*$ as follows:

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	a	0	0	a
b	0	a	b	0	a	b
c	0	0	0	c	c	c
d	0	0	a	c	c	d
1	0	a	b	c	d	1

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	b	b	d	1	1
b	b	b	b	1	1	1
c	c	d	1	c	d	1
d	d	1	1	d	1	1
1	1	1	1	1	1	1

$*$	0	a	b	c	d	1
	1	d	c	b	a	0

Then $(A, \oplus, \odot, *, 0, 1)$ is an MV-algebra [7]. It is clear that $I = \{0, c\}$ is an ideal but I is not a p-ideal, since $a \odot (b^* \oplus a) \odot c^* = 0 \in I$ and $c \in I$ while $a \notin I$.

Proposition 2.4. *Every p-ideal is an ideal.*

Proof. Let I be a p-ideal of A, it is clear that $0 \in I$. Suppose that $y \leq x$ and $x \in I$, for all $x, y \in A$. By Lemma 1.2, we imply that $y \odot x^* = 0$, by setting $z = 0$ in the definition of p-ideal, we obtain $y \odot (0^* \oplus y) \odot x^* = y \odot x^* = 0 \in I$ and $x \in I$, it follows that $y \in I$.

Now, let $x, y \in I$. We prove that $x \oplus y \in I$. Since $x^* \wedge y \leq y \in I$, hence $x^* \wedge y \in I$. So $x^* \wedge y = x^* \odot (x \oplus y) \geq (x \oplus y) \odot (y^* \oplus (x \oplus y)) \odot x^*$ and $(x \oplus y) \odot (y^* \oplus (x \oplus y)) \odot x^* \in I$. Since I is a p-ideal of A and $x \in I$, it follows that $x \oplus y \in I$. ■

The following proposition gives a characterization of p-ideals of A.

Proposition 2.5. *The following conditions are equivalent for any ideal I:*

(i) *I is a p-ideal,*

(ii) *For all $x, y \in A$, $x \odot (y^* \oplus x) \in I$ implies $x \in I$,*

(iii) *If $x^2 \in I$, then $x \in I$.*

Proof. (i) $\rightarrow$ (ii) Suppose that I is a p-ideal of A and $x \odot (y^* \oplus x) \in I$, since $x \odot (y^* \oplus x) \odot 0^* \in I$ and $0 \in I$, we apply the fact that I is p-ideal of A and obtain the result.

(ii) $\rightarrow$ (iii) We obtain the result by setting $y = 1$ in the equation (ii).

(iii) $\rightarrow$ (i) Suppose that $y \odot (z^* \oplus y) \odot x^* \in I$ and $x \in I$. By ideal property, we conclude that $y \odot (z^* \oplus y) \leq x \vee (y \odot (z^* \oplus y)) = x \oplus (x^* \odot y \odot (z^* \oplus y)) \in I$. Hence $y \odot (z^* \oplus y) \in I$. On the other hand, $y \odot y = y \odot (1^* \oplus y) \leq y \odot (z^* \oplus y) \in I$. Since I is an ideal, we obtain $y^2 = y \odot y \in I$ and we apply the hypothesis and obtain $y \in I$. ■

Theorem 2.6. *A proper ideal I is a p-ideal if and only if I is a Boolean ideal of A.*

Proof. Let I be a Boolean ideal of A. By Proposition 2.5, it sufficient to show that if $x^2 \in I$, then $x \in I$. On the other hand, we have for all $x \in A$, $x^2 = x \odot x \in I$ and $x \wedge x^* \in I$. Since I is an ideal, $(x \wedge x^*) \oplus x^2 = x \odot (x^* \oplus x^*) \oplus x^2 = (x \odot (x^2)^*) \oplus x^2 \in I$. On the other hand, $x \leq x^2 \vee x \in I$, thus $x \in I$.

Conversely, suppose that I is a p-ideal. Let $x \in A$. Setting $t = x \wedge x^*$, we show that $t \in I$. Since $t \leq x$, we have $x^* \wedge x \leq x^* \leq t^*$ and then $t \leq t^*$ or $t^2 = t \odot t = 0 \in I$. So since I is a p-ideal, by Proposition 2.5, we imply that $t \in I$. Thus I is Boolean ideal of A. ■

By the above theorem, the extension theorem of p-ideals is obtained from the following result:

Theorem 2.7. *Let I_1 and I_2 two ideals of A such that $I_1 \subseteq I_2$. If I_1 is a p-ideal, then so is I_2.*

3. Fuzzy p-ideals in MV-algebras

Definition 3.1. Let μ be a fuzzy set in A. μ is called a fuzzy p-ideal if it satisfies

(i) $\mu(0) \geq \mu(x)$,

(ii) For all $x, y, z \in A$, $\mu(y \odot (z^* \oplus y) \odot x^*) \wedge \mu(x) \leq \mu(y)$.

The following example shows that fuzzy p-ideals exist.

Example 3.2. Consider Example 2.2. Define a fuzzy set μ in A by

$$\mu(1) = \mu(b) = \mu(a) = t_1,$$
$$\mu(0) = t_2 \quad (0 \leq t_1 < t_2 \leq 1).$$

We can easily verify that μ is a fuzzy p-ideal of A.

Theorem 3.3. *Every fuzzy p-ideal is a fuzzy ideal.*

Proof. Let μ be a fuzzy p-ideal in A. Taking $z = 0$ in Definition 3.1 (ii), we get

$$\mu(y \odot x^*) \wedge \mu(x) \leq \mu(y), \text{ for all } x, y \in A.$$

Hence μ is a fuzzy ideal. ■

Remark 3.4. The converse of the above theorem may not be true.

Let $A = \{0, a, b, c, d, 1\}$, where $0 < a, b < c < 1$ and $0 < b < d < 1$.

Define $\oplus$, $\odot$ and $*$ as follows:

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	0	0	b	b
c	0	a	0	a	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	a	c	c	1	1
b	b	c	d	1	d	1
c	c	c	1	1	1	1
d	d	1	d	1	d	1
1	1	1	1	1	1	1

$*$	0	a	b	c	d	1
	1	d	c	b	a	0

Then $(A, \oplus, \odot, *, 0, 1)$ is an MV-algebra [7].

Define a fuzzy set μ in A by

$$\mu(1) = \mu(a) = \mu(b) = \mu(c) = \mu(d) = t_2,\ \mu(0) = t_1\ (0 \leq t_2 < t_1 \leq 1).$$

Routine calculation shows that μ is a fuzzy ideal. But μ is not a fuzzy p-ideal, because

$$\mu(b \odot (d^* \oplus b) \odot 0^*) \wedge \mu(0) = t_1 \nleq \mu(b) = t_2.$$

The following Theorem gives a characterization of fuzzy p-ideals.

Theorem 3.5. *Let μ be a fuzzy ideal in A. The following are equivalent:*

(i) *μ is a fuzzy p-ideal,*

(ii) *For all $x, y \in A$, $\mu(x \odot (y^* \oplus x)) \leq \mu(x)$,*

(iii) *For all $x, y \in A$, $\mu(x^2) = \mu(x)$.*

Proof. (i) $\rightarrow$ (ii) Suppose that μ is a fuzzy p-ideal of A and we have

$$\mu(x \odot (y^* \oplus x)) = \mu(x \odot (y^* \oplus x) \odot 0^*) \wedge \mu(0) \leq \mu(x).$$

(ii) $\rightarrow$ (iii) By setting $y = 1$ in equation (ii), we obtain $\mu(x^2) \leq \mu(x)$. Also, since $x^2 \leq x$, then $\mu(x) \leq \mu(x^2)$. Thus, $\mu(x) = \mu(x^2)$.

(iii) $\rightarrow$ (i) We show that $\mu(y \odot (z^* \oplus y) \odot x^*) \wedge \mu(x) \leq \mu(y)$.

By ideal property, we get

$$y \odot (z^* \oplus y) \leq x \vee (y \odot (z^* \oplus y)) = x \oplus (x^* \odot y \odot (z^* \oplus y)).$$

Hence we have

$$\begin{aligned} \mu(y) &= \mu(y^2) = \mu(y \odot (1^* \oplus y)), \\ &\geq \mu(y \odot (z^* \oplus y)), \\ &\geq \mu(x \oplus (x^* \odot y \odot (z^* \oplus y)), \\ &= \mu(x) \wedge \mu(x^* \odot y \odot (z^* \oplus y)). \end{aligned}$$

Thus μ is a fuzzy p-ideal of A. ∎

Now, we describe the transfer principle [9] for fuzzy p-ideals in terms of level subsets:

Theorem 3.6. *Let μ be a fuzzy set in A. μ is a fuzzy p-ideal if and only if for each $t \in [0, 1]$, μ_t is either empty or a p-ideal of A.*

Proof. Let μ be a fuzzy p-ideal and for each $t \in [0, 1]$, $\mu_t \neq \emptyset$. We assume that $x_0 \in \mu_t$, i.e., $\mu(x_0) \geq t$. Since μ is a fuzzy p-ideal, $\mu(0) \geq \mu(x_0) \geq t$. On the other hand $0 \in \mu_t$. Next, if $y \odot (z^* \oplus y) \odot x^* \in \mu_t$ and $x \in \mu_t$, then $\mu(y \odot (z^* \oplus y) \odot x^*) \geq t$ and $\mu(x) \geq t$. Hence $\mu(y) \geq \mu(y \odot (z^* \oplus y) \odot x^*) \wedge \mu(x) \geq t$. Thus $\mu(y) \geq t$. Therefore, $y \in \mu_t$. This results μ_t is a p-ideal of A.

Conversely, let for each $t \in [0, 1]$, μ_t is either empty or a p-ideal in A. Since $x \in \mu_{\mu(x)}$, for any $x \in A$, we have $\mu_{\mu(x)}$ is a p-ideal of A. Thus $0 \in \mu_{\mu(x)}$. Hence $\mu(0) \geq \mu(x)$, for all $x \in A$.

Now, let $t = \mu(y \odot (z^* \oplus y) \odot x^*) \wedge \mu(x)$. Then $y \odot (z^* \oplus y) \odot x^* \in \mu_t$ and $x \in \mu_t$. Since μ_t is a p-ideal, $y \in \mu_t$. It follows that $\mu(y) \geq t = \mu(y \odot (z^* \oplus y) \odot x^*) \wedge \mu(x)$. Thus μ is a fuzzy p-ideal of A. ∎

Corollary 3.7. *Let I be a nonempty set of A. I is a p-ideal of A if and only if χ_I is a fuzzy p-ideal of A.*

Proof. Let I be a p-ideal of A. We will prove that χ_I is a fuzzy p-ideal. By Theorem 3.5, we show that $\chi_I(x^2) = \chi_I(x)$, for all $x \in A$.

If $x \in I$, then $x^2 = x \odot x \leq x$. Hence $x^2 \in I$. Thus $\chi_I(x^2) = \chi_I(x) = 1$.

If $x \notin I$, then since I is a p-ideal of A, by Proposition 2.5, we obtain $x^2 \notin I$. Hence $\chi_I(x^2) = \chi_I(x) = 0$.

Conversely, let χ_I be a fuzzy p-ideal. It is sufficient to show that by Proposition 2.5, if $x^2 \in I$, then $x \in I$.

If $x^2 \in I$, then $1 = \chi_I(x^2) = \chi_I(x)$. Hence $\chi_I(x) = 1$. It follows that $x \in I$. Thus by Proposition 2.5, I is a p-ideal of A. ∎

The following theorem, shows the relation between fuzzy p-ideals and fuzzy Boolean ideals in A.

Theorem 3.8. *Every fuzzy ideal μ is fuzzy Boolean ideal if and only if μ is fuzzy p-ideal in A.*

Proof. Let μ be a fuzzy Boolean ideal of A. Then $\mu(x \wedge x^*) = \mu(0)$. By Theorem 1.12 (1) and Lemma 1.8 (3), we have:

$$\begin{aligned}
\mu(x^2) &= \mu(0) \wedge \mu(x^2)\\
&= \mu(x \wedge x^*) \wedge \mu(x^2)\\
&= \mu((x \wedge x^*) \oplus x^2)\\
&= \mu((x \odot (x^* \oplus x^*)) \oplus (x \odot x))\\
&= \mu((x \odot (x^2)^*) \oplus x^2)\\
&= \mu(x^2 \vee x) \leq \mu(x).
\end{aligned}$$

Hence $\mu(x^2) \leq \mu(x)$. Also, since $\mu(x) \leq \mu(x^2)$, we get $\mu(x^2) = \mu(x)$. Thus by Theorem 3.5, we obtain μ is a fuzzy p-ideal of A.

Conversely, let μ be a fuzzy p-ideal and $x \in A$. Setting $t = x \wedge x^*$, we show that $\mu(t) = \mu(0)$. Since $t \leq x$, $t = x^* \wedge x \leq x^* \leq t^*$ and then $t \leq t^*$ or $t^2 = t \odot t = 0$. Hence

$$\mu(t^2) = \mu(0) \tag{1}$$

Also, since μ is a fuzzy p-ideal, $\mu(t^2) = \mu(t)$. It follows from (1) that $\mu(0) = \mu(t)$. Therefore, μ is a fuzzy Boolean ideal of A. ∎

The extension theorem of fuzzy p-ideals is obtained from the following result:

Theorem 3.9. *Let μ, ν be two fuzzy ideals which satisfy $\mu \leq \nu$, $\mu(0) = \nu(0)$. If μ is a fuzzy p-ideal, so is v.*

In general, it is not difficult to see the following:

Theorem 3.10.

(i) *Let μ_i $(i \in \Gamma)$ be a fuzzy p-ideal. Then $\wedge_{i\in\Gamma}\mu_i$ is a fuzzy p-ideal of A.*

(ii) *If $\wedge_{i\in\Gamma}\mu_i$ is a fuzzy p-ideal of A, then by Theorem 3.9 and $\wedge_{i\in\Gamma}\mu_i \leq \mu_i$, we get that μ_i, for all $i \in \Gamma$ is a fuzzy p-ideal of A.*

(iii) *Let μ_i, for all $i \in \Gamma$ be a fuzzy p-ideal of A. Then, by Theorem* 3.9, $\vee_{i\in\Gamma}\mu_i$ *is a fuzzy p-ideal of A.*

Remark 3.11. Let $f : X \to Y$ be onto MV-homomorphism. Then prove that preimage of a fuzzy p-ideal μ under f is also a fuzzy p-ideal of A.

Proof. Suppose that μ is a fuzzy p-ideal of Y. By Theorem 3.5, we have

$$f^{-1}(\mu)(x^2) = \mu(f(x^2)) = \mu((f(x))^2) = \mu(f(x)) = f^{-1}(\mu)(x).$$

It follows from Theorem 3.5 that $f^{-1}(\mu)$ is a fuzzy p-ideal of X. ■

Proposition 3.12. *Let $f : X \to Y$ be an onto MV-homomorphism. The image $f(\mu)$ of a fuzzy p-ideal μ with a sup-property is also a fuzzy p-ideal of A.*

Proof. By Theorem 3.5, it is sufficient to show that $f(\mu)(y^2) = f(\mu)(y)$, for all $y \in Y$.

Let $y \in Y$ and $x \in f^{-1}(y)$ such that $\mu(x) = sup_{t\in f^{-1}(y)}\mu(t)$. We have

$$f(\mu)(y) = \sup_{t\in f^{-1}(y)} \mu(t) = \mu(x) = \mu(x^2) = \sup_{t\in f^{-1}(y^2)} \mu(t) = f(\mu)(y^2). \qquad ■$$

Theorem 3.8 and [5, Theorem 2.8] state that relations among fuzzy implicative, fuzzy Boolean and fuzzy p-ideals in A.

Corollary 3.13. *Let μ be a fuzzy ideal of A. The following are equivalent:*

(1) *μ is a fuzzy implicative ideal,*

(2) *μ is a fuzzy Boolean ideal,*

(3) *μ is a fuzzy p-ideal,*

(4) *$A/\mu_{\mu(0)}$ is a Boolean algebra.*

4. Normalizations of fuzzy p-ideals

Definition 4.1. A fuzzy p-ideal μ of A is said to be normal if there exists $x \in A$ such that $\mu(x) = 1$.

Example 4.2. Let $A = \{0, a, b, 1\}$ be an MV-algebra in Example 2.2. Then the fuzzy set μ in A defined by $\mu(0) = 1$ and $\mu(a) = \mu(b) = \mu(1) = t$, $(0 \leq t \leq 1)$ is a normal fuzzy p-ideal of A.

Proposition 4.3. *Given a fuzzy p-ideal μ of A, let μ^+ be a fuzzy set in A defined by $\mu^+(x) = \mu(x) + 1 - \mu(0)$, for all $x \in A$. Then μ^+ is a normal fuzzy p-ideal of A which contains μ*

Proof. Let μ be a fuzzy p-ideal of A. We have $\mu^+(0) = \mu(0)+1-\mu(0) = 1 \geq \mu(x)$, for all $x \in A$. For any $x, y, z \in A$, by Theorem 3.5 (ii) and Theorem 1.12 (1), we have

$$\begin{aligned}
& \mu^+(y \odot (z^* \oplus y) \odot x^*) \wedge \mu^+(x) \\
= \; & [\mu(y \odot (z^* \oplus y) \odot x^*) + 1 - \mu(0)] \wedge (\mu(x) + 1 - \mu(0)) \\
= \; & [\mu(y \odot (z^* \oplus y) \odot x^*) \wedge \mu(x)] + 1 - \mu(0) \\
= \; & \mu((y \odot (z^* \oplus y) \odot x^*) \oplus x) + 1 - \mu(0) \\
= \; & [\mu(x \vee (y \odot (z^* \oplus y)] + 1 - \mu(0) \\
\leq \; & \mu(y \odot (z^* \oplus y)) + 1 - \mu(0) \\
\leq \; & \mu(y) + 1 - \mu(0).
\end{aligned}$$

Hence μ^+ is a normal fuzzy p-ideal of A and obviously $\mu \subseteq \mu^+$. ∎

Corollary 4.4. *If there is $x \in A$ such that $\mu^+(x) = 0$, then $\mu(x) = 0$.*

Corollary 4.5. $(\mu^+)^+ = \mu^+$, *for any fuzzy p-ideal of A. If μ is normal, then $\mu^+ = \mu$.*

Remark 4.6. Let μ be a fuzzy p-ideal of A. If there is a fuzzy p-ideal ν of A satisfying $\nu^+ \subseteq \mu$, then μ is normal.

Proof. Assume that there is a fuzzy p-ideal ν of A such that $\nu^+ \subseteq \mu$. Then $1 = \nu^+(0) \leq \mu(0)$, and so $\mu(0) = 1$. Hence μ is normal. ∎

Given a fuzzy p-ideal of A, we construct a new normal fuzzy p-ideal of A.

Theorem 4.7. *Let μ be a fuzzy p-ideal of A and let $f : [0, \mu(0)] \to [0, 1]$ be an increasing function. Let $\mu_f : A \to [0, 1]$ be a fuzzy set in A define by $\mu_f = f(\mu(x))$, for all $x \in A$. Then μ_f is a fuzzy p-ideal of A. In particular, if $f(\mu(0)) = 1$, then μ_f is normal and if $f(t) \geq t$, for all $t \in [0, \mu(0)]$, then $\mu \subseteq \mu_f$.*

Proof. Since $\mu(0) \geq \mu(x)$, for all $x \in A$ and f is increasing, we have $\mu_f(0) = f(\mu(0)) \geq f(\mu(x)) = \mu_f(x)$ for all $x \in A$. For any $x, y, z \in A$, we get

$$\begin{aligned}
& \mu_f(y \odot (z^* \oplus y) \odot x^*) \wedge \mu_f(x) \\
= \; & f(\mu(y \odot (z^* \oplus y) \odot x^*) \wedge f(\mu(x)) \\
= \; & f(\mu(y \odot (z^* \oplus y) \odot x^*) \wedge \mu(x)) \\
\leq \; & f(\mu(y)) \\
= \; & \mu_f(y).
\end{aligned}$$

Hence μ_f is a fuzzy p-ideal of A. If $f(\mu(0)) = 1$, then clearly μ_f is normal. Assume that $f(t) \geq t$, for all $t \in [0, \mu(0)]$. Then $\mu_f(x) = f(\mu(x)) \geq \mu(x)$, for all $x \in A$, which proves $\mu \subseteq \mu_f$. ∎

Let $N(A)$ denote the set of all normal fuzzy p-ideals of A.

Theorem 4.8. *Let $\mu \in N(A)$ be nonconstant such that it is a maximal element of the poset $(N(A), \subseteq)$. Then μ takes only the values 0 and 1.*

Proof. Since μ is normal, we have $\mu(0) = 1$. Let $x \in A$ such that $\mu(x) \neq 1$. It is sufficient to show that $\mu(x) = 0$. If not, then there exists $a \in A$ such that $0 < \mu(a) < 1$. Define a fuzzy set ν in A by $\nu(x) = (1/2)\{\mu(x) + \mu(a)\}$, for all $x \in A$. Clearly, ν is well defined, and we get

$$\nu(0) = 1/2\{\mu(0) + \mu(a)\} = 1/2\{1 + \mu(a)\} \geq 1/2\{\mu(x) + \mu(a)\} = \nu(x) \quad \forall x \in A.$$

Let $x, y, z \in A$. Then

$$\begin{aligned} \nu(y) &= 1/2\{\mu(y) + \mu(a)\} \\ &\geq 1/2\{[\mu(y \odot (z^* \oplus y) \odot x^*) \wedge \mu(x)] + \mu(a)\} \\ &= 1/2[(\mu(y \odot (z^* \oplus y) \odot x^*) + \mu(a)] \wedge 1/2(\mu(x) + \mu(a)) \\ &= \nu(y \odot (z^* \oplus y) \odot x^*) \wedge \nu(x). \end{aligned}$$

Hence ν is a fuzzy p-ideal of A. By Proposition 4.3, ν^+ is a maximal fuzzy p-ideal of A, where ν^+ is defined by $\nu^+(x) = \nu(x) + 1 - \nu(0)$, for all $x \in A$. Note that

$$\begin{aligned} \nu^+(a) &= \nu(a) + 1 - \nu(0) \\ &= 1/2\{\mu(a) + \mu(a)\} + 1 - 1/2\{\mu(0) + \mu(a)\} \\ &= 1/2\{\mu(a) + 1\} > \mu(a) \end{aligned}$$

and $\nu^+(a) < 1 = \nu^+(0)$. It follows that ν^+ is nonconstant and μ is not a maximal element of $(N(A), \subseteq)$. This is a contradiction. ∎

Definition 4.9. A nonconstant fuzzy p-ideal μ of A is called maximal if μ^+ is a maximal element of the poset $N(A)$.

Theorem 4.10. *A maximal fuzzy p-ideal μ of A is normal and takes only the values* 0 *and* 1.

Proof. Let μ be a maximal fuzzy p-ideal μ of A. Then μ^+ is a nonconstant maximal element of the poset $N(A)$ and by Theorem 4.8, μ^+ takes only the values 0 and 1. Clearly $\mu^+(x) = 1$ if and only if $\mu(x) = \mu(0)$ and $\mu^+(x) = 0$ if and only if $\mu(x) = \mu(0) - 1$. But $\mu \subseteq \mu^+$ (by Theorem 4.3). So $\mu^+(x) = 0$ implies that $\mu(x) = 0$, consequently $\mu(0) = 1$. Therefore μ is normal. ∎

Theorem 4.11. *Let μ be a fuzzy p-ideal of A, $\mu(1) \neq 0$ and $\overline{\mu}$ be the fuzzy set of A defined by $\overline{\mu}(x) = \mu(x)/\mu(1)$ for all $x \in A$. Then $\overline{\mu}$ is a normal fuzzy p-ideal of A and $\mu \subseteq \overline{\mu}$.*

Proof. Let $x, y \in A$. We have

$$\overline{\mu}(0) = \mu(0)/\mu(1) \geq \mu(x)/\mu(1) = \overline{\mu}(x).$$

Also, we have

$$\begin{aligned}\overline{\mu}(y) &= \mu(y)/\mu(1)\\ &\leq [\mu(y \odot (z^* \oplus y) \odot x^*) \wedge \mu(x)]/\mu(1)\\ &= [\mu(y \odot (z^* \oplus y) \odot x^*)/\mu(1)] \wedge \mu(x)/\mu(1)\\ &= \overline{\mu}(y \odot (z^* \oplus y) \odot x^*) \wedge \overline{\mu}(x).\end{aligned}$$

Hence $\overline{\mu}$ is a fuzzy p-ideal of A. Clearly, $\overline{\mu}$ is normal and $\mu \subseteq \overline{\mu}$. ∎

5. Conclusion

MV-algebras were introduced by C. Chang [1] in 1958 in order to provide an algebraic proof for the completeness theorem of the Łukasiewicz infinite valued propositional logic.

In this paper, we defined the concept of p-ideals and given characterization of them. We proved that p-ideals equivalent to Boolean ideals in MV-algebras.

We introduced the notion of fuzzy p-ideals of MV-algebras and described the transfer principle for fuzzy p-ideals in terms of level subsets.

We have also presented several characterizations and many important properties of fuzzy p-ideals in MV-algebras. Moreover, we obtained the extension theorem of fuzzy p-ideals in MV-algebras. We showed that in any MV-algebra, the concept of fuzzy p-ideals are equivalent to fuzzy Boolean ideals and are equivalent to fuzzy implicative ideals. Finally, by using the notion of maximal and normal fuzzy p-ideals, we showed that under certain conditions a fuzzy p-ideal is two valued and takes the values 0 and 1.

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References

[1] Chang, C.C., *Algebraic analysis of many valued logic*, Trans. Amer. Math. Soc., 88 (1958), 467-490.

[2] Cignoli, R., D'Ottaviano, I.M.L., Mundici, D., *Algebraic Foundations of Many-Valued Reasoning*, Kluwer Academic, Dordrecht, 2000.

[3] Dymek, G., *Fuzzy prime ideals of Pseudo-MV-algebras*, Soft comput, 12 (2008), 365-372.

[4] Hoo, C.S., *Fuzzy ideals of BCI and MV-algebras*, Fuzzy Sets and Systems, 62 (1994), 111-114.

[5] Hoo, C.S., *Fuzzy implicative and Boolean ideals of MV-algebras*, Fuzzy Sets and Systems, 66 (1994), 315-327.

[6] Hoo, C.S., Sessa, S., *Implicative and Boolean ideals of MV-algebras*, Math. Japon., 39 (1994), 215-219.

[7] Iorgulescu, A., *Algebras of logic as BCK algebras*, Academy of Economic Studies Bucharest, Romania, 2008.

[8] Iseki, K., Tanaka, S., *Ideal theory of BCK-algebras*, Math. Japon., 21 (1976), 351-366.

[9] Dudek, K., *On the transfer principle in fuzzy theory*, Mathw. Soft Comput., 12 (2005), 4-55.

[10] Mundici, D., *Interpretation of AFC*-algebras in Łukasiewicz sentential calculus*, J. Funct. Anal., 65 (1986), 15-63.

[11] Piciu, D., *Algebras of fuzzy logic*, Ed. Universitaria Craiova, 2007.

[12] Xi, O., *Fuzzy BCK-algebras*, Math. Japon., 36 (1991), 935-942.

[13] Zadeh, A., *Fuzzy set*, Inform. Control., 8 (year), 338-353.

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MODULES WHOSE PRIMARY-LIKE SPECTRA WITH THE ZARISKI-LIKE TOPOLOGY ARE NOETHERIAN SPACES

Hosein Fazaeli Moghimi[1]
Fatemeh Rashedi

Department of Mathematics
University of Birjand
P.O. Box 97175-615, Birjand
Iran
e-mails: hfazaeli@birjand.ac.ir
fatemehrashedi@birjand.ac.ir

Abstract. Let R be a commutative ring with identity and M be a unital R-module. The primary-like spectrum $Spec_L(M)$ is the collection of all primary-like submodules Q of M such that M/Q is a primeful R-module. The Zariski-like topology on $Spec_L(M)$, denoted $\mathcal{T}$, is described by taking the set $\eta = \{\nu(N) \mid N \text{ is a submodule of } M\}$ as the set of closed sets of $Spec_L(M)$, where $\nu(N) = \{Q \in Spec_L(M) \mid \sqrt{(N : M)} \subseteq \sqrt{(Q : M)}\}$. We establish necessary and sufficient conditions for topological space $(Spec_L(M), \mathcal{T})$ to be a Noetherian space. We show that if M is a finitely generated R-module and $|Spec_L(M)| < \infty$, then the combinatorial dimension of $(Spec_L(M), \mathcal{T})$ and the Krull dimension of $R/Ann(M)$ are equal. In particular, for the Noetherian space $(Spec_L(M), \mathcal{T})$ of zero combinatorial dimension the set of irreducible components is finite, and its elements have the form $\nu(pM)$ for some minimal prime ideal $p \supseteq Ann(M)$.

Keywords: primary-like spectrum, Z-Radical of a submodule, Noetherian spectrum, combinatorial dimension.

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1. Introduction

Throughout this article, R is a commutative ring with $1 \neq 0$ and M is a unitary R-module. For any ideal I of R containing $Ann(M)$, $\overline{I}$ and $\overline{R}$ will denote $I/Ann(M)$ and $R/Ann(M)$, respectively. Also the spectrum $Spec(R)$ of a ring R, consists of all prime ideals of R, will be considered as a topological space in which the closed sets are of the form $V(I) = \{p \in Spec(R) \mid I \subseteq p\}$, where I is an ideal of R (see, for example, [4], [9]).

[1]Corresponding author

Definition 1.1. For M as an R-module and P, Q, N its submodules, we define the following:

(1) The colon ideal of M into N is $(N : M) = \{r \in R \mid rM \subseteq N\} = Ann(M/N)$. Dually, the colon submodule of M into an ideal I of R is $(N : I) = \{m \in M \mid Im \subseteq N\}$. In the case that $I = Rr$, we write $(N : r)$.

(2) P is prime if $P \neq M$, and whenever $rm \in P$, $r \in R$ and $m \in M$, then $m \in P$ or $r \in (P : M)$ [20];

(3) The prime spectrum (or simply, the spectrum) of M, denoted by $Spec(M)$, is the set of all prime submodules of M. Also if p is a prime ideal of R, we let $Spec_p(M) = \{P \in Spec(M) \mid (P : M) = p\}$ [20];

(4) The radical of N, denoted by $\operatorname{rad} N$, is the intersection of all prime submodules of M which contain N, unless no such primes exist, in which case $\operatorname{rad} N = M$ [23]. The radical of an ideal I of R will be denoted by $\sqrt{I}$;

(5) Q is primary-like if $Q \neq M$ and $rm \in Q$ implies that $r \in (Q : M)$ or $m \in \operatorname{rad} Q$ [12];

(6) M is a primeful R-module if either $M = (0)$ or $Spec_p(M) \neq \emptyset$. N satisfies the primeful property if M/N is a primeful R-module. In this case, $\sqrt{(N : M)} = (\operatorname{rad} N : M)$ [18];

(7) If Q is a primary-like submodule satisfying the primeful property, then $(Q : M)$ is a primary ideal [12, Lemma 2.1]. In this case, Q is p-primary-like, where $p = \sqrt{(Q : M)} = (\operatorname{rad} Q : M)$;

(8) The primary-like spectrum of M denoted by $Spec_L(M)$ is the set of all primary-like submodules of M satisfying the primeful property [12]. If p is a prime ideal of R, we set $\mathcal{X}_p = \{Q \in Spec_L(M) \mid \sqrt{(Q : M)} = p\}$;

(9) M is a multiplication module if for every submodule N of M, there exists an ideal I of R such $N = IM$. In this case, we can take $I = (N : M)$ [11].

In the literature, there are many different generalizations of the Zariski topology on the prime spectrum of a ring to modules ([2], [7], [10], [19], [22]. In the following lemma, we introduce one of them.

Lemma 1.2. *Let M be an R-module. Then, for submodules N, N' and $\{N_i \mid i \in I\}$ of M, we have*

(1) $\nu(0) = Spec_L(M)$ *and* $\nu(M) = \emptyset$.

(2) $\bigcap_{i\in I} \nu(N_i) = \nu\left(\sum_{i\in I}(N_i : M)M\right)$.

(3) $\nu(N) \cup \nu(N') = \nu(N \cap N')$.

Proof. (1) and (3) are straightforward.

(2) follows from the following implications:

$$
\begin{aligned}
Q \in \bigcap_{i\in I} \nu(N_i) \ &\Rightarrow\ \sqrt{(Q:M)} \supseteq \sqrt{(N_i:M)}\ \forall i \in I \\
&\Rightarrow\ \sqrt{(Q:M)} \supseteq (N_i:M)\ \ \forall i \in I \\
&\Rightarrow\ \sqrt{(Q:M)} \supseteq \sum_{i\in I}(N_i:M) \\
&\Rightarrow\ \sqrt{(Q:M)}M \supseteq (\sum_{i\in I}(N_i:M))M \\
&\Rightarrow\ (\sqrt{(Q:M)}M:M) \supseteq \left(\left(\sum_{i\in I}(N_i:M)\right)M:M\right) \\
&\Rightarrow\ ((\operatorname{rad} Q:M)M:M) \supseteq \left(\left(\sum_{i\in I}(N_i:M)\right)M:M\right) \\
&\Rightarrow\ (\operatorname{rad} Q:M) \supseteq \left(\left(\sum_{i\in I}(N_i:M)\right)M:M\right) \\
&\Rightarrow\ \sqrt{(Q:M)} \supseteq \sqrt{\left(\left(\sum_{i\in I}(N_i:M)\right)M:M\right)} \\
&\Rightarrow\ Q \in \nu\left(\left(\sum_{i\in I}(N_i:M)\right)M\right).
\end{aligned}
$$

For the reverse inclusion, we have

$$
\begin{aligned}
Q \in \nu\left(\left(\sum_{i\in I}(N_i:M)\right)M\right) \ &\Rightarrow\ \sqrt{(Q:M)} \supseteq \sqrt{\left(\left(\sum_{i\in I}(N_i:M)\right)M:M\right)} \\
&\Rightarrow\ \sqrt{(Q:M)} \supseteq \left(\left(\sum_{i\in I}(N_i:M)\right)M:M\right) \\
&\Rightarrow\ \sqrt{(Q:M)} \supseteq ((N_i:M)M:M)\ \ \forall i \in I \\
&\Rightarrow\ \sqrt{(Q:M)} \supseteq (N_i:M)\ \ \forall i \in I \\
&\Rightarrow\ \sqrt{(Q:M)} \supseteq \sqrt{(N_i:M)}\ \ \forall i \in I \\
&\Rightarrow\ Q \in \bigcap_{i\in I} \nu(N_i).
\end{aligned}
$$

■

Let $\eta(M)$ denotes the collection of all subsets $\nu(N)$ of $Spec_L(M)$. Lemma 1.2 shows that $\eta(M)$ satisfies the axioms for the closed subsets of a topological space on $Spec_L(M)$, called Zariski-like topology and denoted by $\mathcal{T}$.

Definition 1.3. For a topological space X we define the following:

(1) X is Noetherian provided that the open (respectively, closed) subsets of X satisfy the ascending (respectively, descending) chain condition[9, § 4.2];

(2) X is irreducible if the intersection of two non-empty open subsets of X is non-empty [9, §4.1];

(3) We consider strictly decreasing (or strictly increasing) chain $Y_0, Y_1, \cdots, Y_r$ of length r of irreducible closed subsets Y_i of X. The supremum of the lengths, taken over all such chains, is called the combinatorial dimension of X and denoted by $dimX$. For the empty set, the combinatorial dimension of $\emptyset$ is defined to be -1 [17].

Definition 1.4. In this paper we define and use the following notions:

(1) Let $\eta^*(M)$ denotes the collection of all subsets

$$\nu^*(N) = \{Q \in Spec_L(M) \mid \mathrm{rad}\, N \subseteq Q\},$$

of $Spec_L(M)$. It is easily seen that $\eta^*(M)$ contains the empty set and $Spec_L(M)$, and $\eta^*(M)$ is closed under arbitrary intersections. We shall say that M is a top-like module, if $\eta^*(M)$ is closed under finite unions, for in this case $\eta^*(M)$ induces a topology $\mathcal{T}^*$ on $Spec_L(M)$.

(2) Z-radical (resp. Z*-radical) of a submodule N of M, denoted by $\sqrt[Z]{N}$ (resp. $\sqrt[Z^*]{N}$), to be the intersection of all members of $\nu(N)$ (resp. $\nu^*(N)$);

(3) A submodule N of M is a Z-radical (resp. Z*-radical) submodule if $\sqrt[Z]{N} = N$ (resp. $\sqrt[Z^*]{N} = N$);

(4) If $Spec_L(M) \neq \emptyset$, the mapping $\phi : Spec_L(M) \to Spec(\overline{R})$ such that $\phi(Q) = \overline{\sqrt{(Q : M)}}$ for every $Q \in Spec_L(M)$, is called the natural map of $Spec_L(M)$

(5) A finitely generated module M is quasi-Laskerian if every submodule of M is the intersection of a finite number of primary-like submodules. A ring R is a Laskerian ring if R is a quasi-Laskerian R-module [14];

(6) An R-module M is a ZFG-module if for every submodule N of M we have $\sqrt[Z]{N} = \sqrt[Z]{IM}$ for some finitely generated ideal I of R;

(7) An R-module M is a FIC-module if every closed subset of $Spec_L(M)$ relative to the Zariski-like topology $\mathcal{T}$ has a finite number of irreducible components.

In recent years, the study of modules whose spectra have a Zariski-topology has grown in various directions. From an algebraic view, the varieties of submodules (closed sets related to a Zariki topology) forms a semimodule which also is called a Zariski-space (see for example [13], [21], [24]). Some of authors have investigated the interplay between algebraic properties of a module one hand and the topological properties of its spectrum on the other hand (see for example [1], [2], [3], [7], [8], [10], [17], [19], [22], [25]). In the present work, we study modules whose primary-like spectrums equipped with the Zariski-like topology are Noetherian spaces. For this purpose we need to obtain results about Z-Radical and Z*-Radical of submodules that are essential for the later sections of this article. Let M be an R-module. We see that if $(Spec_L(M), \mathcal{T})$ is a Noetherian space, then every Z-radical submodule of M satisfies ACC (Theorem 3.2). Moreover, if M is a top-like R-module and $(Spec_L(M), \mathcal{T}^*)$ is a Noetherian space, then every Z*-radical submodule of M satisfies ACC (Theorem 3.2). We also show that if M is a multiplication R-module and the natural map ϕ is surjective, then M is a ZFG-module if and only if $(Spec_L(M), \mathcal{T})$ is a Noetherian space (Theorem 3.14). Finally, it is proved that if M is a finitely generated R-module with $|Spec_L(M)| < \infty$, then the combinatorial dimension of $Spec_L(M)$ and the Krull dimension of $\overline{R}$ are equal (Theorem 4.6).

2. Z-radical and Z*-radical of submodules

We start this section with some elementary facts about $\nu(N)$ and $\nu^*(N)$.

Lemma 2.1. *Let I be an ideal of R. Let N, N' and $\{N_j \mid j \in J\}$ be submodules of an R-module M. Then the following hold.*

(1) *If $N \subseteq N'$, then $\nu(N') \subseteq \nu(N)$ and $\nu^*(N') \subseteq \nu^*(N)$.*

(2) $\nu^*(N) \subseteq \nu(N)$.

(3) $\nu(IM) = \nu(\sqrt{I}M) = \nu^*(IM) = \nu^*(\sqrt{I}M)$.

(4) $\nu(N) = \nu((N : M)M) = \nu(\sqrt{(N : M)}M) = \nu^*((N : M)M)$
$= \nu^*(\sqrt{(N : M)}M)$.

(5) *If $\sqrt{(N : M)} = \sqrt{(N' : M)}$, then $\nu(N) = \nu(N')$. The converse is also true if both N and N' are primary-like.*

(6) $\nu^*(N) = \nu^*(\operatorname{rad} N)$.

Proof. (1) Clear.

(2) Assume $Q \in \nu^*(N)$. Hence $N \subseteq \operatorname{rad} Q$. Thus $\sqrt{(N : M)} \subseteq \sqrt{(Q : M)}$, i.e., $Q \in \nu(N)$.

(3) We have $\nu^*(\sqrt{I}M) \subseteq \nu^*(IM) \subseteq \nu(IM)$, by (1) and (2). Now we show that $\nu(IM) \subseteq \nu^*(\sqrt{I}M)$. Suppose $Q \in \nu(IM)$. Thus $(\operatorname{rad} Q : M) \supseteq \sqrt{(IM : M)} \supseteq \sqrt{I}$. It follows that $\operatorname{rad} Q \supseteq \sqrt{I}M$, i.e., $Q \in \nu^*(\sqrt{I}M)$ and so $\nu(IM) \subseteq \nu^*(\sqrt{I}M)$. Hence we have $\nu^*(\sqrt{I}M) \subseteq \nu^*(IM) \subseteq \nu(IM) \subseteq \nu^*(\sqrt{I}M)$. Thus the assertion holds when I is replaced by $\sqrt{I}$.

(4) It suffices to show that $\nu(N) = \nu((N : M)M)$ by (3). $Q \in \nu(N)$ if and only if

$\sqrt{(Q:M)} \supseteq \sqrt{(N:M)} = \sqrt{((N:M)M:M)}$ if and only if $Q \in \nu((N:M)M)$. Thus $\nu(N) = \nu((N:M)M)$.

(5) is clear.

(6) $\nu^*(\operatorname{rad} N) \subseteq \nu^*(N)$ by (1). Suppose $Q \in \nu^*(N)$. Hence $N \subseteq \operatorname{rad} Q$. Thus $\operatorname{rad} N \subseteq \operatorname{rad} Q$ and so $Q \in \nu^*(\operatorname{rad} N)$. ■

Proposition 2.2. *Let I be an ideal of R and M an R-module. If N, N' are submodules of M, then the following hold.*

(1) $\sqrt[z]{N} \subseteq \sqrt[z^*]{N}$.

(2) $\sqrt[z]{IM} = \sqrt[z]{\sqrt{I}M} = \sqrt[z^*]{IM} = \sqrt[z^*]{\sqrt{I}M}$.

(3) $\sqrt[z]{N} = \sqrt[z]{(N:M)M} = \sqrt[z]{\sqrt{(N:M)}M} = \sqrt[z^*]{(N:M)M)}$
$= \sqrt[z^*]{\sqrt{(N:M)}M}$.

Proof. (1) By Lemma 2.1(2), $\nu^*(N) \subseteq \nu(N)$. Thus $\sqrt[z]{N} \subseteq \sqrt[z^*]{N}$.

(2) Use Lemma 2.1(3).

(3) It is clear by Lemma 2.1(4). ■

From now on, we use $\mathcal{X}$ to denote $Spec_L(M)$ for short. Let $\mathcal{Y}$ be a subset of $\mathcal{X}$. We will denote the closure of $\mathcal{Y}$ in $\mathcal{X}$ by $\overline{\mathcal{Y}}$ and the intersection of all elements in $\mathcal{Y}$ by $\gamma(\mathcal{Y})$ (note that if $\mathcal{Y} = \emptyset$, then $\gamma(\mathcal{Y}) = M$). It is easy to verify that, if $\mathcal{Y}_1, \mathcal{Y}_2 \subseteq \mathcal{X}$, then $\gamma(\mathcal{Y}_1 \cup \mathcal{Y}_2) = \gamma(\mathcal{Y}_1) \cap \gamma(\mathcal{Y}_2)$.

Lemma 2.3. *Let M be an R-module, $|\mathcal{X}| < \infty$ and $\mathcal{Y} \subseteq \mathcal{X}$. Then $\nu(\gamma(\mathcal{Y})) = \overline{\mathcal{Y}}$. In particular, $\mathcal{Y}$ is closed if and only if $\nu(\gamma(\mathcal{Y})) = \mathcal{Y}$.*

Proof. Suppose $Q \in \mathcal{Y}$. Hence $\gamma(\mathcal{Y}) \subseteq Q$. Therefore $\sqrt{(Q:M)} \supseteq \sqrt{(\gamma(\mathcal{Y}):M)}$. Thus $Q \in \nu(\gamma(\mathcal{Y}))$ and so $\mathcal{Y} \subseteq \nu(\gamma(\mathcal{Y}))$. Next, let $\nu(N)$ be any closed subset of $\mathcal{X}$ containing $\mathcal{Y}$. Then $\sqrt{(Q:M)} \supseteq \sqrt{(N:M)}$ for every $Q \in \mathcal{Y}$ so that $\sqrt{(\gamma(\mathcal{Y}):M)} \supseteq \sqrt{(N:M)}$. Hence, for every $Q' \in \nu(\gamma(\mathcal{Y}))$; $\sqrt{(Q':M)} \supseteq \sqrt{(\gamma(\mathcal{Y}):M)} \supseteq \sqrt{(N:M)}$. Then $\nu(\gamma(\mathcal{Y})) \subseteq \nu(N)$. Thus $\nu(\gamma(\mathcal{Y}))$ is the smallest closed subset of $\mathcal{X}$ containing $\mathcal{Y}$ and so $\nu(\gamma(\mathcal{Y})) = \overline{\mathcal{Y}}$. ■

Lemma 2.4. *Let M be an R-module and N be a submodule of M. If $|\mathcal{X}| < \infty$, then $\nu(\gamma(\nu(N))) = \nu(\sqrt[z]{N}) = \nu(N)$.*

Proof. It is clear by Lemma 2.3. ■

In the following proposition, we list some more properties of $\sqrt[z]{N}$ and $\sqrt[z^*]{N}$ for a submodule N of M.

Proposition 2.5. *Let N, N' be submodules of an R-module M. Then the following hold.*

(1) *If* $\nu(N) \subseteq \nu(N')$, *then* $\sqrt[Z]{N'} \subseteq \sqrt[Z]{N}$. *The converse is true if* $N' \subseteq \sqrt[Z]{N'}$.

(2) *If* $Q \in \nu(N)$, *then* $\sqrt[Z]{N} \subseteq Q$.

(3) *If* $|\mathcal{X}| < \infty$, *then* $\sqrt[Z]{\sqrt[Z]{N}} = \sqrt[Z]{N}$.

(4) $\sqrt[Z]{N \cap N'} = \sqrt[Z]{N} \cap \sqrt[Z]{N'}$.

(5) *If* $\nu^*(N) \subseteq \nu^*(N')$, *then* $\sqrt[Z^*]{N'} \subseteq \sqrt[Z^*]{N}$. *The converse is true if* $N' \subseteq \sqrt[Z^*]{N'}$.

(6) *If* $Q \in \nu^*(N)$, *then* $\sqrt[Z^*]{N} \subseteq Q$.

(7) $\sqrt[Z^*]{N} = \sqrt[Z^*]{\operatorname{rad} N}$.

Proof. (1) Suppose $\nu(N) \subseteq \nu(N')$. Hence $\gamma(\nu(N')) \subseteq \gamma(\nu(N))$ and so $\sqrt[Z]{N'} \subseteq \sqrt[Z]{N}$. Conversely, assume that $Q \in \nu(N)$. Since $N' \subseteq \sqrt[Z]{N'} \subseteq \sqrt[Z]{N}$, then $Q \in \nu(N')$.

(2) Suppose $Q \in \nu(N)$. Hence $\sqrt{(N : M)} \subseteq \sqrt{(Q : M)}$. Thus $\sqrt[Z]{N} \subseteq Q$.

(3) $\nu(\sqrt[Z]{N}) = \nu(N)$, by Lemma 2.4. Therefore $\gamma(\nu(\sqrt[Z]{N})) = \gamma(\nu(N))$. Thus $\sqrt[Z]{\sqrt[Z]{N}} = \sqrt[Z]{N}$.

(4) $\sqrt[Z]{N \cap N'} = \gamma(\nu(N \cap N')) = \gamma(\nu(N) \cup \nu(N')) = \gamma(\nu(N)) \cap \gamma(\nu(N')) = \sqrt[Z]{N} \cap \sqrt[Z]{N'}$, by Lemma 1.2(3).

(5) Suppose $\nu^*(N') \subseteq \nu^*(N)$. Hence $\gamma(\nu^*(N)) \subseteq \gamma(\nu^*(N'))$ and so $\sqrt[Z^*]{N} \subseteq \sqrt[Z^*]{N'}$. Conversely, assume that $Q \in \nu^*(N)$. Since $N' \subseteq \sqrt[Z^*]{N'} \subseteq \sqrt[Z^*]{N}$, then $Q \in \nu^*(N')$.

(6) Assume $Q \in \nu^*(N)$, i.e., $N \subseteq \operatorname{rad} Q$. Thus $\sqrt[Z^*]{N} \subseteq Q$.

(7) By Lemma 2.1(6) $\nu^*(N) = \nu^*(\operatorname{rad} N)$. Hence $\gamma(\nu^*(N)) = \gamma(\nu^*(\operatorname{rad} N))$. Thus $\sqrt[Z^*]{N} = \sqrt[Z^*]{\operatorname{rad} N}$. ■

Proposition 2.6. *Let* N *be a submodule of an* R*-module* M. *If* $N \subseteq \sqrt[Z]{N}$, *then* $\nu(N) = \nu^*(N)$. *In particular,* $\sqrt[Z]{N} = \sqrt[Z^*]{N}$.

Proof. $\nu^*(N) \subseteq \nu(N)$ by Lemma 2.1(2). Suppose $Q \in \nu(N)$. Hence $\sqrt[Z]{N} \subseteq Q$ by Proposition 2.5(2). Thus $N \subseteq Q \subseteq \operatorname{rad} Q$ and so $Q \in \nu^*(N)$. ■

Let N be a submodule of M. Unlike the prime radical case the following example shows that $N \nsubseteq \sqrt[Z]{N}$ may be occurred in general.

Example 2.7. Let V be a vector space over a field F. Then $Spec_L(V) = Spec(V)$ = the set of all proper vector subspaces of V. Suppose W is a non-zero subspace of V. Hence $\sqrt[Z]{W} = 0$. Thus $W \nsubseteq \sqrt[Z]{W}$.

Proposition 2.8. *Let* M *be a finitely generated* R*-module. Then the following hold.*

(1) $\sqrt[Z]{N} \neq M$ *if and only if* $\nu(N) \neq \emptyset$ *if and only if* $N \neq M$.

(2) $\sqrt[Z^*]{N} \neq M$ *if and only if* $\nu^*(N) \neq \emptyset$ *if and only if* $N \neq M$.

Proof. (1) Assume $N \neq M$. Then $(N : M) \neq R$ and so $(N : M) \subseteq p$ for some prime ideal p of R. Since M is finitely generated, M is primeful by [18, Proposition 3.8]. Hence there exists $Q \in Spec(M) \subseteq \mathcal{X}$ such that $\sqrt{(N : M)} \subseteq \sqrt{(Q : M)}$. Thus $Q \in \nu(N)$ and so $\nu(N) \neq \emptyset$. Now suppose $\nu(N) \neq \emptyset$ and $Q \in \nu(N)$. Hence $\sqrt[Z]{N} \subseteq Q \neq M$ by Proposition 2.5(2). If $\sqrt[Z]{N} \neq M$, then $N \neq M$.

(2) Suppose $\sqrt[Z^*]{N} \neq M$. Hence $N \neq M$. Now assume $N \neq M$. Then $(N : M) \neq R$ and so $(N : M) \subseteq p$ for some prime ideal p of R. Since M is primeful, there exists $Q \in Spec(M) \subseteq \mathcal{X}$ such that $N \subseteq \operatorname{rad} Q$. Hence $Q \in \nu^*(N)$. Thus $\nu^*(N) \neq \emptyset$. If $\nu^*(N) \neq \emptyset$ and $Q \in \nu^*(N)$. Hence $\sqrt[Z^*]{N} \subseteq Q \neq M$ by Proposition 2.5(6). ∎

Proposition 2.9. *Let M be a multiplication R-module. Then the following hold.*

(1) $\sqrt[Z]{N} = \sqrt[Z^*]{N}$.

(2) *If $|\mathcal{X}| < \infty$, then* $\sqrt[Z]{\sqrt[Z^*]{N}} = \sqrt[Z^*]{\sqrt[Z]{N}} = \sqrt[Z^*]{N} = \sqrt[Z]{N}$.

Proof. (1) Since M is multiplication, $\nu(N) = \nu^*(N)$ and so $\sqrt[Z]{N} = \sqrt[Z^*]{N}$.
(2) It is clear by (1) and Proposition 2.5(3). ∎

Proposition 2.10. *Let M be an R-module and $Q \in \mathcal{X}_p$ for some prime ideal p of R. Then $\sqrt[Z^*]{Q} = \sqrt[Z^*]{Q + pM}$. Furthermore, if M is a multiplication module, then $\sqrt[Z]{Q} = \sqrt[Z]{Q + pM}$.*

Proof. Since $Q \subseteq Q + pM$, then $\sqrt[Z^*]{Q} \subseteq \sqrt[Z^*]{Q + pM}$. Now, assume $Q_i \in \mathcal{X}_{p_i}$ such that $Q \subseteq \operatorname{rad} Q_i$, $(i \in I)$. Hence $\sqrt{(Q : M)} \subseteq (\operatorname{rad} Q_i : M)$. Therefore $pM \subseteq p_i M$. So $Q + pM \subseteq \operatorname{rad} Q_i + p_i M \subseteq \operatorname{rad} Q_i$. Thus $\sqrt[Z^*]{Q + pM} \subseteq \sqrt[Z^*]{Q}$. Suppose M is a multiplication module. Thus $\sqrt[Z]{Q} = \sqrt[Z]{Q + pM}$ by (1) and Proposition 2.9. ∎

3. Noetherian primary-like spectrum

Recall that a topological space X is a Noetherian space provided that the open (resp. closed) subsets of X satisfy the ascending (resp. descending) chain condition.

Theorem 3.1. *Let M be an R-module and $(\mathcal{X}, \mathcal{T})$ be a Noetherian space. Then every Z-radical submodule of M satisfies ACC.*

Proof. Suppose $(\mathcal{X}, \mathcal{T})$ is a Noetherian space. Let $N_1 \subseteq N_2 \subseteq \cdots$ be an ascending chain of Z-radical submodules of M. Then $\nu(N_1) \supseteq \nu(N_2) \supseteq \cdots$ is a descending chain of closed sets $\nu(N_i)$ of $\mathcal{X}$. Hence there exists a positive integer k such that $\nu(N_m) = \nu(N_k)$ for every $m \geq k$. Thus $\sqrt[Z]{N_m} = \sqrt[Z]{N_k}$ and so $N_m = N_k$ for every $m \geq k$. ∎

Theorem 3.2. *Let M be a top-like R-module. If $(\mathcal{X}, \mathcal{T}^*)$ is a Noetherian space, then every Z^*-radical submodule of M satisfies ACC.*

Proof. The proof is similar to that of Theorem 3.1. ∎

Proposition 3.3. *Let M be a multiplication R-module. Then $(\mathcal{X}, \mathcal{T})$ is a Noetherian space if and only if $(\mathcal{X}, \mathcal{T}^*)$ is a Noetherian space.*

Proof. It follows from the fact that $\nu(N)=\nu^*(N)$ for every submodule N of M. ∎

Recall that if $\mathcal{X} \neq \emptyset$, the mapping $\phi : \mathcal{X} \to Spec(\overline{R})$ such that $\phi(Q) = \overline{\sqrt{(Q : M)}}$ for every $Q \in \mathcal{X}$ is called the natural map of $\mathcal{X}$.

Proposition 3.4. *Let M be an R-module. Then the following hold.*

(1) $\phi^{-1}(V^{\overline{R}}(\overline{I})) = \nu(IM)$*, for every ideal $I \in V(Ann(M))$. Therefore the map ϕ is continuous for the Zariski-like topology on $\mathcal{X}$.*

(2) *If the map ϕ is surjective, then $\phi(\nu(N)) = V^{\overline{R}}(\overline{(N : M)})$ and $\phi(\mathcal{X} - \nu(N)) = Spec(\overline{R}) - V^{\overline{R}}(\overline{(N : M)})$ for every submodule N of M, i.e. the map ϕ is both closed and open.*

Proof. (1) Obvious.

(2) As we have seen in (1), ϕ is a continuous map such that $\phi^{-1}(V^{\overline{R}}(\overline{I})) = \nu(IM)$ for every ideal I of R containing $Ann(M)$. Hence, for every submodule N of M we have $\phi^{-1}(V^{\overline{R}}(\overline{(N : M)})) = \nu((N : M)M) = \nu(N)$. It follows that $\phi(\nu(N)) = \phi \circ \phi^{-1}(V^{\overline{R}}(\overline{(N : M)})) = V^{\overline{R}}(\overline{(N : M)})$ as ϕ is surjective. Similarly, we have $\phi(\mathcal{X} - \nu(N)) = \phi(\phi^{-1}(Spec(\overline{R}) - \phi^{-1}(V^{\overline{R}}(\overline{(N : M)}))) = Spec(\overline{R}) - V^{\overline{R}}(\overline{(N : M)})$. ∎

Theorem 3.5. *Let M be a finitely generated R-module. Then $(\mathcal{X}, \mathcal{T})$ is a Noetherian space if and only if $Spec(\overline{R})$ is a Noetherian space equipped with the Zariski topology.*

Proof. Suppose $(\mathcal{X}, \mathcal{T})$ is a Noetherian space. Assume $V(\overline{I}_1) \supseteq V(\overline{I}_2) \supseteq \cdots$ is a descending chain of closed sets in $Spec(\overline{R})$. Hence by Proposition 3.4(1), $\phi^{-1}(V(\overline{I}_1)) \supseteq \phi^{-1}(V(\overline{I}_2)) \supseteq \cdots$ is a descending chain of closed sets in $\mathcal{X}$. By hypothesis, there exists an i such that $\phi^{-1}(V(\overline{I}_i)) = \phi^{-1}(V(\overline{I}_{i+1}))$. Thus $V(\overline{I}_i) = V(\overline{I}_{i+1})$ because ϕ is surjective. Therefore $Spec(\overline{R})$ is a Noetherian space. Conversely, Assume $\nu(N_1) \supseteq \nu(N_2) \supseteq \cdots$ is a descending chain of closed sets in $\mathcal{X}$. Therefore $\phi(\nu(N_1)) \supseteq \phi(\nu(N_2)) \supseteq \cdots$ is a descending chain of closed sets in $Spec(\overline{R})$. Hence there exists an i such that $\phi(\nu(N_i)) = \phi(\nu(N_{i+1}))$. It implies that $V(\overline{(N_i : M)}) = V(\overline{(N_{i+1} : M)})$ by Proposition 3.4(2). Therefore we have $V(\sqrt{(N_i : M)}) = V(\sqrt{(N_{i+1} : M)})$ and so $\nu(N_i) = \nu(N_{i+1})$. ∎

Lemma 3.6. *Let M be an R-module. If Q is a primary-like submodule of M and N is a submodule of M such that* $\operatorname{rad} Q \cap N = \operatorname{rad}(Q \cap N)$, *then $N \subseteq Q$ or $Q \cap N$ is a primary-like submodule of N.*

Proof. Suppose $N \nsubseteq Q$ and for $n \in N$, $rn \in Q \cap N$ such that $r \notin (Q \cap N : N)$. It implies that $rn \in Q$ and $r \notin (Q : M)$. Since Q is a primary-like submodule of M, we have $n \in \operatorname{rad} Q \cap N$, and so by our assumption $n \in \operatorname{rad}(Q \cap N)$. Thus $Q \cap N$ is a primary-like submodule of N. ∎

Let M be a finitely generated R-module. We recall that M is a quasi-Laskerian if every submodule of M is the intersection of a finite number of primary-like submodules.

Proposition 3.7. *Let M be a quasi-Laskerian R-module. If N is a finitely generated submodule of M such that for every primary-like submodule Q of M we have* $\operatorname{rad} Q \cap N = \operatorname{rad}(Q \cap N)$*, then N and $\frac{M}{N}$ are quasi-Laskerian.*

Proof. Suppose that N' is a submodule of N. Since M is quasi-Laskerian, there exist primary-like submodules Q_i $(1 \leq i \leq n)$ of M such that $N' = \bigcap_{i=1}^{n} Q_i$. Hence $Q_i \cap N$ is a primary-like submodule of N by Lemma 3.6. Thus N is quasi-Laskerian. The scend part follows from [12, Corollary 3.5]. ∎

Proposition 3.8. *Let M be a quasi-Laskerian R-module. Then $Spec(\overline{R})$ is a Noetherian space equipped with the Zariski topology.*

Proof. Since M is finitely generated, $\overline{R} \cong M$. Hence $\overline{R}$ is a Laskerian ring. Thus $Spec(\overline{R})$ is a Noetherian space by [14, Theorem 4]. ∎

Corollary 3.9. *Let M be a quasi-Laskerian R-module. Then $(\mathcal{X}, \mathcal{T})$ is a Noetherian space.*

Proof. By Proposition 3.8, $Spec(\overline{R})$ is a Noetherian space. Thus $(\mathcal{X}, \mathcal{T})$ is a Noetherian space by Theorem 3.5. ∎

Theorem 3.10. *Let M be a finitely generated R-module. Then M is quasi-Laskerian if and only if*

(1) *$Spec(\overline{R})$ is a Noetherian space equipped with the Zariski topology.*

(2) *For every proper submodule N of M, there is a minimal prime ideal p of $\sqrt{(N : M)}$ and an element $r \in R \backslash p$ for which the submodule $(N : r)$ is p-primary-like.*

Proof. Assume M is quasi-Laskerian. Therefore $Spec(\overline{R})$ is a Noetherian space by Proposition 3.8. Suppose N is a proper submodule of M. Hence there exist primary-like submodules $Q_1, \cdots, Q_n$ such that $N = Q_1 \cap \cdots \cap Q_n$. Let p be minimal prime of $\sqrt{(N : M)}$. Then $p = \sqrt{(Q_i : M)}$ for one of the primary-like submodules. Assume $p = \sqrt{(Q_1 : M)}$. Let r be an element in $(\sqrt{(Q_2 : M)} \cap \cdots \cap \sqrt{(Q_n : M)}) \backslash p$; replace r, if necessary, by a power so that $r \in (Q_2 \cap \cdots \cap Q_n) : M)$. Thus $(N : r) = Q_1$. Conversely, suppose M satisfies (1) and (2) and N is a submodule of M. Assume p is a minimal prime of $\sqrt{(N : M)}$ and $r \in R \backslash p$ so that $(N : r) = Q_1$ is p-primary-like. Then $N = Q_1 \cap N_1$, where $N_1 = N + rM$. Since $N \subseteq N_1$, then $\sqrt{(N : M)} \subseteq \sqrt{(N_1 : M)}$. Applying the process repeatedly yields two sequences of submodules $Q_1, \cdots, Q_k$ and $N_1, \cdots, N_k$, where $N_{j-1} = Q_j \cap N_j$ and Q_j is primary-like for $1 \leq j \leq k$. Also $\overline{\sqrt{(N_1 : M)}} \subseteq \cdots \subseteq \overline{\sqrt{(N_k : M)}}$ is a chain of prime ideals of $\overline{R}$. Since $Spec(\overline{R})$ is a Noetherian space, the sequence of radicals terminates; but it can terminate at $\overline{\sqrt{(N_k : M)}}$ only if $N_k = M$. Thus $N = Q_1 \cap \cdots \cap Q_k$ and so M is quasi-Laskerian. ∎

Proposition 3.11. *Let M be an R-module. Then the set*

$$\mathcal{B} = \{\mathcal{X}_r = \mathcal{X} - \nu(rM) \mid r \in R\}$$

forms a base for the topology $\mathcal{T}$ on $\mathcal{X}$.

Proof. If $\mathcal{X} = \emptyset$, then $\mathcal{B} = \emptyset$ and the proposition is trivially true. Hence we assume that $\mathcal{X} \neq \emptyset$ and let $\mathcal{U}$ be any open set in $\mathcal{X}$. Hence $\mathcal{U} = \mathcal{X} - \nu(IM)$ for some ideal I of R. Note that

$$\nu(IM) = \nu\left(\sum_{a_i \in I} a_i M\right) = \nu\left(\sum_{a_i \in I} (a_i M : M)M\right) = \bigcap_{a_i \in I} \nu(a_i M)$$

by Lemma 1.2. Hence $\mathcal{U} = \mathcal{X} - \bigcap_{a_i \in I} \nu(a_i M) = \bigcup_{a_i \in I} \mathcal{X}_{a_i}$. This proves that $\mathcal{B}$ is a base for the topology $\mathcal{T}$ on $\mathcal{X}$. ∎

Proposition 3.12. *Let M be an R-module and the natural map ϕ be surjective. Then $\mathcal{X}_r$ is a quasi-compact subset of $\mathcal{X}$.*

Proof. For any open covering of $\mathcal{X}_r$, there is a family $\{r_\lambda \in R : \lambda \in \Lambda\}$ of elements of R such that $\mathcal{X}_r \subseteq \bigcup_{\lambda \in \Lambda} \mathcal{X}_{r_\lambda}$ by Proposition 3.11. Since the map ϕ is surjective,

$$D_{\overline{r}} = \phi(\mathcal{X}_r) \subseteq \bigcup_{\lambda \in \Lambda} \phi(\mathcal{X}_{r_\lambda}) \subseteq \bigcup_{\lambda \in \Lambda} D_{\overline{r}_\lambda}.$$

It follows that there exists a finite subset Λ' of Λ such that $D_{\overline{r}} \subseteq \bigcup_{\lambda \in \Lambda'} D_{\overline{r}_\lambda}$ as $D_{\overline{r}}$ is quasi-compact and so

$$\mathcal{X}_r = \phi^{-1}(D_{\overline{r}}) \subseteq \bigcup_{\lambda \in \Lambda'} \mathcal{X}_{r_\lambda}.$$

Thus $\mathcal{X}_r$ is quasi-compact. ∎

As noted earlier, an R-module M is a ZFG-module if for every submodule N of M we have $\sqrt[Z]{N} = \sqrt[Z]{IM}$ for some finitely generated ideal I of R. It is easy to see that every multiplication module over a Noetherian ring is a ZFG-module.

Proposition 3.13. *Let N be a submodule of an R-module M and $\sqrt{(N : M)} = \sqrt{I}$ for some finitely generated ideal I of R. Then N is a ZFG-submodule of M.*

Proof. Suppose $\sqrt{(N : M)} = \sqrt{I}$ for some finitely generated ideal I of R. Hence by Proposition 2.2(2,3) we have

$$\sqrt[Z]{N} = \sqrt[Z]{\sqrt{(N : M)}M} = \sqrt[Z]{\sqrt{I}M} = \sqrt[Z]{IM}.$$

Thus N is a ZFG-submodule of M. ∎

Theorem 3.14. *Let M be a multiplication R-module and the natural map ϕ be surjective. Then M is a ZFG-module if and only if $(\mathcal{X}, \mathcal{T})$ is a Noetherian space.*

Proof. Assume that N is a submodule of M. Therefore $\sqrt[Z]{N} = \sqrt[Z]{IM}$ for some finitely generated ideal $I = \sum_{i=1}^{n} Rr_i$ of R if and only if $\nu(N) = \nu(\sum_{i=1}^{n} r_iM) = \nu(\sum_{i=1}^{n}(r_iM : M)M) = \bigcap_{i=1}^{n} \nu(r_iM)$, by Lemma 1.2(2) and Lemma 2.4 if and only if $\mathcal{U} = \mathcal{X} - \nu(N) = \mathcal{X} - \bigcap_{i=1}^{n} \nu(r_iM) = \bigcup_{i=1}^{n}(\mathcal{X} - \nu(r_iM))$ if and only if $\mathcal{U}$ is quasi-compact by proposition 3.12 if and only if $(\mathcal{X}, \mathcal{T})$ is a Noetherian space by [9, P. 123, Proposition 9]. ∎

Corollary 3.15. *Let R be a ring. Then R-module R is a ZFG-module if and only if $Spec_L(R)$ is a Noetherian space.*

Proof. It is clear by Theorem 3.14. ∎

4. Noethrian spectrum, the number of irreducible components and the length of their chains

Recall that a topological space X is irreducible if the intersection of two non-empty open subsets of X is non-empty. Every subset of a topological space consisting of a single point is irreducible and a subset Y of a topological space X is irreducible if and only if its closure is irreducible [9, §4.1]. A maximal irreducible subset Y of X is called an irreducible component of X and it is always closed.

Proposition 4.1. *Let M be an R-module and the natural map ϕ be surjective. If $|\mathcal{X}| < \infty$ and $\mathcal{Y} \subseteq \mathcal{X}$, then $\mathcal{Y}$ is an irreducible closed subset of $\mathcal{X}$ if and only if $\mathcal{Y} = \nu(Q)$ for some $Q \in \mathcal{X}$.*

Proof. Suppose $\mathcal{Y} = \nu(Q)$. Since $\{Q\}$ is an irreducible subset of $\mathcal{X}$, by [9, P. 13, Exercise 20] $\overline{\{Q\}}$ is an irreducible subset of $\mathcal{X}$. Thus $\mathcal{Y} = \nu(Q) = \overline{\{Q\}}$ is an irreducible closed subset of $\mathcal{X}$. Conversely, if $\mathcal{Y}$ is an irreducible closed subset of $\mathcal{X}$, then $\mathcal{Y} = \nu(N)$ for some submodule N of M such that $\sqrt{(\gamma(\nu(N)) : M)} = \sqrt{(\gamma(\mathcal{Y}) : M)} = p$ is a prime ideal of R. Since ϕ is surjective, there exists a p-primary-like submodule $Q \in \mathcal{X}$ such that $\sqrt{(Q : M)} = p$. It follows that $p = \sqrt{(\gamma(\nu(N)) : M)} = \sqrt{(Q : M)}$. Hence $\nu(\gamma(\nu(N))) = \nu(Q)$ by Lemma 2.1(5). Thus $\mathcal{Y} = \nu(Q)$ by Lemma 2.3. Therefore $\nu(Q)$ is an irreducible subset of $\mathcal{X}$. ∎

Proposition 4.2. *Let M be an R module, $|\mathcal{X}| < \infty$ and the natural map ϕ be surjective. Then the correspondence $\nu(Q) \longmapsto \overline{\sqrt{(Q : M)}}$ is a bijection of the set of irreducible components of $\mathcal{X}$ and the set of minimal prime ideals of $\overline{R}$.*

Proof. It is easy to see that the correspondence is well-defined and an injection by Lemma 2.1(5). Suppose $\overline{p} \in Spec(\overline{R})$. Since ϕ is surjective, $\overline{p} = \overline{\sqrt{(Q : M)}}$ for some $Q \in \mathcal{X}$. Thus the correspondence is a surjection. Since each irreducible component of $\mathcal{X}$ is a maximal element of the set $\{\nu(Q) : Q \in \mathcal{X}\}$ by Proposition 4.1, the assertion hold. ∎

We recall that an R-module M is said to be a FIC-module if every closed subset of $\mathcal{X}$ has a finite number of irreducible components. A ring R is said to be a FIC-ring if and only if R-module R is FIC.

Theorem 4.3. *Let M be an R-module. Then the following hold.*

(1) *If M is finitely generated and $|\mathcal{X}| < \infty$, then M is FIC if and only if for every submodule N of M the ideal $\sqrt{(N : M)}$ is contained in a finite number minimal prime ideal of R.*

(2) *If $(\mathcal{X}, \mathcal{T})$ is a Noetherian space, then M is FIC.*

Proof. (1) is a direct result of Proposition 4.2.

(2) Since $(\mathcal{X}, \mathcal{T})$ is a Noetherian space, every closed subset of $\mathcal{X}$ is Noetherian by [9, P. 123, Proposition 8(i)]. Hence every closed subset of $\mathcal{X}$ has a finite number of irreducible components, by [9, P. 124, Proposition 10]. Thus M is FIC. ■

Corollary 4.4. *The following are true.*

(1) *If $|Spec_L(R)| < \infty$, then R is FIC if and only if every ideal I of R, $\sqrt{I}$ is contained in a finite number minimal prime ideal.*

(2) *If $Spec_L(R)$ is a Noetherian space, then R is FIC.*

Proof. By Theorem 4.3 is clear. ■

As it was mentioned, if X is a topological space, we consider strictly decreasing (or strictly increasing) chain $Y_0, Y_1, ..., Y_r$ of length r of irreducible closed subsets Y_i of X. The supremum of the lengths, taken over all such chains, is called the combinatorial dimension of X and denoted by $dimX$. For the empty set, the combinatorial dimension of $\emptyset$ is defined to be -1.

Proposition 4.5. *Let M be a finitely generated R-module and $|\mathcal{X}| < \infty$. Then $\mathcal{X}$ has a chain of irreducible closed subsets of $\mathcal{X}$ of length r if and only if $\overline{R}$ has a chain of prime ideals of length r.*

Proof. Assume that $\mathcal{Y}_0 \subset \mathcal{Y}_1 \subset \cdots \subset \mathcal{Y}_r$ is a strictly increasing chain of irreducible closed subsets $\mathcal{Y}_i$ of $\mathcal{X}$ of length r. By Proposition 4.1, $\mathcal{Y}_i = \nu(Q_i)$ for some $Q_i \in \mathcal{X}$. Hence $\nu(Q_0) \subset \nu(Q_1) \subset \cdots \subset \nu(Q_r)$. Thus $\overline{\sqrt{(Q_0 : M)}} \supset \overline{\sqrt{(Q_1 : M)}} \supset \cdots \supset \overline{\sqrt{(Q_r : M)}}$ is a strictly decreasing chain of prime ideals of $\overline{R}$ of length r. Conversely, suppose $\overline{p}_0 \supset \overline{p}_1 \supset \cdots \supset \overline{p}_r$ is a strictly decreasing chain of prime ideals of $\overline{R}$ of length r. Since M is finitely generated, M is primeful by [18, Theorem 2.2]. Hence there exists $Q_i \in Spec(M) \subseteq \mathcal{X}$ such that $\sqrt{(Q_0 : M)} \supset \sqrt{(Q_1 : M)} \supset \cdots \supset \sqrt{(Q_r : M)}$. Thus $\nu(Q_0) \subset \nu(Q_1) \subset \cdots \subset \nu(Q_r)$ is a strictly increasing chain of irreducible closed subsets $\mathcal{X}$ of length r by proposition 4.1. ■

For a ring R, the Krull dimension of R, $dim(R)$, equals the combinatorial dimension of $Spec(R)$ equipped with the Zariski topology.

Theorem 4.6. *Let M be a finitely generated R-module and $|\mathcal{X}| < \infty$. Then the combinatorial dimension of $\mathcal{X}$ and the Krull dimension of $\overline{R}$ are equal.*

Proof. Use Proposition 4.5. ∎

Corollary 4.7. *Let M be a finitely generated R-module such that $\mathcal{X}$ has combinatorial dimension zero. Then the following hold.*

(1) *Every irreducible closed subset of $\mathcal{X}$ is an irreducible component.*

(2) *If $p \in V(Ann(M))$ and $|\mathcal{X}| < \infty$, then $\mathcal{X}_p = \nu(Q)$ for every $Q \in \mathcal{X}_p$.*

(3) *If $|\mathcal{X}| < \infty$ and $(\mathcal{X}, \mathcal{T})$ is a Noetherian space, then the set of irreducible components of $\mathcal{X}$ is $\{\nu(p_1M), ..., \nu(p_nM)\}$, where the p_i $(1 \leq i \leq n)$ are all the minimal prime containing $Ann(M)$.*

Proof. (1) is obvious.

(2) By Theorem 4.6, $dim(\mathcal{X}) = dim(\overline{R})$. Hence $p = \sqrt{(Q : M)}$ is a maximal ideal of R. If $Q' \in \nu(Q)$, then $\sqrt{(Q' : M)} = \sqrt{(Q : M)} = p$ and so $\nu(Q) \subseteq \mathcal{X}_p$. Now suppose $Q' \in \mathcal{X}_p$. Hence $\sqrt{(Q' : M)} = p = \sqrt{(Q : M)}$. Thus $Q' \in \nu(Q)$ and so $\mathcal{X}_p \subseteq \nu(Q)$.

(3) Since $(\mathcal{X}, \mathcal{T})$ is a Noetherian space with $dim(\mathcal{X}) = 0$, $\overline{R}$ has Noetherian spectrum and $dim(\overline{R}) = 0$ by Theorems 3.5 and 4.6. Hence $Spec(\overline{R})$ has only finitely many elements $\overline{p}_1, \overline{p}_2, \cdots, \overline{p}_n$ each of which is both maximal and minimal prime ideal of $\overline{R}$ by [15, P. 41, Examples 1.4, c) and d)]. Since M is a finitely generated R-module, $(p_iM : M) = p_i$ is a maximal ideal of R. Hence $p_iM \in \mathcal{X}_{p_i}$. So $\nu(p_iM)$ is an irreducible component of $\mathcal{X}$ for every i by (1) and Proposition 4.1. Thus by Proposition 4.2, $\{\nu(p_1M), \cdots, \nu(p_nM)\}$ is the set of all irreducible components of $\mathcal{X}$. ∎

References

[1] Abbasi, A., Hassanzadeh-Lelekaami, D., *Modules and spectral spaces*, Comm. Algebra, 40 (2012), 4111-4129.

[2] Ansari-Toroghy, H., Ovlyaee-Sarmazdeh, R., *On the prime spectrum of a module and Zariski topologies*, Comm. Algebra, 38 (2010), 4461-4475.

[3] Ansari-Toroghy, H., Ovlyaee-Sarmazdeh, R., *On the prime spectrum of X-injective modules,* Comm. Algebra, 38 (2010), 2606-2621.

[4] Atiyah, M.F., McDonald, I.G., *Introduction to commutative algebra*, Addison Weisley Publishing Company, Inc., 1969.

[5] Azizi, A., *Prime submodules and flat modules*, Acta Mathematatica Sinica, English series Jan., 23 (2007), 147-152.

[6] Barnard, A., *Multiplication modules*, J. Algebra, 71 (1981), 174-178.

[7] Behboodi, M., Haddadi, M.R.M., *Classical Zariski topology of modules and spectral spaces*. I, Int. Elec. J. Algebra, 4 (2008), 104-130.

[8] Behboodi, M., Haddadi, M.R.M., *Classical Zariski topology of modules and spectral spaces.* II. Int. Elec. J. Algebra, 4 (2008), 131-148.

[9] Bourbaki, N., *Algebra commutative*, Hermann, Paris, 1972.

[10] Duraivel, T., *Topology on spectrum of modules*, J. Ramanujan Math. Soc., 9 (1994), 25-34.

[11] El-Bast, Z.A., Smith, P.F., *Multiplication modules,* Comm. Algebra, 16 (1988), 755-779.

[12] Fazaeli Moghimi, H., Rashedi, F., *Primary-like submodules satisfying the primeful property*, (to appear).

[13] Fazaeli Moghimi, H., Rashedi, F., *Zariski-like spaces of certain modules*, (to appear).

[14] Gilmer, R., Heinzer, W., *The laskerian property, power series rings and noetherian spectra,* Proc. Amer. Math. Soc., 79 (1980), 13-16.

[15] Kunz, E., *Introduction to commutative algebra and algebraic geometry*, Boston: Birkhuser, 1985.

[16] Low, G.M., Smith, P.F., *Multiplication modules and ideals*, Comm. Algebra, 18 (1990), 4353-4375.

[17] Lu, C.P., *Modules with noetherian spectrum*, Comm. Algebra, 38 (2010), 807-828.

[18] Lu, C.P., *A module whose prime spectrum has the surjective natural map,* Houston J. Math., 33 (2007), 125-143.

[19] Lu, C.P., *The Zariski topology on the prime spectrum of a module*, Houston J. Math., 25 (1999), 417-425.

[20] Lu, C.P., *Prime submodules of modules,* Comment. Math. Univ. St. Pauli, 33 (1984), 61-69.

[21] McCasland, R.L., Moore, M.E., Smith, P.F. *An introduction to Zariski spaces over Zariski topologies,* Rocky Mountain J. Math., 28 (1998), 1358- 1369.

[22] McCasland, R.L., Moore, M.E., Smith, P.F. *On the spectrum of a module over a commutative ring,* Comm. Algebra, 25 (1997), 79-103.

[23] Moore, M.E., Smith, S.J., *Prime and radical submodules of modules over commutative rings,* Comm. Alegbra, 30 (2002), 5073-5064.

[24] Nikseresht, A., Azizi, A., *Zariski spaces of modules*, J. Pure Appl. Algebra, 217 (2013), 1187-1194.

[25] Ohm, J., Pendleton, R.L., *Rings with Noetherian spectrum*, Duke Math. J., 35 (1968), 631-639.

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ON THE EXISTENCE OF CATEGORICAL UNIVERSAL COVERINGS

Ali Pakdaman[1]

Department of Mathematics
Faculty of Sciences
Golestan University
P.O. Box 155, Gorgan
Iran
e-mail: a.pakdaman@gu.ac.ir

Hamid Torabi
Behrooz Mashayekhy

Department of Pure Mathematics
Center of Excellence in Analysis on Algebraic Structures
Ferdowsi University of Mashhad
P.O. Box 1159-91775, Mashhad
Iran
e-mails: h.torabi@ferdowsi.um.ac.ir
bmashf@um.ac.ir

Abstract. In this paper, we study necessary and sufficient conditions for the existence of categorical universal coverings using open covers of a given space X. As some applications, first we show that a first countable Peano space has a categorical universal covering or has an uncountable fundamental group. Second, we prove that the one point union $X_1 \vee X_2 = \frac{X_1 \cup X_2}{x_1 \sim x_2}$ has a categorical universal covering if and only if both X_1 and X_2 have categorical universal coverings.

1. Introduction and motivation

A covering map is a continuous map $p : \widetilde{X} \longrightarrow X$ if for every $x \in X$ there exists an open subset U of X with $x \in U$ such that U is *evenly covered* by p, that is, $p^{-1}(U)$ is a disjoint union of open subsets of $\widetilde{X}$ each of which is mapped homeomorphically onto U by p. Unlike modern nomenclature, the term universal covering space will always means a categorical covering space that is a covering map $p : \widetilde{X} \longrightarrow X$ with the property that for every covering map $q : \widetilde{Y} \longrightarrow X$ with a path connected space $\widetilde{Y}$ there exists a unique (up to equivalence) covering $f : \widetilde{X} \longrightarrow \widetilde{Y}$ such that $q \circ f = p$.

Nowadays, lots of literatures can be found about the covering spaces and their relations with fundamental groups and almost all of them proceed on the classifying of covering spaces, using the universal covering spaces. Also, the important role of the universal covering spaces in the geometry causes that finding features

[1]Corresponding author

of a given space which guarantee the existence of the universal covering space can be a challenge.

Simply connected covering spaces are examples of universal coverings that have been studied more and partially sufficient. As can be seen in many textbooks, locally path connectedness and semi-locally simply connectedness of a given space X is equivalent to the existence of simply connected universal covering spaces [6], [15]. But for the spaces that do not have these nice local behaviors, the existence of simply connected universal coverings is not possible. The question that naturally arises here is the following: Can we provide conditions that ensure the existence of (categorical) universal coverings for spaces with a bad local behavior?

In this regard, the deep connection between fundamental groups and covering spaces becomes more evident. Recently, with the emergence of new subgroups of the fundamental group that will be born in the absence of semi-locally simply connectedness, studying the existence of universal coverings is more accessible. For example, the authors [10, 16, 7] have introduced universal covering spaces of some these locally complicated spaces.

Our first main result of this paper, Theorem 2.8, introduces equivalent conditions, from various viewpoints, for the existence of universal coverings. The main idea is working with the Spanier groups with respect to open covers of a given space X which have been introduced in [15] and named in [4]. The importance of these groups and their intersection which is named Spanier group, $\pi_1^{sp}(X,x)$, is studied by H. Fischer et al. in [4] in order to modify of the definition of semi-locally simply connectedness. As a corollary of our first main result, Corollary 3.13, we show that all the universal coverings are Spanier covering. A Spanier covering is a covering $p : \widetilde{X} \longrightarrow X$ with $p_*\pi_1(\widetilde{X},\tilde{x}) = \pi_1^{sp}(X,x)$ which is universal as we have shown in [7].

Among the recent works studying the universal coverings which we are aware of, we can point out to the following:

G.R. Conner and J.W. Lamoreaux [3]: They studied the existence of simply connected universal covering spaces for separable metric spaces and subsets of the Euclidean plane.

J.W. Cannon and G.R. Conner [2]: They studied the relation of the simply connected universal covering of a separable, connected, locally path connected, one-dimensional metric space with algebraic properties of its fundamental group.

C. Sormani and G. Wei [12], [13]: They studied the existence of universal cover for Gromov-Hausdorff limit of a sequence of manifolds.

H. Fischer, A. Zastrow [5]: They introduced a generalized universal covering which enjoys most of the usual properties, with the possible exception of evenly covered neighborhood.

V. Berestovskii, C. Plaut [1]: They develop a generalized covering space theory for a class of uniform spaces called coverable spaces (different from what will be defined here). Coverable spaces include all geodesic metric spaces, connected and locally path connected compact topological spaces. Several of the main results of this paper were already proved in [18], but not in the generality of the current paper.

J. Wilkins [18]: He studied universal coverings of compact geodesic spaces. Here, it should be mentioned that there are some overlaps between our results and his. However, his approach to these result is by using discrete homotopy theory and concerns himself with compact geodesic spaces while we use continuous path counterparts and the result are more general.

Another result of this paper, Theorem 2.9 (which owes its proof to [2, Theorem 4.4]) says that countability of $\frac{\pi_1(X,x)}{\pi_1^{sp}(X,x)}$ guaranties existence of the universal covering for a connected, locally path connected first countable space X. We can consider this theorem as a weakened version of Mycielski's conjecture [8] that is proved by Shelah [11] and Pawlikowski [9]. In fact, Shelah used very sophisticated model theory results and proved the Mycielski conjecture which state that: *Fundamental group of a compact metric space which is connected and locally path connected is either finitely generated or has the power of the continuum.* Pawlikowski has a follow-up result which replaces the model theory by sophisticated constructive set theory. Using the paragraph preceding Lemma 2 in [9], we can restate this Theorem (conjecture) as follow: *If X is compact metric space which is connected and locally path connected and $\pi_1(X,x)$ is countable, then X has simply connected universal covering.* As a consequence of Theorem 2.9, we show that by deletion compact metric hypothesis from Shelah Theorem, we will just lose simply connectedness of universal covering.

Note that for a given space X, having a π-stable open cover (which we will introduce in section 2) is equivalent to the notion of *strongly coverable* introduced by Berestovskii-Plaut in [1] (Definition 86), at least for uniform spaces, which include compact and metrizable spaces. Also, for the case of a uniformly locally path connected space (e.g. a compact locally path connected space), Theorem 3.14 is an immediate consequence of Lemma 87 in [1]. So Theorem 3.14 simply extends this result to spaces that are not necessarily uniformly locally path connected. Also, having a π-stable open cover is equivalent to the finiteness of covering spectrum, introduced by Sormani-Wei in [14] and also finiteness of Plaut-Wilkins notion of critical spectrum [18]. By these similarities, we can say that Theorem 3.14 is a version of Theorem 3.4 of [14] without compact geodesic assumption. The equivalence of Theorem 2.8 (iv) is a direct generalization of equivalence of Theorem 1.1 [18]; we use the phrase "semi-locally Spanier space" to describe what Wilkins calls being "semi-locally r-simply connected", although Wilkins does only consider the metric case. In other words, these definitions are essentially equivalent if the space is metrizable, connected, and locally path connected. In fact, we use "semi-locally Spanier space" because (as we will show) the universal covering spaces are Spanier space. Part (vi) of Theorem 2.8 and the fact that it follows from (v) is essentially a non-metric version of Wilkins' Lemma 5.3 [18]. And finally, Theorem 2.9 is a restatement and generalization (i.e. without compact metric condition) of Wilkins'main result, Theorem 1.1, (1) $\leftrightarrow$(3) [18].

Our last result on the universal covering spaces is about one point union of two space. The Griffiths space is an example of the one point union of two spaces with simply connected universal coverings which hasn't got simply connected universal covering (Example 3.19). At first, we present a weakened Seifert-van Kampen type theorem for the fundamental group of the one point union of two spaces and then, using it and Theorem 2.8, we prove that $X_1 \vee X_2$ has a universal covering if and only if both of X_1 and X_2 have universal coverings.

2. Definitions and terminology

Throughout this article, all the homotopies between two paths are relative to end points, X is a connected and locally path connected space with the base point $x \in X$, and $p : \widetilde{X} \longrightarrow X$ is a covering of X with $\tilde{x} \in p^{-1}(\{x\})$ as the base point of $\widetilde{X}$. For a space X and any $H \leq \pi_1(X,x)$, by $\widetilde{X}_H$ we mean a covering space of X such that $p_*\pi_1(\widetilde{X},\tilde{x}) = H$, where $\tilde{x} \in p^{-1}(x)$ and $p : \widetilde{X}_H \longrightarrow X$ is the corresponding covering map.

E.H. Spanier [15, §2.5] classified path connected covering spaces of a space X using some subgroups of the fundamental group of X, recently named Spanier groups (see [4]). If $\mathcal{U}$ is an open cover of X, then the subgroup of $\pi_1(X,x)$ consisting of all homotopy classes of loops that can be represented by a product of the following type

$$\prod_{j=1}^{n} \alpha_j * \beta_j * \alpha_j^{-1},$$

where the α_j's are arbitrary paths starting at the base point x and each β_j is a loop inside one of the neighborhoods $U_i \in \mathcal{U}$, is called the *Spanier group with respect to* $\mathcal{U}$, and denoted by $\pi(\mathcal{U},x)$ [4, 15]. For two open covers $\mathcal{U},\mathcal{V}$ of X, we say that $\mathcal{V}$ refines $\mathcal{U}$ if for every $V \in \mathcal{V}$, there exists $U \in \mathcal{U}$ such that $V \subseteq U$.

Definition 2.1 We say that an open cover $\mathcal{U}$ of a space X is π-stable if $\pi(\mathcal{U},x) = \pi(\mathcal{V},x)$, for every refinement $\mathcal{V}$ of $\mathcal{U}$ and $x \in X$.

Definition 2.2 [4] The Spanier group of a topological space X, denoted by $\pi_1^{sp}(X,x)$ is $\pi_1^{sp}(X,x) = \bigcap_{\text{open covers } \mathcal{U}} \pi(\mathcal{U},x)$, for an $x \in X$.

Also, we can obtain the Spanier groups as follows: Let $\mathcal{U},\mathcal{V}$ be open coverings of X, and let $\mathcal{U}$ be a refinement of $\mathcal{V}$. Then since $\pi(\mathcal{U},x) \subseteq \pi(\mathcal{V},x)$, there exists an inverse limit of these Spanier groups, defined via the directed system of all open covers with respect to refinement and it is $\pi_1^{sp}(X,x)$ ([4]).

In the next definition, we follow [7]:

Definition 2.3

(i) A space X is called Spanier space if $\pi_1(X,x) = \pi_1^{sp}(X,x)$, for $x \in X$.

(ii) A covering $p : \widetilde{X} \longrightarrow X$ is called Spanier covering if $\widetilde{X}$ is a Spanier space.

A desirable fact in the category of coverings of a space X is the existence of $\widetilde{X}_H$, for every subgroup $H \leq \pi_1(X,x)$. We characterize spaces with this property as follows.

Definition 2.4 We call a topological space X a coverable space if $\widetilde{X}_H$ exists, for every subgroup $H \leq \pi_1(X,x)$ with $\pi_1^{sp}(X,x) \leq H$.

Note that the above notion does not depend on the point x. Also, since the image subgroups of all the coverings contain $\pi_1^{sp}(X,x)$ ([7]), eliminating the condition $\pi_1^{sp}(X,x) \leq H$ from the above definition is meaningless.

Definition 2.5

(i) A point $x \in X$ is called regular if X is semi-locally simply connected at x.

(ii) A non-regular point x is called wild if for every open neighborhood U of x there is a loop α in U such that $[\alpha] \notin \pi_1^{sp}(X,x)$.

(iii) A non-regular point is called tame if it is not wild.

For example, the common point of shrinking circles in the Hawaiian Earring is a wild point and the common point of shrinking circles in the Harmonic Archipelago is a tame point.

Remark 2.6 The readers should compare the above definition with Definition 4.5 of [18]. A little change in terminology make two definitions equivalent. In fact, by the results of [7], it is an easy exercise to show that a loop α in X belongs to $\pi_1^{sp}(X,x)$ if and only if it belongs to the image subgroup of every covering of X.

The majority of basic algebraic topology books who study covering theory, introduce semi-locally simply connected spaces which are famous because of having simply connected universal covering. Precisely, as Cannon and Conner mentioned in [2, Lemma 7.8], existence of the covering space $\widetilde{X}_H$ of X for $H \leq \pi_1(X,x)$ is equivalent to that every point $y \in X$ has an open neighborhood U such that $i_*\pi_1(U,y) \leq H$, where $i : U \hookrightarrow X$ is the inclusion. This coincidence is seen in [10, 16, 7], where the authors introduced three type of new categorical universal coverings: small covering, small generated covering and Spanier covering. Therein, the equivalent condition for the existence of these coverings are named, respectively: semi-locally small loop space, semi-locally small generated space and semi-locally Spanier space.

Definition 2.7 We call a space X a semi-locally Spanier space if and only if for each $x \in X$ there exists an open neighborhood U of x such that $i_*\pi_1(U,x) \leq \pi_1^{sp}(X,x)$.

The following theorems are main results of the paper.

Theorem 2.8 *For a connected and locally path connected space X, the following statements are equivalent.*

(i) *X is coverable.*

(ii) *X has a universal covering space.*

(iii) *X has a π-stable open cover.*

(iv) *X is a semi-locally Spanier space.*

(v) *X has no wild point.*

(vi) *$\pi_1^{sp}(X,x)$ is an open subgroup of $\pi_1^{\tau}(X,x)$.*

After some topological criterions for the existence of universal coverings, an algebraic criteria is given as follows. Also, this theorem is a generalization of Theorem 2.1 in [3].

Theorem 2.9 *A connected, locally path connected and first countable space X has a universal covering if $\frac{\pi_1(X,x)}{\pi_1^{sp}(X,x)}$ is countable. The converse is true when X is separable metric.*

Proposition 3.17, the main technical result of the paper, is a Seifert-van Kampen type theorem for the fundamental group of the one point union $X_1 \vee X_2$. Using this and Theorem 2.8, we prove that

Theorem 2.10 *Let X be the one point union $X_1 \vee X_2 = \frac{X_1 \cup X_2}{x_1 \sim x_2}$, where $\{x_1\}$ and $\{x_2\}$ are closed in X_1 and X_2, respectively. Then X has a universal covering if and only if X_1 and X_2 admit universal coverings.*

3. Propositions and proofs of the main results

Spanier in his brilliant book has used open covers to study the existence of covering spaces. Therein, the main theorem is the following.

Theorem 3.1 ([15, §2.5 Theorems 12,13]) *Let X be a connected, locally path connected space and $H \leq \pi_1(X,x)$, for $x \in X$. Then there exists a covering $p : \widetilde{X} \longrightarrow X$ such that $p_*\pi_1(\widetilde{X},\tilde{x}) = H$ if and only if there exists an open cover $\mathcal{U}$ of X for which $\pi(\mathcal{U},x) \leq H$.*

For two open covers $\mathcal{U},\mathcal{V}$ of X, we say that $\mathcal{V}$ refines $\mathcal{U}$ if for every $V \in \mathcal{V}$, there exists $U \in \mathcal{U}$ such that $V \subseteq U$. Using the properties of open covers and the definition of the Spanier groups with respect to open covers, we have the following facts which have been also remarked in [15].

Proposition 3.2 *Let $\mathcal{U},\mathcal{V}$ be open covers of a space X. Then the following statements hold.*

(i) *If $\mathcal{V}$ refines $\mathcal{U}$, then $\pi(\mathcal{V},x) \subseteq \pi(\mathcal{U},x)$, for every $x \in X$.*
(ii) *$\pi(\mathcal{U},x)$ is a normal subgroup of $\pi_1(X,x)$.*
(iii) *If α is a path in X, then $\varphi_{[\alpha]}(\pi(\mathcal{U},\alpha(0))) = \pi(\mathcal{U},\alpha(1))$, where $\varphi_{[\alpha]}([\beta]) = [\alpha^{-1} * \beta * \alpha]$.*

As the first observation, we have the following proposition.

Proposition 3.3 *For a connected and locally path connected space X, let $H, K \leq \pi_1(X,x)$. Then $\widetilde{X}_H$ and $\widetilde{X}_K$ exist if and only if $\widetilde{X}_{H\cap K}$ exists.*

Proof. By Theorem 3.1, existence of $\widetilde{X}_H$ and $\widetilde{X}_K$ implies the existence of open covers $\mathcal{U}$ and $\mathcal{V}$ of X such that $\pi(\mathcal{U},x) \leq H$ and $\pi(\mathcal{V},x) \leq K$. Let $\mathcal{U}\cap\mathcal{V} = \{U \cap V | U \in \mathcal{U},\ V \in \mathcal{V}\}$ which is a refinement of $\mathcal{U}$ and $\mathcal{V}$. Hence $\pi(\mathcal{U}\cap\mathcal{V},x) \subseteq \pi(\mathcal{U},x) \subseteq H$ and $\pi(\mathcal{U}\cap\mathcal{V},x) \subseteq \pi(\mathcal{V},x) \subseteq K$ which implies that $\pi(\mathcal{U}\cap\mathcal{V},x) \subseteq H \cap K$. Therefore, there exists $\widetilde{X}_{H\cap K}$. The converse is trivial. ∎

The above theorem shows that intersections of open covers of a space X are important in the existence of new coverings of X. So it is interesting to find the impression of the intersection of all open covers. For this, we use the Spanier groups.

Proposition 3.4 *([7]). If $p : \widetilde{X} \longrightarrow X$ is a covering of X, then $\pi_1^{sp}(X,x) \leq p_*\pi_1(\widetilde{X},\tilde{x})$, for every $x \in X$.*

It should be mentioned that the above proposition holds for $\pi_1^{s}(X,x)$ and $\pi_1^{sg}(X,x)$ (because of the inclusions $\pi_1^{s}(X,x) \leq \pi_1^{sg}(X,x) \leq \pi_1^{sp}(X,x)$) whose role in covering theory has studied in [10, 16].

With a little change in terminology, the following result is well-known in the classical covering theory.

Corollary 3.5 *Every connected, locally path connected and semi-locally simply connected space is coverable.*

Proposition 3.6 *Let X be a connected, locally path connected and coverable space. Then $\pi_1^{sp}(X,x)$ is trivial if and only if X is semi-locally simply connected.*

Proof. Since X is coverable, there exists a covering $p : \widetilde{X} \longrightarrow X$ such that $p_*\pi_1(\widetilde{X},\tilde{x}) = \pi_1^{sp}(X,x) = 1$ and hence $\widetilde{X}$ is simply connected which implies that X is semi-locally simply connected. The converse holds by Proposition 3.4. ■

The Hawaiian Earring space, HE, is a famous example of a space which is not semi-locally simply connected. Also, the Spanier group of the Hawaiian Earring space is trivial since if $\mathcal{U}_n$'s are open covers of the Hawaiian Earring by open disk with diameter $1/n$, for every $n \in \mathbb{N}$, then $\pi_1^{sp}(HE,0) \leq \bigcap_{n\in\mathbb{N}} \pi(\mathcal{U}_n,0) = 1$. Hence we have the following corollary.

Corollary 3.7 *The Hawaiian Earring space is not coverable.*

Proposition 3.8 *A space X is coverable if and only if $\widetilde{X}_{\pi_1^{sp}(X,x)}$ exists.*

Proof. The necessity comes from the definition. For the sufficiency, let $\pi_1^{sp}(X,x) \leq H \leq \pi_1(X,x)$. By Theorem 3.1, since $\widetilde{X}_{\pi_1^{sp}(X,x)}$ exists, there is an open cover $\mathcal{U}$ of X such that $\pi(\mathcal{U},x) \leq \pi_1^{sp}(X,x)$. Hence $\widetilde{X}_H$ exists. ■

Lemma 3.9 ([15, §2.5 Lemma 11]). *If $p : \widetilde{X} \longrightarrow X$ is a covering such that $p_*\pi_1(\widetilde{X},\tilde{x}) = H$ and $\mathcal{U}$ is the open cover of X by evenly covered open neighborhoods, then $\pi(\mathcal{U},x) \leq H$.*

Proposition 3.10

(i) *If $\mathcal{U}$ and $\mathcal{V}$ are two π-stable open covers of X, then $\pi(\mathcal{U},x) = \pi(\mathcal{V},x)$.*

(ii) *The open cover $\mathcal{U}$ of X is π-stable if and only if $\pi_1^{sp}(X,x) = \pi(\mathcal{U},x)$.*

(iii) *The covering space $\widetilde{X}_{\pi_1^{sp}(X,x)}$ exists if and only if there exists a π-stable open covering $\mathcal{U}$ of X.*

Proof. The first step comes from the fact that $\mathcal{U}\cap\mathcal{V}$ is the refinement of $\mathcal{U}$ and $\mathcal{V}$ which implies that $\pi(\mathcal{U},x)=\pi(\mathcal{U}\cap\mathcal{V},x)=\pi(\mathcal{V},x)$. For (ii), let $\mathcal{U}$ be a π-stable open cover of X. By definition $\pi_1^{sp}(X,x)\leq\pi(\mathcal{U},x)$. For the reverse containment, let $\mathcal{V}$ be an arbitrary open cover of X. Then $\mathcal{U}\cap\mathcal{V}$ is a refinement of $\mathcal{V}$ and hence $\pi(\mathcal{U},x)=\pi(\mathcal{U}\cap\mathcal{V},x)\leq\pi(\mathcal{V},x)$. Therefore $\pi(\mathcal{U},x)\leq\pi_1^{sp}(X,x)$, as desired. The converse is trivial by definitions and part (i) of Proposition 3.2. For (iii), assume $p:\widetilde{X}_{\pi_1^{sp}(X,x)}\longrightarrow X$ is a covering and $\mathcal{U}$ is the open cover of X by evenly covered open neighborhoods. Since $p_*\pi_1(\widetilde{X}_{\pi_1^{sp}(X,x)},\tilde{x})=\pi_1^{sp}(X,x)$, Lemma 3.9 implies that $\pi(\mathcal{U},x)\subseteq\pi_1^{sp}(X,x)$ and hence the result holds by (ii). The converse holds by (ii) and Theorem 3.1. ∎

The following theorem shows the importance of a universal covering for the existence of other coverings and vice versa.

Theorem 3.11 *A space X has a universal covering if and only if X is coverable.*

Proof. If X is coverable, then by the definition, $\widetilde{X}_{\pi_1^{sp}(X,x)}$ exists. By Proposition 3.4, $\widetilde{X}_{\pi_1^{sp}(X,x)}$ is a universal covering space and hence the result holds. Conversely, assume that $p:\widetilde{X}\longrightarrow X$ is a universal covering of X and $p_*\pi_1(\widetilde{X},\tilde{x})=H$. We claim that for every open cover $\mathcal{U}$ of X, $H\leq\pi(\mathcal{U},x)$. For, if $q:\widetilde{X}_{\pi(\mathcal{U},x)}\longrightarrow X$ is the covering such that $q_*\pi_1(\widetilde{X}_{\pi(\mathcal{U},x)},x')=\pi(\mathcal{U},x)$, for $x'\in q^{-1}(x)$, then by the universal property of $p:\widetilde{X}\longrightarrow X$

$$H=p_*\pi_1(\widetilde{X},\tilde{x})\leq q_*\pi_1(\widetilde{X}_{\pi(\mathcal{U},x)},x')=\pi(\mathcal{U},x).$$

Hence $H\leq\pi_1^{sp}(X,x)$ which implies that $H=\pi_1^{sp}(X,x)$. Thus the covering space $\widetilde{X}_{\pi_1^{sp}(X,x)}=\widetilde{X}$ exists and therefore by Proposition 3.8, X is coverable. ∎

Proposition 3.12 *A space X is semi-locally Spanier space if and only if there exists an open cover $\mathcal{U}$ of X such that $\pi(\mathcal{U},x)=\pi_1^{sp}(X,x)$, for every $x\in X$.*

Proof. Use Proposition 3.2 (iii) and the definition of $\pi(\mathcal{U},x)$. ∎

Proof of Theorem 2.8.
(i) $\Leftrightarrow$ (ii): Theorem 3.11.
(i) $\Leftrightarrow$ (iii): Use Proposition 3.8 and Proposition 3.10, (iii).
(i) $\Leftrightarrow$ (iv): Use Proposition 3.12, Proposition 3.10, (iii) and Proposition 3.8.
(iv) $\Leftrightarrow$ (v): By definition of wild point, the existence of wild point x causes X not to be semi-locally Spanier space at x and vice versa.
(i) $\Leftrightarrow$ (vi): Use [17, Theorem 2.1, Corollary 3.9]. ∎

By [7], if $p:\widetilde{X}\longrightarrow X$ is a covering such that $p_*\pi_1(\widetilde{X},\tilde{x})=\pi_1^{sp}(X,x)$, then $\widetilde{X}$ is a Spanier space and hence p is a Spanier covering. Therefore, we have

Corollary 3.13 *All the universal covering spaces of connected and locally path connected spaces are Spanier space.*

Recall that for a covering $p : \widetilde{X} \longrightarrow X$, the cardinal number of any fiber, $|p^{-1}(\{x\})|$ is equal to the index of $p_*\pi_1(\widetilde{X},\tilde{x})$ in $\pi_1(X,x)$, $[\pi_1(X,x); p_*\pi_1(\widetilde{X},\tilde{x})]$.

Proof of Theorem 2.9. By Theorem 2.8, it suffices to show that X is semi-locally Spanier space. Let y be fixed but arbitrary and $B_1 \supseteq B_2 \supseteq ...$ be a countable local basis at y. We will denote by G_n the image of the natural map $\pi_1(B_n, y) \longrightarrow \pi_1(X, y)$. By [2, Theorem 4.4], the sequence $G_1\pi_1^{sp}(X,y) \supseteq G_2\pi_1^{sp}(X,y) \supseteq \cdots$ is eventually constant. We can choose $k \in \mathbb{N}$ large enough so that $G_k\pi_1^{sp}(X,y) = G_{k+1}\pi_1^{sp}(X,y) = \cdots$ and claim that $G_k \leq \pi_1^{sp}(X,y)$ which implies that X is a semi-locally Spanier space. Let $[\alpha] \in G_k$ and $\mathcal{U}$ be an arbitrary open cover of X. If $U_y \in \mathcal{U}$ contains y, there exists $m > k$ such that $B_m \subseteq U_y$ and then $\mathcal{U}_m := \mathcal{U} \cup \{B_m\}$ is a refinement of $\mathcal{U}$. Obviously, $[\alpha] \in G_k\pi_1^{sp}(X,y)$ which implies $[\alpha] \in G_m\pi_1^{sp}(X,y)$ since $m > k$. Hence $[\alpha] = [\alpha_m][\gamma]$, where $[\alpha_m] \in G_m$ and $\gamma \in \pi_1^{sp}(X,y)$. Therefore $[\alpha] = [\alpha_m * \gamma] \in \pi(\mathcal{U}_m, y) \leq \pi(\mathcal{U}, y)$.

For the converse, let $p : \widetilde{X} \longrightarrow X$ be a universal covering of X and assume by contradiction that $\frac{\pi_1(X,x)}{\pi_1^{sp}(X,x)}$ is uncountable. Then $p^{-1}(\{x\})$ is an uncountable subset of $\widetilde{X}$, which is a separable metric space by [3, Theorem 4.1]. Hence $p^{-1}(\{x\})$ contains a limit point of itself, contradicting the local homeomorphism property of covering maps. ■

A restatement of the Mycielski's conjecture that is proved by Shelah [11] and by Pawlikowski [9] is that a connected, locally path connected, compact metric space with countable fundamental group has simply connected covering (which is universal covering). As a consequence of Theorem 2.9, we have the following theorem which shows that in Mycielski's conjecture we can replace compact metric hypothesis by a weaker one first countablility and get universal covering instead of simply connected covering.

Theorem 3.14 *A connected, locally path connected and first countable space X has a universal covering or $\pi_1(X,x)$ is uncountable.*

By [2, Theorem 5.1], any free factor group of the fundamental group of a separable, locally path connected metric space has countable rank and hence is countable. Thus we have

Corollary 3.15 *A connected, locally path connected separable metric space X has a universal covering if $\frac{\pi_1(X,x)}{\pi_1^{sp}(X,x)}$ is free.*

By the following corollary which is an explicit consequence of Theorems 2.8 and 2.9, we are able to say easily that the fundamental group of a space with at least one wild point, like Hawaiian Earring, is uncountable.

Corollary 3.16 *Let X be connected, locally path connected and first countable. If X has a wild point, then $\pi_1(X,x)$ is uncountable.*

In the sequel, we concentrate on the fundamental group and the universal covering space of one point unions. At first, we introduce the following Seifert-van Kampen type formulation for the fundamental group of one point unions. It should be mentioned that according to Seifert-van Kampen theorem, the fundamental group of the one point union of two spaces is naturally isomorphic to the free product of their fundamental groups provided that each of them is first countable and locally simply connected. But for the general spaces, this fails.

Proposition 3.17 *Let X be the one point union $X_1 \vee X_2 = \frac{X_1 \cup X_2}{x_1 \sim x_2}$ where $\{x_1\}$ and $\{x_2\}$ are closed in X_1 and X_2, respectively. If U_i is a neighborhood of x_i in X_i for $i = 1, 2$, then $\pi_1(X, *)$ is generated by $i_*\pi_1(X_1, x_1), j_*\pi_1(X_2, x_2)$ and $k_*\pi_1(U_1 \vee U_2, *)$, where $i : X_1 \hookrightarrow X$, $j : X_2 \hookrightarrow X$ and $k : U_1 \vee U_2 \hookrightarrow X$ are inclusions and $*$ is the common point.*

Proof. Let $\alpha : [0, 1] \longrightarrow X$ be a loop at $* \in X$. First we define inductively $a_i \in \alpha^{-1}(*)$ such that $0 = a_0 < a_1 < a_2 ... < a_n \leq 1$ and $\alpha([a_{i-1}, a_i])$ is a subset of X_1 or X_2 or $U_1 \vee U_2$. Since $\{x_1\}$ and $\{x_2\}$ are closed in X_1 and X_2 respectively, $\{\alpha^{-1}(X_1 \backslash \{x_1\}), \alpha^{-1}(X_2 \backslash \{x_2\}), \alpha^{-1}(U_1 \vee U_2)\}$ is an open cover for the compact set I. Let $\lambda > 0$ be the Lebesgue number for this cover. Choose $N \in \mathbb{N}$ such that $1/N \leq \lambda$. Put $a_0 = 0$. Suppose a_{i-1} has been chosen suitably. Now, we obtain a_i properly as follows:

If $a_{i-1} = 1$, then put $a_i = 1$. If $a_{i-1} \neq 1$ and $(\alpha^{-1}(*)) \cap (a_{i-1}, min\{a_{i-1} + (1/N), 1\}] \neq \emptyset$, then consider a_i to be the maximum of the compact set $\{(\alpha^{-1}(*)) \cap [a_{i-1}, min\{a_{i-1} + (1/N), 1\}]\}$. In this case since $a_i - a_{i-1} \leq (1/N)$, we have $\alpha([a_{i-1}, a_i])$ is a subset of X_1 or X_2 or $U_1 \vee U_2$. If $a_{i-1} \neq 1$ and $(\alpha^{-1}(*)) \cap (a_{i-1}, min\{a_{i-1} + (1/N), 1\}] = \emptyset$, then $a_{i-1} + (1/N) < 1$ and put $a_i = min\{(\alpha^{-1}(*)) \cap [a_{i-1} + (1/N), 1]\}$. In this case $\alpha((a_{i-1}, a_i)) \subseteq X \backslash \{*\} = (X_1 \backslash \{x_1\}) \cup (X_2 \backslash \{x_2\})$. Hence $(a_{i-1}, a_i) \subseteq \alpha^{-1}(X_1 \backslash \{x_1\}) \cup \alpha^{-1}(X_1 \backslash \{x_1\})$. Since (a_{i-1}, a_i) is connected and two sets $\alpha^{-1}(X_i \backslash \{x_i\})$ are disjoint open sets, $\alpha([a_{i-1}, a_i])$ is a subset of X_1 or X_2.

Now we show that if $n \geq 2$ and $a_i \neq 1$, then $a_i - a_{i-2} \geq 1/N$. If $(\alpha^{-1}(*)) \cap (a_{i-2}, min\{a_{i-2} + (1/N), 1\}] = \emptyset$, then $a_{i-1} - a_{i-2} \geq 1/N$. If $(\alpha^{-1}(*)) \cap (a_{i-2}, min\{a_{i-2} + (1/N), 1\}] \neq \emptyset$, then $(a_{i-1}, a_{i-2} + (1/N)] \cap \alpha^{-1}(*) = \emptyset$. Therefore $a_i \geq a_{i-2} + 1/N$ and since N is fixed, there exists $k \in \mathbb{N}$ such that $a_k = 1$. Hence $[\alpha] = [\alpha \circ \beta_1][\alpha \circ \beta_2]...[\alpha \circ \beta_k]$, where $\beta_i : I \to [a_{i-1}, a_i]$ is an increasing linear homeomorphism for $i = 1, 2, \cdots, k$. Note that for every $1 \leq i \leq k$, $Im(\alpha \circ \beta_i)$ is a subset of X_1 or X_2 or $U_1 \vee U_2$. ∎

Lemma 3.18 *Let X be a topological space with a base point x.*

- (i) *If U is an element of an open cover $\mathcal{U}$ of X which contains x, then $[\alpha] \in \pi(\mathcal{U}, x)$, for every loop $\alpha : I \longrightarrow U$ at x.*
- (ii) *If Y is a subspace of X, then $[\alpha] \in \pi_1^{sp}(X, x)$, for every loop α such that $[\alpha] \in \pi_1^{sp}(Y, x)$.*

Proof. (i) Let α be a loop at x in U. By the definition of $\pi(\mathcal{U}, x)$, $[\alpha] \in \pi(\mathcal{U}, x)$ since $U \in \mathcal{U}$.

(ii) Let $[\alpha] \in \pi_1^{sp}(Y, x)$ and $\mathcal{U} = \{U_i | i \in I\}$ be an open cover for X. Then $\{U_i \cap Y | i \in I\}$ is an open cover for Y. By the definition there are paths α_j and loops β_j at $\alpha_j(1)$ in $U_{i_j} \cap Y$ such that α is homotopic to $\alpha_1 * \beta_1 * \alpha_1^{-1} * \alpha_2 * \beta_2 * \alpha_2^{-1} ... \alpha_n * \beta_n * \alpha_n^{-1}$ in Y relative to $\{0,1\}$. Therefore $[\alpha] \in \pi(\mathcal{U}, x)$ which implies that $[\alpha] \in \pi_1^{sp}(X, x)$. ∎

Proof of Theorem 2.10. Assume that X_1 and X_2 have universal coverings. Then they are semi-locally Spanier and there is a neighborhood V_j of x_j in X_j such that $(i_j)_* \pi_1(V_j, x_j) \leq \pi_1^{sp}(X_j, x_j)$, for $j = 1,2$ and the inclusions $i_j : V_j \hookrightarrow X_j$. We show that $k_* \pi_1(V_1 \vee V_2, \bar{x}) \leq \pi_1^{sp}(X, \bar{x})$, where $k : V_1 \vee V_2 \hookrightarrow X$ is the inclusion and $\bar{x}$ is the equivalence class of x_1 and x_2 in X. Let $\mathcal{U}$ be an open cover of X and $W \in \mathcal{U}$ contains $\bar{x}$. There exists open set $U_1 \vee U_2 \subseteq W \in \mathcal{U}$ such that U_j is a neighborhood of x_j in X_j, for $j = 1,2$. By Proposition 3.17 $\pi_1(V_1 \vee V_2, \bar{x})$ is generated by $(l_1)_* \pi_1(V_1, x_1), (l_2)_* \pi_1(V_2, x_2)$ and $(l_3)_* \pi_1((U_1 \cap V_1) \vee (U_2 \cap V_2), \bar{x})$, where $l_j : V_j \longrightarrow V_1 \vee V_2$, for $j = 1,2$ and $l_3 : (U_1 \cap V_1) \vee (U_2 \cap V_2) \longrightarrow V_1 \vee V_2$ are inclusions. Since $(i_j)_* \pi_1(V_j, x_j) \leq \pi_1^{sp}(X_j, x_j)$, part (ii) of Lemma 3.18 implies that $(k_j)_*(i_j)_* \pi_1(V_j, x_j) \leq \pi_1^{sp}(X, \bar{x}) \leq \pi(\mathcal{U}, \bar{x})$, where $k_j : X_j \hookrightarrow X$, for $j = 1,2$. Hence $k_* \pi_1(V_1 \vee V_2, \bar{x}) \leq \pi(\mathcal{U}, \bar{x})$ since $k_j \circ i_j = k \circ l_j$ for $j = 1,2$ and $k_*(l_3)_* \pi_1((U_1 \cap V_1) \vee (U_2 \cap V_2), \bar{x}) \subseteq \pi(\mathcal{U}, \bar{x})$ by part (i) of Lemma 3.18.

For the converse, let $x \in X_1$ and assume $q : X \longrightarrow X_1$ is a continuous map that is identity on X_1 and constant on X_2. Let U be the open neighborhood of x in X where $i_* \pi_1(U, x) \leq \pi_1^{sp}(X, x)$, for $i : U \hookrightarrow X$. Then $V = q(U)$ is open in X_1. We claim that $j_* \pi_1(V, x) \leq \pi_1^{sp}(X_1, x)$, where $j : V \hookrightarrow X_1$. Let $\alpha : I \longrightarrow V$ be a loop and $\mathcal{V}$ be an open cover of X_1. Then $\mathcal{U} = q^{-1}(\mathcal{V})$ is an open cover of X and hence $[\alpha] \in \pi_1^{sp}(X, x) \leq \pi(\mathcal{U}, x)$. Therefore $q_*([\alpha]) \in q_*(\pi(\mathcal{U}, x)) = \pi(\mathcal{V}, x)$ which implies that $[\alpha] \in \pi_1^{sp}(X_1, x)$. ∎

In the following example, we show that Theorem 2.10 does not hold for simply connected universal coverings (a covering $p : \widetilde{X} \longrightarrow X$ in which $\widetilde{X}$ is simply connected).

Example 3.19 The cone on the Hawaiian Earring is a connected, locally path connected and semi-locally simply connected space and so the identity map is a simply connected covering which is a universal covering. But the double cone on the Hawaiian Earring does not have a simply connected universal covering space since it is not semi-locally simply connected but by Theorem 2.10 it has a categorical universal covering space which is a Spanier space.

References

[1] Berestovskii, Plaut, C., *Uniform Universal Covers of Uniform Spaces,* Topology and its Applications, 154 (2007), 1748–1777.

[2] Cannon, J.W., Conner, G.R., *On the fundamental groups of one-dimensional spaces,* Topology and its Applications, 153 (2006), 2648–2672.

[3] Conner, G.R., Lamoreaux, J.W., *On the existence of universal covering spaces for metric spaces and subsets of the Euclidean plane,* Fundamenta Mathematicae, 187 (2) (2005), 95–110.

[4] Fischer, H., Repovš, D., Virk, Z., Zastrow, A., *On semilocally simply connected spaces,* Topology and its Applications, 158 (2011), 397–408.

[5] Fischer, H., Zastrow, A., *Generalized universal coverings and the shape group,* Fundamenta Mathematicae, 197 (2007), 167–196.

[6] Lubkin, S., *Theory of covering spaces,* Transactions of the American Mathematical Society, 104 (1962), 205–238.

[7] Mashayekhy, B., Pakdaman, A., Torabi, H., *Spanier spaces and covering theory of non-homotopically path Hausdorff spaces,* Georgian Mathematical Journal, 20 (2013), 303–317.

[8] Mycielski, J., *Independent sets in topological algebras,* Fundamenta Mathematicae, 55 (1964), 139–147.

[9] Pawlikowski, J., *The fundamental group of a compact metric space,* Proceeding of the American Mathematical Society, 126 (1998), 3083–3087.

[10] Pakdaman, A., Torabi, H., Mashayekhy, B., *Small loop spaces and covering theory of non-homotopically Hausdorff spaces,* Topology and its Application, 158 (2011), 803–809.

[11] Shelah, S., *Can the fundamental (homotopy) group of a space be the rationals?*, Proceeding of the American Mathematical Society, 103 (2) (1988), 627–632.

[12] Sormani, C., Wei, G., *Hausdorff Convergence and Universal Covers,* Transactions of the American Mathematical Society, 353 (2001,) 3585–3602.

[13] Sormani, C., Wei, G., *Universal Covers for Hausdorff Limits of Noncompact Spaces,* Transactions of the American Mathematical Society, 356 (3) (2004), 1233–1270.

[14] Sormani, C., Wei, G., *The covering spectrum of a compact length space,* J. Differential Geom., 67 (2004), 35–77.

[15] Spanier, E.H., *Algebraic Topology,* McGraw-Hill, New York, 1966.

[16] Torabi, H., Pakdaman, A., Mashayekhy, B., *Topological fundamental groups and small generated coverings,* Mathematica Slovaca, 65 (2015), 1153-1164.

[17] Torabi, H., akdaman, A., Mashayekhy, B., *On the Spanier groups and covering and semicovering spaces,* arXiv:1207.4394v1.

[18] Wilkins, J., *The revised and uniform fundamental groups and universal covers of geodesic spaces,* Topology and its Applications, 160 (2013), 812–835.

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PSEUDO SEMI B-FREDHOLM AND GENERALIZED DRAZIN INVERTIBLE OPERATORS THROUGH LOCALIZED SVEP

Abdelaziz Tajmouati
Mohammed Karmouni
Mbark Abkari

Sidi Mohamed Ben Abdellah University
Faculty of Sciences Dhar Al Mahraz
Laboratory of Mathematical Analysis and Applications
Fez
Morocco
e-mails: abdelaziz.tajmouati@usmba.ac.ma
mohammed.karmouni@usmba.ac.ma
mbark.abkari@usmba.ac.ma

Abstract. In this paper, we define and study the pseudo upper and lower semi B-Fredholm of bounded operators in a Banach space. In particular, we prove equality up to $S(T)$ between the left generalized Drazin spectrum and the pseudo upper semi B-Fredholm spectrum, $S(T)$ is the set where T fails to have the SVEP. Also, we prove that both of the left and the right generalized Drazin operators are invariant under additive commuting power finite rank perturbations and some perturbations for the pseudo upper and lower semi B-Fredholm operators are given. As applications, we investigate some classes of operators as the supercyclic and multiplier operators.

Keywords: pseudo upper semi B-Fredholm, pseudo lower semi B-Fredholm, left generalized Drazin, right generalized Drazin, Single-valued extension property.
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1. Introduction

Throughout, X denotes a complex Banach space and $\mathcal{B}(X)$ denotes the Banach algebra of all bounded linear operators on X, we denote by T^*, $N(T)$, $R(T)$, $R^\infty(T) = \bigcap_{n\geq 0} R(T^n)$, $K(T)$, $H_0(T)$, $\rho(T)$, $\sigma_{ap}(T)$, $\sigma_{su}(T)$, $\sigma_p(T)$ and $\sigma(T)$, respectively the adjoint, the null space, the range, the hyper-range, the analytic core, the quasi-nilpotent part, the resolvent set, the approximate point spectrum, the surjectivity spectrum, the point spectrum and the spectrum of T.

Next, let $T \in \mathcal{B}(X)$, T has the single valued extension property at $\lambda_0 \in \mathbb{C}$ (SVEP) if for every open neighborhood $U \subseteq \mathbb{C}$ of λ_0, the only analytic function

$f : U \longrightarrow X$ which satisfies the equation $(T - zI)f(z) = 0$ for all $z \in U$ is the function $f \equiv 0$. T is said to have the SVEP if T has the SVEP for every $\lambda \in \mathbb{C}$. Obviously, every operator $T \in \mathcal{B}(X)$ has the SVEP at every $\lambda \in \rho(T)$, then T and T^* have the SVEP at every point of the boundary $\partial(\sigma(T))$ of the spectrum. In particular, T and T^* have the SVEP at every isolated point of the spectrum. We denote by $S(T)$ the open set of $\lambda \in \mathbb{C}$ where T fails to have SVEP at λ, and we say that T has SVEP if $S(T) = \emptyset$. It is easy to see that $S(T) \subset \sigma_p(T)$ ([1], [11]).

An operator $T \in \mathcal{B}(X)$ is said to be decomposable if for any open covering U_1, U_2 of the complex plane $\mathbb{C}$, there are two closed T-invariant subspaces X_1 and X_2 of X such that $X_1 + X_2 = X$ and $\sigma(T|X_k) \subset U_k$, $k = 1, 2$. Note that T is decomposable implies that T and T^* have the SVEP.

A bounded linear operator is called an upper semi-Fredholm (resp, lower semi-Fredholm) if $dim N(T) < \infty$ and $R(T)$ is closed (resp, $codim R(T) < \infty$). T is semi-Fredholm if it is a lower or upper semi-Fredholm. The index of a semi Fredholm operator T is defined by $ind(T) = dim N(T) - codim R(T)$.

T is a Fredholm operator if is lower and upper semi-Fredholm, and T is called a Weyl operator if it is a Fredholm of index zero.

The upper, lower and semi-Fredholm spectra of T are closed and defined by

$$\begin{aligned}
\sigma_{uF}(T) &= \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not an upper semi-Fredholm operator}\} \\
\sigma_{lF}(T) &= \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not a lower semi-Fredholm operator}\} \\
\sigma_{sF}(T) &= \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not a semi-Fredholm operator}\}
\end{aligned}$$

The essential and Weyl spectra of T are closed and defined by

$$\begin{aligned}
\sigma_e(T) &= \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not a Fredholm operator}\} \\
\sigma_W(T) &= \{\lambda \in \mathbb{C} : \ T - \lambda I \ \text{ is not a Weyl operator}\}
\end{aligned}$$

Now, consider a class of operators, introduced and studied by Berkani et al. in a series of papers which extends the class of semi-Fredholm operators [5]–[8]. For every $T \in \mathcal{B}(X)$ and a nonnegative integer n, let us denote by T_n the restriction of T to $R(T^n)$ viewed as a map from the space $R(T^n)$ into itself (we set $T_0 = T$).

An operator $T \in \mathcal{B}(X)$ is said to be upper (lower) semi B-Fredholm, if for some integer $n \geq 0$ the range $R(T^n)$ is closed and T_n is an upper (lower) semi-Fredholm operator. Moreover, if T_n is a Fredholm operator, then T is called a B-Fredholm operator. A semi B-Fredholm operator is an upper or a lower semi B-Fredholm operator. It is easily seen that every nilpotent operator, as well as any idempotent bounded operator is B-Fredholm. The class of B-Fredholm operators contains the class of Fredholm operators as a proper subclass.

Let $T \in \mathcal{B}(X)$, according to [6, Proposition 2.6], T is a B-Fredholm operator if and only if there exists (X_1, X_2) a pair of T-invariant closed subspaces of X, such that $X = X_1 \oplus X_2$ and $T = T_1 \oplus T_2$ where T_1 is Fredholm and T_2 is nilpotent. The upper, lower and B-Fredholm spectra are defined by

$$\begin{aligned}
\sigma_{uBF}(T) &= \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not upper B-Fredholm}\} \\
\sigma_{lBF}(T) &= \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not lower B-Fredholm}\} \\
\sigma_{BF}(T) &= \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not B-Fredholm}\}
\end{aligned}$$

Also $T \in \mathcal{B}(X)$ is a B-Weyl operator if there exists (X_1, X_2) a pair of T-invariant closed subspaces of X, such that $X = X_1 \oplus X_2$ and $T = T_1 \oplus T_2$ where T_1 is Weyl operator and T_2 is nilpotent. The B-Weyl spectrum is defined by

$$\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not B-Weyl}\}.$$

Let $T \in \mathcal{B}(X)$, the ascent of T is defined by $a(T) = \min\{p \in \mathbb{N} : N(T^p) = N(T^{p+1})\}$, if such p does not exists we let $a(T) = \infty$. Analogously the descent of T is $d(T) = \min\{q \in \mathbb{N} : R(T^q) = R(T^{q+1})\}$, if such q does not exists we let $d(T) = \infty$ [12]. It is well known that if both $a(T)$ and $d(T)$ are finite then $a(T) = d(T)$ and we have the decomposition $X = R(T^p) \oplus N(T^p)$ where $p = a(T) = d(T)$.

The descent and ascent spectra of $T \in \mathcal{B}(X)$ are defined by

$$\begin{aligned} \sigma_{des}(T) &= \{\lambda \in \mathbb{C}, T - \lambda I \text{ has not finite descent}\} \\ \sigma_{acc}(T) &= \{\lambda \in \mathbb{C}, T - \lambda I \text{ has not finite ascent}\} \end{aligned}$$

Let $T \in \mathcal{B}(X)$, T is said to be a Drazin invertible if there exists a positive integer k and an operator $S \in \mathcal{B}(X)$ such that

$$ST = TS, \ \ T^{k+1}S = T^k \ \text{ and } S^2T = S,$$

which is also equivalent to the fact that $T = T_1 \oplus T_2$, where T_1 is invertible and T_2 is nilpotent. It is well known that T is Drazin invertible if it has a finite ascent and descent.

Define two sets $LD(X)$ and $RD(X)$ as [4], [15]:

$$\begin{aligned} LD(X) &= \{T \in \mathcal{B}(X) : a(T) < \infty \text{ and } R(T^{a(T)+1}) \text{ is closed }\} \\ RD(X) &= \{T \in \mathcal{B}(X) : d(T) < \infty \text{ and } R(T^{d(T)+1}) \text{ is closed }\} \end{aligned}$$

An operator $T \in \mathcal{B}(X)$ is said to be left (resp. right) Drazin invertible if $T \in LD(X)$ (resp. $T \in RD(X)$). The left and right Drazin invertible spectra are defined by:

$$\begin{aligned} \sigma_{lD}(T) &= \{\lambda \in \mathbb{C}, \ T - \lambda I \notin LD(X)\} \\ \sigma_{rD}(T) &= \{\lambda \in \mathbb{C}, \ T - \lambda I \notin RD(X)\} \end{aligned}$$

and we have [4], [5]:

$$\sigma_D(T) = \sigma_{lD}(T) \cup \sigma_{rD}(T)$$

The concept of Drazin invertible operators has been generalized by Koliha [13]. In fact, $T \in \mathcal{B}(X)$ is generalized Drazin invertible if and only if $0 \notin acc(\sigma(T))$ ($acc(\sigma(T))$ is the set of all points of accumulation of $\sigma(T)$), which is also equivalent to the fact that $T = T_1 \oplus T_2$ where T_1 is invertible and T_2 is quasi-nilpotent. The generalized Drazin invertible spectrum defined by

$$\sigma_{gD}(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not generalized Drazin invertible}\}$$

In [14], the authors introduced and studied a new concept of left and right generalized Drazin inverse of bounded operators as a generalization of left and right

Drazin invertible operators. In fact, an operator $T \in \mathcal{B}(X)$ is said to be left generalized Drazin invertible if $H_0(T)$ is closed and complemented with a subspace M in X such that $T(M)$ is closed which equivalent to $T = T_1 \oplus T_2$ such that T_1 is left invertible and T_2 is quasi-nilpotent see [14, Proposition 3.2].

An operator $T \in \mathcal{B}(X)$ is said to be right generalized Drazin invertible if $K(T)$ is closed and complemented with a subspace N in X such that $N \subseteq H_0(T)$ which equivalent to $T = T_1 \oplus T_2$ such that T_1 is right invertible and T_2 is quasi-nilpotent see [14, Proposition 3.4]. The left and right generalized Drazin spectra of $T \in \mathcal{B}(X)$ are defined by:

$$\begin{aligned} \sigma_{lgD}(T) &= \{\lambda \in \mathbb{C},\ T - \lambda I \text{ is not left generalized Drazin}\} \\ \sigma_{rgD}(T) &= \{\lambda \in \mathbb{C},\ T - \lambda I \text{ is not right generalized Drazin}\} \end{aligned}$$

From [14], we have:

$$\sigma_{gD}(T) = \sigma_{lgD}(T) \cup \sigma_{rgD}(T)$$

$$\sigma_{rgD}(T) \subset \sigma_{rD}(T)$$

$$\sigma_{lgD}(T) \subset \sigma_{lD}(T)$$

$$\lambda \in \sigma_{lgD}(T) \Longleftrightarrow \lambda \in acc(\sigma_{ap}(T))$$

$$\lambda \in \sigma_{rgD}(T) \Longleftrightarrow \lambda \in acc(\sigma_{su}(T))$$

This paper is organized as follows. In Sections 2 and 3, we introduce and study the class of pseudo upper semi B-Fredholm and pseudo lower semi B-Fredholm, and we show that the pseudo upper semi B-Fredholm and pseudo lower semi B-Fredholm spectra, for a bounded linear operator on a Banach space, are compact in the complex plane. Also, we prove equality up to $S(T)$ between the left generalized Drazin spectrum and the pseudo upper semi B-Fredholm spectrum. Some applications are given in Section 4. In Section 5, we prove that the left and the right generalized Drazin spectra of an operator are invariant under additive commuting power finite rank perturbations. Some sufficient conditions are given to assure that the pseudo upper semi B-Fredholm, pseudo lower semi B-Fredholm and pseudo B-Fredholm are stable under additive commuting power finite rank and nilpotent perturbations.

2. Preliminaries

More recently, B-Fredholm and B-Weyl operators were generalized to pseudo B-Fredholm and pseudo B-Weyl [9], [18]. Precisely, T is a pseudo B-Fredholm operator if there exists (X_1, X_2) a pair of T-invariant closed subspaces of X, such that $X = X_1 \oplus X_2$ and $T = T_1 \oplus T_2$ where $T_1 = T_{|X_1}$ is a Fredholm operator and $T_2 = T_{|X_2}$ is a quasi-nilpotent operator. The pseudo B-Fredholm spectrum is defined by

$$\sigma_{pBF}(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not pseudo B-Fredholm}\}.$$

An operator T is a pseudo B-Weyl operator if there exists (X_1, X_2) a pair of T-invariant closed subspaces of X, such that $X = X_1 \oplus X_2$ and $T = T_1 \oplus T_2$, where $T_1 = T_{|X_1}$ is a Weyl operator and $T_2 = T_{|X_2}$ is a quasi-nilpotent operator. The pseudo B-Weyl spectrum is defined by

$$\sigma_{pBW}(T) = \{\lambda \in \mathbb{C} : \; T - \lambda I \text{ is not pseudo B-Weyl}\}.$$

$\sigma_{pBW}(T)$ and $\sigma_{pBF}(T)$ is not necessarily non empty. For example, the quasi-nilpotent operator has empty pseudo B-Weyl and B-Fredholm spectrum. Evidentl, $\sigma_{pBF}(T) \subset \sigma_{pBW}(T) \subset \sigma(T)$.

In the following, we define the pseudo upper semi B-Fredholm, pseudo lower semi B-Fredholm and pseudo semi B-Fredholm of a bounded operator as a generalization of semi B-Fredholm and we give some fundamental results concerning these operators.

Definition 2.1 An operator $T \in \mathcal{B}(X)$ is said to be pseudo upper semi B-Fredholm if there exists two T-invariant closed subspaces X_1 and X_2 of X such that $X = X_1 \oplus X_2$ and $T_{|X_1}$ is upper semi-Fredholm operator and $T_{|X_2}$ is quasi-nilpotent.

Definition 2.2 An operator $T \in \mathcal{B}(X)$ is said to be pseudo lower semi B-Fredholm if there exists two T-invariant closed subspaces X_1 and X_2 of X such that $X = X_1 \oplus X_2$ and $T_{|X_1}$ is lower semi-Fredholm operator and $T_{|X_2}$ is quasi-nilpotent.

Definition 2.3 We say that $T \in \mathcal{B}(X)$ is pseudo semi B-Fredholm if T is pseudo lower semi B-Fredholm or pseudo upper semi B-Fredholm.

It is clear that T is a pseudo B-Fredholm operator if and only if T is a pseudo lower semi B-Fredholm operator and pseudo upper semi B-Fredholm operator.

The pseudo upper semi B-Fredholm, pseudo lower semi B-Fredholm and pseudo semi B-Fredholm spectra are defined by

$$\begin{aligned}
\sigma_{pBuF}(T) &= \{\lambda \in \mathbb{C} : \; T - \lambda I \text{ is not a pseudo upper semi B-Fredholm}\} \\
\sigma_{pBlF}(T) &= \{\lambda \in \mathbb{C} : \; T - \lambda I \text{ is not a pseudo lower semi B-Fredholm}\} \\
\sigma_{pBsF}(T) &= \{\lambda \in \mathbb{C} : \; T - \lambda I \text{ is not a pseudo semi B-Fredholm}\}
\end{aligned}$$

Therefore,

$$\sigma_{pBF}(T) = \sigma_{pBuF}(T) \cup \sigma_{pBlF}(T)$$

and

$$\sigma_{pBsF}(T) = \sigma_{pBuF}(T) \cap \sigma_{pBlF}(T)$$

It is easy to see that T is pseudo upper semi B-Fredholm if and only if T^* is pseudo lower semi B-Fredholm. Then:

$$\sigma_{pBuF}(T) = \sigma_{pBlF}(T^*) \text{ and } \sigma_{pBF}(T) = \sigma_{pBF}(T^*)$$

$\sigma_{pBsF}(T)$, $\sigma_{pBuF}(T)$ and $\sigma_{pBlF}(T)$ are not necessarily non empty. For example, the quasi-nilpotent operator has empty pseudo upper semi B-Fredholm, pseudo lower semi B-Fredholm and pseudo semi B-Fredholm spectrum.

Example 1 Let T_1 be defined on $l^2(\mathbb{N})$ by

$$T_0(x_1, x_2, ...) = (x_1, 0, x_2, 0, x_3, 0, ...)$$

T_0 is injective with closed range of infinite-codimension. Consider the operator T_2 defined on $l^2(\mathbb{N})$ as

$$T_2(x_1, x_2, ...) = (x_2/2, x_3/3, ...).$$

T_2 is a quasi-nilpotent operator. We have $T = T_0 \oplus T_2$ is a pseudo upper semi B-Fredholm operator. Note that $0 \in \sigma_{pBuF}(T_0)$, but $0 \notin \sigma_{pBlF}(T_0)$.

Example 2 Let T_1 be defined on $l^2(\mathbb{N})$ by:

$$T_1(x_1, x_2, ...) = (x_2, x_3, ...)$$

is surjective, but not injective, then is a lower semi-Fredholm operator. Let $T = T_1 \oplus T_2$, T_2 be as in Example 1. Then T is a pseudo lower semi B-Fredholm operator.

3. The class of pseudo semi B-Fredholm operators

Denote the open disc centered at λ_0 with radius ε in $\mathbb{C}$ by $D(\lambda_0, \epsilon)$ and

$$D^*(\lambda_0, \epsilon) = D(\lambda_0, \epsilon) \backslash \{\lambda_0\}.$$

The following theorem establishes that if T is a pseudo semi B-Fredholm operator, then $\lambda I - T$ is semi Fredholm in an open punctured neighborhood of 0.

Theorem 3.1 *Let $T \in \mathcal{B}(X)$ a pseudo semi B-Fredholm, then there exists a constant $\epsilon > 0$ such that $\lambda I - T$ is semi Fredholm for all $\lambda \in \mathbb{D}^*(0, \epsilon)$.*

Proof. If T is a pseudo semi B-Fredholm operator, then there exists two closed T-invariant subspaces X_1 and X_2 such that $X = X_1 \oplus X_2; T_{|X_1}$ is semi Fredholm, $T_{|X_2}$ is quasi-nilpotent and $T = T_{|X_1} \oplus T_{|X_2}$.

If $X_1 = \{0\}$, then T is quasi-nilpotent, then for all $\lambda \neq 0$ $\lambda I - T$ is invertible, hence $T - \lambda I$ is semi Fredholm for all $\lambda \neq 0$.

If $X_1 \neq \{0\}$, thus $T_{|X_1}$ is semi Fredholm, then there exists $\epsilon > 0$ such that $(T - \lambda I)_{|X_1}$ is semi Fredholm for all $\lambda \in D(0, \epsilon)$.

As $T_{|X_2}$ is quasi-nilpotent, then $(T - \lambda I)_{|X_2}$ is invertible for all $\lambda \neq 0$, hence $(T - \lambda I)_{|X_2}$ is semi Fredholm. Therefore, $(T - \lambda I)_{|X_1}$ is semi Fredholm for all $\lambda \in \mathbb{D}(0, \epsilon)$ and $(T - \lambda I)_{|X_2}$ is semi Fredholm for all $\lambda \neq 0$, hence $(T - \lambda I)$ is semi Fredholm for all $\lambda \in \mathbb{D}^*(0, \epsilon)$. ∎

From Theorem 3.1, we derive the following corollary.

Corollary 3.1 *Let $T \in \mathcal{B}(X)$, then $\sigma_{pBuF}(T), \sigma_{pBlF}(T), \sigma_{pBsF}(T)$ are compact subsets of $\mathbb{C}$.*

Moreover, $\sigma_{uF}(T) \setminus \sigma_{pBuF}(T)$, $\sigma_{lF}(T) \setminus \sigma_{pBlF}(T)$, $\sigma_{sF}(T) \setminus \sigma_{pBsF}(T)$ consist of at most countably many isolated points.

Since $\sigma_{pBuF}(T) \subset \sigma_{uBF}(T) \subset \sigma_{uF}(T)$ and $\sigma_{pBlF}(T) \subset \sigma_{lBF}(T) \subset \sigma_{lF}(T)$ the following corollary hold:

Corollary 3.2 *Let $T \in \mathcal{B}(X)$, then $\sigma_{uBF}(T) \setminus \sigma_{pBuF}(T)$, $\sigma_{lBF}(T) \setminus \sigma_{pBlF}(T)$ consist of at most countably many isolated points.*

Recall that $T \in \mathcal{B}(X)$ is said to be Kato operator or semi-regular if $R(T)$ is closed and $N(T) \subseteq R^{\infty}(T)$. Denote by $\rho_K(T) : \rho_K(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is Kato}\}$ the Kato resolvent and $\sigma_K(T) = \mathbb{C} \backslash \rho_K(T)$ the Kato spectrum of T. An operator $T \in \mathcal{B}(X)$ admit a generalized Kato decomposition, abbreviated as GKD if there exists two T-invariant closed subspaces X_1 and X_2 of X such that $X = X_1 \oplus X_2$ and $T_{\restriction X_1}$ is semi-regular (or a Kato) operator and $T_{\restriction X_2}$ is quasi-nilpotent. It is easy to see that every pseudo semi B-Fredholm is a pseudo Fredholm. According to [10, Theorem 2.2], the following proposition hold.

Proposition 3.1 *Let $T \in \mathcal{B}(X)$ a pseudo semi B-Fredholm operator. Then there exists a constant $\epsilon > 0$ such that $T - \lambda I$ is a Kato operator, for all $\lambda \in \mathbb{D}^*(0, \epsilon)$.*

As a consequence of the preceding Proposition, we have:

Corollary 3.3 *Let $T \in \mathcal{B}(X)$, $\sigma_K(T) \setminus \sigma_{pBsF}(T)$ consist of at most countably many isolated points.*

Define $pBuW(X)$ and $pBlW(X)$:

> $T \in pBuW(X)$ if there exists two T-invariant closed subspaces X_1 and X_2 of X where $X = X_1 \oplus X_2$ and $T_{\restriction X_1}$ is upper semi Fredholm of $ind(T_{\restriction X_1}) \leq 0$ and $T_{\restriction X_2}$ is quasi-nilpotent.
>
> $T \in pBlW(X)$ if there exists two T-invariant closed subspaces X_1 and X_2 of X where $X = X_1 \oplus X_2$ and $T_{\restriction X_1}$ is lower semi Fredholm of $ind(T_{\restriction X_1}) \geq 0$ and $T_{\restriction X_2}$ is quasi-nilpotent. The corresponding spectra of these sets are defined by:
>
> $$\sigma_{pBuW}(T) = \{\lambda \in \mathbb{C}, T - \lambda I \notin pBuW(X)\}$$
> $$\sigma_{pBlW}(T) = \{\lambda \in \mathbb{C}, T - \lambda I \notin pBlW(X)\}$$
>
> Then $\sigma_{pBuF}(T) \subseteq \sigma_{pBuW}(T) \subseteq \sigma_{lgD}(T)$ and $\sigma_{pBlF}(T) \subseteq \sigma_{pBlW}(T) \subseteq \sigma_{rgD}(T)$.

Remark 1 We have $\sigma_{pBuF}(T) \subset \sigma_{lgD}(T)$, this inclusion is proper. Indeed, let L be the unilateral left shift operator defined on the Hilbert $l^2(\mathbb{N})$:

$$L(x_1, x_2, ...) = (x_2, x_3, ...).$$

Since $\sigma_{ap}(L) = \{\lambda \in \mathbb{C}; |\lambda| \leq 1\}$, then

$$\sigma_{lgD}(L) = acc(\sigma_{ap}(L)) = \{\lambda \in \mathbb{C}; |\lambda| \leq 1\}$$

Observe that L is an upper semi-Fredholm operator, then $0 \notin \sigma_{pBuF}(L)$. This shows that the inclusion $\sigma_{pBuF}(T) \subset \sigma_{lgD}(T)$ is proper. Then it is naturel to ask about the defect set $\sigma_{lgD}(T) \backslash \sigma_{pBuF}(T)$, where T is a bounded operator.

In the following theorem, we get a characterization of this defect set.

Theorem 3.2 *Let $T \in \mathcal{B}(X)$. Then:*

$$\sigma_{lgD}(T) = \sigma_{pBuF}(T) \cup S(T) = \sigma_{pBuW}(T) \cup S(T)$$

Proof. $S(T) \subseteq \sigma_{lgD}(T)$ and $\sigma_{pBuF}(T) \subseteq \sigma_{lgD}(T)$, hence

$$\sigma_{pBuF}(T) \cup S(T) \subseteq \sigma_{lgD}(T).$$

Indeed, let $\lambda \notin \sigma_{lgD}(T)$. Then $T - \lambda I$ is a left generalized Drazin invertible, and then $H_0(T - \lambda I)$ is closed, by [3, Theorem 1.7] T has the SVEP at λ, hence $S(T) \subseteq \sigma_{lgD}(T)$.

Conversely, let $\lambda \notin \sigma_{pBuF}(T) \cup S(T)$, then $T - \lambda I$ is a pseudo upper semi B-Fredholm, then there exists two closed subspaces T-invariant X_1 and X_2 such that $X = X_1 \oplus X_2$, $(T - \lambda I)_{|X_1}$ is upper semi Fredholm and $(T - \lambda I)_{|X_2}$ is quasi-nilpotent. Since T has SVEP at λ then $T_{|X_1}$ and $T_{|X_2}$ have the SVEP at λ. Since $(T - \lambda I)_{|X_1}$ is upper semi Fredholm, then $(T - \lambda I)_{|X_1}$ admits a (GKD), since $T_{|X_1}$ has SVEP at λ, by [10, Theorem 3.5], we have $\lambda \notin acc(\sigma_{ap}(T_{|X_1})) = \sigma_{lgD}(T_{|X_1})$, then $(T - \lambda I)_{|X_1}$ is left generalized Drazin invertible, hence $X_1 = X_1' \oplus X_1''$ with $(T - \lambda I)_{|X_1'}$ is left invertible and $(T - \lambda I)_{|X_1''}$ is quasi-nilpotent, thus

$$X = X_1' \oplus X_1'' \oplus X_2$$

with $(T - \lambda I)_{|X_1'}$ is left invertible and $(T - \lambda I)_{|X_1'' \oplus X_2}$ is quasi-nilpotent, therefore, $T - \lambda I$ is left generalized Drazin invertible. ∎

By duality, we get a similar result for the right Drazin invertible spectrum.

Theorem 3.3 *Let $T \in \mathcal{B}(X)$. Then:*

$$\sigma_{rgD}(T) = \sigma_{pBlF}(T) \cup S(T^*) = \sigma_{pBlW}(T) \cup S(T^*)$$

Proof. $\sigma_{rgD}(T) = \sigma_{lgD}(T^*) = \sigma_{pBuF}(T^*) \cup S(T^*) = \sigma_{plBF}(T) \cup S(T^*)$, then

$$\sigma_{rgD}(T) = \sigma_{pBlF}(T) \cup S(T^*)$$ ∎

Remark 2 Theorems 3.2 and 3.3 extend [5, Theorem 2.1] and [5, Theorem 2.2].

From the preceding theorems, we get the following corollaries.

Corollary 3.4 *Let $T \in \mathcal{B}(X)$. Then*

$$\sigma_{gD}(T) = \sigma_{pBF}(T) \cup S(T) \cup S(T^*)$$

Corollary 3.5 *Let $T \in \mathcal{B}(X)$.*

If T has the SVEP then:

(1) $$\sigma_{lgD}(T) = \sigma_{pBuF}(T) = \sigma_{pBuW}(T)$$

If T^* has the SVEP then:

$$\sigma_{rgD}(T) = \sigma_{pBlF}(T) = \sigma_{pBlW}(T) \tag{2}$$

If both T and T^* have the SVEP, then all the spectra in (1) *and* (2) *coincide and are equal to the pseudo B-Fredholm, pseudo B-Weyl and generalized Drazin spectra.*

Example 3 Let T be the unilateral weighted shift on $l^2(\mathbb{N})$ defined by:

$$Te_n = \begin{cases} 0, & \text{if } n = p! \text{ for some } p \in \mathbb{N} \\ e_{n+1} & \text{otherwise.} \end{cases}$$

The adjoint operator of T is:

$$T^*e_n = \begin{cases} 0 & \text{if } n = 0 \ \textit{or} \ n = p! + 1 \text{ for some } p \in \mathbb{N} \\ e_{n-1} & \text{otherwise.} \end{cases}$$

We have $\sigma(T) = \overline{D(0,1)}$ the unit closed disc. The point spectrum of T and T^* are

$$\sigma_p(T) = \sigma_p(T^*) = \{0\}.$$

Hence T and T^* have the SVEP. Then $\sigma_{ap}(T) = \sigma_{su}(T) = \sigma(T)$, hence

$$\sigma_{gD}(T) = \sigma_{lgD}(T) = \sigma_{rgD}(T) = \overline{D(0,1)}$$

From Corollary 3.5, we have

$$\sigma_{pBuF}(T) = \sigma_{pBlF}(T) = \sigma_{pBF}(T) = \sigma_{gD}(T) = \sigma_{lgD}(T) = \sigma_{rgD}(T) = \overline{D(0,1)}$$

4. Applications

A bounded linear operator T is called supercyclic provided there is some $x \in X$ such that the set $\{\lambda T^n,\ \lambda \in \mathbb{C},\ n = 0, 1, 2, ...\}$ is dense in X. It is well known that, if T is supercyclic, then

$$\sigma_p(T^*) = \{0\} \quad \text{or} \quad \sigma_p(T^*) = \{\alpha\}$$

for some nonzero $\alpha \in \mathbb{C}$. Since an operator with countable point spectrum has SVEP, then we have the following:

Proposition 4.2 *Let $T \in \mathcal{B}(X)$, a supercyclic operator. Then:*

$$\sigma_{rgD}(T) = \sigma_{pBlF}(T)$$

Since every hyponormal operator T on a Hilbert space has the single valued extension property, we have

Proposition 4.3 *Let T a hyponormal operator on a Hilbert space. Then:*

$$\sigma_{lgD}(T) = \sigma_{pBuF}(T)$$

In particular, if T is auto-adjoint then

$$\sigma_{gD}(T) = \sigma_{pBF}(T).$$

Let A be a semi-simple commutative Banach algebra.

The mapping $T : A \longrightarrow A$ is said to be a multiplier of A if $T(x)y = xT(y)$ for all $x, y \in A$.

It is well known each multiplier on A is a continuous linear operator and that the set of all multiplier on A is a unital closed commutative subalgebra of $B(A)$ [11, Proposition 4.1.1]. Also the semi-simplicity of A implies that every multiplier has the SVEP (see [11, Proposition 2.2.1]). According to Corollary 3.5, we have

Proposition 4.4 *Let T be a multiplier on semi-simple commutative Banach algebra A, then the following assertions are equivalent*

(1) *T is pseudo upper semi B-Fredholm.*

(2) *T is left generalized Drazin invertible.*

Now, if assume, in additiona,l that A is regular and Tauberian (see [11, Definition 4.9.7]), then every multiplier T^* has SVEP. Hence, we have the following Proposition.

Proposition 4.5 *Let T be a multiplier on semi-simple regular and Tauberian commutative Banach algebra A, then the following assertions are equivalent*

(1) *T is pseudo B-Fredholm .*

(2) *T is generalized Drazin invertible.*

Let G a locally compact abelian group, with group operation $+$ and Haar measure μ, let $L^1(G)$ consist of all $\mathbb{C}$-valued functions on G integrable with respect to Haar measure and $M(G)$ the Banach algebra of regular complex Borel measures on G. We recall that $L^1(G)$ is a regular semi-simple Tauberian commutative Banach algebra. Then we have the following:

Corollary 4.6 *Let G be a locally compact abelian group, $\mu \in M(G)$. Then every convolution operator $T_\mu : L^1(G) \longrightarrow L^1(G)$, $T_\mu(k) = \mu \star k$ is pseudo B-Fredholm if and only if is generalized Drazin invertible.*

Remark 3 Proposition 4.5 and Corollary 4.6 generalize [5, Proposition 3.4] and [5, Corollary 3.3]. These results also generalize some results in [8].

5. Perturbation

Let $\mathcal{F}(X)$ denote the ideal of finite rank operators on X. In the following, we show that both $\sigma_{lgD}(T)$ and $\sigma_{rgD}(T)$ are stable under additive commuting power finite rank operator.

Proposition 5.6 *Suppose that $F \in \mathcal{B}(X)$ satisfies $F^n \in \mathcal{F}(X)$ for some $n \in \mathbb{N}$ and that $T \in \mathcal{B}(X)$ commutes with F. Then we have*

$$\sigma_{lgD}(T) = \sigma_{lgD}(T+F) \text{ and } \sigma_{rgD}(T) = \sigma_{rgD}(T+F)$$

Proof. According to [19, Theorem 2.2], we have $acc(\sigma_{ap}(T)) = acc(\sigma_{ap}(T+F))$. Then $\lambda \in \sigma_{lgD}(T)$ if and only if $\lambda \in acc(\sigma_{ap}(T))$ if and only if $\lambda \in acc(\sigma_{ap}(T+F))$ if and only if $\lambda \in \sigma_{lgD}(T+F)$. So $\sigma_{lgD}(T+F) = \sigma_{lgD}(T)$.
By duality, we have $\sigma_{rgD}(T) = \sigma_{rgD}(T+F)$. ■

As a consequence of Proposition 5.6, we have the following corollary.

Corollary 5.7 *Suppose that $F \in \mathcal{B}(X)$ satisfies $F^n \in \mathcal{F}(X)$ for some $n \in \mathbb{N}$ and that $T \in \mathcal{B}(X)$ commutes with F. Then we have*

$$\sigma_{gD}(T) = \sigma_{gD}(T+F)$$

The following example illustrates that the approximate point spectrum $\sigma_{ap}(.)$ in general is not preserved under commuting finite rank perturbations.

Example 4 Let $A \in \mathcal{B}(l^2)$ defined by:

$$A(x_1, x_2, ...) = (0, x_1, x_2, ...).$$

Let $0 < \varepsilon < 1$, $F_\varepsilon \in \mathcal{B}(l^2)$ be a finite rank operator defined by:

$$F_\varepsilon(x_1, x_2, ...) = (-\varepsilon x_1, 0, 0, ...).$$

Let $T = A \oplus I$ and $F = 0 \oplus F_\varepsilon$. Then F is a finite rank operator and $TF = FT$. But $\sigma_{ap}(T) = \{\lambda \in \mathbb{C}, |\lambda| = 1\}$, $\sigma_{ap}(T+F) = \{\lambda \in \mathbb{C}, |\lambda| = 1\} \cup \{1-\varepsilon\}$.

Using Corollary 3.5, Proposition 5.6 and Corollary 5.7, we can prove the following corollary.

Corollary 5.8 *Suppose that $F \in \mathcal{B}(X)$ satisfies $F^n \in \mathcal{F}(X)$ for some $n \in \mathbb{N}$ and that $T \in \mathcal{B}(X)$ commutes with F.*

(1) *If T has the SVEP, then $\sigma_{pBuF}(T) = \sigma_{pBuF}(T+F)$*

(2) *If T^* has the SVEP, then $\sigma_{pBlF}(T) = \sigma_{pBlF}(T+F)$*

(3) *If T and T^* have the SVEP, then $\sigma_{pBF}(T) = \sigma_{pBF}(T+F)$*

Let $T \in \mathcal{B}(X)$, Q a quasi-nilpotent such that $QT = TQ$, from [18, Proposition 2.9], we have $\sigma_{lgD}(T+Q) = \sigma_{lgD}(T)$ and $\sigma_{rgD}(T+Q) = \sigma_{rgD}(T)$.

Proposition 5.7 *Let $T \in \mathcal{B}(X)$, N a nilpotent operator commutes with T then:*

$$\sigma_{pBuF}(T+N) \cup S(T) = \sigma_{pBuF}(T) \cup S(T)$$

Proof. From Theorem 3.2, we have $\sigma_{lgD}(T) = \sigma_{pBuF}(T) \cup S(T)$, then

$$\sigma_{lgD}(T+N) = \sigma_{pBuF}(T+N) \cup S(T+N)$$

since $\sigma_{lgD}(T+N) = \sigma_{lgD}(T)$ and $S(T+N) = S(T)$. Hence

$$\sigma_{pBuF}(T+N) \cup S(T) = \sigma_{pBuF}(T) \cup S(T) \qquad \blacksquare$$

By duality, we have the following proposition.

Proposition 5.8 *Let $T \in \mathcal{B}(X)$, N a nilpotent operator commutes with T then:*

$$\sigma_{pBlF}(T+N) \cup S(T^*) = \sigma_{pBlF}(T) \cup S(T^*)$$

As a consequence of Proposition 5.7 and Proposition 5.8, the following corollaries hold.

Corollary 5.9 *Let $T \in \mathcal{B}(X)$, N a nilpotent operator commute with T then:*

$$\sigma_{pBF}(T+N) \cup S(T) \cup S(T^*) = \sigma_{pBF}(T) \cup S(T) \cup S(T^*)$$

Corollary 5.10 *Let $T \in \mathcal{B}(X)$, N a nilpotent operator commutes with T.*

(1) *If T has SVEP, then $\sigma_{pBuF}(T+N) = \sigma_{pBuF}(T)$*

(2) *If T^* has SVEP, then $\sigma_{pBlF}(T+N) = \sigma_{pBlF}(T)$*

(3) *If T and T^* have SVEP, then $\sigma_{pBF}(T+N) = \sigma_{pBF}(T)$*

Remark 4 Let $T \in \mathcal{B}(X)$, by the same argument of [18, Theorem 2.12] and [4, Theorem 4.1], we can prove that :

$$\bigcap_{F \in \mathcal{F}(X)} \sigma_{lgD}(T+F) \subseteq \sigma_{pBuW}(T)$$

$$\bigcap_{F \in \mathcal{F}(X)} \sigma_{rgD}(T+F) \subseteq \sigma_{pBlW}(T)$$

We would like to finish this work with the following questions.

Questions: Is it true that:

$$\bigcap_{F \in \mathcal{F}(X)} \sigma_{lgD}(T+F) = \sigma_{pBuW}(T)$$

$$\bigcap_{F \in \mathcal{F}(X)} \sigma_{rgD}(T+F) = \sigma_{pBlW}(T) \ ?$$

References

[1] Aiena, P., *Fredholm and Local Spectral Theory with Applications to Multipliers,* Kluwer.Acad.Press, 2004.

[2] Aiena, P., Miller, T., Neumann, M., *On a localized single-valued extension property,* Proc. R. Ir. Acad., 104 (1) (2004), 17-34.

[3] Aiena, P., Biondi, M.T., *Ascent, descent, quasi-nilpotent part and analytic core of operators,* Mat. Vesnik, (2002) 54, 57-70.

[4] Aiena, P., Sanabria, J.E., *On left and right poles of the resolvent,* Acta Sci. Math. (Szeged), 74 (2008), 669-687

[5] Amouch, M., Zguitti, H., *B-Fredholm and Drazin invertible operators through localized SVEP,* Math. Bohemica, 136 (2011), 39-49.

[6] Berkani, M., *On a class of quasi-Fredholm operators,* Integral Equations Operator Theory, 34 (1999), 244-249.

[7] Berkani, M., *Index of B-Fredholm operators and generalization of a Weyl theorem,* Proc. Amer. Math. Soc., 130 (2002), 1717-1723.

[8] Berkani, M., Arroud, A., *B-Fredholm and spectral properties for multipliers in Banach algebras,* Rend. Circ. Mat. Palermo, 55 (2006), 385-397.

[9] Boasso, E., *Isolated spectral points and Koliha-Drazin invertible elements in quotient Banach algebras and homomorphism ranges,* Mathematical Proceedings of the Royal Irish Academy, vol. 115A, no. 2 (2015), 1-15.

[10] Jiang,Q., Zhong, H., *Generalized Kato decomposition, single-valued extension property and approximate point spectrum,* J. Math. Anal. Appl., 356 (2009), 322-327.

[11] Laursen,K.B., Neumann, M.M., *An introduction to Local Spectral Theory,* London Mathematical Society Monograph, New series, vol. 20, Clarendon Press, Oxford, 2000.

[12] Lay, D., Taylor, A., *Introduction to functional analysis,* J. Wiley and Sons, New York, 1980.

[13] Koliha, J.J., *A generalized Drazin inverse,* Glasgow Math. J., 38 (1996), 367-81.

[14] Kouider, M.H., Benharrat, M., Messirdi, B., *Left and right generalized Drazin invertible operators,* Linear and Multilinear Algebra, 63 (8) (2015), 1635-1648.

[15] Müller, V., Mbekhta, M., *On the axiomatic theory of spectrum II,* Studia Mathematica, 119 (1996), 129-147.

[16] Müller, V., *Spectral Theory of Linear Operators and Spectral Systems in Banach Algebras,* 2nd edition. Oper. Theory Advances and Applications, vol. 139, 2007.

[17] Schmoeger, C., *On isolated points of the spectrum of a bounded linear operator,* Proc. Amer. Math. Soc., 117 (1993), 715-719.

[18] Zariouh, H., Zguitti, H., *On pseudo B-Weyl operators and generalized drazin invertible for operator matrices,* Linear and Multilinear Algebra, 64 (7) (2016), 1245-1257.

[19] Zeng, Q., Zhong, H., Yan, K., *An extension of a result of Djordjevic and its applications*, Linear and Multilinear Algebra, 64 (2) (2016), 247-257.

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INJECTIVITY OF G-NOMINAL SETS

Khadijeh Keshvardoost

Department of Mathematics
Shahid Beheshti University, G.C.
Tehran 19839
Iran
e-mail: kh_keshvardoost@sbu.ac.ir

Abstract. In this paper, we consider injectivity for nominal sets, G-sets for a subgroup G of the group $\mathrm{Perm}(\mathbb{D})$ of permutations on an infinite countable set $\mathbb{D}$, with finitely supported elements. Also, we study injectivity of nominal sets with respect to monomorphisms whose domains or codomains are single-orbit. Furthermore, we define the category of G_C-sets, for finite subsets C of $\mathbb{D}$, where G_C is the set of elements of G fixing the elements of C, and study injectivity in such categories.

Keywords: S-set, nominal set, single-orbit nominal set, injectivity.
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1. Introduction and preliminaries

The theory of nominal sets originates from the work of Fraenkel in 1922, and developed by Mostowski in the 1930s, and it is also known as the FM set theory. The FM set theory is an axiomatic set theory which provides a mathematical model for names in syntax. At that time, they used nominal sets to prove the independence of the axiom of choice with the other axioms (in the classical Zermelo-Fraenkel (ZF) set theory). In computer science, Gabbay and Pitts rediscovered nominal sets to properly model the syntax of formal systems involving variable binding operations, see [13]. Since then, nominal sets have become a lively topic in semantics. Bojanczyk et al., used nominal sets in automata theory as a framework for describing automata on data words [3]. Bojanczyk defined the monoids in the category of nominal sets (also called nominal monoids) and used them in the study of languages over infinite alphabets [3]. The theory of syntactic monoids for languages of data words represents the same theory as the theory of finite monoids in the category of nominal sets, and under certain conditions, a language of data words is definable in first-order logic if and only if its syntactic monoid is aperiodic, [4]. Shinwell and Pitts used nominal techniques in order to implement a functional programming language incorporating facilities for manipulating syntax involving names and binding operations [20]. Bojanczyk et al., have also defined computation in nominal sets by presenting a basic functional programming language, [5]. Alexandru and Ciobanu have presented nominal algebraic structures in terms of finitely supported objects, [2]. Nominal techniques have also been used in game theory [1], in logic ([12],[16]), in topology [15], in domain theory ([19],[21]), and in proof theory [22]. Also, in [17], nominal sets have been used to model language constructs for hiding the identity of a name outside a given scope. Because of the importance of nominal sets we are going to study injectivity, which is one of the central notions in many branches of mathematics, in this category. For injectivity, see, for example, [10, 8, 14].

1.1. The category G-Set

In this subsection, we briefly recall the ingredients of the basic category G-**Set**, of sets with the actions of a group G on them, which are needed in this paper to define nominal sets. For more information see [9], [14].

Definition 1.1. Let G be a group (or a monoid) with 1 as its identity. Then, a set X together with a function

$$\lambda : G \times X \to X,$$

called the *action* of G on X (or the G-action), is called a (left) G-*set* if (denoting $\lambda(g, x)$ by gx) $(gh)x = g(hx)$ and $1x = x$, for every $x \in X$ and $g, h \in G$.

Remark 1.2. A G-set X can be regarded as a unary algebra with the family of unary operations

$$L_g : X \to X, \ L_g(x) = gx,$$

for $g \in G$, called the *left translations*, such that $L_s \circ L_t = L_{st}$ and $L_1 = id_X$.

Example 1.3.

1. For a group G, any set X with the identity action of G on it is a G-set, called a *discrete* G-set.
2. Each group G can clearly be considered as a G-set with the action given by its binary operation $G \times G \to G$. Note that the unary algebra related to this G-set is $(G; (L_g)_{g \in G})$, where $L_g : G \to G$ is defined by $L_g(h) = gh$.
3. **This is a prime example in this paper**. Let $\mathbb{D}$ be an infinite countable set and $G = \mathrm{Perm}\mathbb{D}$ be the group of all permutations (bijective maps) on $\mathbb{D}$. Then, $\mathbb{D}$ itself is a G-set, with the canonical action as the evaluation, defined by
$$(\forall \pi \in \mathrm{Perm}\mathbb{D}) \ (\forall d \in \mathbb{D}), \quad \pi \cdot d = \pi(d).$$

Definition 1.4.

1. A *homomorphism* (also called an *equivariant map* or a *G-map*) from a G-set X to a G-set Y is a function f from X to Y such that for each $x \in X$ and $g \in G$, $f(gx) = gf(x)$.

2. Let Y be a subset of a G-set X. We say that Y is a *sub-G-set* of X if for all $\pi \in G$, $\pi Y \subseteq Y$; which is equivalent to $\pi Y = Y$, since G is a group.

Since id_X and the composition of two G-maps are clearly G-maps, we have the category G-**Set** of all G-sets and G-maps between them.

Notice that, the class of all G-sets is equational, and so the category G-**Set** is complete and cocomplete (that is, has all products, equalizers, pullbacks, coproducts, coequalizers, and pushouts). In fact, limits and colimits in this category are computed as in the category **Set** of sets, with natural actions. Also, monomorphisms (epimorphisms) in G-**Set** are exactly one-one (onto) G-maps. Therefore, we do not distinguish between monomorphisms of G-sets and inclusions, and call a G-set X containing (an isomorphic copy of) a G-set Y an extension of X (for more information, see [9], [14]).

We now recall some other notions about G-sets X, needed in the sequel.

Definition 1.5.

1. Let X be a G-set. For each $x \in X$, the set

$$G_x \doteq \mathrm{fix}_G x = \{g \in G \mid gx = x\}$$

is a subgroup of G, called the *fixed* or the *stabilizer* subgroup of x in G. Also, for $C \subseteq X$,

$$G_C \doteq \mathrm{Fix}_G C = \{g \in G \mid (\forall x \in C)\ gx = x\} = \bigcap_{x \in C} G_x$$

is the *fixed* or the *stabilizer* of C in G.

2. An element x of a G-set X is called a *zero element* if for all $g \in G$, $gx = x$; equivalently, $G_x = \mathrm{fix}_G x = G$.

3. The class $[x] = \{gx \mid g \in G\} = Gx$ of the equivalence relation

$$x \sim x' \text{ if there exists } g \in G \text{ such that } gx = x'.$$

is called the *orbit* of x and is denoted by $\mathrm{Orb}_G x$ or $\mathrm{Orb}\ x$, if G is known.

4. A G-set X is called *orbit finite* if $X/\sim$ is finite and it is called *single-orbit* if $X/\sim$ is singleton.

5. A G-set X is called *indecomposable* G-set if it is not the coproduct (disjoint union) of two proper sub-G-sets.

Remark 1.6. Recall that for a category $\mathfrak{C}$ and a subclass $\mathcal{M}$ of monomorphisms in $\mathfrak{C}$, the category $\mathfrak{C}$ is said to satisfy the $\mathcal{M}$-*transferability property* if for all $f \in \mathfrak{C}$ and $m \in \mathcal{M}$ with a common domain there is a commutative diagram

$$\begin{array}{ccc} \bullet & \xrightarrow{m} & \bullet \\ f\big\downarrow & & \big\downarrow g \\ \bullet & \xrightarrow{u} & \bullet \end{array}$$

with $u \in \mathcal{M}$.

Notice that, since pushouts exist in the category M-**Set**, the above condition is equivalent to pushouts transfer monomorphisms; that is, the pushout map corresponding to a morphism in $\mathcal{M}$ again belongs to $\mathcal{M}$. For more information see [9], [11], [14].

2. The categories of G-nominal sets and $G_{\mathcal{P}_f(\mathbb{D})}$-nominal sets

As before, let $\mathbb{D}$ be an infinite countable set, whose elements are called *directions* (or *atomic names*, *data values*) and Perm$\mathbb{D}$ be the group of all permutations (bijection maps) on $\mathbb{D}$. A permutation $\pi \in \mathrm{Perm}\mathbb{D}$ is *finitary* if $\{d \in \mathbb{D} \mid \pi d \neq d\}$ is finite. Clearly, the set $\mathrm{Perm}_f\mathbb{D}$ of all finitary permutations is a subgroup of Perm$\mathbb{D}$. *In this paper we* **fix** *the infinite countable set* $\mathbb{D}$ *and a subgroup* G *of* Perm$(\mathbb{D})$*, which is denoted by* $G \leq \mathrm{Perm}(\mathbb{D})$.

Recall (Example 1.3(3)) that the set $\mathbb{D}$ is itself a G-set with the *canonical action* given by the evaluation

$$\forall \pi \in G,\ d \in \mathbb{D}, \pi \cdot d = \pi(d).$$

We denote this action by $\pi \cdot d = \pi(d)$ and if X is any other G-set, we use the usual juxtaposition notation πx for the action of $\pi \in G$ on $x \in X$. The following definition gives the interplay between these two actions by introducing the notion of a, so called, *support*, which is the centeral notion to define *G-nominal sets.*

Definition 2.1. Let $G \leq \mathrm{Perm}\mathbb{D}$. Consider $\mathbb{D}$ to be the canonical G-set, and let X be any G-set. A subset $C \subseteq \mathbb{D}$ is called "a" G-*support* for an element $x \in X$ if whenever $\pi \in G$ fixes all the elements of C then it fixes x. That is,

$$\mathrm{Fix}_G C = G_C \subseteq G_x = \mathrm{fix}_G x,$$

or, in other words, for all $\pi \in G$,

$$(\forall c \in C, \pi \cdot c = \pi(c) = c) \quad \Rightarrow \quad \pi x = x.$$

If C is finite (possibly empty) then we say that x is *finitely G-supported.*

Remark 2.2.

1. Let X be a G-set and x a finitely G-supported element of X with a support C. Then $\{x\}$ is a G_C-set.
2. If H is a subgroup of G and X a G-set then X is also an H-set.
3. Let C and B be two finite subsets of $\mathbb{D}$ and $C \subseteq B$. Then G_B is a subgroup of G_C.

Definition 2.3.

1. Consider a finite subset $C \subseteq \mathbb{D}$. A *G_C-nominal set* is a G_C-set every element of which has some finite support. In other words, for all $x \in X$, there exists a finite $E \subseteq \mathbb{D}$ such that
$$\mathrm{Fix}_{G_C} E \subseteq \mathrm{fix}_{G_C} x.$$
If $C = \emptyset$ then $G_C = G_\emptyset = G$ and a *G_C-nominal set* is called a G-nominal set.
2. Let X be a G_C-nominal set and Y a G_B-nominal set. A map $f : X \to Y$, for which there is a finite set $E \supseteq C \cup B$ such that for every $\pi \in G_E$ and $x \in X$ we have $f(\pi x) = \pi f(x)$, is called a *G_E-nominal map* or simply a *nominal map.* Notice that since $E \supseteq C \cup B$, by Remark 2.2, we get that X and Y are G_E-sets.
3. The class of G-nominal sets together with G-maps between them form a category, denoted by G-**Nom**.

Let $\mathcal{P}_{\mathrm{f}}(\mathbb{D})$ be the set of all finite subsets of $\mathbb{D}$. In the following lemma it is shown that the class of all G_C-nominal sets, $C \in \mathcal{P}_{\mathrm{f}}(\mathbb{D})$ and the maps between them (see Definition 2.3(2)) form a category, denoted by $G_{\mathcal{P}_{\mathrm{f}}(\mathbb{D})}$-**Nom**. Also, the category G-**Nom** is a full subcategory of $G_{\mathcal{P}_{\mathrm{f}}(\mathbb{D})}$-**Nom**.

Lemma 2.4. *The class $\{G_C$-nominal sets $\}_{C \in \mathcal{P}_{\mathrm{f}}(\mathbb{D})}$ and the morphisms between them form a category.*

Proof. Let A be a finite subset of $\mathbb{D}$ and X a G_A-nominal set. Then, clearly the identity map $id_X : X \to X$ is a G_A-nominal map. Now, let Y be a G_B-nominal set and Z a G_C-nominal set. Suppose $f : X \longrightarrow Y$ is a G_{D_1}-nominal map and $g : Y \longrightarrow Z$ a G_{D_2}-nominal map where D_1 and D_2 are finite subsets which contain $A \cup B$ and $B \cup C$, respectively. Now, let $E = D_1 \cup D_2$. Then, E is a finite set and for all $\pi \in G_E$, we have
$$\begin{aligned} \pi(gf)(x) &= g(\pi f(x)) \\ &= g(f(\pi x)), \end{aligned}$$
where the first equality is because $G_E \subseteq G_{D_2}$ and the second one is because $G_E \subseteq G_{D_1}$ (see Remark 2.2(3)). ■

Example 2.5.

1. As we said above, the set $\mathbb{D}$ is itself a G-set. In this case, for every $d \in \mathbb{D}$, $\operatorname{supp} d = \{d\}$.

2. The set $\mathrm{Perm_f}(\mathbb{D})$, with the action by conjugation $(\pi, \pi') \rightsquigarrow \pi\pi'\pi^{-1}$ and $\operatorname{supp} \pi = \{d \in \mathbb{D} | \ \pi d \neq d\}$, is a G-nominal set.

3. The set $\mathbb{D} \setminus \{d\}$ where $d \in \mathbb{D}$, is a $G_{\{d\}}$-nominal set.

4. Note that, if X is a G-set, then the power set $\mathcal{P}(X) = \{Y | \ Y \subseteq X\}$ is also a G-set with the natural G-action

$$* : G \times \mathcal{P}(X) \to \mathcal{P}(X), \ \pi * Y \doteq \pi Y = \{\pi y | \ y \in Y\}.$$

Now, if X is a G-nominal set, we denote by $\mathcal{P}_{\mathrm{fs}}(X)$ the set of all subsets Y of X which are finitely G-supported relative to the action $*$. Thus, $Y \in \mathcal{P}_{\mathrm{fs}}(X)$ if and only if

$$(\exists C \subseteq \mathbb{D}), \text{ if } \pi \in G_C \text{ then } \pi * Y = \pi Y = Y,$$

which is again equivalent to Y being a G_C-set.

Clearly, $\mathcal{P}_{\mathrm{fs}}(X)$ is a G-set, under the restriction of the action $*$ to it and, since every element of this G-set is finitely G-supported, it is itself a G-nominal set. Thus, finitely supported subsets of X are the elements of $\mathcal{P}_{\mathrm{fs}}(X)$.

4. In particular, the set $\mathcal{P}_{\mathrm{f}}(\mathbb{D})$, of finite subsets of $\mathbb{D}$, is a G-nominal set with the above action $\pi * C = \{\pi \cdot c = \pi(c) | \ c \in C\}$ and $\operatorname{supp} C = C$, for $C \in \mathcal{P}_{\mathrm{f}}(\mathbb{D})$.

Definition 2.6.

1. Let Y be a G_C-nominal set and $X \subseteq Y$. If there is some finite set $E \supseteq C$ such that for all $\pi \in G_E$ we have $\pi X = X$ then we say that X is a *sub-G_E-nominal set of Y*. Notice that G_E-nominal map $X \hookrightarrow Y$, is called *G_E-inclusion nominal map* and denoted by i. Clearly, every G-subset X of a G-nominal set Y is a G-nominal set, called a *sub-G-nominal set* of Y, where $\operatorname{supp}_Y y = \operatorname{supp}_X y$, for $y \in X$.

2. Let X be a G-nominal set. A subset $Y \subseteq X$ is *uniformly G-supported* if there exists a finite subset $C \subseteq \mathbb{D}$ which supports each $x \in Y$.

3. An element of a G-set X which is G-supported by the empty set $\emptyset$ is a zero element.

Example 2.7. Suppose X is a G_C-nominal set and Y a G_B-nominal set. Consider a G_E-nominal map $f : X \to Y$, where $E \supseteq B \cup C$. Let $\ker f = \{(x, x') \in X \times X \mid f(x) = f(x')\}$. Then, $\ker f$ is a sub-G_E-nominal set of X. This is because, for all $\pi \in G_E$ we have $f(\pi x) = \pi f(x) = \pi f(x') = f(\pi x')$.

Lemma 2.8. *The nominal set X is indecomposable if and only if it is single-orbit.*

Proof. Let $G \leq \mathrm{Perm}\mathbb{D}$ and X be an indecomposable nominal set and $x \in X$. Then $Gx \subseteq X$. We show $X = Gx$. Suppose $y \in X \setminus (Gx)$. Thus for all $\pi \in G$, we have $\pi y \in X \setminus (Gx)$. Since if there is π in G such that $\pi y \in Gx$, then $y \in Gx$, which is impossible. Therefore, $X = Gx \cup (X \setminus Gx)$, which contradicts by the assumption that X is indecomposable. The converse is clear. ∎

Remark 2.9.

1. Let X be a G-set and $x \in X$. If there exists a finite set G-supporting x, then there exists a least finite one, with respect to the subset inclusion (see [17]). This G-support is called "the" *G-support* of x and is denoted by $\mathrm{supp}_X x$, or simply by $\mathrm{supp}\, x$. In fact,
$$\mathrm{supp}_X\, x = \bigcap \{C \subseteq \mathbb{D} |\ C \text{ is a finite support of } x\}.$$
2. If X and Y are nominal sets, so is their cartesian product $X \times Y$ as a G-sets and with
$$\mathrm{supp}_{X\times Y}\,(x, y) = \mathrm{supp}_X\, x \,\cup\, \mathrm{supp}_Y\, y.$$
3. Every G-nominal set is also isomorphic to the coproduct of the single-orbit sub-G-nominal sets $O \in X/G$. That is, $X \cong \coprod_{O\in X/G} O$ (see [17]).

3. Injectivity in G-Nom

It is well-known (see, for example, Exercise 1.9 of [14]) that a G-set is injective if and only if it has a zero element. We show that the same holds for the injectivity of G-nominal sets with respecct to monomorphisms (see Theorem 3.3). In this section, we also study the injectivity of G-nominal sets with respect to monomorphims into single-orbit G-nominal and monomorphisms of single-orbit G-nominal sets as their domains and show that every G-nominal set is injective with respect to these classes of monomorphisms(see Theorem 3.9 and Lemma 3.6).

First, we recall some facts needed in this section. In the category of G-nominal sets, monomorphisms are injective maps (see [17]). Pushouts are calculated as in the category G-**Set**. In fact, consider the following pushout situation in the category of G-nominal sets,

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ {\scriptstyle f}\downarrow & & \\ B & & \end{array} \qquad \text{(I)}$$

where A, B, C are G-nominal sets and f, g are equivariant maps. Notice that the coproduct of two G-nominal sets is a G-nominal set, so $B \sqcup C \in$G-**Nom** (see Section 2.2 of [17]). Now, take $O = (B \sqcup C)/\sim$ where $\sim$ is the congruence relation on $B \sqcup C$ generated by all pairs $(\tau_B f(a), \tau_C g(a))$, $a \in A$, $\tau_B : B \to B \sqcup C$ and $\tau_C : C \to B \sqcup C$. Since $B \sqcup C \in$ G-**Nom** and $\sim$ is a congruence on $B \sqcup C$, O is also a G-nominal set. And therefore, as a G-**Set** (see [14, 9]), O is the pushout of the diagram (I).

Also, recall the following:

Definition 3.1.

1. For a subclass $\mathcal{M}$ of monomorphisms in a category $\mathfrak{C}$, an object $Z \in \mathfrak{C}$ is called $\mathcal{M}$-*injective* if for each $\mathcal{M}$-*morphism* $i : X \to Y$ and any morphism $f : X \to Z$ there exists a morphism $g : Y \to Z$ such that $gi = f$:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & Y \arrow[dl, "g"] \\ Z & \end{tikzcd}$$

   If $\mathcal{M}$ is the class of monomorphisms, then $\mathcal{M}$-injective objects are simply called *injective* objects.

2. An object Z of a category $\mathfrak{C}$ is called $\mathcal{M}$-*absolute retract* if it is a retract of each of its $\mathcal{M}$-extensions; that is, for each $\mathcal{M}$-morphism $f : Z \to Y$ there exists a morphism $h : Y \to Z$ such that $hf = id_Z$, in which case h is said to be a *retraction*.

In the following, we recall a fact from the literature about G-sets (see [11], [14]).

Remark 3.2. Every injective G-nominal set contains a zero element (see [14]).

Theorem 3.3.

(i) *Every G-nominal set is injective if and only if it contains a zero element.*

(ii) *A single-orbit G-nominal set is injective if and only if it is trivial* (*singleton*).

(iii) *A uniformly supported G-nominal set is injective if and only if it is discrete.*

(iv) *Injectivity is equivalent to absolute retractness.*

Proof. (i) Suppose Z is a G-nominal set with zero θ. Also, suppose X is a sub-G-nominal set of a nominal set Y. Consider inclusion map $i : X \hookrightarrow Y$ and equivariant map $f : X \to Z$. We show there is an equivariant map $g : Y \to Z$ such that $gi = f$. Define

$$g(y) = \begin{cases} f(y) & \text{if} \quad y \in X \\ \theta & \text{if} \quad y \notin X \end{cases}$$

Notice that $Y \setminus X$ is a sub-G-nominal set of Y. To see this, let $y \in (Y \setminus X)$, and $\pi \in G$. Then $\pi y \in Y \setminus X$. If there is $\pi \in G$ such that $\pi y \in Y$ then

$y = \pi^{-1}(\pi y) \in Y$, which is impossible. Therefore g is a required equivariant map and $gi = f$. The converse holds by Remark 3.2.

(ii) Let X be a single-orbit G-nominal set. Then $X = Gx$, where $x \in X$. Now, by (i), X is injective if and only if X has a zero element θ. Therefore X is injective if and only if $X = G\theta = \{\theta\}$.

(iii) Suppose X is an injective uniformly supported G-nominal set. By (i), X has a zero θ. Since X is uniformly supported, the support of all elements of X is empty. Thus X is discrete. The converse is clear.

(iv) See Remark 1.6 and Proposition 1.2 of [11]. ∎

Lemma 3.4. *For every equivariant map $f : X \longrightarrow Y$ from a single-orbit G-nominal set X to a G-nominal set Y, $f(X)$ is a single-orbit G-nominal set.*

Proof. Note that $f(X)$ is a sub-G-nominal set of X. Now, let $y_1, y_2 \in f(X)$. Then there are $x_1, x_2 \in X$ such that $y_1 = f(x_1)$, and $y_2 = f(x_2)$. Since X is a single-orbit G-nominal set and $x_1, x_2 \in X$, there exists $\pi \in G$ such that $x_1 = \pi x_2$. Hence $y_1 = f(x_1) = f(\pi x_2) = \pi f(x_2) = \pi y_2$, which means that $f(X)$ is single-orbit. ∎

Lemma 3.5. *The only sub-G-nominal sets of a single-orbit G-nominal set X are X and the empty set.*

Proof. Let Y be a single-orbit G-nominal set. Then $Y = Gy$, for $y \in Y$. Suppose X is a non-empty sub-G-nominal set of Y. We show $Y \subseteq X$. Let $z \in Y$. Then there exists $\pi \in G$ such that $z = \pi\, y$. Let $x \in X$. Since $X \subseteq Y$, we get $x \in Y$ and there exists $\pi' \in G$ such that $x = \pi'\, y$. Now, we have $z = \pi\, y = \pi\,(\pi'^{-1}\, x) = (\pi\pi'^{-1})\, x$ and therefore $z \in X$. Thus $X = Y$. ∎

Lemma 3.6. *Every G-nominal set is injective with respect to monomorphisms into single-orbit G-nominal sets.*

Proof. Let Z be a G-nominal set, Y a single-orbit G-nominal set, and X a sub-G-nominal set of Y. Consider the inclusion map $i : X \hookrightarrow Y$. Also, consider the equivariant map $f : X \to Z$. We prove that there exists an equivariant map $g : Y \to Z$ such that $gi = f$. First, notice that since Y is a single-orbit G-nominal set, by Lemma 3.5, we get $X = Y$. Hence taking $g = f$, we get the required (needed) equivariant map. ∎

Lemma 3.7. *Let $f : X \to Y$ be an equivariant map from a single-orbit G-nominal set X to a G-nominal set Y and Z be a sub-G-nominal set of Y. Then* $\mathrm{Imf} \subseteq Z$ *or* $Z \cap \mathrm{Imf} = \emptyset$.

Proof. Let $Z \cap \mathrm{Imf} \neq \emptyset$. Then we show $\mathrm{Imf} \subseteq Z$. Suppose $y \in \mathrm{Imf}$. So there exists $x' \in X$ such that $y = f(x')$. Since $Z \cap \mathrm{Imf} \neq \emptyset$, there is $z \in Z$ and $x \in X$ such that $z = f(x)$. By the assumption, X is a single-orbit G-nominal set and $x, x' \in X$. So there exists $\pi \in G$ such that $\pi x = x'$. Now, $y = f(x') = f(\pi x) = \pi f(x) = \pi z$. Hence $y \in Z$, since Z is a sub-G-nominal set of Y and $z \in Z$. ∎

Lemma 3.8. *Suppose X is a single-orbit sub-G-nominal set of the coproduct of single-orbit G-nominal sets Y_i. Then there exists a unique Y_i such that $X = Y_i$.*

Proof. Suppose $Y = \coprod_{i \in I} Y_i$, and $j : X \hookrightarrow Y$ is an inclusion. We show that there is a unique i_0 such that $X = Y_{i_0}$. Suppose $X \subseteq \bigcup_{j=1}^{l} Y_{i_j}$, where $l \in I$. So, $X = \bigcup_{j=1}^{l}(X \cap Y_{i_j})$, which is impossible. Since X is single-orbit and, by Lemma 2.8, X is an indecomposable G-nominal set. Therefore there is a unique i_0 in I such that $X \subseteq Y_{i_0}$. Now, by Lemma 3.5, $X = Y_{i_0}$. ∎

Theorem 3.9. *Every G-nominal set is injective with respect to monomorphisms with single-orbit G-nominal sets as their domains.*

Proof. Consider the following diagram,

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & Y \\ Z & \end{tikzcd}$$

where X is a single-orbit sub-G-nominal set of Y, Z is a G-nominal set and f is an equivariant map. We prove there exists an equivariant map $g : Y \to Z$ such that $gi = f$. By Remark 2.9(3), $Y \cong \coprod_i Y_i$, where Y_i's are single-orbit sub-G-nominal sets of Y. So the above diagram can be drawn as:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & Y \arrow[r, "k"] & \coprod_{i \in I} Y_i \\ Z & & \end{tikzcd}$$

where k is an isomorphism. Since k is a monomorphism, and monomorphisms are injective maps, so $(ki)(X) \cong X$. Therefore we have:

$$\begin{tikzcd} X \arrow[r, "l"] \arrow[d, "f"'] & (ki(X)) \arrow[r, "j"] & \coprod_{i \in I} Y_i \\ Z & & \end{tikzcd}$$

where l is an isomorphism, j is an inclusion and $ki = jl$. Also, X is a single-orbit sub-G-nominal set of Y and $k \circ i$ is an equivariant map, so, by Lemma 3.4, $k(i(X))$ is a single-orbit sub-G-nominal set of Y. Thus, by Lemma 3.8, $k(i(X)) = k(X)$ is equal to a Y_{i_0}, for some $i_0 \in I$. Now, we define $h : Y \to Z$ as follows:

$$h(y) = \begin{cases} (fl^{-1})(y) & \text{if } \; y \in Y_{i_0} \\ \theta & \text{if } \; y \in (\coprod_{i \in I} Y_i) \setminus Y_{i_0} \end{cases}$$

Notice that, since $(\coprod_{i \in I} Y_i) \setminus Y_{i_0}$ is a G-nominal set, h is an equivariant map and $hj = fl^{-1}$. Take $g = hk$. Thus g is an equivariant map and $gi = (hk)i = h(ki) = h(jl) = (fl^{-1})l = f$, as required. ∎

4. Injectivity in the category $G_{\mathcal{P}_{\mathrm{f}}(\mathbb{D})}$-Nom

In this section, we study injectivity in the categroy $G_{\mathcal{P}_{\mathrm{f}}(\mathbb{D})}$-**Nom**. The category of G-**Nom** is a full subcategory of $G_{\mathcal{P}_{\mathrm{f}}(\mathbb{D})}$-**Nom**.

Lemma 4.1. *Monomorphisms in the category* $G_{\mathcal{P}_{\mathrm{f}}(\mathbb{D})}$-**Nom** *are exactly injective morphisms.*

Proof. Suppose X is a G_C-nominal set and Y a G_B-nominal set. Also, suppose $f : X \longrightarrow Y$ is a monomorphism in the category $G_{\mathcal{P}_{\mathrm{f}}(\mathbb{D})}$-**Nom**. We show f is an injective map.

Consider $\ker f = \{(x, x') \in X \times X \mid f(x) = f(x')\}$. By Example 2.7, $\ker f$ is a sub-G_E-nominal set of X, where E is a finite set which contains $C \cup B$. Consider the projection map $p_x : \ker f \to X$ and $p_{x'} : \ker f \to X$ where $p_x(x, x') = x$, and $p_{x'}(x, x') = x'$. It is clear that p_x and $p_{x'}$ are G_E-nominal maps. Let $x, x' \in \ker f$. Then $f(x) = f(x')$. Now, we have $f(p_x(x, x')) = f(x) = f(x') = f(p_{x'}(x, x'))$. Since f is a monomorphism, we get $x = p_x(x, x') = p_{x'}(x, x') = x'$. Therefore $\ker f = \Delta$, where $\Delta \doteq \{(x, x) \in X \times X\}$. So f is an injective map. ∎

For the counterpart of this lemma in the category of G-**Nom**, where $G = \mathrm{Perm}_{\mathrm{f}}(\mathbb{D})$, see [23].

Lemma 4.2. *Isomorphisms in the category* $G_{\mathcal{P}_{\mathrm{f}}(\mathbb{D})}$-**Nom** *are exactly bijective maps.*

Proof. Let X be a G_C-nominal set and Y a G_B-nominal set. Also, let $f : X \longrightarrow Y$ be an isomorphism in $G_{\mathcal{P}_{\mathrm{f}}(\mathbb{D})}$-**Nom**. Then there exists a map $g : Y \longrightarrow X$ such that $gf = id_X$ and $fg = id_Y$. We show g is a morphism in $G_{\mathcal{P}_{\mathrm{f}}(\mathbb{D})}$-**Nom**. Since f is a morphism in $G_{\mathcal{P}_{\mathrm{f}}(\mathbb{D})}$-**Nom**, there exists a finite subset E containing $C \cup B$ such that for all $x \in X$ and $\pi \in G_E$ we have $f(\pi x) = \pi f(x)$. Suppose $\pi \in G_E$ and $y \in Y$. Since f is surjective, there exists an element $x \in X$ such that $f(x) = y$. Now, for every $\pi \in G_E$,

$$\begin{aligned}
\pi g(y) &= \pi g(f(x)) \\
&= \pi id_X(x) \\
&= \pi x \\
&= id_X(\pi x) \\
&= (gf)(\pi x) \\
&= g(f(\pi x)) \\
&= g(\pi f(x)) \\
&= g(\pi y).
\end{aligned}$$

∎

We recall the definition of a zero element in a G_E-nominal set Z. The element $\theta \in Z$ is a zero if for all $\pi \in G_E$, $\pi\theta = \theta$.

Theorem 4.3. *Let* $Z \in G_{\mathcal{P}_{\mathrm{f}}(\mathbb{D})}$-**Nom**. *Then Z is injective with respect to monomorphisms if and only if it contains a zero.*

Proof. Let $Z \in G_{\mathcal{P}_{\mathrm{f}}(\mathbb{D})}$-**Nom** with zero θ. Then there is a finite set $E \subseteq \mathbb{D}$, such that Z is a G_E-nominal set. Also, suppose Y is a G_B-nominal set and X a sub-G_C-nominal set of Y, where C and B are finite subsets of $\mathbb{D}$ and C contains B. Consider a G_C-inclusion nominal map $i : X \hookrightarrow Y$ and a morphism $f : X \to Z \in G_{\mathcal{P}_{\mathrm{f}}(\mathbb{D})}$-**Nom**. We prove that there exists a morphism $g : Y \to Z \in G_{\mathcal{P}_{\mathrm{f}}(\mathbb{D})}$-**Nom** such that $gi = f$. Since f is a morphism in $G_{\mathcal{P}_{\mathrm{f}}(\mathbb{D})}$-**Nom**, there exists a finite set F containing $C \cup E$ such that for $x \in X$ and $\pi \in G_F$, we have $f(\pi x) = \pi f(x)$. Define $g : Y \to Z$ by

$$g(y) = \begin{cases} f(y) & \text{if} \quad y \in X \\ \theta & \text{if} \quad y \notin X \end{cases}$$

Notice that $F \supseteq E \cup C \supseteq C$. Let $y \in Y$ and $\pi \in G_F$. Then we show that g is a G_F-nominal map. Suppose $y \in X$. Since X is a sub-G_C-nominal set and $G_F \subseteq G_C$, we get X is a sub-G_F-nominal set. Hence

$$\begin{aligned} g(\pi y) &= f(\pi y) \\ &= \pi f(y) \\ &= \pi g(y). \end{aligned}$$

If $y \notin X$, then, since X is a sub-G_C-nominal set and $G_F \subseteq G_C$, for each $\pi \in G_F$ we get $\pi y \notin X$. Thus

$$g(\pi y) = \theta = \pi\, \theta = \pi g(y),$$

where the second equality is because θ is a zero element of Z and $G_F \subseteq G_C$. Also $gi(y) = g(y) = f(y)$, when $y \in Y$.

Conversely, Suppose Z is an injective G_E-nominal set. Consider $Z\dot{\cup}\{\theta\}$ where $\theta \notin Z$ and $\operatorname{supp} \theta = \emptyset$. So $Z\dot{\cup}\{\theta\}$ is a G_E-nominal set. Since Z is injective, there exists a G_E-nominal map $g : Z\dot{\cup}\{\theta\} \to Z$ such that $gi = id$. Now, for all $\pi \in G_E$, we have $\pi g(\theta) = g(\pi\theta) = g(\theta)$. This means $g(\theta)$ is a zero element of Z. ■

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References

[1] Abramsky, S., Ghica, D.R., Luke Ong, C.H., Murawski, A.S., Stark, I.D.B., *Nominal games and full abstraction for the nu-calculus*, 19th Symposium on Logic in Computer Science, 2004, 150-159.

[2] Alexandru, A. and Ciobanu, G., *Nominal groups and their homomorphism theorems*, Fund. Inform., 131 (3-4) (2014), 279-298.

[3] Bojanczyk, M., *Data monoids*, 28th Symposium on Theoretical Aspects of Computer Science, 2011, 105-116.

[4] Bojanczyk, M., *Nominal monoids*, Theory Comput. Syst., 53 (2) (2013), 194-222.

[5] Bojanczyk, M., Braud, L., Klin, B., Lasota, S., *Towards nominal computation*, 39th Symposium on Principles of Programming Languages, 2012, 401-412.

[6] Bojanczyk, M., Klin, B., Lasota, S., *Automata with group actions*, 26th Symposium on Logic in Computer Science, 2011, 355-364.

[7] Bojanczyk, M., Klin, B., Lasota, S., *Automata theory in nominal sets*, Log. Methods Comput. Sci., 10 (3) (2014), 1-44.

[8] Ebrahimi, M.M., Haddadi, M., Mahmoudi, M., *Injectivity in a category, an overview on smallness conditions*, Categ. General Alg. Struct. Appl., 2 (1) (2014), 83-112.

[9] Ebrahimi, M.M., Mahmoudi, M., *The category of M-sets*, Ital. J. Pure Appl. Math., 9 (2001), 123-132.

[10] Ebrahimi, M.M., Mahmoudi, M., *Baer criterion and injectivity of projection algebras*, Semigroup Forum, 71 (2) (2005), 332-335.

[11] Ebrahimi, M.M., Mahmoudi, M., Moghaddasi, Gh., *On the Baer criterion for acts over semigroups*, Comm. Algebra, 35(2) (2007), 3912-3918.

[12] Gabbay, M., Mathijssen, A., *One-and-a-halfth-order logic*, J. Logic Comput., 18 (2008), 521-562.

[13] Gabbay, M., Pitts, A., *A new approach to abstract syntax with variable binding*, Form. Asp. Comput., 13 (3-5) (2002), 341-363.

[14] Kilp, M., Knauer, U., Mikhalev, A., *Monoids, Acts and Categories*, Walter de Gruyter, Berlin, New York, 2000.

[15] Petrisan, D., *Investigations into algebra and topology over nominal sets*, Ph. D. Thesis, University of Leicester, 2011.

[16] Pitts, A., *Nominal logic, a first order theory of names and binding*, Inform. and Comput., 186 (2003), 165-193.

[17] Pitts, A., *Nominal sets, Names and Symmetry in Computer Science*, Cambridge University Press, 2013.

[18] Pitts, A., *Nominal presentations of the cubical sets model of type theory*, LIPIcs. Leibniz Int. Proc. Inform., 2015, 202-220.

[19] Shinwell, M.R., *The fresh approach: functional programming with names and binders*, Ph. D. Thesis, University of Cambridge, Computer Laboratory, 2005.

[20] Shinwell, M.R., Pitts, A., *Fresh objective caml user manual*, Tech. Report UCAM-CL-TR-621, University of Cambridge, 2005.

[21] Turner, D., Winskel, G., *Nominal domain theory for concurrency*, Lecture Notes in Comput. Sci., 2009, 546-560.

[22] Urban, C., *Nominal techniques in isabelle/HOL*, J. Automat. Reason., 40 (2008), 327-356.

[23] Wissmann, Th., *The rational fixed point in nominal sets and its application to infinitary lambda-calculus*, Master Project, Friedrich-Alexander-Universitat Erlangen-Nurnberg, 2014.

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THE STRUCTURE OF FINITE GROUPS WITH c^*-NORMAL SUBGROUPS

Ping Kang[1]

Department of Mathematics
Tianjin Polytechnic University
Tianjin, 300387
People's Republic of China
e-mail: jnkp1980@163.com

Abstract. Let H be a subgroup of a finite group G. H is said to be c^*-normal in G if there exists a normal subgroup K of G such that $G = HK$ and $H \cap K$ is s-quasinormally embedded in G. We fix in every non-cyclic Sylow subgroup P of G some subgroup D satisfying $1 < |D| < |P|$ and study the structure of G under the assumption that every subgroup H of P with $|H| = |D|$ is c^*-normal in G. Some recent results are generalized and unified.

Keywords: c-normal subgroup, s-quasinormally embedded subgroup, saturated formation.

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1. Introduction

All groups considered in this paper are finite. We use conventional notions and notation in [3]. G always means a group, $|G|$ denotes the order of G and $\pi(G)$ denotes the set of all primes dividing $|G|$. Let $\mathscr{F}$ be a class of groups. We call $\mathscr{F}$ a formation, provided that (1) if $G \in \mathscr{F}$ and $H \trianglelefteq G$, then $G/H \in \mathscr{F}$, and (2) if G/M and G/N are in $\mathscr{F}$, then $G/(M \cap N)$ is in $\mathscr{F}$ for any normal subgroups M, N of G. A formation $\mathscr{F}$ is said to be saturated if $G/\Phi(G) \in \mathscr{F}$ implies that $G \in \mathscr{F}$. In this paper, $\mathscr{U}$ will denote the class of all supersolvable groups. Clearly, $\mathscr{U}$ is a saturated formation.

A subgroup H of G is called s-quasinormal (or s-permutable, π-quasinormal) in G provided H permutes with all Sylow subgroups of G, i.e, $HP = PH$ for any Sylow subgroup P of G. This concept was introduced by Kegel in [6] and has been studied extensively by Deskins [2] and Schmidt [14]. More recently, Ballester-Bolinches and Pedraza-Aquilera [1] generalized s-quasinormal subgroups to s-quasinormally embedded subgroups. A subgroup H is said to

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be an s-quasinormally embedded in G if for each prime p dividing the order of H, a Sylow p-subgroup of H is also a Sylow p-subgroup of some s-quasinormal subgroup of G. Clearly, every s-quasinormal subgroup of G is an s-quasinormally embedded subgroup of G, but the converse does not hold. The authors of [18] obtained many interesting results. For example, they prove:

Theorem 1.1 *Let p be the smallest prime dividing the order of a group G and P a Sylow p-subgroup of G. Then G is p-nilpotent if and only if there is a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is s-quasinormally embedded in G.*

As another generalization of the normality, Wang [15] introduced the following concept: A subgroup H of G is called c-normal in G if there is a normal subgroup K such that $G = HK$ and $H \cap K \leq H_G$, where H_G is the normal core of H in G. Many authors consider some subgroups of a Sylow subgroup of a group when investigating the structure of G, such as in [5], [7]-[8], [17], etc. In [5], Jaraden and Skiba provide the following result.

Theorem 1.2 *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$, the class of all supersolvable groups and G a group with E as a normal subgroup of G such that $G/E \in \mathscr{F}$. Suppose that every non-cyclic Sylow subgroup P of $F^*(E)$ has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is c-normal in G, where $F^*(E)$ is the generalized Fitting subgroup of E. Then $G \in \mathscr{F}$.*

There are examples to show s-quasinormally embedded and c-normal are two different properties of subgroups in [16]. There is no inclusion-relationship between the two concepts. Hence it is meaningful to unify and generalize the two concepts and related results.

In [16], the authors introduce a new subgroup embedding property called c^*-normal which is a generalization of both c-normality and s-quasinormal embedding.

Definition 1.1 A subgroup H of a group G is called c^*-normal in G if there exists a normal subgroup K of G such that $G = HK$ and $H \cap K$ is s-quasinormally embedded in G.

It is clear from Definition 1.1 that each of c-normality and s-quasinormal embedding implies weakly c^*-normal. But the converses do not hold.

The aim of this article is to unify and improve above Theorems using c^*-normal subgroups. The following result can unify and generalize some related ones including the above two theorems.

Theorem 1.3 (i.e., Theorem 3.4) *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$, the class of all supersolvable groups and G a group with E as a normal subgroup of G such that $G/E \in \mathscr{F}$. Suppose that every non-cyclic Sylow subgroup P of $F^*(E)$ has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is c^*-normal in G, where $F^*(E)$ is the generalized Fitting subgroup of E. Then $G \in \mathscr{F}$.*

2. Basic definitions and preliminary results

In this section, we collect some known results that are useful later.

Lemma 2.1. ([16]) *Let H be a subgroup of a group G.*

(i) *If H is c^*-normal in G and $H \le M \le G$, then H is c^*-normal in M;*

(ii) *Let $N \lhd G$ and $N \le H$. Then H is c^*-normal in G if and only if H/N is c^*-normal in G/N;*

(iii) *Let p be a set of primes, H a p-subgroup of G and N a normal π'-subgroup of G. If H is c^*-normal in G, then HN/N is c^*-normal in G/N.*

By Corollary 3.2 in [16], we have the following result.

Lemma 2.2. ([16]) *Let p be the smallest prime dividing the order of a group G and P be a Sylow p-subgroup of G. If all maximal subgroups of P are c^*-normal in G. Then G is p-nilpotent.*

Lemma 2.3. *Suppose that H is a p-subgroup for some prime p and H is not s-quasinormally embedded in G. Assume that H is c^*-normal in G. Then G has a normal subgroup M such that $|G : M| = p$ and $G = HM$.*

Proof. By the hypothesis, G has a normal subgroup T such that $HT = G$ and $T \cap H < H$. Hence G has a proper normal subgroup K such that $T \le K$. Since G/K is a p-group, G has a normal maximal subgroup M such that $HM = G$ and $|G : M| = p$. ∎

Lemma 2.4.([9]) *Let H be a nilpotent subgroup of a group G. Then the following statements are equivalent:*

(i) *H is s-quasinormal in G;*

(ii) *$H \le F(G)$ and H is s-quasinormally embedded in G.*

Lemma 2.5. *Suppose that $H \le O_p(G)$ and that H is c^*-normal in G. Then H is weakly s-permutable in G.*

Proof. By the hypothesis, G has a normal subgroup B such that $HB = G$ and $H \cap B$ is s-quasinormally embedded in G. Note that $H \cap B \le H \le O_p(G)$, then by Lemma 2.4 $H \cap B$ is s-quasinormal in G, and thus $H \cap B \leq H_{sG}$. Hence H is weakly s-permutable in G. ∎

By Lemma 2.11 of [13] and Lemma 2.5, we have the following.

Lemma 2.6. *Let N be an elementary abelian normal p-subgroup of a group G. If there exists a subgroup D in N such that $1 < |D| < |N|$ and every subgroup H of N with $|H| = |D|$ is c^*-normal in G, then there exists a maximal subgroup M of N such that M is normal in G.*

By Lemma 2.12 of [13] and Lemma 2.5, we have the following.

Lemma 2.7. *Let $\mathscr{F}$ be a saturated formation containing all nilpotent groups and let G be a group with solvable $\mathscr{F}$-residual $P = G^{\mathscr{F}}$. Suppose that every maximal subgroup of G not containing P belongs to $\mathscr{F}$. Then P is a p-group for some prime p. In addition, if every cyclic subgroup of P with prime order or order 4 (if $p = 2$ and P is non-abelian) not having a supersolvable supplement in G is c^*-normal in G, then $|P/\Phi(P)| = p$.*

The generalized Fitting subgroup $F^*(G)$ of G is the unique maximal normal quasinilpotent subgroup of G. Its definition and important properties can be found in [4, X, 13]. We would like to give the following basic facts we will use in our proof.

Lemma 2.8. ([4, X, 13]) *Let G be a group and M a subgroup of G.*

(i) *If M is normal in G, then $F^*(M) \leq F^*(G)$;*

(ii) *$F^*(G) \neq 1$ if $G \neq 1$; in fact, $F^*(G)/F(G) = Soc(F(G)C_G(F(G))/F(G))$;*

(iii) *$F^*(F^*(G)) = F^*(G) \geq F(G)$; if $F^*(G)$ is solvable, then $F^*(G) = F(G)$.*

Lemma 2.9. ([13]) *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$, the class of all supersolvable groups and G a group with E as a normal subgroup of G such that $G/E \in \mathscr{F}$. Suppose that every non-cyclic Sylow subgroup P of $F^*(E)$ has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is weakly s-permutable in G, where $F^*(E)$ is the generalized Fitting subgroup of E. Then $G \in \mathscr{F}$.*

3. Main results

In this section, we will prove our main results.

Theorem 3.1 *Let p be the smallest prime dividing the order of a group G and P be a Sylow p-subgroup of G. If P has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is c^*-normal in G. Then G is p-nilpotent.*

Proof. Suppose that the theorem is false and let G be a counterexample of minimal order. We will derive a contradiction in several steps.

Step 1. $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, then by Lemma 2.1, $G/O_{p'}(G)$ satisfies the hypotheses of the theorem. Thus $G/O_{p'}(G)$ is p-nilpotent by the choice of G. Then G is p-nilpotent, a contradiction.

Step 2. $|D| > p$.

Suppose that $|D| = p$. Then by Lemma 2.1, G is a minimal non-p-nilpotent group, so $G = [P]Q$, where $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$, $p \neq q$. Set $\Phi = \Phi(P)$ and let X/Φ be subgroup of P/Φ of order p, $x \in X\backslash\Phi$ and $L = \langle x\rangle$. Then L is order p or 4. L is c^*-normal in G. Lemma 2.7 implies that $|P/\Phi| = p$, it follows that G is p-nilpotent, a contradiction.

Step 3. $|P : D| > p$.

By Lemma 2.3.

Step 4. If $N \leq P$ and N is minimal normal in G, then $|N| \leq |D|$.

Suppose that $|N| > |D|$. Since $N \leq O_p(G)$, N is elementary abelian. By Lemma 2.6, N has a maximal subgroup which is normal in G, contrary to the minimality of N.

Step 5. Suppose that $N \leq P$ and N is minimal normal in G. Then G/N is p-nilpotent.

If $|N| < |D|$, G/N satisfies the hypotheses of the theorem by Lemma 2.1. Thus G/N is p-nilpotent by the minimal choice of G. So we may suppose that $|N| = |D|$ by Step 4. Let $N \leq K \leq P$ such that $|K/N| = p$. By Step 2, N is non-cyclic, so K is also non-cyclic, it follows that K has a maximal subgroup $L \neq N$ and $K = LN$. So L is c^*-normal in G (note that $|L| = |D|$), it follows that $K/N = LN/N$ is c^*-normal in G/N. If P/N is abelian, then G/N satisfies hypothesis. Next suppose that that P/N is a non-abelian 2-group. So every subgroup of P of order $2|D|$ is c^*-normal in G. In this case one can show as above that every subgroup X of P containing N and such that $|X : N| = 4$ is c^*-normal in G. Therefore G/N also satisfies the hypothesis. By the minimal choice of G, G/N is p-nilpotent.

Step 6. $O_p(G) = 1$.

Suppose that $O_p(G) \neq 1$. Take a minimal normal subgroup N of G contained in $O_p(G)$. By Step 5, G/N is p-nilpotent. It is easy to see that N is the unique minimal normal subgroup of G contained in $O_p(G)$. Furthermore, $O_p(G) \cap \Phi(G) = 1$. Hence $O_p(G)$ is an elementary abelian p-group. On the other hand, G has a maximal subgroup M such that $G = MN$ and $M \cap N = 1$. It is easy to deduce that $O_p(G) \cap M = 1$, $N = O_p(G)$ and $M \cong G/N$ is p-nilpotent. Then G can be written as $G = N(M \cap P)M_{p'}$, where $M_{p'}$ is the normal p-complement of M. Pick a maximal subgroup S of $M_p = P \cap M$. Then $NSM_{p'}$ is a subgroup of G with index p. Since p is the minimal prime in $\pi(G)$, we know that $NSM_{p'}$ is normal in G. Now by Step 3 and the induction, we have $NSM_{p'}$ is p-nilpotent. Therefore, G is p-nilpotent, a contradiction. Thus $O_p(G) = 1$.

Step 7. The minimal normal subgroup L of G is not p-nilpotent.

If L is p-nilpotent, then it follows from the fact that $L_{p'}$ char $L \lhd G$ that $L_{p'} \leq O_{p'}(G) = 1$. Thus L is a p-group. Whence $L \leq O_p(G) = 1$ by Step 6, a contradiction.

Step 8. G is a non-abelian simple group.

Suppose that G is not a simple group. Take a minimal normal subgroup L of G. Then $L < G$. If $|L|_p > |D|$, then L is p-nilpotent by the minimal choice of G, contrary to Step 7. If $|L|_p \leq |D|$. Take $P_* \geq L \cap P$ such that $|P_*| = p|D|$. Hence P_* is a Sylow p-subgroup of P_*L. Since every maximal subgroup of P_* is of order $|D|$, every maximal subgroup of P_* is weakly c^*-normal in G by hypotheses, thus in P_*L by Lemma 2.1. Now applying Lemma 2.2, we get P_*L is p-nilpotent. Therefore, L is p-nilpotent, contrary to Step 7.

Step 9. All subgroups of P of order $|D|$ and $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) are s-quasinormally embedded in G.

Let $H \leq P$ and $|H| = |D|$ or $2|D|$. If H is not s-quasinormally embedded in G, by Lemma 2.3, there is a normal subgroup M of G such that $|G : M| = p$. By Step 3, M is p-nilpotent, it follows that G is p-nilpotent, a contradiction.

Step 10. The final contradiction.

By Theorem 1.1, G is p-nilpotent, a contradiction. The contradiction completes the proof. ∎

The following corollary is immediate from Theorem 3.1.

Corollary 3.2 *Suppose that G is a group. If every non-cyclic Sylow subgroup of G has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is c^*-normal in G, then G has a Sylow tower of supersolvable type.*

Theorem 3.3 *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$, the class of all supersolvable groups and G a group with E as a normal subgroup of G such that $G/E \in \mathscr{F}$. Suppose that every non-cyclic Sylow subgroup of E has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is c^*-normal in G. Then $G \in \mathscr{F}$.*

Proof. Suppose that P is a non-cyclic Sylow p-subgroup of E, $\forall p \in \pi(E)$. Since P has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is c^*-normal in G by hypotheses, thus in E by Lemma 2.1. Applying Corollary 3.2, we conclude that E has a Sylow tower of supersolvable type. Let q be the maximal prime divisor of $|E|$ and $Q \in \mathrm{Syl}_q(E)$. Then $Q \trianglelefteq G$. Since $(G/Q, E/Q)$ satisfies the hypotheses of the theorem, by induction, $G/Q \in \mathscr{F}$. For any subgroup H of Q with $|H| = |D|$, since $Q \leq O_q(G)$, H is weakly s-permutable in G by Lemma 2.5. Since $F^*(Q) = Q$ by Lemma 2.8, we get $G \in \mathscr{F}$ by applying Lemma 2.9. ∎

Theorem 3.4 *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$, the class of all supersolvable groups and G a group with E as a normal subgroup of G such that $G/E \in \mathscr{F}$. Suppose that every non-cyclic Sylow subgroup of $F^*(E)$ has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is c^*-normal in G. Then $G \in \mathscr{F}$.*

Proof. We distinguish two cases:

Case 1. $\mathscr{F} = \mathscr{U}$.

Let G be a minimal counter-example.

Step 1. Every proper normal subgroup N of G containing $F^*(E)$ (if it exists) is supersolvable.

If N is a proper normal subgroup of G containing $F^*(E)$, then $N/N \cap E \cong NE/E$ is supersolvable. By Lemma 2.8 (iii), $F^*(E) = F^*(F^*(E)) \leq F^*(E \cap N) \leq F^*(E)$, so $F^*(E \cap N) = F^*(E)$. For any Sylow subgroup P of $F^*(E \cap N) = F^*(E)$, P has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is c^*-normal in G by hypotheses, thus in N by Lemma 2.1. So N and $N \cap H$ satisfy the hypotheses of the theorem, the minimal choice of G implies that N is supersolvable.

Step 2. $E = G$.

If $E < G$, then $E \in \mathscr{U}$ by Step 1. Hence $F^*(E) = F(E)$ by Lemma 2.8. It follows that every Sylow subgroup of $F^*(E)$ is normal in G. By Lemma 2.5, every non-cyclic Sylow subgroup of $F^*(E)$ has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is weakly s-permutable in G. Applying Lemma 2.9 for the special case $\mathscr{F} = \mathscr{U}$, $G \in \mathscr{U}$, a contradiction.

Step 3. $F^*(G) = F(G) < G$.

If $F^*(G) = G$, then $G \in \mathscr{F}$ by Theorem 3.3, contrary to the choice of G. So $F^*(G) < G$. By Step 1, $F^*(G) \in \mathscr{U}$ and $F^*(G) = F(G)$ by Lemma 2.8.

Step 4. The final contradiction.

Since $F^*(G) = F(G)$, each non-cyclic Sylow subgroup of $F^*(G)$ has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is weakly s-permutable in G by Lemma 2.5. Applying Lemma 2.9, $G \in \mathscr{U}$, a contradiction.

Case 2. $\mathscr{F} \neq \mathscr{U}$.

By hypotheses, every non-cyclic Sylow subgroup of $F^*(E)$ has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is c^*-normal in G, thus in E Lemma 2.1. Applying Case 1, $E \in \mathscr{U}$. Then $F^*(E) = F(E)$ by Lemma 2.8. It follows that each Sylow subgroup of $F^*(E)$ is normal in G. By Lemma 2.5, each non-cyclic Sylow subgroup of $F^*(E)$ has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is weakly s-permutable in G. Applying Lemma 2.9, $G \in \mathscr{F}$. These complete the proof of the theorem. ∎

The following corollaries are immediate from Theorem 3.4.

Corollary 3.5 (17, Theorem 3.1) *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathscr{F}$. Then $G \in \mathscr{F}$ if and only if every maximal subgroup of any Sylow subgroup of $F^*(H)$ is c-normal in G.*

Corollary 3.6 (17, Theorem 3.2) *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathscr{F}$. Then $G \in \mathscr{F}$ if and only if every cyclic subgroup of any Sylow subgroup of $F^*(H)$ of prime order or order 4 is c-normal in G.*

Corollary 3.7 (12, Theorem 1.1) *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. Suppose that G is a group with a normal subgroup E such that $G/E \in \mathscr{F}$. Then $G \in \mathscr{F}$ if and only if every maximal subgroup of any Sylow subgroup of $F^*(E)$ is π-quasinormally embedded in G.*

Corollary 3.8 (12, Theorem 1.2) *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. Suppose that G is a group with a normal subgroup E such that $G/E \in \mathscr{F}$. Then $G \in \mathscr{F}$ if and only if every cyclic subgroup of any Sylow subgroup of $F^*(E)$ of prime order or order 4 is π-quasinormally embedded in G.*

Corollary 3.9 (10, Theorem 3.4) *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathscr{F}$. Then $G \in \mathscr{F}$ if and only if every maximal subgroup of any Sylow subgroup of $F^*(H)$ is s-quasinormal in G.*

Corollary 3.10 (11, Theorem 3.3) *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathscr{F}$.*

Then $G \in \mathscr{F}$ if and only if every cyclic subgroup of any Sylow subgroup of $F^*(H)$ of prime order or order 4 is s-quasinormal in G.

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References

[1] Ballester-Bolinches, A., Pedraza-Aquilera, M.C., *Sufficient conditions for supersolvability of finite groups*, J. Pure Appl. Algebra, 127 (1998), 113-118.

[2] Deskins, W.E., *On quasinormal subgroups of finite groups*, Math. Z., 82 (1963), 125-132.

[3] Huppert, B., *Endliche Gruppen* I. Springer-Verlag, Berlin-Heidelberg-New York, 1967.

[4] Huppert, B., Blackburn, N., *Finite Groups* III, Springer-Verlag, Berlin-New York, 1982.

[5] Jaraden, J.J., Skiba, A.N., *On c-normal subgroups of finite groups*, Comm. Algebra, 35 (2007), 3776-3788.

[6] Kegel, O.H., *Sylow Gruppen und subnormalteiler endlicher Gruppen*, Math. Z., 78 (1962), 205-221.

[7] Li, D., Guo, X., *The influence of c-normality of subgroups on structure of finite groups*, Comm. Algebra, 26 (1998), 1913-1922.

[8] Li, D., Guo, X., *The influence of c-normality of subgroups on structure of finite groups* II, J. Pure Appl. Algebra, 150 (2000), 53-60.

[9] Li, S., Shen, Z., Liu, J. etc, *The influence of ss-quasinormality of some subgroups on the structure of finite groups*, J. Algebra, 319 (2008), 4275-4287.

[10] Li, Y., Wei, H., Wang, Y., *The influence of π-quasinormality of some subgroups of a finite group*, Arch. Math., 81 (2003), 245-252.

[11] Li, Y., Wang, Y., *The influence of minimal subgroups on the structure of a finite group*, Proc. Amer. Math. Soc., 131 (2002), 337-341.

[12] Li, Y., Wang, Y., *On π-quasinormally embedded subgroups of finite group*, J. Algebra, 281 (2004), 109-123.

[13] Skiba, A.N., *On weakly s-permutable subgroups of finite groups*, J. Algebra, 315 (2007), 192-209.

[14] Schmid, P., *Subgroups permutable with all Sylow subgroups*, J. Algebra, 207 (1998), 285-293.

[15] Wang, Y., *c-normality of groups and its properties*, J. Algebra, 180 (1996), 954-965.

[16] Wei, H., Wang, Y., *On c^*-normality and its properties*, J. Group Theory, 10 (2007), 211-223.

[17] Wei, H., Wang, Y., Li, Y., *On c-normal maximal and minimal subgroups of Sylow subgroups of finite groups* II, Comm. Algebra, 31 (2003), 4807-4816.

[18] Wei, X., Guo, X., *On finite groups with prime-power order S-quasi-normally embedded subgroups*, Monatsh. Math., 162 (3)(2011), 329-339.

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SUBSPACE MIXING AND UNIVERSALITY CRITERION FOR A SEQUENCE OF OPERATORS

A. Tajmouati[1]
Sidi Mohamed Ben Abdellah University
Faculty of Sciences Dhar El Mahraz
Laboratory of Mathematical Analysis and Applications
Fez
Marocco
e-mail: abdelaziz.tajmouati@usmba.ac.ma

M. Amouch
Department of Mathematics
University Chouaib Doukkali
Faculty of Sciences
Eljadida. 24000
Morocco
e-mail: mohamed.amouch@gmail.com

M.R.F. Alhomidi Zakariya
M. Abkari
Sidi Mohamed Ben Abdellah University
Faculty of Sciences Dhar El Mahraz
Laboratory of Mathematical Analysis and Applications
Fez
Marocco
e-mails: zakariya1978@yahoo.fr
mbark.abkari@usmba.ac.ma

Abstract. Let $B(X)$ denote the algebra of all bounded linear operators on an infinite-dimensional separable complex Banach space X and M be a nonzero subspace of X. We will characterize properties of being $d-M$ mixing for a $N \geq 2$ sequence $T_{1,j}, T_{2,j}, ..., T_{N,j}$ of operators in $B(X)$. Also, we will give necessary and sufficient conditions for a $N \geq 2$ sequence $T_{1,j}, T_{2,j}, ..., T_{N,j}$ of operators in $B(X)$ to satisfy $d - M$ universality criterion in terms of d-M topologically transitivity of this sequence.

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[1]Corresponding author

1. Introduction

Let $B(X)$ denote the algebra of all bounded linear operators on a infinite-dimensional separable complex Banach space X.

For $x \in X$, the orbit of x under T is the set $Orb(T,x) = \{T^n x : n \in \mathbb{N}\}$. A vector x is called hypercyclic for T if $Orb(T,x)$ is dense in X and the operator T is said to be hypercyclic if there is some vector $x \in X$ which is hypercyclic. More general, a sequence $(T_n)_{n\geq 0}$ of operators in $B(X)$ is called hypercyclic or universal if $\{T_n(x), n \geq 0\}$ is dense in X for some $x \in X$, in this case x is called universal for the family $(T_n)_{n\geq 0}$, see [9].

In 2007, L. Bernal-González in [3] and J.P. Bès and A. Peris in [4] introduced independently the definition of disjoint hypercyclic for tuple of linear operators. They introduced the concept of diagonally-universality for a tuple of sequences in $B(X)$. They also gave the definition of diagonally universal for a tuple of sequences in $B(X)$.

Recall that the family $(T(t))_{t\geq 0}$ of operators on X is called a strongly continuous semigroup(C_0-semigroup) of operators if:

1. $T(0) = I$;
2. $T(s+t) = T(s)T(t)$ for all $s, t \geq 0$;
3. $\lim\limits_{t\downarrow 0} T(t)x := x$ for every $x \in X$.

The linear operator A defined in

$$D(A) = \{x \in X : \lim_{t\downarrow 0} \frac{T(t)x - x}{t} \text{ exist } \}$$

by

$$Ax = \lim_{t\downarrow 0} \frac{T(t)x - x}{t} = \frac{d^+ T(t)x}{dt} \mid_{t=0} \text{ for } x \in D(A)$$

is the infinitesimal generator of the semigroup $T(t)$ and $D(A)$ is the domain of A, see[10]. A C_0- semigroup $\tau = (T_t)_{t\geq 0}$ of operators in $B(X)$ is called hypercyclic if there exists a vector $x \in X$ such that the orbit of τ, $Orb(\tau, x) = \{T(t)x : t \geq 0\}$ is dense in X. In this case x is called the hypercyclic vector of τ [9].

Definition 1.1 Let $(T_{1,j})_{j=1}^{\infty}, (T_{2,j})_{j=1}^{\infty}, ..., (T_{N,j})_{j=1}^{\infty}$ be an $N \geq 2$ sequences in $B(X)$ and let M be a nonzero subspace of X. We say that the N sequences of operators $(T_{1,j})_{j=1}^{\infty}, (T_{2,j})_{j=1}^{\infty}, ..., (T_{N,j})_{j=1}^{\infty}$ are disjoint or diagonally subspace universal respect to M (in short $d-M$ universal), if there exists a vector $(x, x, ..., x)$ in the diagonal of X^N, such that $\{(T_{1,j}x, T_{2,j}x, ..., T_{N,j}x,), j \in \mathbb{N}\} \cap M^N$ is dense in M^N. We call x a $d - M$ universal vector. We denote by

$$dU((T_{1,j})_{j=1}^{\infty}, (T_{2,j})_{j=1}^{\infty}, ..., (T_{N,j})_{j=1}^{\infty}, M)$$

the set of all $d-M$ universal vectors of the sequences $(T_{1,j})_{j=1}^{\infty}, (T_{2,j})_{j=1}^{\infty}, ..., (T_{N,j})_{j=1}^{\infty}$.

Definition 1.2 Let $(T_{1,j})_{j=1}^{\infty}, (T_{2,j})_{j=1}^{\infty}, ..., (T_{N,j})_{j=1}^{\infty}$ be a $N \geq 2$ sequences in $B(X)$ and let M be a nonzero subspace of X. We say that the N sequences of operators $(T_{1,j})_{j=1}^{\infty}, (T_{2,j})_{j=1}^{\infty}, ..., (T_{N,j})_{j=1}^{\infty}$ are d-M topologically transitive if for any non-empty open V_0, V_1, ...V_N in M there exists $j \geq 0$ so that

$$V_0 \cap T_{1,j}^{-1}(V_1) \cap T_{2,j}^{-1}(V_2) \cap ... \cap T_{N,j}^{-1}(V_N)$$

contains a non-empty open set of M.

Let M a nonzero subspace of X. The notion diagonally subspace universal respect to M (in short $d{-}M$ universal) and the notion of d-M topologically transitive for the sequence $(T_{1,t})_{t\geq 0}, (T_{2,t})_{t\geq 0}, ..., (T_{N,t})_{t\geq 0}, (N \geq 2)$ of a C_0-semigroups of operators on X is studied in [11]. We proved that, if $(T_{1,t})_{t\geq 0}, (T_{2,t})_{t\geq 0}, ..., (T_{N,t})_{t\geq 0}$ is a sequence of C_0-semigroup with generators $A_1, A_2, ..., A_N$ and if there exists $t_0 > 0$ such that $T_{1,t_0}, T_{2,t_0}, ..., T_{N,t_0}$ are surjective and d-universal, then

$$(T_{1,t})_{t\geq 0}, (T_{2,t})_{t\geq 0}, ..., (T_{N,t})_{t\geq 0} \text{ are } d - D(A_j) \text{ universal for all } j = 1, 2, ..., N.$$

Also, we give necessary and sufficient condition for which a sequence

$$(T_{1,t})_{t\geq 0}, (T_{2,t})_{t\geq 0}, ..., (T_{N,t})_{t\geq 0} \text{ with } (N \geq 2)$$

of C_0-semigroup to be d-M topologically transitive.

Definition 1.3 We say that the $N \geq 2$ sequences of operators $(T_{1,j}), (T_{2,j}),..., (T_{N,j})$ are $d - M$ mixing respect to nonempty subset M of X if for any non-empty open subsets $V_0, V_1, ..., V_N$ in M, there exists $n \geq 0$ such that

$$V_0 \cap T_{1,m}^{-1}(V_1) \cap T_{2,m}^{-1}(V_2) \cap ... \cap T_{N,m}^{-1}(V_N)$$

contains a non-empty open set of M for each $m \geq n$.

Definition 1.4 Let M be a nonzero subspace of X and $(T_{1,j})_{j\geq 0}, (T_{2,j})_{j\geq 0}, ...,$ $(T_{N,j})_{j\geq 0}$, $N \geq 2$ sequences of operator in $B(X)$. We say that the sequences $(T_{1,j})_{j\geq 0}, (T_{2,j})_{j\geq 0}, ..., (T_{N,j})_{j\geq 0}$ satisfy the $d{-}M$ universality criterion with respect to some (n_k), if there exist dense subsets $M_0, M_1, ..., M_N$ of M, a strictly increasing sequence of positive integers (n_k), and mapping $S_{l,k} : M_l \to M$, $(1 \leq l \leq N, k \in \mathbb{N})$ such that for each $1 \leq l \leq N$ we have:

1. $T_{l,n_k} \to_{k\to\infty} 0$ pointwise on M_0;
2. $S_{l,k} \to 0$ pointwise on M_l;
3. $(T_{l,n_k} S_{i,k} y_i - \delta_{i,l} y_i) \to_{k\to\infty} 0$ pointwise on M_l;
4. $T_{l,n_k}(M) \subset M \quad (1 \leq l \leq N)$.

Let Let M be a nonzero subspace of X. In this work, we will characterize properties of being $d-M$ mixing for a $N \geq 2$ sequence $T_{1,j}, T_{2,j}, ..., T_{N,j}$ of operators in $B(X)$. Also, we will give necessary and sufficient conditions for a $N \geq 2$ sequence $T_{1,j}, T_{2,j}, ..., T_{N,j}$ of operators in $B(X)$ to satisfies $d-M$ universality criterion in terms of d-M topologically transitivity of this sequence.

2. Main results

We begin with the following result.

Theorem 2.1 *Let $T_{1,j}, T_{2,j}, ..., T_{N,j}$, $N \geq 2$ sequences of operators in $B(X)$ and M a non-empty subspace of X. The following statement are equivalent:*

1. *$(T_{1,j}), (T_{2,j}), ..., (T_{N,j})$ are d-subspace mixing.*

2. *For any nonempty open subsets $V_0, V_1, ..., V_N$ in M, there exists $n \in \mathbb{N} \setminus \{0\}$ such that $V_0 \bigcap \bigcap_{i=1}^{N} T_{i,j}^{-1}(V_i)$ is a relatively nonempty open subset of M for all $j \geq n$.*

3. *For any nonempty open subsets $V_0, V_1, ..., V_N$ in M, there exists $n \in \mathbb{N} \setminus \{0\}$ such that $V_0 \bigcap \bigcap_{i=1}^{N} T_{i,j}^{-1}(V_i) \neq \emptyset$ and $T_{i,j}(M) \subset M$, for all $j \geq n$.*

Proof. (2) $\Rightarrow$ (1) is clear.
(3) $\Rightarrow$ (2) Suppose that $V_0, V_1, ..., V_N$ are $N \geq 2$ nonempty open subset of M, hence by (3) we conclude that there exists $n \in \mathbb{N} \setminus \{0\}$ such that $V_0 \bigcap \bigcap_{i=1}^{N} T_{i,j}^{-1}(V_i) \neq \emptyset$ and $T_{i,j}(M) \subset M$. Since the restricted operator $T_{i,j|M}$ is continuous, then $T_{i,j}^{-1}(V_i)$ is open $\forall \ \ j \geq n \ \ , i = 1, 2, ..., N$. Hence $V_0 \bigcap \bigcap_{i=1}^{N} T_{i,j}^{-1}(V_i)$ is a relatively open nonempty subset.
(1) $\Rightarrow$ (3) Assume that there exist $n \geq 0$such that $V_0 \bigcap \bigcap_{i=1}^{N} T_{i,j}^{-1}(V_i)$ contains a nonempty opens subset of M, then there exists $W \neq \emptyset$ an open subset of M such that $W \subset V_0 \bigcap \bigcap_{i=1}^{N} T_{i,j}^{-1}(V_i)$, hence $W \subset V_0$ and $W \subset \bigcap_{i=1}^{N} T_{i,j}^{-1}(V_i)$ this implies that $Ti, j(W) \subset V_i \ \ \forall j \geq n, \ \ i = 1, 2, ..., N$. Let $x \in M$ and $x_0 \in W$, then there exists r small enough such that $x_0+rx \in W$, hence $T_{i,j}(x_0+rx) \in T_{i,j}(W) \subset V_i \subset M$, $\forall i = 1, 2, ..., N$; $\forall j \geq n$, $T_{i,j}x := \frac{1}{r}T_{i,j}(x_0 + rx) - T_{i,j}(x_0) \in M$, therefore

$$T_{i,j}(x_M) \subset M \quad \text{for all} \ \ j \geq n, i = 1, 2, ..., N. \qquad \blacksquare$$

The following lemma will be used in the sequel.

Lemma 2.1 *Let M be a nonzero subspace of X and $(T_{1,j})_{j \geq 0}$, $(T_{2,j})_{j \geq 0}$,...,$(T_{N,j})_{j \geq 0}$ a $N \geq 2$ sequences satisfying the $d-M$ universal criterion with respect to some (n_k), then $(T_{1,n_k}), (T_{2,n_k}), ..., (T_{N,n_k})$ are $d-M$ mixing. In particular,*

$$(T_{1,j})_{j \geq 0}, (T_{2,j})_{j \geq 0}, ..., (T_{N,j})_{j \geq 0} \text{ are } d-M \text{ universal.}$$

Proof. Let $V_0, V_1, ..., V_N$ be nonempty open subsets of M, let $y_l \in V_l \bigcap M_l$ and $\varepsilon \geq 0$, so that

$$B(y_l, (N+1)\varepsilon) \subset V_l, \quad (0 \leq l \leq N).$$

Since $(T_{1,j})_{j\geq 0}, (T_{2,j})_{j\geq 0}, ..., (T_{N,j})_{j\geq 0}$ is a $N \geq 2$ sequences satisfying the $d-M$ universal criterion with respect to some (n_k), then by Definition 1.4, there exists $n_0 \in \mathbb{N}$ so that $T_{l,n_k}y_0, S_{l,k}y_l$ and $(T_{l,n_k}y_0 S_{i,k}y_i - \delta_{i,l}y_i)$ belong to $B(0,\varepsilon)$ for $k \geq k_0$ and $1 \leq i \leq N$. For each $k \geq k_0$, set $z_k = y_0 + \sum_{i=1}^{N} S_{i,k}y_i$, we have $S_{i,k}y_i \in B(0,\varepsilon)$, this implies that $\sum_{i=1}^{N} S_{i,k}y_i \in B(0, N\varepsilon)$, hence $z_k \in B(y_0,\varepsilon) \subset B(y_0,(N+1)\varepsilon) \subset V_0$ and $T_{l,n_k}z_k = T_{l,n_k}y_0 + \sum_{i=1}^{N} T_{l,n_k}S_{i,k}y_i$. Since $(T_{l,n_k}S_{i,k}y_i - \delta_{i,l}y_i) \in B(0,\varepsilon)$, then there exists $r \in B(0,\varepsilon)$ such that $\sum_{i=1}^{N} T_{l,n_k}\delta_{i,k}y_i = \sum_{i=1}^{N}(r + \delta i, ly_i = \sum_{i=1}^{N} r + y_l$, hence

$$T_{l,n_k}z_k = T_{l,n_k}y_0 + \sum_{i=1}^{N} r + y_l.$$

We have $T_{l,n_k}y_0 \in B(0,\varepsilon)$ and $r \in B(0,\varepsilon)$, then $T_{l,n_k}z_k \in B(y_l,(N+1)\varepsilon) \subset V_l$, this implies implies that $z_k \in T_{l,n_k}^{-1}(V_l)$ for each $1 \leq l \leq N$, so

$$V_0 \cap T_{1,n_k}^{-1}(V_1) \cap T_{2,n_k}^{-1}(V_2) \cap ... \cap T_{N,n_k}^{-1}(V_N) \neq \emptyset \text{ for } k \geq k_0. \qquad \blacksquare$$

In the following theorem, we will give necessary and sufficient conditions for a $N \geq 2$ sequence $T_{1,j}, T_{2,j}, ..., T_{N,j}$ of operators in $B(X)$ to satisfy $d-M$ universality criterion in terms of d-M topologically transitivity of this sequence. Note that that the proof of the following theorem is inspired by Bès and Peris [4, Theorem 2. 7].

Theorem 2.2 *Let M be a nonzero subspace of X, and $(T_{1,j})_{j\geq 0}, (T_{2,j})_{j\geq 0}, ...,$ $(T_{N,j})_{j\geq 0}$, $N \geq 2$, sequences of operator in $B(X)$. The following are equivalent :*

1. *$(T_{1,j})_{j\geq 0}, (T_{2,j})_{j\geq 0}, ..., (T_{N,j})_{j\geq 0}$ satisfy the $d-M$ universality criterion.*

2. *There exists a strictly increasing sequence of positive integers (n_k) such that for each subsequence (n_{k_j}) of (n_k), there exists a dense set of vectors $z \in X$ for which $\{(T_{1,n_{k_j}}z, T_{2,n_{k_j}}z, ..., T_{N,n_{k_j}}z), j \in \mathbb{N}\} \bigcap M^N$ is dense in M^N*

3. *for each $r \in \mathbb{N}$,*

$$T_{1,j} \bigoplus T_{1,j} \bigoplus ... \bigoplus T_{1,j}(r\ time), ..., T_{N,j} \bigoplus T_{N,j} \bigoplus ... \bigoplus T_{N,j}(r\ time)$$

are d-M topologically transitive.

Proof. (1) $\Rightarrow$ (2) Suppose that the sequences $(T_{1,j})_{j\geq 0}, (T_{2,j})_{j\geq 0}, ..., (T_{N,j})_{j\geq 0}$, $N \geq 2$ satisfy the $d-M$ universality criterion, then there exists a increasing sequence of positive integers (n_k) which satisfies conditions of Definition 1.4. If (n_{k_j}) is any subsequence of (n_k), then $(T_{1,j})_{j\geq 0}, (T_{2,j})_{j\geq 0}, ..., (T_{N,j})_{j\geq 0}$ satisfies the $d-M$ universality criterion with respect to it. Hence by lemma 2.1, $T_{1,n_{k_j}}, T_{2,n_{k_j}}, ..., T_{N,n_{k_j}}$, are $d-M$ mixing, so $T_{1,n_{k_j}}, T_{2,n_{k_j}}, ..., T_{N,n_{k_j}}$, are $d-M$ topologically transitive. Thus, $\{(T_{1,n_{k_j}}z, T_{2,n_{k_j}}z, ..., T_{N,n_{k_j}}z), j \in \mathbb{N}\} \bigcap M^N$ is dense in M^N.

(2) $\Rightarrow$ (3) Suppose that there exists a strictly increasing sequence of positive integers (n_k) such that, for any subsequence (n_{k_j}) of (n_k), there exists a dense set of vectors z for which $\{(T_{1,n_{k_j}}z, T_{2,n_{k_j}}z, ..., T_{N,n_{k_j}}z), j \in \mathbb{N}\} \bigcap M^N$ is dense in M^N. Let $r \in \mathbb{N}$ be fixed and for each $l = 0, 1, ..., N$ and $k = 1, 2, ..., r$, let $V_{l,k} \subset M$ be open and nonempty, we have to show that there exist $m \in \mathbb{N}$ so that:

$$\emptyset \neq V_{0,k} \bigcap_{l=1}^{N} T_{l,m}^{-1}(V_{l,k}) \quad (1 \leq k \leq r).$$

Let $(n_{1,k})$ be a subsequence of (n_k), since $dU(T_{1,n_{1,k}}, T_{2,n_{1,k}}, ..., T_{N,n_{1,k}}, M) \bigcap M^N$ is dense in M^N. Then

$$\emptyset \neq V_{0,1} \bigcap_{l=1}^{N} T_{l,n_{1,k}}^{-1}(V_{l,1}) \quad (k \in \mathbb{N}).$$

Next, since $dU(T_{1,n_{1,k}}, T_{2,n_{1,k}}, ..., T_{N,n_{1,k}}, M) \bigcap M^N$ is dense in M^N, then there exist a subsequence $(n_{2,k})$ of $(n_{1,k})$ so that $\emptyset \neq V_{0,2} \bigcap_{l=1}^{N} T_{l,n_{2},k}^{-1}(V_{l,2})$. By the same way and after r steps, we obtain a chain of subsequences $(n_{r,k}) \subset ... \subset (n_{1,k}) \subset (n_k)$, so that

$$\emptyset \neq V_{0,j} \bigcap_{l=1}^{N} T_{l,n_{r,k}}^{-1}(V_{l,j}) \qquad (1 \leq j \leq r) \text{ for all } k \in \mathbb{N},$$

hence we can pick $m := n_{r,1}$.

(3) $\Rightarrow$ (1): Suppose that $(T_{1,j}), (T_{2,j}), ..., (T_{N,j})$ satisfy: for each $r \in \mathbb{N}$ and nonempty open $V_{l,k}(0 \leq l \leq N \ , \ 1 \leq k \leq r)$ of M there exists $m \in \mathbb{N}$ arbitrarily large with

$$V_{0,k} \bigcap_{l=1}^{N} T_{l,m}^{-1}(V_{l,k}) \neq \emptyset \quad (1 \leq k \leq r). \tag{$*$}$$

Let $(A_{0,n})_{n \geq 0}$ be a basis for the topology of M and $\{(A_{1,n}), (A_{2,n}), ..., (A_{N,n})\}$ be a basis of nonempty set for the product topology of M^N. For each $n \in \mathbb{N}$ and $l = 0, 1, 2, ..., N$, let $A_{l,n,0} := A_{l,n}$ and $W_n = B(0, \frac{1}{n})$.

First step:

Denote by $D(A)$ the diameter of nonempty set A.

Let $A_{l,1,1} \subset A_{l,1,0} \quad (1 \leq l \leq N)$ open set such that $D(A_{l,1,1}) < \frac{1}{2}D(A_{l,1,0})$, hence $\overline{A_{l,1,1}} \subset A_{l,1,0}$ by $(*)$, there exists $n_1 > 1$ so that

$$\begin{cases} \emptyset \neq A_{0,1,0} \bigcap \bigcap_{l=1}^{N} T_{l,n_1}^{-1}(W_l); \\ \emptyset \neq W_1 \bigcap T_{l,n_1}^{-1}(A_{l,1,1}) \bigcap \bigcap_{s \neq l} T_{s,n_1}^{-1}(W_l), \quad (1 \leq l \leq N). \end{cases} \tag{$**$}$$

Next, get $A_{0,1,1}$ nonempty open subset of $A_{0,1,0}$ such that $D(A_{0,1,1}) < \frac{1}{2}D(A_{0,1,0})$, then $\overline{A_{0,1,1}} \subset A_{0,1,0}$, this implies that $T_{l,n_1}(\overline{A_{0,1,1}}) \subset W_1 \qquad (1 \leq l \leq N)$ also by $(**)$, we pick $W_{s,1,1} \in W_1 \quad (1 \leq s \leq N)$ so that for each $1 \leq l \leq N$ we have

$$T_{l,n_1} W_{s,1,1} \in \begin{cases} A_{l,1,0}, & \text{if } s = l; \\ W_1, & \text{if } s \neq l. \end{cases}$$

Second step:
for $k = 1, 2$ let $A_{l,k,3-k}$ nonempty open subset of $A_{l,k,2-k}$ such that

$$D(A_{l,k,3-K}) < \frac{1}{3} D(A_{l,k,2-k}),$$

so that $\overline{A_{l,k,3-k}} \subset A_{l,k,3-k}$ and $\overline{A_{l,2,1}} \bigcap \overline{A_{l,1,2}} = \emptyset \quad (1 \leq l \leq N)$. By $(*)$ there exist $n_2 > n_1$ such that

$$\begin{cases} \emptyset \neq A_{0,k,2-k} \bigcap \bigcap_{l=1}^{N} T_{l,n_2}(W_2); \\ \emptyset \neq W_2 \bigcap T_{l,n_2}(A_{l,k,3-k}) \bigcap \bigcap_{s \neq l} T_{s,n_2}(W_2), \quad (1 \leq l \leq N) \quad (k = 1, 2). \end{cases}$$

Next, for $k = 1, 2$ and $l = 1, 2, ..., N$, we get $W_{l,k,3-k} \in W_2$ and nonempty open subset $A_{0,k,3-k}$ of $A_{0,k,2-k}$ such that

$$D(A_{0,k,3-K}) < \frac{1}{3} D(A_{0,k,2-k}),\ T_{l,n_2}(\overline{A_{0,k,3-k}}) \subset W_2$$

and for: $1 \leq s \leq N$,

$$T_{l,n_2} W_{s,k,3-k} \in \begin{cases} A_{l,k,3-k} & if s = l \\ W_2; & if s \neq l. \end{cases}$$

If we continue this process inductively by $(*)$, on each step, we obtain an increasing sequence of positives integer $1 < n_1 < n_2 < ....$ and for each $l \in \{1, 2, ..., N\}$ and each $i \in \mathbb{N}$ the nonempty open sets $A_{l,k,i+1-k} \quad (1 \leq k \leq i)$ such that $D(A_{l,k,i+1-k}) < \frac{1}{i+1} D(A_{l,k,i-k})$ and $W_{l,k,i+1-k} \in W_i$ satisfy

1. $\overline{A_{l,k,i+1-k}} \subset A_{l,k,i-k} \subset A_{l,k}$.
2. Each collection $\{\overline{A_{l,k,i+1-k}} : \quad 1 \leq k \leq i\}$ is pairwise disjoint.
3. $T_{l,n_i}(A_{0,k,i+1-k}) \subset W_i$.
4. For $1 \leq s \leq N$, $T_{l,n_i} W_{s,k,i+1-k} \in \begin{cases} A_{l,k,i+1-k}, & \text{if } s = l; \\ W_i, & \text{if } s \neq l. \end{cases}$

For each fixed l, $(0 \leq l \leq N)$ and $m \in \mathbb{N}$ there exists a unique $a_{l,m} \in M$ so that $\{a_{l,m}\} = \bigcap_{j=m+1}^{\infty} \overline{A_{l,m,j-m}}$ note that $a_{l,m} \neq a_{l,n}$ by (2) if $n \neq m$, and that $M_l := \{a_{l,m} : m \in \mathbb{N}\}$ is dense in M. Consider $S_{l,m} : M_l \to M$ is defined by

$$S_{l,m} a_{l,k} := \begin{cases} W_{l,k,m+1-k}, & \text{if } m \geq k, \\ 0, & \text{if } 1 \leq m < k. \end{cases}$$

From (4), $S_{l,k} \to 0, k \to \infty$ point wise on $M_l \quad (1 \leq l \leq N)$. Also, by (4) we have,

$$T_{s,n_m} S_{l,m} a_{l,k} = T_{s,n_m} W_{l,k,m+1-k} \in \begin{cases} A_{l,k,m+1-k}, & \text{if } s = l, \\ W_m, & \text{if } s \neq l. \end{cases}$$

Hence $(T_{s,n_k} s_{l,k} - \delta_{s,l} Id M_l) \to 0$, $k \to \infty$ point wise on $M_l (1 \leq l \leq N)$. we have also $T_{l,n_k} \to 0, \quad k \to \infty$ point wise on $M_0 \quad (1 \leq l \leq N)$. It easy to see that

$T_{l,n_k}(M) \subset M$ for $1 \leq l \leq N, k \in \mathbb{N}$. Finally, $(T_{1,j}), (T_{2,j}), ..., (T_{N,j})$ satisfies the $d - M$ universality criterion. ∎

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References

[1] Ansari, S.I., *Existance of hypercyclic operatos on topological vector space,* J. Funct. Anal., 148 (1997), 384-390.

[2] Bayart, F., Matheron, E., *Dynamics of Linear Operators,* Cambridge Tracts in Mathematics 179, Cambridge University Press, 2009.

[3] Bernal-González, L., *Disjoint hypercyclic operators,* Studia Math, 182 (2) (2007), 113-130.

[4] Bès, J.P., Peris, A., *Disjointness in hypercyclicity.* J. Math. Anal. Appl., 336 (2007), 297-315.

[5] Bès, J.P., Martin, Ö., Sanders, R., *Weighted Shifts And Disjoint Hypercyclicity,* J. Operator. Theory, (72) (2014), 15-40.

[6] Madore, B.F., Matínez-Avendaño, R.A., *Subspace Hypercyclicity,* J. Math. Anal. Appl., (2011), 502-511.

[7] Costakis, G., Peris, A., *Hypercyclic semigroups and somewhere dense orbits,* C.R. Math. Acad. Sci. Paris, 335 (2002), 895-898.

[8] Engel, K.-J., Nagel, R., *One-Parameter Semigroups for Linear Evolution Equations,* Springer-Verlag New York, 2000.

[9] Erdmann, K.G., Peris, A., *Linear Chaos* , Universitext. Springer, London, 2011.

[10] Pazy, A., *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag New York, 1983.

[11] Tajmouati, A., Amouch, M., Alhomidi Zakaria, M.R.F., Abkari, M., *Universality and transitivity of Semigroups of Operators*, International Journal of Pure and Applied Mathematics, 106 (4) (2016), 1087-1094.

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NOTE ON YOUNG AND ARITHMETIC-GEOMETRIC MEAN INEQUALITIES FOR MATRICES

Yanqiu Wu

School of Mathematics and Statistics
Chongqing Three Gorges University
Chongqing, 404100
P.R. China
e-mail: wuyanqiu0516@126.com

Abstract. In this short note, we prove that the conjecture of singular value Young inequality holds when $j = n$. Meanwhile, we also present a refinement of the arithmetic-geometric mean inequality for unitarily invariant norms.

Keywords: singular values; unitarily invariant norms; Young inequality; arithmetic-geometric mean inequality.
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1. Introduction

Let M_n be the space of $n \times n$ complex matrices. We shall always denote the singular values of A by $s_1(A) \geq \cdots \geq s_n(A) \geq 0$. If $A \in M_n$ has real eigenvalues, we label them as $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$. Let $\|\cdot\|$ denote any unitarily invariant norm on M_n.

Let A, B be positive semidefinite. Ando proved in [1] that if $v \in [0, 1]$, then

$$s_j\left(A^v B^{1-v}\right) \leq s_j\left(vA + (1-v)B\right), \ j = 1, ..., n.$$

This is Young inequality for singular values. Recently, Lin [2] posed the following.

Conjecture. *Let A, B be positive semidefinite and $v \in [0, 1]$. Then*

$$s_j\left(A^v B^{1-v}\right) \leq s_j\left(vA^{1/2} + (1-v)B^{1/2}\right)^2, \ j = 1, ..., n.$$

If the conjecture holds, then it is a strength of Ando's inequality. In this short note, we prove that the conjecture holds when $j = n$.

Let A, B be positive semidefinite. Bhatia and Kittaneh [3] proved that

$$\|AB\| \leq \frac{1}{4}\left\|(A+B)^2\right\|, \tag{1.1}$$

which is a arithmetic-geometric mean inequality for unitarily invariant norms. In this short note, we also obtain a refinement of inequality (1.1).

2. Main results

In this section, we first prove that Lin's conjecture holds when $j = n$.

Theorem 2.1. *Let A, B be positive semidefinite and $v \in [0,1]$. Then*

$$s_n\left(A^v B^{1-v}\right) \le s_n\left(vA^{1/2} + (1-v)\, B^{1/2}\right)^2.$$

Proof. This is obviously true if either A, or B is not invertible. So assume A and B are invertible. Then

$$\begin{aligned} s_n\left(A^v B^{1-v}\right) &= \lambda_n^{1/2}\left(B^{1-v}A^{2v}B^{1-v}\right) \\ &= \lambda_1^{-1/2}\left(B^{v-1}A^{-2v}B^{v-1}\right) \end{aligned} \tag{2.1}$$

On the other hand, we have

$$\begin{aligned} \lambda_1^{1/2}\left(B^{v-1}A^{-2v}B^{v-1}\right) &= s_1\left(A^{-v}B^{v-1}\right) \\ &\ge \lambda_1\left(A^{-v}B^{v-1}\right) \end{aligned} \tag{2.2}$$

It follows from (2.1) and (2.2) that

$$\begin{aligned} s_n\left(A^v B^{1-v}\right) &\le \lambda_1^{-1}\left(A^{-v}B^{v-1}\right) \\ &= \lambda_1^{-1}\left(A^{-v/2}B^{v-1}A^{-v/2}\right) \\ &= \lambda_n\left(A^{v/2}B^{1-v}A^{v/2}\right) \\ &= s_n^2\left(A^{v/2}B^{(1-v)/2}\right). \end{aligned}$$

So, Lemma 2.1 and the last inequality complete the proof. ∎

Next, we give a refinement of inequality (1.1). To do this, we need the following lemma [3].

Lemma 2.2. *Let A, B be positive semidefinite. Then*

$$\left\|A^{1/2}(A+B)\, B^{1/2}\right\| \le \frac{1}{2}\left\|(A+B)^2\right\|$$

Theorem 2.2. *Let A, B be positive semidefinite. Then*

$$\|AB\| + \left(\int_{1/2}^{3/2} \left\|A^v B^{2-v} + A^{2-v}B^v\right\| dv - 2\,\|AB\|\right) \le \frac{1}{4}\left\|(A+B)^2\right\|.$$

Proof. It is known [4, p.265)]that the function

$$g(r) = \left\|A^r B^{1-r} + A^{1-r}B^r\right\|$$

is convex on $[0,1]$. Replacing A by A^2, B by B^2, and $2r$ by v, we know that the function

$$f(v) = \left\|A^v B^{2-v} + A^{2-v} B^v\right\|$$

is convex on $[0,2]$, it follows that this function is also convex on $\left[\frac{1}{2}, \frac{3}{2}\right]$. Therefore, if $v \in \left[\frac{1}{2}, 1\right]$, then by the convexity of the function of $f(v)$, we have

$$f\left(\lambda \times \frac{1}{2} + (1-\lambda) \times 1\right) \le \lambda f\left(\frac{1}{2}\right) + (1-\lambda) f(1). \tag{2.3}$$

Let

$$v = \lambda \times \frac{1}{2} + (1-\lambda) \times 1.$$

Then, we know that inequality (2.3) is equivalent to

$$f(v) \le (2-2v) f\left(\frac{1}{2}\right) + (2v-1) f(1),$$

which yields

$$\int_{1/2}^{1} f(v)\, dv \le \frac{1}{4}\left(f\left(\frac{1}{2}\right) + f(1)\right). \tag{2.4}$$

On the other hand, if $v \in \left[1, \frac{3}{2}\right]$, then by the convexity of the function of $f(v)$, we have

$$f\left(\lambda \times 1 + (1-\lambda) \times \frac{3}{2}\right) \le \lambda f(1) + (1-\lambda) f\left(\frac{3}{2}\right),$$

which implies

$$\int_{1}^{3/2} f(v)\, dv \le \frac{1}{4}\left(f(1) + f\left(\frac{3}{2}\right)\right). \tag{2.5}$$

It follows from (2.4) and (2.5) that

$$\int_{1/2}^{3/2} f(v)\, dv \le \frac{1}{2}\left(f(1) + f\left(\frac{1}{2}\right)\right),$$

which is equivalent to

$$\|AB\| + \left(\int_{1/2}^{3/2} \left\|A^v B^{2-v} + A^{2-v} B^v\right\| dv - 2\|AB\|\right) \le \frac{1}{2}\left\|A^{1/2}(A+B) B^{1/2}\right\|.$$

So, Lemma 2.2 and the last inequality complete the proof. ∎

Remark 2.1. By the convexity of the function of $f(v)$, we know that

$$2\|AB\| \le \left\|A^v B^{2-v} + A^{2-v} B^v\right\|$$

and hence

$$\int_{1/2}^{3/2} \left\| A^v B^{2-v} + A^{2-v} B^v \right\| dv - 2 \left\| AB \right\| \geq 0.$$

So Theorem 2.2 is a refinement of the arithmetic-geometric mean inequality (1.1).

References

[1] Ando, T., *Matrix Young inequality*, Oper. Theory Adv. Appl., 75 (1995), 33-38.

[2] Lin, M., *On Drury's solution of Bhatia & Kittaneh's question*, Linear Algebra Appl. In press.

[3] Bhatia, F. Kittaneh, F., *Notes on matrix arithmetic-geometric mean inequalities*, Linear Algebra Appl., 308 (2000), 203-211.

[4] Bhatia, R., *Matrix Analysis*, Springer-Verlag, New York, 1997.

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FINITE GROUPS HAVING EXACTLY 42 ELEMENTS OF MAXIMAL ORDER

Zhangjia Han[1]

School of Applied Mathematics
Chengdu University of Information Technology
Chengdu, Sichuan 610225
P.R. China

Longhui Zhang

Sichuan Vocational and Technical College
Sichuan 629000
P.R. China

Abstract. Let G be a finite group, $M(G)$ denotes the number of elements of maximal order of G. In this note a finite group G with $M(G) = 42$ is determined.

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1. Introduction

All groups considered are finite. In this paper, P_r denotes Sylow r-subgroup of G for a prime r, $A \rtimes B$ denotes the semidirect product of A and B. For some natural number m and n, C_n^m always denotes the direct product of m cyclic groups of order n. All unexplained notations are standard and can be found in [3].

For a finite group G, we denote by $M(G)$ the number of elements of maximal order of G, and the maximal element order in G by $k = k(G)$. There is a topic related to one of Thompson's Conjectures:

Thompson's Conjecture. *Let G be a finite group. For a positive integer d, define $G(d) = |\{x \in G | \text{the order of } x \text{ is } d\}|$. If S is a solvable group, $G(d) = S(d)$ for $d = 1, 2, \ldots$, then G is solvable.*

Recently, some authors have investigated this topic in several articles(see [2], [5], [6], [7]). In particular, in [1] the authors gave a complete classification of the finite group with $M(G) = 30$, and the finite group with $M(G) = 24$ are classified in [4]. In this paper, we consider a finite group G satisfying $M(G) = 42$. Our main result of this paper is:

Main Theorem. *Suppose G is a finite group having exactly* 42 *elements of maximal order. Then G is solvable and one of the following holds:*

[1]Corresponding author. E-mail address: hzjmm11@163.com

(1) *if* $k = 6$*, then* $|G| = 2^\alpha \cdot 3^\beta$*, where* $2 \leq \alpha \leq 7$ *and* $1 \leq \beta \leq 5$*;*

(2) *if* $k = 14$*, then one of the following holds:*

  (I) $G = C_2^3 \times C_7$ *or* $G = (C_2^3 \times C_7) \rtimes C_6$*;*

  (II) $|G| = 2^\alpha \cdot 3^\beta \cdot 7$*, where* $2 \leq \alpha \leq 4$ *and* $1 \leq \beta \leq 2$*;*

(3) *if* $k = 18$*,* $|G| = 2^\alpha \cdot 3^\beta$*, where* $1 \leq \alpha \leq 4$ *and* $2 \leq \beta \leq 4$*;*

(4) *if* $k \in \{43, 49, 86, 98\}$*, then* $C_G(x) = \langle x \rangle \trianglelefteq G$*. Therefore,* $G/C_G(x) \lesssim Aut(C_k)$*, where* $o(x) = k$*.*

By the above theorem, we have:

Corollary. *Thompson's Conjecture holds if* G *has exactly* 42 *elements of maximal order.*

2. Preliminaries

The following lemma reveals the relationship of $M(G)$ and k.

Lemma 2.1 [7, Lemma 1] *Suppose* G *has exactly* n *cyclic subgroups of order* l*, then the number of elements of order* l *(denoted by* $n_l(G)$*) is* $n_l(G) = n\phi(l)$*, where* $\phi(l)$ *is the Euler function of* l*. In particular, if* n *denotes the number of cyclic subgroups of* G *of maximal order* k*, then* $M(G) = n\phi(k)$*.*

By the above lemma, we have:

Lemma 2.2 *If* $M(G) = 42$ *and* k *is maximal element order of* G*, then possible values of* n*,* k *and* $\phi(k)$ *are given in following table:*

n	$\phi(k)$	k
42	1	2
21	2	$3, 4, 6$
14	3	*null*
7	6	$7, 14, 18$
6	7	*null*
3	14	*null*
2	21	*null*
1	42	$43, 49, 86, 98$

In proving our main theorem, the following two results will be frequently used.

Lemma 2.3 [1, Lemma 6] *If* k *is prime, and the number of elements of maximal order* k *is* m*, then* k *divides* $m + 1$*.*

Lemma 2.4 [1, Lemma 8] *If the number of elements of maximal order* k *is* m*, then there exists a positive integer* α *such that* $|G|$ *divides* mk^α*.*

Lemma 2.5 [6, Lemma 2.5] *Let* P *be a* p*-group with order* p^t*, where* p *is a prime, and* t *is a positive integer. Suppose* $b \in Z(P)$*, where* $o(b) = p^u = k$ *with* u *a positive integer. Then* P *has at least* $(p-1)p^{t-1}$ *elements of order* k*.*

3. Proof of Main Theorem

By the hypothesis $M(G) = 42$, then $k \neq 2, 3, 7, 43$ by [1, Lemma 6], and $k \neq 4$ by [1, Corollary 2]. In the following we prove our theorem case by case for the remaining possible values of k.

Case 1. $k = 6$. In this case $|G| = 2^\alpha 3^\beta$, where $\alpha > 0$ and $\beta > 0$ by Lemma 2.4. Let x be an element of order 6. Then $|C_G(\langle x\rangle)| = 2^u \cdot 3^v$. Since there exists no element of order 9 or 4 in $C_G(x)$, we have $v \leq 3$ and $u \leq 4$ by $M(G) = 42$. Since G has exactly 21 cyclic subgroups of order 6, we have $|G : N_G(\langle x\rangle)| = 1, 2, 3, 4, 6, 8, 9, 12, 16$ or 18. If there is an element y of order 6 in G such that $|G : N_G(\langle y\rangle)| = 12, 16$ or 18, then there exists another element z of order 6 in G such that $|G : N_G(\langle z\rangle)| = 1, 2, 3, 4, 8$ or 9. That is to say, G always has an element x of order 6 such that $|G : N_G(\langle x\rangle)| = 1, 2, 3, 4, 8$ or 9. Therefore $|G|\Big|2^7 \cdot 3^5$ since $|G| = |G : N_G(\langle x\rangle)| \cdot |N_G(\langle x\rangle) : C_G(\langle x\rangle)| \cdot |C_G(\langle x\rangle)|$. Thus (1) follows.

Case 2. $k = 14$. By Lemma 2.4, we may assume that $|G| = 2^\alpha \cdot 3^\beta \cdot 7^\gamma$, where $\alpha, \gamma > 0$ and $\beta = 0$ or 1. Let x be an element of order 14. Then $|C_G(\langle x\rangle)| = 2^u \cdot 7^v$. Since there exists no element of order 49 or 4 in $C_G(x)$, we have $v = 1$ and $u \leq 3$ by Lemma 2.5. Since G has exactly 7 cyclic subgroups of order 14, we have $|G : N_G(\langle x\rangle)| = 1, 2, 3, 4, 6$ or 7. If there is an element y of order 14 in G such that $|G : N_G(\langle y\rangle)| = 2, 4$ or 6, then there exists another element z of order 14 in G such that $|G : N_G(\langle z\rangle)| = 1$ or 3. That is to say, G always has an element x of order 6 such that $|G : N_G(\langle x\rangle)| = 1, 3$ or 7.If there exists an element a of order 14 in G satisfies $|G : N_G(\langle a\rangle)| = 1$, then $\langle a\rangle$ is normal in G, and hence $\langle a^7\rangle$ lies in $Z(G)$, the center of G and $\langle a^2\rangle$ is normal in G, which implies that P_7, the Sylow 7-subgroup of G is normal in G and is of order 7. Thus all elements of order 14 lie in $C_G(a)$ and hence $C_G(a) = C_2^3 \times C_7$. Since $N_G(\langle a\rangle)/C_G(\langle a\rangle) \leq C_6$, we have $G = C_2^3 \times C_7$ or $G = (C_2^3 \times C_7) \rtimes C_6$. Thus (2) follows.If there exists an element a of order 14 in G satisfies $|G : N_G(\langle a\rangle)| = 3$, then we get $|G| = 2^\alpha \cdot 3^\beta \cdot 7$, where $2 \leq \alpha \leq 4$ and $1 \leq \beta \leq 2$ since $|G| = |G : N_G(\langle a\rangle)| \cdot |N_G(\langle a\rangle) : C_G(\langle a\rangle)| \cdot |C_G(\langle a\rangle)|$. Thus (2) follows.If there exists an element a of order 14 in G satisfies $|G : N_G(\langle a\rangle)| = 7$, then $|P_7| = 7^2$, a contradiction to [1, Lemma 7].

Case 3. $k = 18$. By Lemma 2.4, we may assume that $|G| = 2^\alpha \cdot 3^\beta \cdot 7^\gamma$, where $\alpha, \beta > 0$ and $\gamma = 0$ or 1. If $\gamma = 0$, then G is a $\{2, 3\}$-group and $|G| = 2^\alpha \cdot 3^\beta$. Since the number of cyclic subgroups of order 18 in G is 7, it follows that $|G : N_G(\langle x\rangle)| = 1, 2, 3, 4$ or 6 for some element x of order 18. If $|G : N_G(\langle x\rangle)| = 4$ or 6, then there must be another element y of order 18 such that $|G : N_G(\langle y\rangle)| = 1, 2$ or 3. Hence there is always an element x of order 18 such that $|G : N_G(\langle x\rangle)| = 1, 2$ or 3. Let $|C_G(\langle x\rangle)| = 2^u \cdot 3^v$. It is easy to see the Sylow 2-subgroups of $C_G(x)$ are abelian and $u = 1$, $2 \leq v \leq 3$ or $u = 2$, $v = 2$. So we get $|G|\Big|2^4 \cdot 3^4$ since $|G| = |G : N_G(\langle x\rangle)| \cdot |N_G(\langle x\rangle) : C_G(\langle x\rangle)| \cdot |C_G(\langle x\rangle)|$. Thus (3) follows.If $\gamma = 1$, there exists an element x of order 18 such that $|G : N_G(\langle x\rangle)| = 7$ and hence all the cyclic subgroups of order 18 are conjugate in G and also the centralizers. Let $|C_G(x)| = 2^u \cdot 3^v$. By Lemma2.5 and our assumption, we have $u = 1$, $2 \leq v \leq 3$ or $u = 2$, $v = 2$.If $u = 1$, $v = 3$ or $u = 2$, $v = 2$, then $C_G(x)$ contains 18 elements of order 18. Choose $y \in G\backslash C_G(x)$ be an element of order 18, then $C_G(y)$ also contains

18 elements of order 18. We prove that for every $y \in G \backslash C_G(x)$ with $o(y) = 18$, $C_G(y) \cap C_G(x)$ contains no elements of order 18. Otherwise, there is $z \in C_G(x) \cap C_G(y)$ with $o(z) = 18$. Since $C_G(x)$ and $C_G(y)$ are abelian, we have $C_G(y) \leq C_G(z)$ and $C_G(x) \leq C_G(z)$. Noting that all centralizers of cyclic subgroups of order 8 are conjugate, we know that $C_G(x), C_G(y)$ and $C_G(z)$ are also conjugate. Hence $C_G(x) = C_G(y) = C_G(z)$, a contradiction. Thus if there exists an element a of order 18 such that $a \in G \backslash C_G(x) \cup C_G(y)$, then $C_G(x) \cup C_G(y) \cup C_G(a)$ contains 54 elements of order 18, a contradiction. If $n_{18}(G) = n_{18}(C_G(x) \cup C_G(y))$, then $M(G) = n_{18}(C_G(x) \cup C_G(6)) = 36$, a contradiction too.If $u = 1$, $v = 2$, then $|C_G(x)| = 18$ and $|G| = 2^{\alpha+1} \cdot 3^{\beta+2} \cdot 7$, where $\alpha = 0$ or 1 and $\beta = 0$ or 1. Suppose that $\alpha = 0$. Then there exists a normal subgroup M of G such that $|M| = 3^{\beta+2} \cdot 7$. Now by Sylow's Theorem we can easily know $P_7 \trianglelefteq G$, where $P_7 \in Syl_7(G)$, which implies that G has elements of order 21, a contradiction. Therefore $\alpha = 1$. If $\beta = 0$, then $|G| = 2^2 \cdot 3^2 \cdot 7$. If that $P_7 \trianglelefteq G$, then G has elements of order 21, a contradiction. Suppose that P_7 is not normal in G. Then $|G : N_G(P_7)| = 2^2 \cdot 3^2$ by Sylow's Theorem and hence $N_G(P_7) = C_G(P_7)$. Now by the Burnside Theorem we know that there exists a normal subgroup N of G such that $|N| = 2^2 \cdot 3^2$. It is easy to see that either Sylow 2-subgroup of N or Sylow 3-subgroup of N is normal in N and hence normal in G. Thus $|N_G(P_7)| > 7$, a contradiction. Suppose that P_7 is normal in G. Then G has elements of order 21, a contradiction too. Thus $\beta \neq 0$. If $\beta = 1$, then $|G| = 2^2 \cdot 3^3 \cdot 7$. By the same argument as above we can get that $|G : N_G(P_7)| = 2^2 \cdot 3^2$ and hence $|N_G(P_7)| = 21$. Obviously, G is not a simple group, so $F(G)$, the Fitting subgroup of G is not trivial. If $|F(G)|_2 \neq 1$, then $|N_G(P_7)|_2 \neq 1$, a contradiction. It implies that $|F(G)| = 3^\gamma$, where $\gamma > 0$. Now P_7 acts on $F(G)$. Then we have $7 \bigm| |Aut(F(G))| \bigm| (3^3-1)(3^3-3)(3^3-9)$, a contradiction.

Case 4. $k \in \{43, 49, 86, 98\}$. Let x be an element of order k. Then $C_G(x) = \langle x \rangle \trianglelefteq G$. Therefore, $G/C_G(x) \lesssim Aut(C_k)$ and $C_G(x) \cong C_k$. Thus (4) follows.

References

[1] Chen, G.Y., Shi, W.J., *Finite groups with 30 elements of maximal order*, Appl. Categor. Struct., 16 (2008), 239-247.

[2] Du, X.L., Jiang, Y.Y., *On finite groups with exact 4p elements of maximal order are solvable*, Chin. Ann. Math., 25A(5)(2004), 607-612 (in Chinese).

[3] Gorenstein, D., *Finite Groups*, Harper & Row Press, New York, 1968.

[4] Jiang, Q., Shao, C., *Finite groups with* 24 *elements of maximal order*, Front. Math. China, 5 (4) (2010), 665-678.

[5] Jiang, Y., *Finite groups with* $2p^2$ *elements of maximal order are solvable*, Chin. Ann. Math., 21A (1) (2000), 6l-64 (in Chinese).

[6] Jiang, Y., *A theorem of finite groups with* $18p$ *elements having maximal order*, Alg. Coll., 15 (2) (2008), 317-329.

[7] Yang, C., *Finite groups based on the numbers of elements of maximal order*, Chin. Ann. Math., 14A (5) (1993), 561-567 (in Chinese).

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ON REVERSE AM-GM INEQUALITIES FOR n OPERATORS

Xingkai Hu[1]
Ying Sun

Faculty of Science
Kunming University of Science and Technology
Kunming, Yunnan 650500
P.R. China

Abstract. In this paper, we generalize some operator inequalities due to Fu and He [Linear Multilinear Algebra, 63 (2015), 571-577] as follows: Let $A_i\,(i = 1, ..., n)$ be positive operators on a Hilbert space with $0 < m \leq A_i \leq M$. Then for every positive unital linear map Φ,

$$\Phi^{2p}\left(\frac{A_1 + \cdots + A_n}{n}\right) \leq \left[\frac{(M+m)^{2p}}{4M^p m^p}\right]^2 \Phi^{2p}[G(A_1, \cdots, A_n)], \quad 1 \leq p < \infty,$$

and

$$\Phi^{2p}\left(\frac{A_1 + \cdots + A_n}{n}\right) \leq \left[\frac{(M+m)^{2p}}{4M^p m^p}\right]^2 G^{2p}[\Phi(A_1), \cdots, \Phi(A_n)], \quad 1 \leq p < \infty,$$

where $G(A_1, ..., A_n)$ is Ando-Li-Mathias geometric mean.

Keyword: operator inequalities; AM-GM inequality; positive linear maps.
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1. Introduction

Throughout this paper, let M, m be scalars, I be the identity operator and $\mathcal{B}(\mathcal{H})$ be the set of all bounded linear operators on a Hilbert space $(\mathcal{H}, < \cdot, \cdot >)$. The operator norm is defined by $\|\cdot\|$. We write $A \geq 0$ if the operator A is positive. If $A - B \geq 0$, then we say that $A \geq B$. Besides, for $A, B > 0$, the geometric mean $A\sharp B$ is defined by

$$A\sharp B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}.$$

A linear map Φ is positive if $\Phi(A) \geq 0$ whenever $A \geq 0$. It is said to be unital if $\Phi(I) = I$.

[1]Corresponding author. E-mail address: huxingkai84@163.com

We denote the Ando-Li-Mathias geometric mean [1] for $A_1, ..., A_n > 0$ by $G(A_1, ..., A_n)$. When $n = 2$, $G(A_1, A_2) = A_1 \sharp A_2$, but, there is no explicit formula for $G(A_1, ..., A_n)$ in terms of $A_1, ..., A_n$ when $n \geq 3$. However, we only need two basic properties of such a mean:

$$G(A_1^{-1}, ..., A_n^{-1}) = G^{-1}(A_1, ..., A_n), \tag{1.1}$$

$$G(A_1, ..., A_n) \leq \frac{A_1 + \cdots + A_n}{n}. \tag{1.2}$$

Let A and B be positive operators on a Hilbert space with $0 < m \leq A$, $B \leq M$ and Φ be positive unital linear map. Fu and He [2, Theorem 4] showed that the following reverse AM-GM inequalities for two operators:

$$\Phi^p\left(\frac{A_1 + A_2}{2}\right) \leq \left[\frac{(M+m)^2}{4^{\frac{2}{p}} Mm}\right]^p \Phi^p(A_1 \sharp A_2), \quad 2 \leq p < \infty, \tag{1.3}$$

$$\Phi^p\left(\frac{A_1 + A_2}{2}\right) \leq \left[\frac{(M+m)^2}{4^{\frac{2}{p}} Mm}\right]^p [\Phi(A_1) \sharp \Phi(A_2)]^p, \quad 2 \leq p < \infty. \tag{1.4}$$

Replacing p by $2p$ in (1.3) and (1.4), we have

$$\Phi^{2p}\left(\frac{A_1 + A_2}{2}\right) \leq \left[\frac{(M+m)^{2p}}{4M^p m^p}\right]^2 \Phi^{2p}(A_1 \sharp A_2), \quad 1 \leq p < \infty, \tag{1.5}$$

$$\Phi^{2p}\left(\frac{A_1 + A_2}{2}\right) \leq \left[\frac{(M+m)^{2p}}{4M^p m^p}\right]^2 [\Phi(A_1) \sharp \Phi(A_2)]^{2p}, \quad 1 \leq p < \infty. \tag{1.6}$$

Let $0 < m \leq A_i \leq M$ $(i = 1, ..., n)$. J.I. Fujii et al. [3, (12)] showed a reverse AM-GM inequality for n operators:

$$\frac{A_1 + \cdots + A_n}{n} \leq \frac{(M+m)^2}{4Mm} G(A_1, ..., A_n). \tag{1.7}$$

Recently, Lin [4, Theorem 3.2] showed that the reverse AM-GM inequality (1.7) can be squared:

$$\left(\frac{A_1 + \cdots + A_n}{n}\right)^2 \leq \left[\frac{(M+m)^2}{4Mm}\right]^2 G^2(A_1, ..., A_n). \tag{1.8}$$

The inequality (1.7) can be regarded as a counterpart to (1.2). By (1.7), it is easy to obtain the following reverse operator AM–GM inequality:

$$\Phi\left(\frac{A_1 + \cdots + A_n}{n}\right) \leq \frac{(M+m)^2}{4Mm} \Phi[G(A_1, ..., A_n)].$$

Are the following inequalities true for $p \geq 1$?

$$\Phi^{2p}\left(\frac{A_1+\cdots+A_n}{n}\right) \leq \left[\frac{(M+m)^{2p}}{4M^pm^p}\right]^2 \Phi^{2p}[G(A_1,...,A_n)],$$

$$\Phi^{2p}\left(\frac{A_1+\cdots+A_n}{n}\right) \leq \left[\frac{(M+m)^{2p}}{4M^pm^p}\right]^2 G^{2p}[\Phi(A_1),...,\Phi(A_n)].$$

We will answer this question in the next section.

In this paper, we will present some operator inequalities for n operators which are generalizations of (1.5) and (1.6).

2. Main results

We begin this section with the following lemmas.

Lemma 1. [5] *Let $A, B > 0$. Then the following norm inequality holds:*

$$\|AB\| \leq \frac{1}{4}\|A+B\|^2. \tag{2.1}$$

Lemma 2. [6] *Let A and B be positive operators. Then for $1 \leq r < \infty$,*

$$\|A^r + B^r\| \leq \|(A+B)^r\|. \tag{2.2}$$

Lemma 3. [7] *Let A be a positive operator on a Hilbert space. Then for every positive unital linear map Φ,*

$$\Phi(A^{-1}) \geq \Phi^{-1}(A). \tag{2.3}$$

Theorem 1. *Let $0 < m \leq A_i \leq M (i = 1, ..., n)$. Then for every positive unital linear map Φ,*

$$\Phi^{2p}\left(\frac{A_1+\cdots+A_n}{n}\right) \leq \left[\frac{(M+m)^{2p}}{4M^pm^p}\right]^2 \Phi^{2p}[G(A_1,...,A_n)],\ 1 \leq p < \infty. \tag{2.4}$$

Proof. Inequality (2.4) is equivalent to

$$\left\|\Phi^{p}\left(\frac{A_1+\cdots+A_n}{n}\right)\Phi^{-p}[G(A_1,...,A_n)]\right\| \leq \frac{(M+m)^{2p}}{4M^pm^p},\ 1 \leq p < \infty. \tag{2.5}$$

Compute

$$\begin{aligned}
&\left\|\Phi^p\left(\frac{A_1+\cdots+A_n}{n}\right)M^pm^p\Phi^{-p}[G(A_1,...,A_n)]\right\|\\
&\quad\le\frac{1}{4}\left\|\Phi^p(\frac{A_1+\cdots+A_n}{n})+M^pm^p\Phi^{-p}[G(A_1,...,A_n)]\right\|^2 && (\text{by}\,(2.1))\\
&\quad\le\frac{1}{4}\left\|\Phi\left(\frac{A_1+\cdots+A_n}{n}\right)+Mm\Phi^{-1}[G(A_1,...,A_n)]\right\|^{2p} && (\text{by}\,(2.2))\\
&\quad\le\frac{1}{4}\left\|\Phi\left(\frac{A_1+\cdots+A_n}{n}\right)+Mm\Phi[G^{-1}(A_1,...,A_n)]\right\|^{2p} && (\text{by}\,(2.3))\\
&\quad=\frac{1}{4}\left\|\Phi\left[\frac{A_1+\cdots+A_n}{n}+MmG^{-1}(A_1,...,A_n)\right]\right\|^{2p}\\
&\quad\le\frac{1}{4}(M+m)^{2p}. && (\text{by}\,(1.8))
\end{aligned}$$

That is

$$\left\|\Phi^p\left(\frac{A_1+\cdots+A_n}{n}\right)\Phi^{-p}[G(A_1,...,A_n)]\right\|\le\frac{(M+m)^{2p}}{4M^pm^p}.$$

Thus, (2.5) holds. This completes the proof. ∎

Remark 1. When $n=2$, by (2.4) we obtain the operator inequality (1.5). Thus, (2.4) is a generalization of (1.5).

Theorem 2. *Let $0<m\le A_i\le M$ $(i=1,...,n)$. Then for every positive unital linear map Φ,*

$$(2.6)\quad \Phi^{2p}\left(\frac{A_1+\cdots+A_n}{n}\right)\le\left[\frac{(M+m)^{2p}}{4M^pm^p}\right]^2G^{2p}[\Phi(A_1),...,\Phi(A_n)],\ 1\le p<\infty.$$

Proof. Inequality (2.6) is equivalent to

$$(2.7)\quad \left\|\Phi^p\left(\frac{A_1+\cdots+A_n}{n}\right)G^{-p}[\Phi(A_1),...,\Phi(A_n)]\right\|\le\frac{(M+m)^{2p}}{4M^pm^p},\ 1\le p<\infty.$$

Compute

$$\begin{aligned}
&\left\|\Phi^p\left(\frac{A_1+\cdots+A_n}{n}\right)M^pm^pG^{-p}[\Phi(A_1),...,\Phi(A_n)]\right\| \\
&\le \frac{1}{4}\left\|\Phi^p\left(\frac{A_1+\cdots+A_n}{n}\right)+M^pm^pG^{-p}[\Phi(A_1),...,\Phi(A_n)]\right\|^2 && (\text{by}\,(2.1)\,) \\
&\le \frac{1}{4}\left\|\Phi\left(\frac{A_1+\cdots+A_n}{n}\right)+MmG^{-1}[\Phi(A_1),...,\Phi(A_n)]\right\|^{2p} && (\text{by}\,(2.2)\,) \\
&= \frac{1}{4}\left\|\frac{\Phi(A_1)+\cdots+\Phi(A_n)}{n}+MmG[\Phi^{-1}(A_1),...,\Phi^{-1}(A_n)]\right\|^{2p} && (\text{by}\,(1.1)\,) \\
&\le \frac{1}{4}\left\|\frac{\Phi(A_1)+\cdots+\Phi(A_n)}{n}+Mm\frac{\Phi^{-1}(A_1)+\cdots+\Phi^{-1}(A_n)}{n}\right\|^{2p} && (\text{by}\,(1.2)\,) \\
&\le \frac{1}{4}\left\|\frac{\Phi(A_1)+\cdots+\Phi(A_n)}{n}+Mm\frac{\Phi(A_1^{-1})+\cdots+\Phi(A_n^{-1})}{n}\right\|^{2p} && (\text{by}\,(2.3)\,) \\
&\le \frac{1}{4}(M+m)^{2p}. && (\text{by}\,[8,2.3]\,)
\end{aligned}$$

That is

$$\left\|\Phi^p\left(\frac{A_1+\cdots+A_n}{n}\right)G^{-p}[\Phi(A_1),...,\Phi(A_n)]\right\| \le \frac{(M+m)^{2p}}{4M^pm^p}.$$

Thus, (2.7) holds. This completes the proof. ∎

Remark 2. When $n=2$, by (2.6) we obtain the operator inequality (1.6). Thus, (2.6) is a generalization of (1.6).

References

[1] Ando, T., Li, C.-K., Mathias, R., *Geometric means*, Linear Algebra Appl., 385 (2004), 305-334.

[2] Fu, X., He, C., *Some operator inequalities for positive linear maps*, Linear Multilinear Algebra, 63 (2015), 571-577.

[3] Fujii, J. I., Fujii, M., Nakamura, M., Pečarić, J., Seo, Y., *A reverse of the weighted geometric mean due to Lawson-Lim*, Linear Algebra Appl., 427 (2007), 272-284.

[4] Lin, M., *Squaring a reverse AM-GM inequality*, Stud. Math., 215 (2013), 187-194.

[5] Bhatia, R., Kittaneh, F., *Notes on matrix arithmetic-geometric mean inequalities*, Linear Algebra Appl., 308 (2000), 203-211.

[6] Ando, T., Zhan, X., *Norm inequalities related to operator monotone functions*, Math. Ann., 315 (1999), 771-780.

[7] Bhatia, R., *Positive definite matrices*, Princeton University Press, Princeton, 2007.

[8] Lin, M., *On an operator Kantorovich inequality for positive linear maps*, J. Math. Anal. Appl., 402 (2013), 127-132.

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CHARACTERISTIC SEMIMODULES

Barbora Batíková

Department of Mathematics
CULS
Kamýcká 129, 165 21 Praha 6 – Suchdol
Czech Republic
e-mail: batikova@tf.czu.cz

Tomáš Kepka

Department of Algebra
MFF UK
Sokolovská 83, 186 75 Praha 8
Czech Republic
e-mail: kepka@karlin.mff.cuni.cz

Petr Němec[1]

Department of Mathematics
CULS
Kamýcká 129, 165 21 Praha 6 – Suchdol
Czech Republic
e-mail: nemec@tf.czu.cz

Abstract. In the paper, a particular class of semimodules typical for additively idempotent semirings possessing at least two right multiplicatively absorbing elements is investigated.

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The present note is a direct continuation of [1] and the reader is fully referred to [1] as concerns notation, terminology and further references. Here, we introduce and study a certain type of (left) semimodules that are typical for additively idempotent semirings possessing at least two right multiplicatively absorbing elements.

1. Preliminaries

Let $A = A(*)$ be a groupoid. An element $a \in A$ is called *left (right) neutral* if $a*x = x$ ($x*a = x$) for all $x \in A$, and *left (right) absorbing* if $a*x = a$ ($x*a = a$) for all $x \in A$. If $A = A(+)$ then $0_A \in A$ ($o_A \in A$) means that 0_A (o_A) is (the

[1]Corresponding author.

unique) left and right neutral (absorbing) element of $A(+)$ and $0_A \notin A$ ($o_A \notin A$) denotes the fact that $A(+)$ has no (left and right) neutral (absorbing) element.

Similarly, if $A = A(\cdot)$, then $1_A \in A$ means that 1_A is (the unique) left and right neutral element of $A(\cdot)$.

A semiring is a non-empty set equipped with two associative binary operations that are usually written as addition and multiplication. The addition is commutative and the multiplication distributes over the addition. Given a semiring S, a (left S-)semimodule (${}_SM =$) M is a commutative semigroup $M(+)$ together with a scalar multiplication $S \times M \to M$ such that $(a+b)x = ax + bx$, $a(x+y) = ax + ay$ and $a(bx) = (ab)x$ for all $a, b \in S$ and $x, y \in M$. If S is a semiring then $R = \underline{R}(S) = \{\, a \in S \mid Sa = \{a\}\,\}$ denotes the set of right multiplicatively absorbing elements. If $a \in \underline{R}(S)$ then $a + a = aa + aa = (a+a)a = a$ and $a(b+b) = ab + ab = ab$ for every $b \in S$. Consequently, the semiring S is additively idempotent, provided that the right semimodule $\underline{R}(S)_S$ is faithful, i.e., for all $a, b \in S$, $a \neq b$, there is at least one $x \in \underline{R}(S)$ with $xa \neq xb$.

The semiring S is called (*congruence-*)*simple* if it has just two congruence relations (then these are id_S and $S \times S$ and $|S| \geq 2$). If S is simple and the ideal $\underline{R}(S)$ contains at least two elements then $\underline{R}(S)_S$ is faithful and S is additively idempotent (see [1, 7.1]).

Throughout the paper, all semirings and semimodules are assumed to be additively idempotent. It means that the respective additive semigroups $M(+)$ are semilattices, where the basic order relation is given by $\alpha \leq \beta$ iff $\alpha + \beta = \beta$.

2. Characteristic semimodules (a)

Let S be a non-trivial semiring and M be a (left S-)semimodule. The semimodule M will be called

- *precharacteristic* if $|M| \geq 2$, $0_M \in M$, $o_M \in M$, $S0_M = \{0_M\}$ and $So_M = \{o_M\}$ (in this case, we put $N = M \setminus \{o_M\}$ and $L = M \setminus \{0_M\}$);
- *characteristic* if M is faithful (i.e., for all $a, b \in S$, $a \neq b$, there is $x \in M$ with $ax \neq bx$), precharacteristic and there is a mapping $\underline{\varepsilon} : N \to S$ such that $\underline{\varepsilon}(x)y = 0_M$ and $\underline{\varepsilon}(x)z = o_M$ for all $x, y, z \in M$, $y \leq x$, $z \nleq x$.

In the rest of this section, assume that M is a characteristic semimodule.

2.1 Lemma. *$|M| \geq 3$ and $|N| \geq 2$.*

Proof. Since $|S| \geq 2$ and M is faithful, there are $a, b \in S$ and $x \in M$ such that $ax \neq bx$. Then $x \neq 0_M$, $x \neq o_M$ and $|M| \geq 3$. ∎

2.2 Proposition. *The semimodule M is simple.*

Proof. Let α be a congruence of the semimodule M. If $(0_M, o_M) \in \alpha$ then $\alpha = M \times M$. If $(x, o_M) \in \alpha$ for at least one $x \in N$ then $(0_M, o_M) = (\underline{\varepsilon}(x)x, \underline{\varepsilon}(x)o_M) \in \alpha$. If $x, y \in N$ are such that $x \nleq y$ and $(x, y) \in \alpha$ then $(0_M, o_M) = (\underline{\varepsilon}(y)y, \underline{\varepsilon}(y)x) \in \alpha$. ∎

2.3 Proposition. *$\underline{\varepsilon}$ is an injective mapping of N into $R = \underline{R}(S)$ and $|R| \geq 2$.*

Proof. We have $a\underline{\varepsilon}(x)y = \underline{\varepsilon}(x)y$ for all $a \in S$, $x \in N$ and $y \in M$. Since M is faithful, we get $a\underline{\varepsilon}(x) = \underline{\varepsilon}(x)$ and $\underline{\varepsilon}(x) \in R$. If $\underline{\varepsilon}(x_1) = \underline{\varepsilon}(x_2)$ then $0_M = \underline{\varepsilon}(x_1)x_1 = \underline{\varepsilon}(x_2)x_1$ and so $x_1 \le x_2$. Symmetrically, $x_2 \le x_1$ and we get $x_1 = x_2$. ∎

2.4 Lemma.

(i) $R = \{\, a \in S \mid aM = \{0_m, o_M\}\,\}$.

(ii) $\underline{\varepsilon}(0_M) = o_S = o_R \in R$.

(iii) $o_S L = \{o_M\}$.

Proof. (i) If $a \in R$ and $x \in M$ are such that $ax \in N$ then $ax = \underline{\varepsilon}(ax)ax = 0_M$. Thus $aM = \{0_M, o_M\}$. Conversely, if $a \in S$ is such that $aM = \{0_M, o_M\}$ then $bax = ax$ for all $b \in S$ and $x \in M$. Since M is faithful, we get $ba = a$ and $a \in R$.
(ii) We have $(\underline{\varepsilon}(0_M) + a)x = \underline{\varepsilon}(0_M)x + ax = o_M + ax = o_M = \underline{\varepsilon}(0_M)x$ for every $x \in L$ and $a \in S$. Of course, $(\underline{\varepsilon}(0_M)+a)0_M = 0_M = \underline{\varepsilon}(0_M)0_M$. Since M is faithful, we get $\underline{\varepsilon}(0_M) + a = \underline{\varepsilon}(0_M)$ and $\underline{\varepsilon}(0_M) = o_S$. By 2.3, $o_S \in R$.
(iii) If $x \in L$ then $o_S x = \underline{\varepsilon}(0_M)x = o_M$, since $x \not\le 0_M$. ∎

2.5 Proposition. *The right S-semimodule R_S is faithful.*

Proof. Let $a, b \in S$, $a \ne b$. Since the left semimodule ${}_SM$ is faithful, there is $x \in M$ with $ax \ne bx$ and we can assume that $bx \not\le ax$. Then $ax \in N$, $c = \underline{\varepsilon}(ax) \in R$ by 2.3 and $cax = 0_M \ne o_M = cbx$. Consequently, $ca \ne cb$. ∎

2.6 Proposition. *The semiring S is simple if and only if $R + S = S$ and the right semimodule R_S is simple.*

Proof. First, assume that S is a simple semiring. The relation $\alpha = ((R+S) \times (R+S)) \cup \mathrm{id}_S$ is a congruence of S and $R \times R \subseteq \alpha$. Since $|R| \ge 2$ by 2.3, we have $\alpha \ne \mathrm{id}_S$, and hence $\alpha = S \times S$ and $R + S = S$. Let σ be a congruence of the right semimodule R_S. Define a relation β on S by $(a, b) \in \beta$ iff $(ca, cb) \in \sigma$ for every $c \in R$. Then β is a congruence of the semiring S and $\sigma = \beta \cap (R \times R)$. Since S is simple, we have either $\beta = \mathrm{id}_S$ and $\sigma = \mathrm{id}_R$ or $\beta = S \times S$ and $\sigma = R \times R$.

Now, assume that $R + S = S$ and R_S is simple. Let $\varrho \ne \mathrm{id}_S$ be a congruence of the semiring S and $(a, b) \in \varrho$, $a \ne b$. Since R_S is faithful by 2.5, we have $ca \ne cb$ for at least one $c \in R$, and hence $\gamma = \varrho \cap (R \times R) \ne \mathrm{id}_R$. Clearly, γ is a congruence of R_S, so that $\gamma = R \times R$ and $R \times R \subseteq \varrho$. If $d \in S$ then $d = e + f$ for some $e \in R$ and $f \in S$, $(e, o_S) \in \varrho$ (see 2.4(ii)) and $(d, o_S) = (e + f, o_S + f) \in \varrho$. Thus $\varrho = S \times S$. ∎

2.7 Lemma. *Let $a \in S$ and $x \in M$. Then $ax = 0_M$ iff $x \in N$ and $a \le \underline{\varepsilon}(x)$.*

Proof. If $ax = 0_M$ then $x \ne o_M$ and $(a + \underline{\varepsilon}(x))y = \underline{\varepsilon}(x)y$ for every $y \in M$. Since ${}_SM$ is faithful, $a + \underline{\varepsilon}(x) = \underline{\varepsilon}(x)$ and $a \le \underline{\varepsilon}(x)$. Conversely, if $a \le \underline{\varepsilon}(x)$, $x \in N$, then $ax \le \underline{\varepsilon}(x)x = 0_M$. ∎

2.8 Lemma. *Let $x, y \in N$. Then $x \le y$ iff $\underline{\varepsilon}(y) \le \underline{\varepsilon}(x)$.*

Proof. If $x \le y$ then $\underline{\varepsilon}(y)x = 0_M$ and $\underline{\varepsilon}(y) \le \underline{\varepsilon}(x)$ by 2.7. Conversely, if $\underline{\varepsilon}(y) \le \underline{\varepsilon}(x)$ then $\underline{\varepsilon}(y)x \le \underline{\varepsilon}(x)x = 0_M$, and hence $x \le y$. ∎

2.9 Lemma. *If $a \in R \setminus \{o_S\}$ then $a \le \underline{\varepsilon}(x)$ for at least one $x \in K = M \setminus \{0_M, o_M\}$.*

Proof. Since $a \ne o_S = \underline{\varepsilon}(0_M)$ (see 2.4(ii)), there is $x \in K$ with $ax = 0_M$ (use 2.4). Now, 2.7 applies. ∎

2.10 Proposition. *The following conditions are equivalent for $a \in S$:*

(i) $a = 0_S$.

(ii) $a = 0_R$.

(iii) $a + \underline{\varepsilon}(x) = \underline{\varepsilon}(x)$ *for every* $x \in N$.

(iv) $aN = \{0_M\}$.

If $o_N \in N$ then these conditions are equivalent to $ao_N = 0_M$.

Proof. (i) implies (iii) and (ii) implies (iii) trivially.
(iii) implies (iv). $0_M = \underline{\varepsilon}(x)x = (\underline{\varepsilon}(x) + a)x = \underline{\varepsilon}(x)x + ax = ax$ for every $x \in N$.
(iv) implies (i) and (ii). First, $a \in R$ by 2.4(i). Further, $(a+b)x = ax + bx = 0_M + bx = bx$ for all $b \in S$ and $x \in N$. Then $(a+b)y = by$ for every $y \in M$ and, since M is faithful, we have $a + b = b$. Thus $a = 0_S = 0_R$. ∎

2.11 Lemma. *Let $0_S \in S$. Then:*

(i) $0_S \in R$.

(ii) $N + N = N$.

(iii) $\{\, a \in S \mid 0_S a = 0_S \,\} = \{\, a \in S \mid aN \subseteq N \,\}$.

(iv) $0_S = \underline{\varepsilon}(w)$ *for* $w \in N$ *iff* w *is the greatest element of* N.

(v) *For all* $a, b \in \underline{\varepsilon}(N)$ *there is* $c \in \underline{\varepsilon}(N)$ *with* $c \le a$ *and* $c \le b$.

(vi) *If* $c \in S$ *is such that* $0_S c = 0_S$ *then for every* $a \in \underline{\varepsilon}(N)$ *there is* $b \in \underline{\varepsilon}(N)$ *with* $bc \le a$.

(vii) *If* $c \in S$ *and* $x \in N$ *are such that* $0_S c = \underline{\varepsilon}(x)$ *then* $cx \in N$ *and* $0_S c = \underline{\varepsilon}(cx)c$.

(viii) *If* $0_S \notin \underline{\varepsilon}(N)$ *then the set* N *has no maximal element.*

(ix) *If* $c \in S$ *and* $u \in N$ *then* $\underline{\varepsilon}(u)c = 0_S$ *iff* $cN \le u$ *(then* $0_S c = 0_S$*).*

(x) *If* $c \in S$ *and* $v \in N$ *are such that* cv *is the greatest element of* cN *then* $\underline{\varepsilon}(cv)c = 0_S$.

(xi) *If* w *is the greatest element of* N *and* $c \in S$ *then* $\underline{\varepsilon}(cw)c = 0_S$ *and* cw *is the greatest element of* cN.

Proof. (i) Use 2.10(i),(ii).
(ii) Use 2.10(iv).
(iii) If $0_S a = 0_S$ then $0_S ax = 0_S x = 0_M$ for every $x \in N$ (see 2.10(iv)), and hence $ax \in N$. Conversely, if $aN \subseteq N$ then $0_S aN \subseteq 0_S N = \{0_M\}$, and hence $0_S a = 0_S$ by 2.10.

(iv) If $0_S = \underline{\varepsilon}(w)$ then $\underline{\varepsilon}(w)x = 0_S x = 0_M$ and $x \le w$ for every $x \in N$. Conversely, if w is the greatest element of N then $\underline{\varepsilon}(w)N = \{0_M\}$ and $\underline{\varepsilon}(w) = 0_S$ by 2.10.
(v) We have $a = \underline{\varepsilon}(u)$ and $b = \underline{\varepsilon}(v)$, $u, v \in N$. Now, $u + v \in N$ by (ii) and we put $c = \underline{\varepsilon}(u + v)$ (use 2.8).
(vi) We have $a = \underline{\varepsilon}(v)$, $v \in N$. If $cv = o_M$ then $0_S v = 0_S cv = 0_S o_M = o_M$, and hence $Sv = \{o_M\}$, a contradiction with $av = 0_M$. Thus $cv \in N$ and we put $b = \underline{\varepsilon}(cv)$ (use 2.7).
(vii) We have $0_S cx = \underline{\varepsilon}(x)x = 0_M$, and hence $cx \in N$. Furthermore, $\underline{\varepsilon}(cx)cx = 0_M$, and so $\underline{\varepsilon}(cx)c \le \underline{\varepsilon}(x) = 0_S c$ by 2.7. Since $0_S \le \underline{\varepsilon}(cx)$, we have $0_S c \le \underline{\varepsilon}(cx)c$. Thus $0_S c = \underline{\varepsilon}(cx)c$.
(viii) $N+N = N$ by 2.11(ii). If w is maximal in N and $x \in N$ then $w \le w+x \in N$, $w = w+x$, $x \le w$, and therefore w is the greatest element of N. By (iv), $0_S = \underline{\varepsilon}(w)$.
(ix), (x) and (xi). Use 2.10. ∎

2.12 Proposition. *The following conditions are equivalent for $e \in S$:*

(i) *$e = 1_S$ is multiplicatively neutral in S.*

(ii) *e is right multiplicatively neutral in S.*

(iii) *$\underline{\varepsilon}(x)e = \underline{\varepsilon}(x)$ for every $x \in N$.*

(iv) *$ey = y$ for every $y \in M$.*

Proof. (i) implies (ii) and (ii) implies (iii) trivially.
(iii) implies (iv). First, $\underline{\varepsilon}(x)ex = \underline{\varepsilon}(x)x = 0_M$ and $ex \le x$. Further, $0_M = \underline{\varepsilon}(ex)ex = \underline{\varepsilon}(ex)x$, so that $x \le ex$. Thus $x = ex$ for every $x \in N$. Of course, $eo_M = o_M$ anyway.
(iv) implies (i). Clearly, $aey = ay$ and $eay = ay$ for all $a \in S$ and $y \in M$. Since M is faithful, we get $ae = a = ea$. ∎

2.13 Lemma. *Assume that $Sv = M$ for at least one $v \in M$. If $e \in S$ is a left multiplicatively neutral element the $e = 1_S$ is multiplicatively neutral.*

Proof. We have $eav = av$ for every $a \in S$. By 2.12(i),(iv), we get $e = 1_S$. ∎

2.14 Lemma. *Assume that the semiring S is simple. If G is a subsemimodule of M then G is faithful if and only if $G \nsubseteq \{0_M, o_M\}$.*

Proof. If $x \in G \setminus \{0_M, o_M\}$ then $\underline{\varepsilon}(x)x = 0_M \ne o_M = \underline{\varepsilon}(0_M)x = o_S x$. ∎

2.15 Remark. Let ${}_SG$ be a faithful subsemimodule of ${}_SM$. Then $G \nsubseteq \{0_M, o_M\}$ and if $u \in G \setminus \{0_M, o_M\}$ then $\underline{\varepsilon}(u)M = 0_M \in G$. Of course, $o_S u = o_M$, and hence $\{0_M, o_M\} \subseteq G$. Now, it is clear that the semimodule ${}_SG$ is characteristic, too. We have $o_G = o_M$ and $\underline{\varepsilon}(G \setminus \{o_M\}) \subseteq \underline{\varepsilon}(M \setminus \{o_M\})$.

2.16 Remark. (i) Assume that $o_N \in N$ and that the set $G = M \setminus \{o_N\} = (N \setminus \{o_N\}) \cup \{o_M\}$ is a subsemimodule of ${}_SM$. If ${}_SG$ is not a faithful semimodule then there are elements $a_1, b_1 \in S$ such that $a_1 \ne b_1$ and $a_1 u = b_1 u$ for every $u \in G$. According to 2.5, there is $c \in R$ such that $a = ca_1 \ne cb_1 = b$. Of course, $a, b \in R$. We have $au = bu$ for every $u \in G$ and, since ${}_SM$ is faithful, we have

$ao_N \neq bo_N$. Using 2.4(i), we can assume that $o_M = ao_N$ and $0_M = bo_N$. Then $bN = \{0_M\}$ and $b = 0_S$ by 2.10. We have $(a + c)u = cu$ for all $c \in S$ and $u \in G$. If $d \in R \setminus \{0_S\}$ then $do_N = o_M$ (use 2.4(i) and 2.10). Consequently, $(a + d)v = dv$ for all $d \in R \setminus \{0_S\}$ and $v \in M$. It follows that $a + d = d$ and $a \leq d$. Thus a is the smallest element of the set $R \setminus \{0_S\}$. If $a = \underline{\varepsilon}(w)$ for some $w \in N$ then w is the greatest element of the set $N \setminus \{o_N\}$.

(ii) Assume that $0_S \in S$ and that the set $R \setminus \{0_S\}$ has the smallest element, say a. Then $o_N \in N$ and $ax = 0_S x$ for every $x \in M \setminus \{o_N\}$.

2.17 Lemma. *Assume that $R + S = S$. If the semiring S has a right multiplicatively neutral element then $0_S \in S$.*

Proof. If $e \in S$ is right multiplicatively neutral then $e = a + b$, $a \in R$, $b \in S$, and $c = ce = c(a + b) = ca + cb = a + cb$. Thus $a = 0_S$. ∎

2.18 Remark.

(i) If the right S-semimodule S_S is simple then the semiring S is simple and the right S-semimodule R_S is simple, too (see 2.6).

(ii) Assume that R_S is simple and define a relation α on R by $(a, b) \in \alpha$ iff $ac = bc$ for every $c \in S$. Clearly, *alpha* is a congruence of R_S, and hence either $\alpha = \mathrm{id}_R$ or $\alpha = R \times R$. If the latter is true then $ac = bc$ for all $a, b \in R$, $c \in S$, and it follows easily that every congruence of the additive semigroup $R(+)$ is a congruence of the semimodule R_S. Consequently, $R(+)$ is a simple semilattice, $|R| = 2$ and $|S| = 2$ or $|S| = 3$.

(iii) Assume that R_S is simple and $|S| \geq 3$. Let $e \in S$ be a left multiplicatively neutral element. Then $(ae, a) \in \alpha$ for every $a \in R$.

Let $\alpha = \mathrm{id}_R$. Then $ae = a$, and so $abe = ab$ for all $a \in R$ and $b \in S$. Define a relation ϱ on S by $(c, d) \in \varrho$ iff $ac = ad$ for every $a \in R$. It is easy to see that ϱ is a congruence of the semiring S and $\varrho \cap (R \times R) = \mathrm{id}_R$. Thus $\varrho \neq S \times S$. Of course, if $\varrho = \mathrm{id}_S$ then $e = 1_S$ is multiplicatively neutral in S.

Finally, if $\alpha = R \times R$ then $|R| = 2$, $|S| = 3$ and $1_S \in S$. Thus $e = 1_S$.

3. Characteristic semimodules (b)

3.1. Assume that $\underline{\varepsilon}(N) = R$.

3.1.1 Lemma. *Let $a, b \in R$ be such that $P = P_{a,b} = \{\, c \in R \mid c \leq a, c \leq b \,\} \neq \emptyset$. Then $o_P \in P$.*

Proof. We have $a = \underline{\varepsilon}(u)$, $b = \underline{\varepsilon}(v)$, $u, v \in N$. If $c \in P$, $c = \underline{\varepsilon}(w)$, $w \in N$, then (by 2.3) $u \leq w$, $v \leq w$, and hence $u+v \leq w$. Now, $u+v \in N$ and $o_P = \underline{\varepsilon}(u+v)$. ∎

3.1.2 Lemma. *Let $a \in R$ and $b \in S$ be such that $Q = Q_{a,b} = \{c \in R \mid cb \leq a\} \neq \emptyset$. Then $o_Q \in Q$.*

Proof. We have $a = \underline{\varepsilon}(u)$, $u \in N$, and if $c \in Q$ then $c = \underline{\varepsilon}(w)$, $w \in N$. Now, $\underline{\varepsilon}(w)b \leq \underline{\varepsilon}(u)$, and so $\underline{\varepsilon}(w)bu \leq \underline{\varepsilon}(u)u = 0_M$. Then $\underline{\varepsilon}(w) \leq \underline{\varepsilon}(bu)$ and we see that $o_Q = \underline{\varepsilon}(bu)$. ∎

3.1.3. Using 3.1.1, 3.1.2 and [1, 7.2], we get a left S-semimodule ${}^{1}_{S}R = R^{+}(*, \circ))$ defined on $R^{+} = R \cup \{\omega\}$, where $\omega \notin S$, the "addition" $*$ for $a, b \in R$ is defined as $a * b = o_P$ if $P = P_{a,b} \neq \emptyset$ and $a * b = \omega$ otherwise, and the "scalar multiplication" $\circ$ for $a \in S$, $x \in R^{+}$ is defined as $a \circ x = o_Q$ if $x \in R$ and $Q_{a,x} \neq \emptyset$, and $a \circ x = \omega$ otherwise. Let $\varphi : M \to R^{+}$ be defined by $\varphi(x) = \underline{\varepsilon}(x)$ for $x \in N$ and $\varphi(o_M) = \omega$. We have $\varphi(N) = R$ and, due to 2.3, φ is a bijective mapping of M onto R^{+}. We check that, in fact, φ is an isomorphism of the left S-semimodules.

Let $x, y \in M$. We have to show that $\varphi(x + y) = \varphi(x) * \varphi(y)$. If $o_M \in \{x, y\}$ then $\omega \in \{\varphi(x), \varphi(y)\}$ and $\varphi(x + y) = \varphi(o_M) = \omega = \varphi(x) * \varphi(y)$. If $x, y \in N$ and $o_M = x + y$ then $P_{a,b} = \emptyset$, where $a = \underline{\varepsilon}(x)$ and $b = \underline{\varepsilon}(y)$ (use 2.8), and then $\varphi(x + y) = \varphi(o_M) = \omega = a * b = \varphi(x) * \varphi(y)$. Finally, if $x + y \in N$ then $P_{a,b} \neq \emptyset$ and $\varphi(x + y) = \underline{\varepsilon}(x + y) = o_P = a * b = \underline{\varepsilon}(x) * \underline{\varepsilon}(y) = \varphi(x) * \varphi(y)$ (see 3.1.1 and its proof).

Let $a \in S$ and $x \in M$. We have to show that $\varphi(ax) = a \circ \varphi(x)$. If $x = o_M$ then $\varphi(ax) = \varphi(o_M) = \omega = a \circ \omega = a \circ \varphi(x)$. If $x \in N$ and $Q_{\varphi(x),a} = Q_{\underline{\varepsilon}(x),a} = \emptyset$ then $a \circ \varphi(x) = a \circ \underline{\varepsilon}(x) = \omega$ and either $ax = o_M$, $\varphi(ax) = \omega$, or $ax \in N$, $0_M = \underline{\varepsilon}(ax)ax$ and $\underline{\varepsilon}(ax) \in Q_{\underline{\varepsilon}(x),a}$, a contradiction. Assume, finally, that $x \in N$ and $Q_{\underline{\varepsilon}(x),a} \neq \emptyset$. Then $ax \in N$ and $\varphi(ax) = \underline{\varepsilon}(ax) = a \circ \underline{\varepsilon}(x) = a \circ \varphi(x)$ (see 3.1.2 and its proof).

3.2. Assume that $0_S \in S$ (then $0_S \in R$) and $\underline{\varepsilon}(N) = R \setminus \{0_S\}$.

3.2.1 Lemma. *Take $a, b \in R$ and put $P = P_{a,b} = \{\, c \in R \mid c \leq a, c \leq b \,\}$. Then $0_S \in P$ and $o_P \in P$.*

Proof. We can assume that $a \neq 0_S \neq b$. Now we can proceed similarly as in the proof of 3.1.1. ∎

3.2.2 Lemma. *Let $a \in R$, $a \neq 0_S$, $b \in S$ and $Q = Q_{a,b} = \{\, c \in R \mid cb \leq a \,\}$. Then:*

(i) *$Q \neq \emptyset$ iff $0_S b \leq a$.*

(ii) *If $Q \neq \emptyset$ then $o_Q \in Q$.*

Proof. (i) This is clear.
(ii) We have $a = \underline{\varepsilon}(u)$ for some $u \in N$. If $Q = \{0_S\}$ then $o_Q = 0_S$. If $c \in Q \setminus \{0_S\}$ then $c = \underline{\varepsilon}(v)$, $v \in N$, $\underline{\varepsilon}(v)b \leq \underline{\varepsilon}(u)$, $\underline{\varepsilon}(v)bu = 0_M$ and $c = \underline{\varepsilon}(v) \leq \underline{\varepsilon}(bu)$ by 2.7. Of course, $bu \in N$, $\underline{\varepsilon}(bu)bu = 0_M$ and $\underline{\varepsilon}(bu)b \leq \underline{\varepsilon}(u) = a$. Thus $o_Q = \underline{\varepsilon}(bu)$. ∎

3.2.3 Lemma. *For all $a, b \in R \setminus \{0_S\}$ there is $c \in R \setminus \{0_S\}$ with $c \leq a$ and $c \leq b$.*

Proof. We have $a = \underline{\varepsilon}(x)$ and $b = \underline{\varepsilon}(y)$ for $x, y \in N$. Then $x + y \in N$ by 2.11(ii) and we put $c = \underline{\varepsilon}(x + y)$. ∎

3.2.4 Lemma. *If $a \in S$ is such that $0_S a = 0_S$ then for every $b \in R \setminus \{0_S\}$ there is $c \in R \setminus \{0_S\}$ with $ca \leq b$.*

Proof. See 2.11(vi). ∎

3.2.5 Lemma. *If $a \in S$ is such that $0_S a \neq 0_S$ then $0_S a = ba$ for at least one $b \in R \setminus \{0_S\}$.*

Proof. See 2.11(vii). ∎

3.2.6. Using 3.2.1, 3.2.5 and [1, 5.5 and 5.6], we get a left S-semimodule ${}^2_S R = R(*, \triangle)$, where the "scalar multiplication" $\triangle$ for $a \in S$ and $x \in R$ is defined as $a \triangle x = o_Q$ if $x \in R$, $x \neq 0_S$ and $Q_{a,x} \neq \emptyset$, and $a \triangle x = 0_S$ otherwise. Let $\varphi : M \to R$ be defined by $\varphi(x) = \underline{\varepsilon}(x)$ for $x \in N$ and $\varphi(o_M) = 0_S$. Proceeding similarly as in 3.1.3, one checks easily that φ is an isomorphism of the semimodule ${}_S M$ onto the semimodule ${}^2_S R$.

3.3 Theorem. *The following conditions are equivalent:*

(i) *There is a characteristic semimodule ${}_S M$ such that $\underline{\varepsilon}(M \setminus \{o_M\}) = R$.*

(ii) *The ordered set $R^+ = R \cup \{\omega\}$, where ω is the smallest element of the set, is a lattice and $o_Q \in Q = Q_{a,b} = \{\, c \in R \mid cb \leq a \,\}$ for all $a \in R$ and $b \in S$ such that $Q \neq \emptyset$.*

Moreover, if these conditions are satisfied then ${}_S M \cong {}^1_S R$.

Proof. i) implies (ii). See 3.1.
(ii) implies (i). See [1, 7.2]. ∎

3.4 Theorem. *The following conditions are equivalent:*

(i) *$0_S \in S$ and there is a characteristic semimodule ${}_S M$ such that $\underline{\varepsilon}(M \setminus \{o_M\}) = R \setminus \{0_S\}$.*

(ii) *$0_R \in R$, the ordered set R is a lattice, $o_Q \in Q = Q_{a,b} = \{\, c \in R \mid cb \leq a \,\}$ for all $a \in R \setminus \{0_S\}$ and $b \in S$ such that $Q \neq \emptyset$ and the assertions* 3.2.3, 3.2.4 and 3.2.5 *are true.*

Moreover, if the two equivalent conditions are satisfied then ${}_S M \cong {}^2_S R$.

Proof. (i) implies (ii). See 3.2.
(ii) implies (i). See [1, 7.3]. ∎

3.5 Remark. Assume that the equivalent conditions of 3.4 are satisfied. Let $a_0 \in R$ and $b \in S$ be such that $Q = Q_{a_0,b} \neq \emptyset$ and $o_Q \notin Q$.

We have $a_0 = 0_S$ (see 3.4(ii)) and $0_S \in Q = \{\, c \in R \mid cb = 0_S \,\}$. Of course, $Q \subseteq Q_{a,b} = \{\, c \in R \mid cb \leq a \,\}$ for every $a \in R$. If $a \in R_1 = R \setminus \{0_S\}$ then $o_{Q_{a,b}} \in Q_{a,b}$ (see 3.4(ii)), and hence $Q \neq Q_{a,b}$ and there is $c_a \in R$ with $0_S \neq c_a b \leq a$.

Now, assume that $R_2 b = 0_S$, where $R_2 = R \setminus \{o_S\}$. Since $o_Q \notin Q$, we have $Q = R_2$, $a_1 = o_S b \neq 0_S$, $Q_{a,b} = R$ for every $a \in R_1$ and $Rb \leq a$. But $Rb = \{0_S, a_1\}$ and it follows that a_1 is the smallest element of the set R_1. By 3.4(i), $a_1 = \underline{\varepsilon}(w)$ for some $w \in N$. According to 2.8, we see that w is the greatest element of the set N. By 2.11(iv), $a_1 = 0_S$, a contradiction.

We have proved that $c_0 b \neq 0_S$ for at least one $c_0 \in R \setminus \{o_S\}$.

3.6 Remark. Let $0_S \in S$ and let M be a characteristic semimodule. Let $b \in S$ be such that $R_2 b = \{0_S\}$, $R_2 = R \setminus \{o_S\}$ (cf. 3.5). We have $0_S N = \{0_M\}$, and hence $R_2 bN = \{0_M\}$. Consequently, $bN \subseteq N$. For every $x \in N$, we get $R_2 \le \underline{\varepsilon}(bx)$ by 2.7. If $\underline{\varepsilon}(bx) = o_S$ then $bx = 0_M$. On the other hand, if $\underline{\varepsilon}(bx) \in R_2$ then $\underline{\varepsilon}(bx)$ is the greatest element of the set R_2 and bx is the smallest element of the set $M \setminus \{0_M\}$ (use 2.8).

Of course, if $b = 0_S$ then $R_2 b = Sb = S0_S = \{0_S\}$. Assume, henceforth, that $b \neq 0_S$. We have $0_S N = \{0_M\}$ and, since the semimodule M is faithful and $b \neq 0_S$, we see that $bN \neq \{0_M\}$. As we have proved, the set $R_2 = R \setminus \{o_S\}$ has the greatest element a_0 and $a_0 = \underline{\varepsilon}(v)$, where v is the smallest element in $M \setminus \{0_M\}$. Besides, if $x \in N$ then either $bx = 0_M$ or $bx = v$. Thus $bM = \{0_M, v, o_M\}$.

4. Characteristic semimodules (c)

Let M ($= {}_S M$) be a characteristic (left S-)semimodule, $N = M \setminus \{o_M\}$ and $R = \underline{R}(S)$.

4.1 Lemma. *Let $a \in R$ and $A = A_a = \{\, x \in M \mid ax = 0_M \,\}$. Then:*

(i) $0_M \in A \subseteq N$ *and $A(+)$ is a subsemilattice of $M(+)$.*

(ii) *If $w \in A$ is maximal in A then w is the greatest element of A and $a = \underline{\varepsilon}(w)$.*

(iii) $a \le \underline{\varepsilon}(x)$ *for every $x \in A$.*

(iv) *If $a \notin \underline{\varepsilon}(N)$ then $a < \underline{\varepsilon}(x)$ for every $x \in A$.*

Proof. The assertion (i) is obvious. If w is maximal in A then $w = o_A$, $\underline{\varepsilon}(w)y = 0_M = ay$ for every $y \le w$ and $\underline{\varepsilon}(w)z = o_M$ for every $z \not\le w$. Since $z \notin A$, we have $az \neq 0_M$, and hence $az = o_M$ by 2.4(i). Thus $\underline{\varepsilon}(w)u = au$ for every $u \in M$ and $a = \underline{\varepsilon}(w)$, since ${}_S M$ is faithful. We have proved (i) and (iii), (iv) follow from 2.7.∎

Consider the following three conditions:

(α) If $x_1 < x_2 < x_3 < \dots$ is an infinite strictly increasing sequence of elements from M then for every $x \in N$ there is $i \ge 1$ with $x \le x_i$.

(β) If $x_1 < x_2 < x_3 < \dots$ is an infinite strictly increasing sequence of elements from M then for every $x \in N \setminus \{o_N\}$ there is $i \ge 1$ with $x \le x_i$.

(γ) If $a_1 > a_2 > a_3 > \dots$ is an infinite strictly decreasing sequence of elements from R then for every $a \in R \setminus \{0_S\}$ there is $i \ge 1$ with $a \ge a_i$.

4.2 Proposition. *Assume that (α) or (γ) is true. Then either $\underline{\varepsilon}(N) = R$ or $0_S \in S$ and $\underline{\varepsilon}(N) = R \setminus \{0_S\}$.*

Proof. Let $a \in R \setminus \underline{\varepsilon}(N)$ and $A = A_a$ (see 4.1). By 4.1(ii), the set A has no maximal element, and hence there is at least one infinite strictly increasing sequence $x_1 < x_2 < x_3 < \dots$ of elements from A. By 2.8, we get the infinite strictly decreasing sequence $\underline{\varepsilon}(x_1) > \underline{\varepsilon}(x_2) > \underline{\varepsilon}(x_3) > \dots$ of elements from R. By

4.1(iv), $\underline{\varepsilon}(x_i) > a$. Now, if (γ) is true, we get $a = 0_S$. On the other hand, if (α) is true then $A = N$, $aN = \{0_M\}$ and $a = 0_S$ by 2.10. ∎

4.3 Lemma. *(α) implies (γ) and (γ) implies (β). If $o_N \notin N$ then the conditions (α), (β) and (γ) are equivalent.*

Proof. Use 4.2 and 2.11(iv). ∎

4.4 Proposition. *Assume that (β) is true. Then just one of the following three cases takes place:*

(i) *$\underline{\varepsilon}(N) = R$.*

(ii) *$0_S \in S$ and $\underline{\varepsilon}(N) = R \setminus \{0_S\}$.*

(iii) *$o_N \in N$, $0_S \in S$, the set $R \setminus \{0_S\}$ has the smallest element a_0, $\underline{\varepsilon}(N) = R \setminus \{a_0\}$, $a_0(N \setminus \{o_N\}) = \{0_M\}$ and $a_0 o_N = o_M$.*

Proof. Assume that neither (i) or (ii) is true. Let $a \in R \setminus \underline{\varepsilon}(N)$. By 4.1, the set $A = A_a$ has no maximal element. By 4.2, the condition (α) is not satisfied. But (β) is, and so $o_N \in N$. We have $A \subseteq N$, $o_N \notin A$, and therefore $A \subseteq N' = N \setminus \{o_N\}$. Using (β), we conclude that $A = N'$, $aN = \{0_M\}$. If $ao_N = 0_N = 0_M$ then $a = 0_S$ by 2.10 and $\underline{\varepsilon}(N) = R \setminus \{0_S\}$, a contradiction. Thus $ao_N \neq 0_N$ and it follows from 2.4(i) that $ao_n = o_M$. By 4.1(iv), $a < \underline{\varepsilon}(x)$ for every $x \in N'$. But $\underline{\varepsilon}(N) = R \setminus \{a\}$ and $\underline{\varepsilon}(N') = R \setminus \{0_S, a\}$. It follows that a is the smallest element of the set $R \setminus \{0_S\}$. ∎

4.5 Lemma. *Let G be a proper faithful subsemimodule of the semimodule M. Then:*

(i) *$\{0_M, o_M\} \subset G$ and $|M| \geq 4$.*

(ii) *If $w \in M \setminus G$, $a = \underline{\varepsilon}(w)$, then the set $A_a \cap G = \{\, x \in G \mid ax = 0_M \,\}$ has no maximal element.*

Proof. Since ${}_SG$ is faithful, we have $G \nsubseteq \{0_M, o_M\}$. If $u \in G$, $u \neq 0_M, o_M$, then $\underline{\varepsilon}(u)u = 0_M \in G$ and $o_S u = o_M \in G$, so that $\{0_M, o_M\} \subset G$ and $|M| \geq 4$. Furthermore, if v is a maximal element of the set $A_a \cap G$ then $\underline{\varepsilon}(w) = a = \underline{\varepsilon}(v)$, $w = v$ and $w \in G$, a contradiction. ∎

4.6 Proposition. *Assume that (β) is true. Let ${}_SG$ be a proper faithful subsemimodule of ${}_SM$. Then:*

(i) *$o_N \in N$ and $\underline{\varepsilon}(o_N) = 0_S \in S$.*

(ii) *M is infinite.*

(iii) *$\underline{\varepsilon}(N) = R$.*

(iv) *$G = M \setminus \{o_N\}$.*

Proof. By 4.5, $\{0_M, o_M\} \subset G$ and if $w \in M \setminus G$, $a = \underline{\varepsilon}(w)$, then the set $B = \{\, x \in G \mid ax = 0_M \,\}$ has no maximal element. Clearly, $B \subseteq N' = N \setminus \{o_N\}$ and there is an infinite strictly increasing sequence $x_1 < x_2 < x_3 < \dots$ of elements from B. For every $x \in N'$ there is $i \geq 1$ with $x \leq x_i$ and it follows that $aN' = \{0_M\}$. If $aN = \{0_M\}$ then $a = 0_S \in S$ by 2.10 and, by 2.11(iv), $w = o_N$. Besides, $\underline{\varepsilon}(N) = R$ by 4.4. Assume, therefore, that $aN \neq \{0_M\}$. Then $o_N \in N$ and $ao_N = o_M$ by 2.4(i). But $aw = \underline{\varepsilon}(w)w = 0_M$, and therefore $w \neq o_N$, $w \in N'$ and there is $j \geq 1$ with $w \leq x_j$. Since $w \notin G$, we have $w < x_j$ and $b = \underline{\varepsilon}(x_j) < \underline{\varepsilon}(w) = a$ (use 2.8). On the other hand, $(a+b)u = 0_M + bu = bu$ for every $u \in N'$ and $(a+b)o_M = o_M = bo_M$. Since $x_j < o_N$, we have $b \neq 0_S$, $bN \neq \{0_M\}$ and $bo_N = o_M$. Thus $(a+b)v = bv$ for every $v \in M$, $a + b = b$ and $a \leq b$, a contradiction. ∎

4.7 Remark. Another proof of 4.6 is available here. First, by 2.15, ${}_SG$ is a characteristic semimodule and $\underline{\varepsilon}(F) \subseteq \underline{\varepsilon}(N)$, $F = G \setminus \{o_M\}$ (we have $o_M = o_G$). If $\underline{\varepsilon}(F) = \underline{\varepsilon}(N)$ then $F = N$, since $\underline{\varepsilon}$ is injective, and we get $G = M$. Now, if $G \neq M$ then $\underline{\varepsilon}(F) \subset \underline{\varepsilon}(N)$ and it follows from 4.4 that $0_S \in S$ and $\underline{\varepsilon}(N) = R$. Then $0_S = \underline{\varepsilon}(o_N)$, $o_N \in N$, and if $\underline{\varepsilon}(F) = R \setminus \{0_S\}$ then $G = M \setminus \{o_N\}$. On the other hand, if $\underline{\varepsilon}(F) = R \setminus \{a_0\}$, a_0 being the smallest element of $R \setminus \{0_S\}$, then $a_0 \leq \underline{\varepsilon}(u)$ for every $u \in N \setminus \{o_N\}$, and so $a_0 u = 0_M$.Thus $a_0 v = 0_S v$ for every $v \in G$ and, since G is faithful, we get $a_0 = 0_S$, a contradiction.

4.8 Proposition. *If (α) is satisfied then no proper subsemimodule of ${}_SM$ is faithful.*

Proof. Let, on the contrary, ${}_SG$ be a proper faithful subsemimodule of ${}_SM$. By 4.6, $o_N \in N$ and $G = M \setminus \{o_N\}$. By 4.5(ii), there is at least one infinite strictly increasing sequence $x_1 < x_2 < x_3 < \dots$ of elements from G. Since (α) is true, we have $o_N \leq x_i$ for some $i \geq 1$. But then $x_i = o_N$, a contradiction. ∎

4.9 Proposition. *Assume that (β) is true and that a subsemimodule G of ${}_SM$ is faithful whenever $G \nsubseteq \{0_M, o_M\}$* (e.g., if S is simple - see 2.14)*. Then M has at most five distinct subsemimodules and, namely, if H is a subsemimodule then either $H = \{0_M\}$ or $H = \{o_M\}$ or $H = \{0_M.o_M\}$ or $H = M$ or $o_N \in N$, $|M| \geq 4$ and $H = M \setminus \{o_N\}$ (M is infinite in this case).*

Proof. The result follows easily from 4.6. ∎

4.10 Remark. Consider the situation from 4.9.

(i) Put $F = \{\, x \in M \mid Sx \subseteq \{0_M, o_M\} \,\}$. Then F is a subsemimodule of M and $\{0_M, o_M \subseteq F$. If $F = M$ then $SM = \{0_M, o_M\}$ and $S = R$ by 2.4(i). If $o_N \in N$, $|M| \geq 4$ and $F = M \setminus \{o_N\}$ then F is faithful and infinite, $SF = \{0_M, o_M\}$ and $S = R$ again (notice that $F = M \setminus \{o_N\}$ is a characteristic semimodule, too).

(ii) Finally, assume that $F = \{0_M, o_M\}$ (e.g., if $S \neq R$ or if S is simple and $|S| \geq 3$). If $x \in M \setminus \{0_M, o_M\}$ then $x \notin F$, and hence $Sx = M$ or $Sx = M \setminus \{o_N\}$. Consequently, $Sx = M$ for every $x \in M \setminus \{0_M, o_M\}$, provided that either $o_N \notin N$ or $o_N \in N$ and $M \setminus \{o_N\}$ is not a subsemimodule.

(iii) Assume that $F = \{0_M, o_M\}$, $o_N \in N$, $|M| \geq 4$ and $G = M \setminus \{o_N\}$ is a subsemimodule. Then G is an (infinite) characteristic semimodule and $Sx = G$ for every $x \in G \setminus \{0_M, o_M\}$.

4.11 Proposition. *Assume that the semiring S is simple, $|S| \geq 3$ and (α) is true. Then:*

(i) *The semimodule ${}_SM$ has just four subsemimodules and these are $\{0_M\}$, $\{o_M\}$, $\{0_M, o_M\}$ and M.*

(ii) *$Sx = M$ for every $x \in M \setminus \{0_M, o_M\}$.*

Proof. See 2.14, 4.8, 4.9 and 4.10. ∎

In the remaining part of this section, assume that $K = M \setminus \{0_M, o_M\} \subseteq Sx$ for every $x \in K$. Let $\alpha \neq \mathrm{id}_R, R \times R$ be a congruence of the right semimodule R_S. Put $A = \{\, a \in R \mid (a, o_S) \notin \alpha \,\}$ and $B = R \setminus A$.

4.12 Lemma. *For every $x \in K$ there is $a \in R$ with $\underline{\varepsilon}(x) < a$ and $(\underline{\varepsilon}(x), a) \in \alpha$.*

Proof. There is a pair $(b, c) \in \alpha$ such that $b < c$. Since ${}_SM$ is faithful, $bu \neq cu$ for at least one $u \in M$. Of course, $bu < cu$, and hence $bu = 0_M$ and $cu = o_M$ follows from 2.4(i). Consequently, $u \in K$ and $u = dx$, $d \in S$. Now, $bdx = bu = 0_M$ and $(bd + \underline{\varepsilon}(x))y \leq (bd + \underline{\varepsilon}(x))x = 0_M$ for $y \in M$, $y \leq x$. If $z \in M$, $z \nleq x$, then $(bd + \underline{\varepsilon}(x))z = o_M$. Thus $(bd + \underline{\varepsilon}(x))v = \underline{\varepsilon}(x)v$ for every $v \in M$ and we see that $bd \leq \underline{\varepsilon}(x)$. We have $(\underline{\varepsilon}(x), a) = (bd+\underline{\varepsilon}(x), cd+\underline{\varepsilon}(x)) \in \alpha$, where $a = cd+\underline{\varepsilon}(x) \in R$, $\underline{\varepsilon}(x) \leq a$. Since $ax = cdx + \underline{\varepsilon}(x)x = cu = o_M \neq 0_M = \underline{\varepsilon}(x)x$, we get $\underline{\varepsilon}(x) < a$. ∎

4.13 Lemma.

(i) *$A \neq \emptyset \neq B$, $A \cap B = \emptyset$ and $A \cup B = R$.*

(ii) *The set B is a block of the congruence α.*

(iii) *If a_0 is maximal in A then $a_0 \notin \underline{\varepsilon}(N)$.*

Proof. Only (iii) needs a proof. If $a_0 = \underline{\varepsilon}(v)$, $v \in N$, then $v \in K$, since $a_0 \neq o_S = \underline{\varepsilon}(0_M)$, and, by 4.12, $(a_0, a) \in \alpha$ for some $a \in R$, $a_0 < a$. Since a_0 is maximal in A, we have $a \notin A$ and $(a, o_S) \in \alpha$. Since $(a_0, a) \in \alpha$, we have $(a_0, o_S) \in \alpha$, a contradiction with $a_0 \in A$. ∎

4.14 Lemma. *Let a_0 be maximal in A. Put $C = \{\, x \in M \mid a_0x = 0_M \,\}$. Then:*

(i) *$0_M \in C \subseteq N$ and $C(+)$ is a subsemilattice of $M(+)$.*

(ii) *The set C has no maximal element.*

(iii) *$a_0 < \underline{\varepsilon}(x)$ for every $x \in C$.*

(iv) *If $C = N$ then $a_0 = 0_S$ and $\alpha = \alpha_1 = (R_1 \times R_1) \cup \mathrm{id}_R$, where $R_1 = R \setminus \{0_S\}$.*

Proof. If $w \in C$ is maximal in C then $a_0 = \underline{\varepsilon}(w)$ by 4.1(ii), a contradiction with 4.13 (iii). Thus C has no maximal element and we have $a_0 < \underline{\varepsilon}(x)$ for every $x \in C$ by 2.7. If $C = N$ then $a_0 = 0_S$ by 2.10, $A = \{0_S\}$ and $\alpha = \alpha_1$. ∎

4.15 Lemma. *Assume that (β) is true and let a_0 be maximal in A. Then just one of the following three cases holds:*

1. $0_S \in S$ *and* $\alpha = \alpha_1$ (see 4.14(iv)).
2. $0_S \in S$, *the set* $R_1 = R \setminus \{0_S\}$ *has the smallest element* a_0 *and* $\alpha = (R_1 \setminus \{a_0\}) \times (R_1 \setminus \{a_0\}) \cup \mathrm{id}_R$.
3. $0_S \in S$, R_1 *has the smallest element* a_0 *and* $\alpha = (R_1 \setminus \{a_0\}) \times (R_1 \setminus \{a_0\}) \cup (\{a_0, 0_S\} \times \{a_0, 0_S\})$.

Proof. We have $a_0 \notin \underline{\varepsilon}(N)$ by 4.13(iii), and hence $0_S \in S$ by 4.4. Now, assume that $\alpha \neq \alpha_1$ (see 4.14(iv)). Then $a_0 \neq 0_S$ and, by 4.4, $o_N \in N$, a_0 is the smallest element of $R_1 = R \setminus \{0_S\}$ and $\underline{\varepsilon}(N) = R \setminus \{a_0\}$. Consequently, $B = R \setminus \{0_S, a_0\}$ and the rest is clear. ∎

4.16 Lemma. *Assume that* (γ) *is true and let* a_0 *be maximal in* A. *Then* $0_S \in S$ *and* $\alpha = \alpha_1$.

Proof. Combining 4.13(iii) and 4.2, we get $a_0 = 0_S$. The rest is clear. ∎

4.17 Lemma. *Assume that* (β) *is true and the set* A *has no maximal element. Then:*

(i) *The set* $A' = A \cap \underline{\varepsilon}(N)$ *is infinite, has no maximal element and the set* $A \setminus A'$ *contains at most one element.*

(ii) *For every* $a \in A'$ *there is an infinite strictly increasing sequence* $a_1 < a_2 < a_3 < \dots$ *of elements from* A' *such that* $a_1 = a$ *and all the elements* a_i *belong to the same block of* α.

(iii) $a_i = \underline{\varepsilon}(x_i)$, $x_i \in N$ *and* $x_1 > x_2 > x_3 > \dots$.

Proof. (i) See 4.4.
(ii) We have $a_1 = a = e(x_1)$, $x_1 \in K$. By 4.12, there is $a_2 \in R$ with $a_1 < a_2$ and $(a_1, a_2) \in \alpha$. Then $a_2 \in A$ and, since $a_1 \leq a_2$, we have $a_2 \in A'$ (use 4.4). The rest is clear. ∎

Consider the following two conditions:

(δ) If $x_1 > x_2 > x_3 > \dots$ is an infinite strictly decreasing sequence of elements from M then for every $x \in M \setminus \{0_M\}$ there is $i \geq 1$ with $x \geq x_i$.

(ε) If $a_1 < a_2 < a_3 < \dots$ is an infinite strictly increasing sequence of elements from R then for every $a \in R \setminus \{o_S\}$ there is $i \geq 1$ with $a \leq a_i$.

4.18 Lemma.

(i) (ε) *implies* (δ).

(ii) *If* (β) *is true then the conditions* (δ) *and* (ε) *are equivalent.*

Proof. Use 4.4. ∎

4.19 Lemma. *Assume that* (ε) *is true and the set* A *has no maximal element. Then* $B = \{o_S\}$.

Proof. Since A has no maximal element, there is an infinite strictly increasing sequence $a_1 < a_2 < a_3 < \dots$ of elements from A. If $b \in B \setminus \{o_S\}$ then $b \le a_i$ for some $i \ge 1$. Now $(a_i, o_S) = (a_i + b, a_i + o_S) \in \alpha$, a contradiction. ∎

4.20 Lemma. *Assume that (β) and (δ) (or (γ) and (ε)) are true and the set A has no maximal element. Then just one of the following two cases holds:*

(1) $\alpha = \alpha_2 = (R_2 \times R_2) \cup \mathrm{id}_R$, *where* $R_2 = R \setminus \{o_S\}$.

(2) $0_S \in S$ *and* $\alpha = \alpha_3 = (R_3 \times R_3) \cup \mathrm{id}_R$, *where* $R_3 = R \setminus \{0_S, o_S\}$.

Proof. The conditions (β) and (ε) are satisfied (see 4.3 and 4.18). By 4.19, $B = \{o_S\}$. Let $a, b \in A' = A \cap \underline{\varepsilon}(N)$. By 4.17(ii), there are infinite strictly increasing sequences $a = a_1 < a_2 < a_3 < \dots$ and $b = b_1 < b_2 < b_3 < \dots$ of elements from A' such that all a_i belong to one block of α and the same is true for the elements b_i. Using (ε), we find $i, j \ge 1$ such that $b \le a_i$ and $a \le b_j$. Then $(a + b, a_i) = (a + b, a_i + b) \in \alpha$, $(a + b, b_j) = (a + b, a + b_j) \in \alpha$, $(a_i, b_j) \in \alpha$ and, finally $(a, b) \in \alpha$. Consequently, $A' \times A' \subseteq \alpha$. To finish the proof, we have to use 4.4. If $\underline{\varepsilon}(N) = R$ then $A' = A = R \setminus \{o_S\}$ and (1) is true. If $0_S \in S$ and $\underline{\varepsilon}(N) = R \setminus \{0_S\}$ then $A' = A \setminus \{0_S\}$ and either (1) or (2) is true. Finally, if $\underline{\varepsilon}(N) = R \setminus \{a_0\}$ then $A' = A \setminus \{a_0\}$, $(0_S, a) \in \alpha$ for every $a \in A'$, $a > 0_S$, and we get $(a_0, a) = (0_S + a_0, a + a_0) \in \alpha$. Thus $\alpha = \alpha_2$ in this case. ∎

4.21 Proposition. *Assume that the conditions (γ) and (ε) (or (α) and (δ)) are true. Then just one of the following three cases holds:*

(1) $0_S \in S$ *and* $\alpha = \alpha_1 = (R_1 \times R_1) \cup \mathrm{id}_R$, $R_1 = R \setminus \{0_S\}$.

(2) $\alpha = \alpha_2 = (R_2 \times R_2) \cup \mathrm{id}_R$, $R_2 = R \setminus \{0_S\}$.

(3) $0_S \in S$ *and* $\alpha = \alpha_3 = (R_3 \times R_3) \cup rmid_R$, $R_3 = R \setminus \{0_S, o_S\}$.

Proof. First, if (α) and (δ) are true then (β), (γ) and (ε) follow from 4.3 and 4.18. Now, if the set A has at least one maximal element, then $\alpha = \alpha_1$ and $0_S \in S$ is proved in 4.16. On the other hand, if there is no maximal element in A, then $\alpha = \alpha_2, \alpha_3$ is proved in 4.20. ∎

5. Main results (summary)

5.1 Let S be a non-trivial semiring and M be a characteristic (left S-)semimodule. By 2.3, the mapping $\underline{\varepsilon}$ is an injective mapping of $N = M \setminus \{o_M\}$ into $R = \underline{R}(S)$ and we have $|R| \ge 2$, $|N| \ge 2$ and $|M| \ge 3$.

5.1.1 Theorem.

(i) *The (left S-)semimodule M is simple.*

(ii) *The right S-semimodule R_S is faithful and $o_S = o_R \in R$.*

(iii) *The semiring S is simple if and only if $R + S = S$ and the semimodule R_S is simple.*

(iv) $0_S \in S$ *if and only if* $0_R \in R$ *(and then* $0_S = 0_R$*).*

(v) $a = 0_S$ *if and only if* $aN = \{0_M\}$.

(vi) $\underline{\varepsilon}(w) = 0_S$ *if and only if* w *is the greatest element of the set* N.

(vii) *If* $e \in S$ *is a right multiplicatively neutral element of* S *then* $e = 1_S$ *is multiplicatively neutral. Moreover, if* $R + S = S$ *then* $0_S \in S$.

(viii) *If* $Sx = M$ *for at least one* $x \in M$ *and if* $e \in S$ *is a left multiplicatively neutral element then* $e = 1_S$ *is multiplicatively neutral.*

(ix) *If the semiring* S *is simple,* $|S| \geq 3$ *and if* $e \in S$ *is left multiplicatively neutral then* $e = 1_S$ *is multiplicatively neutral.*

Proof. See 2.2, 2.4(ii), 2.5, 2.6, 2.10, 2.11, 2.12, 2.13, 2.17 and 2.18. ∎

5.1.2 Theorem. *Assume that the condition* (α) *is true. Then:*

(i) *Either* $\underline{\varepsilon}(N) = R$ *or* $0_S \in S$ *and* $\underline{\varepsilon}(N) = R \setminus \{0_S\}$.

(ii) *No proper subsemimodule of* M *is faithful.*

(iii) *If* S *is simple then* $Sx = M$ *for every* $x \in M \setminus \{0_M, o_M\}$.

Proof. See 4.2, 4.8 and 4.11. ∎

5.1.3 Remark. Assume that the condition (α) is satisfied. Then either $\underline{\varepsilon}(N) = R$ or $0_S \in S$ and $\underline{\varepsilon}(N) = R \setminus \{0_S\}$. Now, let M' be any characteristic semimodule. According to 4.3, condition (γ) is fulfilled, and hence we can use 4.2 to show that $\underline{\varepsilon}(N') = R$ or $\underline{\varepsilon}(N') = R \setminus \{0_S\}$, where $N' = M' \setminus \{o_{M'}\}$.

(i) If $\underline{\varepsilon}(N) = R = \underline{\varepsilon}(N')$ then ${}_SM \cong {}^1_SR \cong {}_SM'$, and so ${}_SM \cong {}_SM'$ (see 3.3 and [1, 7.2]).

(ii) If $0_S \in S$ and $\underline{\varepsilon}(N) = R \setminus \{0_S\} = \underline{\varepsilon}(N')$ then ${}_SM \cong {}^2_SR \cong {}_SM'$, and so ${}_SM \cong {}_SM'$ (see 3.4 and [1, 7.3]).

(iii) Assume that $0_S \in S$, $\underline{\varepsilon}(N) = R$ and $\underline{\varepsilon}(N') = R \setminus \{0_S\}$. By 3.3 and 3.4, we have $M \cong {}^1_SR$ and $M' \cong {}^2_SR$. By [1, 7.3.7], the semimodule ${}^2_SR$ (or M') is isomorphic to a (proper) subsemimodule of ${}^1_SR$ (or M). But this is a contradiction with 5.1.2(ii).

(iv) Assume, finally, that $0_S \in S$, $\underline{\varepsilon}(N) = R \setminus \{0_S\}$ and $\underline{\varepsilon}(N') = R$. Again, we have $M \cong {}^2_SR$, $M' \cong {}^1_SR$ and M is isomorphic to a (proper) subsemimodule of M'. But M' satisfies (β), and hence $o_{N'} \in N'$ and $G = M' \setminus \{o_{N'}\}$ (see 4.6).

5.2. Let S be a semiring such that $|R| \geq 2$.

5.2.1 Theorem. *Assume that the condition* (γ) *is satisfied. The following two conditions are equivalent:*

(i) *There is a characteristic semimodule* M *such that* $0_S \in \underline{\varepsilon}(M) \setminus \{o_M\}$ *in case when* $0_S \in S$.

(ii) *Condition* 3.3(ii) *is satisfied.*

Proof. Combine 3.3 and 4.2. ∎

5.2.2 Theorem. *Assume that the condition* (γ) *is satisfied and* $0_S \in S$. *The following two conditions are equivalent:*

(i) *There is a characteristic semimodule* M *such that* $0_S \notin \underline{\varepsilon}(M) \setminus \{o_M\}$.

(ii) *Condition* 3.4(ii) *is satisfied.*

Proof. Combine 3.4 and 4.2. ∎

5.2.3 Remark. Assume that (γ) is true. If $0_S \notin S$ then there is (up to isomorphism) at most one characteristic semimodule. If $0_S \in S$ then there are (up to isomorphism) at most two characteristic semimodules.

5.2.4 Remark. Assume that $0_S \in S$. Clearly, condition 3.3(ii) implies condition 3.4(ii).

Now, assume that (ε) and 3.4(ii) are true (in fact, if (ε) is true then both $R = \underline{R}(S)$ and $\underline{R}(S)^+$ are lattices). If $b \in S$ is such that $o_Q \notin Q = Q_{0_S,b} = \{c \in R \mid cb = o_S\}$ then (ε) yields $Q = R_2 = R \backslash \{o_S\}$. But this gives a contradiction (see 3.5). It follows that the condition 3.3(ii) is true.

References

[1] Batíková, B., Kepka, T., Němec, P., *On how to construct left semimodules from the right ones*, Ital. J. Pure Appl. Math., 32 (2014), 561–578.

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THE IMPACT OF NONLINEAR INCIDENCE RATE AND REMOVABLE STORAGE MEDIA ON VIRAL PREVALENCE

Shaoheng Wang
Yuming Feng

Key Laboratory for Nonlinear Science and System Structure
School of Mathematics and Statistics
Chongqing Three Gorges University
Chongqing, 404100
P.R. China
e-mails: wshcher@163.com
yumingfeng25928@163.com

Abstract. In this paper, a new computer virus propagation model is proposed by introducing a generalized nonlinear incidence rate into the generic $SLBRS$ model. Theoretical analysis and numerical simulations show that, under some moderate conditions, the proposed model admits a globally asymptotically stable viral equilibrium. The impact of nonlinear incidence rate and removable storage media on viral prevalence is also illustrated.

Keywords: computer virus; propagation model; nonlinear incidence rate; equilibrium; global stability.

1. Introduction

Since the appearance of the first computer virus, human beings have blown the horn of a prolonged battle between the viruses and human intelligence. Nowadays, the endless war has been increasingly cruel. For one thing, multifarious computer viruses, as well as the various ways they attack computers, pose unprecedented threat to information security. For another, due to the rapid and widespread popularization of the Internet, the spreading ability of viruses has been highly enhanced. Currently, antivirus software is the major means of resisting computer viruses. Yet, in view of the serious situation and the fact that the development of new antivirus software always lags behind the emergence of new viruses, it is desperately necessary to explore the spreading behavior of viruses over the Internet. Thus, based on the appealing analogy between computer viruses and their biological counterparts, quite a few epidemic models of computer viruses, ranging from conventional models such as SIS models [1], [2], SIR models [3], [4], $SIRS$ models [5], [6], $SLBS$ models [7]-[12], to unconventional models such as delayed models [13]-[20], impulsive models [7], [21], stochastic models [7], [22] and the network-based models [12], [23], [24] have been proposed.

That the $SLBS$ model [9] was firstly presented by Yang et al based on the fact that a latent computer possesses infectivity marked a significant breakthrough in exploring the laws governing the spread of computer viruses. Afterwards, a considerable amount of work is done to extend the $SLBS$ model. Considering that an infected computer would have temporary immunity when it is cured by installing the latest version of antivirus software, Yang et al proposed an $SLBRS$ model [25]. However, this model not only ignored the influence of reinstalling operating system, but also neglected the impact of antivirus software. What'more, the assumption that all newly connected computers are all virus-free is unrealistic. Thus, a more reasonable $SLBRS$ model is studies in Ref. [26].

Unfortunately, the above-mentioned model assumed a bilinear infection rate which should be amended and overlooked the impact of removable storage media. About modifying the incidence rate, there are good reasons. On one hand, with the number of infected computers increasing and the countermeasures being strengthened, the contacts between infected computers and susceptible ones would tend to saturate, leading to a nonlinear incidence rate [27]. On the other hand, there always exists some unknown nonlinear factors in the transmission of infections [28]. As a result, this paper is intended to introduce a generalized nonlinear incidence rate. Furthermore, a susceptible computer could also be infected by an infected removable storage device such as a USB flash disk and a mobile hard disk. Therefore, the influence of the removable storage devices should be incorporated into a computer virus propagation model.

In view of foregoing statements, a new computer virus propagation model (see Figure 1) is proposed based on the $SLBRS$ model reported in Ref. [26].

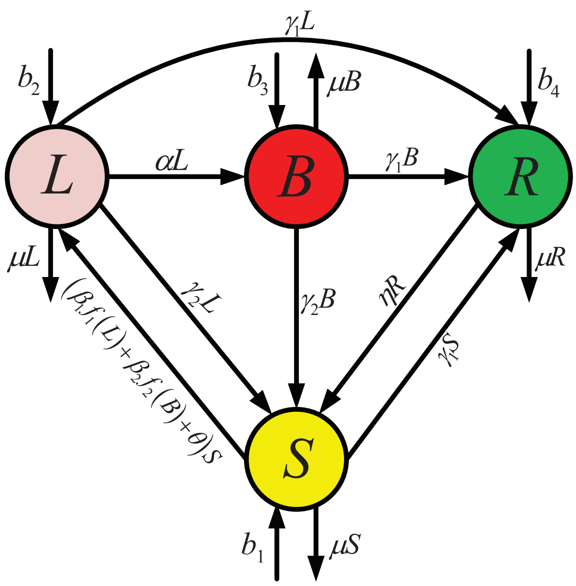

Figure 1: The state transition diagram of the new model.

We shall show by theoretical analysis that, under some moderate conditions, the proposed model admits a globally asymptotically stable viral equilibrium. Numerical simulations will be presented to verify our results, and moreover, the effect of nonlinear incidence rate and removable storage media on viral prevalence is also examined.

The rest of the paper is organized as follows. Section 2 presents the mathematical model. Section 3 proves the existence and global stability of the equilibrium, respectively. Numerical simulations are displayed in Section 4. Section 5 summarizes the work.

2. Model formulation

For the purpose of modeling, the following hypotheses are made.

($H1$) The rates that susceptible, latent, breaking, and recovered computers enter the Internet are b_1, b_2, b_3 and b_4, respectively.

($H2$) Every internal computer leaves the Internet with constant probability μ.

($H3$) Due to the influence of infected removable storage media, every susceptible computer gets infected with constant probability θ.

($H4$) Every latent computer breaks out with constant probability α.

($H5$) Due to the effect of antivirus software, every infected computer is recovered with constant probability γ_1, and every susceptible computer gets immunity with γ_1, too.

($H6$) Every infected computer becomes susceptible with constant probability γ_2 on account of reinstalling operating system.

($H7$) Every recovered computer loses immunity with constant probability η.

($H8$) Due to possible contacts with latent (resp. breaking-out) computers, at time t every susceptible computer gets infected with probability $\beta_1 f_1(L)$ (resp. $\beta_2 f_2(B)$). Here, functions f_i (i = 1, 2) have continuous second derivatives. Clearly, f_i start at the origin (i.e. $f_i(0) = 0$). Furthermore, it is assumed that f_i are strictly increasing (i.e. $f_i' > 0$) and strictly concave (i.e. $f_i'' < 0$).

Based on these hypotheses, we have a new computer virus propagation model which can be represented by the differential equations

$$
(1) \qquad \begin{cases} \dot{S} = b_1 + \gamma_2 L + \gamma_2 B + \eta R - \gamma_1 S - \mu S - \beta_1 f_1(L)S - \beta_2 f_2(B)S - \theta S, \\ \dot{L} = b_2 - \gamma_1 L - \gamma_2 L - \mu L - \alpha L + \beta_1 f_1(L)S + \beta_2 f_2(B)S + \theta S, \\ \dot{B} = b_3 + \alpha L - \gamma_1 B - \gamma_2 B - \mu B, \\ \dot{R} = b_4 + \gamma_1 S + \gamma_1 L + \gamma_1 B - \mu R - \eta R, \end{cases}
$$

where S, L, B, R represent, at time t, the average numbers of susceptible, latent, breaking, and recovered computers, respectively.

Let $N = S + L + B + R, b = b_1 + b_2 + b_3 + b_4,\ N^* = \frac{b}{\mu},\ R^* = \frac{\gamma_1 N^* + b_4}{\gamma_1 + \mu + \eta}$. Then, adding up the four equations of system (1) and simplifying, one can get $\frac{\mathrm{d}N(t)}{\mathrm{d}t} = b - \mu N(t)$, implying $\lim_{t\to\infty} N(t) = N^*$. Likewise, $\lim_{t\to\infty} R(t) = R^*$. Thus, system (1) can be written as the following limiting system [29]:

$$(2)\qquad \begin{cases} \dot{L} = b_2 - (\gamma_1 + \gamma_2 + \alpha + \mu)L + (\beta_1 f_1(L) + \beta_2 f_2(B) + \theta)(N^* - R^* - L - B), \\ \dot{B} = b_3 + \alpha L - (\gamma_1 + \gamma_2 + \mu)B. \end{cases}$$

Bellow we mainly consider the existence, uniqueness and global stability of the equilibrium point in regard to the positively invariant region: $\Omega = \{(L, B) \in R^2_+ : L + B \leq N^*\}$.

3. Theoretical analysis

3.1. Equilibrium

Theorem 1 *System* (2) *has a unique* (*viral*) *equilibrium* $E^*(L^*, B^*)$, *where* $L^*{=}x^*$, $B^* = \frac{b_3 + \alpha x^*}{\gamma_1 + \gamma_2 + \mu}$, *and* x^* *is the unique positive zero of the function*

$$\begin{aligned} g(x) \;&=\; b_2 - (\gamma_1 + \gamma_2 + \alpha + \mu)x \\ &\quad + \left[\beta_1 f_1(x) + \beta_2 f_2\left(\frac{b_3 + \alpha x}{\gamma_1 + \gamma_2 + \mu}\right) + \theta\right]\left(N^* - R^* - x - \frac{b_3 + \alpha x}{\gamma_1 + \gamma_2 + \mu}\right), \end{aligned}$$

where $x \in \left[0, \frac{(\gamma_1 + \gamma_2 + \mu)N^* - b_3}{\gamma_1 + \gamma_2 + \alpha + \mu}\right]$.

Proof. First of all, we would show that function g has at least one zero. As

$$g(0) = b_2 + \left[\beta_1 f_1(0) + \beta_2 f_2\left(\frac{b_3}{\gamma_1 + \gamma_2 + \mu}\right) + \theta\right]\left(N^* - R^* - \frac{b_3}{\gamma_1 + \gamma_2 + \mu}\right) > 0$$

and

$$g\left(\frac{(\gamma_1 + \gamma_2 + \mu)N^* - b_3}{\gamma_1 + \gamma_2 + \alpha + \mu}\right) = -(\gamma_1 + \gamma_2)N^* - b_1 - b_4 < 0,$$

it follows that g does have at least one (positive) zero. Furthermore, we have

$$\begin{aligned} &g'\left(\frac{(\gamma_1 + \gamma_2 + \mu)N^* - b_3}{\gamma_1 + \gamma_2 + \alpha + \mu}\right) = -(\gamma_1 + \gamma_2 + \alpha + \mu) \\ &-\frac{\gamma_1{+}\gamma_2{+}\alpha + \mu}{\gamma_1{+}\gamma_2{+}\mu}\left[\beta_1 f_1\left(\frac{(\gamma_1{+}\gamma_2{+}\mu)N^* - b_3}{\gamma_1{+}\gamma_2{+}\alpha{+}\mu}\right) + \beta_2 f_2\left(\frac{b_3{+}\alpha N^*}{\gamma_1{+}\gamma_2{+}\mu{+}\alpha}\right) + \theta\right] < 0 \end{aligned}$$

and

$$\begin{aligned} g''(x) = &\left[\beta_1 f_1''(x){+}\beta_2\left(\frac{\alpha}{\gamma_1{+}\gamma_2{+}\mu}\right)^2 f_2''\left(\frac{b_3{+}\alpha x}{\gamma_1{+}\gamma_2{+}\mu}\right)\right]\left(N^*{-}R^*{-}x{-}\frac{b_3{+}\alpha x}{\gamma_1{+}\gamma_2{+}\mu}\right) \\ &- 2\frac{\gamma_1 + \gamma_2 + \alpha + \mu}{\gamma_1 + \gamma_2 + \mu}\left[\beta_1 f_1'(x) + \beta_2 \frac{\alpha}{\gamma_1 + \gamma_2 + \mu} f_2'\left(\frac{b_3 + \alpha x}{\gamma_1 + \gamma_2 + \mu}\right)\right] < 0, \end{aligned}$$

where $x \in \left[0, \dfrac{(\gamma_1+\gamma_2+\mu)N^* - b_3}{\gamma_1+\gamma_2+\alpha+\mu}\right]$.

Then, we make the following discussion.

Case 1. $g'(0) \le 0$. Then, g is strictly decreasing and, thus, has a unique (positive) zero.

Case 2. $g'(0) > 0$. Let $\overline{x}{=}max\left\{x \in \left[0, \frac{(\gamma_1+\gamma_2+\mu)N^*-b_3}{\gamma_1+\gamma_2+\alpha+\mu}\right] : g'(x) \ge 0\right\}$. Then, g is strictly increasing in $[0, \overline{x}]$ and strictly decreasing in $\left[\overline{x}, \frac{(\gamma_1+\gamma_2+\mu)N^*-b_3}{\gamma_1+\gamma_2+\alpha+\mu}\right]$, implying that g has a unique (positive) zero in $\left[\overline{x}, \dfrac{(\gamma_1+\gamma_2+\mu)N^*-b_3}{\gamma_1+\gamma_2+\alpha+\mu}\right]$.

In conclusion, g has a unique (positive) zero. The proof is complete. ∎

3.2. The local stability of the equilibrium

Lemma 1 *E^* is locally asymptotically stable with respect to Ω.*

Proof. Let $S^* = N^* - R^* - L^* - B^*$. The Jacobian matrix of system (2) evaluated at E^* is

$$J_{E^*} = \begin{pmatrix} k_1 & k_2 \\ \alpha & -(\gamma_1+\gamma_2+\mu) \end{pmatrix},$$

where

$$k_1 = -(\gamma_1+\gamma_2+\alpha+\mu) + \beta_1 f_1'(L^*)S^* - [\beta_1 f_1(L^*) + \beta_2 f_2(B^*) + \theta],$$
$$k_2 = \beta_2 f_2'(B^*)S^* - [\beta_1 f_1(L^*) + \beta_2 f_2(B^*) + \theta].$$

The corresponding characteristic equation is

$$\lambda^2 + a_1\lambda + a_2 = 0,$$

where

$$a_1 = (\gamma_1+\gamma_2+\mu) - k_1,$$
$$a_2 = -k_1(\gamma_1+\gamma_2+\mu) - \alpha k_2.$$

Let $F(x) = f_1'(x)x - f_1(x)$. As $F(0) = 0$ and $F'(x) = f_1''(x)x \le 0$, we have $F(L^*) < 0$, namely $f_1'(L^*)L^* < f_1(L^*)$. Furthermore, from the second equation of system (1), we get

$$S^* < \frac{(\gamma_1+\gamma_2+\mu+\alpha)L^*}{\beta_1 f_1(L^*) + \beta_2 f_2(B^*) + \theta}.$$

Therefore, we gain

$$f_1'(L^*)S^* < \frac{(\gamma_1+\gamma_2+\mu+\alpha)L^* f_1'(L^*)}{\beta_1 f_1(L^*) + \beta_2 f_2(B^*) + \theta} < \frac{(\gamma_1+\gamma_2+\mu+\alpha)f_1(L^*)}{\beta_1 f_1(L^*) + \beta_2 f_2(B^*) + \theta}$$
$$< \frac{\gamma_1+\gamma_2+\mu+\alpha}{\beta_1}.$$

Thus, $\beta_1 f_1'(L^*)S^* < \gamma_1 + \gamma_2 + \mu + \alpha$, implying $a_1 > 0$. What's more, the proof of Theorem 1 implies that $g'(L^*) < 0$, which is equivalent to $a_2 > 0$.

Hence, it follows from the Hurwitz criterion [30] that the two roots of equation (9) have negative real parts. The claimed result follows. ∎

3.3. The global stability of the equilibrium

In this section, we are about to show the global stability of the equilibrium. First, we have the following two lemmas.

Lemma 2 *System* (2) *has no periodic orbit in the interior of* Ω.

Proof. Let

$$\begin{aligned}
h_1(L,B) &= b_2 - (\gamma_1+\gamma_2+\alpha+\mu)L \\
&\quad +[\beta_1 f_1(L) + \beta_2 f_2(B) + \theta](N^* - R^* - L - B), \\
h_2(L,B) &= b_3 + \alpha L - (\gamma_1+\gamma_2+\mu)B, \\
D(L,B) &= 1/L.
\end{aligned}$$

Then, as the proof of Lemma 1 implies that $f_1'(L)L < f_1(L)$, we get

$$\begin{aligned}
\frac{\partial(Dh_1)}{\partial L} + \frac{\partial(Dh_2)}{\partial B} &= -\frac{b_2}{L^2} - \frac{\gamma_1+\gamma_2+\mu}{L} \\
&\quad +\frac{1}{L^2}\{\beta_1(N^* - R^* - B)[Lf_1'(L) - f_1(L)] \\
&\quad -(\beta_2 f_2(B)+\theta)(N^* - R^* - B) - \beta_1 L^2 f_1'(L)\} \\
&< 0.
\end{aligned}$$

According to the Bendixson-Dulac criterion [30], system (2) admits no periodic orbit in the interior of Ω. ∎

Lemma 3 *System* (2) *has no periodic orbit that passes through a point on* $\partial\Omega$*, the boundary of* Ω.

Proof. Suppose there is a periodic orbit Γ that passes through a point $(\overline{L}, \overline{B})$ on $\partial\Omega$. Considering the smoothness of all orbits, we infer that $(\overline{L}, \overline{B})$ can not be any point of $(0,0), (0,N^*)$ and $(N^*,0)$. Then, $(\overline{L}, \overline{B})$ must be a noncorner point on $\partial\Omega$. There are three possibilities. By the truth that Γ must be tangent to $\partial\Omega$ at $(\overline{L}, \overline{B})$ if it exists, we deny all of the following possibilities with reduction to absurdity.

(1) $0 < \overline{L} < N^*, \overline{B} = 0$. Then $\mathrm{d}B/\mathrm{d}t|_{(\overline{L},\overline{B})} = b_3 + \alpha\overline{L} > 0$, implying that Γ is not tangent to $\partial\Omega$ at this point.

(2) $0 < \overline{B} < N^*, \overline{L} = 0$. Then $\mathrm{d}L/\mathrm{d}t|_{(\overline{L},\overline{B})} = b_2 + (\beta_2 f_2(\overline{B}) + \theta)(N^* - R^* - \overline{B}) > 0$, implying that Γ is not tangent to $\partial\Omega$ at this point.

(3) $\overline{L} + \overline{B} = N^*, \overline{L} \neq 0, \overline{B} \neq 0$. Then $\mathrm{d}(L+B)/\mathrm{d}t|_{(\overline{L},\overline{B})} = -b_1 - b_4 - (\gamma_1+\gamma_2)N^* < 0$, implying that Γ is not tangent to $\partial\Omega$ at this point.

On account of the above discussions, we come to a conclusion that system (2) has no periodic orbit that passes through a point on $\partial\Omega$. ■

The main result of this paper is displayed as follows.

Theorem 2 *E^* is globally asymptotically stable with respect to Ω.*

Proof. On the basis of the generalized *Poincaré-Bendixson* theorem [30], one can reach the claimed result easily by combining Lemmas 1-3. ■

Remark 1 Theorem 2 implies that it would be practically impossible to eradicate computer viruses on the Internet. Figures 2-3 verify the obtained result.

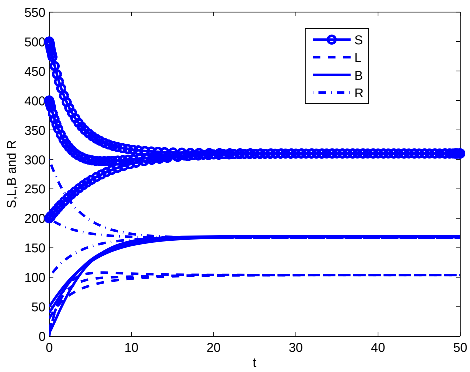

Figure 2: Time plots of S, L, B, R for a common system with three different initial conditions, where $f_1(L) = L/(1 + L)$, $f_2(B) = B/(1 + B)$.

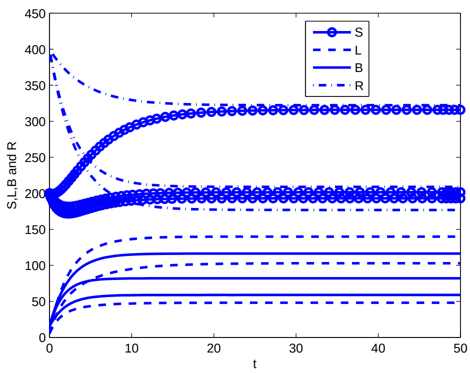

Figure 3: Time plots of S, L, B, R for three different systems with a common initial condition, where $f_1(L) = L/(1 + L)$, $f_2(B) = B/(1 + B)$.

Remark 2 From Figure 3, we can clearly see that the steady number of the infected computers varies with model parameters, implying that viral spread could be controlled below an acceptable level by adjusting the corresponding parameters. Specifically, in real life, we should update our antivirus software timely, run a virus scan when some removable storage devices are connected to our computers and do not click unknown links etc.

4. Numerical Simulations

This section examines the effect of nonlinear incidence rate as well as the removable storage media on the spread of computer viruses. Figure 4 describes the impact of some specific nonlinear incidence rates on viral spreading, from which it can be seen that computer viruses are suppressed in the case of the nonlinear incidence rate. It is logical to draw that a relatively flat incidence rate contributes to the containment of computer viruses. Figure 5 shows that computer viruses transmission levels are increasing with θ. This implies that the removable storage devices had better be scanned when they are connected to the computer.

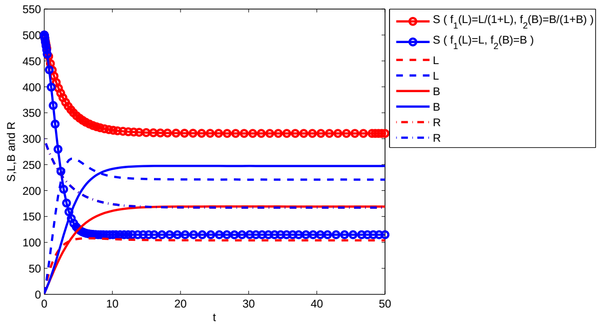

Figure 4: The comparison between the nonlinear and linear incidence rate.

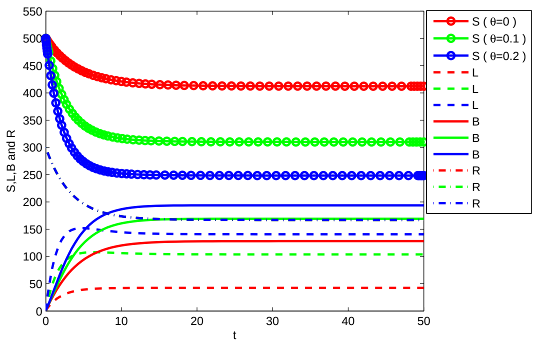

Figure 5: The effect of the infected removable storage media on viral prevelance, where $f_1(L) = L/(1+L)$, $f_2(B) = B/(1+B)$.

Actually, there are quite a few nonlinear functions satisfying the assumption (H8). For example, $f(x) = kx^m \ \ (k > 0,\ 0 < m < 1)$ also fits the postulate. So carrying out more numerical simulations to get more accurate results would be a part of our next work. In addition, it is definitely undeniable that the existing incidence rates don't necessarily meet the mentioned condition. In other words, we need to consider those incidence rates which don't satisfy the hypothesis in subsequent research work.

5. Conclusions

This paper has examined the impact of nonlinear incidence rate as well as the removable storage media on viral prevalence based on an improved $SLBRS$ model.

It has been shown that the new model has a globally asymptotically stable viral equilibrium. And by setting a specific nonlinear incidence rate we have deduced that a relatively flat incidence rate conduces to the containment of computer viruses. The influence of removable storage media has also been illustrated. On this basis, several useful suggestions have been given.

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References

[1] Kephart, J.O., White, S.R., *Directed-graph epidemiological models of computer viruses*, in: Proc. 1991 IEEE Symp. Security Privacy, 1991, 343-359.

[2] Billings, L., Spears, W.M., Schwartz, I.B., *A unified prediction of computer virus spread in connected networks*, Phys. Lett., A 297 (3-4) (2002), 261-266.

[3] Ren, J., Yang, X., Zhu, Q., Yang, L.X., Zhang, C., *A novel computer virus model and its dynamics*, Nonlinear Anal. – Real, 13 (1) (2012), 376-384.

[4] Zhu, Q., Yang, X., Ren, J., *Modeling and analysis of the spread of computer virus*, Commun. Nonlinear Sci. Numer. Simul., 17 (12) (2012), 5117-5124.

[5] Gan, C., Yang, X., Liu, W., Zhu, Q., Zhang, X., *Propagation of computer virus under human intervention: a dynamical model*, Discrete Dyn. Nat. Soc., 2012 (2012). Article ID 106950.

[6] Gan, C., Yang, X., Liu, W., Zhu, Q., *A propagation model of computer virus with nonlinear vaccination probability*, Commun. Nonlinear Sci. Numer. Simul. http://dx.doi.org/10.1016/j.cnsns.2013.06.018.

[7] Yang, X., Yang, L.-X., *Towards the epidemiological modeling of computer viruses*, Discrete Dyn. Nat. Soc., 2012 (2012). Article ID 259671.

[8] Yang, L.-X., Yang, X., Wen, L., Liu, J., *Propagation behavior of virus codes in the situation that infected computers are connected to the Internet with positive probability*, Discrete Dyn. Nat. Soc., 2012 (2012). Article ID 693695.

[9] Yang, L.-X., Yang, X., Wen, L., Liu, J., *A novel computer virus propagation model and its dynamics*, Int. J. Comput. Math., 89 (17) (2012), 2307-2314.

[10] Yang, L.-X., Yang, X., *The spread of computer viruses under the influence of removable storage devices*, Appl. Math. Comput., 219 (8) (2012), 3914-3922.

[11] Yang, L.-X., Yang, X., Wen, L., *A computer virus model with graded cure rates*, Nonlinear Anal. – Real, 14 (1) (2013), 414-422.

[12] Yang, L.-X., Yang, X., Liu, J., Zhu, Q., Gan, C., *Epidemics of computer viruses: a complex-network approach*, Appl. Math. Comput., 219 (16) (2013), 8705-8717.

[13] Mishra, B.K., Jha, N., *Fixed period of temporary immunity after run of anti-malicious software on computer nodes*, Appl. Math. Comput., 190 (2) (2007), 1207-1212.

[14] Mishra, B.K., Saini, D.K., *SEIRS epidemic model with delay for transmission of malicious objects in computer network*, Appl. Math. Comput., 188 (2) (2007), 1476-1482.

[15] Han, X., Tan, Q., *Dynamical behavior of computer virus on Internet*, Appl. Math. Comput., 217 (6) (2010), 2520-2526.

[16] Ren, J., Yang, X., Yang, L.-X., Xu, Y., Yang, F., *A delayed computer virus propagation model and its dynamics*, Chaos Soliton. Fract., 45 (1) (2012), 74-79.

[17] Feng, L., Liao, X., Li, H., Han, Q., *Hopf bifurcation analysis of a delayed viral infection model in computer networks*, Math. Comput. Model., 56 (7-8) (2012), 167-179.

[18] Zhu, Q., Yang, X., Yang, L.-X., Zhang, C., *Optimal control of computer virus under a delayed model*, Appl. Math. Comput., 218 (23) (2012), 11613-11619.

[19] Dong, T., Liao, X., Li, H., *Stability and Hopf bifurcation in a computer virus model with multistate antivirus*, Abstr. Appl. Anal., 2012 (2012). Article ID 841987.

[20] Yang, X., Mishra, B.K., Liu, Y., *Computer viruses: theory, model, and methods*, Discrete Dyn. Nat. Soc., 2012 (2012). Article ID 473508.

[21] Zhang, C., Zhao, Y., Wu, Y., *An impulse model for computer viruses*, Discrete Dyn. Nat. Soc., 2012 (2012). Article ID 260962.

[22] Zhang, C., Zhao, Y., Wu, Y., Deng, S., *A stochastic dynamic model of computer viruses*, Discrete Dyn. Nat. Soc., 2012 (2012). Article ID 264874.

[23] Pastor-Satorras, R., Vespignani, A., *Epidemic spreading in scale-free networks*, Phys. Rev. Lett., 86 (14) (2001) , 3200-3203.

[24] Pastor-Satorras, R., Vespignani, A., *Epidemic dynamics and endemic states in complex networks*, Phys. Rev., E 63 (6) (2001), Article ID 066117.

[25] Yang, M., Zhang, Z., Li, Q., Zhang, G., *An SLBRS model with vertical transmission of computer virus over the internet*, Discrete Dyn. Nat. Soc., 2012; 2012 [Article ID 925648].

[26] Yang, X., Liu, B., Gan, C., *Global stability of an epidemic model of computer virus*, Abstr. Appl. Anal., 2014 (2014). Article ID 456320.

[27] Capasso, V., Serio, G., *A generalization of the Kermack-McKendrick deterministic epidemic model*, Math. Biosci., 42 (1-2) (1978), 43-61.

[28] Korobeinikov, A., *Global properties of infectious disease models with nonlinear incidence*, Bull. Math. Biol., 69 (6) (2007), 1871-1886.

[29] Thieme, H.R., *Asymptotically autonomous differential equations in the plane*, Rocky Mt. J. Math., 24 (1) (1994), 351-380.

[30] Robinson, R.C., *An Introduction to Dynamical Systems: Continuous and Discrete*, Prentice Hall, 2004.

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ON EXTENSIONS OF k-SUBADDITIVE LATTICE GROUP-VALUED CAPACITIES

G. Barbieri

Dipartimento di Matematica e Informatica
University of Udine
via delle Scienze, 206, I-33100 Udine
Italy
e-mail: giuseppina.barbieri@uniud.it

A. Boccuto

Dipartimento di Matematica e Informatica
University of Perugia
via Vanvitelli 1, I-06123 Perugia
Italy
e-mail: antonio.boccuto@unipg.it

Abstract. We prove some theorems on extension of lattice group-valued k-subadditive monotone set functions, continuous and (s)-bounded with respect to a single regulator. Furthermore, we pose some open problems.

Key words: lattice group, capacity, k-subadditivity, continuity from above, continuity from below, (s)-boundedness, extension.

2010 AMS Subject Classifications: 06F20, 06F30, 26E50, 28A12, 28B15, 46G10, 46G12.

1. Introduction

The non-additive set functions have been the object of several studies. Among the related literature, we quote for instance [1], [19], [31], [35] and their bibliographies. In [17], [18] it is dealt with the so-called *M-measures*, that are increasing set functions, continuous from above and from below and compatible with respect to finite suprema and infima, which have several applications, for example to intuitionistic fuzzy events and observables (see also [1], [33]).

Here we prove some extension results for a continuous k-subadditive lattice-group valued capacity, (s)-bounded with respect to a single regulator, defined on a ring $\mathcal{W}$, to the σ-ring $\sigma(\mathcal{W})$ generated by $\mathcal{W}$, extending earlier results proved in [21] and [31]. We first construct a continuous extension by considering unions of suitable increasing sequences and/or intersections of suitable decreasing sequences of sets, using an approach similar to that in [21], and afterwards we prove (s)-boundedness (and continuity) with respect to a single regulator of the found extension, by "approximating" an element of $\sigma(\mathcal{W})$ with a suitable set of $\mathcal{W}$, by means of a technique similar to that used in [11] in the finitely and countably additive cases. Some other results about extensions of finitely additive or modular real-valued, lattice group- or vector lattice-valued measures can be found, for instance, in [3]-[6], [8], [17], [18], [24]-[30], [32], [36], [37].

We often use the tool of (D)-*convergence* in the lattice group setting, which allows us to apply the Fremlin Lemma, by means of which it is possible to replace a sequence of regulators with a single regulator.

In the literature, the study of extensions of set functions is also related to different kinds of limit theorems. For a recent literature about these topics, see also [10]-[14], [20] and, for a comprehensive overview, see for example [15], [31] and their bibliographies. In [16], some kinds of limit theorems are proved for lattice group-valued k-subadditive capacities. Finally, we pose some open problems.

2. Preliminaries

We begin with recalling the following basic concepts on lattice groups (see also [15]).

Definitions 2.1

(a) An abelian partially ordered group $R = (R, +, \leq)$ with neutral element 0 is called a *lattice group* iff it is a lattice (that is $a \vee b$ and $a \wedge b$, the *supremum* and the *infimum* between a and b, respectively, belong to R for any a, $b \in R$) and $a + c \leq b + c$ whenever a, b, $c \in R$ and $a \leq b$.

(b) For every element x of a lattice group R, set $x^+ = x \vee 0$ and $x^- = (-x) \vee 0$. The elements x^+ and x^- are called the *positive* and *negative part* of x, respectively. Given $x \in R$, the *absolute value* $|x|$ of x is defined by $|x| = x \vee (-x)$. It is not difficult to see that $x = x^+ - x^-$ and $|x| = x^+ + x^-$ for every $x \in R$.

(c) A nonempty set $A \subset R$ is said to be *bounded from above* (*from below*, respectively) iff there exists an element $s \in R$ with $a \leq s$ ($a \geq s$, respectively) for every $a \in A$. We say that A is *bounded* iff it is bounded both from above and from below.

(d) A lattice group R is said to be *Dedekind complete* iff every nonempty subset of R, bounded from above (from below, respectively), has supremum (infimum, respectively) in R.

(e) A Dedekind complete lattice group R is said to be *super Dedekind complete* iff for every nonempty set $A \subset R$, bounded from above, there is a finite or countable subset of A having the same supremum as A.

(f) A sequence $(\sigma_p)_p$ in R is called (O)-*sequence* iff it is decreasing and $\bigwedge_{p=1}^{\infty} \sigma_p = 0$.

(g) A bounded double sequence $(a_{t,r})_{t,r}$ in R is a (D)-*sequence* or a *regulator* iff $(a_{t,r})_r$ is an (O)-sequence for every $t \in \mathbb{N}$.

(h) A lattice group R is said to be *weakly σ-distributive* iff $\bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} \Big(\bigvee_{t=1}^{\infty} a_{t,\varphi(t)}\Big) = 0$, for every (D)-sequence $(a_{t,r})_{t,r}$ in R.

(i) A sequence $(x_n)_n$ in R is said to be *order convergent* (or (O)-*convergent*) to x iff there exists an (O)-sequence $(\sigma_p)_p$ in R such that for every $p \in \mathbb{N}$ there is a positive integer n_0 with $|x_n - x| \le \sigma_p$ for each $n \ge n_0$, and in this case we write $(O) \lim\limits_n x_n = x$.

(j) If $(x_n)_n$ is a bounded sequence in R, then set

$$\limsup_n x_n = \bigwedge_{s=1}^{\infty} \Big(\bigvee_{n=s}^{\infty} x_n\Big), \quad \liminf_n x_n = \bigvee_{s=1}^{\infty} \Big(\bigwedge_{n=s}^{\infty} x_n\Big).$$

Note that $(O) \lim\limits_n x_n = x$ if and only if $\limsup\limits_n x_n = \liminf\limits_n x_n = x$ (see also [15]).

(k) A sequence $(x_n)_n$ in R is (D)-*convergent* to x iff there is a (D)-sequence $(a_{t,r})_{t,r}$ in R such that, for every $\varphi \in \mathbb{N}^{\mathbb{N}}$, there is $n^* \in \mathbb{N}$, with $|x_n - x| \le \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}$ whenever $n \ge n^*$, and we write $(D) \lim\limits_n x_n = x$.

(l) We call *sum* of a series $\sum_{n=1}^{\infty} x_n$ in R the limit $(O) \lim\limits_n \sum_{j=1}^{n} x_j$, if it exists in R.

Remarks 2.2

(a) Observe that in every Dedekind complete lattice group R any (O)-convergent sequence is also (D)-convergent, while the converse is true if and only if R is weakly σ-distributive.

(b) Some examples of super Dedekind complete and weakly σ-distributive lattice groups are the space $\mathbb{N}^{\mathbb{N}}$ endowed with the usual componentwise order and the space $L^0(X, \mathcal{B}, \nu)$ of all ν-measurable functions defined on a set function space $(X, \mathcal{B}, \nu)$ with the identification up to ν-null sets endowed with almost everywhere convergence, where ν is a positive, countably additive and σ-finite extended real-valued set function (see also [15]).

We now recall the Fremlin Lemma, which has a fundamental importance in the setting of (D)-convergence, because it allows us to replace a sequence of regulators with a single (D)-sequence.

Lemma 2.3 (see also [23, Lemma 1C], [33, Theorem 3.2.3]) *Let R be any Dedekind complete lattice group and $(a_{t,r}^{(n)})_{t,r}$, $n \in \mathbb{N}$, be a sequence of regulators in R. Then for every $u \in R$, $u \ge 0$ there is a (D)-sequence $(a_{t,r})_{t,r}$ in R with*

$$u \wedge \Big(\sum_{n=1}^{\infty} \Big(\bigvee_{t=1}^{\infty} a_{t,\varphi(t+n)}^{(n)}\Big)\Big) \le \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \quad \textit{for every } \varphi \in \mathbb{N}^{\mathbb{N}}.$$

We now recall the following version of the Fremlin lemma in the setting of (O)-convergence, which allows to replace a countable family of (O)-sequences with a single (O)-sequence, and will be useful in the sequel.

Lemma 2.4 (see also [13, Lemma 2.8]) *Let R be a super Dedekind complete and weakly σ-distributive lattice group, and $\{(\sigma_p^{(n)})_p : n \in \mathbb{N}\}$ be a countable family of (O)-sequences in R, such that the set $\{(\sigma_p^{(n)})_p$: n, $p \in \mathbb{N}\}$ is bounded in R. Then there exists an (O)-sequence $(b_j)_j$, such that for every $j, n \in \mathbb{N}$ there is $p = p(j,n) \in \mathbb{N}$ with $\sigma_p^{(n)} \le b_j$.*

We now recall some fundamental properties of lattice group-valued capacities (see also [16], [19], [31]). From now on, R is a super Dedekind complete and weakly σ-distributive lattice group, G is any infinite set, $\mathcal{P}(G)$ is the family of all subsets of G, $\mathcal{W} \subset \mathcal{P}(G)$ is a ring, $\sigma(\mathcal{W})$ is the smallest sub-σ-ring of $\mathcal{P}(G)$ containing $\mathcal{W}$, and k is a fixed positive integer.

Definitions 2.5

(a) A *capacity* $m : \mathcal{W} \to R$ is a bounded increasing set function with $m(\emptyset) = 0$.

(b) We say that a capacity m is *k-subadditive* on $\mathcal{W}$ iff

$$m(A \cup B) \le m(A) + k\, m(B) \quad \text{whenever } A, B \in \mathcal{W},\ A \cap B = \emptyset. \tag{2.1}$$

(c) When $R = \mathbb{R}$, a 1-subadditive capacity is called also a *submeasure* (see also [15], [21], [22]).

We now recall the following result.

Proposition 2.6 (see [16, Proposition 3.2]) *A capacity m is k-subadditive on $\mathcal{W}$ if and only if*

$$m\Big(\bigcup_{q=1}^{n} E_q\Big) \le m(E_1) + k \sum_{q=2}^{n} m(E_q) \tag{2.2}$$

for each $n \in \mathbb{N}$, $n \ge 2$, and whenever $E_1, E_2, \dots, E_n \in \mathcal{W}$.

Definitions 2.7

(a) Let $E \in \mathcal{W}$. We say that a capacity m is *continuous from above* (*from below*, respectively) *at E* iff

$$\bigwedge_{n=1}^{\infty} m(E_n) = (D) \lim_n m(E_n) = m(E) \tag{2.3}$$

$$\Big(\bigvee_{n=1}^{\infty} m(E_n) = (D) \lim_n m(E_n) = m(E), \text{ respectively } \Big) \tag{2.4}$$

for every decreasing (increasing, respectively) sequence $(E_n)_n$ in $\mathcal{W}$ such that $\bigcap_{n=1}^{\infty} E_n = E \in \mathcal{W}$ $\quad$ ($\bigcup_{n=1}^{\infty} E_n = E \in \mathcal{W}$, respectively).

(b) A capacity m is *continuous from above* (*from below*, respectively) *on* $\mathcal{W}$ iff it is continuous from above (from below, respectively) at every $E \in \mathcal{W}$.

(c) If in (2.3) ((2.4), respectively) it is possible to take the involved (D)-limits with respect to a single regulator, then we say that m is *globally continuous from above* (*globally continuous from below*) *at* E, respectively. Similarly as above, the concepts of *global continuity from above* and *from below on* $\mathcal{W}$ can be formulated.

Note that, when $R = \mathbb{R}$, the concepts of continuity and global continuity are equivalent (see also [15]).

(d) A capacity $m : \mathcal{W} \to R$ is said to be (s)-*bounded on* $\mathcal{W}$ iff

$$(D) \lim_n m(C_n) = 0 \tag{2.5}$$

for every disjoint sequence $(C_n)_n$ in $\mathcal{W}$.

(e) If the (D)-limit in (2.5) can be taken with respect to a single (D)-sequence, then m is said to be *globally* (s)-*bounded on* $\mathcal{W}$.

3. The main results

We begin with giving the following

Proposition 3.1 *Let $m : \mathcal{W} \to R$ be a k-subadditive capacity, continuous from above at $\emptyset$. Then m is continuous from above and from below.*

Proof. We first prove continuity from above. Let $(A_n)_n$ be a decreasing sequence in $\mathcal{W}$, $A := \bigcap_{n=1}^{\infty} A_n$, $A \in \mathcal{W}$, and let $B_n := A_n \setminus A$. We get $B_n \in \mathcal{W}$ for each $n \in \mathbb{N}$, $\bigcap_{n=1}^{\infty} B_n = \emptyset$, and hence

$$(D) \lim_n m(B_n) = \bigwedge_{n=1}^{\infty} m(B_n) = 0.$$

Taking into account monotonicity and k-subadditivity of m (see also Proposition 2.6), we obtain

$$0 \le m(A_n) - m(A) \le k\, m(A_n \setminus A) = k\, m(B_n),$$

and so

$$0 \le \limsup_n (m(A_n) - m(A)) \le k \bigwedge_{n=1}^{\infty} m(B_n) = 0.$$

Therefore $(D) \lim_n (m(A_n) - m(A)) = 0$, namely $(D) \lim_n (m(A_n)) = m(A)$, that is

$$m(A) = (D) \lim_n m(A_n) = \bigwedge_{n=1}^{\infty} m(A_n).$$

Thus, we obtain continuity from above of m.

We now prove continuity from below. Let $(E_n)_n$ be an increasing sequence of elements of $\mathcal{W}$, $E := \bigcup_{n=1}^{\infty} E_n$, $E \in \mathcal{W}$. Let $F_n := E \setminus E_n$, $n \in \mathbb{N}$. Note that $F_n \in \mathcal{W}$ for every $n \in \mathbb{N}$ and that $\bigcap_{n=1}^{\infty} F_n = \emptyset$. Hence, by hypothesis, we get $(D)\lim_n m(F_n) = \bigwedge_{n=1}^{\infty} m(F_n) = 0$. By monotonicity and k-subadditivity of m, we have

$$0 \le m(E) - m(E_n) \le k\, m(E \setminus E_n) = k\, m(F_n),$$

and hence

$$0 \le \limsup_n (m(E) - m(E_n)) \le k \bigwedge_{n=1}^{\infty} m(F_n) = 0.$$

Thus, $(D)\lim_n (m(E) - m(E_n)) = 0$, that is $m(E) = (D)\lim_n m(E_n) = \bigvee_{n=1}^{\infty} m(E_n)$. So, we get that m is continuous from below. ∎

Set now $\mathcal{W}^+ := \{E \subset G$: there exists an (increasing) sequence $(E_n)_n$ in $\mathcal{W}$ with $E = \bigcup_{n=1}^{\infty} E_n\}$, $\mathcal{W}^- := \{E \subset G$: there is a (decreasing) sequence $(E_n)_n$ in $\mathcal{W}$ with $E = \bigcap_{n=1}^{\infty} E_n\}$, and similarly let us define $\mathcal{W}^{+-}$ and $\mathcal{W}^{-+}$. It is not difficult to see that $\mathcal{W}^+$, $\mathcal{W}^-$, $\mathcal{W}^{+-}$ and $\mathcal{W}^{-+}$ are four lattices, $\mathcal{W}^+$ and $\mathcal{W}^{-+}$ are closed under countable (increasing) unions, $\mathcal{W}^-$ and $\mathcal{W}^{+-}$ are closed under countable (decreasing) intersections. We will give extension results for globally (s)-bounded, k-subadditive and continuous R-valued capacities from a ring $\mathcal{W}$ to $\sigma(\mathcal{W})$, extending [11, Theorem 4.4], [21, Theorem 18], [26, Theorem 2.5], [31, Theorems 5.11 and 5.12]. To this aim, we proceed in several steps.

Theorem 3.2 *Let $m_0 : \mathcal{W} \to R$ be a k-subadditive capacity, globally (s)-bounded and continuous from above at $\emptyset$ on $\mathcal{W}$, and define $m_0^+ : \mathcal{W}^+ \to R$ as*

$$m_0^+\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \bigvee_{n=1}^{\infty} m_0(E_n), \tag{3.1}$$

whenever $E \in \mathcal{W}^+$ and $(E_n)_n$ is any increasing sequence in $\mathcal{W}$ with $E = \bigcup_{n=1}^{\infty} E_n$. Then $m_0^+ : \mathcal{W}^+ \to R$ is a k-subadditive capacity, continuous from above and from below on $\mathcal{W}^+$.

Proof. First of all, we prove that the set function m_0^+ in (3.1) is well-defined. Let $E \in \mathcal{W}^+$, $E = \bigcup_{n=1}^{\infty} E_n = \bigcup_{q=1}^{\infty} F_q$, where $(E_n)_n$ and $(F_q)_q$ are any two increasing sequences in $\mathcal{W}$. Since, by Proposition 3.1, m_0 is continuous from below on $\mathcal{W}$, then we get

$$\begin{aligned}\bigvee_{n=1}^{\infty} m_0(E_n) &= \bigvee_{n=1}^{\infty} m_0(E_n \cap E) = \bigvee_{n=1}^{\infty} m_0\Big(\bigcup_{q=1}^{\infty}(E_n \cap F_q)\Big) \\ &= \bigvee_{n=1}^{\infty}\Big(\bigvee_{q=1}^{\infty} m_0(E_n \cap F_q)\Big) = \bigvee_{q=1}^{\infty}\Big(\bigvee_{n=1}^{\infty} m_0(E_n \cap F_q)\Big) \\ &= \bigvee_{q=1}^{\infty} m_0\Big(\bigcup_{n=1}^{\infty}(E_n \cap F_q)\Big) = \bigvee_{q=1}^{\infty} m_0(E \cap F_q) = \bigvee_{q=1}^{\infty} m_0(F_q),\end{aligned}$$

and so m_0^+ is well-defined.

We now prove that m_0^+ is monotone. Let A, $B \in \mathcal{W}^+$, $A \subset B$, $A = \bigcup_{n=1}^{\infty} A_n$, $B = \bigcup_{n=1}^{\infty} B_n$, where $(A_n)_n$ and $(B_n)_n$ are two increasing sequences in $\mathcal{W}$. For each $n \in \mathbb{N}$, set $C_n := A_n \cap B_n$. Note that $(C_n)_n$ is an increasing sequence in $\mathcal{W}$ and $\bigcup_{n=1}^{\infty} C_n = A$. By monotonicity of m_0, for any $n \in \mathbb{N}$ we get

$$m_0(C_n) \le m_0(B_n) \le \bigvee_{n=1}^{\infty} m_0(B_n) = m_0^+(B). \tag{3.2}$$

Taking in (3.2) the supremum as n varies in $\mathbb{N}$, we obtain

$$m_0^+(A) = \bigvee_{n=1}^{\infty} m_0(C_n) \le m_0^+(B).$$

From this and arbitrariness of A and B we get monotonicity of m_0^+.

We now prove that m_0^+ is k-subadditive. To this aim, choose arbitrarily A, $B \in \mathcal{W}^+$, $A = \bigcup_{n=1}^{\infty} A_n$, $B = \bigcup_{n=1}^{\infty} B_n$, where $(A_n)_n$ and $(B_n)_n$ are two increasing sequences in $\mathcal{W}$. For every $n \in \mathbb{N}$, set $D_n := A_n \cup B_n$. Note that $(D_n)_n$ is an increasing sequence in $\mathcal{W}$ and $\bigcup_{n=1}^{\infty} D_n = A \cup B$. By monotonicity and k-subadditivity of m_0 on $\mathcal{W}$ we have

$$\begin{aligned} m_0(D_n) &\le m_0(A_n) + k\, m_0(B_n) \quad \text{for each } n \in \mathbb{N}, \\ m_0^+(A \cup B) &= \bigvee_{n=1}^{\infty} m_0(D_n) \le \bigvee_{n=1}^{\infty} m_0(A_n) + k \bigvee_{n=1}^{\infty} m_0(B_n) \\ &= m_0^+(A) + k\, m_0^+(B). \end{aligned} \tag{3.3}$$

The k-subadditivity of m_0^+ follows from (3.3) and arbitrariness of A and B.

Let $(a_{t,r})_{t,r}$ be a regulator, related to global (s)-boundedness of m_0 on $\mathcal{W}$. We claim that, if $E \in \mathcal{W}^+$ with $E = \bigcup_{n=1}^{\infty} E_n$, where $(E_n)_n$ is any increasing sequence in $\mathcal{W}$, then

$$(D) \lim_n m_0^+(E \setminus E_n) = 0 \tag{3.4}$$

with respect to $(a_{t,r})_{t,r}$. From this it will follow that for each $E \in \mathcal{W}^+$ and $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is a set $E^- \in \mathcal{W}$ with

$$E^- \subset E \text{ and } m_0^+(E \setminus E^-) \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}. \tag{3.5}$$

Indeed, if $(E_n)_n$ is any increasing sequence in $\mathcal{W}$, then for each $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is $\overline{n} \in \mathbb{N}$ with

$$m_0(E_{n+p} \setminus E_n) \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \tag{3.6}$$

whenever $n \geq \overline{n}$ and $p \in \mathbb{N}$. Otherwise there exist an element $\varphi \in \mathbb{N}^{\mathbb{N}}$ and two sequences $(n_i)_i$, $(p_i)_i$ in $\mathbb{N}$, with $n_{i+1} > n_i + p_i$ and

$$m_0(E_{n_i+p_i} \setminus E_{n_i}) \not\leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}$$

for every $i \in \mathbb{N}$, getting a contradiction with global (s)-boundedness of m_0 on $\mathcal{W}$ with respect to the (D)-sequence $(a_{t,r})_{t,r}$. Moreover note that, since

$$\bigcup_{p=1}^{\infty} (E_{n+p} \setminus E_n) = E \setminus E_n \in \mathcal{W}^+$$

for every $n \in \mathbb{N}$, then

$$m_0^+(E \setminus E_n) = \bigvee_{p=1}^{\infty} m_0(E_{n+p} \setminus E_n) \text{ for each } n. \tag{3.7}$$

Taking in (3.6) the supremum as p tends to $+\infty$ and keeping fixed n, from (3.6) and (3.7) we obtain

$$m_0^+(E \setminus E_n) \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \tag{3.8}$$

for each $n \geq \overline{n}$, that is the claim.

We now prove that m_0^+ is continuous from below on $\mathcal{W}^+$. Let $(A_l)_l$ be an increasing sequence in $\mathcal{W}^+$, and set $A = \bigcup_{l=1}^{\infty} A_l$. For each $l \in \mathbb{N}$ there is an increasing sequence $(A_l^n)_n$ in $\mathcal{W}$ with $\bigcup_{n=1}^{\infty} A_l^n = A_l$. For each l, $n \in \mathbb{N}$, set $B_l^n := \bigcup_{s=1}^{l} A_s^n$. It is not difficult to see that the sequences $(B_l^n)_n$ and $(B_l^n)_l$ are in $\mathcal{W}$, are increasing and $A_l = \bigcup_{n=1}^{\infty} B_l^n$ for any $l \in \mathbb{N}$. For each $n \in \mathbb{N}$, set $C_n = B_n^n$. It is not difficult to see that $(C_n)_n$ is an increasing sequence in $\mathcal{W}$ and $A = \bigcup_{n=1}^{\infty} C_n$. By (3.4), for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is $\bar{l} \in \mathbb{N}$ with

$$m_0^+(A) - m_0(C_l) \le k\, m_0^+(A \setminus C_l) \le k \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}. \tag{3.9}$$

Since $A_l \supset C_l$ and m_0^+ is monotone, we get

$$m_0^+(A) - m_0^+(A_l) \le m_0^+(A) - m_0^+(C_l). \tag{3.10}$$

From (3.9) and (3.10) it follows that $(D)\lim_l m_0^+(A_l) = \bigvee_{l=1}^{\infty} m_0^+(A_l) = m_0^+(A)$, namely (global) continuity from below of m_0^+ on $\mathcal{W}^+$ with respect to the regulator $(a_{t,r})_{t,r}$.

We now prove that m_0^+ is continuous from above at $\emptyset$ on $\mathcal{W}^+$. Let $(E_n)_n$ be any decreasing sequence in $\mathcal{W}^+$ with $\bigcap_{n=1}^{\infty} E_n = \emptyset$, and choose arbitrarily $\varphi \in \mathbb{N}^{\mathbb{N}}$. By (3.5), there is a sequence in $(F_n)_n$ in $\mathcal{W}$ with $F_n \subset E_n$ and

$$m_0^+(E_n \setminus F_n) \le \bigvee_{t=1}^{\infty} a_{t,\varphi(t+n)} \tag{3.11}$$

for every $n \in \mathbb{N}$. Set $H_n = \bigcap_{j=1}^{n} F_j$, $n \in \mathbb{N}$. Note that $H_n \in \mathcal{W}$ and $H_n \subset E_n$ for each $n \in \mathbb{N}$, and hence $\bigcap_{n=1}^{\infty} H_n = \emptyset$. By continuity from above at $\emptyset$ of m_0 on $\mathcal{W}$, we have

$$\bigwedge_{n=1}^{\infty} m_0(H_n) = (D)\lim_n m_0(H_n) = 0. \tag{3.12}$$

Taking into account monotonicity and k-subadditivity of m_0^+, we get:

$$\begin{aligned} m_0^+(E_n \setminus H_n) &= m_0^+\Big(\bigcup_{j=1}^{n}(E_n \setminus F_j)\Big) \le m_0^+\Big(\bigcup_{j=1}^{n}(E_j \setminus F_j)\Big) \\ &\le k \sum_{j=1}^{n} m_0^+(E_j \setminus F_j) \le k \sum_{j=1}^{\infty}\Big(\bigvee_{t=1}^{\infty} a_{t,\varphi(t+j)}\Big). \end{aligned}$$

Moreover, by construction, $m_0^+(E_n \setminus H_n) \le u$, where

$$u = \bigvee_{A \in \mathcal{W}} m_0(A). \tag{3.13}$$

By Lemma 2.3, we find a regulator $(b_{t,r})_{t,r}$ with

$$u \wedge \Big(k^2 \sum_{n=1}^{\infty} \Big(\bigvee_{t=1}^{\infty} a_{t,\varphi(t+n)}\Big)\Big) \le \bigvee_{t=1}^{\infty} b_{t,\varphi(t)} \text{ for each } \varphi \in \mathbb{N}^{\mathbb{N}}$$

and thus, thanks to k-subadditivity of m_0^+,

$$m_0^+(E_n) - m_0(H_n) \le k\, m_0^+(E_n \setminus H_n) \le \bigvee_{t=1}^{\infty} b_{t,\varphi(t)},$$

namely $m_0^+(E_n) \le m_0(H_n) + \bigvee_{t=1}^{\infty} b_{t,\varphi(t)}$. Hence, taking into account (3.12),

$$\begin{aligned} 0 \ &\le \ \limsup_n m_0^+(E_n) \le \limsup_n m_0(H_n) + \bigvee_{t=1}^{\infty} b_{t,\varphi(t)} \\ &= \ (D)\lim_n m_0(H_n) + \bigvee_{t=1}^{\infty} b_{t,\varphi(t)} = \bigvee_{t=1}^{\infty} b_{t,\varphi(t)}. \end{aligned} \tag{3.14}$$

From (3.14), arbitrariness of $\varphi \in \mathbb{N}^{\mathbb{N}}$ and weak σ-distributivity of R we obtain

$$0 \le \limsup_n m_0^+(E_n) \le \bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} \Big(\bigvee_{t=1}^{\infty} b_{t,\varphi(t)}\Big) = 0.$$

Hence, $(D)\lim_n m_0^+(E_n) = 0$, and so we get (global) continuity from above at $\emptyset$ of m_0^+ on $\mathcal{W}^+$ with respect to the (D)-sequence $(b_{t,r})_{t,r}$.

Now, we prove continuity from above of m_0^+ on $\mathcal{W}^+$ in the general case. Let $(E_n)_n$ be a decreasing sequence in $\mathcal{W}^+$, with $\bigcap_{n=1}^{\infty} E_n = E \in \mathcal{W}^+$. There is an increasing sequence $(V_n)_n$ in $\mathcal{W}$, with $E := \bigcup_{n=1}^{\infty} V_n \in \mathcal{W}^+$ and $(D)\lim_n m_0(V_n) = \bigvee_{n=1}^{\infty} m_0(V_n) = m_0^+(E)$, and so the sequence $(E_n \setminus V_n)_n$ is in $\mathcal{W}^+$, decreasing and $\bigcap_{n=1}^{\infty}(E_n \setminus V_n) = \emptyset$. Then, by the previous step, we get $(D)\lim_n m_0^+(E_n \setminus V_n) = 0$ with respect to the regulator $(b_{t,r})_{t,r}$. Moreover, thanks to monotonicity and k-subadditivity of m_0^+, we have

$$0 \le m_0^+(E_n) - m_0(V_n) \le k\, m_0^+(E_n \setminus V_n),$$

and hence $(D)\lim_n (m_0^+(E_n) - m_0(V_n)) = 0$. So,

$$m_0^+(E) = (D)\lim_n m_0^+(E_n) = \bigwedge_{n=1}^{\infty} m_0^+(E_n).$$

Thus, m_0^+ is (globally) continuous from above on $\mathcal{W}^+$ with respect to the regulator $(k\, b_{t,r})_{t,r}$. ■

Theorem 3.3 *Let $\mathcal{W}^* = \{A \subset G$: there is $D \in \mathcal{W}^+$ with $D \supset A\}$. For each $A \in \mathcal{W}^*$, set $m_0^*(A) = \bigwedge\Big\{m_0^+(D)\colon D \supset A,\ D \in \mathcal{W}^+\Big\}$. Then m_0^* is a k-subadditive capacity on $\mathcal{W}^*$, such that for every $A \in \mathcal{W}^*$ there exists a set $D \in \mathcal{W}^{+-}$, $D \supset A$, with $m_0^*(A) = m_0^*(D)$. Moreover, if $(A_n)_n$ is any decreasing sequence in $\mathcal{W}^+$ with $A = \bigcap_{n=1}^{\infty} A_n$, then $m_0^*(A) = \bigwedge_{n=1}^{\infty} m_0^+(A_n)$.*

Proof. It is not difficult to check that m_0^* is a k-subadditive capacity on $\mathcal{W}^*$. We now claim that for each $A \in \mathcal{W}^*$ there is $D \in \mathcal{W}^{+-}$, $D \supset A$, with

$$m_0^*(A) = m_0^*(D). \tag{3.15}$$

Choose arbitrarily $A \in \mathcal{W}^*$. Since R is super Dedekind complete, there is a sequence $(C_n)_n$ in $\mathcal{W}^+$, with $C_n \supset A_n$ for every $n \in \mathbb{N}$ and $m_0^*(A) = \bigwedge_{n=1}^{\infty} m_0^+(C_n)$. For each $n \in \mathbb{N}$, set

$$D_n = \bigcap_{i=1}^{n} C_i. \tag{3.16}$$

Then $D_n \in \mathcal{W}^+$, $D_n \supset A$ and the sequence $(D_n)_n$ is decreasing.

Put $D = \bigcap_{n=1}^{\infty} D_n = \bigcap_{n=1}^{\infty} C_n$. We get $D \supset A$. By monotonicity of m_0^*, we have

$$\begin{aligned} m_0^*(A) \le m_0^*(D) &\le \bigwedge_{n=1}^{\infty} m_0^+(D_n) = (D)\lim_n m_0^+(D_n) \\ &\le \bigwedge_{n=1}^{\infty} m_0^+(C_n) = m_0^*(A). \end{aligned} \tag{3.17}$$

Thus, all inequalities in (3.17) are equalities, and so we get (3.15). Furthermore, note that, by (3.17), we find a (D)-sequence $(w_{t,r})_{t,r}$ such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is $\overline{n} \in \mathbb{N}$ with

$$m_0^+(D_n) \le m_0^+(A) + \bigvee_{t=1}^{\infty} w_{t,\varphi(t)} \tag{3.18}$$

whenever $n \geq \overline{n}$. Taking in (3.18) the set $A^+ = D_{\overline{n}}$, we get that for every $A \in \mathcal{W}^*$ and $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is $A^+ \in \mathcal{W}^+$, $A^+ \supset A$, with

$$m_0^+(A^+) \leq m_0^+(A) + \bigvee_{t=1}^{\infty} w_{t,\varphi(t)}. \tag{3.19}$$

We now prove that, if $(A_n)_n$ is any decreasing sequence in $\mathcal{W}^+$ and $A = \bigcap_{n=1}^{\infty} A_n$, then

$$\bigwedge_{n=1}^{\infty} m_0^+(A_n) = (D) \lim_n m_0^+(A_n) = m_0^*(A). \tag{3.20}$$

Indeed, if $(D_q)_q$ is a decreasing sequence in $\mathcal{W}^+$ associated with A as in (3.16), then, by (3.17), monotonicity of m_0^+ and continuity from above of m_0^+ on $\mathcal{W}^+$, we get

$$\begin{aligned} \bigwedge_{n=1}^{\infty} m_0^+(A_n) &\leq \bigwedge_{n=1}^{\infty}\Big(\bigwedge_{q=1}^{\infty} m_0^+(A_n \cup D_q)\Big) = \bigwedge_{q=1}^{\infty}\Big(\bigwedge_{n=1}^{\infty} m_0^+(A_n \cup D_q)\Big) \\ &= \bigwedge_{q=1}^{\infty} m_0^+(D_q) = m_0^*(A) \leq \bigwedge_{n=1}^{\infty} m_0^+(A_n), \end{aligned}$$

and so we obtain (3.20). This ends the proof. ■

We now prove the following result about the existence of extensions of continuous k-subadditive capacities.

Theorem 3.4 *Let $m_0 : \mathcal{W} \to R$ be a globally (s)-bounded k-subadditive capacity, continuous from above at $\emptyset$. Then there exists a (unique) k-subadditive capacity $m : \sigma(\mathcal{W}) \to R$, continuous from above and from below, with $m(A) = m_0(A)$ for every $A \in \mathcal{W}$.*

Proof. Let $\mathcal{S} := \{A \in \mathcal{W}^*$ such that there are $E \in \mathcal{W}^{+-}$, $F \in \mathcal{W}^{-+}$ with $F \subset A \subset E$ and $m_0^*(E \setminus F) = 0\}$. We begin with proving that for every $A \in \mathcal{S}$ there is a regulator $(\gamma_{t,r})_{t,r}$ such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there are $D \in \mathcal{W}^+$ and $H \in \mathcal{W}^-$ with $H \subset A \subset D$ and

$$m_0^+(D \setminus H) \leq \bigvee_{t=1}^{\infty} \gamma_{t,\varphi(t)}. \tag{3.21}$$

Choose arbitrarily $A \in \mathcal{S}$ and, in correspondence with A, let E and F be two sets satisfying the conditions in the definition of $\mathcal{S}$. There are a decreasing sequence $(E_n)_n$ in $\mathcal{W}^+$ and an increasing sequence $(F_n)_n$ in $\mathcal{W}^-$ with $E = \bigcap_{n=1}^{\infty} E_n$, $F = \bigcup_{n=1}^{\infty} F_n$.

Note that $E_n \setminus F_n \in \mathcal{W}^+$ and $E \setminus F = \bigcap_{n=1}^{\infty}(E_n \setminus F_n)$. Then, by (3.20), we get

$$\bigwedge_{n=1}^{\infty} m_0^+(E_n \setminus F_n) = (D)\lim_n m_0^+(E_n \setminus F_n) = m_0^*(E \setminus F) = 0, \tag{3.22}$$

that is there exists a (D)-sequence $(\gamma_{t,r})_{t,r}$ such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is $\overline{n} \in \mathbb{N}$ with

$$m_0^+(E_n \setminus F_n) \leq \bigvee_{t=1}^{\infty} \gamma_{t,\varphi(t)} \tag{3.23}$$

for each $n \geq \overline{n}$. Taking in (3.23) $D = E_{\overline{n}}$ and $H = F_{\overline{n}}$, we obtain (3.21).

We now prove that $\mathcal{S}$ is a σ-ring containing $\mathcal{W}$. First of all, it is readily seen that $\mathcal{W} \subset \mathcal{S}$. Now, choose arbitrarily $A_1,\ A_2 \in \mathcal{S}$, and let $E_1,\ E_2 \in \mathcal{W}^{+-}$, F_1, $F_2 \in \mathcal{W}^{-+}$ be with $F_i \subset A_i \subset E_i$ and $m_0^*(E_i \setminus F_i) = 0$, $i = 1, 2$. It is not difficult to check that $E_1 \cup E_2,\ E_1 \setminus F_2 \in \mathcal{W}^{+-}$, $F_1 \cup F_2,\ F_1 \setminus E_2 \in \mathcal{W}^{-+}$. Taking into account also monotonicity and k-subadditivity of m_0^*, we get

$$\begin{array}{rcl} & & F_1 \cup F_2 \subset A_1 \cup A_2 \subset E_1 \cup E_2, \\ & & F_1 \setminus E_2 \subset A_1 \setminus A_2 \subset E_1 \setminus F_2, \\ 0 & \leq & m_0^*((E_1 \cup E_2) \setminus (F_1 \cup F_2)) \leq m_0^*((E_1 \setminus F_1) \cup (E_2 \setminus F_2)) \\ & \leq & m_0^*(E_1 \setminus F_1) + k\, m_0^*(E_2 \setminus F_2) = 0, \\ 0 & \leq & m_0^*((E_1 \setminus F_2) \setminus (F_1 \setminus E_2)) \leq m_0^*((E_1 \setminus F_1) \cup (E_2 \setminus F_2)) \\ & \leq & m_0^*(E_1 \setminus F_1) + k\, m_0^*(E_2 \setminus F_2) = 0. \end{array} \tag{3.24}$$

Thus, all inequalities in (3.24) are equalities, and hence $A_1 \cup A_2,\ A_1 \setminus A_2 \in \mathcal{S}$. Therefore, $\mathcal{S}$ is a ring. So, in order to prove that $\mathcal{S}$ is a σ-ring it will be enough to show that, if $(A_n)_n$ is an increasing sequence in $\mathcal{S}$ and $A = \bigcup_{n=1}^{\infty} A_n$, then $A \in \mathcal{S}$. To this aim, for technical reasons it is more advisable to proceed dealing with (O)-convergence rather than (D)-convergence. By (3.22), to every $n \in \mathbb{N}$ it is possible to associate two sequences $(E_{h,n})_h$ and $(F_{h,n})_h$ in $\mathcal{W}^+$ and $\mathcal{W}^-$, respectively, with $F_{h,n} \subset A_n \subset E_{h,n}$ for every h and n, and

$$\begin{array}{rcl} \bigwedge_{h=1}^{\infty} m_0^+(E_{h,n} \setminus F_{h,n}) & = & (O)\lim_h m_0^+(E_{h,n} \setminus F_{h,n}) \\ & = & (D)\lim_h m_0^+(E_{h,n} \setminus F_{h,n}) = 0. \end{array} \tag{3.25}$$

Hence, taking into account monotonicity and k-subadditivity of m_0^+, for every $n \in \mathbb{N}$ there is an (O)-sequence $(\sigma_p^{(n)})_p$ such that for every $p \in \mathbb{N}$ there exists $\overline{h} \in \mathbb{N}$ with

$$m_0^+\Big(\bigcup_{i=1}^{n}(E_{h,i} \setminus F_{h,i})\Big) \leq k \sum_{i=1}^{n} m_0^+(E_{h,i} \setminus F_{h,i}) \leq \sigma_p^{(n)} \tag{3.26}$$

for every $h \geq \overline{h}$. Let u be as in (3.13). Without loss of generality, we can assume that $\sigma_p^{(n)} \leq u$ for each n, $p \in \mathbb{N}$ since, by construction, $m_0^+(A) \leq u$ for every $A \in \mathcal{W}^+$. Thus, taking into account that R is super Dedekind complete and weakly σ-distributive, by Lemma 2.4 there is an (O)-sequence $(b_j)_j$ such that for every j and $n \in \mathbb{N}$ there is $p \in \mathbb{N}$ with $\sigma_p^{(n)} \leq b_j$. From this and (3.26) it follows that for every n and $j \in \mathbb{N}$ there is $h' \in \mathbb{N}$ with

$$m_0^+\Big(\bigcup_{i=1}^{n}(E_{h,i} \setminus F_{h,i})\Big) \leq k \sum_{i=1}^{n} m_0^+(E_{h,i} \setminus F_{h,i}) \leq b_j \tag{3.27}$$

for every $h \geq h'$. Passing to the supremum as n varies in $\mathbb{N}$ in (3.26), taking into account continuity from below of m_0^+, from (3.27) we obtain

$$m_0^+\Big(\bigcup_{i=1}^{\infty}(E_{h,i} \setminus F_{h,i})\Big) \leq b_j \tag{3.28}$$

whenever $h \geq h'$. Let now $E_h := \bigcup_{n=1}^{\infty} E_{h,n}$, $E := \bigcap_{h=1}^{\infty} E_h$, $F_h := \bigcap_{n=1}^{\infty} F_{h,n}$, $F := \bigcup_{h=1}^{\infty} F_h$. It is not difficult to check that $E_h \in \mathcal{W}^+$, $F_h \in \mathcal{W}^-$ for all $h \in \mathbb{N}$, $E \in \mathcal{W}^{+-}$, $F \in \mathcal{W}^{-+}$, $F \subset A \subset E$,

$$E \setminus F \subset E_h \setminus F_h \subset \bigcup_{n=1}^{\infty}(E_{h,n} \setminus F_{h,n}) \text{ for every } h \in \mathbb{N}. \tag{3.29}$$

From (3.27), (3.28), (3.29), positivity and monotonicity of m_0^* we obtain

$$0 \leq m_0^*(E \setminus F) \leq m_0^*(E_h \setminus F_h) \leq m_0^+\Big(\bigcup_{n=1}^{\infty}(E_{h,n} \setminus F_{h,n})\Big) \leq b_j. \tag{3.30}$$

By arbitrariness of j, we get $m_0^*(E \setminus F) = 0$. Thus, E and F are the required sets associated with A and satisfying the conditions in the definition of $\mathcal{S}$. Thus, $A \in \mathcal{S}$, and so we deduce that $\mathcal{S}$ is a σ-ring. Since $\mathcal{S} \supset \mathcal{W}$, then $\mathcal{S} \supset \sigma(\mathcal{W})$.

Now we prove that m_0^* is continuous from above at $\emptyset$ on $\mathcal{S}$. Let $(A_n)_n$ be a decreasing sequence in $\mathcal{S}$, with $\bigcap_{n=1}^{\infty} A_n = \emptyset$. By (3.21), taking into account monotonicity and k-subadditivity of m_0^*, for every $n \in \mathbb{N}$ there exists a (D)-sequence $(v_{t,r})_{t,r}$ such that for each $\varphi \in \mathbb{N}^{\mathbb{N}}$, in correspondence with A_n, there is $C_n \in \mathcal{W}^+$ with $C_n \supset A_n$ and

$$m_0^*\Big(\bigcup_{i=1}^{n}(C_i \setminus A_i)\Big) \leq k \sum_{i=1}^{n} m_0^*(C_i \setminus A_i) \leq \bigvee_{t=1}^{\infty} v_{t,\varphi(t+n)}^{(n)}. \tag{3.31}$$

For each $n \in \mathbb{N}$, set $D_n := \bigcap_{i=1}^{n} C_i$. Then $(D_n)_n$ is a decreasing sequence in $\mathcal{W}^+$ and $D_n \supset A_n$ for each $n \in \mathbb{N}$.

Indeed, by induction, if $D_n \supset A_n$, then $D_{n+1} = D_n \cap C_{n+1} \supset A_n \cap A_{n+1} = A_{n+1}$, since $(A_n)_n$ is decreasing. By monotonicity of m_0^*, from (3.31) it follows that

$$m_0^*\Big(\bigcup_{i=1}^n (D_i \setminus A_i)\Big) \le \bigvee_{t=1}^\infty v^{(n)}_{t,\varphi(t+n)}. \tag{3.32}$$

By (3.19), in correspondence with $D_n \setminus A_n$ there are a regulator $(w^{(n)}_{t,r})_{t,r}$ and a set $I_n \in \mathcal{W}^+$, $I_n \supset D_n \setminus A_n$, with

$$k \sum_{i=1}^n m_0^+(I_i) \le k \sum_{i=1}^n m_0^+(D_i \setminus A_i) + \bigvee_{t=1}^\infty w^{(n)}_{t,\varphi(t+n)},$$

and hence, taking into account monotonicity and k-subadditivity of m_0^+,

$$m_0^+\Big(\bigcup_{i=1}^n I_i\Big) \le k \sum_{i=1}^n m_0^+(I_i) \le \bigvee_{t=1}^\infty v^{(n)}_{t,\varphi(t+n)} + \bigvee_{t=1}^\infty w^{(n)}_{t,\varphi(t+n)}. \tag{3.33}$$

Let $x_{t,r} := 2(v_{t,r} + w_{t,r})$, t, $r \in \mathbb{N}$, and u be as in (3.13). By virtue of Lemma 2.3, we find a (D)-sequence $(\tau_{t,r})_{t,r}$ with

$$u \wedge \Big(\sum_{n=1}^\infty \Big(\bigvee_{t=1}^\infty x^{(n)}_{t,\varphi(t+n)}\Big)\Big) \le \bigvee_{t=1}^\infty \tau_{t,\varphi(t)} \text{ for every } \varphi \in \mathbb{N}^{\mathbb{N}}.$$

Since, by construction, $m_0^+\Big(\bigcup_{i=1}^n I_i\Big) \le u$, then $m_0^+\Big(\bigcup_{i=1}^n I_i\Big) \le \bigvee_{t=1}^\infty \tau_{t,\varphi(t)}$ for each $n \in \mathbb{N}$.

If $I = \bigcup_{n=1}^\infty I_n$, then $I \in \mathcal{W}^+$, $I \supset \bigcup_{n=1}^\infty (D_n \setminus A_n)$ and, by continuity from below of m_0^+, we get

$$m_0^+(I) = \bigvee_{n=1}^\infty m_0^+\Big(\bigcup_{i=1}^n I_i\Big) \le \bigvee_{t=1}^\infty \tau_{t,\varphi(t)}. \tag{3.34}$$

As $\bigcap_{n=1}^\infty A_n = \emptyset$, then

$$\bigcap_{n=1}^\infty D_n = \Big(\bigcap_{n=1}^\infty D_n\Big) \setminus \Big(\bigcap_{n=1}^\infty A_n\Big) \subset \bigcup_{n=1}^\infty (D_n \setminus A_n) \subset I$$

and so, from (3.20), (3.34) and monotonicity of m_0^* we obtain

$$0 \le \bigwedge_{n=1}^\infty m_0^*(A_n) \le \bigwedge_{n=1}^\infty m_0^+(D_n) = m_0^*\Big(\bigcap_{n=1}^\infty D_n\Big) \le m_0^+(I) \le \bigvee_{t=1}^\infty \tau_{t,\varphi(t)}. \tag{3.35}$$

By arbitrariness of $\varphi \in \mathbb{N}^{\mathbb{N}}$ and weak σ-distributivity of R, we obtain

$$0 \le \bigwedge_{n=1}^{\infty} m_0^*(A_n) \le \bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} \Big(\bigvee_{t=1}^{\infty} \tau_{t,\varphi(t)} \Big) = 0,$$

that is $\bigwedge_{n=1}^{\infty} m_0^*(A_n) = 0$. So we get that m_0^* is continuous from above at $\emptyset$ on $\mathcal{S}$. Continuity from above and from below of m_0^* on $\mathcal{S}$ follows from continuity from above at $\emptyset$ of m_0^* on $\mathcal{S}$ and Proposition 3.1. Thus, the restriction $m : \sigma(\mathcal{W}) \to R$ of m_0^* satisfies the thesis of the theorem. ∎

Now, we are in position to prove our main result on extensions of lattice group-valued k-subadditive capacities.

Theorem 3.5 *Let $\mathcal{W}$, $\sigma(\mathcal{W})$ and m be as in Theorem 3.4. Then m is globally (s)-bounded, and there exists a regulator $(c_{t,r})_{t,r}$ such that for each $A \in \sigma(\mathcal{W})$ and $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is $F \in \mathcal{W}$ with*

$$m(A \triangle F) \le \bigvee_{t=1}^{\infty} c_{t,\varphi(t)}. \tag{3.36}$$

Proof. Let $m_0 : \mathcal{W} \to R$ and $m : \sigma(\mathcal{W}) \to R$ be as in Theorem 3.4, and $(a_{t,r})_{t,r}$ be a regulator, related with global (s)-boundedness of m_0 on $\mathcal{W}$. We begin with proving (3.36). Let $u = \bigvee_{A \in \sigma(\mathcal{W})} m(A)$. By Lemma 2.3, there is a regulator $(A_{t,r})_{t,r}$ with

$$u \wedge \Big(\sum_{n=1}^{\infty} \Big(\bigvee_{t=1}^{\infty} a_{t,\varphi(t+n)} \Big) \Big) \le \bigvee_{t=1}^{\infty} A_{t,\varphi(t)}.$$

Set $b_{t,r} = 2\,(a_{t,r} + A_{t,r})$ and $c_{t,r} = 2\,k\,b_{t,r}$, $t, r \in \mathbb{N}$. We prove that $(c_{t,r})_{t,r}$ satisfies (3.36). To this aim, we first recall that by (3.4), if $H \in \mathcal{W}^+$, $H = \bigcup_{n=1}^{\infty} H_n$, where $(H_n)_n$ is an increasing sequence in $\mathcal{W}$, then for each $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is $\overline{n} \in \mathbb{N}$ with

$$m(H \setminus H_n) \le \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \quad \text{whenever } n \ge \overline{n}. \tag{3.37}$$

Choose arbitrarily an element $B \in \mathcal{W}^{+-}$ and let $(V_n)_n$ be any decreasing sequence in $\mathcal{W}^+$ with $B = \bigcap_{n=1}^{\infty} V_n$. Pick any element $\varphi \in \mathbb{N}^{\mathbb{N}}$. By (3.37), for each $n \in \mathbb{N}$ there exists $E_n \in \mathcal{W}$ such that $E_n \subset V_n$ and

$$m(V_n \setminus E_n) \le \bigvee_{t=1}^{\infty} a_{t,\varphi(t+n)}.$$

Set now $F_n := \bigcap_{i=1}^{n} E_i$, $n \in \mathbb{N}$. Then $(F_n)_n$ is a decreasing sequence in $\mathcal{W}$.

Proceeding analogously as in (3.6), we find a positive integer n_0 with $m(F_n \setminus F_{n+p}) \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}$ for every $n \geq n_0$ and $p \in \mathbb{N}$. Since m is k-subadditive, we get

$$m(V_n \setminus B) \leq m(V_n \setminus V_{n+p}) + k\, m(V_{n+p} \setminus B)$$

for every n, $p \in \mathbb{N}$. It is possible to check that

$$\begin{aligned} V_n \setminus F_n &\subset (V_1 \setminus E_1) \cup \ldots \cup (V_n \setminus E_n), \\ (V_n \setminus V_{n+p}) \triangle (F_n \setminus F_{n+p}) &\subset (V_1 \setminus E_1) \cup \ldots \cup (V_{n+p} \setminus E_{n+p}) \end{aligned}$$

for every n, $p \in \mathbb{N}$, and hence

$$m(V_n \setminus V_{n+p}) \leq m(F_n \setminus F_{n+p}) + u \wedge \Big(k \sum_{h=1}^{n+p} m(V_h \setminus E_h)\Big) \leq k \Big(\bigvee_{t=1}^{\infty} a_{t,\varphi(t)} + \bigvee_{t=1}^{\infty} A_{t,\varphi(t)}\Big),$$

for each $n \geq n_0$ and $p \in \mathbb{N}$. Thus, we get

$$m(V_n \setminus B) \leq k \Big(\bigvee_{t=1}^{\infty} a_{t,\varphi(t)} + \bigvee_{t=1}^{\infty} A_{t,\varphi(t)}\Big) + k\, m(V_{n+p} \setminus B)$$

for every $n \geq n_0$ and $p \in \mathbb{N}$. Letting p tend to $+\infty$ and taking into account continuity from above of m, we obtain $m(V_n \setminus B) \leq k \bigvee_{t=1}^{\infty} b_{t,\varphi(t)}$ for any $n \geq n_0$. Taking $H = V_{n_0}$, we get that for every $B \in \mathcal{W}^{+-}$ and $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exists a set $H \in \mathcal{W}^{+}$ with $H \supset B$ and

$$m(H \setminus B) \leq k \bigvee_{t=1}^{\infty} b_{t,\varphi(t)}. \tag{3.38}$$

Moreover by (3.37), in correspondence with H and φ, there is a set $A \in \mathcal{W}$, $A \subset H$, with

$$m(H \setminus A) \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}. \tag{3.39}$$

From (3.38) and (3.39), monotonicity and k-subadditivity of m, it follows that

$$\begin{aligned} m(A \triangle B) &\leq m((H \setminus A) \cup (H \setminus B)) \leq m(H \setminus A) + k\, m(H \setminus B) \leq \\ &\leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} + k \bigvee_{t=1}^{\infty} b_{t,\varphi(t)} \leq \bigvee_{t=1}^{\infty} c_{t,\varphi(t)}. \end{aligned}$$

By (3.15), since $\sigma(\mathcal{W}) \subset \mathcal{W}^{*}$, we get that for every $E \in \sigma(\mathcal{W})$ there is $B \in \mathcal{W}^{+-}$ with $B \supset E$ and $m(E) = m(B)$. Thus, if E is any element of $\sigma(\mathcal{W})$ and A is as in (3.39), then, using monotonicity and k-subadditivity of m, we obtain

$m(A\triangle E) \le m(A\triangle B) + k\, m(B\triangle E) = m(A\triangle B) \le \bigvee_{t=1}^{\infty} c_{t,\varphi(t)}$, and hence (3.36) is proved.

Now, we prove global (s)-boundedness of m. Let u, $(a_{t,r})_{t,r}$ and $(c_{t,r})_{t,r}$ be as above. By Lemma 2.3, there is a regulator $(C_{t,r})_{t,r}$ with

$$u \wedge \sum_{n=1}^{\infty}\Big(\bigvee_{t=1}^{\infty} c_{t,\varphi(t+n)}\Big) \le \bigvee_{t=1}^{\infty} C_{t,\varphi(t)}.$$

Put

$$d_{t,r} := 2k(C_{t,r} + a_{t,r}),\ \ t,r \in \mathbb{N}. \tag{3.40}$$

We prove that m is globally (s)-bounded on $\sigma(\mathcal{W})$ with respect to $(d_{t,r})_{t,r}$. Choose arbitrarily a disjoint sequence $(H_n)_n$ in $\sigma(\mathcal{W})$ and an element $\varphi \in \mathbb{N}^{\mathbb{N}}$. By (3.36), for each $n \in \mathbb{N}$ there is a set $F_n \in \mathcal{W}$ with

$$m(H_n \triangle F_n) \le \bigvee_{t=1}^{\infty} c_{t,\varphi(t+n)}.$$

Set $F_1^* := F_1$, $F_n^* := F_n \setminus \Big(\bigcup_{j=1}^{n-1} F_j\Big)$ for every $n \ge 2$. We get

$$F_n^* \in \mathcal{W} \text{ for every } n \in \mathbb{N} \text{ and } H_n \triangle F_n^* \subset \bigcup_{j=1}^{n} (H_j \triangle F_j). \tag{3.41}$$

From (3.41), monotonicity and k-subadditivity of m we obtain

$$\begin{aligned} m(H_n) &\le m(H_n \triangle F_n^*) + k\, m(F_n^*) \\ &\le u \wedge \Big(k \sum_{j=1}^{n} m(H_j \triangle F_j)\Big) + k\, m(F_n^*) \\ &\le \bigvee_{t=1}^{\infty} C_{t,\varphi(t)} + k\, m(F_n^*). \end{aligned} \tag{3.42}$$

Since the sequence $(F_n^*)_n$ is disjoint, then, by global (s)-boundedness of m on $\mathcal{W}$ with respect to $(a_{t,r})_{t,r}$, for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is $\overline{n} \in \mathbb{N}$ with

$$m(F_n^*) \le \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}. \tag{3.43}$$

From (3.42) and (3.43) it follows that $m(H_n) \le \bigvee_{t=1}^{\infty} d_{t,\varphi(t)}$. From this and arbitrariness of the chosen sequence $(H_n)_n$ we get global (s)-boundedness of m on $\sigma(\mathcal{W})$ with respect to the regulator $(d_{t,r})_{t,r}$.

Furthermore, by construction, taking into account that the set functions m_0^+ and m_0^* are well-defined, using weak σ-distributivity of R, it is not difficult to check that the extension $m : \sigma(\mathcal{W}) \to R$ of m_0 is unique. This ends the proof. ∎

Remark 3.6 Observe that the set function m in Theorem 3.4 is also globally continuous on $\sigma(\mathcal{W})$. Indeed, let $(E_n)_n$ be a decreasing sequence in $\sigma(\mathcal{W})$ with empty intersection, and $(d_{t,r})_{t,r}$ be as in (3.40). Using global (s)-boundedness of m on $\sigma(\mathcal{W})$, analogously as in (3.6) it is possible to see that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is $\overline{n} \in \mathbb{N}$ with

$$m(E_n \setminus E_{n+p}) \leq \bigvee_{t=1}^{\infty} d_{t,\varphi(t)} \tag{3.44}$$

for every $n \in \mathbb{N}$, $n \geq \overline{n}$ and $p \in \mathbb{N}$. Taking into account continuity from below of m on $\sigma(\mathcal{W})$, from (3.44), keeping fixed n and letting p tend to $+\infty$, we get

$$m(E_n) \leq \bigvee_{t=1}^{\infty} d_{t,\varphi(t)} \tag{3.45}$$

for every $n \in \mathbb{N}$, $n \geq \overline{n}$. By arbitrariness of the chosen sequence $(E_n)_n$, from (3.45) we get global continuity from above at $\emptyset$ of m on $\sigma(\mathcal{W})$. Global continuity from above and from below of m on $\sigma(\mathcal{W})$ follows from global continuity from above at $\emptyset$, by proceeding analogously as in Proposition 3.1. ∎

Open problems:

(a) Find some types of extensions for continuous set functions with values in a not necessarily super Dedekind complete or weakly σ-distributive lattice group.

(b) Is the extension found in [4] still valid for lattice group-valued measures on effect algebras or even on pseudo-effect algebras?

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References

[1] Atanassov, K., *Intuitionistic Fuzzy Sets: Theory and Applications,* Physica-Verlag, Heidelberg, 1999.

[2] Avallone, A., Barbieri, G., Vitolo, P., *On the Alexandroff decomposition theorem*, Math. Slovaca, 58 (2) (2008), 185–200.

[3] Avallone, A., De Simone, A., *Extensions of modular functions on orthomodular lattices,* Ital. J. Pure Appl. Math., 9 (2001), 109-122.

[4] Avallone, A., De Simone, A., Vitolo, P., *Effect algebras and extensions of measures*, Boll. U.M.I., 9-B (2006), 423-444.

[5] Barbieri, G., *An extension theorem for modular measures on effect algebras,* Czechoslovak Math. J., 59 (134) (3) (2009), 707-719.

[6] Barbieri, G., *The extension of measures on D-lattices,* Fuzzy Sets Systems, 244 (2014), 123-129.

[7] Bernau, S.J., *Unique representation of Archimedean lattice groups and normal Archimedean lattice rings,* Proc. London Math. Soc., 15 (1965), 599-631.

[8] Boccuto, A., *On Stone-type extensions for group-valued measures*, Math. Slovaca, 45 (1995), 309-315.

[9] Boccuto, A., *Egorov property and weak σ-distributivity in Riesz spaces*, Acta Math. (Nitra), 6 (2003), 61-66.

[10] Boccuto, A., Candeloro, D., *Uniform (s)-boundedness and convergence results for measures with values in complete (ℓ)-groups,* J. Math. Anal. Appl., 265 (2002), 170-194.

[11] Boccuto, A., Candeloro, D., *Convergence and decompositions for (ℓ)-group-valued set functions*, Comment. Math., 44 (1) (2004), 11-37.

[12] Boccuto, A., Candeloro, D., *A survey of decomposition and convergence theorems for (ℓ)-group-valued measures,* Atti Semin. Mat. Fis. Univ. Modena Reggio Emilia, 53 (2005), 243-260.

[13] Boccuto, A., Candeloro, D., *Sobczyk-Hammer decompositions and convergence theorems for measures with values in (ℓ)-groups,* Real Anal. Exchange, 33 (2008), 91-105.

[14] Boccuto, A., Dimitriou, X., *Ideal limit theorems and their equivalence in (ℓ)-group setting,* J. Math. Research, 5 (2) (2013), 43-60.

[15] Boccuto, A., Dimitriou, X., *Convergence theorems for lattice group-valued measures,* Bentham Science Publ., U.A.E., 2015. ISBN 9781681080109

[16] Boccuto, A., Dimitriou, X., *Limit theorems for k-subadditive lattice group-valued capacities in the filter convergence setting*, Tatra Mt. Math. Publ., 67 (2015), 1-21. DOI:10.1515/tmmp-2015-0000

[17] Boccuto, A., Riečan, B., *On extension theorems for measures in (ℓ)-groups,* Math. Slovaca, 60 (1) (2010), 65-74.

[18] Boccuto, A., Riečan, B., Sambucini, A.R., *On the product of M-measures in (ℓ)-groups*, Australian J. Math. Anal. Appl., 7 (1) (2010), Paper no. 9, 8 p.

[19] Boccuto, A., Sambucini, A.R., *The monotone integral with respect to Riesz space-valued capacities*, Rend. Mat. (Roma), 16 (1996), 491-524.

[20] Candeloro, D., Sambucini, A.R., *Filter convergence and decompositions for vector lattice-valued measures,* Mediterranean J. Math., 12 (2015), 621-637. doi: 10.1007/s00009-014-0431-0

[21] Dobrakov, I., *On submeasures* I. Dissertationes Math., 112 (1974), 35 pp.

[22] Drewnowski, L., *Topological rings of sets, Continuous set functions, Integration.* I, II, III, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys., 20 (1972), 269-276, 277-286, 439-445.

[23] Fremlin, D.H., *A direct proof of the Matthes-Wright integral extension theorem*, J. London Math. Soc., 11 (2) (1975), 276-284.

[24] Khurana, S.S., *Extension of group-valued regular Borel measures,* Math. Nachr. , 97 (1980), 159-165; Addendum to the paper: "Extension of group-valued regular Borel measures", Math. Nachr., 104 (1981), 348.

[25] Kundu, S.K., Lahiri, B.K., *On lattice-ordered group-valued measures,* Math. Student, 45 (3) (1977), 1-9 (1979).

[26] Leblanc, L., Fox, G.E., *On the extension of measure by the method of Borel,* Canadian J. Math., 8 (1956), 516-523.

[27] Lipecki, Z., *On unique extensions of positive additive set functions,* Arch. Math., 41 (1983), 71-79.

[28] Lipecki, Z., *On unique extensions of positive additive set functions*, II Arch. Math., 50 (1988), 175-182.

[29] Lipecki, Z., *Maximal and tight extensions of positive additive set functions,* Math. Nachr., 146 (1990), 167-173.

[30] Malyugin, S.A., *Quasi-Radon measures* (Russian), Sib. Math. J., 32 (1991), 812-821.

[31] Pap, E., *Null-Additive Set Functions,* Kluwer Acad. Publishers/Ister Science, Bratislava, 1995.

[32] Rama Bala, D.V., Sastry, K.P.R., *Additive set functions with values in a complete lattice ordered group*, Math. Student, 45 (4) (1977), 7-12 (1979).

[33] Riečan, B., Neubrunn, T., *Integral, Measure and Ordering,* Kluwer Acad. Publ./Ister Science, Dordrecht/Bratislava, 1997.

[34] Saeki, S., *The Vitali-Hahn-Saks theorem and measuroids,* Proc. Amer. Math. Soc., 114 (3) (1992), 775-782.

[35] Wang, Z., Klir, G.J., *Generalized Measure theory,* Springer, Berlin-Heidelberg-New York, 2009.

[36] Weber, H., *Two extension theorems. Modular functions on complemented lattices*, Czechoslovak Math. J., 52 (2002), 55-74.

[37] Wright, J.D.M., *The measure extension problem for vector lattices,* Ann. Inst. Fourier (Grenoble), 21 (1971), 65-85.

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ON Γ-BIACTS AND THEIR GREEN'S RELATIONS

Ali Reza Shabani

Hamid Rasouli[1]

Department of Mathematics
Science and Research Branch
Islamic Azad University
Tehran
Iran
e-mails: ashabani@srbiau.ac.ir
hrasouli@srbiau.ac.ir

Abstract. A well-known generalization of a semigroup S is called the Γ–semigroup. We generalize the notion of biacts over semigroups to Γ–biacts over Γ-semigroups. Green's relations on semigroups and biacts play an important role in these theories. In this paper, we study Green's relations on Γ–biacts.

Keywords and phrases: Green's relations, biact, Γ–semigroup, Γ–biact, Γ–congruence.

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1. Introduction and preliminaries

The concept of Γ−semigroup, as a generalization of the notion of semigroup, was introduced by Sen [10]. Certain algebraic properties of Γ–semigroups have been studied by some authors, for example, one may see [2], [3]. Actions over a semigroup S, S–acts, play an important role in a variety of areas such as theoretical computer science (see [7]). We extended some classical notions of S–acts to $\Gamma - S$–acts in [12]. Green [5] introduced the Green's relations on semigroups in 1951. Green's relations for Γ–semigroups were studied by Chinram and Siammai [2]. Also, Green's relations on biacts have been studied in [8]. A generalization of acts over semigroups to Γ–acts over Γ–semigroups can be found in [11]. In this paper, we generalize the notion of biacts to Γ–biacts and consider Green's relations on Γ–biacts, which are in fact a generalization of Green's relations on biacts. Other classical algebraic structures such as modules can also be generalized to Γ–modules. For more information, see for example [1, 6]. As an application of (ordered) Γ–semigroups in connection with fuzzy sets, we refer to [4, 9].

In the following, we recall certain preliminaries on Γ–semigroups and $\Gamma - S$–acts needed in the sequel.

[1]Corresponding author.

Let X be a non-empty set, $B(X)$ denote the set of relations and $\varepsilon(X)$ the set of equivalence relations on X. Also, the set $\{(x,x) \mid x \in X\}$, the diagonal relation on X, is denoted by Δ_X, and the universal relation $X \times X$ is denoted by ∇_X. If $\rho \in B(X)$, the *transitive closure* of ρ is the relation $\rho^\infty = \bigcup_{i=1}^{\infty} \rho^i \in B(X)$ which is the smallest transitive relation in the poset $(B(X), \subseteq)$ containing ρ. Moreover, $\rho^e = (\rho \bigcup \rho^{-1} \bigcup \Delta_X)^\infty$ is the *equivalence closure* of ρ, that is, an equivalence relation on X generated by ρ (see [8, Theorem I.1.6]). A *lattice* is a poset L for which the meet $a \wedge b$ (the greatest lower bound) and the join $a \vee b$ (the least upper bound) exist for every $a, b \in L$.

Corollary 1.1. [8] *For a non-empty set X, if $\rho \in B(X)$, then $(x,y) \in \rho^e$ if and only if $x{=}y$ or for some $n \in \mathbb{N}$ there exists a sequence of elements $x{=}z_1, z_2, ..., z_n{=}y$ in X such that for every $i \in \{1, 2, ..., n-1\}, (z_i, z_{i+1}) \in \rho \bigcup \rho^{-1}$. In particular, if ρ and σ are equivalence relations on a set X, then in $\varepsilon(X)$ their join $\rho \vee \sigma$ is the relation defined by $x\,(\rho \vee \sigma)\,y$ if and only if there exist $z_1, z_2, ..., z_n \in X$ such that $x = z_1, z_n = y$ and $(z_i, z_{i+1}) \in \tau_i, \tau_i \in \{\rho, \sigma\}, i \in \{1, 2, ..., n-1\}$.*

Definition 1.2. [10] Let S and Γ be non-empty sets. Then S is said to be a Γ*–semigroup* if there exists a mapping $S \times \Gamma \times S \to S$ written as $(s, \gamma, t) \mapsto s\gamma t$, satisfying $(s\gamma t)\,\beta u = s\gamma\,(t\beta u)$ for all $s, t, u \in S$ and $\gamma, \beta \in \Gamma$. An element e in a Γ–semigroup S is called a *left* (*right*) Γ*–identity* if $e\gamma s = s$ ($s\gamma e = s$) for all $s \in S$ and $\gamma \in \Gamma$. By a Γ*–identity* we mean an element of S which is both a left and a right Γ–identity. A Γ–semigroup with a Γ–identity 1 is called a Γ*–monoid*.

Definition 1.3. [12] Let S be a Γ–semigroup with a left Γ–identity e and A be a non-empty set. A mapping $\lambda : S \times \Gamma \times A \to A$ where $(s, \gamma, a) \mapsto s\gamma a := \lambda(s, \gamma, a)$ such that $(s\gamma t)\,\beta a = s\gamma\,(t\beta a)$ and $e\gamma a = a$ for all $a \in A, s, t \in S$ and $\gamma, \beta \in \Gamma$, is called a *left* $\Gamma - S$*–action* and A is said to be a *left* $\Gamma - S$*–act* which is denoted by $\Gamma - {}_SA$. Also, for a Γ–semigroup S with a right Γ–identity e, by a *right* $\Gamma - S$*–act* we mean a non-empty set A together with a mapping $\lambda : A \times \Gamma \times S \to A$ where $(a, \gamma, s) \mapsto a\gamma s := \lambda(a, \gamma, s)$ satisfying the properties $a\gamma\,(s\beta t) = (a\gamma s)\,\beta t$ and $a\gamma e = a$ for all $a \in A, s, t \in S$ and $\gamma, \beta \in \Gamma$. We denote a right $\Gamma - S-$ act by $\Gamma - A_S$.

Remark 1.4. If S is a Γ–monoid with Γ–identity 1 and $\Gamma - {}_SA$ is a left $\Gamma - S$–act, then for every $s, t \in S, a \in A, \gamma, \beta \in \Gamma$, we have $s\gamma t = s\beta t$ and $s\gamma a = s\beta a$. Indeed, $s\gamma t = (s\beta 1)\gamma t = s\beta(1\gamma t) = s\beta t$; and $s\gamma a = (s\beta 1)\gamma a = s\beta(1\gamma a) = s\beta a$. Therefore, it is more interesting to consider left $\Gamma - S$–acts for a Γ–semigroup S with a left Γ–identity (not a Γ–identity) and, likewise, right $\Gamma - S$–acts for a Γ–semigroup S with a right Γ–identity (not a Γ–identity).

2. Γ–biacts and some basic properties

The purpose of this section is to introduce the structure of Γ–biacts and investigate some of their properties.

Definition 2.1. [8] Let T and S be monoids. A $T-S$*–biact* $_TA_S$ is a non-empty set A equipped with a left T-action $T\times A\to A, (t,a)\mapsto ta$, satisfying $(t_1t_2)a=t_1(t_2a)$ for all $t_1,t_2\in T, a\in A$, and a right S-action $A\times S\to A, (a,s)\mapsto as$, satisfying $a(s_1s_2)=(as_1)s_2$ for all $s_1,s_2\in S, a\in A$, for which $(ta)s=t(as)$ holds for all $t\in T, s\in S, a\in A$. For a $T-S$–biact $_TA_S$, a relation $\rho\in B(A)$, i.e. $\rho\subseteq A\times A$, is called $T-S$*–compatible* if $(a,b)\in\rho$ implies that $(tas,tbs)\in\rho$ for all $t\in T, a,b\in A$ and $s\in S$. Moreover, an equivalence relation $\rho\in\varepsilon(A)$ which is $T-S$–compatible is called a $T-S$*–congruence* on $_TA_S$. The set of all $T-S$–congruences on $_TA_S$ is denoted by $\mathbf{Con}(_TA_S)$.

Definition 2.2. Let $\Gamma-{}_TA$ be a left $\Gamma-T$–act and $\Gamma-A_S$ be a right $\Gamma-S$–act. We call A a $\Gamma-T-S$*–biact*, or simply a Γ*–biact*, and write $\Gamma-{}_TA_S$, if for all $t\in T$,$s\in S, a\in A$ and $\gamma,\beta\in\Gamma$,$(t\gamma a)\beta s=t\gamma(a\beta s)$.

From now on, $\Gamma-{}_TA_S$ stands for a $\Gamma-T-S$–biact where T and S are Γ–semigroups with a left and a right Γ–identity, respectively (see Remark 1.4), unless otherwise stated. If no confusion arises, we may use the same symbol 1 for a left Γ–identity and a right Γ–identity.

Remark 2.3. Every $T-S$–biact $_TA_S$ over semigroups T and S with a left identity and a right identity, respectively, can be made into a $\Gamma-T-S$–biact over the induced left Γ–semigroup T with a left Γ–identity by setting $t\gamma t':=tt', t,t'\in T$, and right Γ–semigroup S with a right Γ–identity by defining $s\gamma s':=ss', s,s'\in S$. Define mappings $T\times\Gamma\times A\to A$ by $t\gamma a=ta$ and $A\times\Gamma\times S\to A$ by $a\beta s=as$ for all $t\in T, a\in A, s\in S$ and $\gamma,\beta\in\Gamma$. It is easily seen that $_TA_S$ is a $\Gamma-T-S$–biact. Conversely, let A be a $\Gamma-T-S$–biact where T is a Γ–semigroup with a left Γ–identity and S is a Γ–semigroup with a right Γ–identity. Fix an element γ in Γ. First note that T and S are semigroups with the operations $tt':=t\gamma t'$ and $ss':=s\gamma s'$ for all $t,t'\in T$ and $s,s'\in S$ respectively. We define $T\times A\to A$ by $ta:=t\gamma a$ and $A\times S\to A$ by $as:=a\gamma s$ for all $t\in T, a\in A, s\in S$. Then one can show that A is a $T-S$–biact.

Example 2.4. Let $S=T=\{4n+3\mid n\in\mathbb{N}\}, \Gamma=\{4n+1\mid n\in\mathbb{N}\}$ and $A=\{4n\mid n\in\mathbb{N}\}$. Under the usual addition of natural numbers, S and T are Γ–semigroups and A is a $\Gamma-T-S$–biact, but not a $T-S$–biact.

Definition 2.5. Let $\Gamma-{}_TA_S$ be a $\Gamma-T-S$–biact. A relation $\rho\in B(A)$, i.e. $\rho\subseteq A\times A$, is called $\Gamma-T-S$*– compatible* if $(a,b)\in\rho$ implies that $(t\gamma a\beta s, t\gamma b\beta s)\in\rho$ for all $t\in T$, $a,b\in A, s\in S$ and $\gamma,\beta\in\Gamma$. For a $\Gamma-T-S$–biact $\Gamma-{}_TA_S$, an equivalence relation $\rho\in\varepsilon(A)$ which is $\Gamma-T-S$–compatible is called a $\Gamma-T-S$*–congruence*, or simply a Γ*–congruence*, on $\Gamma-{}_TA_S$. We denote the set of all Γ–congruences on $\Gamma-{}_TA_S$ by $\mathbf{Con}(\Gamma-{}_TA_S)$. Clearly, under the usual inclusion of relations, $\mathbf{Con}(\Gamma-{}_TA_S)$ is a poset.

Remark 2.6. If $|S|=1$, we have a definition of a $\Gamma-T$–compatible relation and a $\Gamma-T$–congruence on $\Gamma-{}_TA$; and if $|T|=1$, we have that of a $\Gamma-S$–compatible relation and a $\Gamma-S$–congruence on $\Gamma-A_S$.

Lemma 2.7. *For a $\Gamma-T-S$–biact $\Gamma-{}_TA_S$ and a relation $\rho \in B(A)$ (or $\rho \in \varepsilon(A)$), ρ is $\Gamma-T-S$–compatible (or a $\Gamma-T-S$–congruence) on $\Gamma-{}_TA_S$ if and only if ρ is both $\Gamma-T$–compatible (or a $\Gamma-T$–congruence) on $\Gamma-{}_TA$ and $\Gamma-S$–compatible (or a $\Gamma-S$–congruence) on $\Gamma-A_S$.*

Proof. We need only to show the assertion for the case $\rho \in B(A)$.

Necessity. Suppose that $\rho \in B(A)$ is $\Gamma-T-S$–compatible on $\Gamma-{}_TA_S$ and $(a,b) \in \rho$. For every $\gamma, \beta \in \Gamma$ and $t \in T, s \in S$ we have $(t\gamma a, t\gamma b) = (t\gamma a\beta 1, t\gamma b\beta 1) \in \rho$ and $(a\beta s, b\beta s) = (1\gamma a\beta s, 1\gamma b\beta s) \in \rho$ which means that ρ is both $\Gamma-T$–compatible and $\Gamma-S$–compatible.

Sufficiency. Let $\rho \in B(A)$ be both $\Gamma-T$–compatible and $\Gamma-S$–compatible on $\Gamma-{}_TA_S$, $(a,b) \in \rho$, $t \in T$, $s \in S$ and $\gamma, \beta \in \Gamma$. Then $(t\gamma a, t\gamma b) \in \rho$ by $\Gamma-T$–compatibility, and therefore $((t\gamma a)\beta s, (t\gamma b)\beta s) \in \rho$ by $\Gamma-S$–compatibility. Hence, ρ is $\Gamma-T-S$–compatible on $\Gamma-{}_TA_S$. ∎

Definition 2.8. Let $\Gamma-{}_TA_S$ be a Γ–biact and $\rho \in B(A)$. The relation

$$\rho^c := \{(t\gamma a_1\beta s, t\gamma a_2\beta s) \in A \times A \mid t \in T,\ (a_1,a_2) \in \rho,\ s \in S,\ \gamma,\beta \in \Gamma\}$$

is called the $\Gamma-T-S$*–compatible closure* of ρ. The unique smallest $\Gamma-T-S$–congruence on ${}_TA_S$ containing $\rho \in B(A)$ will be denoted by $\rho^{\#}$ and called the Γ*–congruence closure* of ρ.

Proposition 2.9. *Let $\rho, \sigma \in B(A)$ for a Γ–biact $\Gamma-{}_TA_S$. Then*

(1) $\rho \subseteq \rho^c$.
(2) $(\rho^c)^{-1} = (\rho^{-1})^c$.
(3) $\rho \subseteq \sigma$ *implies that* $\rho^c \subseteq \sigma^c$.
(4) $(\rho^c)^c = \rho^c$.
(5) $(\rho \bigcup \sigma)^c = \rho^c \bigcup \sigma^c$.
(6) $\rho = \rho^c$ *if and only if* ρ *is* $\Gamma-T-S$*–compatible.*

Proof. (1) Take $(a_1,a_2) \in \rho$. Then $(a_1,a_2) = (1\gamma a_1\beta 1, 1\gamma a_2\beta 1) \in \rho^c$ for all $\gamma,\beta \in \Gamma$. Hence, $\rho \subseteq \rho^c$.

(2) Take $(a_1'', a_2'') \in (\rho^c)^{-1}$. So $(a_2'', a_1'') \in \rho^c$ and then $a_2'' = t'\gamma' a_2'\beta' s'$, $a_1'' = t'\gamma' a_1'\beta' s'$ for some $t' \in T$, $s \in S, \gamma', \beta' \in \Gamma$ and $(a_2', a_1') \in \rho$ whence $(a_1', a_2') \in \rho^{-1} \subseteq (\rho^{-1})^c$. Therefore, $a_1' = t\gamma a_1\beta s$, $a_2' = t\gamma a_2\beta s$ for some $t \in T$, $s \in S$, $\gamma,\beta \in \Gamma$ and $(a_1,a_2) \in \rho^{-1}$. Hence, $a_1'' = (t'\gamma' t)\gamma a_1\beta(s\beta' s')$ and $a_2'' = (t'\gamma' t)\gamma a_2\beta(s\beta' s')$ that $t'\gamma' t \in T, s\beta' s' \in S$, i.e. $(a_1'', a_2'') \in (\rho^{-1})^c$. Hence, $(\rho^c)^{-1} \subseteq (\rho^{-1})^c$. Similarly, $(\rho^{-1})^c \subseteq (\rho^c)^{-1}$. Therefore, $(\rho^c)^{-1} = (\rho^{-1})^c$.

(3) Let $\rho \subseteq \sigma$. Take $(a_1'', a_2'') \in \rho^c$. Then $(a_1'', a_2'') =(t'\gamma' a'_1\beta' s', t'\gamma' a'_2\beta' s')$ for some $t' \in T, s' \in S, \gamma', \beta' \in \Gamma$ and $(a_1', a_2') \in \rho$. Therefore, $(a_1', a_2') \in \sigma$ which implies that $(a_1'', a_2'') \in \sigma^c$. Hence, $\rho^c \subseteq \sigma^c$.

(4) By (1), $\rho^c \subseteq (\rho^c)^c$. Conversely, let $(a_1'', a_2'') \in (\rho^c)^c$. Then $(a_1'', a_2'') = (t'\gamma' a'_1\beta' s', t'\gamma' a'_2\beta' s')$ for some $t' \in T, s' \in S, \gamma', \beta' \in \Gamma$ and $(a_1', a_2') \in \rho^c$. Then $(a_1', a_2') = (t\gamma a_1\beta s, t\gamma a_2\beta s)$ for some $t \in T, s \in S, \gamma, \beta \in \Gamma$,$(a_1,a_2) \in \rho$. Hence, $a''_1 = (t'\gamma' t)\gamma a_1\beta(s\beta' s')$ and $a''_2 = (t'\gamma' t)\gamma a_2\beta(s\beta' s')$, i.e. $(a_1'', a_2'') \in \rho^c$. Hence, $(\rho^c)^c \subseteq \rho^c$. Therefore, $(\rho^c)^c = \rho^c$.

(5) Using (3), we have $\rho^c \subseteq (\rho\bigcup\sigma)^c$ and $\sigma^c \subseteq (\rho\bigcup\sigma)^c$, and therefore $\rho^c\bigcup\sigma^c \subseteq (\rho\bigcup\sigma)^c$. Conversely, suppose that $(a_1', a_2') \in (\rho\bigcup\sigma)^c$. Then $a_1' = t\gamma a_1\beta s$,$a_2' = t\gamma a_2\beta s$ for some $t \in T, s \in S, \gamma, \beta \in \Gamma$ and $(a_1, a_2) \in \rho\bigcup\sigma$. Thus, $(a_1, a_2) \in \rho$ or $(a_1, a_2) \in \sigma$, and hence $(a_1', a_2') \in \rho^c$ or $(a_1', a_2') \in \sigma^c$. Thus, $(a_1', a_2') \in \rho^c\bigcup\sigma^c$. Hence, $(\rho\bigcup\sigma)^c \subseteq \rho^c\bigcup\sigma^c$. Therefore, $(\rho\bigcup\sigma)^c = \rho^c\bigcup\sigma^c$.

(6) Let first $\rho = \rho^c$. Then $(a_1, a_2) \in \rho$ implies that $(t\gamma a_1\beta s, t\gamma a_2\beta s) \in \rho^c = \rho$, for all $t \in T, s \in S$ and $\gamma, \beta \in \Gamma$. Thus, ρ is $\Gamma - T - S$–compatible. Conversely, if ρ is a $\Gamma - T - S$–compatible relation and $(a_1', a_2') \in \rho^c$, then $a_1' = t\gamma a_1\beta s$, $a_2' = t\gamma a_2\beta s$ for some $t \in T, s \in S$, $(a_1, a_2) \in \rho, \gamma, \beta \in \Gamma$. Therefore, $(a_1', a_2') = (t\gamma a_1\beta s, t\gamma a_2\beta s) \in \rho$ by $\Gamma - T - S$–compatibility. Thus, $\rho^c \subseteq \rho$. But, by (1), $\rho \subseteq \rho^c$. Therefore, $\rho = \rho^c$. ∎

Lemma 2.10. *Let $\Gamma -{}_TA_S$ be a Γ–biact. If the relation $\rho \in B(A)$ is $\Gamma - T - S$–compatible, then ρ^n is also $\Gamma - T - S$–compatible for any $n \in \mathbb{N}$.*

Proof. Let $(a_1, a_2) \in \rho^n$. Then there exist $b_1, b_2, \ldots, b_{n-1} \in A$ such that $(a_1, b_1), (b_1, b_2), ..., (b_{n-1}, a_2) \in \rho$. Since ρ is $\Gamma{-}T{-}S$–compatible, $(t\gamma a_1\beta s, t\gamma b_1\beta s)$, $(t\gamma b_1\beta s, t\gamma b_2\beta s), ..., (t\gamma b_{n-1}\beta s, t\gamma a_2\beta s) \in \rho$ for all $t \in T$, $s \in S$ and $\gamma, \beta \in \Gamma$, and so $(t\gamma a_1\beta s, t\gamma a_2\beta s) \in \rho^n$. ∎

Definition 2.11. Let Γ–${}_TA_S$ be a Γ–biact and $\rho \in B(A)$. If $(a_1', a_2') \in (\rho\bigcup\rho^{-1})^c$, or equivalently, $a_1' = t\gamma a_1\beta s$ and $a_2' = t\gamma a_2\beta s$ for some $t \in T, s \in S$, $\gamma, \beta \in \Gamma$ and $(a_1, a_2) \in \rho$ or $(a_2, a_1) \in \rho$, then we say that a_1' *is connected with* a_2' *by an elementary* $\Gamma - T - S - \rho$*–transition*, and use the notation $a_1' \to a_2'$.

Theorem 2.12. *Let Γ–${}_TA_S$ be a Γ–biact and $\rho \in B(A)$. Then $\rho^{\#} = (\rho^c)^e$.*

Proof. Obviously, $\rho \subseteq \rho^c \subseteq (\rho^c)^e$. We show that $(\rho^c)^e \in \mathbf{Con}(\Gamma -{}_TA_S)$. In view of [8, Theorem I.1.6], $(\rho^c)^e = \theta^\infty$ where $\theta = \rho^c\bigcup(\rho^c)^{-1}\bigcup\Delta_A$. Let $(a_1, a_2) \in (\rho^c)^e$. Then $(a_1, a_2) \in \theta^n$ for some $n \in \mathbb{N}$. Using Proposition 2.9(2) and (5), and the clear fact $\Delta_A^c = \Delta_A$, we get

$$\theta = \rho^c\bigcup(\rho^{-1})^c\bigcup\Delta_A^c = \left(\rho\bigcup\rho^{-1}\bigcup\Delta_A\right)^c = \theta^c.$$

Therefore, by Proposition 2.9(6), θ is $\Gamma - T - S$–compatible and then so is θ^n by Lemma 2.10. Thus, $(t\gamma a_1\beta s, t\gamma a_2\beta s) \in \theta^n \subseteq (\rho^c)^e$ for every $t \in T$, $s \in S$, $\gamma, \beta \in \Gamma$. Hence, $(\rho^c)^e$ is a Γ– congruence on Γ–${}_TA_S$ containing ρ. Let σ be a Γ–congruence on Γ–${}_TA_S$ containing ρ. Then, by using Proposition 2.9(3) and (6), we get $\rho^c \subseteq \sigma^c = \sigma$ and so $(\rho^c)^e \subseteq \sigma^e = \sigma$. Hence, $\rho^{\#} = (\rho^c)^e$. ∎

Corollary 2.13. *Let $\rho \in B(A)$ for a Γ–biact $\Gamma{-}{}_TA_S$, $a_1, a_2 \in A$. Then $(a_1, a_2) \in \rho^{\#}$ if and only if $a_1 = a_2$ or for some $n \in \mathbb{N}$ there is a sequence $a_1 = z_1 \to z_2 \to \cdots \to z_n = a_2$ of elementary $\Gamma - T - S - \rho$–transitions connecting a_1 to a_2.*

Proof. Using Theorem 2.12, $(a_1, a_2) \in \rho^{\#}$ if and only if $(a_1, a_2) \in (\rho^c)^e$, and by Corollary 1.1, if and only if $a_1 = a_2$ or for some $n \in \mathbb{N}$ there exists a sequence of elements $a_1 = z_1, z_2, ..., z_n = a_2$ in A such that for every $i \in \{1, 2, ..., n{-}1\}$,

$(z_i, z_{i+1}) \in \rho^c \bigcup (\rho^c)^{-1} = (\rho \bigcup \rho^{-1})^c$ by Proposition 2.9(2) and (5) so that $z_i \rightarrow z_{i+1}$, which gives the required sequence $a_1 = z_1 \rightarrow z_2 \rightarrow \cdots \rightarrow z_n = a_2$ of elementary $\Gamma - T - S - \rho$–transitions. ∎

In what follows we shall often use a more explicit version of Corollary 2.13 in the case of $|T| = 1$, i.e. in the case of right $\Gamma - S$–acts.

Lemma 2.14. *Let $\Gamma - A_S$ be a right $\Gamma - S$–act and $\rho \in B(A)$. Then for any $a, b \in A$, $(a, b) \in \rho^{\#}$ if and only if $a = b$ or there exist $p_1, ..., p_n, q_1, ..., q_n \in A$, $w_1, ..., w_n \in S$, $\gamma_1, \gamma_2, ..., \gamma_n \in \Gamma$, where for $i = 1, ..., n$, $(p_i, q_i) \in \rho$ or $(q_i, p_i) \in \rho$, such that*

$$a = p_1\gamma_1 w_1, q_2\gamma_2 w_2 = p_3\gamma_3 w_3, ..., q_n\gamma_n w_n = b.$$
$$q_1\gamma_1 w_1 = p_2\gamma_2 w_2, q_3\gamma_3 w_3 = p_4\gamma_4 w_4, ...$$

Proof. Using Corollary 2.13, we have $(a, b) \in \rho^{\#}$ if and only if $a = b$ or for some $n \in \mathbb{N}$ there is a sequence $a = z_1 \rightarrow z_2 \rightarrow \cdots \rightarrow z_n = b$ of elementary $\Gamma - S - \rho$–transitions connecting a to b. If $a = b$, it is clear. If $a = z_1 \rightarrow z_2 \rightarrow \cdots \rightarrow z_n = b$, then $a = z_1 = p_1\gamma_1 w_1$, $z_2 = q_1\gamma_1 w_1$, such that $(p_1, q_1) \in \rho$ or $(q_1, p_1) \in \rho$ and $z_2 = p_2\gamma_2 w_2 = q_1\gamma_1 w_1$,$z_3 = q_2\gamma_2 w_2$ such that $(p_2, q_2) \in \rho$ or $(q_2, p_2) \in \rho$. Continuing the same way, we get $q_{n-1}\gamma_{n-1}w_{n-1} = p_n\gamma_n w_n$ and $q_n\gamma_n w_n = z_n = b$, for some $p_1, ..., p_n, q_1, ..., q_n \in A$,$w_1, ..., w_n \in S$, $\gamma_1, \gamma_2, ..., \gamma_n \in \Gamma$. ∎

Proposition 2.15. *Let $\varepsilon \in \varepsilon(A)$ for a Γ–biact Γ–${}_TA_S$. Then*

$$\varepsilon^b := \{(a_1, a_2) \in A \times A \mid (t\gamma a_1\beta s, t\gamma a_2\beta s) \in \varepsilon \text{ for all } t \in T, s \in S, \gamma, \beta \in \Gamma\}$$

is the largest Γ–congruence on Γ–${}_TA_S$ contained in ε.

Proof. Taking $t = 1$ and $s = 1$ we see that $\varepsilon^b \subseteq \varepsilon$. Clearly, ε^b is an equivalence relation. If $(a_1, a_2) \in \varepsilon^b$ and $t' \in T$, $s' \in S$, $\gamma', \beta' \in \Gamma$, then we have

$$(t\gamma(t'\gamma' a_1\beta' s')\beta s, t\gamma(t'\gamma' a_2\beta' s')\beta s) = ((t\gamma t')\gamma' a_1\beta'(s'\beta s),\ (t\gamma t')\gamma' a_2\beta'(s'\beta s)) \in \varepsilon$$

for all $t \in T, s \in S$, $\gamma, \beta \in \Gamma$ and so $(t'\gamma' a_1\beta' s', t'\gamma' a_2\beta' s') \in \varepsilon^b$. This means that $\varepsilon^b \in \mathbf{Con}(\Gamma - {}_TA_S)$. If $\sigma \in \mathbf{Con}(\Gamma - {}_TA_S)$ and $\sigma \subseteq \varepsilon$, then for all $a_1, a_2 \in A$, let $(a_1, a_2) \in \sigma$ so that for all $t \in T$, $s \in S$, $\gamma, \beta \in \Gamma$ we have $(t\gamma a_1\beta s, t\gamma a_2\beta s) \in \sigma \subseteq \varepsilon$. Thus, $(a_1, a_2) \in \varepsilon^b$ and then $\sigma \subseteq \varepsilon^b$, i.e. ε^b is the largest Γ–congruence on Γ–${}_TA_S$ contained in ε. ∎

Remark 2.16. [8] For a $T - S$–biact ${}_TA_S$, the poset $\mathbf{Con}({}_TA_S)$ is a lattice and for any $\rho, \sigma \in \mathbf{Con}({}_TA_S)$, $\rho \wedge \sigma$ is $\rho \bigcap \sigma$ and $\rho \vee \sigma$ is $(\rho \bigcup \sigma)^{\#} = (\rho \bigcup \sigma)^e$ where $(\rho \bigcup \sigma)^{\#}$ denotes the $T - S$–congruence closure of $\rho \bigcup \sigma$. Similarly, the poset $\varepsilon(A)$ of all equivalence relations on the set A as a subposet of $B(A)$ is also a lattice and for any $\rho, \sigma \in \varepsilon(A)$, $\rho \wedge \sigma$ is $\rho \bigcap \sigma$ and $\rho \vee \sigma$ is $(\rho \bigcup \sigma)^e$.

Proposition 2.17. *Let Γ–${}_TA_S$ be a Γ–T–S–biact. Then the poset $\mathbf{Con}(\Gamma - {}_TA_S)$ is a lattice and for any $\rho, \sigma \in \mathbf{Con}(\Gamma - {}_TA_S)$, $\rho \wedge \sigma = \rho \bigcap \sigma$ and $\rho \vee \sigma = (\rho \bigcup \sigma)^{\#} = (\rho \bigcup \sigma)^e$.*

Proof. Let $\rho, \sigma \in \mathbf{Con}(\Gamma - {}_TA_S)$. It is easily seen that $\rho \bigcap \sigma$ and $(\rho \bigcup \sigma)^{\#}$ are the meet and the join of ρ, σ in $\mathbf{Con}(\Gamma - {}_TA_S)$, respectively. Then $\mathbf{Con}(\Gamma - {}_TA_S)$ is a lattice. It remains to show that $(\rho \bigcup \sigma)^{\#} = (\rho \bigcup \sigma)^e$. Applying Theorem 2.12, we have $(\rho \bigcup \sigma)^{\#} = ((\rho \bigcup \sigma)^c)^e = (\rho^c \bigcup \sigma^c)^e = (\rho \bigcup \sigma)^e$ in which the last two identities follow from Proposition 2.9(5) and (6). ∎

Theorem 2.18. *Let $\rho, \sigma \in \mathbf{Con}(\Gamma - {}_TA_S)$ for a $\Gamma - T - S$–biact $\Gamma - {}_TA_S$. Then $\rho \vee \sigma = (\rho \bigcup \sigma)^\infty = (\rho \circ \sigma)^\infty$. This means that if $a_1, a_2 \in A$, then $(a_1, a_2) \in \rho \vee \sigma$ if and only if for some $n \in \mathbb{N}$ there exist elements $b_1, b_2, ..., b_{n-1} \in A$ such that $(a_1, b_1) \in \tau_1, (b_1, b_2) \in \tau_2, ..., (b_{n-1}, a_2) \in \tau_n$, where $\tau_i \in \{\rho, \sigma\}$, $i = 1, ..., n$.*

Proof. Consider any $\rho, \sigma \in \mathbf{Con}(\Gamma - {}_TA_S)$. By using Proposition 2.17, we have $\rho \vee \sigma = (\rho \bigcup \sigma)^e = [(\rho \bigcup \sigma) \bigcup (\rho \bigcup \sigma)^{-1} \bigcup \Delta_A]^\infty = (\rho \bigcup \sigma)^\infty$ of which last equality follows from the symmetry and reflexivity properties of $\rho \bigcup \sigma$. We claim that $(\rho \bigcup \sigma)^\infty = (\rho \circ \sigma)^\infty$. To this end, first note that since ρ, σ are reflexive, $\rho, \sigma \subseteq \rho \circ \sigma \subseteq (\rho \circ \sigma)^\infty$ and so $\rho \bigcup \sigma \subseteq (\rho \circ \sigma)^\infty$. This implies that $(\rho \bigcup \sigma)^\infty \subseteq (\rho \circ \sigma)^\infty$. For the reverse inclusion, we have $\rho, \sigma \subseteq \rho \bigcup \sigma \subseteq (\rho \bigcup \sigma)^\infty$ so that $\rho \circ \sigma \subseteq (\rho \bigcup \sigma)^\infty \circ (\rho \bigcup \sigma)^\infty \subseteq (\rho \bigcup \sigma)^\infty$ which the last inclusion follows from the transitivity property of $(\rho \bigcup \sigma)^\infty$. Then $(\rho \circ \sigma)^\infty \subseteq (\rho \bigcup \sigma)^\infty$, as claimed. The second assertion is an easy consequence of the identity $\rho \vee \sigma = (\rho \bigcup \sigma)^\infty$ in the first one. ∎

Corollary 2.19. *For a $\Gamma - T - S$–biact $\Gamma - {}_TA_S$, if $\rho, \sigma \in \mathbf{Con}(\Gamma - {}_TA_S)$ are such that $\rho \circ \sigma = \sigma \circ \rho$, then $\rho \vee \sigma = \rho \circ \sigma$.*

Proof. By the assumption, $(\rho \circ \sigma)^i = \rho^i \circ \sigma^i$ for all $i \in \mathbb{N}$. On the other hand, since ρ, σ are reflexive and transitive, $\rho^i = \rho, \sigma^i = \sigma$ for all $i \in \mathbb{N}$. Then $(\rho \circ \sigma)^i = \rho \circ \sigma$. Hence, using Theorem 2.18, we get $\rho \vee \sigma = (\rho \circ \sigma)^\infty = \bigcup\limits_{i=1}^{\infty} (\rho \circ \sigma)^i = \rho \circ \sigma$. ∎

3. Green's relations on Γ–biacts

This section is devoted to study Green's relations on Γ–biacts.

Definition 3.1. [8] Let ${}_TA_S$ be a biact. The *Green's equivalences* on ${}_TA_S$ are defined by the following rules:

$(a_1, a_2) \in {}_T\mathcal{L}$ if and only if $Ta_1 = Ta_2$,
$(a_1, a_2) \in \mathcal{R}_S$ if and only if $a_1S = a_2S$,
$(a_1, a_2) \in {}_T\mathcal{J}_S$ if and only if $Ta_1S = Ta_2S$,

for all $a_1, a_2 \in A$. Further,

${}_T\mathcal{H}_S := {}_T\mathcal{L} \wedge \mathcal{R}_S = {}_T\mathcal{L} \bigcap \mathcal{R}_S$,
${}_T\mathcal{D}_S := {}_T\mathcal{L} \vee \mathcal{R}_S = ({}_T\mathcal{L} \bigcup \mathcal{R}_S)^e$.

Definition 3.2. Let $\Gamma - {}_TA_S$ be a Γ–biact. We define *Green's relations* on $\Gamma - {}_TA_S$ as follows:

$(a_1, a_2) \in {}_T\mathcal{L}$ if and only if $T\Gamma a_1 = T\Gamma a_2$,
$(a_1, a_2) \in \mathcal{R}_S$ if and only if $a_1\Gamma S = a_2\Gamma S$,

$(a_1, a_2) \in {}_T\mathcal{J}_S$ if and only if $T\Gamma a_1\Gamma S = T\Gamma a_2\Gamma S$,
for all $a_1, a_2 \in A$. Note that it is clear that ${}_T\mathcal{L}, \mathcal{R}_S$ and ${}_T\mathcal{J}_S$ are equivalence relations on the set A. Thus, in view of Remark 2.16, we also define
${}_T\mathcal{H}_S := {}_T\mathcal{L} \wedge \mathcal{R}_S = {}_T\mathcal{L} \bigcap \mathcal{R}_S \in \varepsilon(A)$,
${}_T\mathcal{D}_S := {}_T\mathcal{L} \vee \mathcal{R}_S = ({}_T\mathcal{L} \bigcup \mathcal{R}_S)^e \in \varepsilon(A)$.

Lemma 3.3. *In terms of the previous definition we have ${}_T\mathcal{L} \in$ **Con**$(\Gamma - A_S)$ and $\mathcal{R}_S \in$ **Con**$(\Gamma - {}_TA)$.*

Proof. Let $a_1, a_2 \in A, (a_1, a_2) \in {}_T\mathcal{L}$. Take $s \in S$ and $\gamma \in \Gamma$. Then $T\Gamma a_1 = T\Gamma a_2$ and so $T\Gamma(a_1\gamma s) = (T\Gamma a_1)\gamma s = (T\Gamma a_2)\gamma s = T\Gamma(a_2\gamma s)$, i.e. $(a_1\gamma s, a_2\gamma s) \in {}_T\mathcal{L}$. This means that ${}_T\mathcal{L}$ is a $\Gamma - S-$ congruence on A_S. The proof for $\mathcal{R}_S$ is similar. ∎

Theorem 3.4. *Let $\Gamma - {}_TA_S$ be a Γ–biact. If $\rho \in$ **Con**$(\Gamma - {}_TA)$ and $\rho \subseteq \mathcal{R}_S$, $\lambda \in$ **Con**$(\Gamma - A_S)$ and $\lambda \subseteq {}_T\mathcal{L}$, then $\lambda \circ \rho = \rho \circ \lambda$. In particular, ${}_T\mathcal{L} \circ \mathcal{R}_S = \mathcal{R}_S \circ {}_T\mathcal{L}$.*

Proof. Let $(a_1, a_2) \in \lambda \circ \rho$. So there exists $a_3 \in A$ with $a_1\lambda a_3\rho a_2$. Since $\lambda \subseteq {}_T\mathcal{L}$ and $\rho \subseteq \mathcal{R}_S$, we get $T\Gamma a_1 = T\Gamma a_3$, $a_3\Gamma S = a_2\Gamma S$. Then $a_3 = t_1\gamma_1 a_1$, $a_2 = a_3\beta_3 s_3$, $a_1 = t_3\gamma_3 a_3$ and $a_3 = a_2\beta_2 s_2$ for some $t_1, t_3 \in T$, $s_2, s_3 \in S$ and $\gamma_1, \gamma_3, \beta_2, \beta_3 \in \Gamma$. Let $d = a_1\beta_3 s_3$. Then $d = t_3\gamma_3 a_3\beta_3 s_3 = t_3\gamma_3 a_2$. Now $a_3\rho a_2$ implies that $(t_3\gamma_3 a_3)\,\rho\,(t_3\gamma_3 a_2)$. Thus, $a_1\rho d$. Also, $a_1\lambda a_3$ gives that $(a_1\beta_3 s_3)\,\lambda\,(a_3\beta_3 s_3)$ and then $d\lambda a_2$. Hence, $a_1\,(\rho \circ \lambda)\,a_2$ which follows that $\lambda \circ \rho \subseteq \rho \circ \lambda$. Analogously, the reverse inclusion also holds. Since ρ and λ are arbitrary in ${}_T\mathcal{L}$ and $\mathcal{R}_S$ respectively, using Lemma 3.3, ${}_T\mathcal{L} \circ \mathcal{R}_S = \mathcal{R}_S \circ {}_T\mathcal{L}$. ∎

Remark 3.5. Since ${}_T\mathcal{L}$ and $\mathcal{R}_S$ commute by Theorem 3.4, it follows from Corollary 2.19 that ${}_T\mathcal{D}_S = {}_T\mathcal{L} \vee \mathcal{R}_S = {}_T\mathcal{L} \circ \mathcal{R}_S = \mathcal{R}_S \circ {}_T\mathcal{L}$. It is clear that $|T| = 1$ implies that ${}_T\mathcal{J}_S = \mathcal{R}_S$ and $|S| = 1$ implies that ${}_T\mathcal{J}_S = {}_T\mathcal{L}$. Moreover, we have ${}_T\mathcal{D}_S \subseteq {}_T\mathcal{J}_S$. Indeed, first note that ${}_T\mathcal{L} \subseteq {}_T\mathcal{J}_S$ and $\mathcal{R}_S \subseteq {}_T\mathcal{J}_S$ whence ${}_T\mathcal{L} \bigcup \mathcal{R}_S \subseteq {}_T\mathcal{J}_S$. Since ${}_T\mathcal{J}_S \in \varepsilon(A)$, ${}_T\mathcal{D}_S = {}_T\mathcal{L} \vee \mathcal{R}_S = ({}_T\mathcal{L} \bigcup \mathcal{R}_S)^e \subseteq ({}_T\mathcal{J}_S)^e = {}_T\mathcal{J}_S$.

Here we generalize the notion of periodic semigroup to the Γ–semigroups which is needed in the sequel.

Definition 3.6. [8] A *monogenic* (*cyclic*) semigroup is a semigroup generated by a singleton. A semigroup is called *periodic* if all of its monogenic subsemigroups are finite.

Definition 3.7. Let S be a Γ–semigroup and $\gamma \in \Gamma$. An element $e \in S$ is called a *γ–idempotent* if $e_\gamma^2 = e$ where e_γ^2 means $e\gamma e$. A subset T of a S is called a *γ–subsemigroup* of S if for every $x, y \in T, x\gamma y \in T$. A Γ–semigroup S is said to be *periodic* if all of its monogenic γ–subsemigroups are finite for every $\gamma \in \Gamma$. Here, a monogenic γ–subsemigroup of S generated by $s \in S$ is denoted by $\langle s \rangle_\gamma$, and $\langle s \rangle_\gamma = \left\{ s_\gamma^n \mid n \in \mathbb{N} \right\}$ where $s_\gamma^1 = s$, $s_\gamma^2 = s\gamma s$ $, ..., s_\gamma^n = s_\gamma^{n-1}\gamma s$.

Lemma 3.8. [8] *Every finite semigroup includes an idempotent element.*

Lemma 3.9. *Among the powers s_γ^n of elements of a periodic Γ–semigroup S for $\gamma \in \Gamma$, there is a γ–idempotent.*

Proof. Let $s \in S$ and $\gamma \in \Gamma$. Consider the monogenic γ–subsemigroup $\langle s \rangle_\gamma$. For every $x, y \in \langle s \rangle_\gamma$, define $xy := x\gamma y \in \langle s \rangle_\gamma$. Then $\langle s \rangle_\gamma$ is made into a semigroup by this operation. Since S is periodic, $\langle s \rangle_\gamma$ is a finite Γ–semigroup and then a finite semigroup. Then, using Lemma 3.8, there is an idempotent element e in the semigroup $\langle s \rangle_\gamma$. Thus, there exists $k \in \mathbb{N}$, $e = s^k_\gamma$. Using the operation, $e = e^2 = e\gamma e = e^2_\gamma$. Then $e = s^k_\gamma$ is a γ–idempotent element of S. ∎

Notation 3.10. Let S be a Γ–semigroup, $s_1, s_2 \in S$, $\gamma, \beta \in \Gamma$. Then we put $(s_1\gamma s_2)^2_\beta := (s_1\gamma s_2)\beta(s_1\gamma s_2)$.

Theorem 3.11. *Let $\Gamma - {}_TA_S$ be a Γ–biact over periodic Γ–semigroups T and S. Then on $\Gamma - {}_TA_S$ we have ${}_T\mathcal{D}_S = {}_T\mathcal{J}_S$.*

Proof. In view of Remark 3.5, it suffices to prove that ${}_T\mathcal{J}_S \subseteq {}_T\mathcal{D}_S$. Assume that $a_1, a_2 \in A$ and $(a_1, a_2) \in {}_T\mathcal{J}_S$, i.e. $T\Gamma a_1\Gamma S = T\Gamma a_2 \Gamma S$. Thus, $a_1 = t_2\alpha a_2\lambda s_2$ and $a_2 = t_1\gamma a_1\beta s_1$ for some $s_1, s_2 \in S$, $t_1, t_2 \in T$ and $\alpha, \lambda, \gamma, \beta \in \Gamma$. Then

$$\begin{aligned} a_1 &= t_2\alpha\left(t_1\gamma a_1\beta s_1\right)\lambda s_2 = \left(t_2\alpha t_1\right)\gamma a_1\beta\left(s_1\lambda s_2\right) = \left(t_2\alpha t_1\right)\gamma t_2\alpha a_2\lambda s_2\beta\left(s_1\lambda s_2\right) \\ &= \left(t_2\alpha t_1\right)\gamma t_2\alpha t_1\gamma a_1\beta s_1\lambda s_2\beta\left(s_1\lambda s_2\right) = \left(t_2\alpha t_1\right)^2_\gamma\gamma a_1\beta\left(s_1\lambda s_2\right)^2_\beta = \cdots \end{aligned}$$

Analogously, we obtain

$$a_2 = \left(t_1\gamma t_2\right)\alpha a_2\lambda\left(s_2\beta s_1\right) = \left(t_1\gamma t_2\right)^2_\alpha\alpha a_2\lambda\left(s_1\beta s_2\right)^2_\lambda = \cdots$$

Since T and S are periodic Γ–semigroups, we can find $m \in \mathbb{N}$ such that $(t_2\alpha t_1)^m_\gamma$ is a γ– idempotent by Lemma 3.9. Let now $c = t_1\gamma a_1 \in A$. Then

$$\begin{aligned} a_1 &= \left(t_2\alpha t_1\right)^m_\gamma\gamma a_1\beta\left(s_1\lambda s_2\right)^m_\beta = \left(t_2\alpha t_1\right)^m_\gamma\gamma\left(t_2\alpha t_1\right)^m_\gamma\gamma a_1\beta\left(s_1\lambda s_2\right)^m_\beta \\ &= \left(t_2\alpha t_1\right)^m_\gamma\gamma a_1 = \left(\left(t_2\alpha t_1\right)^{m-1}_\gamma\gamma t_2\right)\alpha\left(t_1\gamma a_1\right) = \left(\left(t_2\alpha t_1\right)^{m-1}_\gamma\gamma t_2\right)\alpha c. \end{aligned}$$

Therefore, $(a_1, c) \in {}_T\mathcal{L}$. Moreover, we have $c\beta s_1 = t_1\gamma a_1\beta s_1 = a_2$, and, using Lemma 3.9, if we choose $n \in \mathbb{N}$ such that $(s_2\beta s_1)^n_\lambda$ is a λ–idempotent, then we get

$$\begin{aligned} c &= t_1\gamma a_1 = t_1\gamma(t_2\alpha t_1)^{n+1}_\gamma\gamma a_1\beta(s_1\lambda s_2)^{n+1}_\beta = (t_1\gamma t_2)^{n+1}_\alpha\alpha(t_1\gamma a_1\beta s_1)\lambda(s_2\beta s_1)^n_\lambda\lambda s_2 \\ &= (t_1\gamma t_2)^{n+1}_\alpha\alpha a_2\lambda(s_2\beta s_1)^{2n}_\lambda\lambda s_2 = ((t_1\gamma t_2)^{n+1}_\alpha\alpha a_2\lambda(s_2\beta s_1)^{n+1}_\lambda)\lambda(s_2\beta s_1)^{n-1}_\beta\lambda s_2 \\ &= a_2\lambda(s_2\beta s_1)^{n-1}_\beta\lambda s_2. \end{aligned}$$

Hence, $(c, a_2) \in \mathcal{R}_S$ and so, using Remark 3.5, $(a_1, a_2) \in {}_T\mathcal{L} \circ \mathcal{R}_S = {}_T\mathcal{D}_S$. ∎

Definition 3.12. Let $\rho \in \mathbf{Con}(\Gamma - {}_TA_S)$ for a Γ–biact $\Gamma - {}_TA_S$. The set $\frac{\Gamma - {}_TA_S}{\rho} = \{[a]_\rho \mid a \in A\}$ with the left $\Gamma - T$–action $t\gamma[a]_\rho := [t\gamma a]_\rho$ and the right $\Gamma - S$–action $[a]_\rho\gamma s := [a\gamma s]_\rho$ for every $t \in T, s \in S$ and $\gamma \in \Gamma$ is clearly a Γ–biact which is called the *factor* Γ*–biact* of $\Gamma - {}_TA_S$ by ρ.

Proposition 3.13. *Let $\Gamma - {}_TA_S$ be a Γ–biact and $\rho \in \mathbf{Con}(\Gamma - {}_TA_S)$. Then*

(i) *If $\rho \subseteq {}_T\mathcal{L}$, then for all $a, b \in A$, $a \; {}_T\mathcal{L} \; b$ if and only if $[a]_\rho \; {}_T\mathcal{L} \; [b]_\rho$ in $\frac{\Gamma - {}_TA_S}{\rho}$.*

(ii) *If* $\rho \subseteq \mathcal{R}_S$*, then for all* $a, b \in A$*,* $a\ \mathcal{R}_S\ b$ *if and only if* $[a]_\rho\ \mathcal{R}_S\ [b]_\rho$ *in* $\frac{\Gamma - {}_TA_S}{\rho}$.

(iii) *If* $\rho \subseteq {}_T\mathcal{H}_S$*, then for all* $a, b \in A$*,* $a\ {}_T\mathcal{H}_S\ b$ *if and only if* $[a]_\rho\ {}_T\mathcal{H}_S\ [b]_\rho$ *in* $\frac{\Gamma - {}_TA_S}{\rho}$.

Proof. (i) Let $a, b \in A$. If $a\ {}_T\mathcal{L}\ b$, then there exist $t, u \in T$ and $\gamma, \beta \in \Gamma$ such that $a = t\gamma b$ and $b = u\beta a$. Then $[a]_\rho = [t\gamma b]_\rho = t\gamma[b]_\rho$ and $[b]_\rho = [u\beta a]_\rho = u\beta[a]_\rho$. Therefore, $T\Gamma[a]_\rho = T\Gamma[b]_\rho$ which means that $[a]_\rho\ {}_T\mathcal{L}\ [b]_\rho$ in $\frac{\Gamma - {}_TA_S}{\rho}$. Conversely, assume that $[a]_\rho\ {}_T\mathcal{L}\ [b]_\rho$ in $\frac{\Gamma - {}_TA_S}{\rho}$. Then $T\Gamma[a]_\rho = T\Gamma[b]_\rho$ so that there exist $t, u \in T$ and $\gamma, \beta \in \Gamma$ such that $[a]_\rho = t\gamma[b]_\rho = [t\gamma b]_\rho$ and $[b]_\rho = u\beta[a]_\rho = [u\beta a]_\rho$, i.e. $a\rho\,(t\gamma b)$ and $b\rho\,(u\beta a)$. Since $\rho \subseteq {}_T\mathcal{L}$, $a\ {}_T\mathcal{L}\ t\gamma b$ and $b\ {}_T\mathcal{L}\ u\beta a$. Then $T\Gamma a = T\Gamma\,(t\gamma b)$ and $T\Gamma b = T\Gamma\,(u\beta a)$. This implies that $a \in T\Gamma\,(t\gamma b) = (T\Gamma t)\gamma b \subseteq T\Gamma b$ and $b \in T\Gamma\,(u\beta a) = (T\Gamma u)\beta b \subseteq T\Gamma a$. Therefore, $T\Gamma a = T\Gamma b$, i.e. $a\ {}_T\mathcal{L}\ b$.

(ii) It is similar to (i).

(iii) Let $a, b \in A$. Assume that $\rho \subseteq {}_T\mathcal{H}_S$. Since ${}_T\mathcal{H}_S = {}_T\mathcal{L} \bigcap \mathcal{R}_S$, $\rho \subseteq {}_T\mathcal{L}$ and $\rho \subseteq \mathcal{R}_S$. Using (i) and (ii), $a\ {}_T\mathcal{H}_S\ b$ if and only if $a\ {}_T\mathcal{L}\ b$ and $a\ \mathcal{R}_S\ b$ if and only if $[a]_\rho\ {}_T\mathcal{L}\ [b]_\rho$ and $[a]_\rho\ \mathcal{R}_S\ [b]_\rho$ if and only if $[a]_\rho\ {}_T\mathcal{H}_S\ [b]_\rho$ in $\frac{\Gamma - {}_TA_S}{\rho}$. ∎

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References

[1] Ameri, A., Sadeghi, R., *Gamma modules*, Ratio Mathematica, 20 (2010), 127-147.

[2] Chinram, R., Siammai, P., *On Green's relations for* Γ*–semigroups and reductive* Γ*– semigroups*, Int. J. Algebra, 2(4) (2008), 187-195.

[3] Chinram, R., Tinpun, K., *Isomorphism theorems for* Γ*–semigroup and ordered* Γ*–semigroup*, J. Math., 7(2) (2009), 231-241.

[4] Feng, Y., Corsini, P., (λ, μ)*-Fuzzy interior ideals of ordered* Γ*–semigroups*, Algebra Discrete Math., 16(1) (2013), 61-70.

[5] Green, J.A., *On the structure of semigroup*, Ann.Math., 54(1951), 163-172.

[6] Hossain, Z., *Gamma modules with gamma rings of gamma endomorphisms*, Int. J. Sci. Tech., 6(3) (2013), 346-354.

[7] Howie, J.M., *Automata and Languages*, Oxford University Press, Oxford, 1991.

[8] Kilp, M., Knauer, U., Mikhalev, A.V., *Monoids, Acts and Categories*, de Gruyter, Berlin, 2000.

[9] Li, B., Feng, Y., *Intuitionistic* (λ, μ)*-fuzzy sets in* Γ*–semigroups*, J. Inequal. Appl., (2013), 2013:107 doi:10.1186/1029-242X-2013-107.

[10] Sen, M.K., *On* Γ*–semigroups*, Proceedings of the International Conference on Algebra and its Application, Decker publication, New York, 1981, 301-308.

[11] Sen, M.K., Seth, A., *Radical of* Γ*–semigroup*, Bull. Calcutta Math. Soc. 80(3), (1988), 189-196.

[12] Shabani, A.R., Rasouli, H., *On gamma S-acts*, The 45th Annual Iranian Mathematics Conference, 2014.

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THE ANALYSIS OF A VARIABLE-VISCOSITY FLUID FLOW BETWEEN PARALLEL POROUS PLATES WITH NON-UNIFORM WALL TEMPERATURE

A.R. Hassan[1]
1. Department of Mathematical Sciences
University of South Africa
Pretoria 0003
South Africa
2. Department of Mathematics
Tai Solarin University of Education
Ijagun, Ogun State
Nigeria

R. Maritz
Department of Mathematical Sciences
University of South Africa
Pretoria 0003
South Africa

Abstract. This paper examined the effectiveness of porosity on a variable-viscosity channel flow with non-uniform wall temperature. The flow is considered to be a steady, incompressible and the fluid viscosity varies linearly with temperature. The analytic expressions are obtained seeking asymptotic solutions for fluid velocity and temperature and these expressions are used to derive and obtain solutions for entropy generation rate and Bejan numbers with variations in other physical parameters present in the fluid flow.

Keywords: variable-viscosity, non-uniform wall temperature, entropy generation rate, Bejan number and porous medium.

1. Introduction

Studies involving fluid flow between parallel porous plates have been investigated by a number of researchers [1]–[11] because of its wide range of engineering applications such as electronic cooling, thermal insulation, crude oil extraction and nuclear reactor. Others, for example, include [12] which considered the unsteady viscous fluid flow with porous medium in the presence of radiation and chemical reaction, [13] conducted the experimental investigation of the permeability and inertia effect on fluid flow through homogeneous porous media and [14] presented a theoretical study of the fluid flow and transfer through a porous medium channel bounded by a permeable parallel walls with equal suction or equal injection.

[1]Corresponding Author. E-mail: anthonyhassan72@yahoo.co.uk, hassaar@unisa.ac.za, Tel: +27847111628

In all these studies, it is worthy to note the influence and effect of porosity on velocity, temperature, pressure distribution and other physical properties within the flow channel.

Meanwhile, [15] in his entropy-generation analysis for variable-viscosity channel flow with non-uniform temperature analyzed that most fluids used in engineering and industrial systems can be subjected to extreme conditions, such as high temperature, pressure and shear rate without considering the effect of porosity. In addition to that, [16] also pointed out that fluid flow is often accompanied with heat transfer which is an integral part of natural convection flow and belongs to the class of problems in boundary layer theory which occurs in various physical phenomena such as fire engineering, combustion modeling, nuclear reactor, heat exchangers, etc. without considering such fluid flowing through porous medium.

But there is need to find out the property of the porous medium which measures the capacity and ability of the formation to transmit fluids. This is a very important property that controls the directional movement and the flow rate of the fluid in any formation. As a result of that, there is need to investigate and examine the effectiveness of porosity on a variable-viscosity channel flow with non-uniform wall temperature which was not considered in [15]. Hence, the analytic expressions for fluid velocity and temperature profiles are determined by seeking asymptotic solutions and these expressions are used to derive and obtain solutions for entropy generation rate and Bejan numbers with variations in other physical parameters present in the fluid flow regime.

In the rest of this paper, the problem is formulated in Section 2. The governing equations are solved using ADM and the entropy generation rate was determined in Section 3. A presentation of analytical results of the problem are shown in tables and graphs in Section 4 and Section 5 gives the concluding remarks.

2. Mathematical model

Consider the steady flow of an incompressible fluid flowing in the $x-$ direction through parallel porous plates of width, (a) and length, (L) with a nonuniform wall temperature under the action of a constant pressure gradient as shown in Figure 1.

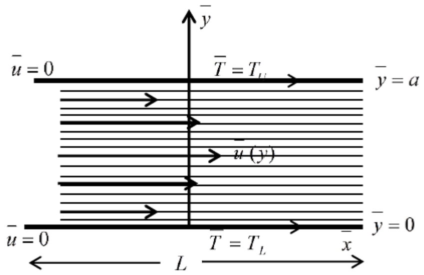

Figure 1: The geometry of the problem

The temperature dependent viscosity ($\overline{\mu}$) as described in [15] can be expressed as

$$\overline{\mu} = \mu_0 \left[1 - \beta \left(\overline{T} - T_L\right)\right], \tag{1}$$

where μ_0 is the fluid dynamic viscosity at the lower wall temperature T_l, β is the viscosity – variation parameter and T is the fluid temperature. Neglecting the consumption of the reactant, the continuity, momentum (along x and y axes) and energy equations governing the fluid flow in nondimensionless form may be written following [15], [17] as:

$$\frac{\partial \overline{u}}{\partial \overline{x}} + \frac{\partial \overline{v}}{\partial \overline{y}} = 0 \tag{2}$$

$$\rho \left[\overline{u}\frac{\partial \overline{u}}{\partial \overline{x}} + \overline{v}\frac{\partial \overline{u}}{\partial \overline{y}}\right] = -\frac{\partial \overline{p}}{\partial \overline{x}} + 2\frac{\partial}{\partial \overline{x}}\left(\overline{\mu}\frac{\partial \overline{u}}{\partial \overline{x}}\right) + \frac{\partial}{\partial \overline{y}}\left[\overline{\mu}\left(\frac{\partial \overline{u}}{\partial \overline{y}} + \frac{\partial \overline{v}}{\partial \overline{x}}\right)\right] - \frac{\overline{\mu}}{K}\overline{u} \tag{3}$$

$$\rho \left[\overline{u}\frac{\partial \overline{v}}{\partial \overline{x}} + \overline{v}\frac{\partial \overline{v}}{\partial \overline{y}}\right] = -\frac{\partial \overline{p}}{\partial \overline{y}} + 2\frac{\partial}{\partial \overline{y}}\left(\overline{\mu}\frac{\partial \overline{v}}{\partial \overline{y}}\right) + \frac{\partial}{\partial \overline{x}}\left[\overline{\mu}\left(\frac{\partial \overline{u}}{\partial \overline{y}} + \frac{\partial \overline{v}}{\partial \overline{x}}\right)\right] \tag{4}$$

$$\begin{aligned} &\rho c_p \left[\overline{u}\frac{\partial \overline{T}}{\partial \overline{x}} + \overline{v}\frac{\partial \overline{T}}{\partial \overline{y}}\right] \\ &= k\left[\frac{\partial^2 \overline{T}}{\partial \overline{x}^2} + \frac{\partial^2 \overline{T}}{\partial \overline{y}^2}\right] + \overline{\mu}\left[2\left(\frac{\partial \overline{u}}{\partial \overline{x}}\right)^2 + 2\left(\frac{\partial \overline{v}}{\partial \overline{y}}\right)^2 + \left(\frac{\partial \overline{u}}{\partial \overline{y}} + \frac{\partial \overline{v}}{\partial \overline{x}}\right)^2\right] + \frac{\overline{\mu}}{K}\overline{u}^2 \end{aligned} \tag{5}$$

where ρ is the fluid density, K is Darcy's permeability constant, k is the thermal conductivity, p is the pressure, c_p is the specific heat at constant pressure, u represent the axial velocity, v is the normal velocity, also, x and y are distances measured in the streamline and normal direction respectively. The additional last term in (3) and (5) is due to the effect of porosity and modification of Brinkman form of Darcys law of porosity as described in [18]

Introducing the following non dimensional variables and parameters in equations (1)–(5) as follows:

$$\begin{gathered} y = \frac{\overline{y}}{\epsilon L}, \quad x = \frac{\overline{x}}{L}, \quad u = \frac{\overline{u}}{U}, \quad v = \frac{\overline{v}}{\epsilon U}, \quad \epsilon = \frac{a}{L}, \quad \mu = \frac{\overline{\mu}}{\mu_0}, \\ T = \frac{\overline{T} - T_l}{T_u - T_l}, \quad p = \frac{\epsilon^2 L \overline{p}}{\mu_0 U}, \quad \alpha = \beta\left(T_u - T_l\right) \quad Br = \frac{\mu_0 U^2}{k\left(T_u - T_l\right)}, \\ -\frac{\partial p}{\partial x} = G, \quad Pe = \frac{\rho c_p L U}{k}, \quad Re = \frac{\rho U L}{\mu_0} \quad \text{and} \quad \gamma = \frac{a^2}{K}, \end{gathered} \tag{6}$$

where ϵ is the channel aspect ratio, (a) is channel width, (L) is the channel characteristic length, T_u is the upper wall temperature, T_l is the lower wall temperature, α represents viscosity - variation parameter, p is the fluid pressure, μ_0 is the fluid dynamic viscosity, G is the constant pressure gradient, U is the velocity scale, and Br is the Brinkmann number. Also, Re is the Reynolds number, Pe is the Peclet number and γ represents porous permeability parameter.

With the introduction of (6) into (1)–(5), the governing equations for the fluid flow in dimensionless form as in [15] may be written as:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{7}$$

$$\epsilon^2 Re\left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right] = G + 2\epsilon^2\left(\mu\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y} + \epsilon^2\frac{\partial v}{\partial x}\right)\right] - \gamma\mu u \tag{8}$$

$$\epsilon^4 Re\left[u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right] = -\frac{\partial p}{\partial y} + 2\epsilon^2\left(\mu\frac{\partial v}{\partial y}\right) + \epsilon^2\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y} + \epsilon^2\frac{\partial v}{\partial x}\right)\right] \tag{9}$$

$$\epsilon^2 Pe\left[u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right] = \epsilon^2\frac{\partial^2 T}{\partial x^2} + \mu Br\left[\varphi + \gamma u^2\right] \tag{10}$$

where

$$\varphi = 2\epsilon^2\left(\frac{\partial u}{\partial x}\right)^2 + 2\epsilon^2\left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial y} + \epsilon^2\frac{\partial v}{\partial x}\right)^2. \tag{11}$$

Since the channel aspect ratio is very small, that is, $0 < \epsilon \leq 1$, the lubrication approximation based on asymptotic simplification of the governing equations (7)–(11) is invoked and obtain the following:

$$\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) - \gamma\mu u + G + \bigcirc(\epsilon^2) = 0 \tag{12}$$

$$-\frac{\partial p}{\partial y} + \bigcirc(\epsilon^2) = 0 \tag{13}$$

$$\frac{\partial^2 T}{\partial y^2} + \mu Br\left[\left(\frac{\partial u}{\partial y}\right)^2 + \gamma\mu u^2\right]\bigcirc(\epsilon^2) = 0 \tag{14}$$

where $\mu = 1 - \alpha T$ with the following boundary conditions at the upper wall of the channel as:

$$u = 0, \quad T = 1 \qquad \text{at } y = 1 \tag{15}$$

and at the lower wall of the channel as:

$$u = 0, \quad T = 0 \qquad \text{at } y = 0 \tag{16}$$

It is worthy to note that (15) and (16) indicate that the temperatures at both upper and lower walls are fixed and different.

3. Method of solution

Following [15], we solve equations (12)–(14) subject to the boundary conditions in (15) and (16), based on assumption that the variations in the fluid viscosity is very small and seek an asymptotic solutions for the velocity and temperature of the fluid flow within porous medium in this form:

$$u = u_0 + \alpha u_1, \qquad T = T_0 + \alpha T_1 \tag{17}$$

Substituting (17) into (12) - (14), the following equations are obtained in the following orders:
Order zero (α^0)

$$\frac{\partial^2 u_0}{\partial y^2} - \gamma u_0 + G = 0, \qquad \frac{\partial^2 T_0}{\partial y^2} + Br\left[\left(\frac{\partial u_0}{\partial y}\right)^2 + \gamma u_0^2\right] = 0 \tag{18}$$

with the boundary conditions

$$u_0 = 0, \quad T_0 = 1 \quad \text{at } y = 1 \quad \text{and} \quad u_0 = 0, \quad T_0 = 0 \quad \text{at } y = 0 \tag{19}$$

Order one (α^1)

$$\frac{\partial^2 u_1}{\partial y^2} - T_0\frac{\partial^2 u_0}{\partial y^2} - \frac{\partial u_0}{\partial y}\frac{\partial T_0}{\partial y} - \gamma\left(u_1 - u_0 T_0\right) = 0, \tag{20}$$

$$\frac{\partial^2 T_1}{\partial y^2} + Br\left[2\frac{\partial u_0}{\partial y}\frac{\partial u_1}{\partial y} - T_0\left(\frac{\partial u_0}{\partial y}\right)^2 + \gamma\left(2u_0u_1 - T_0u_0^2\right)\right] = 0 \tag{21}$$

subject to the boundary conditions

$$u_1 = 0, \quad T_1 = 0 \quad \text{at } y = 1 \quad \text{and} \quad u_1 = 0, \quad T_1 = 0 \quad \text{at } y = 0. \tag{22}$$

Equations (18)–(22) are thereby coded on Mathematica software package, then substituting the results back into (17) to obtain the solutions for the velocity and temperature profiles which are hereby discussed in the next section because of the large volume of outputs.

4. Entropy generation

Entropy generation is a measure of the account of irreversibility associated with the real process. It is a measure of disorderliness of a system. In order to preserve the quality of energy in a fluid flow process or at least to reduce the entropy generation, it is also important to study the distribution of the entropy generation within the fluid volume. The entropy production is due to heat transfer and the combined effects of fluid friction and Joules dissipation. Following [15], [16], [19], [20], the general equation for the entropy generation per unit volume in the presence of porous medium is given by:

$$S^m = \frac{k}{T_l^2}\left(\frac{\partial \overline{T}}{\partial \overline{y}}\right)^2 + \frac{\overline{\mu}}{T_l}\left(\frac{\partial \overline{u}}{\partial \overline{y}}\right)^2 + \frac{\overline{\mu}}{T_l K}\overline{u}^2. \tag{23}$$

The first term in (23) is the irreversibility due to heat transfer, the second term is the entropy generation due to viscous dissipation and the last term is the local entropy generation due to the effects of porosity. We express the entropy generation number in dimensionless form using the existing dimensionless variables and parameter in (6) as:

$$N_s = \frac{S^m a^2 T_l^2}{k(T_u - T_l)^2} = \left(\frac{\partial T}{\partial y}\right)^2 + \frac{Br}{\Omega}\left[\mu\left(\frac{\partial u}{\partial y}\right)^2 + \gamma u^2\right]. \tag{24}$$

We let the first term, $\left(\frac{\partial T}{\partial y}\right)^2$ be assigned N_1 which is the irreversibility due to heat transfer and the second term, $\frac{Br}{\Omega}\left[\mu\left(\frac{\partial u}{\partial y}\right)^2 + \gamma u^2\right]$ be referred to as N_2, which is the entropy generation due to the effects of viscous dissipation and porosity of the flow regime. Also, $\Omega = \frac{T_u - T_l}{T_l}$ is the wall temperature parameter. We defined

$$\phi = \frac{N_1}{N_2} \tag{25}$$

as the irreversibility distribution ratio. The expression (25) shows that heat transfer dominates when $0 < \phi < 1$ and fluid friction dominates when $\phi > 1$. This is used to determine the contribution of heat transfer in many engineering designs. As an alternative to irreversibility parameter, the Bejan number (Be) is defined as

$$Be = \frac{N_1}{N_s} = \frac{1}{1+\phi} \qquad \text{where } 0 \le Be \le 1. \tag{26}$$

5. Discussion of results

In this section, the effectiveness of porosity on a variable-viscosity fluid flow with non-uniform wall temperature together with other important flow parameters are presented and discussed.

Tables (1) and (2) show the effects of porosity on velocity and temperature profiles respectively. It is noticed that there is reduction in the fluid velocity as the porosity parameter (γ) increases with the maximum at the centreline of the fluid flow while the fluid temperature increases generally from lower plate to upper plate but an increment is noticed around the centreline as the porosity parameter increases but a reduction is noticed around both upper and lower surfaces as less efficient packing occur in the interior centreline of the channel.

Table 1: Effect of porosity on velocity profile

$G = 1, Br = 10, \alpha = 0.1$			
y	$u(y)\|_{\gamma=0.1}$	$u(y)\|_{\gamma=0.5}$	$u(y)\|_{\gamma=1}$
0	0	0	0
0.1	0.0457074812	0.0441201095	0.0422986555
0.2	0.0818163056	0.0787948482	0.0753293076
0.3	0.1080885602	0.1039304097	0.0991628394
0.4	0.1243223076	0.1194280309	0.1138177099
0.5	**0.1303270245**	**0.1251614367**	**0.1192404521**
0.6	0.1259088886	0.1209633679	0.1152939695
0.7	0.1108657006	0.1066202469	0.1017520109
0.8	0.0849913686	0.0818746647	0.0782992783
0.9	0.0480900265	0.0464359858	0.0445376403
1	0	0	0

Table 2: Effect of porosity on temperature profile

$G = 1, Br = 10, \alpha = 0.1$			
y	$T(y)\|_{\gamma=0.1}$	$T(y)\|_{\gamma=0.5}$	$T(y)\|_{\gamma=1}$
0	0	0	0
0.1	0.1324173788	0.1314741222	0.1303639983
0.2	0.2481120524	0.2473560560	0.2463829314
0.3	0.3540722466	0.3540473065	0.3538154817
0.4	**0.4553874610**	**0.4560804967**	**0.4565928550**
0.5	**0.5552065540**	**0.5562102886**	**0.5570476095**
0.6	**0.6546952371**	**0.6554518644**	**0.6560340920**
0.7	0.7529839977	0.7530596886	0.7529383744
0.8	0.8471031944	0.8464430053	0.8455749361
0.9	0.9319078047	0.9310179742	0.9299661899
1	1	1	1

Figures 2 to 4 display the velocity profile respectively with variations in porous permeability parameter (γ), viscosity variation parameter (α) and pressure gradient (G).

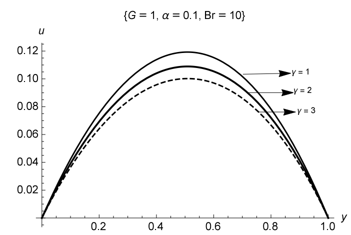

Figure 2: Velocity profile for variations in γ

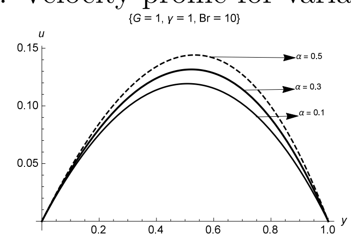

Figure 3: Velocity profile for variations in α

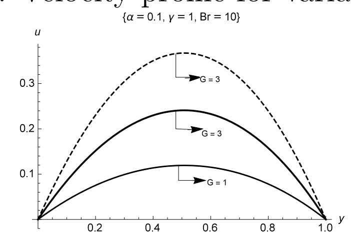

Figure 4: Velocity profile for variations in G

The general observation shows that the velocity of the fluid flow reduces as the porous permeability parameter (γ) increases while the velocity increases as both the viscosity variation parameter (α) and pressure gradient (G) increase. In Figure 2, the presence of porous medium in the flow has the tendency to increase the resistance of the fluid motion in the channel. In Figure 3, an increase in the viscosity variation parameter (α) also leads to greater viscosity distribution and hence increases the fluid velocity and the greater the pressure gradient, the greater the discharge rate of fluid depending on the porous material as shown in Figure 4.

The effects of porous permeability parameter (γ), viscosity variation parameter (α), viscous dissipation parameter (Br) and pressure gradient (G) on the temperature profiles are respectively displayed in Figures 5 to 8. A reduction is noticed as the porous permeability parameter (γ) increases while an increase in temperature is observed with increasing values of viscosity variation parameter (α), viscous dissipation parameter (Br) and pressure gradient (G).

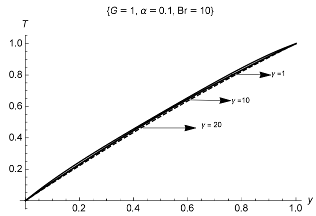

Figure 5: Temperature Profile for variations in γ

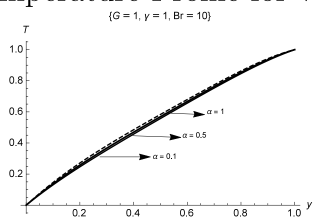

Figure 6: Temperature Profile for variations in α

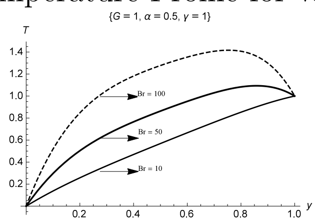

Figure 7: Temperature Profile for variations in Br

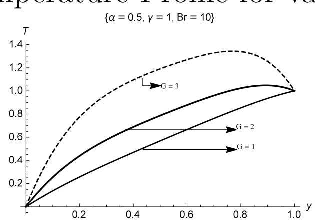

Figure 8: Temperature Profile for variations in G

Figures 9 to 13 respectively display the entropy generation rate with variations in porous permeability parameter (γ), wall temperature parameter (Ω), viscosity variation parameter (α), viscous dissipation parameter (Br) and pressure gradient (G).

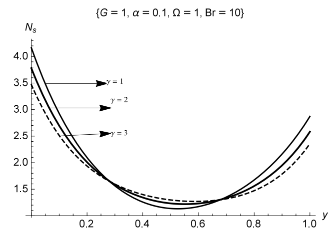

Figure 9: Entropy generation rate for variations in γ

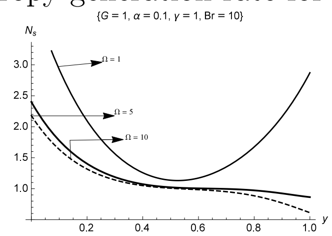

Figure 10: Entropy generation rate for variations in Ω

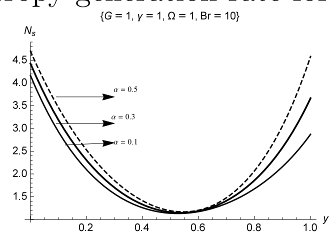

Figure 11: Entropy generation rate for variations in α

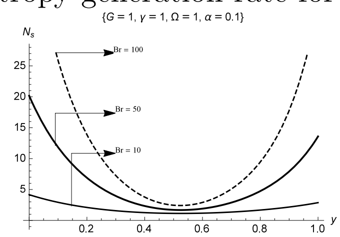

Figure 12: Entropy generation rate for variations in Br

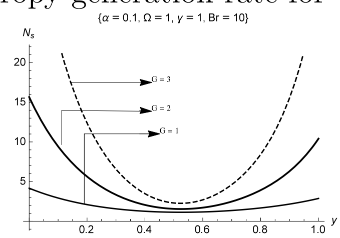

Figure 13: Entropy generation rate for variations in G

Generally, the flow is active and maintains equilibrium around the core centreline region of the flow system. In Figure 9, the entropy generation rate decreases with increasing values of porous permeability parameter (γ) near the walls and otherwise around the centreline region of the channel. Also, the entropy generation rate decreases with increasing values of wall temperature parameter (Ω) in Figure 10, while the entropy generation rate increases with increasing values of viscosity variation parameter (α), viscous dissipation parameter (Br) and pressure gradient (G) respectively in Figures 11, 12 and 13.

Bejan rates of the fluid flow are presented in Figures 14 to 16 for variations in porous permeability parameter (γ), wall temperature parameter (Ω) and viscosity variation parameter (α).

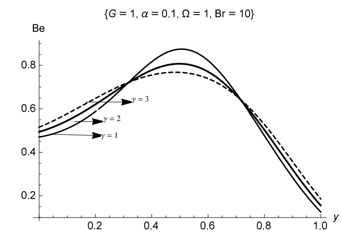

Figure 14: Bejan Number for variations in γ

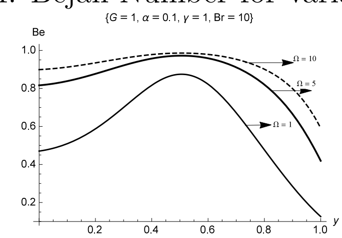

Figure 15: Bejan Number for variations in Ω

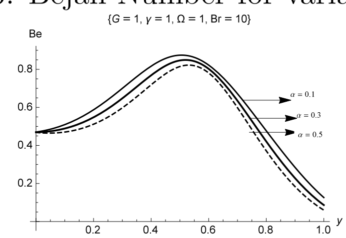

Figure 16: Bejan Number for variations in α

On a general note, it shows that heat-transfer irreversibility dominates around the centreline region of the channel while fluid friction irreversibility dominates at both upper and lower regions. The effects of porous permeability parameter (γ) is interestingly shown in Figure 14 as increasing values result into an increase in the dominant effect of fluid friction irreversibility near the walls and a reduction in the dominant effect of the heat-transfer irreversibility around centreline region. Also, in Figure 15, as increasing values of wall temperature parameter (Ω) result into an increase in the dominant effect across the channel while an increasing values of viscosity variation parameter (α) result into a reduction in the dominant effect across the channel as shown in Figure 16.

6. Conclusion

A review of the effectiveness of porosity on a variable-viscosity channel flow with non-uniform wall temperature has been investigated. The analytic expressions are obtained seeking asymptotic solutions for fluid velocity and temperature and these expressions are used to derive and obtain solutions for entropy generation rate and Bejan numbers (Be). The result shows that there is reduction in the fluid velocity as the porosity parameter (γ) increases while the fluid temperature increases from lower plate to upper plates but an increment is noticed around the centreline as the porosity parameter increases. Also, the entropy generation rate decreases with increasing values of porous permeability parameter (γ) near the walls and otherwise around the centreline region of the channel, and for Bejan number (Be), increasing values of porous permeability parameter (γ) result into increasing the dominant effect of fluid friction irreversibility near the walls and a reduction in the dominant effect of the heat-transfer irreversibility around centreline region.

References

[1] Attia, H.A., *On the effectiveness of porosity on unsteady flow and heat transfer between parallel porous plates with exponential decaying pressure gradient*, Journal of Porous Media, 10 (3) (2007), 319-326.

[2] Attia, H.A., *Effect of porosity on unsteady Couette flow with heat transfer in the presence of uniform suction and injection*, Kragujevac Journal of Science, 31 (1) (2009), 11-16.

[3] Attia, H.A., Abdeen, A.M., *On the effectiveness of variation of physical variables on steady flow between parallel plates with heat transfer in a porous medium*, Journal of Theoretical and Applied Mechanics, 51 (1) (2013), 53-61.

[4] Chauhan, D.S., Rastogi, P., *Heat transfer and entropy generation in MHD flow through a porous medium past a stretching sheet*, Int. J. Energ. Tech., 3 (2011), 1-13.

[5] Cortell, R., *Flow and heat transfer of a fluid through a porous medium over a stretching surface with internal heat generation/absorption and suction/ blowing*, Fluid Dynamics Research, 37 (4) (2005), 231-245.

[6] Jha, B.K., Ajibade, A.O., *Free convective flow of heat generating/ absorbing fluid between vertical porous plates with periodic heat input*, International Communications in Heat and Mass Transfer, 36 (6) (2009), 624-631.

[7] Alam, M.S., Rahman, M.M., Samad, M.A., *Dufour and Soret effects on unsteady MHD free convection and mass transfer flow past a vertical porous plate in a porous medium*, Nonlinear Analysis: Modelling and Control, 11 (3) (2006), 217-226.

[8] Samad, M.A., Mansur-Rahman, M., *Thermal radiation interaction with unsteady MHD flow past a vertical porous plate immersed in a porous medium*, Journal of Naval Architecture and Marine Engineering, 3 (1) (2006), 7-14.

[9] Baoku, I.G., Israel-Cookey, C., Olajuwon, B.I., *Magnetic field and thermal radiation effects on steady hydromagnetic Couette flow through a porous channel*, Surveys in Mathematics and its Applications, 1 (5) (2012), 215-228.

[10] Zeng, Z., Grigg, R., *A criterion for non-Darcy flow in porous media*, Transport in porous media, 63 (1) (2006), 57-69.

[11] Khaled, A.-R.A., Vafai, K., *The role of porous media in modeling flow and heat transfer in biological tissues*, International Journal of Heat and Mass Transfer, 46 (26) (2003), 4989-5003.

[12] Mishra, N.K., Sharma, V.K., Rajput, D., *Unsteady viscous fluid flow with porous medium in the presence of radiation and chemical reaction*, Chinese Journal of Physics, 52 (1) (2014), 185-191.

[13] Ziabasharhagh, M., Mosallat, F., Shahnazari, M.R., *Experimental investigation of the permeability and inertial effect on fluid flow through homogeneous porous media*, Iranian Journal of Chemistry and Chemical Engineering, 27 (2) (2008), 33-38.

[14] Lan, X.K., Khodadadi, J.M., *Fluid flow and heat transfer through a porous medium channel with permeable walls*, International journal of heat and mass transfer, 36 (8) (1993), 2242-2245.

[15] Makinde, O.D., *Entropy-generation analysis for variable-viscosity channel flow with non-uniform wall temperature*, Applied Energy, 85 (5) (2008), 384-393.

[16] Hassan, A.R., Gbadeyan, J.A., *Entropy generation analysis of a reactive hydromagnetic fluid flow through a channel*, U. P. B.Sci Bull. Series A, 77 (2) (2015), 285-296.

[17] Hassan, A.R., Gbadeyan, J.A., *A reactive hydromagnetic internal heat generating fluid flow through a channel*, International Journal of Heat and Technology, 33 (3) (2015), 43-50.

[18] Makinde, O.D., *Thermal ignition in a reactive viscous flow through a channel filled with a porous medium*, Journal of heat transfer, 128 (6) (2006), 601-604.

[19] Woods, L.C., *The thermodynamics of fluid systems*, Oxford University Press, Oxford, 1975.

[20] Bejan, A., *Entropy generation minimization: the method of thermodynamic optimization of finite-size systems and finite-time processes*, CRC Press, 1995.

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EXISTENCE AND UNIQUENESS SOLUTION OF A BOUNDARY VALUE PROBLEMS FOR INTEGRO-DIFFERENTIAL EQUATION WITH PARAMETER

R.N. Butris

Department of Mathematics
Collage of Basic Education
University of Duhok
Duhok, Kurdistan Region
Iraq
e-mail: raad.khlka@yahoo.com

A.Sh. Rafeeq

Department of Mathematics
Faculty of Science
University of Zakho
Duhok, Kurdistan Region
Iraq
e-mail: ava77h@yahoo.com

Abstract. In this paper, we investigate the existence and uniqueness of the solution to a boundary value problem for integro-differential equation with parameter by using Schauder's fixed point theorem.

Keywords: existence and uniqueness, boundary value problem, integro-differential equation with parameter, Schauder's fixed point theorem.

1. Introduction

For many years, the problems with parameter have been studied and some of them considered as mathematical models of physical systems. Existence of solutions have conditions that are important in analysis theorems and results are often obtained by using fixed-point theorems (Banach, Schauder), by successive approximations, or lower and upper solutions of constructing monotone iterations [6].

The following initial value problem were considered in [12]

$$\dot{x}(t) = \frac{\alpha\tau}{T} + f(t, x(t-\tau), \tau), 0 \leq t \leq T, \underline{\tau} \leq \tau \leq \overline{\tau}, \tag{1.1}$$

with $x(t) = \varphi(t, \tau), (\underline{\tau} > 0), -\overline{\tau} \leq t \leq 0, \underline{\tau} \leq \tau \leq \overline{\tau}$

This boundary value problem used to find those numbers τ in $[\underline{\tau}, \overline{\tau}]$ for which problem (1.1) has a solution which satisfies the condition

$$x(T) = x_T. \tag{1.2}$$

In view of Seidov [13], there are some control parameter in which a certain physical process the initial state, speed and phase coordinate of each point of a controlled object depends on τ value, so we must choice a value of τ such that the object assume a given state at a given time. For related result on the problems with parameters, the reader is referred to [3].

Interesting application of the method also connected with the study of differential equations unresolved with respect to the highest derivative. Furthermore, it suggested that these investigations stimulated the appearance of series of works on the study of such equations see [9]. In [15], the T-periodic boundary problem for the integro-differential equation

$$\frac{dx}{dt} = f(t, x(t), \frac{dx}{dt}, \int_0^t \varphi(t, s, \frac{dx}{ds})ds \tag{1.3}$$

was considered and for some other related concepts and results one can see [1], [10], [14].

In this paper, we consider the initial value problem

$$\frac{dx}{dt} = A\lambda + f\left(t, x(t), \lambda, \sum_{i=1}^{\infty}\left(\int_0^t G(t,s)\dot{x}(s)\ ds\right)^i\right) \tag{1.4}$$

with $x(0) = x_0$, where $\dot{x}(t) = \frac{dx(t)}{dt}$, $0 \le t \le T$, A is $(n \times n)$ matrix and λ is a vector of parameters such that $||\lambda|| \le \rho, (\rho > 0)$. We will consider sufficient conditions for solvability of the problem (1.4) and for convergence of certain iterative processes to a solution, and also we will use Schauder-Tychonoff fixed point theorem to discuss the existence of the solution of the problem (1.4) and (1.2).

2. Preliminaries

Throughout this work we suppose that $f(t, x, \lambda, y)$ is continuous in the domain $[0, T] \times D \times ||\lambda|| \le \rho \times D_1$, where $y = \sum_{i=1}^{\infty}(\int_0^t G(t,s)\dot{x}(s)\ ds)^i$, $D : ||x - x_0|| \le r$, $D_1 : ||y|| \le d$ and D and D_1 are closed bounded subset in Euclidean space R^n. Also assume that $f(t, x, \lambda, y)$ satisfies the inequalities:

$$||f(t, x, \lambda, y)|| \le M \tag{2.1}$$

$$||f(t, x_1, \lambda_1, y_1) - f(t, x_2, \lambda_2, y_2)|| \le K_1||x_1 - x_2|| + K_2||\lambda_1 - \lambda_2|| + K_3||y_1 - y_2|| \tag{2.2}$$

where $y_n = \sum_{i=1}^{\infty}(\int_0^t G(t,s)\dot{x}_n(s)\ ds)^i$, $n = 1, 2, forall\ t \in [0, T]$ and M,K_1,K_2,K_3 are positive constants. Also, $G(t, s)$ is $(n \times n)$ continuous positive matrix such that

$$\int_0^t ||G(t,s)||\ ds \le K,\ (K > 0), \quad S_1 = \sum_{i=1}^{\infty} K^i M^{i-1} < d,$$

$$S_2 = \sum_{i=1}^{\infty} \sum_{j=1}^{i} iK^j M^{j-1} < \infty \tag{2.3}$$

for all $t \in [0, T]$. We define the nonempty sets as follows:

$$D_f = D - r, \ \ D_{1f} = D_1 - r_1, \tag{2.4}$$

where $r = ||A||\rho T + MT$ and $r_1 = ||A||\rho + ||M||$ and suppose that the greatest eigenvalue q_{max} of the matrix

$$Q = \begin{bmatrix} K_1 T & (||A|| + K_2)T & K_3 T S_2 \\ \frac{1}{||A||} K_1 & \frac{1}{||A||}(||A|| + K_2) & \frac{1}{||A||} K_3 S_2 \\ K_1 & ||A|| + K_2 & K_3 S_2 \end{bmatrix}$$

does not exceed unity, that is,

$$q_{max}(Q) = \frac{(a+b+c) + \sqrt{(a+b+c)^2 - 4(ab+ac+bc)}}{2} < 1 \tag{2.5}$$

where $a = K_1 T, b = \frac{1}{||A||}(||A|| + K_2), c = K_3 S_2$.

Furthermore, we prove the existence of a solution $x(t)$ on $[0, T]$ by using Schauder-Tychonoff fixed point theorem for the equations (1.4) and (1.2), where $t \in [0, T]$ and $f(t, x, \lambda, y)$ satisfy the following hypotheses:

(i) It is continuous positive real valued function on $[0, T] \times D \times ||\lambda|| \leq \rho \times D_1$.

(ii) It is a non increasing in $x(t)$ for each fixed point $t \in [0, T]$.

We define $B = C^1[0, T]$ is a space of all continuous and continuous derivative, bounded functions $x(t)$ on $[0, T]$ with the norm $||x|| = max_{t \in [0,T]}|x(t)|$, also define a closed and convex subset X of B as

$$X = \{x \in B; ||x(t) - x_0|| \leq r, t \in [0, T]\}$$

Definition 1 A pair (x^*, λ^*) is called a solution of the problem (1.4) and (1.2), if the function $x^*(t) \in D$ is defined for $t \in [0, T]$ satisfies the system (1.4) and conditions (1.2) for $||\lambda^*|| < \rho$.

Definition 2 [11] A function ϕ defined on an open interval (a, b) is said to be convex if, for each $x, y \in (a, b)$ and each μ, $0 \leq \mu \leq 1$, we have

$$\phi(\mu x + (1 - \mu)y \leq \mu\phi(x) + (1 - \mu)\phi(y).$$

Definition 3 [11] A sequence $\{f_n\}$ in a normed linear space is said to converge to an element f in the space if given $\epsilon > 0$, there is an N_0 such that for all $n \geq N_0$, we have $|f_n(x) - f(x)| < \epsilon$.

Definition 4 [5] Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions defined on an interval Ω, we say that $\{f_n\}_{n=1}^{\infty}$ is equicontinuous on Ω, if for any positive number ϵ there exists a positive number δ_ϵ (depends on ϵ only) such that $|f_n(x_1) - f_n(x_2)| < \epsilon$, $n \geq 1$ whenever $|x_1 - x_2| < \delta_\epsilon$ for all $x_1, x_2 \in \Omega$.

Definition 5 [4] If T maps V into itself and x_0 is an element of V such that $T(x_0) = x_0$, the we say that x_0 is a fixed point of T.

Definition 6 [5] The set Ω is said to be compact if every open cover of Ω has a finite subcover.

Definition 7 [5] The closure of Ω is the intersection of all close sets which contain Ω.

Theorem 1 [4] (The Arzela Ascoli Theorem) *Let F be equicontinuous, uniformly bounded family of real valued function f on the interval $\Omega = [0, T]$. Then F contains a uniformly convergent sequence function f_n, converging to a function $f \in C(\Omega)$ where $C(\Omega)$ denotes the space of all continuous bounded functions on Ω. Thus any sequence in F contains a uniformly bounded convergent subsequence on Ω and consequently F has a compact closure in $C(\Omega)$.*

Theorem 2 [2] (The Mean Value Theorem) *Let a and b be real numbers such that $a < b$. If $f : [a, b] \to R$ is continuous on $[a, b]$, and f is differentiable at each point in (a, b), then there is a number c in (a, b) such that $f(b) - f(a) = f'(c)(b - a)$.*

Theorem 3 [7] (Schauder-Tyconoff Fixed Point Theorem) *Let B be a locally convex topological vector space. Let Y be a compact, convex subset of B and T^* a continuous map of Y into itself. Then T^* has a fixed point $y \in Y$, i.e., $T^*_y = y$.*

Lemma 1 [8] *Let $v(x)$ and $y(x)$ be non negative continuous functions on I=$[0, \infty)$ for which the inequality $y(x) \leq c + \int_{x_0}^{x} v(t)y(t)dt$, for all $x \in I$, holds, where c is a non negative constant. Then, $y(x) \leq c e^{\int_{x_0}^{x} v(t)dt}$, for all $x \in I$.*

3. Main results

Firstly, we start from $x_0 \in D_f$ and $||\lambda_0|| \leq \rho$, and constructed by the following iteration for equation (1.4) and (1.2):

$$x_{n+1}(t) = x_0 + \int_0^t \left(A\lambda_n + f\left(s, x_n(s), \lambda_n, \sum_{i=1}^{\infty} \left(\int_0^s G(s,\tau)\dot{x}_n(\tau)\, d\tau \right)^i \right) \right) ds \tag{2.6}$$

$$\lambda_{n+1} = \frac{1}{||A||T} \left[x_T - x_0 - \int_0^T f\left(t, x_n(t), \lambda_n, \sum_{i=1}^{\infty} \left(\int_0^t G(t,s)\dot{x}_n(s)\, ds \right)^i \right) dt \right] \tag{2.7}$$

$$\dot{x}_{n+1}(t) = A\lambda_n + f(t, x_n(t), \lambda_n, \sum_{i=1}^{\infty} \left(\int_0^t G(t,s)\dot{x}_n(s)\, ds)^i \right) \tag{2.8}$$

for all $0 \leq t \leq T$, $||\lambda_{n+1}|| \leq \rho$, $y_n = \sum_{i=1}^{\infty} \left(\int_0^t G(t,s)\dot{x}_n(s)\, ds \right)^i$, $n = 0, 1, 2, ...$, we indeed prove the following theorems:

Theorem 4 *Suppose that the function* $f(t,x,\lambda,y)$ *is continuous in the domain* $[0,T]\times D\times ||\lambda||\le\rho\times D_1$, *and satisfy inequalities* (2.1) *and* (2.2), *finally suppose that for all* $||\lambda||\le\rho$ *and all continuous functions* $x(t)$ *on* $[0,T]$ *with* $||x(t)-x_0||\le r$, $||\dot{x}(t)||\le r_1$, *we have*

$$\frac{1}{||A||T}[||x_T-x_0||+MT]\le\rho$$

and $(||A||\rho+M)T=r$, $||A||\rho+M=r_1$, *and condition* (2.5).

Then the solution of problem (1.4) *and* (1.2) *exists and the successive approximations* (2.6), (2.7) *and* (2.8) *converge to a solution rapidly enough so that*

$$\begin{pmatrix} ||x_{n+1}(t)-x^*(t)|| \\ ||\lambda_{n+1}-\lambda^*|| \\ ||\dot{x}_{n+1}(t)-\dot{x}^*(t)|| \end{pmatrix}\le Q^n(E-Q)^{-1}\begin{pmatrix} r \\ 2\rho \\ r_1\end{pmatrix}, \tag{2.9}$$

where E *is an* $n\times n$ *identity matrix.*

Proof. By assumption, we have $||\lambda_0||\le\rho$, $x_0\in D_f$, and from (2.6), we have

$$\begin{aligned}
&||x_{n+1}(t)-x_n(t)||\\
&=\Bigg\|x_0+\int_0^t\left(A\lambda_n+f\left(s,x_n(s),\lambda_n,\sum_{i=1}^{\infty}\left(\int_0^s G(s,\tau)\dot{x}_n(\tau)\,d\tau\right)^i\right)\right)ds\\
&-x_0-\int_0^t\left(A\lambda_{n-1}+f\left(s,x_{n-1}(s),\lambda_{n-1},\sum_{i=1}^{\infty}\left(\int_0^s G(s,\tau)\dot{x}_{n-1}(\tau)\,d\tau\right)^i\right)\right)ds\Bigg\|\\
&\le||A||T||\lambda_n-\lambda_{n-1}||+TK_1||x_n(t)-x_{n-1}(t)||+TK_2||\lambda_n-\lambda_{n-1}||\\
&+TK_3\left\|\sum_{i=1}^{\infty}\left(\int_0^t G(t,s)\dot{x}_n(s)\,ds\right)^i-\sum_{i=1}^{\infty}\left(\int_0^t G(t,s)\dot{x}_{n-1}(s\,ds)^i\right)\right\|\\
&\le TK_1||x_n(t)-x_{n-1}(t)||+(||A||+K_2)T||\lambda_n-\lambda_{n-1}||\\
&+TK_3S_2||\dot{x}_n(t)-\dot{x}_{n-1}(t)||.
\end{aligned}$$

From the difference $\lambda_{n+1}-\lambda_n$ and $\dot{x}_{n+1}(t)-\dot{x}_n(t)$, we have

$$\begin{aligned}
||\lambda_{n+1}-\lambda_n||&\le\frac{1}{||A||}K_1||x_n(t)-x_{n-1}(t)||+\left(\frac{1}{||A||}K_2+1\right)||\lambda_n-\lambda_{n-1}||\\
&+\frac{1}{||A||}K_3S_2||\dot{x}_n(t)-\dot{x}_{n-1}(t)||\\
||\dot{x}_{n+1}(t)-\dot{x}_n(t)||&\le K_1||x_n(t)-x_{n-1}(t)||+(||A||+K_2)||\lambda_n-\lambda_{n-1}||\\
&+K_3S_2||\dot{x}_n(t)-\dot{x}_{n-1}(t)||
\end{aligned}$$

so that

$$\begin{pmatrix} ||x_{n+1}(t)-x_n(t)|| \\ ||\lambda_{n+1}-\lambda_n|| \\ ||\dot{x}_{n+1}(t)-\dot{x}_n(t)|| \end{pmatrix}\le\begin{pmatrix} K_1T & (||A||+K_2)T & K_3TS_2 \\ \frac{1}{||A||}K_1 & \frac{1}{||A||}(||A||+K_2) & \frac{1}{||A||}K_3S_2 \\ K_1 & ||A||+K_2 & K_3S_2\end{pmatrix}\begin{pmatrix} ||x_n(t)-x_{n-1}(t)|| \\ ||\lambda_n-\lambda_{n-1}|| \\ ||\dot{x}_n(t)-\dot{x}_{n-1}(t)||\end{pmatrix}$$

If let $V_{n+1} = \begin{pmatrix} ||x_{n+1}(t) - x_n(t)|| \\ ||\lambda_{n+1} - \lambda_n|| \\ ||\dot{x}_{n+1}(t) - \dot{x}_n(t)|| \end{pmatrix}$, $n = 0, 1, 2, ...$, we have

$$V_{n+1} \leq QV_n \leq ... \leq Q^n V_1, \text{ where } V_1 = \begin{pmatrix} r \\ 2\rho \\ r_1 \end{pmatrix}. \tag{2.10}$$

From (2.10), the following inequality

$$\begin{pmatrix} ||x_{n+k}(t) - x_n(t)|| \\ ||\lambda_{n+k} - \lambda_n|| \\ ||\dot{x}_{n+k}(t) - \dot{x}_n(t)|| \end{pmatrix} \leq \sum_{i=0}^{k-1} Q^{n+i} V_1,$$

holds for all $k > 1$ and $t \in [0, T]$. But the maximum eigenvalue of the matrix Q is assumed to lie within the circle of a unit radius, i.e., $q_{max} < 1$, which implies that

$$\sum_{i=0}^{k-1} Q^{n+i} \leq Q^m \sum_{i=0}^{\infty} Q^i \leq Q^n (E - Q)^{-1}$$

and $\lim_{n\to\infty} Q^n = 0$, then the sequences $\{x_n(t)\}_{n=1}^{\infty}$, $\{\lambda_n\}_{n=1}^{\infty}$, $\{\dot{x}_n(t)\}_{n=1}^{\infty}$ converge. Then let $\lim_{n\to\infty} x_n(t) = x^*(t)$, $\lim_{n\to\infty} \lambda_n = \lambda^*$ and $\lim_{n\to\infty} \dot{x}_n(t) = \dot{x}^*(t)$. If we take the limit in (2.6) and (2.7) we obtain

$$x^*(t) = x_0 + \int_0^t \left(A\lambda^* + f\left(s, x^*(s), \lambda^*, \sum_{i=1}^{\infty} \left(\int_0^s G(s,\tau)\dot{x}^*(\tau)\, d\tau \right)^i \right) \right) ds \tag{2.11}$$

$$\lambda^* = \frac{1}{||A||T} \left(x_T - x_0 - \int_0^T f\left(t, x^*(t), \lambda^*, \sum_{i=1}^{\infty} \left(\int_0^t G(t,s)\dot{x}^*(s)\, ds \right)^i \right) dt \right). \tag{2.12}$$

Mean that the pair $(x^*(t), \lambda^*)$ is a solution of problem (1.4) and (1.2). It is easy to see that the speed of convergence of (2.6) and (2.7) is validly described by (2.5), the theorem is proved. ∎

Remark 1 By analogous arguments, we call shaw that the following iteration process converges to a solution of (1.4) and (1.2)

$$x_{n+1}(t) = x_0 + \int_0^t \left(A\lambda_n + f\left(s, x_n(s), \lambda_n, \sum_{i=1}^{\infty} \left(\int_0^s G(s,\tau)\dot{x}_n(\tau)\, d\tau \right)^i \right) \right) ds,$$

$$\lambda_{n+1} = \frac{1}{||A||T} \left(x_T - x_0 - \int_0^T f(t, x_n(t), \lambda_n, \sum_{i=1}^{\infty} \left(\int_0^t G(t,s)\dot{x}_n(s)\, ds \right)^i) dt \right),$$

for all $0 \leq t \leq T$, $x_0 \in D_f$, $||\lambda|| \leq \rho$, $n = 0, 1, 2, ...$

Secondly, the solution of the initial value problem (1.4) for each $||\lambda|| \leq \rho$ will be denoted by $x(t,\lambda)$, suppose that $||x(t)-x_0|| \leq r$, then we have $||\lambda|| \leq \rho$ and all continuous functions $x(t)$ on $[0,T]$ with $||x(t)-x_0|| \leq r$, $||\dot{x}(t)|| \leq r_1$, we have

$$x(t,\lambda)=x_0+\int_0^t\left(A\lambda+f\left(s,x(s,\lambda),\lambda,\sum_{i=1}^{\infty}\left(\int_0^s G(s,\tau)\dot{x}(\tau,\lambda)\,d\tau\right)^i\right)\right)ds \tag{2.13}$$

with $x(0)=x_0$, and for all $0 \leq t \leq T$, $x_0 \in D_f$.

Lemma 2 *Suppose that the continuous function* $f(t,x,\lambda,y)$ *satisfies conditions* (2.1) *and* (2.2). *Then the solution of* (2.13) *satisfies the bound.*

$$||x(t,\lambda_1)-x(t,\lambda_2)|| \leq e^{K_1(1+K_3S_2L)}[T(||A||+K_2)(1+K_3S_2L)]||\lambda_1-\lambda_2|| \tag{2.14}$$

where $L=\frac{1}{1-K_3S_2}$.

Proof. From (2.13), we have

$$\begin{aligned}
&||x(t,\lambda_1)-x(t,\lambda_2)|| \\
&=\Bigg\|x_0+\int_0^t\left(A\lambda_1+f\left(s,x(s,\lambda_1),\lambda_1,\sum_{i=1}^{\infty}\left(\int_0^s G(s,\tau)\dot{x}(\tau,\lambda_1)\,d\tau\right)^i\right)\right)ds \\
&-x_0-\int_0^t\left(A\lambda_2+f\left(s,x(s,\lambda_2),\lambda_2,\sum_{i=1}^{\infty}\left(\int_0^s G(s,\tau)\dot{x}(\tau,\lambda_2)\,d\tau\right)^i\right)\right)ds\Bigg\|
\end{aligned}$$

so

$$\begin{aligned}
||x(t,\lambda_1)-x(t,\lambda_2)|| &\leq (||A||+K_2)T||\lambda_1-\lambda_2|| \\
&+K_1\int_0^t ||x(s,\lambda_1)-x(s,\lambda_2)||ds \\
&+K_3S_2\int_0^t ||\dot{x}(s,\lambda_1)-\dot{x}(s,\lambda_2)||ds.
\end{aligned} \tag{2.15}$$

Differentiate equation (2.13), then we have

$$\begin{aligned}
||\dot{x}(t,\lambda_1)-\dot{x}(t,\lambda_2)|| &\leq (||A||+K_2)||\lambda_1-\lambda_2||+K_1||x(t,\lambda_1)-x(t,\lambda_2)|| \\
&+K_3S_2||\dot{x}(t,\lambda_1)-\dot{x}(t,\lambda_2)||,
\end{aligned}$$

then

$$||\dot{x}(t,\lambda_1)-\dot{x}(t,\lambda_2)|| \leq (||A||+K_2)L||\lambda_1-\lambda_2||+K_1L||x(t,\lambda_1)-x(t,\lambda_2)||ds. \tag{2.16}$$

Substituting (2.16) in (2.15), we get

$$\begin{aligned}
||x(t,\lambda_1)-x(t,\lambda_2)|| &\leq T(||A||+K_2)(1+K_3S_2L)||\lambda_1-\lambda_2|| \\
&+K_1(1+K_3S_2L)\int_0^t ||x(s,\lambda_1)-x(s,\lambda_2)||ds.
\end{aligned}$$

Therefore, from the lemma 2.10, we obtain

$$||x(t,\lambda_1)-x(t,\lambda_2)|| \leq e^{K_1(1+K_3S_2L)}[T(||A||+K_2)(1+K_3S_2L)]||\lambda_1-\lambda_2||.$$

The lemma is proved. ∎

Remark 2 Solvability of problem (1.4) and (1.2) is equivalent to solvability of the equation

$$\lambda = \frac{1}{||A||T}\left(x_T - x_0 - \int_0^T f\left(t, x(t), \lambda, \sum_{i=1}^{\infty}\left(\int_{-\infty}^{t} G(t,s)\dot{x}(s)\ ds\right)^i\right) dt\right).$$

Here $x(t,\lambda)$ is the solution of (2.13) corresponding to the value $||\lambda|| \leq \rho$.

Theorem 5 *Let the function $f(t,x,\lambda,y)$ of problem* (1.4) *satisfy* (2.1) *and* (2.2) *and hypotheses* (i) *and* (ii) *and all continuous function $x(t)$ with $||x(t)-x_0|| \leq r$, we have*

$$\left\|\frac{1}{||A||T}\left[x_T - x_0 - \int_0^T f\left(t, x(t), \lambda, \sum_{i=1}^{\infty}\left(\int_0^t G(t,s)\dot{x}(s)\ ds\right)^i\right)dt\right]\right\| \leq \rho.$$

Then, problem (1.4) *and* (1.2) *has at least one solution.*

Proof. Let X be a subset of B, for $x(t) \in X$, we define the norm $||x|| = \max_{t\in[0,T]} |x(t)|$ and the map $T^* : B \to B$ defined by

$$T^*_{x(t)} = x_0 + \int_0^t \left(A\lambda + f\left(s, x(s), \lambda, \sum_{i=1}^{\infty}\left(\int_0^s G(s,\tau)\dot{x}(\tau)\ d\tau\right)^i\right)\right) ds$$

for all $0 \leq t \leq T$, $||\lambda|| \leq \rho$ in order to apply Schauder-Tychonoff fixed point theorem, we should prove the following steps:

Step 1. T^* maps X into itself. Since $f(t,x,\lambda,y)$ is continuous on $[0,T]\times D\times||\lambda|| \leq \rho\times D_1$ then $\int_0^t f(t,x,\lambda,y)ds$ is also continuous, so $T^*_{x(t)}$ is continuous, then

$$\begin{aligned} ||x_0|| - r &\leq ||T_{x(t)}|| \\ &= \left| x_0 + \int_0^t \left(A\lambda + f\left(s, x(s), \lambda, \sum_{i=1}^{\infty}\left(\int_0^s G(s,\tau)\dot{x}(\tau)\ d\tau\right)^i\right)\right) ds \right\| . \\ &\leq ||x_0|| + ||A||\rho T + ||M||T = ||x_0|| + r \end{aligned}$$

Thus, $T^*_{x(t)} \in X$, that is, T^* maps X into itself.

Step 2. T^* is continuous mapping on X. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X that converges to $x(t) \in B$, but X is closed subset of B, so $x(t) \in X$ consider

$$\begin{aligned} ||T^*_{x_n(t)} - T^*_{x(t)}|| \leq &(||A|| + K_2)T||\lambda_n - \lambda|| + K_1T||x_n(t) - x(t)||ds \\ &+ K_3S_2T||\dot{x}_n(t) - \dot{x}(t)||ds. \end{aligned}$$

Since $f(t,x,\lambda,y)$ is continuous function and the sequences $\{x_n(t)\}_{n=1}^{\infty}$, $\{\lambda_n\}_{n=1}^{\infty}$ and $\{\dot{x}_n(t)\}_{n=1}^{\infty}$ converge to $x(t)$, λ and $\dot{x}(t)$ respectively, meaning that

$\max_{t\in[0,T]} ||x_n(t) - x_(t)|| \to 0$, $||\lambda_n - \lambda|| \to 0$ and $\max_{t\in[0,T]} ||\dot{x}_n(t) - \dot{x}(t)|| \to 0$, as $n \to \infty$, then

$$\lim_{n\to\infty} ||T^*_{x_n(t)} - T^*_{x(t)}|| = 0.$$

Therefore, T^* is a continuous mapping on X.

Step 3. The closure of $T^*X = \{T^*_{x(t)}; x(t) \in X\}$ is compact.

To prove Step 3, we will prove that the family T^*X is uniformly bounded and equicontinuous, T^*X is uniformly bounded as shown in Step 1. For proving the equicontinuous, since $||x(t) - x_0|| \le r$, and $f(t, x, \lambda, y)$ is non increasing in $x(t)$, hence we have

$$||T^{*'}_{x(t)}|| = ||A\lambda + f\left(t, x(t), \lambda, \sum_{i=1}^{\infty}\left(\int_0^t G(t,s)\dot{x}(s)\ ds\right)^i\right)|| \le ||A||\rho + ||M|| = r_1.$$

Choose $\delta = \frac{\epsilon}{r_1}$, $\epsilon > 0$, for $t_1, t_2 \in [0, T]$, such that $t_1 < t_2$, with $|t_1 - t_2| < \delta$, then by using the mean value theorem, there exists a number $b \in (t_1, t_2)$ such that

$$||T^*_{x(t_1)} - T^*_{x(t_2)}|| = ||z'(b)|||t_1 - t_2| < r_1\delta = \epsilon.$$

This inequality proves that the family T^*X is equicontinuous, since δ is independent of t_1, t_2 and $x(t) \in X$, thus by Ascoli-Arzelà theorem, T^*X has compact closure. In view of Step 1, Step 2 and Step 3, the Schauder-Tychonoff fixed point theorem shows that T^* has at least one fixed $x(t) \in X$ that is $T^*_{x(t)} = x(t)$ for all $t \in [0, T]$ so $x(t)$ is a solution of the problem (1.4) and (1.2). ∎

Theorem 6 *Suppose that the function $f(t, x, \lambda, y)$ satisfies all conditions of Theorem 3.1 and condition(2.5). Then, problem (1.4) and (1.2) has at most one solution in the region $0 \le t \le T$, $||\lambda|| \le \rho$, $||x(t) - x_0|| \le r$.*

Proof. Suppose that the exist two solutions of (1.4) and (1.2) called (u_1, λ_1) and (u_2, λ_2). Then, we have

$$\frac{du_1(t)}{dt} = A\lambda + f\left(t, u_1(t), \lambda, \sum_{i=1}^{\infty}\left(\int_0^t G(t,s)\dot{u}_1(s)\ ds\right)^i\right),\ u_1(0) = u_0,\ 0 \le t \le T$$

$$\frac{du_2(t)}{dt} = A\lambda + f\left(t, u_2(t), \lambda, \sum_{i=1}^{\infty}\left(\int_0^t G(t,s)\dot{u}_2(s)\ ds\right)^i\right),\ u_2(0) = u_0,\ 0 \le t \le T.$$

From the hypotheses, we have

$$\begin{pmatrix} ||u_1(t) - u_2(t)|| \\ ||\lambda_1 - \lambda_2|| \\ ||\dot{u}_1(t) - \dot{u}_2(t)|| \end{pmatrix} \le \begin{pmatrix} K_1T & (||A|| + K_2)T & K_3TS_2 \\ \frac{1}{||A||}K_1 & \frac{1}{||A||}(||A|| + K_2) & \frac{1}{||A||}K_3S_2 \\ K_1 & ||A|| + K_2 & K_3S_2 \end{pmatrix}\begin{pmatrix} ||u_1(t) - u_2(t)|| \\ ||\lambda_1 - \lambda_2|| \\ ||\dot{u}_1(t) - \dot{u}_2(t)|| \end{pmatrix}.$$

Let $W = \begin{pmatrix} ||u_1(t) - u_2(t)|| \\ ||\lambda_1 - \lambda_2|| \\ ||\dot{u}_1(t) - \dot{u}_2(t)|| \end{pmatrix}$, then we have $W \le QW \le ... \le Q^nW$. Since $q_{max} < 1$, we conclude that $\lambda_1 = \lambda_2$ and $u_1(t) = u_2(t)$, which means that the solution of (1.4) and (1.2) is unique. ∎

References

[1] Butris, R.N., *Periodic solutions for nonlinear system of differential equations with pulse action of parameters*, International Journal of Engineering Invention, 1 (12) (2012), 75-87.

[2] Cohn, D.L., *Measure Theory*, Birkhauser, Boston, 1980.

[3] Goma, I.A., *Method of successive approximation in a two-point boundary value problem with parameter*, Ukrainskii Matematicheskii Zhurnal, 29 (6) (1977), 800-807.

[4] Goldberg, R.R., *Method of Real Analysis*, John Wiley and Sons, Inc., USA, 1976.

[5] Hoffman, K., *Analysis in Euclidean Space*, Prentice-Hall, Inc., Englewood, New Jersey, USA, 1975.

[6] Jankowski, T., *Application of the numerical-analytic method to systems of differential equations with parameter*, Ukrainian Mathematical Journal, 54 (4) (2002).

[7] Jenab, B., *Asymptotic Theory of Second Order non linear Ordinary Differential Equation,* M.Sc., Thesis, British Culombia University, Canada, 1985.

[8] Oguntuase, J.A., *On an inequality of Gronwall*, Journal of Inequalities Pure and Applied Mathematics, 2 (1) (2001).

[9] Petrishyn, W.W., Yu, Z.S., *Periodic solutions of non-linear second-order differential equations which are not solvable for the highest derivative*, J. Math. Anal. Appl., 89 (1982), 462-488.

[10] Ronto, M.I., Samoilenko, A.M., Trofimchuk, S.I., *The theory of the numerical-analytic method: Achievements and new trends of development VII*, Ukrainian Mathematical Journal, 51 (9) (1999).

[11] Roydne, H.L., *Real Analysis*, Prentice-Hall of India Private Limited, New Delhi, 110001, 2005.

[12] Seidov, Z.B., *A boundary-value problem for differential equation with a retarded argument*, Ukrainskii Matematicheskii Zhurnal, 25 (6) (1973), 830-834.

[13] Seidov, Z.B., *A Boundary-value problem with control parameters for second-order differential equations with retarded argument*, Ukrainskii Matematicheskii Zhurnal, 28 (5) (1977), 690-695.

[14] Stanek, S., *On functional boundary value problem systems of functional differential equations depending on parameters*, Arch. Math. 64 (1995), 434-443.

[15] Turbaev, B.E., *Periodic solutions to system of differential equations unsolved with respects to the derivative*, Visn, Kyiv Univ. Ser. Mat. Mekh., 27 (1985), 98-104.

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ON HOMOLOGICAL PROPERTIES OF SOME H_v-STRUCTURES

Hülya Inceboz[1]

Department of Mathematics
Adnan Menderes University
Aydın
Turkey
e-mail: hinceboz@adu.edu.tr

Bijan Davvaz

Department of Mathematics
Yazd University
Yazd
Iran
e-mail: davvaz@yazd.ac.ir

Abstract. The main aim of this note is to investigate the fundamental homological properties of various module derivations for H_v-structures and get some functorial relations for these derivation sets as a continuous line of [6].

1. Introduction

Algebraic hyperstructures are a natural generalization of the ordinary algebraic structures with which was first initiated by Marty [7]. After the pioneered work, algebraic hyperstructures have been developed by many researchers. Let H be a nonempty set and $\mathcal{P}^*(H)$ be the family of nonempty subsets of H. Every function $* : H \times H \longrightarrow \mathcal{P}^*(H)$ is called a *hyperoperation* on H and $(H, *)$ is called a *hyperstructure*. The concept of H_v-structures as a larger class than the well-known hyperstructures was introduced by T. Vougiouklis at the Fourth AHA congress [11]. The concept of an H_v-structure constitutes a generalization of the well-known algebraic hyperstructures (hypergroup, hyperring, hypermodule and so on). Actually some axioms concerning the above hyperstructures such as the associative law, the distributive law and so on are replaced by their corresponding weak axioms. Since the quotients of the H_v-structures with respect to the fundamental equivalence relations are always ordinary structures.

[1]Corresponding author.

The hyperstructure $(H, *)$ is called an H_v-*group* if

(1) the $*$ is *weak associative*, i.e., $x * (y * z) \cap (x * y) * z \neq \emptyset$,

(2) the *reproduction axiom* holds, i.e., $a * H = H * a = H$ for every $a \in H$.

We say H is *weak commutative* if for every $x, y \in H$, $x * y \cap y * x \neq \emptyset$.

Since then the study of H_v-structure theory has been pursued in many directions by T. Vougiouklis, B. Davvaz, S. Spartalis, A. Dramalidis, S. Hoskova, and others. For more definitions and applications on H_v-structures one can see the books and survey papers as [2], [3], [5], [10], and [1], [4], [12].

A multivalued system $(R, *, \cdot)$ is called an H_v-*ring* if the following axioms hold:

(1) $(R, *)$ is a weak commutative H_v-group,

(2) $(R, \cdot)$ is a weak associative, i.e., $x \cdot (y \cdot z) \cap (x \cdot y) \cdot z \neq \emptyset$ for every $x, y, z \in R$,

(3) the hyperoperation $\cdot$ is weak distributive with respect to $*$, i.e., for every $x, y, z \in R$, we have $x \cdot (y * z) \cap (x \cdot y * x \cdot z) \neq \emptyset$, $(x * y) \cdot z \cap (x \cdot z * y \cdot z) \neq \emptyset$.

For example, if $(H, +)$ is an H_v-group, then for every hyperoperation $\cdot$ such that $\{x, y\} \subseteq x \cdot y$ for every $x, y \in H$, the hyperstructure $(H, +, \cdot)$ is an H_v-ring. Therefore, we can construct some H_v-rings by a given H_v-group. Let $(H, \cdot)$ be an H_v-group with (left, right) identity elements. Then, H is called *(left, right) reversible* in itself when any relation $c \in a \cdot b$ implies the existence of a left inverse a' of a and a is a right inverse b' of b such that $b \in a' \cdot c$ and $a \in c \cdot b'$. Furthermore we call $(R, *, \cdot)$ is an H_v-*field* if $(R, *, \cdot)$ is a H_v-ring and $(R, \cdot)$ is a (left, right) reversible H_v-group.

Let $(R, *, \cdot)$ be an H_v-ring, $(M, \sharp)$ be a weak commutative H_v-group and there exists an external hyperoperation $\circ : R \times M \longrightarrow P^*(M)$ denoted by $(a, x) \longmapsto a \circ x$ such that for every $a, a_1, a_2 \in R$ and every $x, x_1, x_2 \in M$, we have

(1) $a \circ (x_1 \sharp x_2) \cap ((a \circ x_1) \sharp (a \circ x_2)) \neq \emptyset$,

(2) $(a_1 * a_2) \circ x \cap ((a_1 \circ x) \sharp (a_2 \circ x)) \neq \emptyset$,

(3) $(a_1 . a_2) \circ x \cap a_1 \circ (a_2 \circ x)$.

Then, M is called a *left H_v-module* over R.

A non-empty subset S of M is an H_v-submodule of M if $(S, +)$ is an H_v-subgroup of $(M, +)$ and $RS \subseteq S$. It is clear that an arbitrary ring (module) will be an H_v-ring (H_v-module) if we identify x with $\{x\}$. In the case of an H_v-field instead of an H_v-ring then the H_v-*vector space* is defined. In [9], Vougiouklis defined the concept of H_v-vector space which is a generalization of the concept of vector space. Note that every vector space is a H_v-vector space that is strongly left and right distributive and specially, every field is a H_v-vector space over itself.

Definition 1.1. Let $(R, \cdot, *)$ and $(S, \circ, \Box)$ be two H_v-rings and $(M, \sharp)$ be a weak commutative H_v-group. If M is a right $S - H_v$-module, a left $R - H_v$-module, and $r(sm) \cap (rs)m \neq \emptyset$ for all $r \in R$, $s \in S$ and $m \in M$, then M is called $R - S - H_v$-*bimodule.*

Definition 1.2. Let X and Y be H_v-vector spaces over F. A map $T : X \longrightarrow Y$ is called

(1) H_v-*linear* if and only if
$T(x+y) \cap (T(x)+T(y)) \neq \emptyset$ and $T(a \circ x) \cap (a \circ T(x)) \neq \emptyset$,
for all $x, y \in X$, $\alpha \in F$.

(2) H_v-*antilinear* if and only if
$T(x+y) \cap (T(x)+T(y)) \neq \emptyset$ and $T(a \circ x) \cap (T(x) \circ a) \neq \emptyset$,
for all $x, y \in X$, $a \in F$.

(3) H_v- *strong linear* if and only if
$T(x+y) = T(x)+T(y)$ and $T(a \circ x) = a \circ T(x)$,
for all $x, y \in X$, $a \in F$.

Let $(A, +)$ be an H_v-vector space over an H_v-field $(F, *, \cdot)$. Then A is called an H_v-*algebra* over F if there exists a mapping $\# : A \times A \longrightarrow \mathcal{P}^*(A)$ (images to be denoted by $x\#y$ for $x, y \in A$) such that the following conditions hold:

(1) $((x+y)\#z) \cap (x\#z) + (y\#z) \neq \emptyset$;
(2) $(c \circ x)\#y \cap c \circ (x\#y) \cap x\#(c \circ y) \neq \emptyset$, for all $x, y, z \in A$ and $c \in F$.

Let $(X, +, \circ, K)$ be an H_v-vector space. Suppose that for every $a \in K$, $|a|$ denoted the valuation of a in K. An H_v-*norm* on X is a mapping $\|\cdot\| : X \longrightarrow K$ that for all $a \in K$ and $x, y \in X$ has the following properties:

(1) if $z \in x + y$, then $\|z\| \leq \|x\| + \|y\|$,
(2) $sup\|a \circ x\| =| a | \cdot\|x\|$.

Let A be an H_v-algebra and $(A, \|.\|)$ be a normed H_v-vector space. If $\|x.y\| \leq \|x\| \cdot \|y\|$ for all $x, y \in A$, then A is called a *normed H_v-algebra.*

Let $(M, +_1, \circ_1, \|\cdot\|_1, K)$ and $(N, +_2, \circ_2, \|\cdot\|_2, K)$ be two normed H_v-vector spaces over the same H_v-field K. A mapping $f : M \longrightarrow N$ is called an H_v-*homomorphism* or *weak homomorphism* if for all $x, y \in M$ and $r \in R$, the following relations hold:

$$f(x +_1 y) \cap (f(x) +_2 f(y)) \neq \emptyset \text{ and } f(r\circ_1 x) \cap r\circ_2 f(x) \neq \emptyset.$$

f is called an *inclusion homomorphism* if $f(x +_1 y) \subseteq f(x) +_2 f(y)$ and $f(r\circ_1 x) \subseteq r\circ_2 f(x)$ for all $x, y \in M$ and $r \in R$.

Finally, f is called a *strong homomorphism* if for all $x, y \in M$ and $x \in R$, we have $f(x +_1 y) = f(x) +_2 f(y)$ and $f(r\circ_1 x) = r\circ_2 f(x)$. If there exists a strong one to one homomorphism from M onto N, then M and N are called *isomorphic.*

Let R be a weak-commutative H_v-ring and $\mathbf{H}$ be the set of all H_v-modules and all strong R-homomorphisms. One can show that $\mathbf{H}$ is a category.

Suppose that M and N are two H_v-modules and $M[N]$ is the set of all functions on M with values in N. First, we equip $M[N]$ to appropriate hyperoperations to be an H_v-module. As in [8] the $M[N]$ with the following hyperoperations is an H_v-module

$$\begin{aligned} f+g &= \{h \in M[N] : h(x) \in f(x)+g(x),\ \forall x \in M\}, \\ r \cdot f &= \{k \in M[N] : k(x) \in r \cdot f(x),\ \forall x \in M\}. \end{aligned}$$

For a normed H_v-vector space $(X, +, \circ, \|\cdot\|, K)$, if (x_n) is a sequence in X, then

$$\lim x_n = x \Leftrightarrow \lim \| x_n - x \| = 0.$$

Let $(X, +, \circ, \| \, . \, \|, K)$ be a normed H_v-vector space. A sequence (x_n) in X is said to be a *Cauchy sequence* if for every $\varepsilon > 0$, there is $N \in N$ such that $\| x_n - x_m \| < \varepsilon$, for every $m, n \in N$.

Definition 1.3. Let A be a normed H_v-algebra. If every Cauchy sequence in A has a limit that is also in A, then A is called a *Banach H_v-algebra*.

Definition 1.4. Let S be an H_v-algebra over a commutative H_v-ring K. M is called an *S-H_v-bimodule*, if M is a left and right S-H_v-module such that $s(mt) = (sm)t$, $a(sm) = s(am)$, $am = ma$, for all $s, t \in S$, $a \in K$, $m \in M$.

Definition 1.5. Let U be a Banach H_v-algebra. A Banach H_v-space X which is also an U-H_v-bimodule is called a *Banach U-H_v-bimodule* if there exists a constant $K > 0$ such that $\|\alpha \cdot x\| \leq K\|\alpha\|\|x\|$ and $\|x \cdot \alpha\| \leq K\|\alpha\|\|x\|$ for each $\alpha \in U$ and $x \in X$.

By using a certain type of equivalence relations, we can connect H_v-structures to usual structures. The smallest of these relations are called *fundamental relations* and denoted by β^*, γ^*, ε^*, so that if H is an H_v-group, (H_v-ring, H_v-module over an H_v-ring R) then H/β^* is a group (H/γ^* is a ring, H/ε^* is an R/γ^*-module). The fundamental relation ε^* on an H_v-module M can be defined as follows:

Consider the left H_v-module M over an H_v-ring R. If ϑ denotes the set of all expressions consisting of finite hyperoperations of either on R and M or of the external hyperoperations applying on finite sets of elements of R and M, a relation ε can be defined on M whose transitive closure is the fundamental relation ε^*. The relation ε is defined as follows: for every $x, y \in M$, $x\varepsilon y$ if and only if $\{x, y\} \subseteq u$ for some $u \in \vartheta$.

Suppose that $\gamma^*(r)$ is the equivalence class containing $r \in R$ and $\varepsilon^*(x)$ is the equivalence class containing $x \in M$. On M/ε^* the $\oplus$ and the external product $\odot$ using the γ^* classes in R are defined as follows:

For every $x, y \in M$, and for every $r \in R$,

$$\varepsilon^*(x) \oplus \varepsilon^*(y) = \varepsilon^*(c), \text{ for every } c \in \varepsilon^*(x) + \varepsilon^*(y),$$

$$\gamma^*(r) \odot \gamma^*(x) = \gamma^*(d), \text{ for every } d \in \gamma^*(r) + \gamma^*(x),$$

The kernel of canonical map $\phi : M \longrightarrow M/\varepsilon^*$ is called the *heart* of M and it is denoted by ω_M, i.e., $\omega_M = \{x \in M : \omega(x) = 0\}$, where 0 is the unit element of the group $(M/\varepsilon^*, \oplus)$. One can prove that the unit element of the group $(M/\varepsilon^*, \oplus)$ is equal to ω_M.

Let M and N be two H_v-modules over an H_v-ring R. The kernel of a strong H_v-homomorphism $f : M \longrightarrow N$ is defined as follows:

$$Ker(f) = \{a \in M : f(a) \in \omega_N\}.$$

A function $f : M_1 \longrightarrow M_2$ is called *weak-monic* if for every $m_1, m_2 \in M_1$, $f(m_1) = f(m_2)$ implies $\varepsilon^*(m_1) = \varepsilon^*(m_2)$ and f is called *weak-epic* if for every $m_2 \in M_2$ there exists $m_1 \in M_1$ such that $\varepsilon^*(m_2) = \varepsilon^*(f(m_1))$. Finally f is called *weak-isomorphism* if f is weak-monic and weak-epic.

Let M be an H_v-module and X, Y be non-empty subsets of M. We say X is *weak equal* to Y and write $X \overset{w}{\equiv} Y$ if and only if for every $x \in X$ there exists $y \in Y$ such that $\varepsilon^*(x) = \varepsilon^*(y)$ and for every $y \in Y$ there exists $x \in X$ such that $\varepsilon^*(x) = \varepsilon^*(y)$.

Let $M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \xrightarrow{f_3} \ldots \xrightarrow{f_{n-1}} M_{n-1} \xrightarrow{f_n} M_n$ be a sequence of H_v-modules and strong H_v-homomorphisms. We say this sequence is *exact* if for every $2 \leq i \leq n$, $Im(f_{i-1}) \overset{w}{\equiv} Ker(f_i)$.

Definition 1.6. Let X and Y be normed H_v-vector spaces over F and $T : X \longrightarrow Y$ be an operator. T is said to be H_v-*bounded* if there exists a positive real number K such that we have

$$\|Tx\| \leq K\|x\|, \; x \in X.$$

Definition 1.7. Let X and Y be normed H_v-vector spaces over F and $T : X \longrightarrow Y$ be an operator. Then T is called H_v-*weak linear operator* if T is additive and satisfies

$$(T(Z_{a\circ x}) \cap (a \circ T(x))) \neq \emptyset; \; (a \in F; x \in X)$$

Here z_{aox} for $a \neq 0$ is that element of aox such that $x \in a^{-1} \circ z_{a\circ x}$. So the set of all these elements denoted by $Z_{a\circ x}$.

Let X and Y be H_v-vector spaces over F. Denote the set of all weak linear operators and the set of all bounded weak linear operators from X into Y by $L_w(X, Y)$ and $B_w(X, Y)$, respectively.

Let Y be normal. Then $L_w(X, Y)$ with the following sum and product is a weak H_v-vector space over F.

$(T + S)(x) = T(x) + T(y)$, $T, S \in L_w(X, Y)$, $x \in X$

$a \circ T = \{S \in L_w(X, Y) : Sx \in a \circ Tx, x \in X\}$, $(a \in F, T \in L_w(X, Y))$.

Let Y be normal. Then $L_w(X, Y)$ is a normal weak H_v-vector space and $B_w(X, Y)$ is a sub H_v-vector space of $L_w(X, Y)$.

Let $(X, +_1, \circ_1, \|.\|_1, K)$ and $(Y, +_2, \circ_2, \|.\|_2, K)$ be two normed H_v-vector spaces. For a H_v-bounded strong homomorphism $f : X \longrightarrow Y$, we define the norm of f by

$$\|f\| := sup\left\{ sup\left\| f\left(\frac{1}{\|x\|}\circ_1 x\right)\right\|_2 : 0 \neq x \in X \right\}.$$

Let $(R, +, \cdot)$ be H_v-ring. The function $d : R \longrightarrow R$ is called H_v- derivation if for all $x, y \in R$,

(1) $d(x + y) \cap (d(x) + d(y)) \neq \emptyset$,

(2) $d(x \cdot y) \cap (d(x)y + xd(y)) \neq \emptyset$.

2. H_v-module derivations

2.1. Some structures

Throughout this paper, A and U are Banach H_v-algebras such that A is a Banach U-H_v-bimodule with the compatible actions, as follows:

$$\alpha \cdot (ab) \cap (\alpha \cdot a)b \neq \emptyset,\ (ab) \cdot \alpha \cap a(b \cdot \alpha) \neq \emptyset\ (\alpha \in U,\ a, b \in A)$$

Let X be a Banach A-H_v-bimodule and a Banach U-H_v-bimodule with the compatible actions, that is;

$$\alpha \cdot (a \cdot x) \cap (\alpha \cdot a) \cdot x \neq \emptyset,\ (\alpha \cdot x) \cdot a \cap \alpha(x \cdot a) \neq \emptyset,\ (a \cdot \alpha) \cdot x \cap a \cdot (\alpha \cdot x) \neq \emptyset$$

for every $\alpha \in U, a \in A, x \in X$ and similarly for the right or two-sided actions. Then, we say that X is a *Banach A-U-H_v-module*. If moreover $\alpha \cdot x \cap x \cdot \alpha \neq \emptyset$ $(\alpha \in U, \quad x \in X)$, then X is called a *commutative Banach A-U-H_v-module*. Furthermore if $a \cdot x \cap x \cdot a \neq \emptyset$ for all $x \in X$ and $a \in A$, then X is called a *bi-commutative Banach A-U-H_v-module*. Throughout this paper, by a A-U-H_v-module, we shall always mean a commutative Banach A-U-H_v-module.

Note that in general, A is not a Banach A-U-H_v-module because A does not satisfy the compatibility condition $a(\alpha \cdot b) \cap (a \cdot \alpha)b \neq \emptyset$ for $\alpha \in U$ and $a, b \in A$. But when A is a commutative U-H_v-bimodule and acts on itself by H_v-algebra multiplication from both sides, then it is also a Banach A-U-H_v-module.

If X is a (commutative) Banach A-U-H_v-module, then so is X^*, where the actions of A and U on X^* are defined as follows:

$$\begin{aligned} \langle f \cdot \alpha, x \rangle &= \langle f, \alpha \cdot x \rangle, & \langle f \cdot a, x \rangle &= \langle f, a \cdot x \rangle, \\ \langle \alpha \cdot f, x \rangle &= \langle f, x \cdot \alpha \rangle, & \langle a \cdot f, x \rangle &= \langle f, x \cdot a \rangle, \end{aligned}$$

for each $a \in A, \alpha \in U, x \in X$ and $f \in X^*$.

2.2. Some H_v-maps

Let A and B be Banach H_v-algebras and Banach U-H_v-bimodules with compatible actions, a *U-H_v-module map* is a H_v-bounded H_v-map $h : A \to B$ with

$$\begin{aligned} &h(a \pm b) \cap (h(a) \pm h(b)) \neq \emptyset, \\ &h(\alpha \cdot a) \cap (\alpha \cdot h(a)) \neq \emptyset, \\ &h(a \cdot \alpha) \cap (h(a) \cdot \alpha) \neq \emptyset \quad (\alpha \in U, a, b \in A). \end{aligned}$$

Here h is not necessarily H_v-linear, so it is not necessarily a U-H_v-module homomorphism.

h is called *multiplicative U-H_v-module map* (or called *U-H_v-module morphism*) if $h(ab) \cap h(a)h(b) \neq \emptyset$ $(a, b \in A)$. We denote by $Hom_U(A, B)$ the H_v-metric space of all multiplicative U-H_v-module maps from A into B, with the H_v-metric derived from the usual H_v-linear operator norm $\|.\|$ on $L_U(A, B)$;

the set of all H_v-bounded linear operators from A into B, and denote $Hom_U(A, A)$ by $Hom(A)$.

Let A and U be as above and X be a Banach A-U-H_v-module. A *H_v-module derivation* $D : A \to X$ is a U-H_v-module map such that $D(ab) \cap (D(a) \cdot b + a \cdot D(b)) \neq \emptyset$ for all $a, b \in A$. The set of all H_v-module derivations from A to X is denoted by $Z^U(A, X)$ (abb. $Z(A, X)$). Note that D is not necessarily H_v-linear, but still its H_v-boundedness implies its norm H_v-continuity since D preserves subtraction.

Let f and g be H_v-module derivations from A to X and $\alpha \in U$, so are $f + g$ and αf. Since X is a Banach U-H_v-bimodule, we have that $Z(A, X)$ is a Banach U-H_v-bimodule.

For $x \in X$, define a map by $D_x : A \to X$, $a \longmapsto a \cdot x - x \cdot a$, $a \in A$. When X is a A-U-H_v-module, it is clear that D_x is a H_v-module derivation. H_v-module derivations of this kind are called *inner* and denoted by $InnZ(A, X)$.

An *H_v-Jordan module derivation* $D : A \to X$ is a U-H_v-module map such that $D(a^2) \cap (D(a) \cdot a + a \cdot D(a)) \neq \emptyset$ for all $a \in A$. The set of all Jordan H_v-module derivations from A to X is denoted by $JZ(A, X)$.

The U-H_v-module map $f : A \to X$ is said to be *H_v-Lie module derivation* if the identity

$$f([a, b]) \cap ([f(a), b] + [a, f(b)]) \neq \emptyset$$

holds for all $a, b \in A$. The set of all Lie H_v-module derivations is denoted by $LieZ(A, X)$. Here $[a, b] = ab - ba$.

By a *Brešar generalized H_v-module derivation* (f, D), we mean $f : A \to X$ is a U-H_v-module map such that $f(ab) \cap (f(a) \cdot b + a \cdot D(b)) \neq \emptyset$ for all $a, b \in A$, where D is a H_v-module derivation on A. We denote by $Z^B(A, X)$ is the set of Brešar generalized H_v-module derivation from A to X.

If (f_1, D_1) and (f_2, D_2) are Brešar generalized H_v-module derivations and $\alpha \in U$, then $(f_1 + f_2, D_1 + D_2)$ and $(\alpha f_1, \alpha D_1)$ are also Brešar generalized H_v-module derivations and hence, $Z^B(A, X)$ is a Banach U-H_v-bimodule.

For $x, y \in X$, a U-H_v-module map satisfies the identity

$$f_{x,y} : A \ni a \mapsto (x \cdot a + a \cdot y) \in X$$

for all $a \in A$ is called a *Brešar generalized inner H_v-module derivation.*

For a U-H_v-module map $f : A \to X$ is called a *Brešar generalized Jordan H_v-module derivation* if

$$f(a^2) \cap (f(a) \cdot a + a \cdot D(a)) \neq \emptyset$$

for all $a \in A$. Here D is a Jordan H_v-module derivation. We denote the set of Brešar generalized Jordan H_v-module derivations from A to X by $JZ^B(A, X)$.

The U-H_v-module map $f : A \to X$ is said to be *Brešar generalized Lie H_v-module derivation* if the identity

$$f([a, b]) \cap ([f(a), b] + [a, D(b)]) \neq \emptyset$$

holds for all $a, b \in A$. Here D is a Lie H_v-module derivation. We denote the set by $LieZ^B(A, X)$.

For a U-H_v-module map $f : A \to X$ and an element $x \in X$, a pair (f, x) is called an H_v-*generalized module derivation* in the sense of *Nakajima*, if

$$f(ab) \cap (f(a) \cdot b + a \cdot f(b) + a \cdot x \cdot b) \neq \emptyset$$

for all $a, b \in A$. We denote the set of this type of generalized H_v-module derivations by $Z^G(A, X)$. This is also a Banach U-H_v-bimodule for a Banach A-U-H_v-module X.

For $x, y \in X$, a U-H_v-module map $f_{x,y} : A \to X$ is called H_v-*generalized inner module derivation* if

$$f_{x,y}(ab) \cap (f_{x,y}(a) \cdot b + a \cdot f_{x,y}(b) + a \cdot (-x - y) \cdot b) \neq \emptyset$$

for all $a, b \in A$. We denote this H_v-derivation by $(f_{x,y}, -x - y)$.

A pair (f, x) is called a H_v-*generalized Jordan module derivation* if

$$f(a^2) \cap (f(a) \cdot a + a \cdot f(a) + a \cdot x \cdot a) \neq \emptyset$$

for all $a \in A$. We denote the set of generalized Jordan H_v-module derivations from A to X by $JZ^G(A, X)$.

The pair (f, x) is called a H_v-*generalized Lie module derivation* if the relation

$$f([a, b]) \cap ([f(a), b] + [a, f(b)] + a \cdot x \cdot b - b \cdot x \cdot a) \neq \emptyset$$

holds for all $a, b \in A$ and the set of generalized Lie H_v-module derivations from A to X can be denoted by $LieZ^G(A, X)$.

If $x = 0$, then these definitions lead to the conventional notions of generalized Jordan and Lie H_v-module derivations.

Throughout this paper we use the following notations for the above sets:

- $Z(A, X)$, the set of H_v-module derivations,
- $InnZ(A, X)$, the set of inner H_v-module derivations,
- $JZ(A, X)$, the set of Jordan H_v-module derivations,
- $LieZ(A, X)$, the set of Lie H_v-module derivations,
- $Z^B(A, X)$, the set of Bres̃ar generalized H_v-module derivations,
- $InnZ^B(A, X)$, the set of Bres̃ar generalized inner H_v-module derivations,
- $JZ^B(A, X)$, the set of Bres̃ar generalized Jordan H_v-module derivations,
- $LieZ^B(A, X)$, the set of Bres̃ar generalized Lie H_v-module derivations,
- $Z^G(A, X)$, the set of generalized H_v-module derivations,
- $InnZ^G(A, X)$, the set of generalized inner H_v-module derivations,
- $JZ^G(A, X)$, the set of generalized Jordan H_v-module derivations,
- $LieZ^G(A, X)$, the set of generalized Lie H_v-module derivations.

A U-H_v-module map $f : A \to X$ is said to be *left H_v-module multiplier* if $f(ab) \cap (f(a) \cdot b) \neq \emptyset$ for all $a, b \in A$. We denote by $Mull^U(A, X)$ (abb. $Mull(A, X)$) the set of all left H_v-module multipliers from A to X. Especially if $f(a^2) \cap (f(a) \cdot a) \neq \emptyset$ for all $a \in A$, then f is called *Jordan left H_v-module multiplier* and we denote the set of these maps by $JMull(A, X)$. Furthermore, we can define the set

$$\begin{aligned} &LieMull(A, X) \\ &= \{f \mid f : A \to X,\ H_v\text{-module map and } f[a, b] \cap ([-f(b), a]) \neq \emptyset \text{ for all } a, b \in A\} \end{aligned}$$

which is called the set of *Lie left H_v-module multipliers*.

If f and g are left H_v-module multipliers in all types and $\alpha \in U$, then $f + g$ and αf are also left H_v-module multipliers in all types, hence all the above special sets are Banach U-H_v-bimodules.

At the end of this section, we want to give a well-known lemma which will be used several times in the next sections in our paper.

Lemma 2.1. [8, Theorem 4.7 (Five Short Lemma in H_v-module)] *Let*

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\ \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} & & \downarrow{\scriptstyle \alpha_4} & & \downarrow{\scriptstyle \alpha_5} \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5 \end{array}$$

be a commutative diagram of H_v-modules and H_v-module homomorphisms over an H_v-ring R with exact rows, the followings hold:

(1) *If α_1 is an epimorphism and α_2, α_4 are monomorphisms, then α_3 is a monomorphism;*

(2) *If α_5 is a monomorphism and α_2, α_4 are epimorphisms, then α_3 is a monomorphism*

3. Homological properties of generalized H_v-module derivations

In this section, we first discuss the relation between the U-H_v-bimodules $Z^B(A, X)$ and $Z^G(A, X)$. Now, we give some elementary lemmas which show the relation between H_v-module derivations and our generalized H_v-module derivations.

Lemma 3.1.

(1) *If $(f, x) : A \to X$ is a generalized H_v-module derivation, then there exists an H_v-module derivation $d = f + l_x : A \to X$, where $l_x : A \to X$ is a left multiplication, i.e., $l_x(a) = xa$, such that $f(ab) = f(a)b + ad(b)$ for all $a, b \in A$. Moreover, if $\{x \in X \mid Ax = 0\} = 0$, then d is uniquely determined by f.*

(2) *If $D : A \to X$ is an H_v-module derivation, then for any nonzero element $x \in X$, $(f = D + l_x, -x) : A \to X$ is a generalized H_v-module derivation such that $f \neq D$ and D associates to f.*

(3) *If* $(f,x) : A \to X$ *is a generalized* H_v*-module derivation, then* $(f, f+l_x) : A \to X$ *is a Breš̃ar generalized* H_v*-module derivation.*

(4) *If* A *contains a unit element and* $(f,D) : A \to X$ *is a Breš̃ar generalized* H_v*-module derivation, then* $(f, -f(1)) : A \to X$ *is a generalized* H_v*-module derivation. It means that the notions of generalized* H_v*-module derivations of Nakajima and Breš̃ar coincide when* A *contains an identity element.*

Proof. We only need to check the boundedness of the map $d = f + l_x : A \to X$. Since f is a U-H_v-module map and X is a A-U-H_v-bimodule, then we get

$$\|(f+l_x)(a)\| \leq \|f(a)\| + \|x \cdot a\| \leq M\|a\| + K\|x\|\|a\| = (M + K\|x\|)\|a\|,$$

for each $a \in A$. This means that the map $f + l_x$ isH_v- bounded. The other parts of the proof can be done easily. ∎

Remark 1. Throughout this paper, the most important thing which we have to check is the boundedness of the maps (for U-H_v-module maps). In the next parts of the paper, we have omitted the boundedness of the maps (because all of them are done similarly).

Corollary 3.2. *The following sequence of* U-H_v*-modules* $Z^G(A,X)$ *and* $Z(A,X)$ *is exact and splitting:*

$$0 \to X \xrightarrow{\varphi_1} Z^G(A,X) \xrightarrow{\varphi_2} Z(A,X) \to 0,$$

where $\varphi_1(x) = (l_x, -x)$ *and* $\varphi_2((f,x)) = f + l_x$ *are* U-H_v*-module maps. Hence, we get* $Z^G(A,X) \cong X \oplus Z(A,X)$.

Our aim is to give necessary and sufficient condition for $Z^B(A,X)$ to be isomorphic to $Z^G(A,X)$ as a Banach U-H_v-bimodule when A *does not have a unit element.*

Theorem 3.3. *Suppose that* $\Phi : Z^G(A,X) \to Z^B(A,X)$ *and* $\psi : X \to \mathrm{Mull(A, X)}$ *are* U-H_v*-module morphisms such that* $\Phi((f,x)) = (f, f+l_x)$ *and* $\psi(x) = l_x$*. Then* Φ *is a* U-H_v*-module isomorphism if and only if* ψ *is a* U-H_v*-module isomorphism.*

Proof. We have the following split exact sequence of Banach U-H_v-bimodules:

$$0 \to Mull(A,X) \xrightarrow{\psi_1} Z^B(A,X) \xrightarrow{\psi_2} Z(A,X) \to 0,$$

where $\psi_1(g) = (g, 0)$ and $\psi_2((f,D)) = D$.

Define a map $\psi_2' : Z(A,X) \to Z^B(A,X)$ by $\psi_2'(D) = (D,D)$. Then $\psi_2\psi_2' = id_{Z(A,X)}$, and thus is split exact. This gives the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{\varphi_1} & Z^G(A,X) & \xrightarrow{\varphi_2} & Z(A,X) & \longrightarrow & 0 \\
 & & \downarrow \psi & & \downarrow \Phi & & \downarrow id & & \\
0 & \longrightarrow & Mull(A,X) & \xrightarrow{\psi_1} & Z^B(A,X) & \xrightarrow{\psi_2} & Z(A,X) & \longrightarrow & 0
\end{array}$$

Hence we complete the proof of the theorem by using Five Lemma. ∎

Corollary 3.4. *The following sequence of Banach U-H_v-bimodules $JZ(A,X)$ and $JZ^G(A,X)$, is exact and splitting:*

$$0 \to X \xrightarrow{\psi_X} JZ^G(A,X) \xrightarrow{\phi_X} JZ(A,X) \to 0,$$

where $\psi_X(x) = (l_x, -x)$ and $\phi_X((f,x)) = f + l_x$.

Corollary 3.5. *Suppose that $\Phi : JZ^G(A,X) \longrightarrow JZ^B(A,X)$ and $\psi : X \longrightarrow$ JMull(A, X) are U-H_v-module morphisms such that $\Phi((f,x)) = (f, f + l_x)$ and $\psi(x) = l_x$. Then Φ is a U-H_v-module isomorphism if and only if ψ is a U-H_v-module isomorphism.*

Corollary 3.6. *The following sequence of Banach U-H_v-bimodules $LieZ(A,X)$ and $LieZ^G(A,X)$, is exact and splitting:*

$$0 \to X \xrightarrow{\psi_X} LieZ^G(A,X) \xrightarrow{\phi_X} LieZ(A,X) \to 0$$

where $\psi_X(x) = (l_x, -x)$ and $\phi_X((f,x)) = f + l_x$.

Corollary 3.7. *Suppose that $\Phi : LieZ^G(A,X) \longrightarrow LieZ^B(A,X)$ and $\psi : X \longrightarrow$ LieMull(A, X) are U-H_v-module morphisms such that $\Phi((f,x)) = (f, f + l_x)$ and $\psi(x) = l_x$. Let us define the set*

$$\mathcal{X}(A) = \{x \in X \mid [x,a] = 0 \text{ for all } a \in A\}.$$

If $\mathcal{X}(A) = X$ (If X is a bi-commutative Banach A-U-H_v-module), then Φ is a U-H_v-module isomorphism if and only if ψ is a U-H_v-module isomorphism.

Corollary 3.8. *Let X be a A-U-H_v-module, then the following diagram is commutative and the rows are split exact:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{\psi_1} & InnZ^G(A,X) & \xrightarrow{\psi_2} & InnZ(A,X) & \longrightarrow & 0 \\
 & & \downarrow i_0 & & \downarrow i & & \downarrow i_1 & & \\
0 & \longrightarrow & X & \xrightarrow{\varphi_1} & Z^G(A,X) & \xrightarrow{\varphi_2} & Z(A,X) & \longrightarrow & 0
\end{array}$$

where i_0, i_1, i are the canonical H_v-module injections and $\psi_1(x) = (f_{x,0}, -x)$, $\psi_2(f_{x,y}, -x-y) = f_{x,y} + l_{(-x-y)}$.

Proof. All maps in the above diagram are U-H_v-module maps, and the commutativity of the diagram is easily seen. If $\psi_2(f_{x,y}, -x-y) = 0$, then we see that $f_{x+y,0} = f_{x,y}$. Thus $Ker\psi_2 = Im\psi_1$. The other part is clear by Corollary 3.2 using the definitions of φ_1 and φ_2. ∎

4. Functorial relations

4.1. Functorial relations

Firstly, we give a functorial relation between $Z(A,-)$ and $Z^G(A,-)$ as follows:

Theorem 4.1. *Let X_1 and X_2 be Banach A-U-H_v-odules and $\gamma : X_1 \to X_2$ be a U-H_v-module morphism. Then γ induces a U-H_v-module map*

$$\gamma' : Z^G(A, X_1) \to Z^G(A, X_2)$$

such that

$$\gamma'((f, x_1)) = (\gamma f, \gamma(x_1))$$

and $Z^G(A, -)$ is a covariant functor from the category of Banach A-U-H_v-modules to the category of Banach U-H_v-bimodules.

Proof. The map

$$\gamma' : Z^G(A, X_1) \to Z^G(A, X_2),\ (f, x_1) \mapsto (\gamma f, \gamma(x_1))$$

is a U-hypermodule map.

Let $\mathcal{X}$ be a category of Banach A-U-H_v-modules and $\mathcal{M}$ be a category of Banach U-H_v-bimodules. Define the functor as follows:

$$Z^G(A, -) : \mathcal{X} \to \mathcal{M}, \quad X \mapsto \mathcal{X}(X) = Z^G(A, X).$$

If $\gamma : X_1 \to X_2, \quad \gamma_* : X_2 \to X_3$ are U-H_v-module morphisms, then, for the following map, the first condition is satisfied:

$$\begin{aligned} &Z^G(A, -)(\gamma_* \circ \gamma) : Z^G(A, X_1) \to Z^G(A, X_3), \\ &(f, x_1) \mapsto ((\gamma_* \circ \gamma) \circ f, (\gamma_* \circ \gamma)(x_1)) \end{aligned}$$

On the other hand,

$$\begin{aligned} Z^G(A, -)(\gamma_*)((Z^G(A, -)(\gamma))(f, x_1)) &= Z^G(A, -)(\gamma_*)(\gamma f, \gamma(x_1)) \\ &= (\gamma_* \circ (\gamma f), \gamma_*(\gamma(x_1))). \end{aligned}$$

Thus $Z^G(A, -)(\gamma_*) \circ Z^G(A, -)(\gamma) = Z^G(A, -)(\gamma_* \circ \gamma)$.

For the second condition, we use the map,

$$\begin{aligned} &Z^G(A, -)(1_X) : Z^G(A, X) \to Z^G(A, X), \\ &(f, x_1) \mapsto (1_X f, 1_X(x_1)) = (f, x_1) \end{aligned}$$

Therefore, $Z^G(A, -)(1_X) = 1_{Z^G(A,-)(X)}$. ∎

Theorem 4.2. *Let $\Phi : Z^G(A, -) \to Z(A, -) \oplus F$ be a map of functors where F is the forgetful functor from the category of Banach A-U-H_v-modules to the category of Banach U-H_v-bimodules. Then Φ assigns to each Banach A-U-H_v-module X of $\mathcal{X}$, a U-H_v-module isomorphism $\Phi_X : Z^G(A, X) \to Z(A, -) \oplus X$ of $\mathcal{M}$ such that $\Phi_X((f, x)) = (f + l_x, x)$ where $x \in X$ and $X = X_1, X_2$; in such a way that for every U-H_v-module morphism of Banach A-U-H_v-modules $\gamma : X_1 \to X_2$ of $\mathcal{X}$, the diagram*

$$\begin{array}{ccc} Z^G(A,X_1) & \xrightarrow{\alpha_*} & Z^G(A,X_2) \\ \downarrow \Phi_{X_1} & & \downarrow \Phi_{X_2} \\ Z(A,X_1)\oplus X_1 & \xrightarrow{\overline{\alpha_*}} & Z(A,X_2)\oplus X_2 \end{array}$$

in $\mathcal{M}$ is commutative, where $\Phi_X(d,x) = (\gamma d, d(x))$ and $\overline{\alpha_*}(f,x_1) = (\gamma f, f(x_1))$. Hence we can say that Φ is a natural transformation of functors.

Since Φ_X is an equivalence for every A-U-H_v-module X_1 in $\mathcal{X}$ by Theorem 4.2, we have the following corollary:

Corollary 4.3. *The functors $Z^G(A,-)$ and $Z^G(A,-)\oplus F$ from the category of Banach A-U-H_v-modules to the category of Banach U-H_v-bimodules are naturally equivalent.*

References

[1] Bayon, R., Lygeros, N., *Advanced results in enumeration of hyperstructures*, J. Algebra, 320 (2008), 821-835.

[2] Corsini, P., *Prolegomena of Hypergroup Theory*, Second Edition, Udine, Tricesimo, Italy, Aviani, 1993.

[3] Corsini, P., Leoreanu, V., *Applications of Hyperstructure Theory*, Dordrecht, the Netherlands: Kluwer Academic Publishers (Advances in Mathematics), 2003.

[4] Davvaz, B., *A Brief Survey of The theory of H_v-structures*, Proc. 8th International Congress on Algebraic Hyperstructures and Applications, Greece, 2002, 39-70.

[5] Davvaz, B., Leoreanu-Fotea, V., *Hyperring Theory and Applications*, International Academic Press, 2007.

[6] Inceboz, H., Arslan, B., *On homological properties of some module derivations on Banach algebras*, Bol. Soc. Paran.Mat., 34 (2) (2016), 169-188.

[7] Marty, F., *Sur une généralization de la notion de groupe*, In Proceeding of the 8th Congress des Mathematiciens, Scandinavia, Stockholm, Sweden, 1934, 45-49.

[8] Vaziri, Y., Ghadiri, M., Davvaz, B., *The $M[-]$ and $-[M]$ functors and five short lemma in H_v-modules*, Turkish Journal Of Math., 40 (2016), 397-410.

[9] Vougiouklis, T., *H_v-vector spaces*, Proc. 5th International Congress on Algebraic Hyperstructures and Applications, Romania, 1994, 181-190.

[10] Vougiouklis, T., *Hyperstructures and Their Representations*, Palm Harbor, 1994.

[11] Vougiouklis, T., *The fundamental relation in hyperrings*, The General Hyperfield, Algebraic Hyperstructures and Applications, Teaneck, 1991, 203-211.

[12] Vougiouklis, T., *The Lie-hyperalgebras and their fundamental relations*, Southeast Asian Bulletin of Mathematics, 37 (4), (2013), 601-614.

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AN ANALYTICAL APPROXIMATION TECHNIQUE FOR THE DUFFING OSCILLATOR BASED ON THE ENERGY BALANCE METHOD

M. Alal Hosen[1]
Department of Manufacturing and Material Engineering
International Islamic University Malaysia
Jalan Gombak, 53100 Kuala Lumpur
Malaysia
e-mail: alal_ruet@yahoo.com

M.S.H. Chowdhury
Department of Science in Engineering
International Islamic University Malaysia
Jalan Gombak, 53100 Kuala Lumpur
Malaysia
e-mail: sazzadbd@iium.edu.my

Mohammad Yeakub Ali
Department of Manufacturing and Material Engineering
International Islamic University Malaysia
Jalan Gombak, 53100 Kuala Lumpur
Malaysia
e-mail: mmyali@iium.edu.my

Ahmad Faris Ismail
Department of Mechanical Engineering
International Islamic University Malaysia
Jalan Gombak, 53100 Kuala Lumpur
Malaysia
e-mail: faris@iium.edu.my

Abstract. In this paper, an analytical approximation technique has been presented of obtaining higher-order approximate solutions for highly nonlinear Duffing oscillator based on the energy balance method (EBM). Higher-order approximate natural frequencies have been obtained in a novel analytical way. The accuracy of the solution method is evaluated within as error analysis. It is highly remarkable that using the presented technique, the approximation solutions produces desired results even for large oscillation as compared with the exact ones. Moreover, the solution method yields much better results than existing solutions after using a suitable truncation formula. The presented technique is applied to well-known Duffing oscillator to illustrate its novelty, reliability and wider applicability.

Keywords: Duffing oscillator; Energy balance method; Analytical approximate technique; Truncation Principle.

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[1]Corresponding author

1. Introduction

Considerable attention has been directed towards the study in the field of vibration analysis because this issue is very applicable in dynamics of structures, free and forced vibrations and vibration of elastic plates have been published [1]–[3]. In general, analytical solutions of highly nonlinear oscillators do not always exist and hence most of the researchers have used either approximate analytical techniques or numerical methods to obtain approximate solutions. A few nonlinear systems can be solved explicitly, and numerical methods especially the most well-known Runge-Kutta fourth order method are frequently used to calculated approximate solutions. However, the class of stiff differential equations and chaotic differential equations, the numerical schemes do not always give accurate results, which present big challenge to numerical analysis. In this situation, many researchers have been showed an intensifying interest in the field of analytical approximate techniques. Most of the widely used analytical technique for solving nonlinear equations associated with oscillatory systems is Perturbation Method [4], [5], which is the most versatile tools available in nonlinear analysis of engineering problems, and they are constantly being developed and applied to ever more complex problems. However, the standard perturbation methods have many limitations, and they are not yield for highly nonlinear oscillators. As a result, to overcome the limitations of standard perturbation technique, a large variety of new analytical approximate techniques including Optimal Homotopy Asymptotic Method [6], Homotopy Perturbation Method [7], [8], Modified Homotopy Perturbation Method [9], Modified He's Homotopy Perturbation Method [10], [11], He's Modified Lindsted-Poincare Method [12], Parameterized Perturbation Method [13] commonly used to solve nonlinear systems especially for highly nonlinear oscillators.

Recently, some other approximation techniques such as He's Max-Min Approach Method [14], Elliptic Balance Method [15], Algebraic Method [16], Rational Energy Balance Method [17], He's Frequency-Amplitude Formulation [18]–[20], Iteration Method [21], [22], Variational Approach Method [23]–[25], Harmonic Balance Method [26]–[34], Rational Harmonic Balance Method [35] have been paid much attention to determined periodic solutions of highly nonlinear oscillatory problems. The Energy Balance Method (EBM) and He's Energy Balance Method (HEBM) [36]–[41] are another technique for solving highly nonlinear oscillators. In fact, to the best of our knowledge, there is no clear idea to obtain higher-order approximation solutions in HEBM. Moreover, only first-order approximation has been considered which does not lead much better accuracy. In this study, the higher-order approximate periodic solutions for the well-known Duffing oscillator is studied employing modified energy balance method (MEBM). The presented technique gives much better results than the classical energy balance method, He's energy balance method and other previously existing methods. In adding, a suitable truncation principle has been introduced which produced much better results. Considering the interesting property that the presented technique not only provides accurate results but also it is more convenient and efficient for solving more complex nonlinear problems.

2. Solution approaches

2.1 The basic idea of He's energy balance method

Let us consider a second order nonlinear differential equation is

$$\ddot{x} = -f, \quad \text{with initial condition} \quad x(0) = A_0, \quad \dot{x}(0) = 0, \tag{2.1}$$

in which x and t are represent dimensionless displacement and time variables respectively and $f = f(x, \dot{x})$.

The variational principle of Eq. (2.1) can be easily obtained as follows

$$J(x) = \int_0^t (-\frac{1}{2}\dot{x}^2 + F(x))dt \tag{2.2}$$

where $F(x) = \int f(x, \dot{x})dx$. Its Hamiltonian can be written in the following form

$$H = \frac{1}{2}\dot{x}^2 + F(x) = F(A_0) \tag{2.3}$$

or

$$R(t) = \frac{1}{2}\dot{x}^2 + F(x) - F(A_0) = 0 \tag{2.4}$$

The following trial solution utilized to obtain the natural frequency

$$x = A_0 \cos(\omega t) \tag{2.5}$$

Substituting equation (2.5) into equation (2.4), the following residual equation is reduced as

$$R(t) = \frac{1}{2}A_0^2\omega^2 \sin^2(\omega t) + F(A_0 \cos(\omega t)) - F(A_0) = 0 \tag{2.6}$$

Since equation (2.5) in only an approximation to the exact solution, equation (2.6) cannot be made zero everywhere. Collocation at $\omega t = \frac{\pi}{4}$ gives

$$\omega(A_0) = \frac{2}{A_0}\sqrt{F(A_0) - F(\frac{A_0}{\sqrt{2}})} \tag{2.7}$$

Its period can be determining by using the relation $T = \frac{2\pi}{\omega}$ as

$$T = \frac{2\pi}{\frac{2}{A_0}\sqrt{F(A_0) - F(\frac{A_0}{\sqrt{2}})}} \tag{2.8}$$

2.2 The Modified energy balance method

A general n-th order periodic solution of equation (2.1) is in the form

$$x = A_0(\rho\cos(\omega t) + u\cos(3\omega t) + v\cos(5\omega t) + w\cos(7\omega t) + z\cos(9\omega t) + \cdots), \tag{2.9}$$

where A_0, ρ and ω are constants. If $\rho = 1-u-v-\cdots$, then the solution equation (2.9) readily satisfies the initial conditions given in equation (2.1).

Substituting (2.9) into (2.4) and expanding it in a Fourier series expansion as

$$\frac{1}{2}\dot{x}^2 + F(x) - F(A_0) = b_1\cos(\omega t) + b_3\cos(3\omega t) + b_5\cos(5\omega t) + \cdots \tag{2.10}$$

where $b_1, b_3, \cdots$ will be calculated by using the following integration

$$b_{2n+1} = \frac{4}{\pi}\int_0^{\pi/2} (\frac{1}{2}\dot{x}^2 + F(x) - F(A_0))cos[(2n+1)\varphi]d\varphi; n = 0,1,2,3,\ldots, \tag{2.11}$$

setting $\varphi = \omega t$. Substituting (2.9) into (2.11), the coefficients $b_1, b_3, \ldots$ are determined. Finally, substituting $b_1, b_3, \ldots$ into equation (2.10) and then equating the coefficients of $\cos(\omega t), \cos(3\omega t), \cos(5\omega t), \ldots,$ equal to zero, a set of nonlinear algebraic equations is obtained whose solution provides the unknown natural frequency ω and the others unknown coefficients $u, v, \ldots$ in terms of amplitude A_0. This completes the determination of all related unknowns for the proposed periodic solution given in (2.9).

3. Problem Descriptions

The focused generalized nonlinear oscillator, which is a numerous range of applications in nonlinear sciences and engineering as

$$\ddot{x} + \omega_0^2 x + \varepsilon x^n |\, x\,|^{\alpha-1} + \frac{\gamma x^{m-1}}{\sqrt{x^m+1}} = 0, \quad n = 2k-1, \quad \alpha > 0, \quad m = 2p \tag{3.1}$$

If $\gamma = 0, \alpha = 1$ and $n = 3$, equation (3.1) represents the governing equation of generalized Duffing oscillator is as follows

$$\ddot{x} + \omega_0^2 x + \varepsilon x^3 = 0, \tag{3.2}$$

which is stated in [16]–[18], [36], [37], [39], [41]. Equation (3.2) also represents the free undamped vibration of an orthotropic claimed triangular plate which has been stated in [37].

4. Application of Duffing oscillator

In this section, an example will be presented to illustrate the accuracy, efficiency and its wider applicability of the presented method.

The Variational and Hamiltonian formulations of (3.2) can be obtained as

$$J(x) = \int_0^t \left(-\frac{1}{2}\dot{x}^2 - \frac{\omega_0^2 x^2}{2} - \frac{\varepsilon x^4}{4}\right) dt \tag{4.1}$$

$$H = \frac{1}{2}\dot{x}^2 + \frac{\omega_0^2 x^2}{2} + \frac{\varepsilon x^4}{4} = \frac{\omega_0^2 A_0^2}{2} + \frac{\varepsilon {A_0}^4}{4} \tag{4.2}$$

$$H = \frac{1}{2}\dot{x}^2 + \frac{\omega_0^2 x^2}{2} + \frac{\varepsilon x^4}{4} - \frac{\omega_0^2 A_0^2}{2} - \frac{\varepsilon {A_0}^4}{4} = 0 \tag{4.3}$$

In equation (2.9), the first-order approximation solution is

$$x = A_0 \cos(\omega t) \tag{4.4}$$

Using (4.4) into (4.3) and then substitute into (2.11), $b_1, b_3, \ldots$ are obtained. Finally, substituting $b_1, b_3, \ldots$ into (2.10), then equating the coefficient of $\cos(\omega t)$ equal to zero, the first order natural frequency is obtained as

$$\omega(A_0) = \sqrt{\frac{10\omega_0^2 + 7\varepsilon A_0^2}{10}} \tag{4.5}$$

Therefore, the first-order approximation solution of equation (3.2) is equation (4.4) where ω is given by equation (4.5).

Consider a second-order approximation solution from equation (2.9) is

$$x = A_0 \cos(\omega t) + A_0 u(\cos(3\omega t) - \cos(\omega t)) \tag{4.6}$$

Using (4.6) into (4.3) and then substitute into (2.11), $b_1, b_3, \ldots$ are obtained.

Finally, substituting $b_1, b_3, \ldots$ into (2.10), then equating the coefficient of $\cos(\omega t)$ and $\cos(3\omega t)$ equal to zero, the following nonlinear algebraic equations are

$$\begin{aligned} &\varepsilon A_0^2(-21021 - 54912u + 109824u^2 - 133120u^3 + 71680u^4) \\ &+858((-35 - 112u + 96u^2)\omega_0^2 + (35 + 56u + 368u^2)\omega^2) = 0, \end{aligned} \tag{4.7}$$

$$\begin{aligned} &\varepsilon A_0^2(25311 + 18304u - 89856u^2 + 153600u^3 - 100352u^4) \\ &-286((-147 - 240u + 352u^2)\omega_0^2 + 3(49 - 152u + 208u^2)\omega^2) = 0 \end{aligned} \tag{4.8}$$

The higher order terms of u more than second order terms have no effect on the value of the unknowns u and ω. So, we may ignore more than second order terms of u in equations (4.7)-(4.8); but half of the second order terms are considered. This is called truncation principle (see details in [29]).

Moreover, it is clearly being seen that solution equation (4.6) gives much better results and it saves a lot of calculation. Using the truncation principle, (4.7)-(4.8) take the following form

$$\begin{aligned} \varepsilon A_0^2(-21021 - 54912u + 54912u^2) + 858((-35 - 112u + 48u^2)\omega_0^2 \\ +(35 + 56u + 184u^2)\omega^2) = 0, \end{aligned} \tag{4.9}$$

$$\begin{aligned} \varepsilon A_0^2(25311 + 18304u - 44928u^2) - 286((-147 - 240u + 176u^2)\omega_0^2 \\ +3(49 - 152u + 104u^2)\omega^2) = 0 \end{aligned} \tag{4.10}$$

From (4.9), it can easily written as

$$\omega^2 = \frac{\varepsilon A_0^2(21021 + 54912u - 54912u^2) + 858(35 + 112u - 48u^2)\omega_0^2}{858(35 + 56u + 184u^2)} \tag{4.11}$$

Eliminating ω^2 from equation (4.10) with the help of equation (4.11), the nonlinear algebraic equation of u is as follows

$$\begin{aligned} f(u) : \varepsilon A_0^2(693 - 12320u - 62312u^2 + 63488u^3 + 12288u^4) \\ +176u(-126 - 637u + 176u^2 + 136u^3)\omega_0^2 = 0 \end{aligned} \tag{4.12}$$

Now, applying the iterative homotopy perturbation method (**See Appendix A**) to obtain the value of u from equation (4.12) is

$$u = u_0 + u_1 + u_2 + u_3 + \cdots , \tag{4.13}$$

where u_0 is an initial approximation and the unknowns u_1, u_2, u_3, ... are

$$u_1 = -\frac{f(u_0)}{f'(u_0)}, \tag{4.14}$$

$$u_2 = -\frac{f''(u_0)}{f'(u_0)}\left(\frac{f(u_0)}{f'(u_0)}\right)^2, \tag{4.15}$$

$$u_3 = \frac{1}{f'(u_0)}\left(\frac{1}{6}\left(\frac{f(u_0)}{f'(u_0)}\right)^3\right)f'''(u_0) + \frac{f(u_0)}{f'(u_0)}\left(-\frac{f''(u_0)}{f'(u_0)}\left(\frac{f(u_0)}{f'(u_0)}\right)^2\right), \tag{4.16}$$

and so on.

Substituting the value of u from equation (4.13) into equation (4.11), the second order approximate natural frequency is determined the following

$$\omega(A_0) = \sqrt{\frac{\varepsilon A_0^2(21021 + 54912u - 54912u^2) + 858(35 + 112u - 48u^2)\omega_0^2}{858(35 + 56u + 184u^2)}} \tag{4.17}$$

The third-order approximate solution is in the form as

$$x = A_0\cos(\omega t) + A_0 u(\cos(3\omega t) - \cos(\omega t)) + A_0 v(\cos(5\omega t) - \cos(\omega t)) \tag{4.18}$$

which is easily apply for obtaining third-order approximate solutions in the presented method.

5. Results and discussion

Comparison the first- and second-order approximation solutions of equation (3.2) for $\omega_0 = \varepsilon = 1$ and initial amplitude $A_0 = 10$ corresponding with exact solutions have been shown in Figure 1.

Figure 1: Time history of dynamic response, equation (3.2) for $\omega_0 = \varepsilon = 1$ and $A_0 = 10$.

Using different values of A_0, the approximate natural frequencies has been compared with corresponding exact and previously existing frequencies are listed in Table 1.

Table 1: Comparison the obtained natural frequencies and previously existing results with corresponding exact frequency of equation (3.2) for $\omega_0 = \varepsilon = 1$:

εA_0^2	ω_{ex}	$\omega^{[36]}_{1stEBM}$ $Er(\%)$	$\omega^{[36]}_{2ndEBM}$ $Er(\%)$	$\omega^{[39]}_{1stEBM}$ $Er(\%)$	$\omega^{[39]}_{2ndEBM}$ $Er(\%)$	$\omega_{1stMEBM}$ $Er(\%)$	$\omega_{2ndMEBM}$ $Er(\%)$
0.5	1.1708	1.1726 *0.1537*	1.1702 *0.0512*	1.1619 *0.7601*	1.1702 *0.0512*	1.1619 *0.7601*	1.1708 *0.0000*
1	1.3178	1.3229 *0.3870*	1.3161 *0.1290*	1.3038 *1.0623*	1.3163 *0.1138*	1.3038 *1.0623*	1.3180 *0.0151*
5	2.1504	2.1795 *1.3532*	2.1406 *0.4557*	2.1213 *1.3532*	2.1426 *0.3627*	2.1213 *1.3532*	2.1518 *0.0651*
10	2.8666	2.9155 *1.7058*	2.8500 *0.5790*	2.8284 *1.3325*	2.8535 *0.4569*	2.8284 *1.3325*	2.8690 *0.0837*
100	8.5336	8.7178 *2.1585*	8.4700 *0.7452*	8.4261 *1.2597*	8.4842 *0.5788*	8.4261 *1.2597*	8.5425 *0.1042*
1000	26.8107	27.4044 *2.2144*	26.6055 *0.7653*	26.4764 *1.2468*	26.6519 *0.5923*	26.4764 *1.2468*	26.8394 *0.1070*
5000	59.9157	61.2454 *2.2192*	59.4559 *0.7674*	59.1692 *1.2459*	59.5599 *0.5938*	59.1692 *1.2459*	59.9799 *0.1071*

Note: In Table 1, $\omega_{1stEBM}^{[36]}$, $\omega_{2ndEBM}^{[36]}$, $\omega_{1stEBM}^{[39]}$ and $\omega_{2ndEBM}^{[39]}$ represent first- and second-order approximate natural frequencies previously obtained in [36, 39]. $\omega_{1stMEBM}$ and $\omega_{2ndMEBM}$ denote first- and second-order approximate natural frequencies obtained in present study. ω_{ex} represents the exact frequency which is stated in [36]. $Er(\%)$ denotes the percentage error which has been calculated by the relation $|\frac{\omega_{MEBM}(A_0)-\omega_{ex}(A_0)}{\omega_{ex}(A_0)}| \times 100$.

In regard to the above Figure 1, it can clearly be seen that the presented method is a better applicable and reliable for solving strongly nonlinear oscillators with high precision like the presented problem in Section 3. Seeing in Table 1, comparing the relative errors with previously existing different methods, it is observed that the accuracy of the presented method is much better in the whole range of initial amplitude A_0. It is highly remarkable that the approximate solutions (second-order approximation) obtained by the presented technique is very close to the exact solutions and better than those obtained previously by several authors. The advantages of this method include its analytical simplicity and computational efficiency, and the ability to objectively find better results for many other oscillatory problems arising in nonlinear sciences and engineering.

6. Conclusion

An analytical approximation technique based on the EBM has been presented to determine approximate solutions of the Duffing oscillator. In this problem, the approximate solutions show much better agreement compared with the corresponding exact solutions and previously existing results. High accuracy of the approximate solutions reveals the versatility of the presented technique in solving highly nonlinear problems. Analytical simplicity, computational efficiency and the ability to objectively find better results are the advantages of this method. It can be concluded that the presented technique is a better and efficient alternative than the existing ones for approximating solutions for highly nonlinear oscillatory problems.

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7. APPENDIX A

A higher-order nonlinear algebraic equation is of the form

$$f(x) = 0 \tag{7.1}$$

Consider the nonlinear algebraic equation equation (7.1), and we construct a homotopy $H : R \times [0, 1] \to R$ which satisfies

$$H(x,p) = f(x) - f(x_0) + pf(x_0) = 0, \quad x \in R, \quad p \in [0,1] \tag{7.2}$$

where p is embedding parameter, x_0 is an initial approximation of equation (7.1). Hence, it is obvious that

$$H(x,0) = f(x) - f(x_0) = 0 \tag{7.3}$$

$$H(x,1) = f(x) = 0 \tag{7.4}$$

and the changing process of p from 0 to 1, refers to $H(x,p)$ from $H(x,0)$ to $H(x,1)$. Applying the perturbation technique (**See details in [22]**), due to the fact that $0 \leq p \leq 1$ can be considered as a small parameter, we can assume that the solution of equation (7.2) can be express as a series in p

$$x = x_0 + x_1 p + x_2 p^2 + x_3 p^3 + \cdots \tag{7.5}$$

Where $p \to 1$, equation (7.2) corresponds to equation (7.1) and equation (7.5) becomes the approximate solution of equation (7.1), that is [22].

$$x = \lim_{p \to 1} = x_0 + x_1 + x_2 + x_3 + \cdots \tag{7.6}$$

and in [22] the unknowns are

$$x_1 = -\frac{f(x_0)}{f'(x_0)}, \tag{7.7}$$

$$x_2 = -\frac{f''(x_0)}{f'(x_0)} \left(\frac{f(x_0)}{f'(x_0)} \right)^2, \tag{7.8}$$

$$x_3 = \frac{1}{f'(x_0)} \left(\frac{1}{6} \left(\frac{f(x_0)}{f'(x_0)} \right)^3 \right) f'''(x_0) + \frac{f(x_0)}{f'(x_0)} \left(-\frac{f''(x_0)}{f'(x_0)} \left(\frac{f(x_0)}{f'(x_0)} \right)^2 \right), \tag{7.9}$$

and so on.

References

[1] Leissa, A.W., *Vibration of Plates*, Acoustical Society of America, Washington DC, 1993.

[2] Chopra, A.k., *Dynamic of Structures, Theory and Application to Earthquake Engineering*, New Jersey: Prentice-Hall, 1095., 1 (1) (2016), 15-25.

[3] Dimarogonas, A.D., Haddad, S., *Vibration for Engineers*, Prentice-Hall, Englewood Cliffs, New Jersey, 1992.

[4] Nayfeh, A.H., *Perturbation Methods*, J. Wiley, New York, 1973.

[5] Azad, A.K., Hosen, M.A., Rahman, M.S., *A perturbation technique to compute initial amplitude and phase for the Krylov-Bogoliubov-Mitropolskii method*, Tamkang Journal of Mathematics, 43 (4) (2012), 563-575.

[6] Marincaa, V., Herisanu, N., *Determination of periodic solutions for the motion of a particle on a rotating parabola by means of the optimal homotopy asymptotic method*, Journal of Sound and Vibration, 329 (9) (2010), 1450-1459.

[7] Belendez, A., *Homotopy perturbation method for a conservative $x^{1/3}$ force nonlinear oscillator*, Computers & Mathematics with Applications, 58 (11-12) (2009), 2267-2273.

[8] Leung, A.Y.T., Guo, Z., *Homotopy perturbation for conservative Helmholtz-Duffing oscillators*, Journal of Sound and Vibration, 325 (1) (2009), 287-296.

[9] Suleman, M., Wu, Q., *Comparative solution of nonlinear quintic cubic oscillator using modified homotopy perturbation method*, Advances in Mathematical Physics, 2015.

[10] Belndez, A., Pascual, C., Belndez, T., Hernndez, A., *Solution for an anti-symmetric quadratic nonlinear oscillator by a modified He's homotopy perturbation method*, Nonlinear Analysis: Real World Applications, 10 (1) (2009), 416-427.

[11] Belndez, A., Pascual, C., Ortuno, M., Belndez, T., Gallego, S., *Application of a modified He's homotopy perturbation method to obtain higher-order approximations to a nonlinear oscillator with discontinuities*, Nonlinear Analysis: Real World Applications, 10 (2) (2009), 601-610.

[12] Ozis, T., Yildirim, A., *Determination of periodic solution for a $u^{1/3}$ force by He's modified Lindstedt-Poincar method*, Journal of Sound and Vibration, 301 (1) (2007), 415-419.

[13] He, J., *Some new approaches to Duffing equation with strongly and high order nonlinearity (II) parametrized perturbation technique*, Communications in Nonlinear Science and Numerical Simulation, 4 (1) (1999), 81-83.

[14] Azami, R., Ganji, D.D., Babazadeh, H., Dvavodi, A.G., Ganji, S.S., *He's Max-min method for the relativistic oscillator and high order Duffing equation*, International journal of modern physics B, 23 (32) (2009), 5915-5927.

[15] Elas-Ziga, A., Rodrguez, C.A., Martnez-Romero, O., *Application of the elliptic balance method to a nonlinear singular oscillator*, Applied Mathematics and Computation, 218 (22) (2012), 11112-11117.

[16] Mirgolbabaee, H., Ledari, S.T., Gaji, D.D., *New approach method for solving Duffing-type nonlinear oscillator*, Alexandria Engineering Journal, 55 (2) (2016), 1695-1702.

[17] Daeichin, M., Ahmadpoor, M.A., Askari, H., Yildirim, A., *Rational energy balance method to nonlinear oscillators with cubic term*, Asian-European Journal of Mathematics, 6 (02) (2013), 1350019.

[18] Younesian, D., Askari, H., Saadatnia, Z., Yazdi, M.K., *Frequency analysis of strongly nonlinear generalized Duffing oscillators using He's frequency-amplitude formulation and He's energy balance method*, Computers & Mathematics with Applications, 59 (9) (2010), 3222-3228.

[19] Zhao, L., *He's frequency-amplitude formulation for nonlinear oscillators with an irrational force*, Computers & Mathematics with Applications, 58 (11) (2009), 2477-2479.

[20] Cai, X.C., Wu, W.Y., *He's frequency formulation for the relativistic harmonic oscillator*, Computers & Mathematics with Applications, 58 (11) (2009), 2358-2359.

[21] Haque, B.I., Alam, M.S., Rahman, M.M., *Modified solutions of some oscillators by iteration procedure*, Journal of the Egyptian Mathematical Society, 21 (2) (2013), 142-147.

[22] Javidi, M., *Iterative methods to nonlinear equations*, Applied Mathematics and Computation, 193 (2) (2007), 360-365.

[23] He, J.H., *Variational approach for nonlinear oscillators*, Chaos, Solitons & Fractals, 34 (5) (2007), 1430-1439.

[24] Yildirim, A., Askari, H., Yazdi, M.K., Khan, Y., *A relationship between three analytical approaches to nonlinear problems*, Applied Mathematics Letters, 25 (11) (2012), 1729-1733.

[25] Solon, M.P., Esguerra, J.P.H., *Periods of relativistic oscillators with even polynomial potentials*, Physics Letters A, 372 (44) (2008), 6608-6612.

[26] Mickens R.E., *Oscillations in planar dynamic systems (Vol. 37)*, World Scientific, Singapore, 1996.

[27] Mickens, R.E., *A generalization of the method of harmonic balance*, Journal of Sound and Vibration, 111 (3) (1986), 515-518.

[28] Mickens, R.E., *Periodic solutions of the relativistic harmonic oscillator*, Journal of sound and vibration, 212 (5) (1998), 905-908.

[29] Hosen, M.A., Rahman, M.S., Alam, M.S., Amin, M.R., *An analytical technique for solving a class of strongly nonlinear conservative systems*, Applied Mathematics and Computation, 218 (9) (2012), 5474-5486.

[30] Hosen, M.A., *Approximate solutions of the equation of motion's of the rigid rod which rocks on the circular surface without slipping*, Ain Shams Engineering Journal, 5 (3) (2014), 895-899.

[31] Belndez, A., Pascual, C., *Harmonic balance approach to the periodic solutions of the (an) harmonic relativistic oscillator*, Physics Letters A, 371 (4) (2007), 291-299.

[32] Belndez, A., Hernndez, A., Belndez, T., lvarez, M.L., Gallego, S., Ortuo, M., Neipp, C., *Application of the harmonic balance method to a nonlinear oscillator typified by a mass attached to a stretched wire*, Journal of Sound and Vibration, 302 (4) (2007), 1018-1029.

[33] Sun, W.P., Wu, B.S., Lim, C.W., *Approximate analytical solutions for oscillation of a mass attached to a stretched elastic wire*, Journal of sound and vibration, 300 (3) (2007), 1042-1047.

[34] Wu, B.S., Lim, C.W., He, L.H., *A new method for approximate analytical solutions to nonlinear oscillations of nonnatural systems*, Nonlinear Dynamics, 32 (1) (2003), 1-13.

[35] Yamgou, S.B., Bogning, J.R., Jiotsa, A.K., Kofan, T.C., *Rational harmonic balance-based approximate solutions to nonlinear single-degree-of-freedom oscillator equations*, Physica Scripta, 81 (3) (2010), 035003.

[36] Durmaz, S., Demirbag, S.A., Kaya, M.O., *High order He's energy balance method based on collocation method*, International Journal of Nonlinear Sciences and Numerical Simulation, 11 (Supplement) (2010), 1-6.

[37] Askari, H., Saadatnia, Z., Esmailzadeh, E., Younesian, D., *Multi-frequency excitation of stiffened triangular plates for large amplitude oscillations*, Journal of Sound and Vibration, 333 (22) (2014), 5817-5835.

[38] Younesian, D., Askari, H., Saadatnia, Z., Yazdi, M.K., *Analytical approximate solutions for the generalized nonlinear oscillator*, Applicable Analysis, 91 (5) (2012), 965-977.

[39] Durmaz, S., Kaya, M.O., *High-order energy balance method to nonlinear oscillators*, Journal of Applied Mathematics, 2012.

[40] Jamshidi, N., Ganji, D.D., *Application of energy balance method and variational iteration method to an oscillation of a mass attached to a stretched elastic wire*, Current Applied Physics, 10 (2) (2010), 484-486.

[41] Younesian, D., Askari, H., Saadatnia, Z., Yazdi, M.K., *Frequency analysis of strongly nonlinear generalized Duffing oscillators using He's frequency-amplitude formulation and He's energy balance method*, Computers & Mathematics with Applications, 59 (9) (2010), 3222-3228.

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OD-CHARACTERIZABILITY OF THE SYMMETRIC GROUP $\mathbb{S}_{27}$

G.R. Rezaeezadeh[1]

Shahrekord University
Shahrekord
Iran
e-mail: rezaeezadeh@sci.sku.ac.ir

M.R. Darafsheh

School of Mathematics
Statistics and Computer Science
College of Science
University of Tehran
Tehran
Iran
e-mail: darafsheh@ut.ac.ir

Abstract. Let G be a finite group with degree pattern $D(G)$. A finite group G is called k-fold OD-characterizable if there are exactly k non-isomorphic groups H such that $|G| = |H|$ and $D(G) = D(H)$. In this paper our purpose is to correct an earlier paper and prove that the symmetric group on 27 letters is 38-OD-characterizable.

Keywords: OD-characterization, symmetric group, prime group, degree pattern.
2000 Mathematics Subject Classification: 20D05, 20D06, 20D08.

1. Introduction and preliminaries

The set of all the prime divisors of the order of a finite group G is denoted by $\pi(G)$ and the set of elements order in G is denoted by $\pi_e(G)$. The Gruenberg-Kegel graph or the prime graph of G, denoted by $\Gamma(G)$, is a graph with vertex set $\pi(G)$ and two distinct vertices p and q are joined by an edge if and only if $pq \in \pi_e(G)$, and in this case we write $p \sim q$, otherwise we will write $p \nsim q$ which means that G does not have an element of order pq. The number of connected components of $\Gamma(G)$ is denoted by $t(G)$ and these connected components are denoted by $\pi_i(G)$, $1 \leq i \leq t(G)$. If $2 \in \pi(G)$, then $\pi_1(G)$ is the connected component of $\Gamma(G)$ containing 2. If n is a natural number, then $\pi(n)$ is the set of all primes dividing n, hence $|G| = m_1 m_2 \cdots m_{t(G)}$ where m_i is a positive integer with $\pi(m_i) = \pi_i$, $1 \leq i \leq t(G)$. The numbers m_i, $1 \leq i \leq t(G)$ are called the order components of G,

[1]Corresponding author.

we write $OC(G) = \{m_1, ..., m_{t(G)}\}$ for the set of order components of G. The set of prime graph components of G is denoted by $T(G) = \{\pi_1(G), \pi_2(G), ..., \pi_{t(G)}(G)\}$.

If the order of G is $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, with p_i's distinct prime numbers and α_i positive integers, then for $p \in \{p_1, ..., p_k\}$, the degree of p in $\Gamma(G)$ is the number of vertices adjacent to p and is denoted by $deg(p)$, i.e., $deg(p) = |\{p_i | p \sim p_i\}|$. We choose an ordering such that $p_1 < p_2 < \cdots < p_k$ and define $D(G) = (deg(p_1), deg(p_2), ..., deg(p_k))$ as the degree pattern of G. Let n be a positive integer. A group H is called n-fold OD-characterizable if there are exactly n non-isomorphic groups G such that $|G| = |H|$ and $D(G) = D(H)$. Usually a 1-fold OD-characterizable group is called an OD-characterizable group.

For the first time the prime graphs of a finite group was defined in [5] and its significance can be found in many recent researches on finite groups. In [1] it is shown that the alternating groups $\mathbb{A}_p$, where p and $p-2$ are prime numbers are OD-characterizable, it is also shown that all the sporadic simple groups, certain groups of Lie type are OD-characterizable, but the projective symplectic group $SP_6(3)$ is 2-fold OD-characterizable. In [10] it is shown that all the simple K_4-groups, except $\mathbb{A}_{10}$ are OD-characterizable, whereas $\mathbb{A}_{10}$ is 2-fold OD-characterizable. In [7] it is proved that all the simple groups of order less than 10^8 except $\mathbb{A}_{10}$ and $U_4(2)$ are OD-characterizable while $\mathbb{A}_{10}$ and $U_4(2)$ are 2-fold OD-characterizable.

In [9] it is proved that $\mathbb{A}ut(L_2(49))$ is 9-fold OD-characterizable, and in [8] the authors prove that the alternating group of degree 16 is OD-characterizable. In [3] the authors studied OD-characterizability of the alternating and symmetric groups on 27 letters. Although their results for the alternating group is correct but there is an error in the results stated for the symmetric group. OD-characterizability of $\mathbb{S}_{27}$ is also studied in [2] but the conclusion is in error. In [4] we found some results on OD-characterizability of the symmetric group on 27 letters. In this paper our aim is to give a complete proof of OD-characterizability of the group $\mathbb{S}_{27}$. Our main result is:

Theorem 1.1. *Let G be a finite group such that $|G| = |\mathbb{S}_{27}|$ and $D(G) = D(\mathbb{S}_{27})$. Then G is 38 OD-characterizable.*

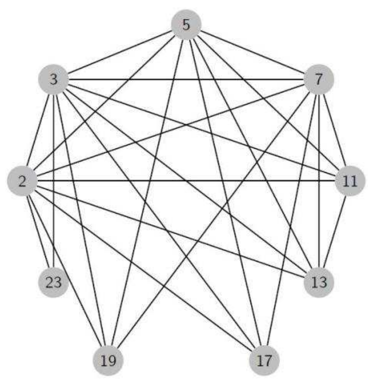

Figure 1: The prime graph of $\mathbb{S}_{27}$

For the proof we need to know about non-abelian simple groups with a prime divisor at most 19. Using [6] we listed these groups in the Table 1 of [3] which is used frequently in this paper.

2. Proof of the main result

The order and degree pattern of $\mathbb{S}_{27}$ are as follows:

$$|\mathbb{S}_{27}| = 2^{23}.3^{13}.5^{6}.7^{3}.11^{2}.13^{2}.17.19.23 \text{ and } D(\mathbb{S}_{27}) = (8,8,7,7,5,5,4,4,2).$$

Let G be a finite group such that $|G| = |\mathbb{S}_{27}|$ and $D(G) = D(\mathbb{S}_{27})$. Since deg(2)=8, the vertex 2 is joined to all other vertices of $\Gamma(G)$, hence $\Gamma(G)$ is a connected graph. Obviously $\Gamma(G) = \Gamma(\mathbb{S}_{27})$ and that $\pi_e(G) \supseteq \{2,3,5,7,11,13,17,19,23,6,10,14,22, 26,34,38,46,15,27,33,39,51,57,69,35,55,65,85,95,77,91, 119,133,143\}$. We also have $\Gamma(\mathbb{S}_{27}) = \Gamma(\mathbb{A}_{27}) = \Gamma(\mathbb{S}_{26})$.

Lemma 2.1. *Let K be the maximal normal solvable subgroup of G. Then K is a $\{2,3\}$-group. In particular G is a non-solvable group.*

Proof. Let p be a prime divisor of $|K|$ and S_p be a Sylow p-subgroup of K. By the Frattini argument, we have $G = KN_G(S_p)$. We consider the following cases:
Case (i) $p = 23$. Since S_{23} is a cyclic 23-group, $\overline{N} = N_G(S_{23})/C_G(S_{23})$ is isomorphic to a subgroup of $\mathbb{Z}_{22}$, implying that $|\overline{N}|$ is a divisor of 22. Since G contains elements of order 46 and 69, we deduce that $C_G(S_{23})$ is a $\{2,3,23\}$-group, and from $G = KN_G(S_{23})$ we obtain $19 \mid |K|$. By assumption K is a solvable group, hence by considering a $\{19,23\}$-Hall subgroup of K we obtain $19 \sim 23$, a contradiction.
Case (ii) $p = 17$ or 19. Let S_p be a Sylow p-subgroup of K, $p = 17$ or 19. Then, similar to case (i), $G = KN_G(S_p)$ and $\overline{N} = N_G(S_p)/C_G(S_p)$ is a subgroup of $\mathbb{Z}_{16}$ or $\mathbb{Z}_{18}$. But using the prime graph $\Gamma(G)$, we see that only $2,3,5,7$ are joined to 17 and 19. We deduce that in any case $23 \mid |K|$ implying that K contains an element of order $23p$, $p = 17$ or 19, a contradiction.
Case (iii) $p = 11$ or 13. In this case, a Sylow p-subgroup S_p of K may have order p or p^2. If S_p is cyclic, then $|\mathbb{A}ut(S_p)| = 11.10$ or 13.12, for the respective cases $p = 11$ or 13. Since $G = KN_G(S_p)$ and $|\overline{N}| = |N_G(S_p)/C_G(S_p)|$ divides 11.10 or 13.12, we deduce that in any case $23 \mid |K|$. Therefore K contains element of order 23.11 or 23.13, a contradiction. Next suppose that a Sylow p-subgroup S_p of K is not cyclic, hence it is of the form $\mathbb{Z}_p \times \mathbb{Z}_p$, $p = 11$ or 13. But $\mathbb{A}ut(S_{13}) = GL_2(13)$ is a group of order $2^5.3^2.7.13$ and $\mathbb{A}ut(S_{11}) = GL_2(11)$ is a group of order $2^4.3.5^2.11$. But $deg(13) = deg(11) = 5$ and 13 is joined to the vertices $2,3,5,7,11$, whereas 11 is joined to the vertices $2,3,5,7,13$. Therefore, $N_G(S_p)$ is a $\{2,3,5,7,11,13\}$-group, $p = 11$ or 13. Hence, $19 \mid |K|$, a contradiction.
Case (iv) $p = 7$. In this case, a Sylow 7-subgroup of K has order 7, 7^2 or 7^3. If $|S_7| = 7$ or 7^2, then using the same techniques as in case (i) and (iii) we obtain a contradiction. Therefore, suppose $|K| = 7^3$. From $G = KN_G(S_7)$ and the fact that $23 \nmid |K|$, we deduce that $23 \mid |N_G(S_7)|$, so S_7 is normalized by an element σ

of order 23. Since G has no element of order 23.7, $\langle\sigma\rangle$ should act fixed-point freely on S_7, implying $23 \mid 7^3 - 1$, a contradiction.

Case (v) $p = 5$. In this case, $|S_5| = 5^\alpha$, $1 \leq \alpha \leq 6$. By case (i), we have $23 \nmid |K|$ and from $G = KN_G(S_5)$ we deduce that S_5 is normalized by an element of order 23, hence $23 \mid 5^\alpha - 1$. But considering all the numbers $1 \leq \alpha \leq 6$, we obtain a contradiction.

Therefore, we have proved that $|K|$ can only be divisible by 2 and 3 and the Lemma is proved. ■

Lemma 2.2. *G/K is an almost simple group, $S \leq G/K \leq \mathbb{A}ut(S)$ where S is a simple group.*

Proof. We set $\overline{G} = G/K$ and $\overline{S} = soc(\overline{G})$ where $soc(\overline{G})$ denotes the socle of $\overline{G}$, i.e the subgroup of $\overline{G}$ generated by the set of all the minimal normal subgroups of $\overline{G}$. We have $\overline{S} \leq \overline{G} \leq \mathbb{A}ut(\overline{S})$ and $\overline{S} \cong S_1 \times S_2 \times \cdots \times S_n$; where S_i's are finite non-abelian simple groups. We show that $n = 1$. Assume on the contrary $n \geq 2$. If $23 \mid |\overline{S}|$, then the order of one S_i is divisible by 23. We assume $23 \mid |S_1|$. Since $23 \sim 2$ and $23 \sim 3$, the S_i's, $i \geq 2$ must be $\{2,3\}$-groups contradicting the simplicity of S_i. Therefore $23 \nmid |\overline{S}|$ and by Lemma 2.1, $23 \mid |\overline{G}|$. From $N_G(\overline{S})/C_G(\overline{S}) = \overline{G}/C_G(\overline{S}) \leq \mathbb{A}ut(\overline{S})$ we deduce that $23 \in \pi(\overline{G}) \subseteq \pi(\mathbb{A}ut(\overline{S}))$. But $|\mathbb{A}ut(\overline{S})| = |\overline{S}|.|Out(\overline{S})|$, hence $23 \mid |Out(\overline{S})|$. Let $P_1, \cdots, P_k$ be non-isomorphic simple groups among $S_1, \cdots, S_n$ such that $\overline{S} \cong P_1^{t_1} \times \cdots \times P_k^{t_k}$, $t_1 + \cdots + t_k = n$. We have $Out(S) = Out(P_1^{t_1}) \times \cdots \times Out(P_k^{t_k})$ and from $23 \nmid |\overline{S}|$ we obtain $23 \nmid |P_i|$ for all $1 \leq i \leq k$, so P_i's are simple $\{2,3,5,7,11,13,17,19\}$-group and by Table 1 in [2] we obtain $23 \nmid |Out(P_i)|, 1 \leq i \leq k$. But $\mathbb{A}ut(P_i^{t_i}) \cong P_i wr \mathbb{S}_{t_i}$ and from $23 \mid |\mathbb{A}ut(P_1^{t_1})| = |P_1|^{t_1}.t_1!$ we obtain $23 \mid t_1!$ which implies $t_1 \geq 23$. Therefore $(23!)_2 \times 2^{23} = 2^{46}$ must divide the order of G, a contradiction. Hence $n = 1$, and $\overline{S}$ is a simple group with $\overline{S} \leq \overline{G} \leq \mathbb{A}ut(\overline{S})$, the Lemma is proved. ■

Lemma 2.3. *$\overline{S}$ is isomorphic to $\mathbb{A}_{26}$ or $\mathbb{A}_{27}$.*

Proof. By Lemma 2.2, $\overline{S} \leq \overline{G} \leq \mathbb{A}ut(\overline{S})$, where $\overline{S}$ is a simple group such that the largest prime factor of $|\overline{S}|$ is 23. Hence by $|G| = |\mathbb{S}_{27}|$ and the fact that $\{13,17,19,23\} \cap \pi(Out(\overline{S})) = \emptyset$, we obtain $13^2.17.19.23 \mid |\overline{S}|$. Now by [5], $\overline{S} \cong \mathbb{A}_{26}$ or $\mathbb{A}_{27}$. ■

Lemma 2.4. *If $\overline{S} \cong \mathbb{A}_{27}$, then there are 3 possibilities for G such that $|G| = |\mathbb{S}_{27}|$ and $OD(G) = OD(\mathbb{S}_{27})$.*

Proof. If $\overline{S} \cong \mathbb{A}_{27}$ then $\mathbb{A}_{27} \leq G/K \leq \mathbb{S}_{27}$. If $G/K \cong \mathbb{S}_{27}$, then $K = 1$ and $G \cong \mathbb{S}_{27}$. If $G/K \cong \mathbb{A}_{27}$, then $|K| = 2$, hence $G \cong \mathbb{Z}_2.\mathbb{A}_{27}$, which gives us 2 possibilities for G. The Lemma is proved. ■

Lemma 2.5. *If $\overline{S} \cong \mathbb{A}_{26}$, then there are 35 non-isomorphic groups G with $|G| = |\mathbb{S}_{27}|$ and $OD(G) = OD(\mathbb{S}_{27})$.*

Proof. If $\overline{S} \cong \mathbb{A}_{26}$, then from $\mathbb{A}_{26} \leq G/K \leq \mathbb{S}_{26}$ we obtain $G/K \cong \mathbb{A}_{26}$ or $\mathbb{S}_{26}$.

Case (i) $G/K \cong \mathbb{A}_{26}$. In this case $|K| = 54$. It can be shown that $G \cong K \times \mathbb{A}_{26}$ or $G \cong P \times (2.\mathbb{A}_{26})$ where P is the unique subgroup of order 27 in K. Since there are 15 non-isomorphic groups of order 54 and 5 non-isomorphic groups of order 27, we obtain 20 possibilities for G.

Case (ii) $G/K \cong \mathbb{S}_{26}$. In this case, $|K| = 27$. It can be shown that G has a normal subgroup H isomorphic to $K \times \mathbb{A}_{26}$ such that $G/H \cong \mathbb{Z}_2$. This will give us 15 possibilities for G. Hence, altogether, we obtain 35 groups with the property mentioned in the theorem. ∎

Main theorem. *There are* 38 *non-isomorphic finite groups* G *with* $|G| = |\mathbb{S}_{27}|$ *and* $OD(G) = OD(\mathbb{S}_{27})$.

Proof. This follows from Lemmas 2.1-2.5. ∎

Remark 2.1. The structures of the 35 groups mentioned in Lemma 2.5 may be described in terms of the semidirect product.

Case(i) $G/K \cong \mathbb{A}_{26}, |K| = 54$.

Consider the semidirect product $K \rtimes \mathbb{A}_{26}$. Then, we have a homomorphism $\varphi : \mathbb{A}_{26} \to \mathbb{A}ut(K)$. Since $\mathbb{A}_{26}$ is a simple group, we obtain $Ker\varphi = \mathbb{A}_{26}$ implying that $G \cong K \times \mathbb{A}_{26}$. There are 15 groups of order 54, hence there are 15 possibilities for G.

Case (ii) $G/K \cong \mathbb{S}_{26}, |K| = 27$.

Let us consider the semidirect product $K \rtimes \mathbb{S}_{26}$ corresponding to the homomorphism $\varphi : \mathbb{S}_{26} \to \mathbb{A}ut(K)$. Considering the normal subgroups of $\mathbb{S}_{26}$ we have $ker\varphi = \mathbb{S}_{26}$ or $\mathbb{A}_{26}$. If $Ker\varphi = \mathbb{S}_{26}$, then $G \cong K \times \mathbb{S}_{26}$ and we obtain 5 possibilities for G. If $Ker\varphi = \mathbb{A}_{26}$, then $\mathbb{S}_{26}/\mathbb{A}_{26} \cong \varphi(\mathbb{S}_{26}) \leq \mathbb{A}ut(K)$. But $\mathbb{S}_{26}/\mathbb{A}_{26} \cong \mathbb{Z}_2$, implying that $\varphi(\mathbb{S}_{26})$ corresponds to an involution (an element of order 2) in $\mathbb{A}ut(K)$. But there are 5 non-isomorphic groups of order 27.

1) $K \cong \mathbb{Z}_{27}$, then $|\mathbb{A}ut(K)| = 18$ has only one involution.
2) $K \cong \mathbb{Z}_3 \times \mathbb{Z}_9$, then $\mathbb{A}ut(K) = \mathbb{Z}_2 \times \mathbb{Z}_2$, which has 3 involutions.
3) $K \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, then $\mathbb{A}ut(K) = GL_3(3)$ and it can be verified that this group has 7 conjugacy classes of involutions.
4) K is a group of order 27 with element of order 9, then $\mathbb{A}ut(K) \cong V_3(3) \rtimes GL_2(3)$, where $V_3(3)$ is the vector space of dimension 3 over the field with 3 elements. It can be verified that $V_3(3) \rtimes GL_2(3)$ has 3 conjugacy classes of involutions.
5) K is a group of order 27 with elements of order 9, then $\mathbb{A}ut(K) = V_3(3) \rtimes AGL_1(3)$, where $AGL_1(3)$ is the affine group in dimension 1 over the field with 3 elements. This group has only one conjugacy classes of involutions. Considering the non-isomorphic groups obtained from cases $1-4$, we obtain 15 groups. Hence altogether we obtain 35 non-isomorphic groups from the semidirect product $K \rtimes \mathbb{A}_{26}$, $|K| = 54$ and $K \rtimes \mathbb{S}_{26}, |K| = 27$.

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References

1. Darafsheh, M.R., Moghaddamfar, A.R., Zokayi, A.R., *A characterization of finite simple groups by the degrees of vertices of their prime graphs*, Alg. Colloq., 12 (2005), 431-442.

2. Liu, Shitian, *A characterization of some groups by their orders and degree patterns*, Wseas Transactions on Mathematics, Vol. 13 (2014), 586-594.

3. Mahmiani, A., Darafsheh, M.R., *On OD-characterizibility of a certain alternating and symmetric group*, Italian Journal of Pure and Applied Mathematics, 32 (2014), 7-14.

4. Moghaddamfar, A.R., *On alternating and symmetric groups which are quasi OD-characterizable*, arXiv: 1504.00273v2.

5. Williams, J.S., *Prime graph components of finite groups*, J. Algebra, 69 (1981), 487-513.

6. Zavarnitsine, A., *Finite simple groups with narrow prime spectrum*, Siberian Electronic Math. Reports, 6 (2009), 1-12.

7. Zhang, L., Shi, W., *OD-characterization of all simple groups whose orders are less than* 10^8, Front. Math. China, 3 (2008), 461-494.

8. Zhang, L., Shi, W., Wang, L., Shao, C., *OD-characterization of* $\mathbb{A}_{16}$, J. Suzhou Univ., 24 (2008), 7-10.

9. Zhang, L., Shi, W., *OD-characterization of almost simple groups related to* $L_2(49)$, Arch. Math. (Brno), 44 (2008), 191-199.

10. Zhang, L., Shi, W., *OD-characterization of simple* K_4*-groups*, Alg. Colloq., 16 (2009), 275-282.

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A NOTE ON EXTRINSIC FRAME HOMOGENEITY OF HYPERQUADRICS

Parviz Ahmadi
Masoud Hasani

Department of Mathematics
University of Zanjan
P.O. Box 45371-38791 Zanjan
Iran
e-mails: p.ahmadi@znu.ac.ir
masoud.hasani@znu.ac.ir

Abstract. Let p_i and q_i belong to a hyperquadric Q and $(e_{1_i},\dots,e_{n_i})$ and $(f_{1_i},\dots,f_{n_i})$ be orthonormal frames in $T_{p_i}Q$ and $T_{q_i}Q$, respectively, where $1 \leqslant i \leqslant m$. We study sufficient and necessary conditions for existence of an isometry $\varphi : \mathbb{R}^{n+1}_\nu \to \mathbb{R}^{n+1}_\nu$ such that $\varphi(Q) \subset Q$, $\varphi(p_i) = q_i$ and $d\varphi(e_{j_i}) = f_{j_i}$.

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2010 MSC: 53A35, 53B99.

1. Introduction

A semi-Riemannian manifold M is said to be frame-homogeneous provided any frame on M can be carried to any other by the differential map of an isometry of M. If M is a connected frame-homogeneous Riemannian manifold with, $\dim M \geqslant 2$, then it is homothetic to S^n, $\mathbb{P}^n$, $\mathbb{R}^n$ or $\mathbb{H}^n$ (see [1, p.259]). For indefinite metrics, the list is longer (see [2]).

Let M be a semi-Riemannian manifold and N be a semi-Riemannian hypersurface in M. Then N is called extrinsically frame-homogeneous if any frame on N can be carried to any other by the differential map of a pair isometry $\varphi : (M,N) \to (M,N)$. Proposition 4.30 of [1] asserts that hyperquadrics in $\mathbb{R}^n_\nu$ are extrinsically frame-homogeneous. Assume that Q is a hyperquadric in $\mathbb{R}^n_\nu$. Here we consider frames $(e_{1_i},\dots,e_{n_i})$ and $(f_{1_i},\dots,f_{n_i})$ in $T_{p_i}Q$ and $T_{q_i}Q$ respectively, where $1 \leqslant i \leqslant m$, and then we study conditions on the points and on the frames which insure the existence of a pair isometry $\varphi : (\mathbb{R}^n_\nu, Q) \to (\mathbb{R}^n_\nu, Q)$ with the further property that its differential map carries each frame $(e_{1_i},\dots,e_{n_i})$ to $(f_{1_i},\dots,f_{n_i})$, respectively (see Theorem 3.3). Then we conclude the extrinsic frame-homogeneity of hyperquadrics as a special case of the theorem.

2. Preliminaries

Throughout the following $\mathbb{R}^n_\nu$ denotes the n-dimensional real vector space $\mathbb{R}^n$ with a scalar product of signature $(\nu, n-\nu)$ given by

$$\langle x, y\rangle = -\sum_{i=1}^{\nu} x_i y_i + \sum_{j=\nu+1}^{n} x_j y_j, \tag{2.1}$$

where $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$. Let q be the associated quadratic form to the above scalar product, i.e. $q(x) = \langle x, x\rangle$. For $\varepsilon \in \{-1, 1\}$ and fixed real number $r > 0$, let $Q = q^{-1}(\varepsilon r)$. Then Q is a semi-Riemannian hypersurface of $\mathbb{R}^n_\nu$ with sign ε. These hypersurfaces are called the hyperquadrics of $\mathbb{R}^n_\nu$. An orthonormal basis for a tangent space T_xQ is called a frame on Q at x. For any frame $(e_1, \dots, e_n)$ on Q, we assume that $e_1, \dots, e_\nu$ are time-like and $e_{\nu+1}, \dots, e_n$ are space-like.

Let us identify an $n \times n$ real matrix A with the linear operator $A : \mathbb{R}^n_\nu \to \mathbb{R}^n_\nu$ such that

$$(Ax)_i = \sum_j A_{ij} x_j, \quad 1 \leqslant i \leqslant n.$$

By this identification, composition of functions becomes the matrix multiplication. Hence, the set (group) of all linear isometries $\mathbb{R}^n_\nu \to \mathbb{R}^n_\nu$ is the same as the set $O_\nu(n)$ of all matrices $A \in GL(n, \mathbb{R})$ that preserve the scalar product defined in (2.1). Clearly, $O_\nu(n)$ is a closed subgroup of $GL(n, \mathbb{R})$ and so is itself a Lie group. Also if $x \in \mathbb{R}^n_\nu$, the translation $T_x : \mathbb{R}^n_\nu \to \mathbb{R}^n_\nu$ sending each v to $v + x$ is an isometry. Since $T_x \circ T_y = T_{x+y} = T_y \circ T_x$, the set of all translations of $\mathbb{R}^n_\nu$ is an abelian subgroup of $Iso(\mathbb{R}^n_\nu)$ isomorphic to the additive group $\mathbb{R}^n$. In fact, each isometry of $\mathbb{R}^n_\nu$ has a unique expression as T_aA, with $a \in \mathbb{R}^n_\nu$ and $A \in O_\nu(n)$ (see [1, p.240]), where the group multiplication is given by

$$T_aA.T_bB = T_{a+Ab}AB.$$

By this multiplication $\mathbb{R}^n$ is a normal subgroup of $Iso(\mathbb{R}^n_\nu)$. Define the one-to-one and onto function

$$\varphi : O_\nu(n) \times \mathbb{R}^n \to Iso(\mathbb{R}^n_\nu),$$

by $\varphi(A, a) = T_aA$. If we define the multiplication on $O_\nu(n) \times \mathbb{R}^n$ as follows

$$(A, a).(B, b) = (AB, a + Ab),$$

and denote the resulting group by $O_\nu(n) \ltimes \mathbb{R}^n$, then

$$\varphi : O_\nu(n) \ltimes \mathbb{R}^n \to Iso(\mathbb{R}^n_\nu)$$

is an isomorphism between Lie groups. Then the action of each isometry (A, a) on $\mathbb{R}^n_\nu$ is defined as follows

$$(A, a)x = Ax + a, \quad x \in \mathbb{R}^n_\nu.$$

3. Main result

The main result of this section is Theorem 3.3. To prove the theorem first we recall the following lemma from [1, p.234].

Lemma 3.1 *Let* $E = \begin{pmatrix} -I_\nu & 0 \\ 0 & I_{n-\nu} \end{pmatrix}$, *where* I_ν *denotes the* $\nu \times \nu$ *identity matrix.The following conditions on an* $n \times n$ *real matrix are equivalent.*

(1) $A \in O_\nu(n)$.
(2) *]* $A^{tr} = EA^{-1}E$.
(3) *The columns (rows) of* A *form an orthonormal basis for* $\mathbb{R}^n_\nu$ *(first* ν *vectors timelike).*
(4) A *carries one (hence every) orthonormal basis for* $\mathbb{R}^n_\nu$ *to an orthonormal basis.*

Lemma 3.2 *Let* $Q = q^{-1}(\varepsilon r^2)$, *where* r *is a fixed positive real number and* $\varepsilon \in \{-1, 1\}$. *If* $\varphi : \mathbb{R}^{n+r}_\nu \to \mathbb{R}^{n+1}_\nu$ *is an isometry such that* $\varphi(Q) \subset Q$, *then* φ *is a linear isometry.*

Proof. Let $\varphi : \mathbb{R}^{n+1}_\nu \to \mathbb{R}^{n+1}_\nu$ be an isometry such that $\varphi(Q) \subset Q$. By the fact that $Iso(\mathbb{R}^{n+1}_\nu) = O_\nu(n+1) \times \mathbb{R}^{n+1}$, there exists $A \in O_\nu(n+1)$ and $a \in \mathbb{R}^{n+1}$ such that $\varphi = (A, a)$. We need only to show that $a = 0$. For arbitrary $x \in Q$;

$$q(Ax + a) = q(x). \tag{3.1}$$

Since $A \in Iso(\mathbb{R}^{n+1}_\nu)$, so $A(Q) = Q$. Hence (3.1) implies that

$$\langle y, a \rangle = \frac{-1}{2} q(a);\ \forall y \in Q,$$

which is obviously impossible unless $a = 0$. ∎

Theorem 3.3 *Let* p_i *and* q_i *belong to* $Q = q^{-1}(\varepsilon r^2)$, *where* $1 \leq i \leq m$ *and* $\varepsilon \in \{-1, 1\}$. *Let* $(e_{1_i}, \ldots, e_{n_i})$ *and* $(f_{1_i}, \ldots, f_{n_i})$ *be orthonormal frames in* $T_{p_i}Q$ *and* $T_{q_i}Q$, *respectively. Then there is an isometry* $\varphi : \mathbb{R}^{n+1}_\nu \to \mathbb{R}^{n+1}_\nu$ *such that*

$$\varphi(Q) \subseteq Q, \quad \varphi(p_i) = q_i, \quad \text{and} \quad d\varphi(e_{j_i}) = f_{j_i}, \tag{$*$}$$

if and only if

$$\langle \widetilde{e}_{j_i}, \widetilde{e}_{k_i} \rangle = \langle \widetilde{f}_{j_i}, \widetilde{f}_{k_i} \rangle,\ \langle p_i, p_l \rangle = \langle q_i, q_l \rangle,\ \langle \widetilde{e}_{j_l}, p_i \rangle = \langle \widetilde{f}_{j_l}, q_i \rangle, \tag{$**$}$$

where $1 \leqslant i, l \leqslant m$, $1 \leqslant j, k \leqslant n$ *and* $\widetilde{e}_{j_i}$ *and* $\widetilde{f}_{j_l}$ *denote the elements of* $\mathbb{R}^{n+1}_\nu$ *canonically corresponding to* e_{j_i} *and* f_{j_i}, *respectively.*

Proof. First assume that $\varepsilon = +1$. To simplify the notation, let $\widetilde{e}_{(n+1)_i} = \frac{p_i}{r}$ and $\widetilde{f}_{(n+1)_i} = \frac{q_i}{r}$, where $1 \leq i \leq m$. Let $A_i = [\widetilde{e}_{1_i}, \cdots, \widetilde{e}_{(n+1)_i}]$ and $B_i = [\widetilde{f}_{1_i}, \cdots, \widetilde{f}_{(n+1)_i}]$ be two matrices with columns $\widetilde{e}_{j_i}$ and $\widetilde{f}_{j_i}$. Since the columns of A_i and B_i are orthonormal, so they belong to $O_\nu(n+1)$. Hence, by Lemma 3.2, the statement $(*)$ is equivalent to the following;
There is $A \in O_\nu(n+1)$ such that

$$AA_i = B_i\ , \quad 1 \leq i \leq m. \tag{$*'$}$$

And this holds if and only if

$$A_iB_i^{-1} = A_lB_l^{-1}\ , \quad 1 \le i, l \le m,$$

equivalently

$$A_l^{-1}A_i = B_l^{-1}B_i\ , \quad 1 \le i, l \le m. \tag{3.2}$$

Let $E = \begin{pmatrix} -I_\nu & 0 \\ 0 & I_{n+1-\nu} \end{pmatrix}$, where I_ν denotes the $\nu \times \nu$ identity matrix. Then, by Lemma 3.1, one gets that $A_i^{-1} = EA_i^{tr}E$. Hence (3.2) holds if and only if

$$A_l^{tr}EA_i = B_l^{tr}EB_i, \quad 1 \le i, l \le m,$$

which is equivalent to $(**)$.

In the case that $\varepsilon = -1$, let $\widetilde{e}_{0_i} = \frac{p_i}{r}$, $\widetilde{f}_{0_i} = \frac{q_i}{r}$, $A_i = [\widetilde{e}_{0_i}, \widetilde{e}_{1_i}, \ldots, \widetilde{e}_{n_i}]$ and $B_i = [\widetilde{f}_{0_i}, \widetilde{f}_{1_i}, \ldots, \widetilde{f}_{n_i}]$, then follow the above proof. ∎

As an immediate consequence of Theorem 3.3, one gets the following corollary. Hence the theorem is a generalization of Proposition 4.30 of [1, p.113].

Corollary 3.4 *Let $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ be (tangent) frames respectively on points $p, q \in Q = q^{-1}(\varepsilon r^2)$, where $\varepsilon \in \{-1, 1\}$. Then there is a unique isometry $\varphi : \mathbb{R}^{n+1}_\nu \to \mathbb{R}^{n+1}_\nu$ carry Q isometricaly to itself, with $\varphi(p) = q$ and $d\varphi(e_i) = f_i$, for $1 \le i \le n$. Furthermore, this isometry should be linear.*

Another interesting consequence is the following corollary which is coincide with one's intuition.

Corollary 3.5 *Let p_i and q_i belong to S^n, where S^n denotes the standard n-dimensional sphere and $1 \leqslant i \leqslant m$. Let $(e_{1_i}, \ldots, e_{n_i})$ and $(f_{1_i}, \ldots, f_{n_i})$ be orthonormal frames in $T_{p_i}S^n$ and $T_{q_i}S^n$, respectively. Then there is an isometry $\varphi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ such that*

$$(*) \qquad \varphi(S^n) \subseteq S^n, \quad \varphi(p_i) = q_i, \quad \text{and} \quad d\varphi(e_{j_i}) = f_{j_i},$$

if and only if

$$(**) \qquad \theta(p_i, p_l) = \theta(q_i, q_l) \quad \text{and} \quad \theta(\widetilde{e}_{k_i}, p_l) = \theta(\widetilde{f}_{j_l}, q_i),$$

where $1 \leqslant i, l \leqslant m$, $1 \leqslant j, k \leqslant n$, $\theta(x, y)$ is the angle between two vectors x and y, and $\widetilde{e}_{j_i}$ and $\widetilde{f}_{j_l}$ denote the elements of $\mathbb{R}^{n+1}$ canonically corresponding to e_{j_i} and f_{j_i}, respectively.

Proof. Since $\langle \widetilde{e}_{j_i}, \widetilde{e}_{k_i} \rangle = \delta_{jk}$ and $\langle p_i, p_l \rangle = \cos\theta(p_i, p_l)$, the corollary is a direct consequence of Theorem 3.3. ∎

References

[1] O'Neill, B., *Semi-Riemannian Geometry With Applications to Relativity*, Academic Press, July 12, 1983.

[2] Wolf, J.A., *Spaces of Constant Curvature*, Publish or Perish, Berkley, California, 2011.

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SOME GEOMETRIC AGGREGATION OPERATORS BASED ON PICTURE FUZZY SETS AND THEIR APPLICATION IN MULTIPLE ATTRIBUTE DECISION MAKING

Chunyong Wang[1]
College of Science
Hezhou University
Hezhou, Guangxi 542800
P.R. China
e-mail: wangchunyong2006@sina.com

Xiaoqiang Zhou
College of Mathematics
Hunan Institute of Science and Technology
Yueyang, Hunan 414000
P.R. China
e-mail: zxq0923@163.com

Huonian Tu
Guangxi University of Finance and Economics
Nanning, Guangxi 530003
P.R. China
e-mail: tuhuonian2015@163.com

Shengda Tao
College of Science
Hezhou University
Hezhou, Guangxi 542800
P.R. China
e-mail: sdtao3@163.com

Abstract. In this paper, we investigate the multiple attribute decision making (MADM) problems with picture fuzzy information. Firstly, concepts and some operational laws of picture fuzzy sets are introduced. Then, we develop some picture fuzzy geometric operators and discuss their basic properties. Next, we apply the proposed operators to deal with multiple attribute decision making problems under picture fuzzy environment. Finally, an illustrative example is given to demonstrate the practicality and effectiveness of our proposed method.

Keywords: picture fuzzy set, picture fuzzy number, picture fuzzy geometric aggregation operator, multiple attribute decision making.

[1]Corresponding author.

1. Introduction

Fuzzy set (FS), proposed by Zadeh in 1965 [24], has achieved a great success in various fields since it appears. Since then, many new theories treating imprecision and uncertainty have been introduced. As an extension of FS, the intuitionistic fuzzy set (IFS) which was proposed by Atanassov [1], [2] has received much attention. As a powerful tool to deal with vagueness and uncertainty, the intuitionistic fuzzy set has been widely used from the application's view point in many fields, such as logic programming [3], decision making [4], [11]– [14], [20]– [22], pattern recognition [9], [10], medical diagnosis [8], [16], and cluster analysis [17], [18].

The prominent characteristic of intuitionistic fuzzy set is that it assigns to each element a membership degree and a nonmembership degree. Although the intuitionistic fuzzy set is a powerful tool to deal with uncertainty, it still has inherent drawbacks. Here, we can give voting as an example. The voters may be divided into four groups of those who: vote for, abstain, vote against, refusal of the voting. Nevertheless, the IFS only care of those who vote for or vote against, and consider those who abstain and refusal are equivalent.

In fact, it is more reasonable to consider those who abstain as they vote for and vote against at the same degree. When some one refuse to vote, we can interpret that he is not concerned about the current election. Following this way, Cuong [5], [6] generalized IFS to picture fuzzy sets (PFSs). As a result, models which base on picture fuzzy set may be adequate in situations when we face human opinions involving more answers of the type: yes, abstain, no, refusal.

In data analysis, information may be gathered in the form of PFSs rather than IFSs.

In order to deal with these information, we need to develop new methods. Singh [15] proposed the weighted correlation coefficient to calculate the degree of correlation between the picture fuzzy sets aiming at clustering different objects.

However, aggregation operators are more commonly used in multi-attribute decision making problems. As a new generation of IFS, PFS still has not any aggregation operators yet.

Thus, in this paper, we aim to introduce new operations on picture fuzzy numbers and develop some new aggregation operators on PFSs.

To facilitate our discussion, the remainder of this paper is organized as follows.

In the next section, we review some basic concepts related to picture fuzzy set. New operations on PFSs are studied in Section 3.

In Section 4, some picture fuzzy geometric operators are introduced. Some properties of these new operators are also investigated.

In Section 5, we develop a method for multiple attribute decision making based on new operators under picture fuzzy environment.

An illustrative example is also given to show the effectiveness of the developed approach in Section 6.

In Section 7, we give the conclusion and some remarks.

2. Preliminaries

Let us first review some basic concepts related to PFS.
FS was first proposed by Zadeh [24] in 1965 as the following.

Definition 2.1 [24]. Let X be an universe of discourse, then a fuzzy set is defined as: $A = \{\langle x, \mu_A(x)\rangle | x \in X\}$ which is characterized by a membership function $\mu_A : X \to [0,1]$, where μ_A denotes the degree of membership of the element x to the set A.

Atanassov [1] generalized FS to IFS as below.

Definition 2.2 [1]. An IFS in X is given by

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}$$

which is characterized by a membership function $\mu_A : X \to [0,1]$ and a non-membership function $\nu_A : X \to [0,1]$, with the condition

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1, \forall x \in X,$$

where the numbers $\mu_A(x)$ and $\nu_A(x)$ represent the degree of membership and the degree of non-membership of the element x to the set A, respectively.

If $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x) = 0, \forall x \in X$, then the IFS is reduced to a common fuzzy set. To aggregate intuitionistic preference information, Xu [19] defined operations as below.

Definition 2.3 [19] Let $\alpha = (\mu_\alpha, \nu_\alpha)$ and $\beta = (\mu_\beta, \nu_\beta)$ be two intuitionistic fuzzy numbers, then

(1) $\alpha \cdot \beta = (\mu_\alpha \mu_\beta, \nu_\alpha + \nu_\beta - \nu_\alpha \nu_\beta)$;

(2) $\alpha^\lambda = (\mu_\alpha^\lambda, 1 - (1 - \nu_\alpha)^\lambda), \lambda > 0$.

Voting was given as a good example in [7]. Human voters can be divided into four groups: vote for, abstain, vote against, refusal of the voting. In order to describe this situation, Cuong et. al. [5], [6], [7] generalized FS and IFS to the picture fuzzy set as the new concept for computaional intelligence problems.

Definition 2.4 [7]. An picture fuzzy set A on a universe X is an object of the form

$$A = \{\langle x, \mu_A(x), \eta_A(x), \nu_A(x)\rangle | x \in X\}$$

where $\mu_A(x)$ $(x \in [0,1])$ is called the *degree of positive membership of x in A*, $\eta_A(x)$ $(x \in [0,1])$ is called the *degree of neutral membership of x in A* and $\nu_A(x)$ $(x \in [0,1])$ is called the *degree of negative membership of x in A*. Besides, μ_A, η_A, ν_A satisfy the following condition

$$\forall x \in X, \quad \mu_A(x) + \eta_A(x) + \nu_A(x) \leq 1.$$

Then, for $x \in X$, $\rho_A(x) = 1 - \mu_A(x) - \eta_A(x) - \nu_A(x)$ could be called the *degree of refusal membership of x in A*.

When $\eta_A(x) = 0, \forall x \in X$, the PFS is reduced to IFS. In the voting, those who are abstain can be interpreted as: on one hand, they vote for; on the other hand, they vote against. Meanwhile, those who are refusal of the voting can be explained as they are not care about this voting.

For convenience, we call $\alpha = (\mu_\alpha, \eta_\alpha, \nu_\alpha, \rho_\alpha)$ a picture fuzzy number (PFN), where $\mu_\alpha \in [0,1]$, $\eta_\alpha \in [0,1]$, $\nu_\alpha \in [0,1]$, $\rho_\alpha \in [0,1]$ and $\mu_\alpha + \eta_\alpha + \nu_\alpha + \rho_\alpha = 1$. Sometimes, we omit ρ_α and denote a PFN as $\alpha = (\mu_\alpha, \eta_\alpha, \nu_\alpha)$ for short.

For every two PFSs A and B, Cuong et. al. [7] also defined some operations as following.

(1) $A \subseteq B$ iff $\mu_A(x) \leq \mu_B(x)$ and $\eta_A(x) \leq \eta_B(x)$ and $\nu_A(x) \geq \nu_B(x), \forall x \in X$;

(2) $A = B$ iff $A \subseteq B$ and $B \subseteq A$;

(3) $A \cup B = \{(x, max(\mu_A(x), \mu_B(x)), min(\eta_A(x), \eta_B(x)), min(\nu_A(x), \nu_B(x))) | x \in X\}$;

(4) $A \cap B = \{(x, min(\mu_A(x), \mu_B(x)), min(\eta_A(x), \eta_B(x)), max(\nu_A(x), \nu_B(x))) | x \in X\}$;

(5) $coA = \bar{A} = \{(x, \nu_A(x), \eta_A(x), \mu_A(x)) | x \in X\}$.

Several properties of these operations were also discussed in [5]:

(1) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$;

(2) $\bar{\bar{A}} = A$;

(3) Operations $\cap$ and $\cup$ are commutative, associative and distributive;

(4) Operations $\cap$, *co* and $\cup$ satisfy the De Morgan law.

3. New operations on PFSs

To aggregate picture fuzzy preference information, some new operators will be introduced in this section. In order to generate Definition 2.3 to PFNs, let us first explain it from a view point of probability. We also choose voting as a realistic example. For an IFN $\alpha = (\mu_\alpha, \nu_\alpha)$, it represents that the ratio who vote for α is μ_α and the ratio who vote against α is ν_α in a decision conference. When we consider both α and β, by the multiplication formula of probability $P(A \cap B) = P(A)P(B)$, we can get the ratio who vote for both α and β as $\mu_{\alpha\beta} = \mu_\alpha \cdot \mu_\beta$. Then, by the complementary formula of probability and $A \cap B = \overline{\bar{A} \cup \bar{B}}$, we can get the ratio who vote against α or vote against β as $\nu_\alpha + \nu_\beta - \nu_\alpha \nu_\beta = 1 - (1 - \nu_\alpha)(1 - \nu_\beta)$. Next, by explaining

$$\alpha^\lambda = \underbrace{\alpha \cdot \alpha \cdot \dots \cdot \alpha}_{\lambda},$$

we can obtain (2) in Definition 2.3.

Table 1. Explain Definition 2.3 from the view point of probability (The joint probability)

	vote for α	vote against α
vote for β	$\mu_\alpha\mu_\beta$	$\nu_\alpha\mu_\beta$
vote against β	$\mu_\alpha\nu_\beta$	$\nu_\alpha\nu_\beta$

In PFS theory, voters are divided into four groups: vote for (its ratio is denoted as μ), abstain (its ratio is denoted as η), vote against (its ratio is denoted as ν), refusal (its ratio is denoted as ρ).

In order to combine two PFNs $\alpha = (\mu_\alpha, \eta_\alpha, \nu_\alpha, \rho_\alpha)$ and $\beta = (\mu_\beta, \eta_\beta, \nu_\beta, \rho_\beta)$, we can construct the joint probability as Table 2.

Table 2. Generate Definition 2.3 from the view point of probability (The joint probability)

	vote for α	abstain for α	vote against α	refusal for α
vote for β	$\mu_\alpha\mu_\beta$	$\eta_\alpha\mu_\beta$	$\nu_\alpha\mu_\beta$	$\rho_\alpha\mu_\beta$
abstain for β	$\mu_\alpha\eta_\beta$	$\eta_\alpha\eta_\beta$	$\nu_\alpha\eta_\beta$	$\rho_\alpha\eta_\beta$
vote against β	$\mu_\alpha\nu_\beta$	$\eta_\alpha\nu_\beta$	$\nu_\alpha\nu_\beta$	$\rho_\alpha\nu_\beta$
refusal for β	$\mu_\alpha\rho_\beta$	$\eta_\alpha\rho_\beta$	$\nu_\alpha\rho_\beta$	$\rho_\alpha\rho_\beta$

To compute $\mu_{\alpha\cdot\beta}$, we come to choose those who vote for both α and β, then $\mu_{\alpha\cdot\beta} = \mu_\alpha\mu_\beta + \eta_\alpha\mu_\beta + \mu_\alpha\eta_\beta = (\mu_\alpha + \eta_\alpha)(\mu_\beta + \eta_\beta) - \eta_\alpha\eta_\beta$. Similarly, those who are abstain for α and abstain for β can be viewed as abstain for α and β. That is $\eta_{\alpha\cdot\beta} = \eta_\alpha\eta_\beta$. We can also choose those who vote against α or vote against β as $\nu_{\alpha\cdot\beta} = \nu_\alpha\eta_\beta + \nu_\alpha\mu_\beta + \eta_\alpha\nu_\beta + \mu_\alpha\nu_\beta + \nu_\alpha\nu_\beta + \rho_\alpha\nu_\beta + \nu_\alpha\rho_\beta = 1-(1-\nu_\alpha)(1-\nu_\beta)$. The rest products in Table 2 are considered as $\rho_{\alpha\cdot\beta} : \rho_{\alpha\cdot\beta} = \mu_\alpha\rho_\beta + \eta_\alpha\rho_\beta + \rho_\alpha\mu_\beta + \rho_\alpha\eta_\beta + \rho_\alpha\rho_\beta$. Also by explaining $\alpha^\lambda = \underbrace{\alpha \cdot \alpha \cdot \cdots \cdot \alpha}_{\lambda}$, we can obtain that

$$\alpha^\lambda = ((\mu_\alpha + \eta_\alpha)^\lambda - \eta_\alpha^\lambda, \eta_\alpha^\lambda, 1 - (1 - \nu_\alpha)^\lambda).$$

After simplify these expressions, we can introduce the following definition.

Definition 3.1 Let $\alpha = (\mu_\alpha, \eta_\alpha, \nu_\alpha)$ and $\beta = (\mu_\beta, \eta_\beta, \nu_\beta)$ be two picture fuzzy numbers, then

(1) $\alpha \cdot \beta = ((\mu_\alpha + \eta_\alpha)(\mu_\beta + \eta_\beta) - \eta_\alpha\eta_\beta, \eta_\alpha\eta_\beta, 1 - (1 - \nu_\alpha)(1 - \nu_\beta))$;

(2) $\alpha^\lambda = ((\mu_\alpha + \eta_\alpha)^\lambda - \eta_\alpha^\lambda, \eta_\alpha^\lambda, 1 - (1 - \nu_\alpha)^\lambda), \lambda > 0$.

When $\eta_\alpha = \eta_\beta = 0$, the above definition is reduced to Definition 2.3.

Based on Definition 3.1, we can verify the following properties easily.

Theorem 3.2 *Let $\alpha = (\mu_\alpha, \eta_\alpha, \nu_\alpha, \rho_\alpha)$, $\beta = (\mu_\beta, \eta_\beta, \nu_\beta, \rho_\beta)$ and $\gamma = (\mu_\gamma, \eta_\gamma, \nu_\gamma, \rho_\gamma)$ be three picture fuzzy numbers, then*

(1) $\alpha \cdot \beta = \beta \cdot \alpha$;

(2) $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$;

(3) $\alpha^{\lambda_1+\lambda_2} = \alpha^{\lambda_1} \cdot \alpha^{\lambda_2}$;

(4) $(\alpha \cdot \beta)^\lambda = \alpha^\lambda \cdot \beta^\lambda$.

In order to compare two PFNs, we introduce the following comparison laws.

Definition 3.3 Let $\alpha = (\mu_\alpha, \eta_\alpha, \nu_\alpha, \rho_\alpha)$ be a picture fuzzy numbers, then a score function S can be defined as $S(\alpha) = \mu_\alpha - \nu_\alpha$ and the accuracy function H is given by $H(\alpha) = \mu_\alpha + \eta_\alpha + \nu_\alpha$, where $S(\alpha) \in [-1, 1]$ and $H(\alpha) \in [0, 1]$. Then, for two picture fuzzy numbers α and β

(i) if $S(\alpha) > S(\beta)$, then α is superior to β, denoted by $\alpha \succ \beta$;

(ii) if $S(\alpha) = S(\beta)$, then

  (1) $H(\alpha) = H(\beta)$, implies that α is equivalent to β, denoted by $\alpha \sim \beta$;

  (2) $H(\alpha) > H(\beta)$, implies that α is superior to β, denoted by $\alpha \succ \beta$.

We also use voting as a good example to explain the above definition, where $S(\alpha) = \mu_\alpha - \nu_\alpha$ represents goal difference and $H(\alpha) = \mu_\alpha + \eta_\alpha + \nu_\alpha$ can be interpreted as the effective degree of voting. When $S(\alpha)$ increases, we can know that there are more people who vote for α and people who vote against α become less. When $H(\alpha)$ increases, we can know that there are more people who vote for or against α and people who refuse to vote become less. So, $H(\alpha)$ depicts the effective degree of voting.

4. Picture fuzzy geometric operators

In this section, we will define some picture fuzzy geometric aggregation operators based on the geometric mean.

Definition 4.1 Let p_j $(j = 1, 2, ..., n)$ be a collection of PFNs, then we define the picture fuzzy weighted geometric (PFWG) operator as below:

$$PFWG_w(p_1, p_2, ..., p_n) = \prod_{j=1}^{n} p_j^{w_j}$$

where $w = (w_1, w_2, ..., w_n)^T$ be the weight vector of p_j $(j = 1, 2, ..., n)$, and $w_j > 0$, $\sum_{j=1}^{n} w_j = 1$.

According to the operational laws of PFNs, we can get the following theorem.

Theorem 4.2 *Let $p_j = (\mu_j, \eta_j, \nu_j, \rho_j)$ $(j = 1, 2, ..., n)$ be a collection of PFNs, then their aggregated value by using the PFWG operator is also a PFNs, and*

$$PFWG_w(p_1, p_2, \cdots, p_n) = (\prod_{i=1}^{n}(\mu_i + \eta_i)^{w_i} - \prod_{i=1}^{n}\eta_i^{w_i}, \prod_{i=1}^{n}\eta_i^{w_i}, 1 - \prod_{i=1}^{n}(1 - \nu_i)^{w_i}).$$

Proof. By using mathematics induction on n, we prove Theorem 4.2 as follows.

(1) For $n = 2$:

With the operational laws of PFNs, we can get

$$p_1^{w_1} = (\mu_1 + \eta_1)^{w_1} - \eta_1^{w_1}, \eta_1^{w_1}, 1 - (1 - \nu_1)^{w_1}),$$
$$p_2^{w_2} = (\mu_2 + \eta_2)^{w_2} - \eta_2^{w_2}, \eta_2^{w_2}, 1 - (1 - \nu_2)^{w_2}).$$

Then, it follows that

$$p_1^{w_1} \cdot p_2^{w_2} = (\mu_1 + \eta_1)^{w_1}(\mu_2 + \eta_2)^{w_2} - \eta_1^{w_1}\eta_2^{w_2}, \eta_1^{w_1}\eta_2^{w_2}, 1 - (1 - \nu_1)^{w_1}(1 - \nu_2)^{w_2}).$$

Thus, Theorem 4.2 holds.

(2) If Theorem 4.2 holds for $n = k$, that is,

$$PFWG_w(p_1, p_2, \cdots, p_k) = (\prod_{i=1}^{k}(\mu_i + \eta_i)^{w_i} - \prod_{i=1}^{k}\eta_i^{w_i}, \prod_{i=1}^{k}\eta_i^{w_i}, 1 - \prod_{i=1}^{k}(1 - \nu_i)^{w_i}),$$

then, when $n = k + 1$, by the operational laws in Theorem 3.2, we have

$$\begin{aligned}
\prod_{i=1}^{k+1} p_i^{w_i} &= \prod_{i=1}^{k} p_i^{w_i} \cdot p_{k+1}^{w_{k+1}} \\
&= \left(\prod_{i=1}^{k}(\mu_i + \eta_i)^{w_i} - \prod_{i=1}^{k}\eta_i^{w_i}, \prod_{i=1}^{k}\eta_i^{w_i}, 1 - \prod_{i=1}^{k}(1 - \nu_i)^{w_i}\right) \\
&\cdot((\mu_{k+1} + \eta_{k+1})^{w_{k+1}} - \eta_{k+1}^{w_{k+1}}, \eta_{k+1}^{w_{k+1}}, 1 - (1 - \nu_{k+1})^{w_{k+1}}) \\
&= \prod_{i=1}^{k+1}(\mu_i + \eta_i)^{w_i} - \prod_{i=1}^{k+1}\eta_i^{w_i}, \prod_{i=1}^{k+1}\eta_i^{w_i}, 1 - \prod_{i=1}^{k+1}(1 - \nu_i)^{w_i}),
\end{aligned}$$

i.e., Theorem 4.2 holds for $n = k + 1$. Thus, by the principle of mathematical induction Theorem 4.2 holds for all n. Obviously,

$$\prod_{i=1}^{n}(\mu_i + \eta_i)^{w_i} - \prod_{i=1}^{n}\eta_i^{w_i}, \prod_{i=1}^{n}\eta_i^{w_i}, 1 - \prod_{i=1}^{n}(1 - \nu_i)^{w_i} \in [0, 1],$$
$$\left(\prod_{i=1}^{n}(\mu_i + \eta_i)^{w_i} - \prod_{i=1}^{n}\eta_i^{w_i}\right) + \prod_{i=1}^{n}\eta_i^{w_i} + \left(1 - \prod_{i=1}^{n}(1 - \nu_i)^{w_i}\right) \leq 1,$$

and the result of $PFWG_w(p_1, p_2, \cdots, p_n)$ is also a PFN.

Theorem 4.3 (Idempotency). *Let $p_j = (\mu_j, \eta_j, \nu_j)$ $(j = 1, 2, ..., n)$ be a collection of PFNs. If $p_1 = p_2 = \cdots = p_n = p$, then*

$$PFWG_w(p_1, p_2, ..., p_n) = p.$$

Proof. Let $p_1 = p_2 = \cdots = p_n = p = (\mu, \eta, \nu)$. By Theorem 4.2, we obtain

$$\begin{aligned} PFWG_w(p_1, p_2, \cdots, p_n) &= \left(\prod_{i=1}^{n} (\mu_i + \eta_i)^{w_i} - \prod_{i=1}^{n} \eta_i^{w_i}, \prod_{i=1}^{n} \eta_i^{w_i}, 1 - \prod_{i=1}^{n} (1 - \nu_i)^{w_i} \right) \\ &= \left(\prod_{i=1}^{n} (\mu + \eta)^{w_i} - \prod_{i=1}^{n} \eta^{w_i}, \prod_{i=1}^{n} \eta^{w_i}, 1 - \prod_{i=1}^{n} (1 - \nu)^{w_i} \right) \\ &= ((\mu + \eta)^{\sum_{i=1}^{n} w_i} - \eta^{\sum_{i=1}^{n} w_i}, \eta^{\sum_{i=1}^{n} w_i}, 1 - (1 - \nu)^{\sum_{i=1}^{n} w_i}). \end{aligned}$$

As $\sum_{i=1}^{n} w_i = 1$, we have

$$\widetilde{p} = PFWG_w(p_1, p_2, \cdots, p_n) = (\mu, \eta, \nu),$$

which completes the proof.

Theorem 4.4 (Boundedness). *Let $p_j = (\mu_j, \eta_j, \nu_j, \rho_j)$ $(j = 1, 2, ..., n)$ be a collection of PFNs. If $\eta_* = min_j\{\eta_j\}$, $\nu_* = min_j\{\nu_j\}$, $\rho_* = min_j\{\rho_j\}$, $\mu_* = 1 - \eta_* - \nu_* - \rho_*$, $p^* = (\mu_*, \eta_*, \nu_*, \rho_*)$ and $\eta^* = max_j\{\eta_j\}$, $\nu^* = max_j\{\nu_j\}$, $\rho^* = max_j\{\rho_j\}$, $\mu^* = 1 - \eta^* - \nu^* - \rho^*$, $p_* = (\mu^*, \eta^*, \nu^*, \rho^*)$ then*

$$p_* \leq PFWG_w(p_1, p_2, \cdots, p_n) \leq p^*.$$

Proof. By Theorem 4.2, we get

$$PFWG_w(p_1, p_2, \cdots, p_n) = \left(\prod_{i=1}^{n} (\mu_i + \eta_i)^{w_i} - \prod_{i=1}^{n} \eta_i^{w_i}, \prod_{i=1}^{n} \eta_i^{w_i}, 1 - \prod_{i=1}^{n} (1 - \nu_i)^{w_i} \right).$$

From the definition of p_*, p^*, we know

$$(\mu_i + \eta_i)^{w_i} = (1 - \nu_i - \rho_i)^{w_i} \leq 1 - \nu_* - \rho_*$$

$$(\mu_i + \eta_i)^{w_i} = (1 - \nu_i - \rho_i)^{w_i} \geq 1 - \nu^* - \rho^*$$

and

$$\eta_* \leq \eta \leq \eta^*$$

so

$$\prod_{i=1}^{n} (1 - \nu^* - \rho^*)^{w_i} - \prod_{i=1}^{n} \eta^{*w_i} \leq \prod_{i=1}^{n} (\mu_i + \eta_i)^{w_i} - \prod_{i=1}^{n} \eta_i^{w_i} \leq \prod_{i=1}^{n} (1 - \nu_* - \rho_*)^{w_i} - \prod_{i=1}^{n} \eta_*^{w_i}.$$

Using the condition $\sum_{j=1}^{n} w_j = 1$, we acquire

$$\mu^* = 1 - \nu^* - \rho^* - \eta^* \leq \prod_{i=1}^{n} (\mu_i + \eta_i)^{w_i} - \prod_{i=1}^{n} \eta_i^{w_i} \leq 1 - \nu_* - \rho_* - \eta_* = \mu_*.$$

Similarly, we obtain

$$\eta_* \leq \prod_{i=1}^{k} \eta_i^{w_i} \leq \eta^*$$

$$\nu_* \leq 1 - \prod_{i=1}^{n} (1-\nu_i)^{w_i} \leq \nu^*.$$

As

$$S(PFWG_w(p_1, p_2, \cdots, p_n)) = \left(\prod_{i=1}^{n} (\mu_i + \eta_i)^{w_i} - \prod_{i=1}^{n} \eta_i^{w_i} \right) - (1 - \prod_{i=1}^{n} (1-\nu_i)^{w_i})$$

we get

$$(1 - \eta^* - \nu^* - \rho^*) - \nu^* \leq S(PFWG_w(p_1, p_2, \cdots, p_n)) \leq (1 - \eta_* - \nu_* - \rho_*) - \nu_*.$$

In other words,

$$p_* \leq PFWG_w(p_1, p_2, \cdots, p_n) \leq p^*$$

which completes the proof.

Theorem 4.5 (Monotonicity). *Let* $p_j = (\mu_j, \eta_j, \nu_j, \rho_j)$ $(j = 1, 2, ..., n)$ *and* $p'_j = (\mu'_j, \eta'_j, \nu'_j, \rho'_j)$ $(j = 1, 2, ..., n)$ *be two collections of PFNs. If* $\eta_j \leq \eta'_j$, $\nu_j \leq \nu'_j$, $\rho_j \leq \rho'_j$, $\forall 1 \leq j \leq n$, *then*

$$PFWG_w(p_1, p_2, ..., p_n) \geq PFWG_w(p'_1, p'_2, ..., p'_n).$$

Proof. Since $\eta_j \leq \eta'_j,\ \nu_j \leq \nu'_j,\ \rho_j \leq \rho'_j,\ \forall 1 \leq j \leq n$, we obtain that

$$\begin{aligned} \prod_{i=1}^{n} (\mu_i + \eta_i)^{w_i} - \prod_{i=1}^{n} \eta_i^{w_i} &= \prod_{i=1}^{n} (1 - \nu_i - \rho_i)^{w_i} - \prod_{i=1}^{n} \eta_i^{w_i} \\ &\geq \prod_{i=1}^{n} (1 - \nu'_i - \rho'_i)^{w_i} - \prod_{i=1}^{n} \eta_i^{'w_i} = \prod_{i=1}^{n} (\mu'_i + \eta'_i)^{w_i} - \prod_{i=1}^{n} \eta_i^{'w_i}. \end{aligned}$$

Apparently

$$1 - \prod_{i=1}^{n} (1-\nu_i)^{w_i} \leq 1 - \prod_{i=1}^{n} (1-\nu'_i)^{w_i}.$$

Following this way, we have

$$S(PFWG_w(p_1, p_2, ..., p_n)) \geq S(PFWG_w(p'_1, p'_2, ..., p'_n)).$$

Then, we complete the proof.

When we need to weight the ordered positions of the picture fuzzy arguments instead of weighting the arguments themselves, PFWG can be generalized to PFOWG.

Definition 4.6 . Let p_j $(j = 1, 2, ..., n)$ be a collection of PFNs, then we define the picture fuzzy ordered weighted geometric (PFOWG) operator as below:

$$PFOWG_w(p_1, p_2, ..., p_n) = \prod_{j=1}^{n} p_{\sigma(j)}^{w_j}$$

where $w = (w_1, w_2, ..., w_n)^T$ be the weight vector of p_j $(j = 1, 2, ..., n)$, and $w_j > 0$, $\sum_{j=1}^{n} w_j = 1$.

According to the operational laws of PFNs, we can get the following theorems. As their proofs are similar to the ones listed above, we omit them here.

Theorem 4.7 *Let $p_j = (\mu_j, \eta_j, \nu_j, \rho_j)$ $(j = 1, 2, ..., n)$ be a collection of PFNs, then their aggregated value by using the PFWG operator is also a PFNs, and*

$$\begin{aligned} &PFOWG_w(p_1, p_2, ..., p_n) \\ &= \left(\prod_{i=1}^{n} (\mu_{\sigma(i)} + \eta_{\sigma(i)})^{w_i} - \prod_{i=1}^{n} \eta_{\sigma(i)}^{w_i}, \prod_{i=1}^{n} \eta_{\sigma(i)}^{w_i}, 1 - \prod_{i=1}^{n} (1 - \nu_{\sigma(i)})^{w_i} \right). \end{aligned}$$

Theorem 4.8 (Idempotency). *Let p_j $(j = 1, 2, ..., n)$ be a collection of PFNs. If $p_1 = p_2 = \cdots = p_n = p$, then*

$$PFOWG_w(p_1, p_2, ..., p_n) = p.$$

Theorem 4.9 (Boundedness). *Let $p_j = (\mu_j, \eta_j, \nu_j, \rho_j)$ $(j = 1, 2, ..., n)$ be a collection of PFNs. If $\eta_* = min_j\{\eta_j\}$, $\nu_* = min_j\{\nu_j\}$, $\rho_* = min_j\{\rho_j\}$, $\mu_* = 1 - \eta_* - \nu_* - \rho_*$, $p^* = (\mu_*, \eta_*, \nu_*, \rho_*)$ and $\eta^* = max_j\{\eta_j\}$, $\nu^* = max_j\{\nu_j\}$, $\rho^* = max_j\{\rho_j\}$, $\mu^* = 1 - \eta^* - \nu^* - \rho^*$, $p_* = (\mu^*, \eta^*, \nu^*, \rho^*)$ then*

$$p_* \leq PFOWG_w(p_1, p_2, ..., p_n) \leq p^*.$$

Theorem 4.10 (Monotonicity). *Let $p_j = (\mu_j, \eta_j, \nu_j, \rho_j)$ $(j = 1, 2, ..., n)$ and $p'_j = (\mu'_j, \eta'_j, \nu'_j, \rho'_j)$ $(j = 1, 2, ..., n)$ be two collections of PFNs. If $\eta_j \leq \eta'_j$, $\nu_j \leq \nu'_j$, $\rho_j \leq \rho'_j$, $\forall 1 \leq j \leq n$, then*

$$PFWOG_w(p_1, p_2, ..., p_n) \geq PFOWG_w(p'_1, p'_2, ..., p'_n).$$

Theorem 4.11 (Commutativity) *Let p_j $(j = 1, 2, ..., n)$ then*

$$PFOWG_w(p_{\sigma(1)}, p_{\sigma(2)}, ..., p_{\sigma(n)}) = PFOWG_w(p_1, p_2, ..., p_n)$$

where $(\sigma(1), \sigma(2), ..., \sigma(n))$ is any permutation of $(1, 2, ..., n)$.

When we need to weight both the ordered positions of the picture fuzzy arguments and the arguments themselves, PFWG can be generalized to the following picture fuzzy hybrid geometric operator.

Definition 4.12 . Let p_j $(j = 1, 2, ..., n)$ be a collection of PFNs, then we define the picture fuzzy hybrid geometric (PFHG) operator as below:

$$PFHG_w(p_1, p_2, ..., p_n) = \prod_{j=1}^{n} \widetilde{p}_{\sigma(j)}^{w_j}$$

where $w = (w_1, w_2, ..., w_n)^T$ is the associated weighting vector of p_j $(j = 1, 2, ..., n)$ with $w_j > 0$,$\sum_{j=1}^{n} w_j = 1$ and $p_{\sigma(j)}$ is the j-th largest element of the picture fuzzy aruguments ($\widetilde{p}_j = p_j^{n\omega_j}$, ω_j is the weighting vector of picture fuzzy arguments p_j with $\omega_j > 0$, $\sum_{j=1}^{n} \omega_j = 1$ and n is the balancing coefficient).

Based on the operations of the PFNs, we can drive the following theorem which is similar to 4.2.

Theorem 4.13 *Let $p_j = (\mu_j, \eta_j, \nu_j, \rho_j)$ $(j = 1, 2, ..., n)$ be a collection of PFNs, then their aggregated value by using the PFWG operator is also a PFNs, and*

$$\begin{aligned} &PFHG_w(p_1, p_2, ..., p_n) \\ &= \left(\prod_{i=1}^{n} (\widetilde{\mu}_{\sigma(i)} + \widetilde{\eta}_{\sigma(i)})^{w_i} - \prod_{i=1}^{n} \widetilde{\eta}_{\sigma(i)}^{w_i}, \prod_{i=1}^{n} \widetilde{\eta}_{\sigma(i)}^{w_i}, 1 - \prod_{i=1}^{n} (1 - \widetilde{\nu}_{\sigma(i)})^{w_i} \right). \end{aligned}$$

5. An approach to multiple attribute decision making with picture fuzzy information

In this section, we shall utilize the proposed operators to multiple attribute decision making under picture fuzzy environment. As their procedures are similar, we only consider the PFWG operator here.

The following assumptions or notations are used to represent the MADM problems for evaluation of alternatives with picture fuzzy information. Let $A = \{A_1, A_2, ..., A_m\}$ be a set of m alternatives and $G = \{G_1, G_2, ..., G_n\}$ be a set of n attributes. If the decision makers provide values for the alternative A_i under the attribute G_j with anonymity, these values can be considered as a picture fuzzy element p_{ij}. Suppose that the decision matrix $P = (p_{ij})_{m \times n}$ is the picture fuzzy decision matrix, where p_{ij} $(i = 1, 2, ..., m,\ j = 1, 2, ..., n)$ are in the form of PFNs. In the following, we apply the PFWG operator to the MADM problems for evaluation of alternatives with picture fuzzy information.

Step 1. We utilize the decision information given in matrix P, and the PFWG operator

$$\begin{aligned} \widetilde{p}_i &= PFWG_w(p_{i1}, p_{i2}, ..., p_{in}) = \prod_{j=1}^{n} p_{ij}^{w_j} \\ &= \left(\prod_{i=1}^{n} (\mu_i + \eta_i)^{w_i} - \prod_{i=1}^{n} \eta_i^{w_i}, \prod_{i=1}^{n} \eta_i^{w_i}, 1 - \prod_{i=1}^{n} (1 - \nu_i)^{w_i} \right) \end{aligned}$$

to derive the overall preference values $\widetilde{p}_i$ $(i = 1, 2, ..., m)$ of the alternative A_i.

Step 2. Calculate the scores $S(\widetilde{p}_i)$ $(i = 1, 2, ..., m)$ of the overall picture fuzzy values $\widetilde{p}_i$ by Definition 3.3.

Step 3. Rank all the alternatives A_i $(i = 1, 2, ..., m)$ in accordance with the values of $S(\widetilde{p}_i)$ $(i = 1, 2, ..., m)$ and select the best one(s).

Step 4. End.

6. Numerical example

In this section, we will present a numerical example (adapted from [15]) to show evaluation of theses with picture fuzzy information in order to illustrate the proposed method.

Suppose there are five theses A_i $(i = 1, 2, ..., 5)$, and we want to select the best one. Four attributes are selected by experts to evaluate the theses: (1) G_1 is the language of a thesis; (2) G_2 is the innovation; (3) G_3 is the rigor; (4) G_4 is the structure of the thesis. In order to avoid influence each other, the experts are required to evaluate the five theses A_i $(i = 1, 2, ..., 5)$ under the above four attributes in anonymity. The decision matrix $P = (p_{ij})_{5\times 4}$ is presented in Table 3, where p_{ij} $(i = 1, 2, 3, 4, 5,\ j = 1, 2, 3, 4)$ are in the form of PFNs.

Table 3. Picture fuzzy decision matrix P

	G_1	G_2	G_3	G_4
A_1	(0.2,0.3,0.1,0.4)	(0.7,0.1,0.1,0.1)	(0.1,0.2,0.6,0.1)	(0.4,0.1,0.2,0.3)
A_2	(0.4,0.2,0.3,0.1)	(0.1,0.6,0.1,0.2)	(0.3,0.2,0.4,0.1)	(0.3,0.1,0.4,0.2)
A_3	(0.2,0.5,0.1,0.2)	(0.6,0.1,0.1,0.2)	(0.5,0.1,0.2,0.2)	(0.5,0.1,0.3,0.1)
A_4	(0.2,0.3,0.1,0.4)	(0.6,0.2,0.1,0.1)	(0.5,0.3,0.2,0)	(0.5,0,0.3,0.2)
A_5	(0.6,0.1,0.2,0.1)	(0.4,0.2,0.3,0.1)	(0.6,0.1,0.2,0.1)	(0.3,0.4,0.2,0.1)

The information about the attribute weights is known as: $w = (0.2, 0.4, 0.1, 0.3)$.

6.1. The decision making steps

Now, we apply the developed approach to evaluate these theses with picture fuzzy information.

Step 1. Utilize the decision information given in matrix P and

$$\widetilde{p}_i = PFWG_w(p_{i1}, p_{i2}, ..., p_{in}),$$

we have

$$\begin{aligned}
\widetilde{p}_1 &= (0.439853, 0.133514, 0.198915),\\
\widetilde{p}_2 &= (0.302763, 0.252098, 0.272259),\\
\widetilde{p}_3 &= (0.520169, 0.137973, 0.175133),\\
\widetilde{p}_4 &= (0.632456, 0, 0.175133),\\
\widetilde{p}_5 &= (0.458142, 0.2, 0.241609).
\end{aligned}$$

Step 2. Calculate the scores $S(\widetilde{p}_i)$ $(i = 1,2,3,4)$ of the overall picture fuzzy preference values $\widetilde{p}_i$ by Definition 3.3:

$$S(\widetilde{p}_1) = 0.240938,$$
$$S(\widetilde{p}_2) = 0.0305047,$$
$$S(\widetilde{p}_3) = 0.345036,$$
$$S(\widetilde{p}_4) = 0.457323,$$
$$S(\widetilde{p}_5) = 0.216533.$$

Step 3. Rank all the alternatives $A_i(i = 1,2,...,4)$ in accordance with the values of $S(\widetilde{p}_i)$: $A_4 \succ A_3 \succ A_1 \succ A_5 \succ A_2$. Note that $\succ$ means "preferred to". Thus, the best thesis is A_4.

6.2. Comparative analysis

The proposed method has several advantages as below.

First, our method can accommodate situations in which the input arguments are picture fuzzy numbers. As mentioned before, picture fuzzy set is a generalized set containing FS and IFS as its special cases. Thus, our method can be widely used.

Second, we can compare our method with IFWG [23]. Here, we have to translate data in Table 3 into intuitionistic fuzzy numbers (IFNs). For example, the fist PFN $(0.2, 0.3, 0.1, 0.4)$ should be changed into IFN $(0.2, 0.1)$. Then, we omit the process of calculation and list the results in Table 4.

Table 4. Scores for alternatives obtained by $IFWG$ operator

	The overall intuitionistic fuzzy values	Scores	Order
A_1	(0.379196, 0.198915)	0.18028	3
A_2	(0.204767, 0.272259)	-0.0674913	5
A_3	(0.447769, 0.175133)	0.272636	1
A_4	(0.447769, 0.175133)	0.272636	1
A_5	(0.414387, 0.241609)	0.172778	4

Obviously, A_3 and A_4 can not be distinguished by $IFWG$ while they can be distinguished by $PFWG$. This indicates that the picture fuzzy set takes much more information and our method is meaningful.

7. Conclusion

In this paper, we have investigated the multiple attribute decision making (MADM) problems based on the PFWG, PFOWG and PFHG operators with picture fuzzy information. Firstly, some basic concepts related to picture fuzzy set have been reviewed. Then, from the view of probability, some new operations on picture fuzzy sets have been developed. At the same time we have discussed their basic properties. Furthermore, we have discussed the picture fuzzy geometric operators and applied these new picture fuzzy operators to multiple attribute decision making problems in which attribute values take the form of picture fuzzy information. Finally, an illustrative example for evaluation of theses has been given to demonstrate the validity and applicability of the new approach. There are some other generalizations of these basic aggregation operators such as generalized weighted aggregation operator and Bonferroni mean, which can also be used to construct new operators for PFSs. The researches about these new operators may be interesting and meaningful.

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References

[1] Atanassov, K.T., *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems 20 (1986), 87-96.

[2] Atanassov, K.T., *Intuitionistic Fuzzy Sets*, Springer-Verlag, Heidelberg, 1999.

[3] Atanassov, K.T., Georgiev, C., *Intuitionistic fuzzy prolog*, Fuzzy Sets and Systems, 53 (1993), 121-128.

[4] Bhatia, N., Kumar, A., *A new method for sensitivity analysis of fuzzy transportation problems*, Journal of Intelligent and Fuzzy Systems, 25 (2013), 167-175.

[5] Cuong, B.C., *Picture fuzzy sets-first results*. Part 1: Seminar "Neuro-Fuzzy Systems with Applications", Preprint 03/2013, Institute of Mathematics, Hanoi, May 2013.

[6] Cuong, B.C., *Picture fuzzy sets-first results*. Part 2: Seminar "Neuro-Fuzzy Systems with Applications", Preprint 03/2013, Institute of Mathematics, Hanoi, May 2013.

[7] Cuong, B.C., Kreinovich, V., *Picture Fuzzy Sets-a new concept for computational intelligence problems* (2013). Departmental Technical Reports (CS). Paper 809. http://digitalcommons.utep.edu/cs-techrep/809.

[8] De, S.K., Biswas, R., Roy, A.R., *An application of intuitionistic fuzzy sets in medical dianosis*, Fuzzy Sets and Systems, 117 (2001), 209-213.

[9] Hung, W.L., Yang, M.S., *Similartity measures of intuitionistic fuzzy sets based on Hausdorff distance*, Pattern Recognition Letters, 25 (2004), 1603-1611.

[10] Hung, W.L., Yang, M.S., *Similartity measures of intuitionistic fuzzy sets based on LP metric*, International Journal of Approximate Reasoning, 46 (2007), 120-136.

[11] Kumar, A., Kaur, A., *Optimization for different types of transportation problems with fuzzy coefficients in the objective function*, Journal of Intelligent and Fuzzy Systems, 23 (2012), 237-248.

[12] Li, D.F., *Multiattribute decision making models and methods using intuitionistic fuzzy sets*, Journal of Computer and Systems Sciences, 70 (2005), 73-85.

[13] Li, D.F., *Extension of the LINMAP for multiattribute decision making under Atanassov's intuitionistic fuzzy environment*, Fuzzy Optimization and Decision Making, 7 (2008), 17-34.

[14] Liu, H.W., *Multi-criteria decision-making methods based on intuitionistic fuzzy sets*, European Journal of Operational Research, 179 (2007), 220-233.

[15] Singh, P., *Correlation coefficients for picture fuzzy sets*, Journal of Intelligent and Fuzzy Systems, Doi:10.3233/IFS-141338.

[16] Szmidt, E., Kacprzyk, J., *A similartity measure for intuitionistic fuzzy sets and its application in supporting medical diagnostic reasoning*, Lecture Notes in Artificial Intelligence, 3070 (2004), 388-393.

[17] Wang, Z., Xu, Z.S., Liu, S.S., et al., *A netting clustering analysis method under intuitionistic fuzzy environment*, Applied Soft Computing, 11 (2011), 5558-5564.

[18] Xu, Z.S., Chen, J., Wu, J.J., *Clustering algorithm for intuitionistic fuzzy sets*, Information Sciences, 178 (2008), 3775-3790.

[19] Xu, Z.S., Yager, R.R., *Some geometric aggregation operators based on intuitionistic fuzzy set*, International Journal of General Systems, 35 (2006), 417-433.

[20] Xu, Z.S., *Models for multiple attribute decision making with intuitionistic fuzzy information*, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, 15 (2007), 285-297.

[21] Xu, Z.S., *Intuitionistic preference relations and their application in group decision making*, Information Sciences, 177 (2007), 2263-2379.

[22] Xu, Z.S., *Intuitionistic fuzzy aggregation operators*, IEEE Transactions on Fuzzy Systems, 15 (2007), 1179-1187.

[23] Xu, Z.S., Yager, R.R., *Some geometric aggregation operators based on intuitionistic fuzzy sets*, Int. J. Gen. Syst., 35 (2006), 417-433.

[24] Zadeh, L.A., *Fuzzy sets*, Information and Control, 8 (1965), 338-353.

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ON CHARACTERIZATIONS OF BL-ALGEBRAS VIA IMPLICATIVE IDEALS

Yongwei Yang[1]

School of Mathematics and Statistics
Anyang Normal University
Anyang, 455000
China
e-mail: yangyw2010@126.com

Xiaolong Xin

School of Mathematics
Northwest University
Xi'an, 710127
China
e-mail: xlxin@nwu.edu.cn

Abstract. In the paper, we introduce the concept of implicative ideals in BL-algebras by the pseudo implication operation and show some characterizations of ideals. We prove that implicative ideals coincide with Boolean ideals through analyzing the characterizations of implicative ideals. Finally, we consider the concepts of maximal ideals and investigate the relationships among the introduced ideals.

Keywords: BL-algebra, Boolean ideal, implicative ideal, maximal ideal.

1. Introduction

BL-algebras as the algebraic structures for Hájek's basic logic were raised from the continuous t-norm, familiar in the fuzzy logic framework [4]. The main examples of BL-algebras are from the unit interval endowed with continuous t-norms. BL-algebras arising as Lindenbaum BL-algebras from certain logic axioms have close relationships with the quantum structures. In fact, MV-algebras are BL-algebras while BL-algebras with the double negation are MV-algebras. It has been proved that MV-algebras are categorically equivalent to bounded commutative BCK-algebras [15] and a BL-algebra is a particular case of a reversed left BCK-algebra [6]. The study of these algebras have been carried out from both logic and algebraic standpoints.

[1]Corresponding author.

The filter theory plays an important role in studying these algebras. From logic point of view, various filters have natural interpretation as various sets of provable formulas. At present, the filter theory of BL-algebras has been widely studied and some important results are obtained. Hájek [4] introduced the notions of filters and prime filters in BL-algebras and proved the completeness of Basic Logic BL. Turunen studied the filter theory of BL-algebras and proposed the concepts of implicative filters and Boolean filters, which are called deductive systems to emphasize the fact that they correspond to sets of provable formulas and are closed with respect to modus ponens [16], [17]. It turned out that Boolean deductive systems coincide with implicative deductive systems in BL-algebras [17]. In particular, some types of filters such as (positive) implicative filters and fantastic filters [5], integral filters [3] and obstinate filters [1] were introduced and some of their characterizations were presented in [2], [9]. Moreover, based on the fuzzy set theory, the related fuzzy structures of filters in BL-algebras were further investigated [12], [18]. Besides BL-algebras, the filter theories of other algebraic structures such as R_0-algebras [14] and resituated lattices [13], [8], [19] which are closely related to BL-algebras, had been investigated by several researchers.

Since MV-algebras are BL-algebras, it is nature to generalize some notions of MV-algebras to BL-algebras. In MV-algebras, filters and ideals are dual notions, while some papers claimed that the notion of ideals is missing in BL-algebras for lack of a suitable algebraic addition [16], [17]. To fill the gap, Lele and Nganou [11] introduced the notions of ideals, prime ideals and Boolean ideals in BL-algebras and derived some characterizations of them. They also investigated the relationship between ideals and filters by exploiting the set of complements. The results derived from [11] show that Lele and Nganou 's ideals play an important role in the characterizations of BL-algebras, while compared with the filter theory, the notion of ideals has no corresponding deductive types and it is very intricate to introduce other types of ideals, such as implicative ideals etc., based on the original operations. Therefore it is meaningful to introduce new operation and give the deductive type of ideals in BL-algebras.

The aim of the paper is to investigate the ideals further more. We present some characterizations of ideals, prime ideals and Boolean ideals by using pseudo implication operation. It is known that Boolean filters coincide with implicative filters in BL-algebras. In line with the result, we propose the notion of implicative ideals and prove that implicative ideals coincide with Boolean ideals by the derived characterizations of implicative ideals. By introducing maximal ideals, we investigate the relationships among these ideals.

2. Preliminaries

In this section, for purpose of reference, we present some definitions and results about BL-algebras.

Definition 2.1 [4] An algebra $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ is called a BL-algebra if it satisfies the following conditions: for all $x, y, z \in L$,

(BL-1) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice,
(BL-2) $(L, \odot, 1)$ is a commutative monoid,
(BL-3) $x \odot y \leq z$ if and only if $x \leq y \to z$,
(BL-4) $x \odot (x \to y) = x \wedge y$,
(BL-5) $(x \to y) \vee (y \to x) = 1$.

Let L be a BL-algebra. If $x \vee \bar{x} = 1$ for any $x \in L$, then L is called a Boolean algebra, where $\bar{x} = x \to 0$; if L satisfies the double negation, i.e., $\bar{\bar{x}} = x$ for any $x \in L$, then L is called an MV-algebra; if $x^2 = x \odot x = x$ for any $x \in L$, then L is called a Gödel algebra.

The following properties are well known to hold in BL-algebras, we summarize them as follows.

Lemma 2.2 [16], [11] *In any BL-algebra $(L, \wedge, \vee, \odot, \to, 0, 1)$, the following relations hold: for any $x, y, z \in L$,*

(1) $x \leq y$ *if and only if* $x \to y = 1$, $x \odot y = 0$ *if and only if* $x \leq \bar{y}$;

(2) $x \odot (x \to y) \leq y$, $x \odot y \leq x \wedge y$, $x \leq y \to x$;

(3) $x \to y \leq (y \to z) \to (x \to z)$, $y \to x \leq (z \to y) \to (z \to x)$;

(4) $(x \to y) \odot (y \to z) \leq (x \to z)$, $(x \to y) \to (x \to z) \leq x \to (y \to z)$;

(5) $x \to (y \to z) = (x \odot y) \to z = y \to (x \to z)$;

(6) $\bar{0} = 1$, $\bar{1} = 0$, $1 \to x = x$, $x \to 1 = 1$, $x \odot \bar{x} = 0$, $x \leq \bar{\bar{x}}$, $\bar{x} = \bar{\bar{\bar{x}}}$;

(7) *if* $x \vee \bar{x} = 1$, *then* $x \wedge \bar{x} = 0$;

(8) $x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x)$;

(9) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$, $(x \wedge y) \to z = (x \to z) \vee (y \to z)$;

(10) $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$, $x \odot (y \wedge z) = (x \odot y) \wedge (x \odot z)$;

(11) *if* $x \leq y$, *then* $y \to z \leq x \to z$, $z \to x \leq z \to y$, $x \odot z \leq y \odot z$;

(12) $\overline{x \odot y} = x \to \bar{y}$, $\overline{x \vee y} = \bar{x} \wedge \bar{y}$, $\overline{x \wedge y} = \bar{x} \vee \bar{y}$;

(13) $\overline{\overline{x \odot y}} = \bar{\bar{x}} \odot \bar{\bar{y}}$, $\overline{\overline{x \to y}} = \bar{\bar{x}} \to \bar{\bar{y}}$, $\overline{\overline{x \vee y}} = \bar{\bar{x}} \vee \bar{\bar{y}}$, $\overline{\overline{x \wedge y}} = \bar{\bar{x}} \wedge \bar{\bar{y}}$.

In order to introduce the notion of ideals in BL-algebras as a generalization of the existing notion in MV-algebras, Lele and Nganou adopted the pseudo addition operation in a BL-algebra L: $x \oslash y := \bar{x} \to y$, for any $x, y \in L$, and gave the concept of ideals in BL-algebras as follows.

Definition 2.3 [11] Let $(L, \wedge, \vee, \odot, \to, 0, 1)$ be a BL-algebra and I a nonempty subset of L. I is called an ideal if it satisfies: for any $x, y \in L$,

(1) $x, y \in I$ implies $x \oslash y \in I$,

(2) if $x \leq y$ and $y \in I$, then $x \in I$.

From the above definition, it is easy to see that $0 \in I$, and $x \in I$ if and only if $\bar{\bar{x}} \in I$ for any $x \in L$. It is easy to prove that $\{0\}$ is an ideal of L. If a BL-algebra L is not an MV-algebra, then there exists an element $x \in L$ such that $\bar{\bar{x}} \neq x$, i.e., $x \oslash 0 \neq 0 \oslash x$, hence the operation $\oslash$ is not commutative in general.

Let P be an ideal of a BL-algebra L. If $\overline{x \to y} \in I$ or $\overline{y \to x} \in P$ for any $x, y \in L$, then P is called a prime ideal; if $x \wedge \bar{x} \in P$ for any $x \in L$, then P is called a Boolean ideal [11].

Proposition 2.4 [11] *An ideal P of a BL-algebra L is a prime ideal if and only if $x \wedge y \in P$ implies $x \in P$ or $y \in P$, for any $x, y \in L$.*

Theorem 2.5 [11] *Let I be an ideal of a BL-algebra L. Define the relation $\sim_I$ on L by: for any $x, y \in L$,*

$$x \sim_I y \text{ if and only if } \bar{x} \odot y \in I \text{ and } x \odot \bar{y} \in I.$$

Then $\sim_I$ is a congruence relation on L.

Let L be a BL-algebra and I an ideal of L, the set of all congruence classes is denoted by L/I, that is, $L/I := \{[x] | x \in L\}$, where $[x] = \{y \in L | x \sim_I y\}$.

For any $x, y \in L$, the operations $\star$, $\mapsto$, $\sqcap$, $\sqcup$ on L/I are defined as follows: $[x] \sqcap [y] = [x \wedge y]$, $[x] \sqcup [y] = [x \vee y]$, $[x] \star [y] = [x \odot y]$, $[x] \mapsto [y] = [x \to y]$. It follows that $(L/I, \sqcap, \sqcup, \star, \mapsto, [0], [1])$ is a BL-algebra which is called a quotient BL-algebra with respect to I.

Proposition 2.6 [11] *Let L be a BL-algebra and I an ideal of L. Then L/I is an MV-algebra.*

3. Some characterizations of ideals

Compared with filter theory, we observe that the notion of ideals has no corresponding deductive type in BL-algebras. In order to fill the gap, we introduce a new operation and give another description of ideals in BL-algebras.

Let L be a BL-algebra, we define the pseudo implication operation $\rightharpoonup$ by $x \rightharpoonup y := x \odot \bar{y}$, for any $x, y \in L$. It is easy to see that $z \leq x \oslash y$ if and only if $z \rightharpoonup x \leq y$.

We can establish some properties of the operation $\rightharpoonup$ as follows.

Lemma 3.1 *Let L be a BL-algebra, for any $x, y, z \in L$, we have:*

(1) *$x \leq y$ implies $z \rightharpoonup y \leq z \rightharpoonup x$ and $x \rightharpoonup z \leq y \rightharpoonup z$;*

(2) *$(x \rightharpoonup y) \rightharpoonup z = (x \rightharpoonup z) \rightharpoonup y = x \rightharpoonup (y \oslash z)$;*

(3) *$x \rightharpoonup 0 = x$, $0 \rightharpoonup x = 0$, $x \rightharpoonup x = 0$;*

(4) *$(x \rightharpoonup z) \rightharpoonup (y \rightharpoonup z) \leq x \rightharpoonup y$;*

(5) *$(x \rightharpoonup z) \leq (y \rightharpoonup z) \oslash (x \rightharpoonup y)$;*

(6) $x \rightharpoonup y \leq x$, $x \leq y$ *implies* $x \rightharpoonup y = 0$;

(7) $x \rightharpoonup y = 0$ *implies* $\bar{y} \leq \bar{x}$ *and* $x \leq \bar{\bar{y}}$;

(8) $(y \rightharpoonup (y \rightharpoonup x)) \rightharpoonup (x \rightharpoonup (x \rightharpoonup y)) = 0$.

Proof. By Lemma 2.2, routine computations prove the above results. ∎

We would like to point out that the operation $\rightharpoonup$ is not commutative in general, since $1 \rightharpoonup 0 = 1 \neq 0 \rightharpoonup 1 = 0$.

By using the operation $\rightharpoonup$, we can give a concrete description of ideals in BL-algebras as follows.

Lemma 3.2 [11] *Let I be a nonempty subset of a BL-algebra L. Then I is an ideal of L if and only if it satisfies:*

(1) $0 \in I$,

(2) *for any $x, y \in L$, if $x \rightharpoonup y \in I$ and $y \in I$, then $x \in I$.*

Lemma 3.3 *Let I be an ideal of a BL-algebra L. Then the followings hold: for any $x, y, z \in L$,*

(1) $x \rightharpoonup y \in I$ *if and only if* $\bar{y} \rightharpoonup \bar{x} \in I$;

(2) $x \rightharpoonup y \in I$ *if and only if* $\bar{\bar{x}} \rightharpoonup y \in I$;

(3) $(y \rightharpoonup \bar{x}) \rightharpoonup z \in I$ *if and only if* $(\bar{z} \rightharpoonup \bar{y}) \rightharpoonup \bar{x} \in I$.

Proof. (1) Suppose that $x \rightharpoonup y \in I$. Since $(\bar{y} \rightharpoonup \bar{x}) \rightharpoonup (x \rightharpoonup y) = \bar{y} \odot \bar{\bar{x}} \odot \overline{x \odot \bar{y}} = (\bar{y} \wedge \bar{x}) \odot \bar{\bar{x}} = 0 \in I$ and I is an ideal, thus $\bar{y} \rightharpoonup \bar{x} \in I$.

Conversely, suppose that $\bar{y} \rightharpoonup \bar{x} \in I$. Since $(x \rightharpoonup y) \rightharpoonup (\bar{y} \rightharpoonup \bar{x}) = x \odot \bar{y} \odot \overline{\bar{y} \odot \bar{\bar{x}}} = (\bar{y} \wedge \bar{x}) \odot x = 0 \in I$, therefore $x \rightharpoonup y \in I$.

The proofs of (2) and (3) are similar to that of (1). ∎

Theorem 3.4 *Let L be a BL-algebra. A nonempty subset I of L is an ideal if and only if it satisfies the conditions:*

(1) $0 \in I$,

(2) *for any $x, y \in L$, if $y \in I$ and $\overline{\bar{y} \to \bar{x}} \in I$, then $x \in I$.*

Proof. Suppose that I is an ideal of L. By Lemma 3.2, we get $0 \in I$. For any $x, y \in L$, if $y \in I$ and $\overline{\bar{y} \to \bar{x}} \in I$, that is $y \in I$ and $\overline{\overline{x \rightharpoonup y}} \in I$, then $y \in I$ and $x \rightharpoonup y \in I$. Hence $x \in I$, and so I satisfies the conditions (1) and (2).

Conversely, suppose that the nonempty subset I satisfies the conditions (1) and (2). For any $x, y \in L$, if $x \in I$, then $\bar{\bar{x}} \in I$. In fact, since $\overline{\bar{x} \to \bar{\bar{\bar{x}}}} = \overline{\bar{x} \to \bar{x}} = 0 \in I$ and $x \in I$, hence $\bar{\bar{x}} \in I$. Let $x \rightharpoonup y \in I$ and $y \in I$. Then $\overline{\overline{x \rightharpoonup y}} = \overline{\bar{y} \to \bar{x}} \in I$, and so $x \in I$. Therefore I is an ideal. ∎

Let L be a BL-algebra, we denote by $L(x, y) = \{z \in L | z \rightharpoonup x \leq y\}$ for any $x, y \in L$. Our next aim is to establish further characterizations of ideals.

Proposition 3.5 *Let I be a nonempty subset of a BL-algebra L. Then the following conditions are equivalent:*

(1) *I is an ideal of L,*

(2) *$L(x, y) \subseteq I$, for any $x, y \in I$,*

(3) *$(z \rightharpoonup x) \rightharpoonup y = 0$ implies $z \in I$, for any $x, y \in I$ and $z \in L$.*

Proof. (1) $\Rightarrow$ (2) Let I be an ideal. For any $x, y \in I$, if $z \in L(x, y)$, then $z \rightharpoonup x \leq y$. By Definition 2.3 and Lemma 3.2, we have $z \in I$, hence $L(x, y) \subseteq I$.

(2) $\Rightarrow$ (3) It is obvious.

(3) $\Rightarrow$ (1) Assume that $(z \rightharpoonup x) \rightharpoonup y = 0$ implies $z \in I$, for any $x, y \in I$ and $z \in L$. Since I is a nonempty subset, then there exists $x \in I$. By Lemma 3.1, we get that $(0 \rightharpoonup x) \rightharpoonup x = 0$, therefore $0 \in I$. Let $x \rightharpoonup y \in I$ and $y \in I$. It follows from Lemma 3.1, $(x \rightharpoonup (x \rightharpoonup y)) \rightharpoonup y = (x \rightharpoonup y) \rightharpoonup (x \rightharpoonup y) = 0$, hence $x \in I$, and so I is an ideal. ∎

The following result will display that the operation $\rightharpoonup$ plays an important role in the ideal theory of BL-algebras.

Proposition 3.6 *Let P be a proper ideal of a BL-algebra L. Then P is a prime ideal if and only if $x \rightharpoonup y \in P$ or $y \rightharpoonup x \in P$, for any $x, y \in L$.*

Proof. Indeed, P is prime if and only if $\overline{x \rightarrow y} \in P$ or $\overline{y \rightarrow x} \in P$, for all $x, y \in L$. Now, it follows from Lemma 2.2 (13) that $\overline{x \rightarrow y} \in P$ if and only if $\overline{\bar{x} \rightarrow \bar{y}} \in P$ for all $x, y \in L$. The result is now clear since $a \in P$ if and only if $\bar{\bar{a}} \in P$, and $\overline{\bar{x} \rightarrow \bar{y}} = \overline{\overline{y \rightharpoonup x}}$. ∎

Notice the fact that $x \wedge \bar{x} \leq \bar{\bar{x}} \wedge \bar{x}$ for any $x \in L$, we can obtain a condition under which an ideal is a Boolean ideal in BL-algebras.

Remark 3.7 Let I be an ideal of a BL-algebra L. Then I is a Boolean ideal if and only if $\bar{x} \wedge \bar{\bar{x}} \in I$ for any $x \in L$.

4. Implicative ideals

In this section, we introduce the notion of implicative ideals in BL-algebras and investigate some of their properties. Here we obtain that an ideal is Boolean if and only if it is an implicative ideal.

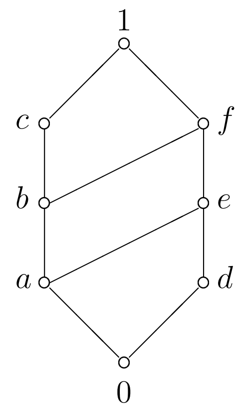
Definition 4.1 A nonempty subset I of a BL-algebra L is called an implicative ideal if it satisfies:

(1) $0 \in I$,

(2) $(x \rightharpoonup y) \rightharpoonup z \in I$ and $y \rightharpoonup z \in I$ imply $x \rightharpoonup z \in I$, for any $x, y, z \in L$.

For better understanding of the above definition, we illustrate it by the following example.

Example 4.2 Let $L = \{0, a, b, c, d, e, f, 1\}$ be a set with Hasse diagram and Cayley tables as follows.

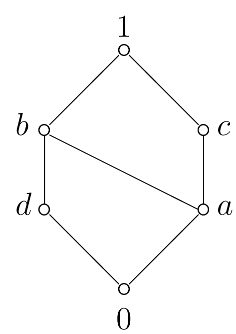


$\odot$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	a	a	a	0	a	a	a
b	0	a	a	b	0	a	a	b
c	0	a	b	c	0	a	b	c
d	0	0	0	0	d	d	d	d
e	0	a	a	a	d	e	e	e
f	0	a	a	b	d	e	e	f
1	0	a	b	c	d	e	f	1

$\to$	0	a	b	c	d	e	f	1
0	1	1	1	1	1	1	1	1
a	d	1	1	1	d	1	1	1
b	d	f	1	1	d	f	1	1
c	d	e	f	1	d	e	f	1
d	c	c	c	c	1	1	1	1
e	0	c	c	c	d	1	1	1
f	0	b	c	c	d	f	1	1
1	0	a	b	c	d	e	f	1

Then $(L, \wedge, \vee, \otimes, \to, 0, 1)$ is a BL-algebra that is not an MV-algebra. It is easy to check that $I = \{0, a, b, c\}$ is an implicative ideal of L.

Example 4.3 Let $L = \{0, a, b, c, d, 1\}$ be a set with Hasse diagram and Cayley tables as follows.



$\otimes$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	a	0	0	a
b	0	a	b	0	a	b
c	0	0	0	c	c	c
d	0	0	a	c	c	d
1	0	a	b	c	d	1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	1	d	1	1
b	c	d	1	c	d	1
c	b	b	b	1	1	1
d	a	b	b	d	1	1
1	0	a	b	c	d	1

According to [7], $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a BL-algebra. It is easy to check that $I = \{0, c, d\}$ is an implicative ideal of L.

The following proposition describes the relationship between implicative ideals and ideals.

Proposition 4.4 *Let I be an implicative ideal of a BL-algebra L. Then I is an ideal, but the converse is not true in general.*

Proof. Suppose that I is an implicative ideal of a BL-algebra L. Let $x, y \in L$. If $x \rightharpoonup y \in I$ and $y \in I$, then $(x \rightharpoonup y) \rightharpoonup 0 = x \rightharpoonup y \in I$ and $y \rightharpoonup 0 = y \in I$. By hypothesis we get $x = x \rightharpoonup 0 \in I$, hence I is an ideal.

For the converse we consider the BL-algebra L of Example 4.3, it is easy to see that $J = \{0, b\}$ is an ideal but not an implicative ideal since $(c \rightharpoonup a) \rightharpoonup a = 0 \in J$ and $a \rightharpoonup a = 0 \in I$ but $c \notin J$. ■

In the following, we give some characterizations of implicative ideals for further discussion.

Theorem 4.5 *Let I be an ideal of a BL-algebra L. Then I is an implicative ideal of L if and only if it satisfies the condition* (PI)*: $(y \rightharpoonup \bar{x}) \rightharpoonup z \in I$ and $x \rightharpoonup y \in I$ imply $x \rightharpoonup z \in I$, for any $x, y, z \in L$.*

Proof. Suppose that I is an implicative ideal. For any $x, y \in L$, let $(y \rightharpoonup \bar{x}) \rightharpoonup z \in I$ and $x \rightharpoonup y \in I$. By Lemma 3.3, we have $(\bar{z} \rightharpoonup \bar{y}) \rightharpoonup \bar{x} \in I$ and $\bar{y} \rightharpoonup \bar{x} \in I$. Hence $\bar{z} \rightharpoonup \bar{x} \in I$, and so $x \rightharpoonup z \in I$.

Conversely, suppose that I satisfies the condition (PI). Let $(x \rightharpoonup y) \rightharpoonup z \in I$ and $y \rightharpoonup z \in I$, that is $x \rightharpoonup (y \oslash z) \in I$ and $y \rightharpoonup z \in I$. By Lemma 3.3 we get $(\bar{y} \rightharpoonup \bar{\bar{z}}) \rightharpoonup \bar{x} \in I$ and $\bar{z} \rightharpoonup \bar{y} \in I$. Therefore $\bar{z} \rightharpoonup \bar{x} \in I$, and so $x \rightharpoonup z \in I$. Hence I is an implicative ideal of L. ■

Theorem 4.6 *Let I be an ideal of a BL-algebra L. Then the following conditions are equivalent:*

(1) *I is an implicative ideal of L;*

(2) *for any $a \in L$, the set $I_a := \{x \in L | x \rightharpoonup a \in I\}$ is an ideal of L.*

Proof. Suppose that I is an implicative ideal. For any $x, y \in L$, if $x \rightharpoonup y \in I_a$ and $y \in I_a$, then $(x \rightharpoonup y) \rightharpoonup a \in I$ and $y \rightharpoonup a \in I$, hence $x \rightharpoonup a \in I$, that is, $x \in I_a$. By Lemma 3.1, $0 \rightharpoonup a = 0 \in I$, then $0 \in I_a$, hence I_a is an ideal.

Conversely, suppose that I_a is an ideal of L for any $a \in L$. For any $x, y, z \in L$, if $(x \rightharpoonup y) \rightharpoonup z \in I$ and $y \rightharpoonup z \in I$, then $x \rightharpoonup y \in I_z$ and $y \in I_z$. Since I_z is an ideal, we obtain $x \in I_z$, that is, $x \rightharpoonup z \in I$. Therefore I is an implicative ideal. ■

As a consequence of Theorem 4.6, we have the following result.

Proposition 4.7 *Let I be an implicative ideal of a BL-algebra L. Then for any $a \in L$, I_a is the least ideal of L containing I and a.*

Proof. Assume that I is an implicative ideal and $a \in L$. By Theorem 4.6, we obtain I_a an ideal. For any $x \in I$, then $x \rightharpoonup a \leq x$. From Proposition 4.4, it follows that I is an ideal of L, hence $x \rightharpoonup a \in I$, that is, $x \in I_a$, and so $I \subseteq I_a$. Notice that $a \rightharpoonup a = 0 \in I$, we obtain $a \in I_a$. If J is an ideal containing I and a, then for any $x \in I_a$, we have $x \rightharpoonup a \in I \subseteq J$. Since J is an ideal of L and $a \in J$, we get $x \in J$, that is $I_a \subseteq J$. Therefore I_a is the least ideal containing I and a. ∎

Next, we discuss some properties of I_a.

Proposition 4.8 *Let I, J be ideals of a BL-algebra L. Then the following statements hold: for any $a, b \in L$,*

(1) $I_a = I$ *if and only if* $a \in I$,

(2) $a \leq b$ *implies* $I_a \subseteq I_b$,

(3) $I \subseteq J$ *implies* $I_a \subseteq J_a$,

(4) $(I \cap J)_a = I_a \cap J_a$, $(I \cup J)_a = I_a \cup J_a$,

(5) *if I is an implicative ideal, then* $I_{a \oslash b} = (I_a)_b$

Proof. (1) Suppose that $I_a = I$. Since $a \rightharpoonup a = 0 \in I$, we have $a \in I_a$ by the definition of I_a, hence $a \in I$.

Conversely, assume that $a \in I$. For any $x \in I$, since $x \rightharpoonup a \leq x \in I$ and I is ideal, we have $x \rightharpoonup a \in I$. Hence $x \in I_a$, and so $I \subseteq I_a$. On the other hand, for any $x \in I_a$, then $x \rightharpoonup a \in I$. Since $a \in I$, we get $x \in I$. Therefore $I_a \subseteq I$ and so $I_a = I$.

(2) Suppose that $a \leq b$. For any $x \in I_a$, then $x \rightharpoonup a \in I$. Since $x \rightharpoonup b \leq x \rightharpoonup a$ and I is an ideal, we have $x \rightharpoonup b \in I$. Therefore $x \in I_b$, and so $I_a \subseteq I_b$.

(3) Assume that $I \subseteq J$. For any $x \in I_a$, we have $x \rightharpoonup a \in I \subseteq J$, then $x \in J_a$, it follows that $I_a \subseteq J_a$.

(4) Since $I \cap J \subseteq I, J$, we have $(I \cap J)_a \subseteq I_a \cap J_a$ by (3). On the other hand, for any $x \in I_a \cap J_a$, we have $x \rightharpoonup a \in I$ and $x \rightharpoonup a \in J$. Hence $x \rightharpoonup a \in I \cap J$, and so $I_a \cap J_a \subseteq (I \cap J)_a$. Thus $(I \cap J)_a = I_a \cap J_a$. $(I \cup J)_a = I_a \cup J_a$ can be proved in the similar way.

(5) Suppose that I is an implicative ideal. By Theorem 4.6, I_a is an ideal. $x \in I_{a \oslash b}$ if and only if $x \rightharpoonup (a \oslash b) = (x \rightharpoonup a) \rightharpoonup b = (x \rightharpoonup b) \rightharpoonup a$ if and only if $x \in (I_a)_b$. Hence $I_{a \oslash b} = (I_a)_b$. ∎

The following results are the characterizations of implicative ideals.

Theorem 4.9 *Let I be a nonempty subset of a BL-algebra L. Then the following conditions are equivalent:*

(1) *I is an implicative ideal of L,*

(2) *I is an ideal and for any $x, y \in L$, $(x \rightharpoonup y) \rightharpoonup y \in I$ implies $x \rightharpoonup y \in I$,*

(3) *I is an ideal and for any $x, y, z \in L$, $(x \rightharpoonup y) \rightharpoonup z \in I$ implies $(x \rightharpoonup z) \rightharpoonup (y \rightharpoonup z) \in L$,*

(4) $0 \in I$, *and if* $((x \rightharpoonup y) \rightharpoonup y) \rightharpoonup z \in I$ *and* $z \in I$, *then* $x \rightharpoonup y \in I$, *for any* $x, y, z \in L$.

Proof. (1) $\Rightarrow$ (2) Let I be an implicative ideal and $(x \rightharpoonup y) \rightharpoonup y \in I$. It follows from Proposition 4.4 that I is an ideal. Observe that $y \rightharpoonup y = 0 \in I$, we get $x \rightharpoonup y \in I$.

(2) $\Rightarrow$ (3) Assume that (2) holds. For any $x, y, z \in L$, let $(x \rightharpoonup y) \rightharpoonup z \in I$. According to Lemma 3.1, we have $((x \rightharpoonup (y \rightharpoonup z)) \rightharpoonup z) \rightharpoonup z = ((x \rightharpoonup z) \rightharpoonup (y \rightharpoonup z)) \rightharpoonup z \leq (x \rightharpoonup y) \rightharpoonup z$. It follows that $((x \rightharpoonup (y \rightharpoonup z)) \rightharpoonup z) \rightharpoonup z \in I$. Therefore $(x \rightharpoonup z) \rightharpoonup (y \rightharpoonup z) = (x \rightharpoonup (y \rightharpoonup z)) \rightharpoonup z \in I$ by hypothesis, and so (3) is valid.

(3) $\Rightarrow$ (4) Suppose that (3) holds. Obviously, $0 \in I$. For any $x, y, z \in L$, if $((x \rightharpoonup y) \rightharpoonup y) \rightharpoonup z \in I$ and $z \in I$, then $(x \rightharpoonup y) \rightharpoonup y \in I$. It follows from hypothesis and Lemma 3.1 that $x \rightharpoonup y = (x \rightharpoonup y) \rightharpoonup (y \rightharpoonup y) \in I$, therefore (4) is valid.

(4) $\Rightarrow$ (1) Assume that (4) is valid. We assert that I is an ideal of L. In fact, for any $x, y \in L$, if $x \rightharpoonup y \in I$ and $y \in I$, then $((x \rightharpoonup 0) \rightharpoonup 0) \rightharpoonup y = x \rightharpoonup y \in I$. By hypothesis, we get $x = x \rightharpoonup 0 \in I$, therefore I is an ideal. Let $(x \rightharpoonup y) \rightharpoonup z \in I$ and $y \rightharpoonup z \in I$. From Lemma 3.1, it follows that $((x \rightharpoonup z) \rightharpoonup z) \rightharpoonup (y \rightharpoonup z) \leq (x \rightharpoonup z) \rightharpoonup y = (x \rightharpoonup y) \rightharpoonup z$. Due to the fact that I an ideal of L, we have $((x \rightharpoonup z) \rightharpoonup z) \rightharpoonup (y \rightharpoonup z) \in I$, therefore $x \rightharpoonup z \in I$, and thus (1) holds. ∎

The extension property for implicative ideals is obtained from the following proposition.

Proposition 4.10 *Let I and J be ideals of a BL-algebra L such that $I \subseteq J$. If I is an implicative ideal, then so is J.*

Proof. Let $x, y, z \in L$ such that $(x \rightharpoonup y) \rightharpoonup z \in J$. Denote $u = (x \rightharpoonup y) \rightharpoonup z$, it follows that $((x \rightharpoonup u) \rightharpoonup y) \rightharpoonup z = ((x \rightharpoonup y) \rightharpoonup z) \rightharpoonup u = 0 \in I$. Since I is an implicative ideal of L, we have $((x \rightharpoonup u) \rightharpoonup z) \rightharpoonup (y \rightharpoonup z) = ((x \rightharpoonup z) \rightharpoonup (y \rightharpoonup z)) \rightharpoonup u \in I \subseteq J$ by Theorem 4.9. Consider that J is an ideal and $u \in J$, we obtain that $(x \rightharpoonup z) \rightharpoonup (y \rightharpoonup z) \in J$. Hence J is an implicative ideal of L. ∎

Now we continue to study the characterizations of implicative ideals.

Theorem 4.11 *Let I be a nonempty subset of a BL-algebra L. Then the following conditions are equivalent:*

(1) *I is an implicative ideal of L,*

(2) $0 \in I$, *and if* $(x \rightharpoonup (y \rightharpoonup x)) \rightharpoonup z \in I$ *and* $z \in I$, *then* $x \in I$, *for any* $x, y, z \in L$,

(3) *I is an ideal and for any $x, y \in L$, $x \rightharpoonup (y \rightharpoonup x) \in I$ implies $x \in I$.*

Proof. (1) $\Rightarrow$ (2) Suppose that I is an implicative ideal. Obviously, $0 \in I$. For any $x, y, z \in L$, if $(x \rightharpoonup (y \rightharpoonup x)) \rightharpoonup z \in I$ and $z \in I$, then $x \rightharpoonup (y \rightharpoonup x) \in I$. Since $((y \rightharpoonup (y \rightharpoonup x)) \rightharpoonup (y \rightharpoonup x)) \rightharpoonup (x \rightharpoonup (y \rightharpoonup x)) = ((y \rightharpoonup (y \rightharpoonup x)) \rightharpoonup (x \rightharpoonup

$(y \rightharpoonup x)) \rightharpoonup (y \rightharpoonup x) \leq (y \rightharpoonup x) \rightharpoonup (y \rightharpoonup x) = 0 \in I$, we get that $(y \rightharpoonup (y \rightharpoonup x)) \rightharpoonup (y \rightharpoonup x) \in I$. Using Theorem 4.9, we obtain that $y \rightharpoonup (y \rightharpoonup x) \in I$. Since $(x \rightharpoonup (x \rightharpoonup y)) \rightharpoonup (y \rightharpoonup (y \rightharpoonup x)) = 0 \in I$, then $x \rightharpoonup (x \rightharpoonup y) \in I$. By Lemma 2.2, it is easy to obtain that $((x \rightharpoonup y) \rightharpoonup z) \rightharpoonup (x \rightharpoonup (y \rightharpoonup x)) = 0$, therefore $(x \rightharpoonup y) \rightharpoonup z \in I$. Notice that $z \in I$, we have $x \rightharpoonup y \in I$. Follows the result $x \rightharpoonup (x \rightharpoonup y) \in I$, we get $x \in I$, and so (2) holds.

(2) $\Rightarrow$ (3) For any $x, y \in L$, if $x \rightharpoonup y \in I$ and $y \in I$, then $(x \rightharpoonup (x \rightharpoonup x)) \rightharpoonup y = x \rightharpoonup y \in I$. By hypothesis, we have $x \in I$, hence I is an ideal. Suppose that $x \rightharpoonup (y \rightharpoonup x) \in I$, then $(x \rightharpoonup (y \rightharpoonup x)) \rightharpoonup 0 \in I$ and $0 \in I$. Therefore $x \in I$, and thus (3) is valid.

(3) $\Rightarrow$ (1) Assume that (3) holds. For any $x, y \in L$, let $(x \rightharpoonup y) \rightharpoonup y \in I$. Routine calculation shows that $((x \rightharpoonup y) \rightharpoonup (x \rightharpoonup (x \rightharpoonup y))) \rightharpoonup ((x \rightharpoonup y) \rightharpoonup y) = 0$, then $(x \rightharpoonup y) \rightharpoonup (x \rightharpoonup (x \rightharpoonup y)) \in I$. By hypothesis, $x \rightharpoonup y \in I$. From Theorem 4.9, it follows that I is an implicative ideal. ∎

To show the relationship between implicative ideals and Boolean ideals in BL-algebras, we give a characterization of implicative ideals as follows.

Theorem 4.12 *Let I be an ideal of a BL-algebra L. Then I is an implicative ideal of L if and only if $(x \oslash x) \rightharpoonup x \in I$ for any $x \in L$.*

Proof. For any $x \in L$, we have $x \rightharpoonup x = 0 \in I$ and $((x \oslash x) \rightharpoonup x) \rightharpoonup x = (\bar{x} \to x) \odot \bar{x} \odot \bar{x} = (\bar{x} \wedge x) \odot \bar{x} = 0 \in I$. Due to the fact that I is an implicative ideal, we get $(x \oslash x) \rightharpoonup x \in I$.

Conversely, suppose that for any $x \in L$, $(x \oslash x) \rightharpoonup x \in I$ holds. Let $x, y, z \in L$ such that $(x \rightharpoonup y) \rightharpoonup z \in I$ and $y \rightharpoonup z \in I$. By Lemma 3.1, it not difficult to obtain that $(x \rightharpoonup (z \oslash z)) \rightharpoonup ((x \rightharpoonup y) \rightharpoonup z) \rightharpoonup (y \rightharpoonup z) = 0$. From Definition 2.3 and Lemma 3.2, it follows that $x \rightharpoonup (z \oslash z) \in I$. Since $(z \oslash z) \rightharpoonup z \in I$ and $x \rightharpoonup z \leq ((z \oslash z) \rightharpoonup z) \oslash (x \rightharpoonup (z \oslash z))$, we get $x \rightharpoonup z \in I$. Hence I is an implicative ideal of L. ∎

Let L be a BL-algebra. For any $x \in L$, we have $(x \oslash x) \rightharpoonup x = (\bar{x} \to x) \odot \bar{x} = \bar{x} \wedge x$. As an application of the above theorem, we have the following result.

Corollary 4.13 *Let I be an ideal of a BL-algebra L. Then I is an implicative ideal if and only if I is a Boolean ideal of L.*

Summarizing the above results, we have the following proposition.

Proposition 4.14 *In a BL-algebra L, the following conditions are equivalent:*

(1) *any ideal I of L is implicative;*

(2) $\{0\}$ *is an implicative ideal;*

(3) *for any $a \in L$, the set $L(a) := \{x \in L | x \rightharpoonup a = 0\}$ is an ideal of L.*

Proof. (1) $\Leftrightarrow$ (2) It is clear by Proposition 4.10.

(2) $\Rightarrow$ (3) For any $a, x, y \in L$, if $x \rightharpoonup y \in L(a)$ and $y \in L(a)$, then $(x \rightharpoonup y) \rightharpoonup a = 0$ and $y \rightharpoonup a = 0$. Since $\{0\}$ is an implicative ideal, we have $x \rightharpoonup a = 0$, that is, $x \in L(a)$. Hence $L(a)$ is an ideal of L.

(3) $\Rightarrow$ (2) For any $x, y \in L$, if $(x \rightharpoonup y) \rightharpoonup y = 0$, then $x \rightharpoonup y \in L(y)$. Since $L(y)$ is an ideal and $y \in L(y)$, then $x \in L(y)$, that is $x \rightharpoonup y = 0$. By Theorem 4.9, $\{0\}$ is an implicative ideal. ∎

Proposition 4.15 *If L is a Boolean algebra or a Gödel algebra, then any ideal of L is an implicative ideal.*

Proof. To prove the proposition, we consider two cases.

Case 1. If L is a Boolean algebra, then $x \vee \bar{x} = 1$ for any $x \in L$. By Lemma 2.2, we have $x \wedge \bar{x} = 0$. It follows that $\{0\}$ is a Boolean ideal. By Corollary 4.13 and Proposition 4.14, any ideal of L is an implicative ideal.

Case 2. Assume that L is a Gödel algebra and I is an ideal of L. For any $x \in L$, we have $(x \oslash x) \rightharpoonup x = (\bar{x} \to x) \odot \bar{x} = (\bar{x} \to x) \odot (\bar{x} \odot \bar{x}) = (\bar{x} \wedge x) \odot \bar{x} = 0$. Notice that I is an ideal, we have $x \oslash x \rightharpoonup x \in I$. From Theorem 4.12, it follows that I is an implicative ideal of L. ∎

Next, we introduce the notion of maximal ideals in BL-algebras and discuss the relations among various ideals.

Definition 4.16 Let L be a BL-algebra. A proper ideal I of L is called a maximal ideal if I is not a proper subset of any proper ideal of L.

It is easy to prove the following result.

Proposition 4.17 *Any maximal ideal of a BL-algebra L is a prime ideal.*

The following example shows that maximal ideals exist.

Example 4.18 Let $(L, \wedge, \vee, \odot, \to, 0, 1)$ be a BL-algebra of Example 4.3. It can be verified that $I = \{0, c, d\}$ is a maximal ideal, and also a prime ideal.

For the purpose of investigating some relationships among various ideals of BL-algebras, we prepare the following results.

Lemma 4.19 *Let M be an ideal of a BL-algebra. Then ideal M is maximal if and only if L/M is simple, i.e, has no proper ideal other than $\{0\}$.*

Lemma 4.20 [11] *An ideal P of a BL-algebra L is a prime ideal if and only if the BL-algebra quotient L/P is an MV-chain.*

Lemma 4.21 [11] *Let L be a BL-algebra, then an ideal I is Boolean if and only if L/I is a Boolean algebra.*

Theorem 4.22 *Let I be a proper ideal of a BL-algebra L. Then the following statements are equivalent:*

(1) *I is a maximal and implicative ideal;*

(2) *$x \notin I$ and $y \notin I$ imply $x \rightharpoonup y \in I$ and $y \rightharpoonup x \in I$ for any $x, y \in L$;*

(3) *if $x \notin I$, then exists some $n > 0$ such that $x^n_{\oslash} := \underbrace{\bar{x} \oslash \cdots \oslash \bar{x}}_{n \text{ times}} \in I$;*

(4) *$x \in I$ or $\bar{x} \in I$ for any $x \in L$;*

(5) *I is a prime and implicative ideal.*

Proof. (1) $\Rightarrow$ (2) Let $x, y \notin I$. It follows from Proposition 4.7 that $I_y = \{z \in L \mid z \rightharpoonup y \in I\}$ is the least ideal containing I and y. Since I is maximal and $y \notin I$, we get $I_y = L$. Thus $x \in I_y$, and so $x \rightharpoonup y \in I$. $y \rightharpoonup x \in I$ can be proved similarly.

(2) $\Rightarrow$ (3) Suppose that $x \notin I$. Since I is a proper ideal, then $1 \notin I$. By hypothesis, we have $x \rightharpoonup 1 = 0 \in I$ and $1 \rightharpoonup x = \bar{x} \in I$. It follows from Definition 2.3 that $x^n_{\oslash} \in I$ for any $n > 0$.

(3) $\Rightarrow$ (4) Assume that (3) holds. For any $x \in L$, if $x \in I$, it is true. Assume $x \notin I$, then there exists some $n > 0$ such that $x^n_{\oslash} \in I$. Since $\bar{x} \leq x^n_{\oslash}$ and I is a proper ideal of L, we have $\bar{x} \in I$. Thus (4) is valid.

(4) $\Rightarrow$ (5) For any $x \in L$, $x \wedge \bar{x} \leq x, \bar{x}$, it follows from hypothesis and Corollary 4.13 that I is an implicative ideal. For any $y \in L$, we have $(y \rightharpoonup x) \rightharpoonup \bar{x} = 0 \in I$ and $(x \rightharpoonup y) \rightharpoonup x = 0 \in I$. Since I is a proper ideal, and $x \in I$ or $\bar{x} \in I$, we have $x \rightharpoonup y \in I$ or $y \rightharpoonup x \in I$. It follows from Proposition 3.6 that I is prime. Hence I is a prime and implicative ideal.

(5) $\Rightarrow$ (1) Let I be a prime and implicative ideal. From Lemmas 4.20 and 4.21, it follows that L/I would be an MV-chain that is also a Boolean algebra. While the only Boolean chain is the two-element Boolean algebra, hence $L/I \cong 2$. According to Lemma 4.19, (1) is clear. ∎

5. Conclusions

In this paper, we gave some characterizations of ideals, prime ideals and Boolean ideals by the pseudo implication operation. Then we introduced the notions of implicative ideals and investigated some characterizations of them. For future work, we could use the pseudo implication operation to investigate the relationships among BL-algebras and other logic algebras, and obtain some logic results.

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References

[1] Borumand Saeid, A., Motamed, S., *A new filter in BL-algebras*, J. Intell. Fuzzy Systems, 27 (6) (2014), 2949–2957.

[2] Borumand Saeid, A., Motamed, S., *Some results in BL-algebras*, Mathematical Logic Quarterly, 55 (2009), 649–658.

[3] Borzooei, R.A., Paad, A., *Integral filters and integral BL-algebras*, Ital. J. Pure Appl. Math., 30 (2013), 303–316.

[4] HÁJek, P., *Metamathematics of Fuzzy Logic*, Kluwer, Dordrecht, 1998.

[5] Haveshki, M., Borumand Saeid, A., Eslami, E., *Some types of filters in BL algebras*, Soft Comput., 10 (2006), 657–664.

[6] Iorgulescu, A., *Some direct ascendents of Wajsberg and MV-algebras*, Sci. Math. Jpn., 57 (2003), 583–647.

[7] Jun, Y.B., Ko, J.M., *Folding theory applied to BL-algebras*, Cent. Eur. J. Math., 2 (2004), 584–592.

[8] Jun, Y.B., Lee, K.J., Park C.H., Roh, E.H., *Filters of residuated lattices based on soft set theory*, Commun. Korean Math. Soc., 30 (2015), 155–168.

[9] Kondo, M., Dudek, W.A., *Filter theory of BL algebras*, Soft Comput., 12 (2008), 419–423.

[10] Kondo, M., *Simple characterization of strict residuated lattices with an involutive negation*, Soft Comput., 17 (2013), 39–44.

[11] Lele, C., Nganou, J.B., *MV-algebras derived from ideals in BL-algebras*, Fuzzy Sets and Systems, 218 (2013), 103–113.

[12] Liu, L., Li, K., *Fuzzy Boolean and positive implicative filters of BL-algebras*, Fuzzy Sets and Systems, 152 (2005), 333–348.

[13] Liu, L., *Generalized intuitionistic fuzzy filters on residuated lattices*, J. Intell. Fuzzy Systems, 28 (2015), 1545C1552.

[14] Li, W., Hu, B.Q., *Prime fuzzy (implicative) MP-filters of R_0-algebras*, J. Intell. Fuzzy Systems, 30 (2016), 1915–1925.

[15] Mundici, M., *MV-algebras are categorically equivalent to bounded commutative BCK-algebras*, Math. Japon., 31 (1986), 889–894.

[16] Turunen, E., *BL-algebras of basic fuzzy logic*, Mathew. Soft Comput., 6 (1999), 49–61.

[17] Turunen, E., *Boolean deductive systems of BL-algebras*, Arch. Math. Logic, 40 (2001), 467–473.

[18] Zhang, X.H., Zhou, H., Mao, X., *IMTL(MV)-filters and fuzzy IMTL(MV)-filters of residuated lattices*, J. Intell. Fuzzy Systems, 26 (2014), 589–596.

[19] VÍTA, M., *Fuzzy t-filters and their properties*, Fuzzy Sets and Systems, 247 (2014), 127–134.

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ON SOME PROPERTIES OF ϕ-MULTIPLIERS

Abdelaziz Tajmouati

Sidi Mohamed Ben Abdellah University
Faculty of Sciences Dhar Al Mahraz
Laboratory of Mathematical Analysis and Applications
Fez
Morocco
e-mail: abdelaziz.tajmouati@usmba.ac.ma

A. El Bakkali

Chouaib Dokkali University
Faculty of Sciences
El Jadida
Morocco
e-mail: aba0101q@yahoo.fr

M.B. Mohamed Ahmed

Sidi Mohamed Ben Abdellah University
Faculty of Sciences Dhar Al Mahraz
Laboratory of Mathematical Analysis and Applications
Fez
Morocco
e-mails: bbaba2012@gmail.com

Abstract. In this paper, we investigate some new properties of ϕ-multipliers studied recently by M.Adib and A.Riazi on faithful Banach algebras. Specially we justifies the existence of Helgason-Wang function for a ϕ-multipliers and give some characterizations. As corollary we obtain some results for classical multipliers.

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1. Introduction and notations

The general theory of multipliers on a faithful Banach algebras was originally introduced by Helgason [4] and has been developed by Wang [8] and Birtal [2]. A good reference for this theory is the monograph of Larsen [5] and Laursen and Neumann [6].

In the following, let $\mathcal{A}$ denote a complex Banach algebra. We recall that a mapping $T : \mathcal{A} \to \mathcal{A}$ is said to be a multiplier if for all $x,\ y \in \mathcal{A}$

$$x(Ty) = (Tx)y.$$

The set of multipliers of $\mathcal{A}$ is denoted by $M(\mathcal{A})$. It is well known that if $\mathcal{A}$ is faithful (see definition below) then $M(\mathcal{A})$ is a closed commutative subalgebra of $\mathcal{B}(\mathcal{A})$ the Banach algebra of all bounded linear operators of $\mathcal{A}$ (see [1], [5], [6]).

We shall say that two elements $x, y \in \mathcal{A}$ are orthogonal whenever $xy = yx = 0$. Given a nonempty subset B of $\mathcal{A}$ the orthogonal of B is defined to be the set

$$B^{\top} := \{x \in \mathcal{A} : xy = yx = 0 \, for\ each\ \ y \in B\}.$$

Trivially, $B^{\top}$ is a closed two sided ideal of $\mathcal{A}$.

Recall that the left annihilator and the right annihilator of B are respectively the sets:

$$lanB = \{x \in A\ :\ xB = \{0\}\}\ \ and\ ranB = \{x \in \mathcal{A}\ :\ Bx = \{0\}\}.$$

We say that $\mathcal{A}$ is faithful (or without order) if $lan\mathcal{A} = \{0\}$ or $ran\mathcal{A} = \{0\}$, $\mathcal{A}$ is semi prime if $\{0\}$ is the unique two-sided ideal J such that $J^2 = \{0\}$ and $\mathcal{A}$ is said semi simple if rad $(\mathcal{A}) = \{0\}$ where rad $\mathcal{A}$ is the (Jacobson) radical of $\mathcal{A}$, see [1], [5], [6].

Note that if $\mathcal{A}$ is semi prime then $\mathcal{A}$ is faithful. Moreover if $x\mathcal{A}x = \{0\}$ then $x = 0$. It is well known that each semi simple algebra is semi prime and therefore faithful.

If $\mathcal{A}$ is a faithful Banach algebra then the linearity and continuity of every $T \in M(\mathcal{A})$ are automatic and for all $x,\ y \in \mathcal{A}$ we have

$$T(xy) = x(Ty) = (Tx)y.$$

Let $\Delta(\mathcal{A})$ denote the set of all maximal regular ideals of a commutative Banach algebra $\mathcal{A}$ and let $\mathcal{A}^*$ denote the dual of $\mathcal{A}$ (see [9]).
Recall that a multiplicative linear functional on a complex Banach algebra $\mathcal{A}$ is a non-zero linear functional $m \in \mathcal{A}^*$ such that $m(xy) = m(x)m(y)$ for all $x, y \in \mathcal{A}$. It is well known that if $\mathcal{A}$ is a commutative Banach algebra $\mathcal{A}$ then

$$rad\, \mathcal{A} = \bigcap_{m \in \Delta(\mathcal{A})} \ker m.$$

Now, let $\widehat{x}$ denotes the Gelfand transform of $x \in \mathcal{A}$ defined by

$$\widehat{x}(m) := m(x)\, for\ each\, m \in \Delta(\mathcal{A}).$$

Recall that $\mathcal{A}$ is semi simple if and only if for every non-zero element x of $\mathcal{A}$ there exists some $m \in \Delta(\mathcal{A})$ such that $\widehat{x}(m) \neq 0$, obviously this condition implies that if $\widehat{x}(m)$ is zero for all $m \in \Delta(\mathcal{A})$ then $x = 0$ [9].

Recently, Riazi and Adib [7] have defined and studied the concept of ϕ-multipliers where ϕ is a homomorphism from $\mathcal{A}$ to $\mathcal{A}$ which is a generalization of multipliers if $\mathcal{A}$ is a faithful Banach algebra.

Definition 1.1 [7] Let $\mathcal{A}$ be a Banach algebra, ϕ be a homomorphism from $\mathcal{A}$ to $\mathcal{A}$ and $T : \mathcal{A} \to \mathcal{A}$ be a continuous linear mapping.

T is said ϕ-multiplier on $\mathcal{A}$ if for all $x, y \in \mathcal{A}$, we have

$$T(xy) = T(x)\phi(y) = \phi(x)T(y).$$

We denote $M_\phi(\mathcal{A})$ the collection of all ϕ-multipliers of $\mathcal{A}$ where ϕ is a homomorphism.

An immediate example of a ϕ-multiplier of a Banach algebra $\mathcal{A}$ is given by $L_a \circ \phi$ where $a \in com(\mathcal{A})$ ($com(\mathcal{A})$ the commutator of $\mathcal{A}$) and $L_a : x \in \mathcal{A} \to ax$ the left multiplication operator by a.

If $\mathcal{A}$ is a commutative Banach algebra with unit u, given a multiplier $T \in M_\phi(\mathcal{A})$ where ϕ is a homomorphism from $\mathcal{A}$ to $\mathcal{A}$, for each $x \in \mathcal{A}$ we have

$$\begin{aligned}(L_{Tu} \circ \phi)x &= (Tu)(\phi(x)) \\ &= \phi(u)(Tx) \\ &= uTx = Tx \ \ (T \in M_\phi(\mathcal{A}),\ \phi \text{ is homomorphism } : \phi(u) = u).\end{aligned}$$

Thus $T = L_{Tu} \circ \phi$.

The most important example of a ϕ-multipliers is obtained when we take $\mathcal{A} = L_1(G)$ the group algebra of a locally compact Abelian group G [3].

In the following section, we investigate some news properties of ϕ-multipliers and We extend some classical results for multipliers.

2. ϕ-Multipliers and their properties

Proposition 2.1 *Let $\mathcal{A}$ be a faithful Banach algebra, ϕ be a bounded onto homomorphism from $\mathcal{A}$ to $\mathcal{A}$ and $T : \mathcal{A} \to \mathcal{A}$ be a mapping.*
T is ϕ-multiplier on $\mathcal{A}$ if and only if for all $x, y \in A$ we have

$$T(x)\phi(y) = \phi(x)T(y).$$

Moreover $T(\mathcal{A})$ is a two-sided ideal in $\mathcal{A}$.

Proof. Directly, is trivial.

Conversely, assume that for all $x, y \in \mathcal{A}$ we have $T(x)\phi(y) = \phi(x)T(y)$. Since $\mathcal{A}$ is faithful then we can consider $\operatorname{ran}\mathcal{A} = \{0\}$. First, for any x, y, $z \in A$ we have

$$\begin{aligned}z[\phi(x)T(y)] &= \phi(z')[\phi(x)T(y)] \quad (\phi \text{ is onto } : \exists\, x' \in \mathcal{A} \,/\, \phi(x') = x) \\ &= \phi(z')[T(x)\phi(y)] \\ &= [\phi(z')T(x)]\phi(y)] \\ &= [T(z')\phi(x)]\phi(y) \\ &= T(z')\phi(xy) \\ &= \phi(z')T(xy) \\ &= zT(xy).\end{aligned}$$

Thus $[\phi(x)T(y) - T(xy)] \in \text{ran } \mathcal{A} = \{0\}$, and therefore the equalities

$$T(xy) = \phi(x)T(y) = T(x)\phi(y).$$

Now, we prove that T is linear, for any x, y, $z \in \mathcal{A}$ and λ, $\mu \in \mathbb{C}$ we have

$$\begin{aligned} z[T(\lambda x + \mu y)] &= \phi(z')T[\lambda x + \mu y] \quad (\phi \text{ is onto } : \exists z' \in \mathcal{A} \,/\, \phi(z') = z) \\ &= T(z')\phi[\lambda x + \mu y] \\ &= T(z')[\lambda\phi(x) + \mu\phi(y)] \\ &= \lambda T(z')\phi(x) + \mu T(z')\phi(y) \\ &= \lambda\phi(z')(Tx) + \mu\phi(z')(Ty) \\ &= z(\lambda Tx + \mu Ty) \qquad (z = \phi(z')), \end{aligned}$$

which implies that $z[T(\lambda x + \mu y) - (\lambda Tx + \mu Ty)] = 0$ for all $z \in \mathcal{A}$ and since ran $\mathcal{A} = \{0\}$ this implies $T(\lambda x + \mu y) = \lambda Tx + \mu Ty$.

To prove that T is bounded let $y, z \in \mathcal{A}$ and $(y_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ such that $\|y_n - y\| \to 0$ and $\|T(y_n) - z\| \to 0$, for $x \in \mathcal{A}$ arbitrary we have

$$\begin{aligned} \|xz - xT(y)\| &= \|xz - xT(y_n) + xT(y_n) - xT(y)\| \\ &= \|xz - xT(y_n) + \phi(x')T(y_n) - \phi(x')T(y)\| \\ &\qquad (\phi \text{ is onto } : \exists\, z' \in \mathcal{A} \,/\, \phi(z') = z) \\ &\leq \|xz - xT(y_n)\| + \|\phi(x')T(y_n) - \phi(x')T(y)\| \\ &\leq \|x\|\|z - T(y_n)\| + \|T(x')\phi(y_n) - T(x')\phi(y)\| \\ &\leq \|x\|\|z - T(y_n)\| + \|T(x')\|\|\phi\|\|y_n - y\| \;\to\; 0 \end{aligned}$$

which implies that $x(z - T(y)) = 0$, since ran $\mathcal{A} = \{0\}$ therefore $z = Ty$ and by the closed graph theorem we conclude that T is a bounded operator.

Finally let $y \in T(\mathcal{A})$ and $z \in \mathcal{A}$, then there exists $x \in \mathcal{A}$ such that $y = T(x)$. Since ϕ is onto then there exists $z' \in \mathcal{A}$ such that $z = \phi(z')$ we obtain that

$$yz = T(x)z = T(x)\phi(z') = T(xz') \in T(\mathcal{A})$$

and

$$zy = zT(x) = \phi(z')T(x) = T(z'x) \in T(\mathcal{A}).$$

Then $T(\mathcal{A})$ is a two-sided ideal in $\mathcal{A}$. ∎

Remark 2.1. In hypothesis of the Proposition 2.1 if we take $\mathcal{A}$ a semi simple Banach algebra it suffice to suppose that ϕ is onto homomorphism because in this case it is automatically continuous by the Johnson's theorem [9].

Theorem 2.1 *Let $\mathcal{A}$ be a semi prime Banach algebra, ϕ be an onto homomorphism. If $T \circ \phi = \phi \circ T$ for all $T \in M_\phi(\mathcal{A})$ then $M_\phi(\mathcal{A})$ is semi prime Banach algebra.*

Proof. Let J be a two-sided ideal of $M_\phi(\mathcal{A})$ such that $J^2 = \{0\}$ and $T \in J$. We have for all $x, y \in \mathcal{A}$:

$$\begin{aligned}(Tx)(Ty) &= (Tx)(T[\phi(y')]) \quad (\phi \textit{ is onto } : \exists\, y' \in \mathcal{A}/\ \ y = \phi(y')) \\ &= (Tx)(\phi[T(y')]) \quad (T \circ \phi = \phi \circ T) \\ &= \phi(x)T^2(y') = 0 \quad\ (T \in M_\phi(\mathcal{A}) \textit{ and } T^2 \in J^2 = \{0\}).\end{aligned}$$

Then the two-sided ideal $T(\mathcal{A})$ in A satisfy $[T(\mathcal{A})]^2 = \{0\}$, since $\mathcal{A}$ is semi prime we conclude that $T(\mathcal{A}) = \{0\}$.Therefore $T = 0$, and hence $J = \{0\}$.
We deduce that $M_\phi(\mathcal{A})$ is semi prime Banach algebra. ∎

As corollary of this theorem, we obtain the classical following result [1, Theorem 1.4.4 ch IV].

Corollary 2.1 *Let $\mathcal{A}$ be a semi prime Banach algebra then $M(\mathcal{A})$ is semi prime Banach algebra.*

Proof. Since $\mathcal{A}$ is semi prime then $M(\mathcal{A}) = M_{Id_\mathcal{A}}(\mathcal{A})$ ($Id_\mathcal{A}$ the identity operator) and by theorem 2.1 we deduce that $M(\mathcal{A})$ is semi prime. ∎

Lemma 2.1 *Let $\mathcal{A}$ be a semi prime Banach algebra, ϕ be an onto homomorphism from $\mathcal{A}$ to $\mathcal{A}$ and $T \in M_\phi(\mathcal{A})$ such that $\phi \circ T = T \circ \phi$. Then we have*

$$\ker(T^2) \subseteq \ker(\phi \circ T).$$

Proof. Let $x \in \ker(T^2)$ and $a \in \mathcal{A}$ arbitrary, then

$$\begin{aligned}(\phi \circ T)(x)a(\phi \circ T)(x) &= (\phi \circ T)(x)\phi(a')(\phi \circ T)(x) \\ &\qquad (\phi \text{ is onto, then } \exists\, a' \text{ such that } \phi(a') = a) \\ &= (T \circ \phi)(x)\phi(a')(\phi \circ T)(x) \quad (\phi \circ T = T \circ \phi) \\ &= [T(\phi(x))\phi(a')]\phi(T(x)) \\ &= [\phi^2(x)T(a')]\phi(T(x)) \\ &\qquad (T \in M_\phi(\mathcal{A}) : T(\phi(x))\phi(a') = \phi^2(x)T(a')) \\ &= \phi^2(x)[T(a')\phi(T(x))] \\ &= \phi^2(x)\phi(a')T^2(x) \\ &\qquad (T \in M_\phi(\mathcal{A}) : T(a')\phi(T(x)) = \phi(a')T^2(x)) \\ &= 0 \qquad (x \in \ker(T^2) : T^2(x) = 0),\end{aligned}$$

which proves that $(\phi \circ T)(x)\mathcal{A}(\phi \circ T)(x) = \{0\}$, and since $\mathcal{A}$ is semi prime then $(\phi \circ T)(x) = 0$ and $x \in \ker(\phi \circ T)$. Finally we obtain $\ker(T^2) \subseteq \ker(\phi \circ T)$. ∎

Theorem 2.2 *Let $\mathcal{A}$ be a semi prime Banach algebra, ϕ be a bijective homomorphism from A to $\mathcal{A}$ and $T \in M_\phi(\mathcal{A})$ such that $\phi \circ T = T \circ \phi$. Then*

$$\ker(T^2) = \ker T.$$

Proof. It's clear that $\ker(T) \subseteq \ker(T^2)$. On the other hand, since ϕ is bijective by Lemma 2.1 we have

$$\ker(T^2) \subseteq \ker(\phi \circ T) = \ker(T).$$

Therefore

$$\ker(T^2) = \ker T.$$

■

Corollary 2.2 *Let $\mathcal{A}$ be a semi prime Banach algebra and $T \in M(\mathcal{A})$ then*

$$\ker(T^2) = \ker T.$$

Proof. Since $\mathcal{A}$ is a semi prime Banach algebra and $T \in M(\mathcal{A})$ then $T \in M_{Id_{\mathcal{A}}}(\mathcal{A})$ and by Theorem 2.2, we obtain that

$$\ker(T^2) = \ker T.$$

■

Theorem 2.3 *Let $\mathcal{A}$ be a semi prime Banach algebra, ϕ be an onto homomorphism from $\mathcal{A}$ to $\mathcal{A}$ and $T \in M_\phi(\mathcal{A})$ such that $\phi \circ T = T \circ \phi$. Then we have*

$$T(\mathcal{A}) \cap \ker(T) \subseteq \ker(\phi) \subseteq \ker(T)$$

Proof. Let $x \in T(\mathcal{A}) \cap \ker(T)$ then $Tx = 0$ and $\exists\, z \in \mathcal{A}$ such that $x = Tz$. We have, for all $a \in \mathcal{A}$,

$$\begin{array}{rcll} \phi(x)a\phi(x) & = & \phi(Tz)\phi(a')\phi(Tz) & (x = Tz \text{ and } \phi \text{ is onto } : \exists\, a' \in \mathcal{A} \,/\, \phi(a') = a) \\ & = & T(\phi(z))\phi(a')\phi(Tz) & (\phi \circ T = T \circ \phi) \\ & = & \phi^2(z)T(a')\phi(Tz) & (T \in M_\phi(\mathcal{A}) : T(\phi(z))\phi(a') = \phi^2(z)T(a')) \\ & = & \phi^2(z)\phi(a')T^2(z) & (T \in M_\phi(\mathcal{A}) : T(a')\phi(Tz) = \phi(a')T^2(z)) \\ & = & 0 & (T^2(z) = T(Tz) = T(x) = 0). \end{array}$$

We conclude that $\phi(x)\mathcal{A}\phi(x) = \{0\}$ and since $\mathcal{A}$ is semi prime then $\phi(x) = 0$ and $x \in \ker \phi$. Therefore, $T(\mathcal{A}) \cap \ker(T) \subseteq \ker \phi$.

Let $x \in \ker \phi$. We have, for all $a \in \mathcal{A}$,

$$\begin{array}{rcll} T(x)aT(x) & = & T(x)\phi(a')T(x) & (\phi \text{ is onto } : \exists\, a' \in \mathcal{A} \,/\, \phi(a') = a)) \\ & = & \phi(x)T(a')T(x) & (T \in M_\phi(\mathcal{A}) : T(x)\phi(a') = \phi(x)T(a')) \\ & = & 0 & (x \in \ker \phi \,:\, \phi(x) = 0). \end{array}$$

We conclude that $T(x)\mathcal{A}T(x) = \{0\}$, since $\mathcal{A}$ is semi prime then $T(x) = 0$ and $x \in \ker T$. Therefore,

$$\ker \phi \subseteq \ker T.$$

■

We obtain assertion (i) of [1, Theorem 1.4.32 Ch IV] as corollary of this theorem.

Corollary 2.3 *Let $\mathcal{A}$ be a semi prime Banach algebra and $T \in M(\mathcal{A})$ then*

$$T(\mathcal{A}) \cap \ker(T) = \{0\}.$$

Proof. Since $\mathcal{A}$ is a semi prime Banach algebra and $T \in M(\mathcal{A})$ then $T \in M_{Id_{\mathcal{A}}}(\mathcal{A})$ and by Theorem 2.3 we obtain $T(\mathcal{A}) \cap \ker(T) \subseteq \ker Id_{\mathcal{A}} = \{0\}$, consequently

$$T(\mathcal{A}) \cap \ker(T) = \{0\}. \qquad \blacksquare$$

Theorem 2.4 *Let $\mathcal{A}$ be a faithful Banach algebra, ϕ be a homomorphism from $\mathcal{A}$ to $\mathcal{A}$ and $T \in M_\phi(\mathcal{A})$. Then we have the following inclusions*

(1) $\overline{T(\mathcal{A})} \subseteq [\phi(\ker(T))]^\top$

(2) $\phi(\ker(T)) \subseteq [T(\mathcal{A})]^\top$.

Proof.

(1) Let $x \in \ker T$ and $y \in \overline{T(\mathcal{A})}$, then $\exists\ (z_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ such that $y = \lim_{n\to\infty} Tz_n$ and we have

$$\phi(x)Tz_n = T(x)\phi(z_n) = 0 \quad (T \in M_\phi(\mathcal{A}) \textit{ and } x \in \ker(T)),$$

and hence $\phi(x)y = \lim_{n\to\infty} \phi(x)Tz_n = 0$ for all $x \in \ker T$ and $y \in \overline{T(\mathcal{A})}$. In same way, we show that $y\phi(x) = 0$ for all $x \in \ker T$ and $y \in \overline{T(\mathcal{A})}$. Therefore, $\overline{T(\mathcal{A})} \subseteq [\phi(\ker(T))]^\top$.

(2) Let $x \in \ker T$ and $y \in T(\mathcal{A})$, then $Tx = 0$ and $\exists\ z \in \mathcal{A}$ such that $y = Tz$ it follows that $\phi(x)y = \phi(x)Tz = T(x)\phi(z) = 0$. In same wa, we show that $y\phi(x) = 0$, therefore, $\phi(\ker(T)) \subseteq [T(\mathcal{A})]^\top$. $\blacksquare$

As corollary of this theorem, we obtain assertion (i) of [1, Theorem 1.4.9, Ch. IV].

Corollary 2.4 *Let $\mathcal{A}$ be a faithful Banach algebra and $T \in M(\mathcal{A})$ then the following inclusions holds*

(1) $\overline{T(\mathcal{A})} \subseteq [\ker T]^\top$

(2) $\ker T \subseteq [T(\mathcal{A})]^\top$.

Proof. Since $\mathcal{A}$ is a faithful Banach algebra and $T \in M(\mathcal{A})$ then $T \in M_{Id_{\mathcal{A}}}(\mathcal{A})$. Theorem 2.4 implies that

(1) $\overline{T(\mathcal{A})} \subseteq [\ker T]^\top$

(2) $\ker T \subseteq [T(\mathcal{A})]^\top$. $\blacksquare$

In the next theorem, we justifies the existence of Helgason-Wang function (for details, see [1], [5] and [6]) of a ϕ-multiplier on $\mathcal{A}$.

Theorem 2.5 *Let $\mathcal{A}$ be a semi simple commutative Banach algebra and ϕ be a homomorphism from $\mathcal{A}$ to $\mathcal{A}$. Then, for each $T \in M_\phi(\mathcal{A})$, there exists an unique continuous function φ_T on $\Delta(\mathcal{A})$ such that the equation*

$$\widehat{Tx}(m) = \varphi_T(m)\widehat{\phi(x)}(m)$$

holds for all $x \in A$ and all $m \in \Delta(\mathcal{A})$.

Moreover, $|\varphi_T(m)| \leq \frac{\|m\|\|T\|}{\|m\circ\phi\|}$ for all $m \in \Delta(\mathcal{A})$.

Proof. For each $m \in \Delta(\mathcal{A})$ take $x \in A$ such that $\widehat{\phi(x)}(m) \neq 0$ and define

$$\varphi_T(m) := \frac{\widehat{Tx}(m)}{\widehat{\phi(x)}(m)}.$$

The definition of φ_T is independent of x because if $y \in \mathcal{A}$ such that $\widehat{\phi(y)}(m) \neq 0$ then since $(Tx)\phi(y) = \phi(x)(Ty)$ we obtain

$$\frac{\widehat{Tx}(m)}{\widehat{\phi(x)}(m)} = \frac{\widehat{Ty}(m)}{\widehat{\phi(y)}(m)}.$$

Hence the function φ_T is well defined.

Suppose $\widehat{\phi(x)}(m) = 0$ and let $y \in \mathcal{A}$ such that $\widehat{\phi(y)}(m) \neq 0$. Then we have

$$\widehat{Tx}(m)\widehat{\phi(y)}(m) = \widehat{\phi(x)}(m)\widehat{Ty}(m) = 0,$$

which implies $\widehat{Tx}(m) = 0$. Hence $\widehat{Tx}(m) = \varphi_T(m)\widehat{\phi(x)}(m)$ for all $x \in \mathcal{A}$ and $m \in \Delta(\mathcal{A})$.

On other hand, since ϕ is a homomorphism from semi-simple commutative Banach $\mathcal{A}$ then by Gelfant theorem ϕ is automatically continuous [9], consequently φ_T is a continuous function on $\Delta(\mathcal{A})$ with the Gelfand topology.

To prove the uniqueness of φ_T, suppose that there is an other complex-valued function defined on $\Delta(\mathcal{A})$ denote ψ for which $\widehat{Tx} = \widehat{\psi(x)}$. Then

$$(\varphi_T(m) - \psi(m))\widehat{\phi(x)}(m) = 0 \text{ for all } x \in \mathcal{A}$$

Therefore, $\varphi_T(m) = \psi(m)$.

Let us denote

$$\|m\| := sup\{|\widehat{x}(m)| : \|x\| = 1\}$$

and

$$\|m \circ \phi\| := sup\{|\widehat{\phi(x)}(m)| : \|x\| = 1\}$$

Because $0 < \|m\| \leq 1$ and $0 < \|m \circ \phi\| \leq \|\phi\| < \infty$, for each $x \in \mathcal{A}$ we have

$$|\varphi_T(m)||\widehat{\phi(x)}(m)| = |\varphi_T(m)\widehat{\phi(x)}(m)| = |\widehat{Tx}(m)| \leq \|m\|\|T\|\|x\|.$$

We obtain, for $x \in \mathcal{A}$, such that $\|x\| = 1$

$$\begin{aligned} |\varphi_T(m)| &\leq \inf_{\|x\|=1} \frac{\|m\|\|T\|}{|\widehat{\phi(x)}(m)|} \\ &\leq \frac{\|m\|\|T\|}{\sup_{\|x\|=1|} |\widehat{\phi(x)}(m)|} \\ &\leq \frac{\|m\|\|T\|}{\|m \circ \phi\|} \end{aligned}$$

So, $|\varphi_T(m)| \leq \frac{\|m\|\|T\|}{\|m\circ\phi\|}$ for all $m \in \Delta(\mathcal{A})$. ∎

Remark 2.2.

1) The function φ_T given by Theorem 2.5 to a ϕ-multiplier T will be called the Helgason-Wang function of T.

2) In Theorem 2.5, if we suppose that ϕ is homomorphism for which there is $\delta > 0$ such that $\delta\|m\| \leq \|m \circ \phi\|$ for all $m \in \Delta(\mathcal{A})$, then the function φ_T is bounded and satisfies $\|\varphi_T\|_\infty \leq \frac{1}{\delta}\|T\|$.

As corollary of this theorem, we obtain Wang's theorem [8], [1, Theorem 1.4.14 Ch IV].

Corollary 2.5 *Let $\mathcal{A}$ be a semi simple commutative Banach algebra. Then for each $T \in M(\mathcal{A})$ there exists an unique bounded continuous function φ_T on $\Delta(\mathcal{A})$ such that the equation*

$$\widehat{Tx}(m) = \varphi_T(m)\widehat{x}(m)$$

holds for all $x \in \mathcal{A}$ and all $m \in \Delta(\mathcal{A})$.

Moreover, $\|\varphi_T\|_\infty \leq \|T\|$ for all $T \in M(\mathcal{A})$.

Proof. Since $\mathcal{A}$ is commutative semi-simple Banach algebra and $T \in M(\mathcal{A})$. Then $T \in M_{Id_\mathcal{A}}(\mathcal{A}) = M(\mathcal{A})$ and, by Theorem 2.5, we obtain the existence of an unique continuous function φ_T on $\Delta(\mathcal{A})$ such that the equation

$$\widehat{Tx}(m) = \varphi_T(m)\widehat{Id_\mathcal{A}(x)}(m) = \varphi_T(m)\widehat{x}(m)$$

holds for all $x \in \mathcal{A}$ and all $m \in \Delta(\mathcal{A})$.

Moreover, by Theorem2.5, we have $|\varphi_T(m)| \leq \frac{\|m\|\|T\|}{\|m\circ Id_\mathcal{A}\|}$ for all $m \in \Delta(\mathcal{A})$. Then we conclude that $\|\varphi_T\|_\infty \leq \frac{\|m\|\|T\|}{\|m\|} = \|T\|$ for all $T \in M_{Id_\mathcal{A}}(\mathcal{A}) = M(\mathcal{A})$. ∎

Theorem 2.6 *Let $\mathcal{A}$ be a commutative semi simple Banach algebra and ϕ be a homomorphism from $\mathcal{A}$ to $\mathcal{A}$. If $T \in M_\phi(\mathcal{A})$ then we have*

$$T(\ker(m \circ \phi)) \subseteq \ker m \textit{ for each } m \in \Delta(\mathcal{A}).$$

Proof. Let $T \in M_\phi(\mathcal{A})$ and $m \in \Delta(\mathcal{A})$. Let $x \in \mathcal{A}$ such that $x \notin \ker(m \circ \phi)$, for every $y \in A$ such that $y \in \ker(m \circ \phi)$ we have

$$\widehat{Ty}(m)\widehat{\phi(x)}(m) = \widehat{\phi(y)}(m)\widehat{Tx}(m) = 0 \quad (\widehat{\phi(y)}(m) = m(\phi(y)) = 0)$$

Since $\widehat{\phi(x)}(m) \neq 0$ this implies $\widehat{Ty}(m) = m(Ty) = 0$ and hence $Ty \in \ker m$. Therefore, $T(\ker(m \circ \phi)) \subseteq \ker m$. ∎

References

[1] Aiena, P., *Fredholm and local spectral theory with aplications to multipliers,* Kluwer, 2004.

[2] Birtel, F.T., *Isomorphism and isometric multipliers,* Proc. Amer. Math. Soc., 13 (1962), 204-210.

[3] Darty, C., Maraz, G., *Multipliers problems,* Proceeding of the third Asian mathematical conference Diliman-Philippines, 2000.

[4] Helgason, S., *Multipliers of Banach algebras,* Ann. of Math., 64 (1956), 240-254.

[5] Larsen, R., *An introduction to the theory of multipliers,* Springer-Verlag, New York, 1971.

[6] Laursen, K.B., Neumann, M.M., *An Introduction to Local Spectral Theory,* Oxford University Press, New York, 2000.

[7] Riazi, A., Adib, M., *φ-multipliers on Banach algebras without order,* Int. Journal of Math. Analysis, 3 (3) (2009), 121-132.

[8] Wang, J.K., *Multipliers of commutative Banach algebras,* Pacific. J. Math., 11 (1961), 1131-1149.

[9] Zelazko, W., *Banach algebras,* Elsevier, Amsterdam, 1973.

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NEW RESULTS ON FIXED POINTS FOR AN INFINITE SEQUENCE OF MAPPINGS IN G-METRIC SPACE

Sh. Al-Sharif
Mathematics Department
Yarmouk University
Jordan
e-mail: sharifa@yu.edu.jo

A. Alahmari
Mathematics Department
Umm Al-Qura University
Saudi Arabia
e-mail: aasqaa@gmail.com

M. Al-Khaleel
A. Salem
Mathematics Department
Yarmouk University
Jordan
e-mails: mojarrah@yahoo.com
aminobedat@yahoo.com

Abstract. In this paper, we prove new results on coincidence and common fixed points for a sequence of mappings satisfying generalized $(\Psi - \Phi)$ contractive conditions in G-metric space. Also we investigate the existence of common fixed point for a sequence of mappings satisfying the almost generalized cyclic weak contractive condition in G-metric space. An example supporting our results is included.

1. Introduction

Fixed point theory is one of the most useful tools in analysis. The first result of fixed point theorem is given by Banach S. [4] by the general setting of complete metric space using which is known as Banach Contraction Principle. This principle has been generalized by many researchers in many ways like by [2], [5], [6], [11], [24]-[26], and so on.

In 2006, Mustafa and Sims [16] introduced a new structure to generalize the usual notion of metric space (X, d). This new structure leads to generalized metric spaces or to what are called G-metric spaces that allow to develop and introduce a new fixed point theory for various mappings. Later, several fixed point theorems were obtained in these new metric spaces for mappings satisfying certain contractive conditions. For example, in [9], [10], [13], [14] some fixed point results and theorems for self mappings satisfying some kind of contractive type conditions on complete G-metric spaces were proved. Abbas et. al. [1] studied common fixed point theorems for three maps in G-metric spaces. For other work on common fixed points using different conditions and considering more than three maps one can see [12, 23] and references therein. Fixed point results for cyclic ϕ-contraction mappings on metric spaces were proved by Pacurar and Rus [20]. Also, in [8], Karapinar obtained a unique fixed point of cyclic weak ϕ- contraction mappings. Whereas, Aydi [3] proved some fixed point theorems in G-metric spaces involving generalized cyclic contractions. Many researchers have also studied what are called coincidence points in G-metric spaces and obtained results for existence and uniqueness for such points, see for example [23], and for a recent work in metric spaces see [7].

In this paper, we prove some results on coincidence and common fixed points for a sequence of mappings in G-metric spaces.

Now, we give first in what follows preliminaries and basic definitions which will be used throughout the paper.

Definition 1.1 [16] Let X be a non-empty set, and let $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties.

(G1) $G(x, y, z) = 0$ if $x = y = z$.

(G2) $0 < G(x, x, y)$ whenever $x \neq y$, for all $x, y \in X$.

(G3) $G(x, x, y) \leq G(x, y, z)$ whenever $y \neq z$, for all $x, y, z \in X$.

(G4) $G(x, y, z) = G(x, z, y) = G(y, x, z) = ...$ (Symmetry in all three variables).

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$. (Rectangle Inequality).

Then the function G is called a generalized metric, or more specifically, a G-metric on X, and the pair (X, G) is called a generalized metric space, G-metric space.

Example 1.2 [13] Let (X, d) be any metric space. Define the mappings G_s and G_m on $X \times X \times X \to \mathbb{R}^+$ by

$$\begin{aligned} G_s(x, y, z) &= d(x, y) + d(y, z) + d(x, z), \\ G_m(x, y, z) &= \max\{d(x, y), d(y, z), d(x, z)\}, \forall x, y, z \in X. \end{aligned}$$

Then (X, G_s) and (X, G_m) are G -metric spaces.

Definition 1.3 [16] Let (X, G) be a G-metric space, and let x_n be a sequence of points of X, a point $x \in X$ is said to be the limit of the sequence x_n, if

$$\lim_{n,m\to\infty} G(x, x_n, x_m) = 0,$$

and one says that the sequence x_n is G-convergent.

Proposition 1.4 [16] *Let (X, G) be a G-metric space, then the following are equivalent.*

(1) x_n *is G-convergent to* x.

(2) $G(x_n, x_n, x) \to 0,$ *as* $n \to \infty$.

(3) $G(x_n, x, x) \to 0,$ *as* $n \to \infty$.

(4) $G(x_m, x_n, x) \to 0,$ *as* $m, n \to \infty$.

Definition 1.5 [13] Let (X, G) be a G-metric space, a sequence x_n is called G-Cauchy if for every $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \epsilon$, for all $n, m, l \geq N$; that is $G(x_n, x_m, x_l) \to 0$, as $n, m, l \to \infty$.

Proposition 1.6 [17] *If (X, G) is a G-metric space, then the following are equivalent.*

(1) *The sequence x_n is G-Cauchy.*

(2) *For every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \epsilon$, for all $n, m \geq N$.*

Definition 1.7 [16] A G-metric space (X, G) is said to be G-complete (or complete G-metric) if every G-Cauchy sequence in (X, G) is G-convergent in (X, G).

Definition 1.8 [16] Let (X, G) and (X', G') be two G-metric spaces, and let $f : (X, G) \to (X', G')$ *be a function. Then* f is said to be G-continuous at a point $a \in X$ if and only if, given $\epsilon > 0$, there exists $\delta > 0$ such that $x, y \in X$; and $G(a, x, y) < \delta$ implies that $G'(f(a), f(x), f(y)) < \epsilon$. A function f is G-continuous at X if and only if it is G-continuous at all $a \in X$.

Theorem 1.9 [16] *Let (X, G) and (X', G') be two G-metric spaces. Then a function $f : X \to X'$ is G-continuous at a point $x \in X$ if and only if it is G sequentially continuous at x, that is; whenever x_n is G-convergent to x, $f(x_n)$ is G-convergent to $f(x)$.*

Definition 1.10 [16] *Let (X, G) be a G-metric space. Then for $x_0 \in X$, $r > 0$, the G-ball with center x_0 and radius r is*

$$B_G(x_0, r) = \{y \in X : G(x_0, y, y) < r\}.$$

Theorem 1.11 [15] *Let (X, G) be a G-metric space. The sequence $x_n \subset X$ is G-convergent to x if it converges to x in the G-metric topology, $T(G)$.*

Definition 1.12 [7] The function $\Psi : [0, \infty) \to [0, \infty)$ is called an altering distance function, if the following properties are satisfied.

(i) Ψ is continuous and non-decreasing.

(ii) $\Psi(t) = 0$ if and only if $t = 0$.

Definition 1.13 [7] Let f, g be two self mappings on partially ordered set $(X, \preceq)$. A pair (f, g) is said

(i) weakly increasing if $fx \preceq g(fx)$ and $gx \preceq f(gx)$ for all $x \in X$.

(ii) partially weakly increasing if $fx \preceq g(fx)$ for all $x \in X$.

Definition 1.14 [7] Let $(X, \preceq)$ be partially ordered set and $f, g, h : X \to X$ are given mappings such that $fX \subseteq hX$ and $gX \subseteq hX$. We say that f and g are weakly increasing with respect to h if and only if for all $x \in X$, we have

$$fx \preceq gy, \ \forall y \in h^{-1}(fx),$$

and

$$gx \preceq fy, \ \forall y \in h^{-1}(gx).$$

If $f = g$, we say that f is weakly with respect to h.

Definition 1.15 [7] Let $(X, \preceq)$ be a partially ordered set and $f, g, h : X \to X$ given mappings such that $f(X) \subseteq h(X)$. We say that (f, g) are partially weakly increasing with respect to h if and only if for all $x \in X$, we have

$$fx \preceq gy, \ \forall y \in h^{-1}(fx).$$

Note that, a pairs f and g is weakly increasing with respect to h if and only if the ordered pair (f, g) and (g, f) are partially weakly increasing with respect to h.

Definition 1.16 [7] Let $(X, d, \preceq)$ be an ordered metric space. We say that X is regular if the following hypothesis holds: if $\{x_n\}$ is a non-decreasing in X with respect to $\preceq$ such that $x_n \to x \in X$ as $n \to \infty$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.

Definition 1.17 [21] Let T and S be two self mappings of a metric space (X, d). T and S are said to be weakly compatible if for all $x \in X$ the equality $Tx = Sx$ implies $TSx = STx$.

2. Main results

Theorem 2.18 *Let (X, G) be a complete ordered G-metric space such that X is regular. Let $T : X \to X$ be a self mappings and $\{f_k\}_{k=1}^{\infty}$ a sequence of mappings of X into itself. Suppose that for every $i, j \in \mathbb{N}$ and all $x, y \in X$ with Tx and Ty are comparable, we have*

$$\begin{aligned}\Psi(G(f_ix, f_jy, f_jy)) &\leq \Psi(M_{i,j}(x, y, y)) - \Phi(M_{i,j}(x, y, y)) \\ &+L \min\{G(f_ix, f_ix, Tx), G(f_jy, f_jy, Ty), G(f_ix, Ty, Ty), G(f_jy, f_jy, Tx)\},\end{aligned} \tag{2.1}$$

where $L \geq 0$,

$$\begin{aligned} M_{i,j}(x, y, y) &= \max\{G(Tx, Ty, Ty), G(Tx, f_ix, f_ix), G(Ty, f_jy, f_jy), \\ &\qquad \alpha[G(Tx, f_jy, f_jy) + G(f_ix, Ty, Ty)]\}, \end{aligned}$$

$0 \leq \alpha \leq \frac{1}{2}$, *and $\Psi : [0, \infty) \to [0, \infty)$ is an altering distance function, and $\Phi : [0, \infty) \to [0, \infty)$ is a continuous function with $\Phi(t) = 0$ if and only if $t = 0$. Assume that T and $\{f_k\}_{k=1}^{\infty}$ satisfy the following hypotheses:*

(i) $f_k X \subseteq TX$, *for every* $k \in \mathbb{N}$.

(ii) TX *is a closed subset of* (X, G).

(iii) (f_i, f_j) *are partially weakly increasing with respect to* T, *for every* $i, j \in \mathbb{N}$ *and* $j > i$.

Then T *and* $\{f_k\}_{k=1}^{\infty}$ *have a coincidence point* $u \in X$; *that is* $f_1 u = f_2 u = ... = Tu$.

Proof. Let x_0 be an arbitrary point in X. Since $f_1 X \subseteq TX$, there exists $x_1 \in X$ such that $Tx_1 = f_1 x_0$. Since $f_2 X \subseteq TX$, there exists $x_2 \in X$ such that $Tx_2 = f_2 x_1$. Continuing this process, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X defined by

$$y_n = f_{n+1} x_n = Tx_{n+1} \ , \forall n \in \mathbb{N} \cup \{0\}. \tag{2.2}$$

By construction, we have $x_{n+1} \in T^{-1}(f_{n+1} x_n)$. Then, using the fact that (f_{n+1}, f_{n+2}) are partially weakly increasing with respect to T, we obtain

$$Tx_{n+1} = f_{n+1} x_n \preceq f_{n+2} x_{n+1} = Tx_{n+2}, \forall \ n \in \mathbb{N} \cup \{0\}.$$

Therefore, we can write

$$Tx_1 \preceq Tx_2 \preceq ... \preceq Tx_{n+1} \preceq Tx_{n+2} \preceq ...,$$

or

$$y_0 \preceq y_1 \preceq ... \preceq y_n \preceq y_{n+1} \cdots . \tag{2.3}$$

We will prove our result in three steps.

Step1. We show that

$$\lim_{n \to \infty} G(y_n, y_{n+1}, y_{n+1}) = 0,$$

and

$$\lim_{n \to \infty} G(y_n, y_n, y_{n+1}) = 0.$$

First case. For every $n \in \mathbb{N}$, let $y_{n-1} \neq y_{n+1}$. Since Tx_n and Tx_{n+1} are comparable, by inequality (2.1), we have

$$\begin{aligned}
&\Psi(G(y_n, y_{n+1}, y_{n+1})) \\
&\quad = \Psi(G(f_{n+1} x_n, f_{n+2} x_{n+1}, f_{n+2} x_{n+1})) \\
&\quad \leq \Psi(M_{n+1,n+2}(x_n, x_{n+1}, x_{n+1})) - \Phi(M_{n+1,n+2}(x_n, x_{n+1}, x_{n+1})) \\
&\qquad +L \min\{G(f_{n+1} x_n, f_{n+1} x_n, Tx_n), G(f_{n+2} x_{n+1}, f_{n+2} x_{n+1}, Tx_{n+1}), \\
&\qquad G(f_{n+1} x_n, Tx_{n+1}, Tx_{n+1}), G(f_{n+2} x_{n+1}, f_{n+2} x_{n+1}, Tx_n)\} \\
&\quad = \Psi(M_{n+1,n+2}(x_n, x_{n+1}, x_{n+1})) - \Phi(M_{n+1,n+2}(x_n, x_{n+1}, x_{n+1})) \\
&\qquad +L \min \left\{ \begin{array}{c} G(y_n, y_n, y_{n-1}), G(y_{n+1}, y_{n+1}, y_n), \\ G(y_n, y_n, y_n), G(y_{n+1}, y_{n+1}, y_{n-1}) \end{array} \right\} \\
&\quad = \Psi(M_{n+1,n+2}(x_n, x_{n+1}, x_{n+1})) - \Phi(M_{n+1,n+2}(x_n, x_{n+1}, x_{n+1}))
\end{aligned} \tag{2.4}$$

where

$$\begin{aligned}
&M_{n+1,n+2}(x_n, x_{n+1}, x_{n+1}) \\
&= \max\left\{\begin{array}{c} G(Tx_n, Tx_{n+1}, Tx_{n+1}), G(Tx_n, f_{n+1}x_n, f_{n+1}x_n), \\ G(Tx_{n+1}, f_{n+2}x_{n+1}, f_{n+2}x_{n+1}), \\ \alpha[G(Tx_n, f_{n+2}x_{n+1}, f_{n+2}x_{n+1}) + G(f_{n+1}x_n, Tx_{n+1}, Tx_{n+1})] \end{array}\right\} \\
&= \max\left\{\begin{array}{c} G(y_{n-1}, y_n, y_n), G(y_{n-1}, y_n, y_n), G(y_n, y_{n+1}, y_{n+1}), \\ \alpha[G(y_{n-1}, y_{n+1}, y_{n+1}) + G(y_n, y_n, y_n)] \end{array}\right\} \\
&= \max\{G(y_{n-1}, y_n, y_n), G(y_n, y_{+1}, y_{n+1}), \alpha G(y_{n-1}, y_{n+1}, y_{n+1})\}.
\end{aligned}$$

Now, we have

$$G(y_{n-1}, y_{n+1}, y_{n+1}) \leq G(y_{n-1}, y_n, y_n) + G(y_n, y_{n+1}, y_{n+1}).$$

Hence, if $G(y_{n-1}, y_n, y_n) \leq G(y_n, y_{n+1}, y_{n+1})$, then we have

$$\begin{aligned}
G(y_{n-1}, y_{n+1}, y_{n+1}) &\leq G(y_{n-1}, y_n, y_n) + G(y_n, y_{n+1}, y_{n+1}) \\
&\leq G(y_n, y_{n+1}, y_{n+1}) + G(y_n, y_{n+1}, y_{n+1}) \\
&= 2G(y_n, y_{n+1}, y_{n+1}).
\end{aligned}$$

and

$$\frac{1}{2}G(y_{n-1}, y_{n+1}, y_{n+1}) \leq G(y_n, y_{n+1}, y_{n+1}).$$

Therefore, for $0 \leq \alpha \leq \frac{1}{2}$, we have

$$\alpha G(y_{n-1}, y_{n+1}, y_{n+1}) \leq G(y_n, y_{n+1}, y_{n+1}).$$

Similarly, one can do the same when $G(y_n, y_{n+1}, y_{n+1}) \leq G(y_{n-1}, y_n, y_n)$. Therefore, for $0 \leq \alpha \leq \frac{1}{2}$, we have always

$$\alpha G(y_{n-1}, y_{n+1}, y_{n+1}) \leq \max\{G(y_n, y_{n+1}, y_{n+1}), G(y_{n-1}, y_n, y_n)\}.$$

It follows that

$$M_{n+1,n+2}(x_n, x_{n+1}, x_{n+1}) = \max\{G(y_{n-1}, y_n, y_n), G(y_n, y_{n+1}, y_{n+1})\}. \tag{2.5}$$

If $G(y_{n-1}, y_n, y_n) < G(y_n, y_{n+1}, y_{n+1})$, then it follows from (2.5) that

$$M_{n+1,n+2}(x_n, x_{n+1}, x_{n+1}) = G(y_n, y_{n+1}, y_{n+1}).$$

Therefore, (2.4) implies that

$$\Psi(G(y_n, y_{n+1}, y_{n+1})) \leq \Psi(G(y_n, y_{n+1}, y_{n+1})) - \Phi(G(y_n, y_{n+1}, y_{n+1})),$$

which implies that $\Phi(G(y_n, y_{n+1}, y_{n+1})) = 0$, and hence we have

$$G(y_n, y_{n+1}, y_{n+1}) = 0,$$

or that $y_{n-1} = y_{n+1}$. This is a contradiction to our assumption that $y_{n-1} \neq y_{n+1}$. Therefore, for any $n \in \mathbb{N}$,

$$G(y_n, y_{n+1}, y_{n+1}) \leq G(y_{n-1}, y_n, y_n). \tag{2.6}$$

It follows that the sequence $\{G(y_n, y_{n+1}, y_{n+1})\}$ is a monotonic non-increasing sequence. Hence, there exists an $r \geq 0$ such that

$$\lim_{n\to\infty} G(y_n, y_{n+1}, y_{n+1}) = r. \tag{2.7}$$

We now prove that $r = 0$. As Ψ and Φ are continuous, and taking the limit on both sides of (2.4), we get

$$\begin{aligned}
&\Psi\left(\lim_{n\to\infty} G(y_n, y_{n+1}, y_{n+1})\right) \\
&\quad\leq \Psi(\max\{\lim_{n\to\infty} G(y_{n-1}, y_n, y_n), \lim_{n\to\infty} G(y_n, y_{n+1}, y_{n+1})\}) \\
&\qquad -\Phi(\max\{\lim_{n\to\infty} G(y_{n-1}, y_n, y_n), \lim_{n\to\infty} G(y_n, y_{n+1}, y_{n+1})\}) \\
&\quad\leq \Psi(\max\{\lim_{n\to\infty} G(y_{n-1}, y_n, y_n), \lim_{n\to\infty} G(y_n, y_{n+1}, y_{n+1})\}).
\end{aligned}$$

Then, by (2.7), we deduce that

$$\begin{aligned}
\Psi(r) &\leq \Psi(\max\{r, r\}) - \Phi(\max\{r, r\}) \\
&\leq \Psi(\max\{r, r\}) = \Psi(r),
\end{aligned}$$

which implies that $\Phi(\max\{r, r\}) = \Phi(r) = 0$; that is $r = 0$. Thus we have

$$\lim_{n\to\infty} G(y_n, y_{n+1}, y_{n+1}) = 0.$$

Second case. There exists an $n \in \mathbb{N}$ such that $y_{n-1} = y_{n+1}$. If there exists an $n \in \mathbb{N}$ such that $y_{n-1} = y_{n+1}$, then by (2.3), we have

$$y_{n-1} = y_n = y_{n+1}.$$

On the other hand, we have

$$\begin{aligned}
&M_{n+1,n+3}(x_n, x_{n+2}, x_{n+2}) \\
&\quad= \max\{G(Tx_n, Tx_{n+2}, Tx_{n+2}), G(Tx_n, f_{n+1}x_n, f_{n+1}x_n), \\
&\qquad G(Tx_{n+2}, f_{n+3}x_{n+2}, f_{n+3}x_{n+2}), \\
&\qquad \alpha[G(Tx_n, f_{n+3}x_{n+2}, f_{n+3}x_{n+2}) \\
&\qquad +G(Tx_{n+2}, Tx_{n+2}, f_{n+1}x_n)]\} \\
&\quad= \max\{G(y_{n-1}, y_{n+1}, y_{n+1}), G(y_{n-1}, y_n, y_n), \\
&\qquad G(y_{n+1}, y_{n+2}, y_{n+2}), \\
&\qquad \alpha[G(y_{n-1}, y_{n+2}, y_{n+2}) + G(y_{n+1}, y_{n+1}, y_n)]\} \\
&\quad= \max\{0, 0, G(y_{n+1}, y_{n+2}, y_{n+2}), \\
&\qquad \alpha[G(y_{n-1}, y_{n+2}, y_{n+2}) + 0]\} \\
&\quad= \max\{0, 0, G(y_n, y_{n+2}, y_{n+2}), \alpha G(y_n, y_{n+2}, y_{n+2})\}.
\end{aligned}$$

Since $0 \leq \alpha \leq \frac{1}{2}$, we have

$$M_{n+1,n+3}(x_n, x_{n+2}, x_{n+2}) = G(y_n, y_{n+2}, y_{n+2}).$$

Since Tx_n and Tx_{n+2} are comparable. By inequality (2.1), we have

$$\begin{aligned}
&\Psi(G(y_n, y_{n+2}, y_{n+2})) \\
&\quad = \Psi(G(f_{n+1}x_n, f_{n+3}x_{n+2}, f_{n+3}x_{n+2})) \\
&\quad \leq \Psi(M_{n+1,n+3}(x_n, x_{n+2}, x_{n+2})) - \Phi(M_{n+1,n+3}(x_n, x_{n+2}, x_{n+2})) \\
&\qquad +L\min\{G(f_{n+1}x_n, f_{n+1}x_n, Tx_n), G(f_{n+3}x_{n+2}, f_{n+3}x_{n+2}, Tx_{n+2}), \\
&\qquad G(f_{n+1}x_n, Tx_{n+2}, Tx_{n+2}), G(f_{n+3}x_{n+2}, f_{n+3}x_{n+2}, Tx_n)\} \\
&\quad = \Psi(G(y_n, y_{n+2}, y_{n+2})) - \Phi(G(y_n, y_{n+2}, y_{n+2})) \\
&\qquad +L\min\{G(y_n, y_n, y_{n-1}), G(y_{n+2}, y_{n+2}, y_{n+1}), \\
&\qquad G(y_n, y_{n+1}, y_{n+1}), G(y_{n+2}, y_{n+2}, y_{n-1})\} \\
&\quad = \Psi(G(y_n, y_{n+2}, y_{n+2})) - \Phi(G(y_n, y_{n+2}, y_{n+2})) + 0 \\
&\quad \leq \Psi(G(y_n, y_{n+2}, y_{n+2})),
\end{aligned}$$

which implies that $\Phi(G(y_n, y_{n+2}, y_{n+2})) \leq 0$; that is $G(y_n, y_{n+2}, y_{n+2}) = 0$, and hence $y_n = y_{n+2}$. Thus, for $k \geq n$, we have $y_k = y_{n-1}$. This implies that

$$\lim_{n\to\infty} G(y_n, y_{n+1}, y_{n+1}) = 0. \tag{2.8a}$$

Now,

$$\begin{aligned}
G(y_n, y_n, y_{n+1}) &= G(y_n, y_{n+1}, y_n) \\
&\leq G(y_n, y_{n+1}, y_{n+1}) + G(y_{n+1}, y_{n+1}, y_n) \\
&= 2G(y_n, y_{n+1}, y_{n+1}),
\end{aligned}$$

and by letting $n \to \infty$ in the above inequality and using (2.8a), we get

$$\lim_{n\to\infty} G(y_n, y_n, y_{n+1}) = 0. \tag{2.8b}$$

Step 2. We claim that $\{y_n\}$ is a Cauchy sequence in X. Suppose the contrary, i.e., $\{y_n\}$ is not a Cauchy sequence. Then, there exists an $\epsilon > 0$ for which we can find two subsequences $\{y_{m(k)}\}$ and $\{y_{n(k)}\}$ of $\{y_n\}$ such that $n(k)$ is the smallest index for which

$$n(k) > m(k) > k, G(y_{m(k)}, y_{n(k)}, y_{n(k)}) \geq \epsilon. \tag{2.9}$$

This means that

$$G(y_{m(k)}, y_{n(k)-1}, y_{n(k)-1}) < \epsilon. \tag{2.10}$$

Therefore, we use (2.9), (2.10), and the rectangle inequality to get

$$\begin{aligned}
\epsilon &\leq G(y_{m(k)}, y_{n(k)}, y_{n(k)}) \\
&\leq G(y_{m(k)}, y_{n(k)-1}, y_{n(k)-1}) + G(y_{n(k)-1}, y_{n(k)}, y_{n(k)}) \\
&< \epsilon + G(y_{n(k)-1}, y_{n(k)}, y_{n(k)}).
\end{aligned}$$

Letting $k \to \infty$ in the above inequality and using (2.8a), we obtain

$$\lim_{k\to\infty} G(y_{m(k)}, y_{n(k)}, y_{n(k)}) = \epsilon. \tag{2.11}$$

Again, using the rectangle inequality we have

$$\left|G(y_{m(k)-1}, y_{n(k)}, y_{n(k)}) - G(y_{m(k)}, y_{n(k)}, y_{n(k)})\right| \le G(y_{m(k)-1}, y_{m(k)}, y_{m(k)})$$

Letting $k \to \infty$ in the above inequality and using (2.8a), (2.11)

$$\lim_{k\to\infty} G(y_{m(k)-1}, y_{n(k)}, y_{n(k)}) = \epsilon. \tag{2.12}$$

On the other hand, we have

$$G(y_{m(k)}, y_{n(k)}, y_{n(k)}) \le G(y_{m(k)}, y_{n(k)+1}, y_{n(k)+1}) + G(y_{n(k)+1}, y_{n(k)}, y_{n(k)}).$$

Letting $k \to \infty$, we have from the above inequality that

$$\epsilon \le \lim_{k\to\infty} G(y_{m(k)}, y_{n(k)+1}, y_{n(k)+1}). \tag{2.13}$$

Also, by rectangle inequality, we have

$$\begin{aligned} G(y_{m(k)}, y_{n(k)}, y_{n(k)}) &\le G(y_{m(k)}, y_{m(k)-1}, y_{m(k)-1}) + G(y_{m(k)-1}, y_{n(k)}, y_{n(k)}) \\ &\le G(y_{m(k)}, y_{m(k)-1}, y_{m(k)-1}) + G(y_{m(k)-1}, y_{n(k)+1}, y_{n(k)+1}) \\ &\quad + G(y_{n(k)+1}, y_{n(k)}, y_{n(k)}). \end{aligned}$$

Letting $k \to \infty$ in the above inequality, and using (2.8b) and (2.11), we obtain

$$\epsilon \le \lim_{k\to\infty} G(y_{m(k)-1}, y_{n(k)+1}, y_{n(k)+1}).$$

Now,

$$\begin{aligned} G(y_{m(k)-1}, y_{n(k)+1}, y_{n(k)+1}) &\le G(y_{m(k)-1}, y_{m(k)}, y_{m(k)}) + G(y_{m(k)}, y_{n(k)+1}, y_{n(k)+1}) \\ &\le G(y_{m(k)-1}, y_{m(k)}, y_{m(k)}) + G(y_{m(k)}, y_{n(k)}, y_{n(k)}) \\ &\quad + G(y_{n(k)}, y_{n(k)+1}, y_{n(k)+1}). \end{aligned}$$

Letting $k \to \infty$ in the above inequality, and using (2.8a) and (2.13), we obtain

$$\lim_{k\to\infty} G(y_{m(k)-1}, y_{n(k)+1}, y_{n(k)+1}) \le \epsilon.$$

So,

$$\lim_{k\to\infty} G(y_{m(k)-1}, y_{n(k)+1}, y_{n(k)+1}) = \epsilon. \tag{2.14}$$

Now, by the rectangle inequality, we have

$$\begin{aligned} G(y_{m(k)}, y_{n(k)+1}, y_{n(k)+1}) &\le G(y_{m(k)}, y_{n(k)}, y_{n(k)}) + G(y_{n(k)}, y_{n(k)+1}, y_{n(k)+1}) \\ &\le G(y_{m(k)}, y_{m(k)-1}, y_{m(k)-1}) + G(y_{m(k)-1}, y_{n(k)}, y_{n(k)}) \\ &\quad + G(y_{n(k)}, y_{n(k)+1}, y_{n(k)+1}). \end{aligned}$$

Letting $k \to \infty$ in the above inequality and using (2.8a) and (2.12) we obtain

$$\lim_{k\to\infty} G(y_{m(k)}, y_{n(k)+1}, y_{n(k)+1}) \le \epsilon. \tag{2.15}$$

Therefore, from (2.12) and (2.15), we have

$$\lim_{k\to\infty} G(y_{m(k)}, y_{n(k)+1}, y_{n(k)+1}) = \epsilon. \tag{2.16}$$

From (2.1), we have

$$\begin{aligned}
&\Psi(G(y_{m(k)}, y_{n(k)+1}, y_{n(k)+1}))\\
&\quad= \Psi(G(f_{m(k)+1}x_{m(k)}, f_{n(k)+2}x_{n(k)+1}, f_{n(k)+2}x_{n(k)+1}))\\
&\quad\le \Psi(M_{m(k)+1,n(k)+2}(x_{m(k)}, x_{n(k)+1}, x_{n(k)+1})\\
&\qquad -\Phi(M_{m(k)+1,n(k)+2}(x_{m(k)}, x_{n(k)+1}, x_{n(k)+1})\\
&\qquad +L\min\{G(Tx_{m(k)}, f_{m(k)+1}x_{m(k)}, f_{m(k)+1}x_{m(k)}),\\
&\qquad G(Tx_{n(k)+1}, f_{n(k)+2}x_{n(k)+1}, f_{n(k)+2}x_{n(k)+1}),\\
&\qquad G(f_{m(k)+1}x_{m(k)}, Tx_{n(k)+1}, Tx_{n(k)+1}),\\
&\qquad G(f_{n(k)+2}x_{n(k)+1}, f_{n(k)+2}x_{n(k)+1}, Tx_{m(k)})\},
\end{aligned} \tag{2.17}$$

where

$$\begin{aligned}
&M_{m(k)+1,n(k)+2}(x_{m(k)}, x_{n(k)+1}, x_{n(k)+1})\\
&\quad= \max\{G(Tx_{m(k)}, Tx_{n(k)+1}, Tx_{n(k)+1}),\\
&\qquad G(Tx_{m(k)}, f_{m(k)+1}x_{m(k)}, f_{m(k)+1}x_{m(k)}),\\
&\qquad G(Tx_{n(k)+1}, f_{n(k)+2}x_{n(k)+1}, f_{n(k)+2}x_{n(k)+1}),\\
&\qquad \alpha[G(Tx_{m(k)}, f_{n(k)+2}x_{n(k)+1}, f_{n(k)+2}x_{n(k)+1})\\
&\qquad +G(f_{m(k)+1}x_{m(k)}, Tx_{n(k)+1}, Tx_{n(k)+1})]\}\\
&\quad= \max\{G(y_{m(k)-1}, y_{n(k)}, y_{n(k)}), G(y_{m(k)-1}, y_{m(k)}, y_{m(k)}),\\
&\qquad G(y_{n(k)}, y_{n(k)+1}, y_{n(k)+1}), \alpha[G(y_{m(k)-1}, y_{n(k)+1}, y_{n(k)+1})\\
&\qquad +G(y_{m(k)}, y_{n(k)}, y_{n(k)})]\},
\end{aligned}$$

and

$$\begin{aligned}
&L\min\{G(Tx_{m(k)}, f_{m(k)+1}x_{m(k)}, f_{m(k)+1}x_{m(k)}),\\
&\qquad G(Tx_{n(k)+2}, f_{n(k)+2}x_{n(k)+1}, f_{n(k)+2}x_{n(k)+1}),\\
&\qquad G(f_{m(k)+1}x_{m(k)}, Tx_{n(k)+1}, Tx_{n(k)+1}),\\
&\qquad G((f_{n(k)+2}x_{n(k)+1}, f_{n(k)+2}x_{n(k)+1}, Tx_{m(k)})\}\\
&= L\min\{G(y_{m(k)-1}, y_{m(k)}, y_{m(k)}), G(y_{n(k)+1}, y_{n(k)+1}, y_{n(k)+1}),\\
&\qquad G(y_{m(k)}, y_{n(k)}, y_{n(k)}), G(y_{n(k)+1}, y_{n(k)+1}, y_{m(k)-1})\}.
\end{aligned}$$

Letting $k \to \infty$ in (2.17) and using (2.8a), (2.8b), (2.11), (2.12), (2.14), (2.16), and the continuity of Ψ and Φ , we get that

$$\begin{aligned}
\Psi(\epsilon) &\le \Psi(\max\{\epsilon, 0, 0, \alpha(\epsilon+\epsilon)\}) - \Phi(\max\{\epsilon, 0, 0, \alpha(\epsilon+\epsilon)\})\\
&\le \Psi(\max\{\epsilon, 0, 0, \alpha(\epsilon+\epsilon)\},
\end{aligned}$$

or

$$\Psi(\epsilon) \leq \Psi(\epsilon) - \Phi(\epsilon) \leq \Psi(\epsilon).$$

Which implies that $\Psi(\epsilon) = 0$ and hence, $\epsilon = 0$, which is a contradiction. Thus, $\{y_n\}$ is a Cauchy sequence in X.

Step 3. In this step, we show the existence of a coincidence point for $\{f_k\}_{k=1}^{\infty}$ and T. From the completeness of (X, G), there exists $v \in X$ such that

$$\lim_{n\to\infty} y_n = v. \tag{2.18}$$

Since TX is a closed subset of (X, G), there exists $u \in X$ such that

$$y_n = Tx_{n+1} \to Tu.$$

Therefore, $v = Tu$. Since $\{y_n\}$ is a non-decreasing sequence and X is regular, it follows from (2.18) that $y_n \preceq v$ for all $n \in \mathbb{N} \cup \{0\}$. Hence,

$$Tx_n = y_{n-1} \preceq v = Tu.$$

Applying inequality (2.1), we get

$$\begin{array}{rcl} \Psi(G(y_n, f_k u, f_k u)) & = & \Psi(G(f_{n+1}x_n, f_k u, f_k u)) \\ & \leq & \Psi(M_{n+1,k}(x_n, u, u)) - \Phi(M_{n+1,k}(x_n, u, u)) + \\ & & L\min\{\, G(f_{n+1}x_n, f_{n+1}x_n, Tx_n), G(f_k u, f_k u, Tu), \\ & & G(f_{n+1}x_n, Tu, Tu), G(f_k u, f_k u, Tx_n)\}, \end{array} \tag{2.19}$$

where

$$\begin{aligned} M_{n+1,k}(x_n, u, u) &= \max\{G(Tx_n, Tu, Tu), G(Tx_n, f_{n+1}x_n, f_{n+1}x_n), G(Tu, f_k u, f_k u) \\ &\quad \alpha[G(Tx_n, f_k u, f_k u) + G(f_{n+1}x_n, Tu, Tu)]\} \\ &= \max\{G(y_{n-1}, v, v), G(y_{n-1}, y_n, y_n), G(v, f_k u, f_k u), \\ &\quad \alpha[G(y_{n-1}, f_k u, f_k u) + G(y_n, v, v)]\}, \end{aligned}$$

and

$$\begin{aligned} &L\min\left\{ \begin{array}{c} G(f_{n+1}x_n, f_{n+1}x_n, Tx_n), G(f_k u, f_k u, Tu), \\ G(f_{n+1}x_n, Tu, Tu), G(Tu, Tu, f_{n+1}x_n) \end{array} \right\} \\ &\quad = L\min\left\{ \begin{array}{c} G(y_{n-1}, y_{n-1}, y_n), G(f_k u, f_k u, v), \\ G(y_n, v, v), G(v, v, y_{n-1}) \end{array} \right\}. \end{aligned}$$

Letting $n \to \infty$ in (2.19) and using (2.18), we obtain

$$\begin{array}{rcl} \Psi(G(v, f_k u, f_k u)) & \leq & \Psi(\max\{0, 0, G(v, f_k u, f_k u), \alpha[G(v, f_k u, f_k u) + 0]\}) \\ & & -\Phi(\max 0, 0, G(v, f_k u, f_k u), \alpha[G(v, f_k u, f_k u) + 0]\}) \\ & & +L\min\{0, G(f_k u, f_k u, v), 0, 0\} \end{array}$$

Since, $0 \leq \alpha \leq \frac{1}{2}$, we have

$$\Psi(G(v, f_k u, f_k u)) = \Psi(G(v, f_k u, f_k u)) - \Phi(G(v, f_k u, f_k u)),$$

which implies that $\Phi(G(v, f_k u, f_k u)) = 0$; that is $G(v, f_k u, f_k u) = 0$, and hence

$$Tu = v = f_k u, \forall k \in \mathbb{N}.$$

Therefore, u is a coincidence point of $\{f_k\}_{k=1}^{\infty}$ and T. ■

The proof of the following theorem is omitted. It is similar to that of Theorem 2.18.

Theorem 2.19 *Let (X, G) be a complete ordered G-metric space such that X is regular. Let $T : X \to X$ be a self mappings and $\{f_k\}_{k=1}^{\infty}$ a sequence of mappings of X into itself. Suppose that for every $i, j \in \mathbb{N}$ and all $x, y \in X$ with Tx and Ty are comparable, we have*

$$\Psi(G(f_i x, f_j y, f_j y)) \leq \Psi(M_{i,j}(x, y, y)) - \Phi((N_{i,j}(x, y, y)),$$

where

$$\begin{aligned} M_{i,j}(x, y, y) \;\; &= \;\; \max\{G(Tx, Ty, Ty), G(Tx, f_i x, f_i x), G(Ty, f_j y, f_j y), \\ &\qquad \alpha[G(Tx, f_j y, f_j y) + G(Ty, Ty, f_i x)]\}, \end{aligned}$$

and $0 \leq \alpha \leq \frac{1}{2}$,

$$N_{i,j}(x, y, y) = \max\{G(Tx, Ty, Ty), G(Tx, f_j y, f_j y), G(Ty, Ty, f_i x)\},$$

and $\Psi : [0, \infty) \to [0, \infty)$ is an altering distance function, and $\Phi : [0, \infty) \to [0, \infty)$ is a continuous function with $\Phi(t) = 0$ if and only if $t = 0$. Assume that T and $\{f_k\}_{k=1}^{\infty}$ satisfy the following hypotheses:

(i) *$f_k X \subseteq TX$, for every $k \in \mathbb{N}$.*

(ii) *TX is a closed subset of (X, G).*

(iii) *(f_i, f_j) are partially weakly increasing with respect to T, for every $i, j \in \mathbb{N}$ and $j > i$.*

Then T and $\{f_k\}_{k=1}^{\infty}$ have a coincidence point $u \in X$; that is $f_1 u = f_2 u = ... = Tu$.

Corollary 2.20 *Let (X, G) be a complete ordered G-metric space such that X is regular. Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of mappings of X into itself. Suppose that for every $i, j \in \mathbb{N}$ and all $x, y \in X$ are comparable, we have*

$$\begin{aligned} \Psi(G(f_i x, f_j y, f_j y)) \;\; &\leq \;\; \Psi(M_{i,j}(x, y, y)) - \Phi((M_{i,j}(x, y, y)) \\ &\qquad + L \min \left\{ \begin{array}{c} G(f_i x, f_i x, x), G(f_j y, f_j y, y), \\ G(f_i x, y, y), G(f_j y, f_j y, x) \end{array} \right\} \end{aligned} \tag{2.20}$$

where

$$\begin{aligned} M_{i,j}(x, y, y) \;\; &= \;\; \max\{G(x, y, y), G(x, f_i x, f_i x), G(y, f_j y, f_j y), \\ &\qquad \alpha[G(x, f_j y, f_j y) + G(y, y, f_i x)]\}, \end{aligned}$$

$0 \leq \alpha \leq \frac{1}{2}$, *and Ψ and Φ are as in Theorem 2.18.*

If (f_i, f_j) is partially weakly increasing, for every $i, j \in \mathbb{N}$, then the sequence of mappings $\{f_k\}_{k=1}^{\infty}$ has a common fixed point $u \in X$; that is, $f_1 u = f_2 u = ... = u$.

Proof. It follows straightforwardly from Theorem 2.18, by taking the mapping $Tx = x$. ∎

Corollary 2.21 *Let (X, G) be a complete ordered G-metric space such that X is regular. Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of mappings of X into itself. Suppose that for every $i, j \in \mathbb{N}$ and all $x, y \in X$ are comparable, we have*

$$\Psi(G(f_ix, f_jy, f_jy)) \leq \Psi(M_{i,j}(x, y, y)) - \Phi((N_{i,j}(x, y, y)) \tag{2.21}$$

where

$$\begin{aligned} M_{i,j}(x, y, y) \quad = \quad & \max\{G(x, y, y), G(x, f_ix, f_ix), G(y, f_jy, f_jy), \\ & \alpha[G(x, f_jy, f_jy) + G(y, y, f_ix)]\}, \end{aligned}$$

and $0 \leq \alpha \leq \frac{1}{2}$,

$$N_{i,j}(x, y, y) = \max\{G(x, y, y), G(x, f_jy, f_jy), G(y, y, f_ix)\},$$

and Ψ and Φ are as in Theorem 2.19.

If (f_i, f_j) is partially weakly increasing, for every $i, j \in \mathbb{N}$, then the sequence of mappings $\{f_k\}_{k=1}^{\infty}$ has a common fixed point $u \in X$; that is, $f_1u = f_2u = ... = u$.

Proof. It follows straightforwardly from Theorem 2.19, by taking the mapping $Tx = x$. ∎

The following examples illustrate Corollary 2.20 and 2.21.

Example 2.22 Let $X = \{0, 1, 2, 3, ...\}$. We define a partial order $\preceq$ on X as $x \preceq y$ if $x \geq y$ for all $x, y \in X$. Let $G : X \times X \times X \to [0, \infty)$ be given by:

$$G(x, y, z) = \begin{cases} 0, & x = y = z, \\ x + y + z, & x \neq y \neq z, \\ x + y, & z = x \text{ or } z = y. \end{cases}$$

Let $f_k : X \to X$ be defined by:

$$f_k(x) = \begin{cases} 0\ , & 0 \leq x < k, \\ x - 1, & x \geq k. \end{cases}$$

For every $k \in \mathbb{N}$, define $\Psi, \Phi : [0, \infty) \to [0, \infty)$ by $\Psi(t) = t^2$ and $\Phi(t) = \ln(1 + t)$. Then we have the following.

(i) $(X, G, \preceq)$ is a complete partially ordered G-metric space.
(ii)$(X, G, \preceq)$ is regular.
(iii) (f_i, f_j) are partially weakly increasing for every $i, j \in \mathbb{N}$ such that $j > i$.
(iv) For any $i, j \in \mathbb{N}$, f_i and f_j satisfy (21), for every $x, y \in X$ with $x \preceq y$.

Thus, by Corollary 2.21, $\{f_k\}_{k=1}^{\infty}$ have a common fixed point. Moreover, 0 is the unique common fixed point of $\{f_k\}_{k=1}^{\infty}$.

Solution. The proof of (i) is clear. We need to show that X is regular, let $\{x_n\}$ be a non-decreasing sequence in X with respect to $\preceq$ such that $x_n \to x$, then there exists $k \in \mathbb{N}$ such that $x_n = x$ for all $n \geq k$. Hence, $(X, G, \preceq)$ is regular.

To prove (iii), let $x \in X$ and $i, j \in \mathbb{N}$ such that $j > i$.If $x < i$, then $f_i x = 0$ and $f_j(f_i x) = 0$. So, $f_i x \preceq f_j(f_i x)$.

If $i \leq x \leq j$, then $f_i x = x - 1$ and $f_j(f_i x) = 0$, so, $f_i x = x - 1 \geq 0 = f_j(f_i x)$ or $f_i x \preceq f_j(f_i x)$.

Finally, if $x \geq j$, then $f_i x = x - 1$ and $f_j(f_i x) = x - 2$, so, $f_i x = x - 1 \geq x - 2 = f_j(f_i x)$ or $f_i x \preceq f_j(f_i x)$. Therefore, (f_i, f_j) are partially weakly increasing.

Now, we prove (iv). By its symmetry and without loss of generality, it suffices to assume that $i \leq j$. Let $x, y \in X$ with $x \preceq y$, so $x \geq y$. Then, the following cases are possible.

Case 1. $x < i$ and $y < j$. Then $f_i x = 0$ and $f_j y = 0$ and hence,

$$G(f_i x, f_j y, f_j y) = 0,$$

$$\begin{aligned} M_{i,j}(x, y, y) &= \max\{G(x, y, y), G(x, 0, 0), G(y, 0, 0), \alpha[G(x, 0, 0) + G(0, y, y)]\} \\ &= \max\{G(x, y, y), x, y\}, \end{aligned}$$

and

$$\begin{aligned} N_{i,j}(x, y, y) &= \max\{G(x, y, y), G(x, 0, 0), G(0, y, y)\} \\ &= \max\{G(x, y, y), x, y\}. \end{aligned}$$

Since, by elementary calculus, $\Psi(t) - \Phi(t) \geq 0$, for every $t \geq 0$, we have

$$\Psi(0) = 0 \leq \Psi(\max\{G(x, y, y), x, y\}) - \Phi(\max\{G(x, y, y), x, y\}),$$

or

$$\Psi(G(f_i x, f_j y, f_j y)) \leq \Psi(M_{i,j}(x, y, y)) - \Phi(G(N_{i,j}(x, y, y)).$$

Case 2. $x \geq i$ and $y < j$. Then $f_i x = x - 1$ and $f_j y = 0$ and hence,

$$G(f_i x, f_j y, f_j y) = x - 1,$$

$$\begin{aligned} M_{i,j}(x, y, y) &= \max\{G(x, y, y), G(x, x-1, x-1), G(y, 0, 0), \\ &\quad \alpha[G(x, 0, 0) + G(x-1, y, y)]\} \\ &= \max\{G(x, y, y), 2x - 1, y, \alpha[x + G(x-1, y, y)]\} \\ &= 2x - 1, \end{aligned}$$

and

$$\begin{aligned} N_{i,j}(x, y, y) &= \max\{G(x, y, y), G(x, 0, 0), G(x-1, y, y)\} \\ &= \max\{G(x, y, y), x, G(x-1, y, y) \\ &= \begin{cases} 2x - 1, & x = y \text{ or } x - 1 = y, \\ x + y, & x \neq y \text{ and } x - 1 \neq y. \end{cases} \end{aligned}$$

If $x = y$ or $x - 1 = y$, then, since by elementary calculus,

$$(x-1)^2 \leq (2x-1)^2 - \ln(1+2x-1)$$

for every $x \geq 1$, we have

$$\Psi(G(f_i x, f_j y, f_j y) \leq \Psi(M_{i,j}(x,y,y)) - \Phi((N_{i,j}(x,y,y)).$$

If $x \neq y$ and $x - 1 \neq y$, then, since $y \leq x - 1$, we have

$$\ln(1+x+y) \leq \ln(1+x+x-1) = \ln(2x) \leq (2x-1)^2 - (x-1)^2,$$

or

$$(x-1)^2 \leq (2x-1)^2 - \ln(1+x+y).$$

Therefore,

$$\Psi(G(f_i x, f_j y, f_j y)) \leq \Psi(M_{i,j}(x,y,y)) - \Phi(G(N_{i,j}(x,y,y)).$$

Case 3. $x \geq i$ and $y \geq j$. Then $f_i x = x - 1$ and $f_j y = y - 1$ and hence

$$G(f_i x, f_j y, f_j y) = \begin{cases} 0, & x = y, \\ x + y - 2, & x \neq y, \end{cases}$$

$$\begin{aligned} M_{i,j}(x,y,y) &= \max\{G(x,y,y), G(x,x-1,x-1), G(y,y-1,y-1), \\ &\qquad \alpha[G(x,y-1,y-1) + G(x-1,y,y)]\} \\ &= \max\{G(x,y,y), 2x-1, 2y-1, \alpha[x+y-1+G(x-1,y,y)]\} \\ &= 2x-1, \end{aligned}$$

and

$$\begin{aligned} N_{i,j}(x,y,y) &= \max\{G(x,y,y), G(x,y-1,y-1), G(x-1,y,y)\} \\ &= \max\{G(x,y,y), x+y-1, G(x-1,y,y)\} \\ &= \begin{cases} 2x-1 & x = y \text{ or } x-1 = y, \\ x+y & x \neq y \text{ and } x-1 \neq y. \end{cases} \end{aligned}$$

If $x = y$, then, since $\Psi(2x-1) - \Phi(2x-1) \geq 0$, for every $x \geq 1$, we have

$$\Psi(0) = 0 \leq \Psi(2x-1) - \Phi(2x-1),$$

or

$$\Psi(G(f_i x, f_j y, f_j y)) \leq \Psi(M_{i,j}(x,y,y)) - \Phi((N_{i,j}(x,y,y)).$$

By elementary calculus, we get $\ln(2x) \leq 2(4x-5)$, for every $x \geq 2$.

If $x - 1 = y$, then we have

$$(2x-3)^2 \leq (2x-1)^2 - \ln(1+2x-1),$$

and, therefore,

$$\Psi(G(f_i x, f_j y, f_j y)) \leq \Psi(M_{i,j}(x,y,y)) - \Phi((N_{i,j}(x,y,y)).$$

If $x \neq y$ and $x - 1 \neq y$, then, since $y \leq x - 1$, we have

$$(x + y - 1)^2 \leq (2x - 1)^2 - \ln(1 + x + y),$$

and, therefore,

$$\Psi(G(f_i x, f_j y, f_j y)) \leq \Psi(M_{i,j}(x, y, y)) - \Phi((N_{i,j}(x, y, y)).$$

Thus, Ψ, Φ and $\{f_k\}_{k=1}^{\infty}$ satisfy all hypotheses of Corollary 2.21 and hence $\{f_k\}_{k=1}^{\infty}$ have a common fixed point. Indeed, 0 is the unique common fixed point of $\{f_k\}_{k=1}^{\infty}$.

Using the calculation above, in all cases one can show that

$$\begin{aligned}\Psi(G(f_i x, f_j y, f_j y)) \leq\ & \Psi(M_{i,j}(x, y, y)) - \Phi((M_{i,j}(x, y, y)). \\ & + L \min\{G(f_i x, f_i x, x), G(f_j y, f_j y, y), \\ & \qquad G(f_i x, y, y), G(f_j y, f_j y, x)\}.\end{aligned}$$

hence, by Corollary 2.20, $\{f_k\}_{k=1}^{\infty}$ have a common fixed point. Indeed, 0 is the unique common fixed point of $\{f_k\}_{k=1}^{\infty}$.

3. Cyclic contractions

In this section, we investigate the existence of common fixed point for a sequence of mappings satisfying the almost generalized cyclic weak contractive condition in G-metric space.

Definition 3.1 [8] Let X be a non-empty set, p be a positive integer, and $T : X \to X$ be a mapping. $X = \cup_{i=1}^{p} A_i$ is said to be a cyclic representation of X with respect to T if

(i) $A_i, i = 1, 2, ..., p$ are non-empty closed sets,

(ii) $T(A_1) \subseteq A_2, ..., T(A_{p-1}) \subseteq A_p, T(A_p) \subseteq A_1$.

Theorem 3.2 *Let (X, G) be a complete G-metric space, and let $A_1, A_2, ..., A_p$ be a non-empty closed subset of X and $X = \cup_{i=1}^{p} A_i$. Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of mappings of X into itself. Suppose that, for every $i, j \in \mathbb{N}$ and all $x, y \in X$, we have*

$$\begin{aligned}\Psi(G(f_i x, f_j y, f_j y)) \leq\ & \Psi(M_{i,j}(x, y, y)) - \Phi(M_{i,j}(x, y, y)) \\ & + L \min\{G(f_i x, f_i x, x), G(f_j y, f_j y, y), \\ & \qquad G(f_i x, y, y), G(f_j y, f_j y, x)\}, \end{aligned} \tag{3.1}$$

where $L \geq 0$, and

$$\begin{aligned}M_{i,j}(x, y, y) =\ & \max\{G(x, y, y), G(x, f_i x, f_i x), G(y, f_j y, f_j y), \\ & \alpha[G(x, f_j y, f_j y) + G(f_i x, y, y)]\},\end{aligned}$$

$0 \le \alpha \le \frac{1}{2}$, $\Psi : [0,\infty) \to [0,\infty)$ *is an altering distance function, and* $\Phi : [0,\infty) \to [0,\infty)$ *is a continuous function with* $\Phi(t) = 0$ *if and only if* $t = 0$.

Assume that $f_k(A_i) \subseteq A_{i+1}$, *for every* $k \in \mathbb{N}$. *Then,* $\{f_k\}_{k=1}^{\infty}$ *have a common fixed point* $u \in \cap_{i=1}^{p} A_i$; *that is,*

$$f_1 u = f_2 u = ... = u.$$

Proof. Let $x_0 \in A_1$, then there exists $x_1 \in A_2$ such that $x_1 = f_1 x_0$, and there exists $x_2 \in A_3$ such that $x_2 = f_2 x_1$. Continuing this process we can construct a sequence $\{x_n\}$ in X defined by

$$x_{n+1} = f_{n+1} x_n. \tag{3.2}$$

We will prove that

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0. \tag{3.3}$$

If, for some l, we have $x_{l+1} = x_l$, then (3.3) follows immediately. So, we can assume that $G(x_n, x_{n+1}, x_{n+1}) > 0$ for all n. Now, for all n, there exists $i = i(n) \in \{1, 2, ..., p\}$ such that $(x_n, x_{n+1}) \in A_i \times A_{i+1}$. Then, from (3.1), we have

$$\begin{aligned}
&\Psi(G(f_n x_{n-1}, f_{n+1}x_n, f_{n+1}x_n)\\
&\quad = \Psi(G(x_n, x_{n+1}, x_{n+1}))\\
&\quad \le \Psi(M_{n,n+1}(x_{n-1}, x_n, x_n)) - \Phi(M_{n,n+1}(x_{n-1}, x_n, x_n))\\
&\qquad +L \min\{G(f_n x_{n-1}, f_n x_{n-1}, x_{n-1}), G(f_{n+1}x_n, f_{n+1}x_n, x_n),\\
&\qquad G(f_n x_{n-1}, x_n, x_n), G(f_{n+1}x_n, f_{n+1}x_n, x_{n-1})\}\\
&\quad = \Psi(M_{n,n+1}(x_{n-1}, x_n, x_n)) - \Phi(M_{n,n+1}(x_{n-1}, x_n, x_n))\\
&\qquad +L \min\{G(x_n, x_n, x_{n-1}), G(x_{n+1}, x_{n+1}, x_n),\\
&\qquad G(x_n, x_n, x_n), G(x_{n+1}, x_{n+1}, x_{n-1})\}\\
&\quad \le \Psi(M_{n,n+1}(x_{n-1}, x_n, x_n)),
\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}
M_{n,n+1}(x_{n-1}, x_n, x_n) &= \max\{G(x_{n-1}, x_n, x_n), G(x_{n-1}, f_n x_{n-1}, f_n x_{n-1}),\\
&\quad G(x_n, f_{n+1}x_n, f_{n+1}x_n),\\
&\quad \alpha[G(x_{n-1}, f_{n+1}x_n, f_{n+1}x_n) + G(f_n x_{n-1}, x_n, x_n)]\}\\
&= \max\{G(x_{n-1}, x_n, x_n), G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1}),\\
&\quad \alpha[G(x_{n-1}, x_{n+1}, x_{n+1}) + G(x_n, x_n, x_n)]\}\\
&= \max\{G(x_{n-1}, x_n, x_n), G(x_n, x_{+1}, x_{n+1}),\\
&\quad \alpha G(x_{n-1}, x_{n+1}, x_{n+1})\}.
\end{aligned}$$

Now, we have

$$G(x_{n-1}, x_{n+1}, x_{n+1}) \le G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1}),$$

Hence, if $G(x_{n-1}, x_n, x_n) \leq G(x_n, x_{n+1}, x_{n+1})$, then we have

$$\begin{aligned} G(x_{n-1}, x_{n+1}, x_{n+1}) &\leq G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1}) \\ &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_n, x_{n+1}, x_{n+1}) \\ &= 2G(x_n, x_{n+1}, x_{n+1}), \end{aligned}$$

and

$$\frac{1}{2} G(x_{n-1}, x_{n+1}, x_{n+1}) \leq G(x_n, x_{n+1}, x_{n+1}).$$

Therefore, for $0 \leq \alpha \leq \frac{1}{2}$, we have

$$\alpha G(x_{n-1}, x_{n+1}, x_{n+1}) \leq G(x_n, x_{n+1}, x_{n+1}).$$

Similarly, one can do the same when $G(x_n, x_{n+1}, x_{n+1}) \leq G(x_{n-1}, x_n, x_n)$. Therefore, for $0 \leq \alpha \leq \frac{1}{2}$, we always have

$$\alpha G(x_{n-1}, x_{n+1}, x_{n+1}) \leq \max\{G(x_n, x_{n+1}, x_{n+1}), G(x_{n-1}, x_n, x_n)\}.$$

Consequently,

$$M_{n,n+1}(x_{n-1}, x_n, x_n) = \max\{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1})\}, \tag{3.5}$$

If $G(x_{n-1}, x_n, x_n) < G(x_n, x_{n+1}, x_{n+1})$, then it follows from (3.5) that

$$M_{n,n+1}(x_{n-1}, x_n, x_n) = G(x_n, x_{n+1}, x_{n+1}).$$

Therefore, equation (3.4) implies that

$$\Psi(G(x_n, x_{n+1}, x_{n+1})) \leq \Psi(G(x_n, x_{n+1}, x_{n+1})) - \Phi(G(x_n, x_{n+1}, x_{n+1})),$$

which implies that $\Phi(G(x_n, x_{n+1}, x_{n+1}))=0$, and hence we have $G(x_n, x_{n+1}, x_{n+1})=0$. This is contradicts our assumption that $G(x_n, x_{n+1}, x_{n+1}) > 0$. Therefore, for any $n \in \mathbb{N}$,

$$G(x_n, x_{n+1}, x_{n+1}) \leq G(x_{n-1}, x_n, x_n) \tag{3.6}$$

which implies that $\{G(x_n, x_{n+1}, x_{n+1})\}$ is a monotonic non-increasing. Then there exists $r \geq 0$ such that

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = r. \tag{3.7}$$

We now prove that $r = 0$. As Ψ and Φ are continuous, and taking the limit on both sides of equation (3.4), we get

$$\begin{aligned} &\Psi(\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1})) \\ &\quad \leq \Psi(\max\{\lim_{n\to\infty} G(x_{n-1}, x_n, x_n), \lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1})\}) \\ &\qquad -\Phi(\max\{\lim_{n\to\infty} G(x_{n-1}, x_n, x_n), \lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1})\}) \\ &\quad \leq \Psi(\max\{\lim_{n\to\infty} G(x_{n-1}, x_n, x_n), \lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1})\}). \end{aligned}$$

Then, by (3.7), we deduce that

$$\begin{aligned}\Psi(r) &\leq \Psi(\max\{r,r\}) - \Phi(\max\{r,r\}) \\ &\leq \Psi(\max\{r,r\}) = \Psi(r),\end{aligned}$$

which implies that $\Phi(\max\{r,r\}) = \Psi(r) = 0$, $r = 0$. Thus, we have

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0.$$

Now, we will prove that $\{x_n\}$ is a Cauchy sequence in (X, G). Suppose the contrary, i.e., $\{x_n\}$ is not a Cauchy sequence. Then, there exists an $\epsilon > 0$ for which we can find two subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ such that $n(k)$ is the smallest index for which $n(k) > m(k) > k$,

$$G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \geq \epsilon, G(x_{m(k)}, x_{n(k)-1}, x_{n(k)-1}) < \epsilon. \tag{3.8}$$

Using equation (3.8) and the rectangle inequality, we get

$$\begin{aligned}\epsilon &\leq G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \\ &\leq G(x_{m(k)}, x_{n(k)-1}, x_{n(k)-1}) + G(x_{n(k)-1}, x_{n(k)}, x_{n(k)}) \\ &< \epsilon + G(x_{n(k)-1}, x_{n(k)}, x_{n(k)}).\end{aligned}$$

Letting $k \to \infty$ in the above inequality and using (3.3), we obtain

$$\lim_{n\to\infty} G(x_{m(k)}, x_{n(k)}, x_{n(k)}) = \epsilon. \tag{3.9}$$

On the other hand, for all k, there exists $j(k) \,\epsilon\, \{1, ..., p\}$ such that

$$n(k) - m(k) + j(k) \equiv 1[p].$$

Then, $x_{m(k)-j(k)}$ (for k large enough; $m(k) > j(k)$) and $x_{n(k)}$ lie in different adjacently labelled sets A_i and A_{i+1} for certain $i\epsilon\{1, ..., p\}$. Using (3.1), we obtain

$$\begin{aligned}&\Psi(G(x_{m(k)-j(k)+1}, x_{n(k)+1}, x_{n(k)+1})) \\ &\quad= \Psi(G(f_{m(k)-j(k)+1}x_{m(k)-j(k)}, f_{n(k)+1}x_{n(k)}, f_{n(k)+1}x_{n(k)})) \\ &\quad\leq \Psi(M_{m(k)-j(k)+1,n(k)+1}(x_{m(k)-j(k)}, x_{n(k)}, x_{n(k)})) \\ &\qquad -\Phi(M_{m(k)-j(k)+1,n(k)+1}(x_{m(k)-j(k)}, x_{n(k)}, x_{n(k)})) \\ &\qquad +L\min\{G(x_{m(k)-j(k)+1}, x_{m(k)-j(k)+1}, x_{m(k)-j(k)}), \\ &\qquad G(x_{n(k)+1}, x_{n(k)+1}, x_{n(k)}), G(x_{m(k)-j(k)+1}, x_{n(k)}, x_{n(k)}), \\ &\qquad G(x_{n(k)+1}, x_{n(k)+1}, x_{m(k)-j(k)})\}\end{aligned} \tag{3.10}$$

for all k. Now, we have

$$\begin{aligned}&M_{m(k)-j(k)+1,n(k)+1}(x_{m(k)-j(k)}, x_{n(k)}, x_{n(k)}) \\ &\quad= \max\{G(x_{m(k)-j(k)}, x_{n(k)}, x_{n(k)}), \\ &\qquad G(x_{m(k)-j(k)}, x_{m(k)-j(k)+1}, x_{m(k)-j(k)+1}), G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}), \\ &\qquad \alpha[G(x_{m(k)-j(k)}, x_{n(k)+1}, x_{n(k)+1}) + G(x_{m(k)-j(k)+1}, x_{n(k)}, x_{n(k)})]\}\end{aligned}$$

for all k. Using the rectangle inequality and (3.3), we get

$$\begin{aligned}&\left|G(x_{m(k)-j(k)},x_{n(k)},x_{n(k)})-G(x_{m(k)},x_{n(k)},x_{n(k)})\right|\\&\quad\le G(x_{m(k)-j(k)},x_{m(k)},x_{m(k)})\\&\quad\le\sum_{l=0}^{j(k)-1}G(x_{m(k)-j(k)+l},x_{m(k)-j(k)+l+1},x_{m(k)-j(k)+l+1})\\&\quad\le\sum_{l=0}^{p}G(x_{m(k)-j(k)+l},x_{m(k)-j(k)+l+1},x_{m(k)-j(k)+l+1})\to 0,\text{as } k\to\infty,\end{aligned}$$

which implies from (3.9) that

$$\lim_{k\to\infty}G(x_{m(k)-j(k)},x_{n(k)},x_{n(k)})=\epsilon. \tag{3.11}$$

Using (3.3), we have

$$\lim_{k\to\infty}G(x_{m(k)-j(k)},x_{m(k)-j(k)+1},x_{m(k)-j(k)+1})=0. \tag{3.12}$$

$$\lim_{k\to\infty}G(x_{n(k)},x_{n(k)+1},x_{n(k)+1})=0. \tag{3.13}$$

Again, using the rectangle inequality, we get

$$\begin{aligned}&\left|G(x_{m(k)-j(k)},x_{n(k)+1},x_{n(k)+1})-G(x_{m(k)-j(k)},x_{n(k)},x_{n(k)})\right|\\&\qquad\le G(x_{n(k)},x_{n(k)+1},x_{n(k)+1}).\end{aligned} \tag{3.14}$$

Letting $k\to\infty$ in the above inequality, using (3.13) and (3.11), we get

$$\lim_{k\to\infty}G(x_{m(k)-j(k)},x_{n(k)+1},x_{n(k)+1})=\epsilon. \tag{3.15}$$

Similarly, we have

$$\begin{aligned}&\left|G(x_{m(k)-j(k)+1},x_{n(k)},x_{n(k)})-G(x_{m(k)-j(k)},x_{n(k)},x_{n(k)})\right|\\&\qquad\le G(x_{m(k)-j(k)+1},x_{m(k)-j(k)},x_{m(k)-j(k)}).\end{aligned}$$

Letting $k\to\infty$, using (3.3) and (3.11), we obtain

$$\lim_{k\to\infty}G(x_{m(k)-j(k)+1},x_{n(k)},x_{n(k)})=\epsilon. \tag{3.16}$$

Letting again $k\to\infty$ in (3.12) and using (3.3), (3.11), (3.12), (3.13), (3.15), (3.16) and the continuity of Ψ and Φ, we get that

$$\begin{aligned}\Psi(\epsilon)&\le\Psi(\max\{\epsilon,0,0,\alpha(\epsilon+\epsilon)\})-\Phi(\max\{\epsilon,0,0,\alpha(\epsilon+\epsilon\})+L\min\{0,0,\epsilon,\epsilon\}\\&\le\Psi(\max\{\epsilon,0,0,\alpha(\epsilon+\epsilon)\}.\end{aligned}$$

Since $0\le\alpha\le\frac{1}{2}$, then

$$\Psi(\epsilon)\le\Psi(\epsilon)-\Phi(\epsilon)+0\le\Psi(\epsilon),$$

which implies that $\Phi(\epsilon) = 0$, and hence that $\epsilon = 0$, which is a contradiction. Thus $\{x_n\}$ is a Cauchy sequence in X.

Finally, we need to show the existence of a common fixed point for $\{f_k\}_{k=1}^{\infty}$. Now, since (X, G) is complete, there exists $u \in X$ such that

$$\lim_{n\to\infty} x_n = u. \tag{3.17}$$

We will prove that

$$u \in \bigcap_{i=1}^{p} A_i. \tag{3.18}$$

Since $X = \bigcup_{i=1}^{p} A_i$, and since $x_0 \in A_1$, we have $\{x_{np}\}_{n\geq 0} \subseteq A_1$. Since A_1 is closed, from (3.17), we get that $u \in A_1$. Again, we have $\{x_{np+1}\}_{n\geq 0} \subseteq A_2$. Since A_2 is closed, from (3.17), we get that $u \in A_2$. Continuing this process, we obtain (3.18). Now, we will prove that u is a common fixed point of $\{f_k\}_{k=1}^{\infty}$. Indeed, from (3.18), since for all n, there exists $i(n) \in \{1, 2, ..., p\}$ such that $x_n \in A_{i(n)}$. Applying (3.1), with $x = u$ and $y = x_n$, we obtain

$$\begin{aligned} \Psi(G(f_k u, x_{n+1}, x_{n+1})) &= \Psi(G(f_k u, f_{n+1}x_n, f_{n+1}x_n)) \\ &\leq \Psi(M_{k,n+1}(u, x_n, x_n)) - \Phi(M_{k,n+1}(u, x_n, x_n)), \\ &\quad + L\min\{G(f_k u, f_k u, u), G(f_{n+1}x_n, f_{n+1}x_n, x_n), \\ &\qquad G(f_k u, x_n, x_n), G(f_{n+1}x_n, f_{n+1}x_n, u)\} \end{aligned} \tag{3.19}$$

where

$$\begin{aligned} M_{k,n+1}(u, x_n, x_n) &= \max\{G(u, x_n, x_n), G(u, f_k u, f_k u), G(x_n, f_{n+1}x_n, f_{n+1}x_n), \\ &\qquad \alpha[G(u, f_{n+1}x_n, f_{n+1}x_n) + G(f_k u, x_n, x_n)]\} \\ &= \max\{G(u, x_n, x_n), G(u, f_k u, f_k u), G(x_n, x_{n+1}, x_{n+1}), \\ &\qquad \alpha[G(u, x_{n+1}, x_{n+1}) + G(f_k u, x_n, x_n)]\}, \end{aligned}$$

and

$$\begin{aligned} &L\min\left\{ \begin{array}{c} G(f_k u, f_k u, u), G(f_{n+1}x_n, f_{n+1}x_n, x_n), \\ G(f_k u, x_n, x_n), G(f_{n+1}x_n, f_{n+1}x_n, u) \end{array} \right\} \\ &= L\min\left\{ \begin{array}{c} G(f_k u, f_k u, u), G(x_{n+1}, x_{n+1}, x_n), \\ G(f_k u, x_n, x_n), G(x_{n+1}, x_{n+1}, u) \end{array} \right\}. \end{aligned}$$

Letting $k \to \infty$ in (3.19) and using (3.17), we obtain

$$\begin{aligned} \Psi(G(f_k u, u, u)) &\leq \Psi(\max\{0, G(u, f_k u, f_k u), 0, \alpha[0 + G(f_k u, u, u)]\}) \\ &\quad - \Phi(\max\{0, G(u, f_k u, f_k u), 0, \alpha[0 + G(f_k u, u, u)]\}), \\ &\quad + L\min\{G(f_k u, f_k u, u), 0, G(f_k u, u, u), 0\}, \end{aligned}$$

or

$$\begin{aligned} \Psi(G(f_k u, u, u)) &\leq \Psi(G(f_k u, f_k u, u)) - \Phi(G(f_k u, u, u)) \\ &\leq \Psi(G(f_k u, f_k u, u)), \end{aligned}$$

which implies that $\Phi(G(f_k u, u, u)) \leq 0$, that is, $G(f_k u, u, u) = 0$ and hence $u = f_k u$, $\forall k \in \mathbb{N}$. Therefore, u is a common fixed point of $\{f_k\}_{k=1}^{\infty}$. ∎

The proof of the following theorem is omitted because it is similar to that of Theorem 3.2.

Theorem 3.3 *Let (X, G) be a complete G-metric space, and let $A_1, A_2, ..., A_p$ be a non-empty closed subset of X and $X = \bigcup_{i=1}^{p} A_i$. Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of mappings of X into itself. Suppose that for every $i, j \in \mathbb{N}$ and all $x, y \in X$, we have*

$$\Psi(G(f_i x, f_j y, f_j y)) \leq \Psi(M_{i,j}(x, y, y)) - \Phi(N_{i,j}(x, y, y))$$

where

$$\begin{aligned} M_{i,j}(x, y, y) &= \max\{G(x, y, y), G(x, f_i x, f_i x), G(y, f_j y, f_j y), \\ &\quad \alpha[G(x, f_j y, f_j y) + G(f_i x, y, y)]\}, \end{aligned}$$

$$N_{i,j}(x, y, y) = \max\{G(x, y, y), G(x, f_j y, f_j y), G(f_i x, y, y)\},$$

where $0 \leq \alpha \leq \frac{1}{2}$. Let $\Psi : [0, \infty) \to [0, \infty)$ be an altering distance function, and $\Phi : [0, \infty) \to [0, \infty)$ be a continuous function with $\Phi(t) = 0$ if and only if $t = 0$.

Assume that $f_k(A_i) \subseteq A_{i+1}$, for every $k \in \mathbb{N}$. Then, $\{f_k\}_{k=1}^{\infty}$ have a common fixed point $u \in \cap_{i=1}^{p} A_i$; that is,

$$f_1 u = f_2 u = ... = u.$$

4. Conclusions

We have proved some results on coincidence and common fixed points for a sequence of mappings satisfying generalized ($\Psi - \Phi$) contractive conditions in G-metric space. The existence of common fixed point for a sequence of mappings satisfying the almost generalized cyclic weak contractive condition in G-metric space is investigated and an example supporting our results is included.

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References

[1] Abbas, M., Nazir, T., Vetro, P., *Common fixed point results for three maps in G-metric spaces*, Faculty of Sciences and Mathematics, University of Nis, Serbia, Filomat 25:4, DOI:10.2298/FIL1104001A, 2011.

[2] Arvanitakis, A.D., *A proof of the generalized Banach contraction conjecture*, Proc. Am. Math. Soc., 131 (12) (2003), 3647-3656.

[3] Aydia, H., Felhib,A., Sahmim, S., *Related fixed point results for cyclic contractions on G-metric spaces and applications*, Filomat, 2015, to appear.

[4] Banach, S., *Sur les opérations dans les ensembles abstraits et leurs applications aux équations intégrales*, Fund Math., 3 (1922), 133–181.

[5] Boyd, D.W., Wong, J.S.W., *On nonlinear contractions*, Proc. Am. Math. Soc., 20 (1969), 458-464.

[6] Choudhury, B.S., Das, K.P., *A new contraction principle in Menger spaces*, Acta Math. Sin., 24 (8) (2008), 1379-1386.

[7] Esmaily, J., Vaezpour, S.M., Rhoades, B.E., *Coincidence and common fixed point theorems for a sequence of mappings in ordered metric spaces*, App. Math. and Comp., 219 (2013), 5684-5692.

[8] Karapinar, E., *Fixed point theory for cyclic weak ϕ-contraction*, Appl. Math. Lett., 24 (2011), 822-825.

[9] Khandaqji, M., Al-Sharif, Sh., Al-Khaleel, M., *Property P and some fixed point results on (Ψ, Φ)-weak contractive G-metric spaces*, Int. J. of Math. and Math. Sci., Volume 2012 (2012), Article ID 675094, 11 pages doi:10.1155/2012/675094.

[10] Mohanta, S.K., *Some fixed point theorems in G-metric spaces*, An. St. Univ. Ovidius Constanţa, 20 (1) (2012), 285-306.

[11] Mongkolkeha, C., Sintunavarat, W., Kumam, P., *Fixed point theorems for contraction mappings in modular metric spaces*, Fixed Point Theory Appl., 93 (2011).

[12] Mustafa, Z., *Common fixed points of weakly compatible mappings in G-metric spaces*, App. Math. Sci., 6 (92) (2012), 4589-4600.

[13] Mustafa, Z., Khandaqji, M., Shatanawi, W., *Fixed point results on complete G-metric spaces,* Studia Sci. Math. Hung., 48 (3) (2011), 304-319.

[14] Mustafa, Z., Obiedat, H., Awawdeh, F., *Some fixed point theorems for mapping on complete G-metric spaces*, Fixed Point Theory and Applications, Volume 2008, Article ID 189870, 12 pages, doi:10.1155/2008/189870.

[15] Mustafa, Z., Shatanawi, W., Bataineh, M., *Existence of fixed point results in G-meteic spaces*, Int. J. of Math. and Math. Sci., vol. 2009, article ID 283028, 10 pages.

[16] Mustafa, Z., Sims, B., *A new approach to generalized metric spaces*, J. Nonlinear and Convex Anal., 7 (2) (2006), 289-297.

[17] Mustafa, Z., Sims, B., *Some remarks concerning D-metric spaces*, Proceedings of the International Conference on Fixed Point Theory and Application: Valencia, Spain, 2003. Yokohama Publishers, Yokohama, 189-198, 2004.

[18] Naidu, S., Rao, K., Sirnivasa Rao, N., *On the concepts of balls in a D-metric space*, Int. J. of Math. and Math. Sci., 1 (2005), 133-141.

[19] Naidu, S., Rao, K., Sirnivasa Rao, N., *On the convergent sequences and fixed point theorems in a D-metric space*, Int. J. of Math. and Math. Sci., 12 (2005), 1969-1988.

[20] Păcurar, M., Rus, I.A., *Fixed point theory for cyclic φ-contractions*, Nonlinear Anal., 72 (2010), 1181-1187.

[21] Parvanch, V., Bracvh, G.G., *Some fixed point theorems in complete metric spaces*, Int. J. Pure Appl. Math., 76 (1) (2012), 1-8. ISSN: 1311-8080.

[22] Petric, M.A., *Fixed points and best proximity points theorems for cyclical contractive operators*, Ph.D. Thesis, North University of Baia Mare, Romania, 2011.

[23] Rao, K.P.R., Sombabu, A., Prasad, J.R., *A common fixed point theorem for six expansive mappings in G-metric spaces*, Kath. Univ. J. of Sci., Eng. and Tech., 7 (I) (2011), 113-120.

[24] Sintunavarat, W., Kumam, P., *Gregus type fixed points for a tangential multi-valued mappings satisfying contractive conditions of integral type*, J. Inequal. Appl., 3 (2011).

[25] Sintunavarat, W., Cho, Y.J., Kumam, P., *Common fixed point theorems for c-distance in ordered cone metric spaces*, Comput. Math. Appl., 62 (2011), 1969-1978.

[26] Suzuki, T., *A generalized Banach contraction principle that characterizes metric completeness*, Proc. Am. Math. Soc., 136 (5) (2008), 1861-1869.

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APPROXIMATION OF CONTIONUOUS FUNCTIONS BY VALLEE-POUSSIN'S SUMS

Rateb Al-Btoush

Department of Mathematics
Faculty of Science
Mutah University
Mutah
Jordan
e-mail: btoush@mutah.edu.jo

Al-Oushoush Nizar Kh.Kh.

Department of Mathematics
Faculty of Science
Balqa' Applied University
Al-Salt
Jordan
e-mail: aloushoush@bau.edu.jo

Abstract. Let $V_{n,m}^{(\alpha,\beta)}(f;x) = \frac{1}{m+1}\sum\limits_{k=n}^{n+m} S_k^{(\alpha,\beta)}(f;x)$ be the Vallee-Poussin's partial sums of Fourier-Jacobi series. In this paper, we study the deviations of $V_{n,m}^{(\alpha,\beta)}(f;x)$ on $[-1,1]$ for continuous function $f(x)$.

1. Introduction

Let $P_n^{(\alpha,\beta)}(x)$ $(n = 0, 1, 2, ...)$ denote the Jacobi orthonormal system of polynomials with weight function

$$(1-x)^\alpha(1+x)^\beta,\ (\alpha > -1, \beta > -1) \text{ on } [-1,1].$$

Furthermore, let

$$\sum_{k=0}^{\infty} C_k(f)P_k^{(\alpha,\beta)}(x), \tag{1.1}$$

be the Fourier-Jacobi series of the function $f(x)$, where

$$C_k(f) = \int_{-1}^{1} (1-t)^\alpha(1+t)^\beta f(t)P_k^{(\alpha,\beta)}(t)dt. \tag{1.2}$$

Denote the Fourier-Jacobi series of partial sums, $S_n^{(\alpha,\beta)}(f;x)$ as

$$S_n^{(\alpha,\beta)}(f;x) = \sum_{k=0}^{n} C_k(f) P_k^{(\alpha,\beta)}(x). \tag{1.3}$$

Denote the Fejer sum of $f(x)$, $\sigma_n^{(\alpha,\beta)}(f;x)$ as

$$\sigma_n^{(\alpha,\beta)}(f;x) = \frac{1}{(n+1)} \sum_{k=0}^{n} S_k(f;x). \tag{1.4}$$

Define the Vallee-Poussin's partial sums of Fourier-Jacobi series,$V_{n,m}^{(\alpha,\beta)}(f;x)$, as

$$V_{n,m}^{(\alpha,\beta)}(f;x) = \frac{1}{m+1} \sum_{k=n}^{n+m} S_k^{(\alpha,\beta)}(f;x). \tag{1.5}$$

An estimation for the deviation of the continuous function $f(x)$, with period 2π from its Fourier sum $S_n(f;x)$ is given in [1], when $f(x)$has a bounded variation and supreme modulus of continuity. This paper generalizes and improves some results in theory of approximation of continuous functions, as those reported in [1]–[10]. The efficient study for approximation by Vallee-Poussin sums has been carried out for several decades. Recently, several studies dealing with the Vallee-Poussin sums have been introduced, see [8]–[10]. The results presented in this paper generalize and improve many results of [1]–[10] and many others in theory of approximation of continuous functions.

Our problem, here, is to study the deviations of $S_n^{(\alpha,\beta)}(f;x)$ on $[-1,1]$ for continuous function of one variables $f(x)$.

In this regard, three theorems have been introduced.

2. Jackson's theorem

We state one of most important theorem in the approximation theory, namely Jackson's theorem, and which is used in our study.

Let $f(x)$ be continuous function on closed interval $[a,b]$. Denote by $E_n(f)$ the best uniform approximation element of a function $f(x)$ by algebraic polynomials of order not exceeded n on $[a,b]$, i.e.,

$$E_n(f) = \inf \left\{ \max_{x\in[a,b]} \left| f(x) - \sum_{k=0}^{\infty} C_k x^k \right| \right\}. \tag{2.1}$$

Jackson's Theorem. *If $\omega(f;\frac{1}{n})$ is a modulus of continuity of a function $f(x)$, then the inequality*

$$E_n(f) \le c\,\omega\left(f; \frac{b-a}{n}\right), \tag{2.2}$$

holds, where c an absolute, and $\omega(f;t)$ is defined as

$$\omega(f;t)=\sup_{\substack{|x_1-x_2|\\ x_1,x_2\in[a,b]}}|f(x_1)-f(x_2)|\,. \tag{2.3}$$

In case of $f(x)\in C[-1,1]$, then

$$E_n(f)\le c\,\omega\left(f;\frac{1}{n}\right). \tag{2.4}$$

3. The main result

Let $P_n^{(\alpha,\beta)}(x)$ $(n=0,1,2,...)$ denote the Jacobi orthonormal system of polynomials with weight function

$$(1-x)^{\alpha}(1+x)^{\beta},\ (\alpha>-1,\beta>-1)\text{ on }[-1,1].$$

We start with the following special cases of the Jacobi polynomial:

1. $\alpha=-\frac{1}{2}$, $\beta=-\frac{1}{2}$, then

$$P_n^{(-\frac{1}{2},-\frac{1}{2})}(x)=\sqrt{\frac{2}{\pi}}\cos(n\cos^{-1}x), \tag{3.1}$$

where

$$P_0^{(-\frac{1}{2},-\frac{1}{2})}(x)=\frac{1}{\sqrt{\pi}},$$

which are called the Tschebyscheff polynomials of the first kind.

2. $\alpha=\frac{1}{2}$, $\beta=\frac{1}{2}$, then

$$P_n^{(\frac{1}{2},\frac{1}{2})}(x)=\sqrt{\frac{2}{\pi}}\frac{\sin\left((n+1)\cos^{-1}x\right)}{\sin\left(\cos^{-1}x\right)}, \tag{3.2}$$

which are called the Tschebyscheff polynomials of the second kind.

3. $\alpha=\frac{1}{2}$, $\beta=-\frac{1}{2}$, then

$$P_n^{(\frac{1}{2},-\frac{1}{2})}(x)=\sqrt{\frac{2}{\pi}}\frac{\sin\frac{2n+1}{2}\cos^{-1}x}{\sin\frac{1}{2}\left(\cos^{-1}x\right)}, \tag{3.3}$$

4. $\alpha=-\frac{1}{2}$, $\beta=\frac{1}{2}$, then

$$P_n^{(-\frac{1}{2},\frac{1}{2})}(x)=\sqrt{\frac{2}{\pi}}\frac{\cos\frac{2n+1}{2}\cos^{-1}x}{\cos\frac{1}{2}\left(\cos^{-1}x\right)}, \tag{3.4}$$

In this work, we will prove the following three theorems:

Theorem 1. *If* $M = \max_{-1\le x\le 1} |f(x)|$ *, then the following inequalities hold*

$$\left|V_{n,m}^{(-\frac{1}{2},-\frac{1}{2})}(f;x)\right| \le CM\left(1+\ln\left(\frac{n+m+1}{m+t1}\right)\right). \tag{3.5}$$

$$\left|V_{n,m}^{(\frac{1}{2},\frac{1}{2})}(f;x)\right| \le \frac{CM}{\sqrt{1-x^2}}\left(1+\ln\left(\frac{n+m+1}{m+1}\right)\right). \tag{3.6}$$

$$\left|V_{n,m}^{(\frac{1}{2},-\frac{1}{2})}(f;x)\right| \le \frac{CM}{\sqrt{1-x}}\left(1+\ln\left(\frac{n+m+1}{m+1}\right)\right). \tag{3.7}$$

$$\left|V_{n,m}^{(-\frac{1}{2},\frac{1}{2})}(f;x)\right| \le \frac{CM}{\sqrt{1+x}}\left(1+\ln\left(\frac{n+m+1}{m+1}\right)\right). \tag{3.8}$$

Proof. First, we consider the case $\alpha = \beta = \frac{1}{2}$ (equation (3.6)). Define

$$P_n(x) = P_n^{(\frac{1}{2},\frac{1}{2})}(x),\ S_n(f;x) = S_n^{(\frac{1}{2},\frac{1}{2})}(f;x), \sigma_n(f;x) = \sigma_n^{(\frac{1}{2},\frac{1}{2})}(f;x),$$
$$V_{n,m}(f;x) = V_{n,m}^{(\frac{1}{2},\frac{1}{2})}(f;x), S_n(f;x) = \sum_{k=0}^{n} C_k(f)P_k(x),$$

where

$$C_k(f) = \int_{-1}^{1} \sqrt{1-t^2} f(t) P_k(t)dt.$$

Using the definition of $P_k(x)$, given in (3.2), we have

$$\begin{aligned} C_k(f) &= \sqrt{\frac{2}{\pi}}\int_{-1}^{1}\sqrt{1-t^2}f(t)\frac{\sin\left((k+1)\cos^{-1}t\right)}{\sin(\cos^{-1}t)}dt \\ &= \sqrt{\frac{2}{\pi}}\int_{-1}^{1} f(t)\sin\left((k+1)\cos^{-1}t\right)dt \qquad (3.9)\\ &= \sqrt{\frac{2}{\pi}}\int_{0}^{\pi} f(\cos t)\sin(k+1)t\sin t dt. \end{aligned}$$

Then

$$\begin{aligned} S_n(f;x) &= \sum_{k=0}^{n} C_k(f)P_k(x) \\ &= \frac{2}{\pi\sin y}\int_0^{\pi} f(\cos t)\sin t\sum_{k=0}^{n}\sin(k+1)t\sin(k+1)y dt, \end{aligned}$$

where $y = \cos^{-1} x$ and, since $\sin\alpha \sin\beta = \frac{1}{2}(\cos(\alpha-\beta) - \cos(\alpha+\beta))$, we obtain

$$
\begin{aligned}
S_n(f;x) &= \sum_{k=0}^{n} C_k(f)P_k(x) \\
&= \frac{2}{\pi \sin y}\int_0^{\pi} f(\cos t)\sin t \sum_{k=0}^{n}(\cos(k+1)(t-y) - \cos(k+1)(t+y)dt.
\end{aligned} \tag{3.10}
$$

Using the well-known quality

$$\frac{1}{2} + \sum_{k=0}^{n} \cos kx = \frac{\sin(n+\frac{1}{2})x}{2\sin\frac{x}{2}},$$

equation (3.10) can be written as

$$S_n(f;x) = \frac{2}{\pi \sin y}\int_0^{\pi} f(\cos t)\sin t \left(\frac{\sin(n+\frac{1}{2}+1)(t-y)}{\sin\frac{1}{2}(t-y)} - \frac{\sin(n+\frac{1}{2}+1)(t+y)}{\sin\frac{1}{2}(t+y)}\right) dt,$$

that is,

$$
\begin{aligned}
S_n(f;x) = \frac{1}{2\pi \sin y}\int_0^{\pi} &(f(\cos(t+y))\sin(t+y) \\
&- f(\cos(t-y))\sin(t-y))\frac{\sin(n+\frac{1}{2}+1)t}{\sin\frac{1}{2}t}dt.
\end{aligned} \tag{3.11}
$$

From equation (1.5), we have

$$
\begin{aligned}
V_{n,m}(f;x) &= \frac{1}{m+1}\sum_{k=n}^{n+m} S_k(f;x) \\
&= \frac{1}{m+1}\left(\sum_{k=0}^{n+m} S_k(f;x) - \sum_{k=0}^{n-1} S_k(f;x)\right).
\end{aligned} \tag{3.12}
$$

To obtain an estimation for $V_{n,m}(f;x)$, first, from the integral representation of $V_{n,m}(f;x)$, let

$$M_{n,m} = \frac{1}{\pi(m+1)}\int_{-\pi}^{\pi} \frac{|\sin\frac{2n+m+1}{2}t \sin\frac{m+1}{2}t|}{2\sin^2\frac{t}{2}}dt,$$

and, if $p = \frac{m+1}{2}$, $rp = \frac{2n+m+1}{2}$, for $p \geq \frac{1}{2}, r \geq 1$, we obtain

$$M_{n,m} = \frac{1}{\pi(m+1)}\int_0^{\pi} \frac{|\sin rpt \sin pt|}{2\sin^2\frac{t}{2}}dt.$$

Since the function $\frac{1}{\sin^2\frac{t}{2}}-\frac{1}{\frac{t^2}{4}}$ is bounded on $[0,\pi]$ and $p\geq\frac{1}{2}$, then

$$M_{n,m}=\frac{2}{\pi p}\int\limits_0^{\pi}\frac{|\sin rpt\sin pt|}{t^2}dt+O(1).$$

Since

$$\int\limits_0^{\pi}\frac{|\sin rpt\sin pt|}{t^2}dt=p\int\limits_0^{p\pi}\frac{|\sin rt\sin t|}{t^2}dt,$$

and

$$\left|\int\limits_0^{p\pi}\frac{|\sin rt\sin t|}{t^2}dt-\int\limits_0^{\pi}\frac{|\sin rt\sin t|}{t^2}dt\right|\leq\int\limits_{\frac{\pi}{2}}^{\infty}\frac{dt}{t^2},$$

then

$$M_{n,m}=\frac{2}{\pi}\int\limits_0^{\pi}\frac{|\sin rt\sin t|}{t^2}dt+O(1),$$

and again, since the function $\frac{1}{\sin^2\frac{t}{2}}-\frac{1}{\frac{t^2}{4}}$ is bounded on $[0,\pi]$ and $p\geq\frac{1}{2}$, then

$$M_{n,m}=\frac{2}{\pi}\int\limits_0^{\pi}\frac{|\sin rt|}{t}dt+O(1).$$

Next, we must show that

$$\int\limits_0^{\pi}\frac{|\sin rt|}{t}dt=\frac{2}{\pi}\ln r+O(1),$$

where $k\leq r<k+1$, for $k\geq 1$. For this, let

$$\begin{aligned}\int\limits_0^{\pi}\frac{|\sin rt|}{t}dt &= \sum_{i=0}^{k-1}(-1)^i\int\limits_{\frac{i\pi}{r}}^{(i\ +1)\frac{\pi}{r}}\frac{|\sin rt|}{t}dt+O(1)\\ &= \sum_{i=0}^{k-1}\int\limits_0^{\frac{\pi}{r}}\frac{\sin rt}{t+\frac{i\pi}{r}}dt+O(1)\\ &= \int\limits_0^{\frac{\pi}{r}}\sin rt\left(\sum_{i=0}^{k-1}\frac{1}{t+\frac{i\pi}{r}}\right)dt+O(1).\end{aligned}$$

And, since for $0\leq t\leq\frac{\pi}{r}$ and from $\sum\limits_{k=0}^{n}\sin(k+\frac{1}{2})x=\frac{\sin^2(n+1)\frac{x}{2}}{\sin\frac{x}{2}}$, we have:

$$\sum_{i=1}^{k-1}\frac{1}{t\ +\frac{i\ \pi}{r}}=\frac{r}{\pi}\left(\sum_{i=1}^{k-1}\frac{1}{i}+O(1)\right)=\frac{r}{\pi}(\ln k+O(1)).$$

Then

$$\int_0^{\pi} \frac{|\sin rt|}{t} dt = \int_0^{\frac{\pi}{r}} \sin rt \left(\frac{r}{\pi}(\ln k + O(1))\right) dt = \frac{2}{\pi} \ln k + O(1) = \frac{2}{\pi} \ln r + O(1).$$

So

$$M_{n,m} = \frac{4}{\pi} \ln r + O(1) = \frac{4}{\pi} \ln \frac{2(n+1)+m+1}{m+1} + O(1) = \frac{4}{\pi} \ln \frac{n+m+1}{m+1} + O(1).$$

Therefore, using the integral form of $V_{n,m}$, we obtain our result for, i.e.,

$$\left|V_{n,m}^{(\frac{1}{2},\frac{1}{2})}(f;x)\right| \le \frac{CM}{\sqrt{1-x^2}} \left(1 + \ln\left(\frac{n+m+1}{m+1}\right)\right).$$

In a similar way, we can prove the other three cases.

Theorem 2. *Suppose that $f(x)$ is continuous function on $[-1,1]$. Then*

$$\left|f(x) - V_{n,m}^{(-\frac{1}{2},-\frac{1}{2})}(f;x)\right| \le CE_n(f) \left(1 + \ln\left(\frac{n+m+1}{m+1}\right)\right). \tag{3.13}$$

$$\left|f(x) - V_{n,m}^{(\frac{1}{2},\frac{1}{2})}(f;x)\right| \le \frac{C}{\sqrt{1-x^2}} E_n(f) \left(1 + \ln\left(\frac{n+m+1}{m+1}\right)\right). \tag{3.14}$$

$$\left|f(x) - V_{n,m}^{(\frac{1}{2},-\frac{1}{2})}(f;x)\right| \le \frac{C}{\sqrt{1-x}} E_n(f) \left(1 + \ln\left(\frac{n+m+1}{m+1}\right)\right). \tag{3.15}$$

$$\left|f(x) - V_{n,m}^{(-\frac{1}{2},\frac{1}{2})}(f;x)\right| \le \frac{C}{\sqrt{1+x}} E_n(f) \left(1 + \ln\left(\frac{n+m+1}{m+1}\right)\right). \tag{3.16}$$

Proof. First, we consider the case $\alpha = \beta = \frac{1}{2}$.

Let $Q_n(x)$ be the best uniform approximation of algebraic polynomial for function $f(x)$ of order not exceeded n on $[-1,1]$, then

$$\begin{aligned} |f(x) - V_{n,m}(f;x)| &\le |f(x) - Q_n(x)| + |Q_n(x) - V_{n,m}(f;x)| \\ &= |f(x) - Q_n(x)| + |V_{n,m}(f - Q_n; x)| \\ &\le E_n(f) + \frac{C}{\sqrt{1-x^2}} E_n(f) \left(1 + \ln\left(\frac{n+m+1}{m+1}\right)\right) \\ &\le \frac{C}{\sqrt{1-x^2}} E_n(f) \left(1 + \ln\left(\frac{n+m+1}{m+1}\right)\right). \end{aligned}$$

In a similar way, we can prove the other three cases.

Theorem 3. *Suppose that $f(x)$ is continuous function on $[-1,1]$, then*

$$\left|f(x)-V_{n,m}^{(-\frac{1}{2},-\frac{1}{2})}(f;x)\right|\le\frac{C}{m+1}\sum_{k=n}^{n+m}E_k(f)\left(1+\ln\frac{k+1}{k-n+1}\right). \tag{3.17}$$

$$\left|f(x)-V_{n,m}^{(\frac{1}{2},\frac{1}{2})}(f;x)\right|\le\frac{C}{(m+1)\sqrt{1-x^2}}\sum_{k=n}^{n+m}E_k(f)\left(1+\ln\frac{k+1}{k-n+1}\right). \tag{3.18}$$

$$\left|f(x)-V_{n,m}^{(\frac{1}{2},-\frac{1}{2})}(f;x)\right|\le\frac{C}{(m+1)\sqrt{1-x}}\sum_{k=n}^{n+m}E_k(f)\left(1+\ln\frac{k+1}{k-n+1}\right). \tag{3.19}$$

$$\left|f(x)-V_{n,m}^{(\frac{-1}{2},\frac{1}{2})}(f;x)\right|\le\frac{C}{(m+1)\sqrt{1+x}}\sum_{k=n}^{n+m}E_k(f)\left(1+1+\ln\frac{k+1}{k-n+1}\right). \tag{3.20}$$

Proof. We start the case $\alpha=\beta=\frac{1}{2}$.

Choose the integer p such that $2^p\le m+1<2^{p+1}$, then

$$\begin{aligned}f(x)-V_{n,m}(f;x) &= \frac{1}{m+1}\sum_{k=n}^{n+m}(f(x)-S_k(f;x))\\ &= \frac{1}{m+1}\Bigg\{(f(x)-S_n(f;x))+\sum_{k=1}^{p}\sum_{i=n+2^{k-1}}^{n+2^k-1}(f(x)-S_i(f;x))\\ &\quad+\sum_{k=n+2^p}^{n+m}(f(x)-S_k(f;x))\Bigg\}.\end{aligned}$$

Since

$$\sum_{k=n}^{n+m}S_k(f;x)=(m+1)V_{n,m}(f;x),$$

we get

$$S_n(f;x)=V_{n,0}(f;x),$$

which yields

$$\sum_{i=n+2^{k-1}}^{n+2^k-1}S_i(f;x)=2^{k-1}V_{n+2^{k-1},2^{k-1}-1}(f;x),$$

and

$$\sum_{i=n+2^p}^{n+m}S_i(f;x)=(m+1-2^p)V_{n+2^p,m-2^p}(f;x),$$

So

$$\begin{aligned} f(x)-V_{n,m}(f;x) &= \frac{1}{m+1}\{(f(x)-V_{n,0}(f;x)) \\ &+\sum_{k=1}^{p}2^{k-1}(f(x)-V_{n+2^{k-1},2^{k-1}-1}(f;x)) \\ &+(m+1-2^p)(f(x)-V_{n+2^p,m-2^p}(f;x)\}. \end{aligned}$$

Applying Theorem 2, we obtain

$$|f(x)-V_{n,0}(f;x)|\leq \frac{C}{\sqrt{1-x^2}}(1+\ln(n+1))E_n(f)$$

$$|f(x)-V_{n+2^{k-1},2^{k-1}-1}(f;x)|\leq \frac{C}{\sqrt{1-x^2}}\left(1+\ln\left(\frac{n+2^k}{2^{k-1}}\right)\right)E_{n+2^{k-1}}(f) \tag{3.21}$$

$$|f(x)-V_{n+2^p,m-2^p}(f;x)|\leq \frac{C}{\sqrt{1-x^2}}\left(1+\ln\left(\frac{n+m+1}{m+1-2^p}\right)\right)E_{n+2^p}(f).$$

Thus

$$\begin{aligned} |f(x)-V_{n,m}(f;x)| \leq & \frac{C}{(m+1)\sqrt{1-x^2}}\{(1+\ln(n+1))E_n(f) \\ & +\sum_{k=1}^{p}2^{k-1}\left(1+\ln(\frac{n+2^k}{2^{k-1}})\right)E_{n+2^{k-1}}(f) \\ & +(m+1-2^p)\left(1+\ln\left(\frac{n+m+1}{m+1-2^p}\right)\right)E_{n+2^p}(f)\right\}. \end{aligned} \tag{3.22}$$

Note that $\forall u,v>0$, we have the inequality $u+v\leq(1+u)(1+v)$ and then $\ln(u+v)\leq\ln(1+u)+\ln(1+v)$. Setting $u=\frac{x}{z}, v=\frac{y}{z}$, we obtain

$$ln\frac{x+y}{z}\leq\ln\left(1+\frac{x}{z}\right)+\ln(1+\frac{y}{z}).$$

So

$$\begin{aligned} I &= \sum_{k=1}^{p}2^{k-1}E_{n+2^{k-1}}(f)\ln\left(\frac{n+2^k}{2^{k-1}}\right) \\ &\leq \sum_{k=1}^{p}2^{k-1}E_{n+2^{k-1}}(f)\ln\left(1+\frac{n}{2^{k-1}}\right)+\sum_{k=1}^{p}2^{k-1}E_{n+2^{k-1}}(f)\ln\left(1+\frac{2^k}{2^{k-1}}\right) \\ &= I_1+I_2. \end{aligned} \tag{3.23}$$

For I_2, as mentioned above

$$I_2\leq C\sum_{k=n}^{n+2^p-1}E_k(f). \tag{3.24}$$

For I_1, note that

$$
\begin{aligned}
I_1 &= \sum_{k=1}^{p} 2^{k-1} E_{n+2^{k-1}}(f) \ln(1+\frac{n}{2^{k-1}}) \\
&= E_{n+1}(f)\ln(1+n) + 2\sum_{k=2}^{p} 2^{k-2} E_{n+2^{k-1}}(f) \ln(1+\frac{n}{2^{k-1}}) \\
&= E_{n+1}(f)\ln(1+n) + 2\sum_{k=2}^{p} \sum_{i=n+2^{k-2}}^{n+2^{k-1}-1} 2^{k-2} E_{n+2^{k-1}}(f) \ln(1+\frac{n}{2^{k-1}}) \\
&= E_{n+1}(f)\ln(1+n) + 2\left\{ \sum_{i=n+1}^{n+1} E_{n+2^1}(f)\ln\left(1+\frac{n}{2^1}\right) + \sum_{i=n+2}^{n+3} E_{n+2^2}(f)\ln\left(1+\frac{n}{2^2}\right) \right. \\
&\quad \left. + \sum_{i=n+4}^{n+7} E_{n+2^3}(f)\ln(1+\frac{n}{2^3}) + \cdots + \sum_{i=n+2^{p-1}}^{n+2^{p-1}-1} E_{n+2^{p-1}}(f)\ln\left(1+\frac{n}{2^{p-1}}\right) \right\} \\
&\leq C \sum_{k=n}^{n+2^p-1} E_k(f)\ln\left(1+\frac{n}{k-n+1}\right).
\end{aligned} \tag{3.25}
$$

Combining the estimates given in (3.25) and (3.24) for I_1 and I_2, we obtain

$$
\begin{aligned}
&\sum_{k=1}^{p} 2^{k-1} E_{n+2^{k-1}}(f) \ln\left(\frac{n+2^k}{2^{k-1}}\right) \\
&\qquad \leq C\left\{ \sum_{k=n}^{n+2^p-1} E_k(f) + \sum_{k=1}^{n+2^p-1} E_k(f) \ln\left(1+\frac{k+1}{k-n+1}\right) \right\}
\end{aligned} \tag{3.26}
$$

Thus, from the choice of p such that $m+1-2^p \leq 2^p$, we have:

$$
(m+1-2^p)E_{n+2^p}(f) \leq \sum_{k=n}^{n+2^p-1} E_k(f).
$$

In addition, note that for any natural numbers α, β such that $1 \leq \alpha \leq \beta - 1$, also we have

$$
\begin{aligned}
\alpha \ln \frac{\beta-\alpha}{\alpha} &= \ln\frac{\beta-\alpha}{\alpha} + \ln\frac{\beta-\alpha}{\alpha} + \cdots + \ln\frac{\beta-\alpha}{\alpha} \\
&\leq \ln\frac{\beta-\alpha}{1} + \ln\frac{\beta-\alpha}{2} + \cdots + \ln\frac{\beta-\alpha}{\alpha} \\
&= \sum_{k=\alpha+1}^{\beta} \ln\frac{\beta-\alpha}{k-\beta+\alpha}
\end{aligned}
$$

therefore,

$$\begin{aligned}(m+1-2^p)\ln\frac{n+m+1}{m+1-2^p} &\le \sum_{k=n+m+2}^{n+2m+2-2^p}\ln\frac{n+m+1}{k-n-m-1}\\ &= \sum_{k=1}^{n+1-2^p}\ln\frac{n+m+1}{k}\\ &= \sum_{k=n}^{n+m-2^p}\ln\frac{n+m+1}{k-n+1}\\ &\le \sum_{k=n}^{n+m-2^p}\ln\left(1+\frac{n}{k-n+1}\right)+\sum_{k=n}^{n+m-2^p}\ln\left(1+\frac{m+1}{k-n+1}\right).\end{aligned}$$

where $\alpha = m+1-2^p,\ \ \beta = 2m+n-2^p$, but

$$\begin{aligned}\sum_{k=n}^{n+m-2^p}\ln\left(1+\frac{m+1}{k-n+1}\right) &= \sum_{k=1}^{n+m-2^p}\ln\left(1+\frac{n}{k-n+1}\right)\\ &\le \sum_{k=1}^{m+1}\ln\left(1+\frac{m+1}{k}\right)\\ &\le C(m+1)\\ &\le C2^p,\end{aligned}$$

where we used

$$\ln(1+x) = \ln x + O\left(\frac{1}{x}\right),\ (x\ge 1),$$

and

$$\sum_{k=1}^{m}\ln\left(\frac{m}{k}\right) = \ln\frac{m^n}{m!} \le \ln e^m \le m \le C2^p,$$

so

$$(m+1-2^p)\ln\frac{n+m+1}{m+1-2^p} \le C\left[2^p+\sum_{k=n}^{n+m-2^p}\ln\left(1+\frac{n}{k-n+1}\right)\right].$$

Since $2^p \le m+1$, then

$$E_n(f)+E_{n+1}(f)+\cdots+E_{n+m}(f) \ge (m+1)E_{n+2^p}(f) \ge 2^pE_{n+2^p}(f),$$

therefore,

$$
\begin{aligned}
&(m+1-2^p)E_{n+2^p}(f)\ln\tfrac{n+m+1}{m+1-2^p}\\
&\quad\leq C\left[2^pE_{n+2^p}(f)+\sum_{k=n}^{n+m-2^p}E_{n+2^p}(f)\ln\left(1+\frac{n}{k-n+1}\right)\right]\\
&\quad\leq C\left[\sum_{k=n}^{n+m}E_k(f)+\sum_{k=n}^{n+m-2^p}E_{n+2^p}(f)\ln\left(1+\frac{n}{k-n+1}\right)\right]\\
&\quad\leq C\left[\sum_{k=n}^{n+m}E_k(f)+\sum_{k=n}^{n+m}E_k(f)\ln\left(1+\frac{n}{k-n+1}\right)\right].
\end{aligned}
\tag{3.27}
$$

Combining all of the above estimates (3.22)–(3.27), we get the desired result. This ends of the proof of the theorem.

References

[1] Stechkin, S.B., *On approximation of continuous functions by Fourier series*, Uspehi Mat. Nauk., 7 (4) (1952), 139-141.

[2] Al-Btoush, R., *On Approximation of bounded variation functions by Fourier–Legendre sums*, J. of MU'TAH Lil-Buhuth Wad-Dirasat, (25) (1) (2010), 155-178.

[3] Abelov, V.A., *Approximation of functions by Fourier-Lagerre sums*, Mat. Zametki, 57) (2) (1995), 163-170.

[4] Nikolski, S.M., *Approximation of functions of many variables and embedding theorems*, Nauka, Moscow, 1977.

[5] Abelov, V.A., Koorbanova, M.M., *Approximation of functions by Vallee–Poussin sums*, Viniti, Moscow, 1987.

[6] Al-Btoush, R., *Approximation of continuous functions by algebraic Polynomials,* Italian Journal of Pure and Applied Mathematics, 24 (2008), 9-18.

[7] Al-Btoush, R., *Approximation of periodic functions by Vallee–Poussin sums*, Hokkaido Mathematical Journal, 30 (2) (2001), 269-282.

[8] Kholshchevnikova, N.N., *On the de la Vallee-Poussin theorem on the uniqueness of the trigonometric series reoresenting a function*, Sb. Math., 187 (5) (1996), 767-784.

[9] Novikov, O.A., Rukaasov, V.I., *Approximation of classes of continuous functions by generalized de la Vallee-Poussin sums*, Ukr. Mat. Zh., 47 (8) (1995), 1069-1079.

[10] Nowakowski, K., Taberski, R., *Estimates for the strong de la Vallee-Poussin means in case of periodic continuous functions of two variables*, Funct. Approx. Comment. Math., 24 (1996), 83-101.

[11] Nahmwoo, Hshm., Hong, B.I., *Degree of approximation by periodic neural network*, Korean Journal. Math., 22 (2) (2014), 307-315.

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MATCHING EXTENSION IN COMPLEMENTARY PRISM OF REGULAR GRAPHS

Pongthep Janseana
Department of Mathematics
Faculty of Science
Silpakorn University
Nakorn Pathom 73000
Thailand
e-mail: jpongthep@yahoo.com

Nawarat Ananchuen[1,2]
Department of Mathematics
Faculty of Science
Silpakorn University
Nakorn Pathom 73000
Thailand
and
Centre of Excellence in Mathematics
CHE, Si Ayutthaya Rd.
Bangkok 10400
Thailand

Abstract. Let $\overline{G}$ denote the complement of a simple graph G. The complementary prism of G, denoted by $G\overline{G}$, is obtained by taking a copy of G and a copy of $\overline{G}$ and then adding a perfect matching that joins corresponding vertices. A connected graph G of order at least $2k+2$ is k-extendable if for every matching M of size k in G, there is a perfect matching in G containing all edges of M. In this paper, we establish some sufficient conditions for the complementary prism of regular graphs to be 2-extendable.

Keywords: matching; extendable; complementsry prism; regular graph.
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1. Introduction

All graphs considered in this paper are finite, undirected and simple. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The complement of G is denoted by $\overline{G}$. For $S \subseteq V(G)$, $G[S]$ denotes the induced subgraph of G by S.

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[2]Corresponding author

A neighbor set of a vertex v in G is denoted by $N_G(v) = \{u \in V(G) | uv \in E(G)\}$. For $v \in V(G)$ and $T \subseteq V(G)$, a neighbor set of a vertex v in T is denoted by $N_T(v) = \{u \in T | uv \in E(G)\}$ and if $X \subseteq V(G)$, $N_G(X)$ denotes $\bigcup_{v \in X} N_G(v)$. The number of components of G, the number of odd components of G and the number of even components of G are denoted by $c(G), c_o(G)$ and $c_e(G)$, respectively. A complete graph of order r is denoted by K_r. For graphs H and G, G is called H-free if G does not contain H as an induced subgraph. A subgraph H is called a clique if $H \cong K_r$, for some r. A set $M \subseteq E(G)$ is called a matching if no two edges of M have a common end vertex. A vertex u is saturated by M if there is an edge in M incident with u. For simplicity, the set of all vertices saturated by M is denoted by $V(M)$. M is called a maximum matching in G if there is no matching N in G of size greater than M. A perfect matching in G is a matching that saturates all vertices of G. A connected graph G of order at least $2k + 2$ is k-extendable if for every matching M of size k in G, there is a perfect matching in G containing all edges of M. A graph G is k-factor-critical if, for every set $S \subseteq V(G)$ with $|S| = k$, the graph $G - S$ contains a perfect matching. For $k = 1$ and $k = 2$, a k-factor-critical graph is also called factor-critical and bicritical, respectively.

The concept of k-extendable graphs was introduced, in 1980, by Plummer [9]. He gave a sufficient condition for a graph to be k-extendable in terms of minimum degree. A fundamental theorem (see Theorem 2.2) that is mainly used in studying matching extension was established. He also proved that 2-extendable non-bipartite graphs are bicritical. Some sufficient conditions for special classes of graphs to be k-extendable have been established (see [8], [10], [14]). For a comprehensive survey of this topic, the reader is referred to Plummer [11]–[13].

The concept of k-factor-critical graphs was introduced, in 1996, by Favaron [6]. She gave a necessary and sufficient condition for a graph to be k-factor-critical and also provided a relationship between n-extendable graphs and k-factor- critical graphs.

A *complementary prism* of G, denoted by $G\overline{G}$, is the graph obtained by taking a copy of G and a copy of its complement $\overline{G}$ and then joining corresponding vertices by an edge. A complementary prism is a specific case of complementary product of graphs introduced by Haynes et al.[3] in 2007. Haynes et al. ([3], [4], [5]) studied some parameters of complementary prism of graphs such as the vertex independence number, the chromatic number and the domination number.

According to the definition of the complementary prism of G, it is easy to see that $G\overline{G}$ contains a perfect matching. A problem that arises is that of investigating properties of G so that $G\overline{G}$ is k-extendable for some k. In [7], Janseana et al. established that if G is a 2-regular, H-free graph where $H \in \{C_3, C_4, C_5\}$, then $G\overline{G}$ is 2-extendable. In this paper, we concentrate on connected r-regular graphs for $r \geq 3$. Let $F = K_{2,3}$ with the addition of an edge shown in Figure 1. We prove that for a connected graph G of order p, if G is either 3-regular, F-free where $p \geq 8$ or r_0-regular where $p \geq 2r_0 + 1 \geq 9$, then $G\overline{G}$ is 2-extendable. We further extend this result to disconnected graphs. We show that if each component G_i of

G is 3-regular, F-free of order at least 8 or r_0-regular of order at least $2r_0+1 \geq 9$, then $G\overline{G}$ is 2-extendable. These results are presented in Section 3. Section 2 contains some preliminary results that we make use of in our work.

Figure 1: The graph F

2. Preliminary results

In this section, we state some results which are used in establishing our results in Section 3. Our first result is a well known theorem for studying the existence of a perfect matching in graphs established by Tutte.

Theorem 2.1. [2] (Tutte's Theorem) *A graph G has a perfect matching if and only if for any $S \subseteq V(G)$, $c_o(G-S) \leq |S|$.*

In 1980, Plummer [9] established the following fundamental theorem on k-extendable graphs.

Theorem 2.2. [9] *Let G be a graph of order $p \geq 2k+2$ and $k \geq 1$. If G is k-extendable, then*
(a) *G is $(k-1)$-extendable, and*
(b) *G is $(k+1)$-connected.*

Ananchuen and Caccetta [1] gave a necessary condition for a neighbor set of a vertex having minimum degree in extendable graphs. They showed that:

Theorem 2.3. [1] *Let G be a k-extendable graph on $p \geq 2k+2$ vertices with $\delta(G) = k+t$, $1 \leq t \leq k \leq p$. If $d_G(u) = \delta(G)$, then the induced subgraph $G[N_G(u)]$ has at most $t-1$ independent edges.*

A neccessary and sufficient condition for a graph to be k-extendable and to be k-factor-critical was provided by Yu [15] and Favaron [6], respectively.

Theorem 2.4. [15] *A graph G is k-extendable $(k \geq 1)$ if and only if for any $S \subseteq V(G)$,*
(a) *$c_o(G-S) \leq |S|$ and*
(b) *$c_o(G-S) = |S| - 2t, (0 \leq t \leq k-1)$ implies that $F(S) \leq t$, where $F(S)$ is the size of a maximum matching in $G[S]$.*

Theorem 2.5. [6] *A graph G is k-factor-critical if and only if $|V(G)| \equiv k(mod\ 2)$ and for any $S \subseteq V(G)$ with $|S| \geq k, c_o(G-S) \leq |S|-k$.*

We now turn our attention to some results concerning complementary prism of graphs.

Theorem 2.6. [7] *For positive integers l and i where $1 \leq i \leq l$, let $G_1, \ldots, G_l$ be components of G. If for each i, $G_i\overline{G}_i$ is a k-extendable graph of order $p_i \geq 2k+2$ for some positive integer k, then $G\overline{G}$ is k-extendable.*

Theorem 2.7. [7] *Let G be a 2-regular, H-free graph where $H \in \{C_3, C_4, C_5\}$, then $G\overline{G}$ is 2-extendable.*

3. Main results

We begin this section by establishing some lemmas concerning complementary prism of graphs and of regular graphs. These results are essential for establishing Theorem 3.10, the main result of our paper. To simplify our discussion of complementary prisms, G and $\overline{G}$ are referred to as subgraph copies of G and $\overline{G}$, respectively, in $G\overline{G}$. For a vertex v of G, there is exactly one vertex of $\overline{G}$ which is adjacent to v in $G\overline{G}$. This vertex is denoted by $\overline{v}$. That is $\{\overline{v}\} = N_{\overline{G}}(v)$. Conversely, v is the only vertex of G which is adjacent to $\overline{v}$. Similarly, for $\phi \neq X = \{x_1, x_2, \ldots, x_k\} \subseteq V(G)$, $\{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k\} \subseteq V(\overline{G})$ is denoted by $\overline{X}$ and vice versa. Clearly, $|X| = |\overline{X}|$.

Lemma 3.1. *Let G be a graph. Then $G\overline{G}$ is even and connected.*

Proof. Clearly, $G\overline{G}$ is even. Let $u, v \in V(G\overline{G})$. It is easy to see that if $u, v \in V(G)(V(\overline{G}))$, then either $uv \in E(G)$ or $u\bar{u}\bar{v}v$ is a $u-v$ path. We may now assume that $u \in V(G)$ and $v \in V(\overline{G})$. Clearly, $uv \in E(G\overline{G})$ if $v = \bar{u}$. So suppose that $v = \bar{w}$ for some $w \in V(G) - \{u\}$. Then either $u\bar{u}\bar{w}$ or $uw\bar{w}$ is a $u-v$ path. This proves that $G\overline{G}$ is connected and completes the proof of our lemma. ∎

For a graph G, it is easy to see that $G\overline{G}$ has a perfect matching. It then follows by Theorem 2.1 that for a cutset $S \subseteq V(G\overline{G})$, $c_o(G\overline{G}-S) \leq |S|$. The next lemma provides a relationship of a cutset and the number of odd components in a complementary prism.

Lemma 3.2. *Let $G\overline{G}$ be a complementary prism and let $S = A \cup \overline{B}$ be a cutset of $G\overline{G}$, where $A \subseteq V(G)$ and $\overline{B} \subseteq V(\overline{G})$. Then*

(a) *$c_o(G\overline{G}-S) = |S| - 2t = |A| + |B| - 2t$, for some $t \geq 0$.*

(b) *$c_o(G\overline{G}-S) \leq c_o(G[B-A]) + c_o(\overline{G}[\overline{A}-\overline{B}]) \leq |A| + |B| - 2|A \cap B|$. Consequently, $|A \cap B| \leq t$.*

(c) *If $c_o(G[B-A]) + c_o(\overline{G}[\overline{A}-\overline{B}]) = |A| + |B| - 2|A \cap B|$, then each component of $G[B-A] \cup \overline{G}[\overline{A}-\overline{B}]$ is singleton. Consequently, $G[A-B]$ is a clique.*

Proof. (a) Since $G\overline{G}$ contains a perfect matching and is of even order, it follows by Theorem 2.1 that there is a non-negative integer t such that $c_o(G\overline{G} - S) = |S| - 2t$, for any cutset $S \subseteq V(G\overline{G})$. Clearly, $|S| = |A| + |B|$. Thus $c(G\overline{G} - S) = |S| - 2t = |A| + |B| - 2t$ as required.

We first observe that $|B-A|+|\overline{A}-\overline{B}| = |B-A|+|A-B| = |A|+|B|-2|A\cap B|$ since $|A| = |A - B| + |A \cap B|$ and $|B| = |B - A| + |A \cap B|$.

(b) Let $C = V(G)-(A\cup B)$. It is easy to see that if $C = \phi$, then $c_o(G\overline{G}-S) = c_o(G[B - A]) + c_o(\overline{G}[\overline{A} - \overline{B}]) \leq |B - A| + |\overline{A} - \overline{B}| = |A| + |B| - 2|A \cap B|$. We now suppose that $C \neq \phi$. Then, by Lemma 3.1, $G\overline{G}[C \cup \overline{C}]$ is even and connected. Thus $c_o(G\overline{G} - S) \leq c_o(G\overline{G} - (S \cup C \cup \overline{C})) = c_o(G[B - A]) + c_o(\overline{G}[\overline{A} - \overline{B}]) \leq |B - A| + |\overline{A} - \overline{B}| = |A| + |B| - 2|A \cap B|$ as required.

(c) follows by the fact that $|B - A| + |A - B| = |A| + |B| - 2|A \cap B|$. ∎

For an induced subgraph H of G, Com_H denotes the set of all components in H. If $X \subseteq V(G)$, then we use Com_X for $Com_{G[X]}$. For a cutset S of $G\overline{G}$, put $A = S\cap V(G)$, $\overline{B} = S\cap V(\overline{G})$ and $C = V(G)-(A\cup B)$. Thus $S = A\cup \overline{B}$. Further, let $T_{B-A} = \{F \mid F$ is an odd component of $G[B - A]$ and $N_G(u) - V(F) \subseteq A$ for all $u \in V(F)\}$. $T_{\overline{A}-\overline{B}} = \{F \mid F$ is an odd component of $\overline{G}[\overline{A} - \overline{B}]$ and $N_{\overline{G}}(\bar{u}) - V(F) \subseteq \overline{B}$ for all $\bar{u} \in V(F)\}$. Finally, let $L = L_G \cup L_{\overline{G}}$, where $L_G = \{F \mid F$ is an odd component in $G[B - A]$ and $N_{G\overline{G}}(V(F)) \cap C \neq \phi\}$ and $L_{\overline{G}} = \{F \mid F$ is an odd component in $\overline{G}[\overline{A} - \overline{B}]$ and $N_{G\overline{G}}(V(F)) \cap \overline{C} \neq \phi\}$. Note that if $C = \phi$, then $L = \phi$. Clearly, $T_{B-A} \cap L_G = \phi$ and $T_{\overline{A}-\overline{B}} \cap L_{\overline{G}} = \phi$. It is easy to see that, if G is connected and $G[B - A]$ contains only odd components, then $Com_{B-A} = T_{B-A} \cup L_G$. Similarly, if $\overline{G}$ is connected and $\overline{G}[\overline{A} - \overline{B}]$ contains only odd components, then $Com_{\overline{A}-\overline{B}} = T_{\overline{A}-\overline{B}} \cup L_{\overline{G}}$. In what follows, the symbols Com_H, S, A, $\overline{B}$, C, T_{B-A}, $T_{\overline{A}-\overline{B}}$, L, L_G and $L_{\overline{G}}$ are referred to these set up.

The next lemma follows from our set up.

Lemma 3.3. *Let G be an r-regular connected graph of order $p \geq 2r + 1$ and $G\overline{G}$ a complementary prism. If $|A| < r$, then T_{B-A} contains no singleton components. Similarly, if $|\overline{B}| < p - r - 1$, then $T_{\overline{A}-\overline{B}}$ contains no singleton components.*

Lemma 3.4. *For $r \geq 3$, let G be a connected r-regular graph of order $p \geq 2r + 1$. Let $A, B, T_{B-A}, T_{\overline{A}-\overline{B}}$ be defined as above. Then*

(a) *If $G[A] = K_r$, then each component of T_{B-A} is of order at least 3.*

(b) *If $|A \cap B| = 1$ and $G[A - B] \cong K_r$, then the number of singleton components in T_{B-A} is at most 1.*

(c) *If $|A \cap B| = 1$ and $G[A - B] \cong K_{r-1}$, then the number of singleton components in T_{B-A} is at most 2.*

Proof. (a) It follows by the fact that G is connected r-regular of order $p \geq 2r+1$.

(b) Suppose to the contrary that T_{B-A} contains two singleton components, say F_1 and F_2 where $V(F_1) = \{y_1\}$ and $V(F_2) = \{y_2\}$. Because $|A \cap B| = 1$, y_1 and y_2 are adjacent to at least $r - 1$ vertices of $A - B$. Since $G[A - B] = K_r$ and $r \geq 3$, it follows that there exists a vertex of $A - B$, say y_3, such that $\{y_1, y_2\} \cup (A - B) \subseteq N_G(y_3)$. Thus $d_G(y_3) \geq r + 1$, a contradiction

(c) By applying similar arguments as in the proof of (b), (c) follows. ∎

Let x be a real number, $\lfloor x \rfloor_e$ denotes the greatest even integer less than or equal to x, that is, $\lfloor x \rfloor_e = 2\lfloor x/2 \rfloor$. Note that if x is an integer and $\lfloor x \rfloor_e = k$ then $x = k$ or $x = k+1$.

Lemma 3.5. *Let $G\overline{G}$ be a complementary prism and $L = L_G \cup L_{\overline{G}}$ be defined as above. Then $c_o(G\overline{G} - S) = c_o(G[B-A]) + c_o(\overline{G}[\overline{A} - \overline{B}]) - \lfloor |L| \rfloor_e$. Consequently, $c_o(G[B-A]) + c_o(\overline{G}[\overline{A} - \overline{B}]) - c_o(G\overline{G} - S) \leq |L| \leq c_o(G[B-A]) + c_o(\overline{G}[\overline{A} - \overline{B}]) - c_o(G\overline{G} - S) + 1$.*

Proof. If $C = \phi$, then $|L| = 0$ and thus $c_o(G\overline{G} - S) = c_o(G[B-A]) + c_o(\overline{G}[\overline{A} - \overline{B}])$ as required. We now suppose that $C \neq \phi$. By Lemmas 3.2(a) and (b), $c_o(G\overline{G} - S) \leq c_o(G[B-A]) + c_o(\overline{G}[\overline{A} - \overline{B}])$. By Lemma 3.1, $G\overline{G}[C \cup \overline{C}]$ is even and connected. So it must be contained in some component of $G\overline{G} - S$, say F. If $x \in V(F) - (C \cup \overline{C})$, then x is in some component of $G[B-A] \cup \overline{G}[\overline{A} - \overline{B}]$, say M. So $V(M) \subseteq V(F)$. If M is odd, then $M \in L$. Note that each odd component of L is a subgraph of F. Hence, $|V(F)|$ has the same parity with $|L|$ and $c_o(G\overline{G} - S) = c_o(G[B-A] \cup \overline{G}[\overline{A} - \overline{B}]) - |L| + \epsilon$, where $\epsilon = 1$ if $|L|$ is odd and $\epsilon = 0$ if $|L|$ is even. So $c_o(G\overline{G} - S) = c_o(G[B-A] \cup \overline{G}[\overline{A} - \overline{B}]) - \lfloor |L| \rfloor_e$. Thus $\lfloor |L| \rfloor_e = c_o(G[B-A] \cup \overline{G}[\overline{A} - \overline{B}]) - c_o(G\overline{G} - S)$. By properties of $\lfloor x \rfloor_e$, our result follows. This proves our lemma. ∎

Lemma 3.6. *If G is an r-regular graph of order $p \geq 2r+1$, then $\overline{G}$ is connected.*

Proof. Note that $\overline{G}$ is $(p-r-1)$-regular graph of order p. Suppose $\overline{G}$ is disconnected. Then each component must have order at least $p-r$. So $p \geq 2(p-r)$ and thus $p \leq 2r$, a contradiction. This proves our lemma. ∎

Lemma 3.7. *Let G be a connected r-regular graph of order $p \geq 2r+1$. Let S be a cutset of $G\overline{G}$. Then $S \cap V(G) \neq \phi$ and $S \cap V(\overline{G}) \neq \phi$.*

Proof. By Lemma 3.6, $\overline{G}$ is connected. Hence, G and $\overline{G}$ are connected. Suppose without loss of generality that $S \cap V(G) = \phi$. So $S \subseteq V(\overline{G})$. Since $G = G\overline{G} - V(\overline{G})$ is connected and each vertex $\overline{u}$ of $V(\overline{G}) - S$ is adjacent to a vertex u in G, it follows that $G\overline{G} - S$ is connected, a contradiction. Hence, $S \cap V(G) \neq \phi$. By a similar argument, $S \cap V(\overline{G}) \neq \phi$. This proves our lemma. ∎

Theorem 3.8. *Let G be a connected r-regular graph of order $p \geq 2r+1$, for some $r \geq 2$. Then $G\overline{G}$ is bicritical. Consequently, $G\overline{G}$ is 1-extendable.*

Proof. Suppose $G\overline{G}$ is not bicritical. By Theorem 2.5, there is a cutset $S \subseteq V(G\overline{G})$, where $|S| \geq 2$ such that $c_o(G\overline{G} - S) > |S| - 2$. It follows by Lemmas 3.2(a) that $c_o(G\overline{G} - S) = |S|$ for $|S| \geq 2$. Note that, by Lemma 3.7, $A = S \cap V(G)$ and $\overline{B} = S \cap V(\overline{G})$ are not empty. Thus $\overline{A}$ and B are not empty. By Lemma 3.2 (b), $A \cap B = \phi$ and thus $c_o(G[B-A]) + c_o(\overline{G}[\overline{A} - \overline{B}]) = c_o(G[B]) + c_o(\overline{G}[\overline{A}])) = c_o(G\overline{G} - S) = |S| = |B| + |\overline{A}|$. By Lemma 3.2(c), each component of $G[B]$ and $\overline{G}[\overline{A}]$ is singleton. Hence, $G[A] \cong K_{|A|}$. Since G is r-regular of order $p \geq 2r+1$, $|A| \leq r+1$. If $|A| = r+1$, then $G[A] \cong K_{r+1}$ is a disconnected component in G, a contradiction. So $1 \leq |A| \leq r$. By Lemmas 3.3 and 3.4(a), no singleton component in $G[B]$ belongs to T_{B-A}. Since each component of $G[B]$ is singleton, $T_{B-A} = \phi$. Because $c_o(G\overline{G} - S) = c_o(\overline{G}[\overline{A}]) + c_o(G[B])$, it follows by Lemma 3.5 that $0 \leq |L| \leq 1$. Since $B \neq \phi$ and $G[B]$ contains only singleton components, it follows that $1 \leq |B| = |T_{B-A}| + |L_G| \leq 1$. Hence, $|B| = |L_G| = 1$. Therefore, $|\overline{B}| = 1 < r \leq p-r-1$. By Lemma 3.3, $T_{\overline{A}-\overline{B}}$ contains no singleton components. Hence, $T_{\overline{A}-\overline{B}} = \phi$. Since each component of $\overline{G}[\overline{A}]$ is singleton, it is contained in $L_{\overline{G}}$. So $|L_{\overline{G}}| = |\overline{A}| = |A| \geq 1$. Therefore, $|L| = |L_G| + |L_{\overline{G}}| \geq 2$, a contradiction. Hence, $G\overline{G}$ is bicritical. It then follows that $G\overline{G}$ is 1-extendable. This proves our theorem. ∎

The next lemma follows by Theorem 2.3.

Lemma 3.9. *Let G be a connected r-regular graph of order $p \geq 2r+1$, for some $r \geq 2$. If G contains a triangle, then $G\overline{G}$ is not r-extendable.*

By Lemma 3.9, if G is a 3-regular graph of order $p \geq 8$ containing a triangle, then $G\overline{G}$ is not 3-extendable. The next theorem provides a sufficient condition for a connected r-regular graph so that $G\overline{G}$ is 2-extendable, for $r \geq 4$. In case $r = 3$, if G contains the graph F in Figure 1 as an induced subgraph, then $\{yz, \bar{w}\bar{x}\}$ cannot be extended to a perfect matching in $G\overline{G}$. Hence, $G\overline{G}$ is not 2-extendable. We next show that the complementary prism of connected 3-regular, F-free graphs and connected r-regular graphs for $r \geq 4$ are 2-extendable.

Theorem 3.10. *Suppose G is a connected graph of order p. If G is either 3-regular, F-free where $p \geq 8$ and F is the graph in Figure 1 or r_0-regular where $p \geq 2r_0 + 1 \geq 9$, then $G\overline{G}$ is 2-extendable.*

Proof. Observe that $\overline{G}$ is $(p-r-1)$-regular where $r \in \{3, r_0\}$ and $p - r - 1 \geq 4$. By Theorem 3.8, $G\overline{G}$ is bicritical. Suppose to the contrary that $G\overline{G}$ is not 2-extendable. Then there is a matching $M \subseteq E(G\overline{G})$ of size two such that $G\overline{G} - V(M)$ contains no perfect matching. By Theorem 2.1, there is a cutset $T \subseteq V(G\overline{G}) - V(M)$ such that $c_o(G\overline{G} - (V(M) \cup T)) > |T|$. Let $S = T \cup V(M)$. Clearly, $|S| \geq 4$. Thus $c_o(G\overline{G} - S) > |S| - 4$. Because $G\overline{G}$ is bicritical, by Theorem 2.5, $c_o(G\overline{G} - S) \leq |S| - 2$. It follows by parity that $c_o(G\overline{G} - S) = |S| - 2$ and $G\overline{G}[S]$ contains a matching of size at least two. Let $A = S \cap V(G)$ and $\overline{B} = S \cap V(\overline{G})$. By Lemma 3.2 (b), $|A \cap B| \leq 1$. Further, by Lemma 3.7, $A \neq \phi$ and $\overline{B} \neq \phi$. So $\overline{A} \neq \phi$ and $B \neq \phi$. We distinguish 2 cases according to $|A \cap B|$.

Case 1. $|A \cap B| = 1$. Put $\{u\} = A \cap B$. By Lemma 3.2(b) $c_o(G\overline{G} - S) = c_o(G[B - A]) + c_o(\overline{G}[\overline{A} - \overline{B}]) = |S| - 2$. By Lemma 3.5, $|L| \leq 1$. Further, by Lemma 3.2(c), each component of $\overline{G}[\overline{A} - \overline{B}] \cup G[B - A]$ is singleton. Thus, $G[A - B]$ is a clique, $|Com_{\overline{A}-\overline{B}}| = |\overline{A} - \overline{B}|$ and $|Com_{B-A}| = |B - A|$. Since G is connected, it is easy to see that if $|A - B| \geq r+1$, then $G[A - B] \cong K_{|A-B|}$ contains a vertex of degree greater than r or $G \cong K_{r+1}$ is a graph of order less than p, a contradiction. Hence, $|A - B| \leq r$.

We first show that $|T_{B-A}| \geq 2$. Suppose to the contrary that $|T_{B-A}| \leq 1$. Since $G[B - A]$ contains only singleton components and $|L_G| \leq |L| \leq 1$, it follows that $|B - A| = |Com_{B-A}| = |T_{B-A}| + |L_G| \leq 2$. Thus $|\overline{B}| = |B| = |B - A| + |B \cap A| \leq 3 < 4 \leq p - r - 1$. By Lemma 3.3, $T_{\overline{A}-\overline{B}}$ contains no singleton components. Thus $T_{\overline{A}-\overline{B}} = \phi$. Consequently, $Com_{\overline{A}-\overline{B}} = T_{\overline{A}-\overline{B}} \cup L_{\overline{G}} = L_{\overline{G}}$. Therefore, $|\overline{A}-\overline{B}| = |L_{\overline{G}}| \leq 1$ since $\overline{G}[\overline{A}-\overline{B}]$ contains only singleton components. So $|A| = |\overline{A}| = |\overline{A} - \overline{B}| + |\overline{A} \cap \overline{B}| \leq 2 < r$. By Lemma 3.3, T_{B-A} contains no singleton components. So $T_{B-A} = \phi$. Since $T_{\overline{A}-\overline{B}} = \phi$ and $T_{B-A} = \phi$, it follows that every odd component of $\overline{G}[\overline{A} - \overline{B}] \cup G[B - A]$ is in L. Because $|L| \leq 1$ and $\overline{G}[\overline{A} - \overline{B}] \cup G[B - A]$ contains only singleton components, it follows that $|\overline{A} - \overline{B}| + |B - A| \leq 1$. Hence, $|S| = |A - B| + |A \cap B| + |\overline{A} \cap \overline{B}| + |\overline{B} - \overline{A}| = |A - B| + 2|A \cap B| + |\overline{B} - \overline{A}| \leq 3 < 4$, contradicting the fact that $|S| \geq 4$. Therefore, $|T_{B-A}| \geq 2$.

Let $D_1, D_2 \in T_{B-A}$. Since $G[B - A]$ contains only singleton components, $D_i \cong K_1$, for $1 \leq i \leq 2$. Put $\{v_i\} = V(D_i)$. By Lemma 3.3, $|A| \geq r$. Consequently, $|A-B| \geq r-1$. Because $|A-B| \leq r$, $r-1 \leq |A-B| \leq r$. Since $G[A-B]$ is clique, $|A \cap B| = 1$ and $|T_{B-A}| \geq 2$, it follows by Lemmas 3.4 (b) and (c) that $|A - B| = r - 1$ and $|T_{B-A}| = 2$. Thus $|A| = |A - B| + |A \cap B| = r$. Because $r - 1 = |A - B| = |\overline{A} - \overline{B}| = |Com_{\overline{A}-\overline{B}}| = |T_{\overline{A}-\overline{B}}| + |L_{\overline{G}}| \leq |T_{\overline{A}-\overline{B}}| + 1$, it follows that $|T_{\overline{A}-\overline{B}}| \geq r - 2 \geq 1$. Thus $T_{\overline{A}-\overline{B}}$ contains a singleton component. By Lemma 3.3, $|\overline{B}| \geq p-r-1 \geq 4$. Therefore, $|\overline{B}-\overline{A}| = |\overline{B}|-|\overline{B}\cap\overline{A}| \geq p-r-2 \geq 3$. On the other hand, $|B-A| = |Com_{B-A}| = |T_{B-A}|+|L_G| \leq 3$. Then $|B-A| = |\overline{B}-\overline{A}| = 3$. Thus $3 = |T_{B-A}| + |L_G| = 2 + |L_G|$. It follows that $L = L_G = \{K_1\}$ and consequently $L_{\overline{G}} = \phi$. Since $|A| = r$, $deg_G v_1 = deg_G v_2 = r$ and $N_G(v_1) = N_G(v_2) \subseteq A$, it follows that $N_G(v_1) = N_G(v_2) = A$.

We now put $\{\bar{w}\} = V(K_1)$ where $K_1 \in T_{\overline{A}-\overline{B}}$. Clearly, $N_{\overline{G}}(\bar{w}) \subseteq \overline{B} - \{\bar{v}_1, \bar{v}_2\}$ since v_1 and v_2 are adjacent to every vertex in A. Because $|\overline{B}| = |\overline{B}-\overline{A}|+|\overline{A}\cap\overline{B}| = 3 + 1 = 4$, $|N_{\overline{G}}(\bar{w})| \leq |\overline{B}| - |\{\bar{v}_1, \bar{v}_2\}| = 2$ thus $\overline{G}$ is t-regular where $t \leq 2$. This contradicts the fact that $\overline{G}$ is $(p - r - 1)$-regular where $p - r - 1 \geq 4$. Therefore, Case 1 cannot occur.

Case 2. $|A \cap B| = 0$. By Lemmas 3.2(a) and (b), $|S| - 2 = c_o(G\overline{G} - S) \leq c_o(\overline{G}[\overline{A}]) + c_o(G[B]) \leq |\overline{A}| + |B| = |S|$. By parity, $c_o(\overline{G}[\overline{A}]) + c_o(G[B]) = |S|$ or $c_o(\overline{G}[\overline{A}]) + c_o(G[B]) = |S| - 2$. We distinguish 2 cases.

Case 2.1. $c_o(\overline{G}[\overline{A}]) + c_o(G[B]) = |S| = |\overline{A}| + |B|$. Clearly, each component of $\overline{G}[\overline{A}] \cup G[B]$ is a singleton. So $G[A] \cong K_{|A|}$. It is easy to see that if $|A| \geq r + 1$, then $G[A]$ contains a vertex of degree greater than r or $G[A]$ is a disconnected component in G, a contradiction. Hence, $|A| \leq r$. By Lemmas 3.3 and 3.4(a), T_{B-A} contains no singleton components. Therefore, $T_{B-A} = \phi$. Thus $|L_G| = |B|$. Because $c_o(G[B]) + c_o(\overline{G}[\overline{A}]) - c_o(G\overline{G} - S) = |S| - (|S| - 2) = 2$, by Lemma 3.5, $2 \leq |L| \leq 3$. Since $B \neq \phi$ and $|B| = |L_G| \leq |L|$, it follows that $1 \leq |B| \leq 3$. Because $|\overline{B}| = |B| \leq 3 < 4 \leq p - r - 1$, by Lemma 3.3, $T_{\overline{A}-\overline{B}}$ contains no singleton components. Thus $T_{\overline{A}-\overline{B}} = \phi$. Hence, $|L_{\overline{G}}| = |\overline{A}| = |A|$. Therefore, $|L| = |L_G| + |L_{\overline{G}}| = |B| + |\overline{A}| = |S|$ and thus $2 \leq |S| \leq 3$ since $2 \leq |L| \leq 3$, contradicting the fact that $|S| \geq 4$. Hence, Case 2.1 cannot occur.

Case 2.2. $c_o(\overline{G}[\overline{A}]) + c_o(G[B]) = |S| - 2 = |\overline{A}| + |B| - 2$. Put $s = |S|$. It is easy to see that $\overline{G}[\overline{A}] \cup G[B]$ contains all singleton components except exactly one non-singleton component which is of order 2 or 3. Hence, $\overline{G}[\overline{A}] \cup G[B]$ is isomorphic to a graph in $\{(s-2)K_1 \cup K_2, (s-3)K_1 \cup P_3, (s-3)K_1 \cup K_3\}$. If $|\overline{A}| \geq r+2 \geq 5$, then $\overline{G}[\overline{A}]$ must contain a singleton component, say F, where $V(F) = \{\overline{u}\}$. It follows that $deg_G u \geq r+1$, a contradiction. Hence, $|A| = |\overline{A}| \leq r+1$. Since $c_o(\overline{G}[\overline{A}]) + c_o(G[B]) - c_o(G\overline{G} - S) = (|S|-2) - (|S|-2) = 0$, by Lemma 3.5, $|L| \leq 1$. We distinguish 2 subcases according to the non-singleton component.

Subcase 2.2.1. The only non-singleton component in $\overline{G}[\overline{A}] \cup G[B]$ is contained in $G[B]$. So $\overline{G}[\overline{A}] \cong |\overline{A}|K_1$ and $G[A] \cong K_{|\overline{A}|} \cong K_{|A|}$. Clearly, $|A| \leq r$ otherwise $G[A]$ is a disconnected component in G. By Lemmas 3.3 and 3.4(a), T_{B-A} contains no singleton components. So every singleton component in $G[B]$ is contained in L_G. Since $|L_G| \leq |L| \leq 1$, $G[B]$ contains at most 1 singleton component. We first show that $T_{\overline{A}-\overline{B}} = \phi$. Suppose this is not the case. Then there is $K_1 \in T_{\overline{A}-\overline{B}}$ since $\overline{G}[\overline{A}]$ contains only singleton components. By Lemma 3.3, $|B| = |\overline{B}| \geq p-r-1 \geq 4$. Because $G[B]$ contains a non-singleton component of order either 2 or 3 and at most 1 singleton component, it follows that $G[B]$ is isomorphic to a graph in $\{K_1 \cup P_3, K_1 \cup K_3\}$. Thus $|B| = 4$ and either $T_{B-A} = \{P_3\}$ or $T_{B-A} = \{K_3\}$, and $L_G = \{K_1\}$. Thus $L_{\overline{G}} = \phi$. So $Com_{\overline{A}} = T_{\overline{A}-\overline{B}} \cup L_{\overline{G}} = T_{\overline{A}-\overline{B}}$. Therefore, each vertex of $\overline{A}$ is adjacent to every vertex of $\overline{B}$ since $\overline{G}$ is $(p-r-1)$-regular and $p-r-1 \geq 4$. It follows that there is no edge joining vertices of A and B. But this contradicts the fact that $T_{B-A} \neq \phi$. Hence, $T_{\overline{A}-\overline{B}} = \phi$ as required.

Therefore, $Com_{\overline{A}} = L_{\overline{G}}$. Since $|L_{\overline{G}}| \leq |L| \leq 1$ and $|\overline{A}| = |A| \neq 0$, it follows that $|Com_{\overline{A}}| = |L_{\overline{G}}| = 1$. Further, $L_G = \phi$ and $\overline{G}[\overline{A}] = K_1$. Thus $Com_B = T_{B-A}$. Because $|A| = |\overline{A}| = 1 < r \leq 3$, by Lemma 3.3, T_{B-A} contains no singleton components. So $G[B]$ contains no singleton components and $G[B]$ is isomorphic to a graph in $\{P_3, K_3\}$ since $|B| = |S| - |A| \geq 3$. Then $G\overline{G}[S] = G[A] \cup \overline{G}[\overline{B}]$ contains a matching of size less than two, contradicting the fact that $G\overline{G}[S]$ contains a matching of size at least two. Hence, Subcase 2.2.1 cannot occur.

Subcase 2.2.2. The only non-singleton component in $\overline{G}[\overline{A}] \cup G[B]$ is contained in $\overline{G}[\overline{A}]$. So $G[B] \cong |B|K_1$. We first show that $T_{B-A} \neq \phi$. Suppose this is not the case. Then $T_{B-A} = \phi$ and thus $Com_B = T_{B-A} \cup L_G = L_G$. Since $B \neq \phi$ and $|L_G| + |L_{\overline{G}}| = |L| \leq 1$, it follows that $|L_G| = 1$ and $|L_{\overline{G}}| = 0$. Consequently, $|B| = 1$ since $G[B] \cong |B|K_1$. Because $|\overline{B}| = |B| = 1 < r$, $T_{\overline{A}-\overline{B}}$ contains no singleton components by Lemma 3.3. Hence, $G[\overline{A}]$ contains exactly one non-singleton component of order 2 or 3. Thus $|A| = |\overline{A}| \leq 3$. It is easy to see that $G\overline{G}[S] = G[A] \cup \overline{G}[\overline{B}]$ contains a matching of size at most one since $|\overline{B}| = 1$. This contradicts the fact that $G\overline{G}[S]$ contains a matching of size at least two. Hence, $T_{B-A} \neq \phi$. Further, $|T_{B-A}| \geq |B| - 1$ since $|L_G| \leq |L| \leq 1$ and $|T_{B-A}| + |L_G| = |B|$.

Because $G[B] \cong |B|K_1$, there exists $K_1 \in T_{B-A}$. By Lemma 3.3, $|A| \geq r$. So $r \leq |A| \leq r+1$. We first suppose that $|A| = r+1$. Let F_t be the non-singleton component of order t in $\overline{G}[\overline{A}]$ and let $\overline{A}_1 = V(F_t)$. Then $2 \leq t \leq 3$ and $\overline{G}[\overline{A}] \cong (r+1-t)K_1 \cup F_t$. It is easy to see that $G[A]$ contains $r+1-t$ vertices of degree r and each vertex of $A_1 = \overline{\overline{A}}_1$ has degree, in $G[A]$, at least $r+1-t$

and at most $r-1$. Let $\{w\} = V(K_1)$ where $K_1 \in T_{B-A}$, then $N_G(w) \subseteq A_1$ and thus $3 \leq r = deg_G(w) \leq t \leq 3$. It then follows that $N_G(w) = A_1$ and $t = r = 3$. Thus $\bar{w}$ is not adjacent to any vertex of $\overline{A}_1$ and $\overline{G}[\overline{A}] \cong K_1 \cup F_3$. Further, each vertex of A_1 has degree at least $|T_{B-A}|+1 = |B|-|L_G|+1 \geq |B|$ since $|L_G| \leq 1$. Thus $|B| \leq 3$ since G is now 3-regular. Because $\overline{G}$ is $(p-r-1)$-regular where $p-r-1 \geq 4$ and each vertex of $V(F_3) = \overline{A}_1$ has degree at most 3 in $\overline{G}[\overline{A} \cup \overline{B}]$ since it must be adjacent to at most one vertex in $\overline{B}$, it follows that $F_3 \in L_{\overline{G}}$. Since $|L_{\overline{G}}| \leq |L| \leq 1$, the only singleton component, K_1, of $\overline{G}[\overline{A}]$ must be in $T_{\overline{A}-\overline{B}}$. By Lemma 3.3, $|\overline{B}| \geq p-r-1 \geq 4$. But this contradicts the fact that $|\overline{B}| = |B| \leq 3$. Therefore, $|A| = r$.

Consequently, for each $w \in V(K_1)$ where $K_1 \in T_{B-A}$, $N_G(w) = A$. Now let $\bar{v} \in \overline{A}$. Then $deg_{\overline{B}}(\bar{v}) \leq |\overline{B}| - |T_{B-A}| = |B| - |T_{B-A}| = |L_G| \leq 1$. Further, $deg_{\overline{A}}(\bar{v}) \leq 2$ since each component of $\overline{G}[\overline{A}]$ has order at most 3. Because $\overline{G}$ is $(p-r-1)$-regular where $p-r-1 \geq 4$, $\bar{v}$ is adjacent to some vertex of $\overline{C}$. Consequently, each odd component of $\overline{G}[\overline{A}]$ is contained in $L_{\overline{G}}$. Because $|\overline{A}| = |A| = r \geq 3$, $\overline{G}[\overline{A}]$ contains a non-singleton component of order either 2 or 3 and $|L_{\overline{G}}| \leq |L| \leq 1$, it follows that $c_o(\overline{G}[\overline{A}]) = 1$. Therefore, $\overline{G}[\overline{A}]$ is isomorphic to a graph in $\{K_1 \cup K_2, P_3, K_3\}$. Hence, $r = |A| = 3$, $|L| = |L_{\overline{G}}| = 1$, $Com_B = T_{B-A} = \{|B|K_1\}$. Further, for $x \in B, y \in A, N_G(x) = A$ and $deg_G(y) = r = 3 \geq |B| = |\overline{B}|$.

We first suppose that $\overline{G}[\overline{A}] \cong K_3$. Then $G[A]$ is independent and thus $\overline{G}[\overline{B}]$ must contain a matching of size at least two since $G\overline{G}[S]$ contains a matching of size at least two. So $|B| = |\overline{B}| \geq 4$. But this contradicts the fact that $|B| = |\overline{B}| \leq 3$. Hence, $\overline{G}[\overline{A}] \neq K_3$. Therefore, $\overline{G}[\overline{A}]$ is isomorphic to a graph in $\{P_3, K_1 \cup K_2\}$. In either case, $G[A]$ contains a maximum matching of size one. Then $2 \leq |\overline{B}| \leq 3$ since $G\overline{G}[A \cup \overline{B}]$ contains a matching of size at least two.

We now suppose that $\overline{G}[\overline{A}] \cong K_1 \cup K_2$. Then $G[A] \cong P_3$ and then the vertex of degree two in P_3 has degree, in G, greater than $r = 3$, again a contradiction. Hence, $\overline{G}[\overline{A}] \neq K_1 \cup K_2$. Consequently, $\overline{G}[\overline{A}] \cong P_3$ and then $G[A] \cong K_1 \cup K_2$. Clearly, $|B| \neq 3$ otherwise $G[A]$ contains a vertex of degree greater than $r = 3$. So $|B| = 2$ and thus $G[A \cup B]$ contains the graph F in Figure 1 as an induced subgraph. But this contradicts our hypothesis that G is 3-regular, F-free graph. This completes the proof of our theorem. ∎

It is clear that a connected 3-regular graph containing F, in Figure 1, as an induced subgraph contains v as a cut vertex. So 2-connected 3-regular graphs are F-free. The next corollary follows by this fact and Theorem 3.10.

Corollary 3.11. *If G is a 2-connected r-regular graph of order $p \geq 2r+1$, for $r \geq 3$, then $G\overline{G}$ is 2-extendable.*

According to Theorems 2.6 and 3.10, we have the following theorem.

Theorem 3.12. *If each component G_i of G is 3-regular, F-free of order at least 8 where F is the graph in Figure 1 or r_0-regular of order at least $2r_0+1 \geq 9$, then $G\overline{G}$ is 2-extendable.*

We conclude our paper by posing following problem.

Problem. Establish sufficient condition for a complementary prism of r-regular graphs to be k-extendable for $r \geq k \geq 3$.

References

[1] ANANCHUEN, N., CACCETTA, L., *On critically k-extendable graphs*, Australasian Journal of Combinatorics, 6 (1992), 39-65.

[2] BONDY, J.A., MURTY, U.S., *Graph Theory with Applcations*, Elsevier Science Publishing Co., Inc, New York, 1976.

[3] HAYNES, T.W., HENNING, M.A., SLATER, P.J., VAN DER MERWE, L.C., *The complementary product of two graphs*, Bulletin of the Institute of Combinatorics and its Applications. 51 (2007) 21-30.

[4] HAYNES, T.W., HENNING, M.A., SLATER, P.J., VAN DER MERWE, L.C., *Domination and total domination in complementary prisms*, Journal of Combinatorial Optimization, 18 (2009), 23-37.

[5] HAYNES,T.W., HOLMES, K.R.S., KOESSLER, D.R., SEWELL, L., *Locating-Domination in Complementary Prisms of Paths and Cycles*, Congressus numerantium, 199 (2009), 45-55.

[6] FAVARON, O., *On n-factor-critical graphs*, Discussiones Math., Graph Theory, 16 (1996), 41-51.

[7] JANSEANA, P., RUEANGTHAMPISAN, S., ANANCHUEN, N., *Extendability of the complementary prism of 2-regular graphs*, (submitted).

[8] GYÖRI, E., PLUMMER, M.D., *The cartesian product of a k-extendable and an l-extendable graph is $(k + l + 1)$-extendable*, Discrete Mathematics, 101 (1992), 87-96.

[9] PLUMMER, M.D., *On n-extendable graphs*, Discrete Mathematics, 31 (1980), 201-210.

[10] PLUMMER, M.D., *Extending matchings in claw-free graphs*, Discrete Mathematics, 125 (1994), 301-307.

[11] PLUMMER, M.D., *Extending matchings in graphs: a survey*, Discrete Mathematics, 127(1994) (1-3) 277-292.

[12] PLUMMER, M.D., *Extending matchings in graphs: an update*, Congressus Numerantium, 116 (1996), 3-32.

[13] Plummer, M.D., *Recent Progress in Matching Extension*, Bolyal Society Mathematical Studies, 19 (2008), 427-454.

[14] Xu, R., Yu, Q., *Degree-sum conditions for k-extendable graphs*, Congressus Numerantium, 163 (2003), 189-195.

[15] Yu, Q., *Characterizations of various matching extensions in graphs*, Australian Journal of Combinatorics, 7 (1993), 55-64.

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STATISTICAL INFERENCE IN PRINCIPAL COMPONENT ANALYSIS BASED ON STATISTICAL THEORY

Yan Liu

Xi'an Eurasia University
Xi'an, Shanxi
China
e-mail: liuyanl023@sina.com

Abstract. Principal component analysis is a diversified statistical method, while statistical inference is the major research subject in modern statistics, whose theories and methods have comprised the core content of mathematical statistics. According to the relevant knowledge of statistical theory and based on a large sample size, this study explored the statistical inference problem when population followed normal distribution. Besides, statistical methods were applied to further analyze the statistical inference problems in principle component analysis under the condition of population with non-normal distribution or small sample size. First, principal component analysis was performed on parameter estimation and hypothesis testing on the condition that population followed multivariate normal distribution. Then under the condition of complex distribution of population, simulated sampling statistical inference method, i.e., Bootstrap method, was used to do interval estimation and discuss over other statistical inference problems of the characteristic values of the correlation coefficient matrixes in principle component analysis, and then the defects of Bootstrap method were adjusted using Bayesian theory.
Keywords: Bootstrap method; Bayesian estimation; principle component analysis; hypothesis testing.

1. Introduction

Principle component analysis can process a problem of high-latitude space by transforming it into a problem of low-latitude space [1], [2], thus to make the problem simpler and more visualized. Moreover, in the process of principal component analysis, each principal component will generate its weight automatically [3], which resists the influence of human factors in evaluation process and makes evaluation results more objective. Principal component analysis can be divided into principal component analysis for population and principal component analysis for samples [4]. In the analysis of practical problems, population covariance matrix and correlation coefficient matrix are generally unknown; therefore, sample covariance matrix and correlation coefficient matrix need to be calculated for the evaluation or hypothesis testing of samples [5]. When practical problems are analyzed using principal component analysis, principal component which can cover population information is selected through changing variable values and moreover further analysis is made on this basis. But, the premise of principal component selection is to suppose that these principal components can cover population information [6]. Therefore, corresponding hypothesis testing is needed to determine whether the processing of follow-up question is feasible.

In terms of the current situation of the development of statistical theory, studies of the statistical inference problem of principal component analysis mainly focus on the interval estimation of characteristic value of covariance matrix, the applicability test of principal component analysis, the test of principal component quantity selection, the test of characteristic root of population covariance matrix, etc [7]. With the development of modern computer technology in recent years, some new methods and ideas, for example, Bootstrap method, Logistic regression method and Jackknife estimation method, are added into statistical inference processing new technology [8, 9]. Bootstrap method and Jackknife estimation methods are two major simulation methods of statistical inference, which have been explored by many scholars in China and abroad. On the basis of previous studies, this study attempted to create resampling samples using Bootstrap method and then made interval estimation on the characteristic root of population covariance matrix based on the obtained samples. Finally, the existing loophole and defects of Bootstrap method were fixed and overcome using Bayesian theory.

2. Principal component analysis under multivariate normal distribution

2.1. Parameter estimation

Parameter estimation mainly includes confidence interval of characteristic root of population covariance matrix, confidence region of characteristic vector and combination confidence region of principal component score.

(1) Confidence interval of characteristic root of population covariance matrix

In the following, principal component analysis performed on parameter estimation on the premise that population followed multivariate normal distribution. Firstly, the approximate distribution of covariance matrix characteristic root of samples was deduced and then confidence interval with a confidence level of $1-\alpha$ was constructed.

If population X followed normal distribution, then the original sample $X_{(i)} = (x_{i1}, x_{i2}, ..., x_{in})$ and $X \sim N_n(\mu, \Sigma)$. The covariance matrix S of samples was the maximum likelihood estimation of population covariance matrix Σ, and, moreover, we have [10]:

$$S = \frac{1}{m-1}\sum_{i=1}^{m}(X_{(i)} - \overline{X}_{(*)})(X_{(i)} - \overline{X}_{(*)})' = (S_{ij})_{n\times n}.$$

As the maximum likelihood estimation result of Σ characteristic root was $\hat{\lambda}_1, \hat{\lambda}_2, ..., \hat{\lambda}_n$, $(\hat{\lambda}_1, \hat{\lambda}_2, ..., \hat{\lambda}_n)$ was determined as being in almost normal distribution. As likelihood function of λ_1 was $L(\lambda_i) = (\lambda_i)^{-\frac{n}{2}}\, e^{\frac{a_i' v a_i}{2\lambda_i}}$ and, moreover, $a_i'\Sigma a_i = \lambda_i$, $v \sim W_n(m-1, \Sigma)$, therefore $a_i' \sim \lambda_i \chi^2(m-1)$.

Because of the mutual independence among $\hat{\lambda}_1, \hat{\lambda}_2, ..., \hat{\lambda}_n$ and the almost normal distribution of $(\hat{\lambda}_1, \hat{\lambda}_2, ..., \hat{\lambda}_n)$, we have:

$$\sqrt{m-2}(\hat{\lambda}_i - \lambda_i) \to N_n\left(0.2\ \frac{\lambda_i^2}{m}\right).$$

Therefore, the confidence interval with a confidence level of $1-\alpha$ was:

$$\frac{\hat{\lambda}_i}{1+\sqrt{\frac{2}{m-2}}\,Z_{\frac{1-\alpha}{2}}} \leq \lambda_i \leq \frac{\hat{\lambda}_i}{1-\sqrt{\frac{2}{m-2}}\,Z_{\frac{1-\alpha}{2}}}.$$

It can be known from the above equation that, the increase of $\hat{\lambda}_i$ could widen the confidence interval of λ_i under the condition of a large sample size. However, when the characteristic value was relatively large, the confidence interval would be wide as well, even though the sample size was relatively large; at this moment, the estimation of characteristic root lost significance.

(2) Confidence region of characteristic vector

In the following content, parameter estimation of characteristic vector $a_1, a_2, ..., a_n$ was performed through building the confidence region of $a_1, a_2, ..., a_n$. If and moreover the maximum likelihood estimation result of Σ was $\hat{\Sigma} = S$, when the characteristic roots of Σ, i.e., $\lambda_1, \lambda_2, ..., \lambda_n$, satisfied the relationship that $\hat{\lambda}_1 \geq \hat{\lambda}_2 \geq \cdots \geq \hat{\lambda}_n$, then the corresponding unit orthotropic characteristic vectors were $\hat{a}_1, \hat{a}_2, ..., \hat{a}_n$ respectively and, moreover, $\hat{a}_i = (\hat{a}_{1i}, \hat{a}_{2i}, ..., \hat{a}_{ni})$. If $\hat{a}_{1i} \geq 0$, then $(\hat{a}_i, \hat{a}_2, ..., \hat{a}_n)$ was determined as being in almost normal distribution based on the maximum likelihood estimation. Suppose

$$A_i = \lambda_i \sum_{k=1}^{n} \frac{\lambda_k}{(\lambda_k - \lambda_i)^2}\ a_k a_k'$$

and, moreover, $\sqrt{n}(\hat{a}_i - a_i) \to N_n(0, A_i)$, then the confidence region with a confidence level of $1-\alpha$ was:

$$\frac{n^2(\hat{a}_i - a_i)(\hat{a}_i - a_i)'}{\lambda_i \sum\limits_{k=1}^{n} \frac{\hat{\lambda}_k a + k a_k'}{(\hat{\lambda}_k - \hat{\lambda}_i)^2}} \leq \left(Z_{\frac{1-\alpha}{2}}\right)^2.$$

(3) Combination confidence region of principal component score

If population X followed normal distribution, then the original sample $X_{(i)} = (x_{i1}, x_{i2}, ..., x_{in})$ and, moreover, $X \sim N_n(\mu, \Sigma)$. If $\Sigma > 0$ and the maximum likelihood estimation result ofΣ was $hat\Sigma = S$, when characteristic roots of Σ, i.e., $\lambda_1, \lambda_2, ..., \lambda_n$, satisfied the relationship that $\hat{\lambda}_1 \geq \hat{\lambda}_2 \geq \cdots \geq \hat{\lambda}_n$, then the corresponding unit orthotropic characteristic vectors were $\hat{a}_1, \hat{a}_2, ..., \hat{a}_n$ and, moreover, $\hat{a}_i = (\hat{a}_{1i}, \hat{a}_{2i}, ..., \hat{a}_{ni})$. If $\hat{a}_{1i} \geq 0$, then the i-th principal component could be expressed as follows according to the principal component analysis

$$Y_i = a_i X = a_{i1} X_i + a_{i2} X_2 + \cdots + a_{in} X_n.$$

According to the confidence interval of characteristic root and the confidence region of characteristic vector of population covariance matrix obtained by parameter estimation previously, the combination confidence region of principal component score could be deduced.

According to $\sqrt{n}(\hat{a}_i - a_i) \to N_n(0, A_i)$, the principal component score approximately followed normal distribution [11]. Therefore, the following calculation was needed

$$E[\hat{Y}_i] = E[a_i' X_{(i)}] = E[a_i' V - \hat{a}_i' X_{(i)}] + E[\hat{a}_i' X_{(i)}] = \hat{a}_i' \mu_i$$
$$D[\hat{Y}] = D[a_i' X_{(i)}] = D[a_i' X_{(i)} - \hat{a}_i' X_{(i)}] = A_i \Sigma$$

It could be known that $\hat{Y}_i - \hat{a}_i' \mu_i \to N_n(0, A_i, \Sigma)$. Thus the confidence region of principal component score was:

$$\frac{n(\hat{Y} - \hat{a}_i'\mu_i)(\hat{Y} - \hat{a}_i'\mu_i)'}{\left[\hat{\lambda}_i \sum\limits_{k=1}^{n} \frac{\hat{\lambda}_k \hat{a}_k \hat{a}_k'}{(\hat{\lambda}_k - \hat{\lambda}_i)^2}\right] \hat{\Sigma}} \leq (Z_{\frac{1-\alpha}{2}})^2$$

2.2. Hypothesis testing

Hypothesis testing mainly includes [12] hypothesis testing on the adaptability of principal component analysis, hypothesis testing of population covariance matrix characteristic root, and hypothesis testing on the selection of principal component number, etc.

(1) Hypothesis testing on the adaptability of principal component analysis

Hypothesis testing on the adaptability of principal component analysis, a common correlation test [13], aims at testing the hypothesis that the correlation coefficients of variables were all equal or not all equal to each other. If the correlation coefficients of n principal components were all equal, then it indicated that, characteristic roots of covariance matrix were all unequal [14]; at that moment, principal component analysis cannot be performed on n principal components. Therefore, the testing of correlation structure should be carried out. The following was an example of Kaiser-Meyer-Olkin (KMO) test.

The calculation formula of KMO [15] was:

$$\mathrm{KMO} = \frac{\sum\limits_{j=1}^{k} \sum\limits_{i \neq j}^{k} r_{ij}^2}{\sum\limits_{j=1}^{k} \sum\limits_{i \neq j}^{k} r_{ij}^2 + \sum\limits_{j=1}^{k} \sum\limits_{i \neq j}^{k} \rho_{ij}^2}$$

where r_{ij} stands for simple correlation and ρ_{ij} stands for partial correlation coefficient. It can be seen from the above formula that, the value of KMO was $[0, 1]$, which fell out of the range of $[0, 0.5)$. Thus, it was unsuitable to carry out a principal component analysis.

(2) Hypothesis testing of population covariance matrix characteristic root

It can be known from statistical theory that, if matrix $A' = A$, then characteristic values of the matrix, i.e., $\lambda_1, \lambda_2, ..., \lambda_n$, were ranked in order of size. Suppose $\lambda_1 > \lambda_2 > \cdots > \lambda_n$ and, moreover, $\gamma_1, \gamma_2, ..., \gamma_n$ as the standard orthogonal characteristic vectors corresponding to characteristic roots, then there is an A for any vector x

$$\max_{x \neq 0} \frac{x'Ax}{x'x} = \lambda_1, \quad \min_{x \neq 0} \frac{x'Ax}{x'x} = \lambda_n.$$

Suppose the covariance matrix of random vector($X = (X_1, X_2, ..., X_m)'$ was Σ, $\lambda_1, \lambda_2, ..., \lambda_n$ were characteristic roots, and $\gamma_1, \gamma_2, ..., \gamma_n$ were standard orthogonal characteristic vectors corresponding to characteristic roots of matrix A, then the i-th principal component was:

$$Y_i = \gamma_{1i}X_1 + \gamma_{2i}X_2 + \cdots + \gamma_{mi}X_m \ (i = 1, 2..., m).$$

At that moment, $\mathrm{Var}(Y_i) = \gamma_i'\Sigma\gamma_i = \lambda_1$ and $\mathrm{cov}(Y_i, Y_j) = \gamma'\Sigma\gamma_j = 0 \ (i \neq j)$. Standard characteristic vectors $\gamma_1, \gamma_2, ..., \gamma_n$ corresponding to nonzero characteristic values of covariance matrix Σ, i.e., $\lambda_1, \lambda_2, ..., \lambda_m$, were taken as coefficient vectors, that was:

$$Y_1 = \gamma_1'X, Y_2 = \gamma_2'X, ..., Y_m = \gamma_m'X,$$

where $Y_1, Y_2, ..., Y_m$ stand for the first principal component, the second principal component,..., and the m-th principal component of vector X. The necessary and sufficient conditions for that the separation of the principal components is equal to vector X were as follows. Firstly, $Y = u'X$ and $u'u = I$, i.e., u was an m-order orthogonal matrix. Secondly, principal components Y were uncorrelated to each other. Thirdly, m components of principal component were ranked according to the size of variance.

Thus X vector and principal component had the following relationship

$$\begin{aligned} Y &= u'X = [u_1', u_2', ..., u_m']^T \cdot X \\ &= \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1m} \\ u_{21} & u_{22} & \cdots & u_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ u_{m1} & u_{m2} & \cdots & u_{mm} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_m \end{bmatrix} = \begin{bmatrix} \gamma_1' \\ \gamma_2' \\ \vdots \\ \gamma_m' \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_m \end{bmatrix} \end{aligned}$$

(3) Hypothesis testing of the selection of principal component number

In instance analysis, whether the selected principal components can satisfy the requirements of application needs to be determined at first; therefore, the selection of principal component number requires hypothesis testing. The selection of principal component number mainly relies on the size of contribution. The common testing methods include Bartlett test, mean method and empirical method [16].

Bartlett test: the method determines the number of principal component by testing whether the characteristic root is 0 [17]. If characteristics roots with low ranks had no statistically significant difference with 0, then they were not considered as principal components. Only principal components whose characteristic roots were not zero were selected.

Mean method: the mean value of characteristic roots, i.e., $\overline{\lambda}$, was calculated at first and then principal components whose characteristic roots were larger than $\overline{\lambda}$ were selected.

Empirical method: empirical method determines the number of principal component according to practical experience. Generally, a component with an accumulated contribution higher than 80% can be selected as a principal component. Suppose λ_i stands for the characteristic value of the i-th principal component, then the contribution rate of the i-th principal component y_i was expressed as $\lambda_i \sum\limits_{i=1}^{m} \lambda_2$. Therefore, the accumulated contribution rate of the first m principal components should be expressed as:

$$\frac{\sum\limits_{i=1}^{m} \lambda_i}{\sum\limits_{i=1}^{n} \lambda_i} \times 100\% \quad (n > m).$$

Generally, to reduce the loss of information and variables as well as simplify problem, the value of m should be able to ensure over 80% of accumulated contribution rate.

It can be known from practical analysis condition that, Bartlett test is easy to be affected by the volume of samples. Many principals can be obtained if the volume of sample is large; otherwise, fewer principals can be obtained. Mean method is usually easy to be affected by extremum, which can make the obtained $\overline{\lambda}$ fail to reflect practical situation. Thus, the empirical method is adopted usually as it can reflect practical situation better and satisfy the requirements of application.

3. Discussion on the inference problem of principal component analysis using the Bootstrap method

The Bootstrap method mainly includes a parametric Bootstrap method and a nonparametric Bootstrap method [18]. The parametric Bootstrap method extracts samples according to population distribution function. The nonparametric Bootstrap method has no requirement on population distribution, i.e., sampling with replacement is performed to obtain samples when population distribution is unknown [19]. For example, when a parameter θ is selected for N experiments, an estimated value $\theta^{(n)}$ $(n = 1, 2, ..., N)$ can be obtained every time; then, some properties of θ including mean value, standard deviation and confidence interval [20] are estimated according to its distribution $F(\theta)$. The procedures of the method are as follows.

Figure 1: Sampling principle of the Bootstrap method

However, the distribution condition of population is unknown generally. Therefore, sampling needs to be carried out for the analysis of statistical inference problems using the nonparametric Bootstrap method. The parameter estimation of principal component analysis has been described before. The following content was about non-parametric Bootstrap method. As there is no need to know the distribution of population in advance when the nonparametric Bootstrap method is used, we suppose the size of samples was small and insufficient for the traditional statistical inference.

The specific procedures were as follows.

(1) Suppose the distribution of population X followed $F(x)$, the number of samples was m, and the original sample $X(i) = (x_{i1}, x_{i2}, ..., x_{ip})$ $(i = 1, 2, ..., n)$.

(2) The covariance matrix $\hat{\Sigma} = S$ of samples was calculated according to $X_{(i)}$, then P characteristic roots, i.e., $\hat{\lambda}_1, \hat{\lambda}_2, ..., \hat{\lambda}_p$, which were not equal with each other were obtained. The first principal component λ_1 was processed by parameter estimation.

(3) Sampling with replacement was carried out among m samples; the sampling repeated for m times or for B times to obtain a sample with a volume of B.

(4) Suppose the sample obtained after B times of sampling as $X_{(B)}$. The characteristic roots of the covariance matrix of samples, i.e., $\lambda_1^{(b)}, \lambda_2^{(b)}, ..., \lambda_p^{(b)}$ $(p = 1, 2, ..., B)$, could be obtained after the covariance and characteristic root were calculated.

(5) Frequency distribution of characteristic roots of group B samples was analyzed. The Bootstrap percentiles method was used to estimate the confidence interval under the confidence level of 1α.

4. Adjustment of Bootstrap based on Bayesian theory

4.1. The problems existing in Bootstrap method

Subsamples obtained in random sampling are very likely to be similar to the original samples or be regenerated samples which deviate greatly from the original samples when the traditional Bootstrap method is used to perform repeated random sampling with replacement, because the observed value of original sample has been determined [21], [22].

The defect will be more obvious when there are fewer original measured values. The phenomenon that probability concentrates on a certain observed value may result in the deviation of the finally estimated $R(X, F)$ distribution from real distribution, which severely affects the accuracy of prediction results.

On the other hand, subsamples obtained when repeated random sampling with replacement is performed on sample $X = (X_1, X_2, ..., X_n)$ are all samples points in original observation samples. At the moment, the properties of complete real distribution at non-original observed value points cannot be obtained and moreover the real distribution characteristics of $R(X, F)$ cannot be obtained, which further lowers the accuracy of prediction results.

4.2 The adjustment of Bootstrap method using the Bayesian theory

In reality, it is common that sample size is not large enough for statistical inference using traditional method. When the original sample is not less than 10, estimated result obtained using Bootstrap method is usually highly accurate. The advantage of the Bayesian idea in combination with the Bootstrap method is that it can solve the problem of parameter estimation in the situation of unknown and complex population distribution or small sample size. Compared to risk estimation using traditional method, this method has high preciseness. The characteristic of the Bayesian theory is that, it determines prior distribution of parameters with unknown distribution using the Bootstrap method and taking prior distribution as the starting point [23] and adjusts the problems existing in Bootstrap method using the Bayesian theory in combination with Bootstrap method.

The specific procedures were as follows.

Firstly, suppose the population $X \sim F(x)$ as an unknown distribution and $\theta(F)$ as a parameter to be estimated. Besides, $(X_{(i)} = (x_{i1}, x_{2i}, ..., x_{in})$ was supposed as n original samples from the population $X \sim F(x)$. Let the empirical distribution function of $X_{(i)}$ $(i = 1, 2, ..., m)$ be:

$$F_i(x) \sim \frac{1}{n} \sum_{i=1}^{n} P(X_i < x).$$

Secondly, before the acquisition of $\hat{\theta}(F)$ by evaluating $\theta(F)$, statistic $T_n = \hat{\theta}(F) - \theta(F)$ needed to be created to obtain its statistical characteristics.

Thirdly, $m-1$ original samples following $U(0, 1)$ were extracted and ranked according to their size, denoted as $u_{(1)}, u_{(2)}, ..., u_{(n-1)}$ $(u_{(0)}, = 0,\ u_{(n)}, = 1)$.

Fourthly, suppose $v_i = u_{(i)} - u_{(i-1)}$ $(i = 1, 2..., n-1)$, let:

$$\begin{aligned} \hat{\theta}_v &= \theta\left(\sum_{i=1}^{n} v_i F_i(x)\right), \\ D_n &= \hat{\theta}_v - \hat{\theta}(F) \end{aligned}$$

$\hat{\theta}_v$ stands for the random weighed statistic. T_n and its statistical characteristics could be obtained through D_n because D_n is the approximate distribution of T_n.

Fifthly, if the number of original samples $X_{(i)} = (x_{i1}, x_{i2}, ..., x_{in})$ was not large enough for the Bayesian estimation, repeated sampling could be adopted to obtain regenerated samples and then $\theta(F)$ was processed by the Bayesian estimation.

If population X followed normal distribution and the number of original samples was small, then we have:

$$X_{(i)} \sim N(\mu, \sigma^2)$$

Suppose $\mu = 0$ and $\theta = \sigma^2$ as parameters to be estimated; moreover, $\theta = \sigma^2 \sim G(\alpha, \beta)$. Then it could be known that $\pi(\theta)\infty\theta^{(\beta-1)}e^{-\frac{\alpha}{\theta}}$ $(\theta, \beta > 0)$.

If the sample extracted from regenerated samples was supposed as $X_{(B)}$ and, moreover, $\hat{\sigma}_B^2 = \hat{\theta}_B$, then the Bayesian estimation result obtained by substituting samples into the above estimation result was as follows.

$$\hat{\theta}_B = E(\theta|x) = \frac{2\alpha + s}{2\beta + n - 2} = \frac{2\beta - 2}{2\beta + n - 2}\, E(\theta) + \frac{n}{2\beta + n - 2}\, \hat{\sigma}_B^2$$

Finally, the Bayesian estimation method was used to perform parameter estimation on characteristic roots of population covariance matrix. The operation was as follows.

If population X followed normal distribution, original sample was $X_{(i)} = (x_{i1}, x_{i2}, ..., x_{in})$, and the volume of samples was small, then $X_{(i)} \sim N(\overline{X}, \hat{\Sigma})$ was thought to follow normal distribution. Then sample $X_{(B)}$ with a volume of B was extracted through regeneration sampling; characteristic roots $\lambda_1^{(b)}, \lambda_2^{(b)}, ..., \lambda_p^{(b)}$ $(b = 1, 2, ..., B)$ of covariance matrix were calculated. The confidence interval of the characteristic root of population covariance matrix, with a confidence level of α, could be obtained according to the upper and lower $\frac{100\alpha}{2}\%$ quantiles of frequency distribution of characteristic roots.

5. Conclusion

In this study, statistical theory was applied to discuss over the statistical inference problem of the population following multivariate normal distribution in principal component analysis under the condition of a large sample size; then non-parametric Bootstrap method was used to explore the statistical inference problem in principal component analysis under the condition of a small sample size; finally, the limitations and bugs of Bootstrap method were corrected using the Bayesian estimation. However, Bootstrap method still has some defects during parameter estimation. Hence, further studies using other methods are needed to improve Bootstrap method.

References

[1] Bro, R., Smilde, A.K., *Analytical Methods*, 6 (9) (2014), 433459.

[2] Ohta, N., *Analytical Chemistry*, 45 (3) (2015), 218-219.

[3] Doukas, H., Papadopoulou, A., Savvakis, N., et al., *Renewable & Sustainable Energy Reviews*, 16 (4) (2012), 1949-1957.

[4] Zheng, X., Levine, D., Shen, J., et al., *Bioinformatics*, 28 (24) (2012), 3326-3328.

[5] Sabokbar, H, F., Roodposhti, M.S., Tazik, E., *Geomorphology*, 226 (2014), 1524.

[6] Moore, B., Moore, B., *Social Science Electronic Publishing*, 61 (100) (2012), 41-64.

[7] Kwak, N., *Cybernetics IEEE Transactions*, 44 (5) (2014), 594-609.

[8] Li, Y., Hsing, T., *Annals of Statistics*, 38 (2010) (2012), 3321-3351.

[9] Garcia-Alvarez, D., Fuente, M.J., Sainz, G.I., *Journal of Process Control*, 22 (3) (2012), 551-563.

[10] Helwig, N.E., Hong, S., Polk, J.D., *Human Movement Science*, 31 (3) (2012), 630-648.

[11] Metsalu, T., Vilo, J., *Nucleic Acids Research*, 43 (1) (2015), 566-70.

[12] Josse, J., Husson, F., *Computational Statistics & Data Analysis*, 56 (6) (2012), 1869-1879.

[13] Abson, D.J., Dougill, A.J., Stringer, L.C., *Applied Geography*, 35 (1-2) (2012), 515-524.

[14] Peng, X., Xin, L., Shi, X., et al., *Food Chemistry*, 151 (20) (2014), 532-8.

[15] Zandi, G., Naysary, B., Sok, Kwan, S., *Journal of Accounting & Management Information Systems*, 12 (3) (2013), 471-488.

[16] Peterson, M.D., Liu, D., Iglayreger, H.B., et al., *Cardiovascular Diabetology*, 11 (1) (2012), 1-10.

[17] Yang, S., Wang, M., Jiao, L., *Information Fusion*, (2012), 13(3):177-184.

[18] Babamoradi, H., Berg, F.V.D., Åsmund, Rinnan, *Chemometrics & Intelligent Laboratory Systems*, 120 (2) (2013), 97-105.

[19] Sharpe, J., Fieller, N., *Journal of Applied Statistics*, (2016), 1-15.

[20] Chen, X., Samson, E., Tocqueville, A., et al., *Journal of Cleaner Production*, 87 (1) (2015), 87-95.

[21] Yaqub, T., Katupitya, J., *Journal of Automation Mobile Robotics & Intelligent Systems*, 2 (6) (2008), 44-53.

[22] She, Y., Li, S., Wu, D., *Journal of the American Statistical Association*, (2014), 41-64.

[23] Jolliffe, I.T., Springer Berlin, 87 (100) (2015), 41-64.

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SOME RESULTS ON SLOWLY CHANGING FUNCTION ORIENTED RELATIVE ORDER, RELATIVE TYPE AND RELATIVE WEAK TYPE OF DIFFERENTIAL MONOMIALS

Sanjib Kumar Datta

Department of Mathematics,University of Kalyani
P.O. Kalyani, Dist-Nadia, PIN-741235
West Bengal
India
e-mail: sanjib_kr_datta@yahoo.co.in

Tanmay Biswas

Rajbari, Rabindrapalli, R.N. Tagore Road
P.O. Krishnagar, Dist-Nadia, PIN-741101
West Bengal
India
e-mail: tanmaybiswas_math@rediffmail.com

Debasmita Dutta

Mohanpara Nibedita Balika Vidyalaya (High)
Block-English Bazar
P.O.-Amrity, Dist-Malda Pin-732208
West Bengal
India
e-mail: debasmita.dut@gmail.com

Abstract. In this paper, we establish the relationship between the relative L-order (relative L^*-order), relative L-type (relative L^*-type) and relative L-weak type (relative L^*-weak type) of a transcendental meromorphic function f with respect to an transcendental entire function g and that of monomial generated by the meromorphic f and entire g.

Keywords and phrases: transcendental entire function, transcendental meromorphic function, relative order (relative lower order), relative type (relative lower type), relative weak type, monomial, slowly changing function.
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1. Introduction, definitions and notations

We denote by $\mathbb{C}$ the set of all finite complex numbers. Let f be an entire function defined on $\mathbb{C}$. The function $M_f(r) = \max_{|z|=r} |f(z)|$ known as maximum modulus function corresponding to f.

When f is meromorphic, $M_f(r)$ can not be defined as f is not analytic. In this situation one may define another function $T_f(r)$ known as Nevanlinna's Characteristic function of f, playing the same role as $M_f(r)$ in the following manner:

$$T_f(r) = N_f(r) + m_f(r)$$

where the function $N_f(r)$ and $m_f(r)$ are respectively the enumerative function and the proximity function corresponding to f. If f is an entire function, then the Nevanlinna's Characteristic function $T_f(r)$ of f reduces to $m_f(r)$. Also for a non-constant entire f, $T_f(r)$ is strictly increasing and continuous function of r and its inverse $T_f^{-1} : (T_f(0), \infty) \to (0, \infty)$ exists and is such that $\lim\limits_{s\to\infty} T_g^{-1}(s) = \infty$.

Furthermore, we called the function $N_f(r,a)$ $\left(\overline{N}_f(r,a)\right)$ as counting function of a-points (distinct a-points) of f. We put

$$N_f(r,a) = \int_0^r \frac{n_f(t,a) - n_f(0,a)}{t} dt + \overline{n}_f(0,a) \log r \ ,$$

where we denote by $n_f(r,a)$ $(\overline{n}_f(r,a))$ the number of a-points (distinct a-points) of f in $|z| \leq r$ and an ∞ -point is a pole of f. In many occasions $N_f(r,\infty)$ and $\overline{N}_f(r,\infty)$ are denoted by $N_f(r)$ and $\overline{N}_f(r)$ respectively. Also we denote by $n_{f|=1}(r,a)$,the number of simple zeros of $f-a$ in $|z| \leq r$. Accordingly, $N_{f|=1}(r,a)$ is defined in terms of $n_{f|=1}(r,a)$ in the usual way and we set

$$\delta_1(a;f) = 1 - \limsup_{r\to\infty} \frac{N(r,a;f \mid= 1)}{T_f(r)} \ \{\text{cf. [9]}\},$$

the deficiency of 'a' corresponding to the simple a-points of f, i.e., simple zeros of $f-a$. In this connection, Yang [8] proved that there exists at most a denumerable number of complex numbers $a \in \mathbb{C} \cup \{\infty\}$ for which

$$\delta_1(a;f) > 0 \text{ and } \sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;f) \leq 4 \ .$$

Further, a meromorphic function $b = b(z)$ is called small with respect to f if $T_b(r) = S_f(r)$ where $S_f(r) = o\{T_f(r)\}$ i.e., $\frac{S_f(r)}{T_f(r)} \to 0$ as $r \to \infty$. Moreover for any transcendental meromorphic function f, we call $P[f] = bf^{n_0}(f^{(1)})^{n_1}...(f^{(k)})^{n_k}$, to be a differential monomial generated by it where $\sum\limits_{i=0}^{k} n_i \geq 1$ (all $n_i \mid i = 0,1,...,k$ are non-negative integers) and the meromorphic function b is small with respect to f. In this connection, the numbers $\gamma_{P[f]} = \sum\limits_{i=0}^{k} n_i$ and $\Gamma_{P[f]} = \sum\limits_{i=0}^{k}(i+1)n_i$ are called the degree and weight of $P[f]$ respectively {cf. [1]}.

In this connection, the following definitions are well known:

Definition 1 The order ρ_f and lower order λ_f of a meromorphic function f are defined as

$$\rho_f = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log r} \ .$$

Somasundaram and Thamizharasi [6] introduced the notions of L-order and L-lower order for entire functions where $L \equiv L(r)$ is a positive continuous function increasing slowly i.e.,$L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant "a". Their definitions are as follows:

Definition 2 [6] The L-order ρ_f^L and the L-lower order λ_f^L of a meromorphic function f are defined as follows:

$$\rho_f^L = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log[rL(r)]} \text{ and } \lambda_f^L = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log[rL(r)]} .$$

The more generalised concept of L-order and L-lower order of meromorphic functions are L^*-order and L^*-lower order respectively which are as follows:

Definition 3 The L^*-order $\rho_f^{L^*}$ and the L^*-lower order $\lambda_f^{L^*}$ of a meromorphic function f are defined by

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log[re^{L(r)}]} \text{ and } \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log[re^{L(r)}]} .$$

Lahiri and Banerjee [5] introduced the definition of relative order of a meromorphic function with respect to an entire function which is as follows:

Definition 4 [5] Let f be meromorphic and g be entire. The relative order of f with respect to g denoted by $\rho_g(f)$ is defined as

$$\begin{aligned} \rho_g(f) &= \inf\{\mu > 0 : T_f(r) < T_g(r^\mu) \text{ for all sufficiently large } r\} \\ &= \limsup_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r} . \end{aligned}$$

The definition coincides with the classical one [5] if $g(z) = \exp z$.

Similarly, one can define the relative lower order of a meromorphic function f with respect to an entire g denoted by $\lambda_g(f)$ in the following manner:

$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r} .$$

Datta and Biswas [2] gave the definition of relative type and relative weak type of a meromorphic function with respect to an entire function g which are as follows:

Definition 5 [2] The relative type $\sigma_g(f)$ of a meromorphic function f with respect to an entire function g are defined as

$$\sigma_g(f) = \limsup_{r\to\infty} \frac{T_g^{-1} T_f(r)}{r^{\rho_g(f)}}, \quad \text{where } 0 < \rho_g(f) < \infty .$$

Similarly, one can define the lower relative type $\overline{\sigma}_g(f)$ in the following way

$$\overline{\sigma}_g(f) = \liminf_{r\to\infty} \frac{T_g^{-1} T_f(r)}{r^{\rho_g(f)}}, \quad \text{where } 0 < \rho_g(f) < \infty .$$

Definition 6 [2] The relative weak type $\tau_g(f)$ of a meromorphic function f with respect to an entire function g with finite positive relative lower order $\lambda_g(f)$ is defined by

$$\tau_g(f) = \liminf_{r\to\infty} \frac{T_g^{-1}T_f(r)}{r^{\lambda_g(f)}} .$$

Analogously, one can define the growth indicator $\overline{\tau}_g(f)$ of a meromorphic function f with respect to an entire function g with finite positive relative lower order $\lambda_g(f)$ as

$$\overline{\tau}_g(f) = \limsup_{r\to\infty} \frac{T_g^{-1}T_f(r)}{r^{\lambda_g(f)}} .$$

In order to prove our results we require the following definitions:

Definition 7 The relative L-order $\rho_g^L(f)$ and the relative L-lower order $\lambda_g^L(f)$ of a meromorphic function f with respect to an entire function g are defined as follows:

$$\rho_g^L(f) = \limsup_{r\to\infty} \frac{\log T_g^{-1}T_f(r)}{\log[rL(r)]} \text{ and } \lambda_g^L(f) = \liminf_{r\to\infty} \frac{\log T_g^{-1}T_f(r)}{\log[rL(r)]} .$$

Definition 8 The relative L-type $\sigma_g^L(f)$ and the relative L-lower type $\overline{\sigma}_g^L(f)$ of a meromorphic function f with respect to an entire function g are defined as follows:

$$\sigma_g^L(f) = \limsup_{r\to\infty} \frac{T_g^{-1}T_f(r)}{[rL(r)]^{\rho_g^L(f)}} \text{ and } \overline{\sigma}_g^L(f) = \liminf_{r\to\infty} \frac{T_g^{-1}T_f(r)}{[rL(r)]^{\rho_g^L(f)}},$$

where $0 < \rho_g^L(f) < \infty$.

Definition 9 The relative L-weak type $\tau_g^L(f)$ of a meromorphic function f with respect to an entire function g with finite positive relative L-lower order $\lambda_g^L(f)$ is defined by

$$\tau_g^L(f) = \liminf_{r\to\infty} \frac{T_g^{-1}T_f(r)}{[rL(r)]^{\lambda_g^L(f)}} .$$

Similarly, one can define the growth indicator $\overline{\tau}_g^L(f)$ of a meromorphic function f with respect to an entire function g with finite positive relative L-lower order $\lambda_g^L(f)$ as

$$\overline{\tau}_g^L(f) = \limsup_{r\to\infty} \frac{T_g^{-1}T_f(r)}{[rL(r)]^{\lambda_g^L(f)}} .$$

The more generalised concept of relative L-order (relative L-lower order), relative L-type (relative L-lower type) and relative L-weak type of meromorphic function with respect to an entire function are relative L^*-order (relative L^*-lower order), relative L^*-type (relative relative L^*-lower type) and relative L^*-weak type respectively which are as follows:

Definition 10 The L^*-order $\rho_f^{L^*}$ and the L^*-lower order $\lambda_f^{L^*}$ of a meromorphic function f are defined by

$$\rho_g^{L^*}(f) = \limsup_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log \left[re^{L(r)}\right]} \text{ and } \lambda_g^{L^*}(f) = \liminf_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log \left[re^{L(r)}\right]} .$$

Definition 11 The relative L^*-type $\sigma_g^{L^*}(f)$ and the relative L^*-lower type $\overline{\sigma}_g^{L^*}(f)$ of a meromorphic function f with respect to an entire function g are defined as follows:

$$\sigma_g^{L^*}(f) = \limsup_{r\to\infty} \frac{T_g^{-1} T_f(r)}{\left[re^{L(r)}\right]^{\rho_g^{L^*}(f)}} \text{ and } \overline{\sigma}_g^{L^*}(f) = \liminf_{r\to\infty} \frac{T_g^{-1} T_f(r)}{\left[re^{L(r)}\right]^{\rho_g^{L^*}(f)}},$$

where $0 < \rho_g^{L^*}(f) < \infty$.

Definition 12 The relative L^*-weak type $\tau_g^{L^*}(f)$ of a meromorphic function f with respect to an entire function g with finite positive relative L^*-lower order $\lambda_g^{L^*}(f)$ is defined by

$$\tau_g^{L^*}(f) = \liminf_{r\to\infty} \frac{T_g^{-1} T_f(r)}{\left[rL(r)\right]^{\lambda_g^{L^*}(f)}} .$$

Similarly, one can define the growth indicator $\overline{\tau}_g^{L^*}(f)$ of a meromorphic function f with respect to an entire function g with finite positive relative L^*-lower order $\lambda_g^{L^*}(f)$ as

$$\overline{\tau}_g^{L^*}(f) = \limsup_{r\to\infty} \frac{T_g^{-1} T_f(r)}{\left[rL(r)\right]^{\lambda_g^{L^*}(f)}} .$$

In this paper, we wish to establish the relationship between the relative L-order (relative L^*-order), relative L-type (relative L^*-type) and relative L-weak type (relative L^*-weak type) of a transcendental meromorphic function f with respect to a transcendental entire function g and that of monomial generated by the transcendental meromorphic f and transcendental entire g. We use the standard notations and definitions in the theory of entire and meromorphic functions which are available in [4] and [7].

2. Lemmas

In this section, we present two lemmas which will be needed in the sequel.

Lemma 1 [3] *Let f be a transcendental meromorphic function of finite order or of non-zero lower order and $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;f) = 4$ and g be a transcendental entire function with regular growth and non zero finite order. Also let $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;g) = 4$. Then*

$$\lim_{r\to\infty} \frac{\log T_{P[g]}^{-1} T_{P[f]}(r)}{\log T_g^{-1} T_f(r)} = 1 .$$

Lemma 2 [3] *Let f be a transcendental meromorphic function of finite order or of non-zero lower order and* $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$ *and g be a transcendental entire function with regular growth and non zero finite type. Also let* $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;g)=4$. *Then*

$$\lim_{r\to\infty}\frac{T_{P[g]}^{-1}T_{P[f]}\left(r\right)}{T_g^{-1}T_f\left(r\right)}=\left(\frac{\Gamma_{P[f]}-(\Gamma_{P[f]}-\gamma_{P[f]})\Theta(\infty;f)}{\Gamma_{P[g]}-(\Gamma_{P[g]}-\gamma_{P[g]})\Theta(\infty;g)}\right)^{\frac{1}{\rho_g}}$$

where$\Theta(\infty;f)=1-\limsup\limits_{r\to\infty}\frac{\overline{N}_f(r)}{T_f(r)}$ *and* $\Theta(\infty;g)=1-\limsup\limits_{r\to\infty}\frac{\overline{N}_g(r)}{T_g(r)}$.

3. Theorems

In this section, we present the main results of the paper.

Theorem 1 *Let f be a transcendental meromorphic function of finite order or of non-zero lower order and* $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$ *and g be a transcendental entire function with regular growth and non zero finite order. Also let* $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;g)=4$. *Then the relative L-order and relative L-lower order order of $P[f]$ with respect to $P[g]$ are same as those of f with respect to g.*

Proof. By Lemma 1 we obtain that,

$$\begin{aligned}
\rho_{P[g]}^{L}\left(P[f]\right) &= \limsup_{r\to\infty}\frac{\log T_{P[g]}^{-1}T_{P[f]}\left(r\right)}{\log\left[rL\left(r\right)\right]}\\
&= \limsup_{r\to\infty}\left\{\frac{\log T_g^{-1}T_f\left(r\right)}{\log\left[rL\left(r\right)\right]}\cdot\frac{\log T_{P[g]}^{-1}T_{P[f]}\left(r\right)}{\log T_g^{-1}T_f\left(r\right)}\right\}\\
&= \limsup_{r\to\infty}\frac{\log T_g^{-1}T_f\left(r\right)}{\log\left[rL\left(r\right)\right]}\cdot\lim_{r\to\infty}\frac{\log T_{P[g]}^{-1}T_{P[f]}\left(r\right)}{\log T_g^{-1}T_f\left(r\right)}\\
&= \rho_g^L\left(f\right)\cdot 1\\
&= \rho_g^L\left(f\right)\ .
\end{aligned}$$

In a similar manner, $\lambda_{P[g]}^{L}\left(P[f]\right)=\lambda_g^L\left(f\right)$. This proves the theorem. ∎

Theorem 2 *Let f be a transcendental meromorphic function of finite order or of non-zero lower order and* $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$ *and g be a transcendental entire function with regular growth and non zero finite order. Also let* $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;g)=4$. *Then the relative L^*-order and relative L^*-lower order order of $P[f]$ with respect to $P[g]$ are same as those of f with respect to g.*

We omit the proof of Theorem 2 because it can be carried out in the line of Theorem 1.

Theorem 3 *Let f be a transcendental meromorphic function of finite order or of non-zero lower order and* $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$ *and g be a transcendental entire function of regular growth having non zero finite type and* $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;g)=4$. *Then the relative L-type and relative L-lower type of $P[f]$ with respect to $P[g]$ are* $\left(\frac{\Gamma_{P[f]}-(\Gamma_{P[f]}-\gamma_{P[f]})\Theta(\infty;f)}{\Gamma_{P[g]}-(\Gamma_{P[g]}-\gamma_{P[g]})\Theta(\infty;g)}\right)^{\frac{1}{\rho_g}}$ *times that of f with respect to g if $\rho_g^L(f)$ is positive finite.*

Proof. From Lemma 2 and Theorem 1 we get that

$$\begin{aligned}\sigma_{P[g]}^L(P[f]) &= \limsup_{r\to\infty}\frac{T_{P[g]}^{-1}T_{P[f]}(r)}{[rL(r)]^{\rho_{P[g]}(P[f])}}\\ &= \lim_{r\to\infty}\frac{T_{P[g]}^{-1}T_{P[f]}(r)}{T_g^{-1}T_f(r)}\cdot\limsup_{r\to\infty}\frac{T_g^{-1}T_f(r)}{[rL(r)]^{\rho_g^L(f)}}\\ &= \left(\frac{\Gamma_{P[f]}-(\Gamma_{P[f]}-\gamma_{P[f]})\Theta(\infty;f)}{\Gamma_{P[g]}-(\Gamma_{P[g]}-\gamma_{P[g]})\Theta(\infty;g)}\right)^{\frac{1}{\rho_g}}\cdot\sigma_g^L(f)\ .\end{aligned}$$

Similarly,

$$\overline{\sigma}_{P[g]}^L(P[f])=\left(\frac{\Gamma_{P[f]}-(\Gamma_{P[f]}-\gamma_{P[f]})\Theta(\infty;f)}{\Gamma_{P[g]}-(\Gamma_{P[g]}-\gamma_{P[g]})\Theta(\infty;g)}\right)^{\frac{1}{\rho_g}}\cdot\overline{\sigma}_g^L(f)\ .$$

Thus the theorem is established. ∎

Theorem 4 *Let f be a transcendental meromorphic function of finite order or of non-zero lower order and* $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$ *and g be a transcendental entire function of regular growth having non zero finite type and* $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;g)=4$. *Then the relative L^*-type and relative L^*-lower type of $P[f]$ with respect to $P[g]$ are* $\left(\frac{\Gamma_{P[f]}-(\Gamma_{P[f]}-\gamma_{P[f]})\Theta(\infty;f)}{\Gamma_{P[g]}-(\Gamma_{P[g]}-\gamma_{P[g]})\Theta(\infty;g)}\right)^{\frac{1}{\rho_g}}$ *times that of f with respect to g if $\rho_g^{L^*}(f)$ is positive finite.*

We omit the proof of Theorem 4 because it can be carried out in the line of Theorem 3.

Now, we state the following two theorems without proof because it can be carried out in the line of Theorem 3 and Theorem 4 respectively.

Theorem 5 *Let f be a transcendental meromorphic function of finite order or of non-zero lower order and* $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$ *and g be a transcendental entire function of regular growth having non zero finite type and* $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;g)=4$. *Then $\tau_{P[g]}^L(P[f])$ and $\overline{\tau}_{P[g]}^L(P[f])$ are* $\left(\frac{\Gamma_{P[f]}-(\Gamma_{P[f]}-\gamma_{P[f]})\Theta(\infty;f)}{\Gamma_{P[g]}-(\Gamma_{P[g]}-\gamma_{P[g]})\Theta(\infty;g)}\right)^{\frac{1}{\rho_g}}$ *times that of* *f with respect to g i.e.,* $\tau_{P[g]}^{L}(P[f]) = \left(\frac{\Gamma_{P[f]}-(\Gamma_{P[f]}-\gamma_{P[f]})\Theta(\infty;f)}{\Gamma_{P[g]}-(\Gamma_{P[g]}-\gamma_{P[g]})\Theta(\infty;g)}\right)^{\frac{1}{\rho_g}} \cdot \tau_{g}^{L}(f)$ *and* $\overline{\tau}_{P[g]}^{L}(P[f]) = \left(\frac{\Gamma_{P[f]}-(\Gamma_{P[f]}-\gamma_{P[f]})\Theta(\infty;f)}{\Gamma_{P[g]}-(\Gamma_{P[g]}-\gamma_{P[g]})\Theta(\infty;g)}\right)^{\frac{1}{\rho_g}} \cdot \overline{\tau}_{g}^{L}(f)$ *when* $\lambda_{g}^{L}(f)$ *is positive finite.*

Theorem 6 *Let f be a transcendental meromorphic function of finite order or of non-zero lower order and* $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$ *and g be a transcendental entire function of regular growth having non zero finite type and* $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;g)=4$. *Then* $\tau_{P[g]}^{L^*}(P[f])$ *and* $\overline{\tau}_{P[g]}^{L^*}(P[f])$ *are* $\left(\frac{\Gamma_{P[f]}-(\Gamma_{P[f]}-\gamma_{P[f]})\Theta(\infty;f)}{\Gamma_{P[g]}-(\Gamma_{P[g]}-\gamma_{P[g]})\Theta(\infty;g)}\right)^{\frac{1}{\rho_g}}$ *times that of f with respect to g i.e.,* $\tau_{P[g]}^{L^*}(P[f]) = \left(\frac{\Gamma_{P[f]}-(\Gamma_{P[f]}-\gamma_{P[f]})\Theta(\infty;f)}{\Gamma_{P[g]}-(\Gamma_{P[g]}-\gamma_{P[g]})\Theta(\infty;g)}\right)^{\frac{1}{\rho_g}} \cdot \tau_{g}^{L^*}(f)$ *and* $\overline{\tau}_{P[g]}^{L^*}(P[f]) = \left(\frac{\Gamma_{P[f]}-(\Gamma_{P[f]}-\gamma_{P[f]})\Theta(\infty;f)}{\Gamma_{P[g]}-(\Gamma_{P[g]}-\gamma_{P[g]})\Theta(\infty;g)}\right)^{\frac{1}{\rho_g}} \cdot \overline{\tau}_{g}^{L^*}(f)$ *when* $\lambda_{g}^{L^*}(f)$ *is positive finite.*

References

[1] Doeringer, W., *Exceptional values of differential polynomials,* Pacific J. Math., 98 (1) (1982), 55-62.

[2] Datta, S.K. and Biswas, A., *On relative type of entire and meromorphic functions*, Advances in Applied Mathematical Analysis, 8 (2) (2013), 63-75.

[3] Datta, S.K., Biswas, T. and Bhattacharyya, S., *On relative order and relative type based growth properties of differential monomials,* J. Indian Math. Soc. (N.S.), 82 (3-4) (2015), 39–52.

[4] Hayman, W.K., *Meromorphic Functions*, The Clarendon Press, Oxford, 1964.

[5] Lahiri, B.K. and Banerjee, D., *Relative order of entire and meromorphic functions*, Proc. Nat. Acad. Sci. India, 61(A) (III) (2004), 339-354.

[6] Somasundaram, D and Thamizharasi, R., *A note on the entire functions of L-bounded index and L-type*, Indian J. Pure Appl. Math., 19 (3) (1988), 284-293.

[7] Valiron, G., *Lectures on the general theory of integral functions*, Chelsea Publishing Company, 1949.

[8] Yang, L., *Value distribution theory and new research on it*, Science Press, Beijing, 1982.

[9] Yi, H.X., *On a result of Singh*, Bull. Austral. Math. Soc., 41 (1990), 417-420.

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A NOVEL STUDY OF SOFT SETS OVER n-ARY SEMIGROUPS

Qiumei Wang
Xiaowu Zhou
Jianming Zhan[1]

Department of Mathematics
Hubei University for Nationalities
Enshi, Hubei Province, 445000
P.R. China

Abstract. In this paper, we show that the regular n-ary semigroups can be described by using idealistic soft n-ary semigroups. The relationships between regular n-ary semigroups and soft regular n-ary semigroups are also discussed. Finally, we introduce quotient n-ary semigroups via soft congruence relations and establish some homomorphisms and related properties with respect to soft congruence relations.

Keyword: n-ary semigroups; soft n-ary semigroups; idealistic soft n-ary semigroups; soft congruence relations;soft homomorphisms.

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1. Introduction

The generalization of classical algebraic structures to n-ary structures was first initiated by Kasner [17] in 1904. In the following decades and nowadays, a number of different n-ary systems have been studied in depth in different contexts. Sioson [20] introduced regular n-ary semigroups and investigated their related properties. Since then, the nature of regular n-ary semigroups were discussed in detail by Dudek [11]. In [6], [7], [8], Dudek proved some results and presented many examples of n-ary groups. Earlier, Crombez et al. [2], [3] gave the generalized rings and named it as (m, n)-rings and introduced their quotient structure. Up till now, the theory of n-ary systems has many applications, for example, application in physics [21], [22] and in automata theory [15], fuzzy sets and rough set theory(see [4], [5], [9], [24]) and so on.

In dealing with uncertainties, many theories have been recently developed, including the theory of probability, theory of fuzzy sets, theory of intuitionistic fuzzy sets and theory of rough sets and so on. Although many new techniques have been developed as a result of theories, yet difficulties are still. The major difficulties posed by these theories are probably due to the inadequacy of parameters. In 1999, Molodtsov [19] initiated

[1]Corresponding author. E-mail: zhanjianming@hotmail.com

the concept of soft set theory, which was a completely new approach for modeling uncertainty and had a rich potential for applications in several directions. Later on, Maji et al. [18] introduced several operations in soft set theory and carried out a detailed theoretical study on soft sets. The algebraic structure of soft sets has been studied by several authors. For examples, Aktaş and Çağman [1] introduced the notion of soft groups and discussed their basic properties. Feng et al. [12] defined the notions of soft semirings, idealistic soft semirings, soft ideals and introduced the algebraic properties of semirings. Other applications of soft set theory in different algebraic structure can be found in [16], [23] and so on. Feng et al. [13] initiated the soft binary relations, some interesting properties of soft equivalence and soft congruence relations are discussed.

In this paper, we first recall some concepts and results on n-ary semigroups and soft sets. In Section 3, we define the notion of soft n-ary semigroups and idealistic soft n-ary semigroups over an n-ary semigroup. Some basic related properties with soft n-ary semigroups and idealistic soft n-ary semigroups are proposed. In Section 4, we show that the regular n-ary semigroups can be described by using idealistic soft n-ary semigroups. Moreover, we discuss relationships between regular n-ary semigroups and soft regular n-ary semigroups. In Section 5, we give the concept of soft congruence relations over an n-ary semigroup and introduce quotient n-ary semigroups via soft congruence relations. Some homomorphisms and related properties with respect to soft congruence relations are proposed.

2. Preliminaries

A non-empty set S together with one n-ary operation $f : S^n \to S$, where $n \geq 2$, is called an n-ary groupoid and is denoted by (S, f). According to the general convention used in the theory of n-ary groupoids, the sequence of elements $x_i, x_{i+1}, ..., x_j$ is denoted by x_i^j. In the case $j < i$, it is the empty symbol. If $x_{i+1} = x_{i+2} = \cdots = x_{i+t} = x$, then we write $\overset{(t)}{x}$ instead of x_{i+1}^{i+t}. In this convention,

$$f(x_1, x_2, ..., x_n) = f(x_1^n),$$

and

$$f(x_1, ..., x_i, \underbrace{x, ..., x}_{t}, x_{i+t+1}, ..., x_n) = f(x_1^i, \overset{(t)}{x}, x_{i+t+1}^n).$$

An n-ary groupoid (S, f) is called (i, j)-associative if

$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1})$$

hold for all $x_1, x_2, ..., x_{2n-1} \in S$. If this identity holds for all $1 \leq i \leq j \leq n$, then we say that the operation f is associative, and (S, f) is called an n-ary semigroup. An n-ary semigroup (S, f) is called idempotent if $f(x, ..., x) = x$ for all $x \in S$.

A non-empty subset H of an n-ary semigroup (S, f) is an n-ary subsemigroup if (H, f) is an n-ary subsemigroup, i.e., if it is closed under the operation f. Throughout this paper, unless otherwise mentioned, S will denote an n-ary semigroup.

Definition 2.1 [11], [20] A non-empty subset I of S is called an i-ideal of S if for every $x_1, ..., x_{i-1}, x_{i+1}, ..., x_n \in S$ with $a \in I$, then $f(x_1^{i-1}, a, x_{i+1}^n) \in I$. I is called an ideal of S if I is an i-ideal for every $1 \leq i \leq n$.

Definition 2.2 [4] Let R be an equivalence relation of S. R is called a congruence of S if $(x_i, y_i)\in R$ implies $(f(x_1^n), f(y_1^n))\in R$ for all $1 \leq i \leq n$ and $x_1, x_2, ..., x_n, y_1, y_2, ..., y_n \in S$.

Definition 2.3 [4] A mapping $\varphi : S \to T$ from S into T is called a homomorphism if $\varphi(f(x_1^n)) = g(\varphi(x_1), \varphi(x_2), ..., \varphi(x_n))$ for all $x_1, x_2, ..., x_n \in S$.

Definition 2.4 [19] A pair (F, A) is called a soft set over U, where $A \subseteq E$ and $F : A \to P(U)$ is a set-valued mapping.

For a soft set (F, A), the set $\mathrm{Supp}(F, A) = \{x \in A | F(x) \neq \emptyset\}$ is called a soft support of (F, A). Thus a null soft set is indeed a soft set with an empty support, and we say that a soft set (F, A) is non-null if $\mathrm{Supp}(F, A) \neq \emptyset$.

Definition 2.5 [19] A soft set (F, A) over S is called an absolute soft set if $F(a) = S$ for all $a \in A$.

Definition 2.6 [14] A soft set (F, A) over S is called a full soft set if $\bigcup\limits_{x\in A} F(x) = S$.

Definition 2.7 [12] Let (F, A) and (G, B) be two soft sets over a common universe U. The inclusion symbol "$\widetilde{\subseteq}$" of (F, A) and (G, B), denoted by $(F, A)\widetilde{\subseteq}(G, B)$, is defined as

(1) $A \subseteq B$;

(2) $F(x) \subseteq G(x)$ for all $x \in A$.

If $(F, A)\widetilde{\subseteq}(G, B)$ and $(G, B)\widetilde{\subseteq}(F, A)$, then we denote $(F, A) = (G, B)$.

3. Soft n-ary semigroups and idealistic soft n-ary semigroups

In this section, we define the notion of soft n-ary semigroups and idealistic soft n-ary semigroups over S. Some basic related properties with soft n-ary semigroups and idealistic soft n-ary semigroups are proposed.

Definition 3.1 Let $(F_1, A_1), (F_2, A_2), ..., (F_n, A_n)$ be soft sets over S. Then the $\widetilde{f}$-product of them, denoted by $\widetilde{f}((F_1, A_1), (F_2, A_2), ..., (F_n, A_n))$, is defined as a soft set $(G, B) = \widetilde{f}((F_1, A_1), (F_2, A_2), ..., (F_n, A_n))$, where $B = \bigcap\{A_i | i = 1, 2, ..., n\} \neq \emptyset$ and $G : B \to P(S)$ defined by $G(a) = f(F_1(a), F_2(a), ..., F_n(a))$ for all $a \in B$.

Definition 3.2 Let (F, A) be a non-null soft set over S. Then (F, A) is called a soft n-ary semigroup over S if $F(a)$ is an n-ary subsemigroup of S for all $a \in \mathrm{Supp}(F, A)$.

Example 3.3 Let $S = \{-i, 0, i\}$ be a set with a ternary operation f as the usual multiplication of complex numbers. Then (S, f) is a ternary semigroup. Let (F, A) be a soft set over S, where $A = \{a, b, c\}$ and $F : A \to P(S)$ be a set-valued function defined by $F(x) = \{y \in S | (x, y) \in R\}$ for all $x \in A$, where $R = \{(a, 0), (c, -i), (c, 0), (c, i)\}$. Then $F(a) = \{0\}, F(b) = \emptyset, F(c) = \{-i, 0, i\}$. Therefore (F, A) is a soft ternary semigroup over S.

Proposition 3.4 *A non-null soft set (F, A) over S is a soft n-ary semigroup if and only if $\widetilde{f}((F, A), ..., (F, A))\widetilde{\subseteq}(F, A)$.*

Proof. Let (F, A) be a soft n-ary semigroup over S, then for all $a \in \mathrm{Supp}(F, A)$, $F(a)$ is an n-ary subsemigroup of S. By Definition 3.1, we denote $\widetilde{f}((F, A), ..., (F, A)) = (G, A)$, where $G : A \to P(S)$ defined by $G(a) = f(F(a), ..., F(a))$ for all $a \in \mathrm{Supp}(F, A)$. $F(a)$ is an n-ary subsemigroup of S, it follows that $f(F(a), ..., F(a)) \subseteq F(a)$, that is $G(a) \subseteq F(a)$ for all $a \in A$. Hence $\widetilde{f}((F, A), \cdots, (F, A)) \widetilde{\subseteq} (F, A)$.

Conversely, if $\widetilde{f}((F, A), ..., (F, A)) \widetilde{\subseteq} (F, A)$, it follows that $f(F(a), ..., F(a)) \subseteq F(a)$ for all $a \in \mathrm{Supp}(F, A)$. This means $F(a)$ is an n-ary subsemigroup of S. By Definition 3.2, (F, A) is a soft n-ary semigroup over S. ∎

Definition 3.5 Let (F, A) be a non-null soft set over S. Then (F, A) is called a j-idealistic soft n-ary semigroup over S, if $F(x)$ is a j-ideal of S for all $x \in \mathrm{Supp}(F, A)$. Moreover, if (F, A) is a j-idealistic soft n-ary semigroup of S for each $j = 1, 2, ..., n$, then (F, A) is called an idealistic soft n-ary semigroup.

Example 3.6 Consider the natural numbers $\mathbb{N}$ with usual multiplication. Let $S = 2\mathbb{N}$. We define the 4-ary operation f, $f(a, b, c, d) = \frac{abcd}{2}$ for all $a, b, c, d \in S$. Then (S, f) is a 4-ary semigroup. Let (F, A) be a soft set over S, where $A = \mathbb{N}$ and $F : A \to P(S)$ is a set-valued function defined by $F(x) = \{4xn | n \in \mathbb{N}\}$ for all $x \in \mathbb{N}$. Since for all $x_1, x_2, x_3 \in S$ and $r \in F(x)$, $f(x_1^{i-1}, r, x_{i+1}^4) \in F(x)$ for $i = 1, 2, 3, 4$. Hence $F(x)$ is an ideal of (S, f). Therefore (F, A) is an idealistic soft 4-ary semigroup of S.

Remark 3.7 If an idealistic soft n-ary semigroup (F, A) satisfying $\widetilde{f}((F, A), ..., (F, A)) = (F, A)$, then (F, A) is called a soft idempotent idealistic soft n-ary semigroup.

Proposition 3.8 *Let (H, E) be an absolute soft set over S. Then, a non-null soft set (F, A) over S is a j-idealistic soft n-ary semigroup if and only if $\widetilde{f}(\overset{(j-1)}{(H, E)}, (F, A), \overset{(n-j)}{(H, E)}) \widetilde{\subseteq} (F, A)$, where $j = 1, 2, ..., n$.*

Proof. Let (F, A) be a j-idealistic soft n-ary semigroup, then $F(a)$ is a j-ideal of S for all $a \in \mathrm{Supp}(F, A)$, it follows that $f(x_1^{k-1}, r, x_{k+1}^n) \in F(a)$ for all $x_1, x_2, ..., x_n \in S$ and all $r \in F(a)$. By Definition 3.1, we denote $\widetilde{f}(\overset{(k-1)}{(H, E)}, (F, A), \overset{(n-k)}{(H, E)}) = (G, A)$, where $G : A \to P(S)$ defined by $G(a) = f(\overset{j-1}{H(a)}, F(a), \overset{n-j}{H(a)})$ for all $a \in \mathrm{Supp}(F, A)$. Since (H, E) is an absolute soft set over S, so $H(a) = S$ for all $a \in \mathrm{Supp}(F, A)$. Hence $G(a) = f(\overset{j-1}{S}, F(a), \overset{n-j}{S}) \subseteq F(a)$. Therefore, $\widetilde{f}(\overset{(j-1)}{(H, E)}, (F, A), \overset{(n-j)}{(H, E)}) \widetilde{\subseteq} (F, A)$.

Conversely, if $\widetilde{f}(\overset{(j-1)}{(H, E)}, (F, A), \overset{(n-j)}{(H, E)}) \widetilde{\subseteq} (F, A)$, then $f(\overset{j-1}{H(a)}, F(a), \overset{n-j}{H(a)}) \subseteq F(a)$ for all $a \in \mathrm{Supp}(F, A)$. Since (H, E) is an absolute soft set over S, so $H(a) = S$ for all $a \in \mathrm{Supp}(F, A)$. Hence $f(\overset{j-1}{S}, F(a), \overset{n-j}{S}) \subseteq F(a)$. This means $F(a)$ is a j-ideal of S. By Definition 3.5, (F, A) is a j-idealistic soft n-ary semigroup over S. ∎

Proposition 3.9 *Let (H, E) be an absolute soft set over S. Then, a non-null soft set (F, A) over S is an idealistic soft n-ary semigroup if (F, A) is a j-idealistic soft n-ary semigroup for all $1 \leq j \leq n$.*

Proof. It is straightforward. ∎

4. The characterizations of regular n-ary semigroups

In this section, we show that the regular n-ary semigroups can be described by using idealistic soft n-ary semigroups. Moreover, we discuss relationships between regular n-ary semigroups and soft regular n-ary semigroups.

Definition 4.1 [11] An element $a \in S$ is called regular if there exist $x_2, x_3, ..., x_{n-1} \in S$ such that $f(a, x_2^{n-1}, a) = a$. S is called regular if every element of S is regular.

Definition 4.2 [20] S is called regular if for all $a \in S$, there exist $x_{ij} \in S$ $(i, j = 1, 2, ..., n)$ such that

$$a = f(f(a, x_{12}^{1n}), f(x_{21}, a, x_{23}^{2n}), ..., f(x_{n1}^{nn-1}, a)).$$

Remark 4.3 In [11], Dudek proved that Definitions 4.1 and 4.2 are equivalent.

Example 4.4 Let $S = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$, where the ternary operation f is the usual matrix multiplication. One can easily show that (S, f) is a regular ternary semigroup.

Lemma 4.5 [20] *The following conditions are equivalent:*

(1) *S is regular,*

(2) $\bigcap_{i=1}^{n} B_i = f(B_1, B_2, ..., B_n)$ *for all i-deals B_i,*

(3) *every ideal is idempotent.*

Theorem 4.6 *S is regular if and only if*

$$(F_1, A_1) \Cap (F_2, A_2) ... \Cap (F_n, A_n) = \widetilde{f}((F_1, A_1), (F_2, A_2), ..., (F_n, A_n))$$

for every j-idealistic soft n-ary semigroup $(F_j, A_j) (j = 1, 2, ..., n)$.

Proof. Let S be a regular n-ary semigroup and (F_j, A_j) be an j-idealistic soft n-ary semigroup over S, respectively. Then $(G, B) = \widetilde{f}((F_1, A_1), (F_2, A_2), ..., (F_n, A_n))$, where $B = \bigcap_{j=1}^{n} A_j$ and G is defined as $G(a) = f(F_1(a), F_2(a), ..., F_n(a))$ for all $a \in B$. Also, $(K, C) = (F_1, A_1) \Cap (F_2, A_2) ... \Cap (F_n, A_n)$, where $C = \bigcap_{j=1}^{n} A_j$ and K is defined as $K(a) = F_1(a) \cap F_2(a) ... \cap F_n(a)$ for all $a \in C$. By Proposition 3.8, we have

$$\widetilde{f}((F_1, A_1), (F_2, A_2), ..., (F_n, A_n)) \widetilde{\subseteq} \widetilde{f}((F_1, A_1), (H, E), ..., (H, E)) \widetilde{\subseteq} (F_1, A_1),$$

$$\widetilde{f}((F_1, A_1), (F_2, A_2), ..., (F_n, A_n)) \widetilde{\subseteq} \widetilde{f}((H, E), (F_2, A_2), ..., (H, E)) \widetilde{\subseteq} (F_2, A_2),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\widetilde{f}((F_1, A_1), (F_2, A_2), ..., (F_n, A_n)) \widetilde{\subseteq} \widetilde{f}((H, E), (H, E), ..., (F_n, A_n)) \widetilde{\subseteq} (F_n, A_n),$$

where (H, E) is an absolute soft set over S.

Thus, $\widetilde{f}((F_1, A_1), (F_2, A_2), ..., (F_n, A_n)) \widetilde{\subseteq} (F_1, A_1) \Cap (F_2, A_2) ... \Cap (F_n, A_n)$.

Let $b \in F_1(a) \cap F_2(a) ... \cap F_n(a)$, then $b \in F_j(a)$. Since S is regular, there exist $x_{ij} \in S(i, j = 1, 2, ..., n)$ such that $b = f(f(b, x_{12}^{1n}), f(x_{21}, b, x_{23}^{2n}), ..., f(x_{n1}^{nn-1}, b))$. By (F_j, A_j) is an j-idealistic soft n-ary semigroup over S, we have $F_j(a)$ is an j-ideal of S, so $f(b, x_{12}^{1n}) \in F_1(a), f(x_{21}, b, x_{23}^{2n}) \in F_2(a), ..., f(x_{n1}^{nn-1}, b) \in F_n(a)$. Hence $b \in f(F_1(a), F_2(a), ..., F_n(a))$. This means $F_1(a) \cap F_2(a) ... \cap F_n(a) \subseteq f(F_1(a), F_2(a), ..., F_n(a))$, for all $a \in \bigcap_{j=1}^{n} A_j$. It implies $(F_1, A_1) \Cap (F_2, A_2) ... \Cap (F_n, A_n) \widetilde{\subseteq} \widetilde{f}((F_1, A_1), (F_2, A_2), ..., (F_n, A_n))$.

Hence $(F_1, A_1) \Cap (F_2, A_2) ... \Cap (F_n, A_n) = \widetilde{f}((F_1, A_1), (F_2, A_2), ..., (F_n, A_n))$ for every j-idealistic soft n-ary semigroup$(F_j, A_j)(j = 1, 2, ..., n)$.

Conversely, suppose that $A_1 = A_2 = \cdots = A_n = S$ and F_j is a function from A_j to $P(S)$. For all $a \in S$, we define $F_1(a) = \{a\} \cup f(a, S, ..., S)$, $F_2(a) = \{a\} \cup f(S, a, S, ..., S)$, ..., $F_n(a) = \{a\} \cup f(S, S, ..., S, a)$. Since

$$\begin{array}{rcl} f(F_1(a), S, ..., S) & = & f(\{a\} \cup f(a, S, ..., S), S, ..., S) \\ & \subseteq & f(a, S, ..., S) \cup f(f(a, S, ..., S), S ..., S) \\ & = & f(a, S, ..., S) \cup f(a, S, ..., S, f(S, ..., S)) \\ & \subseteq & f(a, S, ..., S) \cup f(a, S, ..., S) \\ & = & f(a, S, ..., S). \end{array}$$

Hence $a \in F_1(a) \cap F_2(a) ... \cap F_n(a) = f(F_1(a), F_2(a), ..., F_n(a)) \subseteq f(F_1(a), S, ..., S) \subseteq f(a, S, ..., S)$. In similar way, we have $a \in f(S, a, S, ..., S), ..., a \in f(S, ..., S, a)$. So

$$\begin{aligned} & f(f(a, S, ..., S), f(S, a, S, ..., S), ..., f(S, ..., S, a)) \\ & \quad = f(a, f(S, ..., S), f(a, S, ..., S), ..., f(S, ..., S), a) \\ & \quad \subseteq f(a, S, ..., S, a). \end{aligned}$$

Therefore,

$$\begin{aligned} & f(F_1(a), F_2(a), ..., F_n(a)) \\ & \quad = f(\{a\} \cup f(a, S, ..., S), ..., \{a\} \cup f(S, S, ..., S, a)) \\ & \quad = f(a, ..., a), ..., \cup f(f(a, S, ..., S), f(S, a, S, ..., S), ..., f(S, ..., S, a)) \\ & \quad = f(f(a, S, ..., S), f(S, a, S, ..., S), ..., f(S, ..., S, a)) \\ & \quad \subseteq f(a, S, ..., S, a). \end{aligned}$$

This means $a \in f(a, S, ..., S, a)$. Hence S is regular. ∎

Theorem 4.7 *S is regular if and only if every idealistic soft n-ary semigroup over S is soft idempotent.*

Proof. Let S be a regular n-ary semigroup and (F, A) an idealistic soft n-ary semigroup. Putting $(F, A) = (F_1, A_1) = (F_2, A_2) = ... = (F_n, A_n)$, then by Theorem 4.1, $(F, A) = (F, A) \Cap (F, A) ... \Cap (F, A) = \widetilde{f}((F, A), ..., (F, A))$. Thus (F, A) is soft idempotent.

Conversely, if all (F_j, A_j) are j-idealistic soft n-ary semigroups over S, where $j = 1, 2, ..., n$. Then $(F_1, A_1) \Cap (F_2, A_2) ... \Cap (F_n, A_n)$ is an idealistic soft n-ary semigroup over S and $(F_1, A_1) \Cap (F_2, A_2) ... \Cap (F_n, A_n) \widetilde{\subseteq} (F_j, A_j)$ for each $j = 1, 2, \cdots, n$. This implies that

$$\begin{aligned} & (F_1, A_1) \Cap (F_2, A_2) ... \Cap (F_n, A_n) \\ & \quad = \widetilde{f}((F_1, A_1) \Cap (F_2, A_2) ... \Cap (F_n, A_n), ..., (F_1, A_1) \Cap (F_2, A_2) ... \Cap (F_n, A_n)) \\ & \quad \widetilde{\subseteq} \widetilde{f}((F_1, A_1), (F_2, A_2), ..., (F_n, A_n)). \end{aligned}$$

But $\widetilde{f}((F_1, A_1), (F_2, A_2), ..., (F_n, A_n))\widetilde{\subseteq}(F_1, A_1)\Cap(F_2, A_2)...\Cap(F_n, A_n)$ always holds. Thus, $(F_1, A_1)\Cap(F_2, A_2)...\Cap(F_n, A_n) = \widetilde{f}((F_1, A_1), (F_2, A_2), ..., (F_n, A_n))$. Hence, by Theorem 4.6, S is regular. ■

By Theorems 4.6 and 4.7, we have the following Corollary.

Corollary 4.8 *Then the following conditions are equivalent:*

(1) *S is regular,*

(2) $(F_1, A_1)\Cap(F_2, A_2)...\Cap(F_n, A_n) = \widetilde{f}((F_1, A_1), (F_2, A_2), ..., (F_n, A_n))$ *for every j-idealistic soft n-ary semigroup $(F_j, A_j)(j = 1, 2, ..., n)$,*

(3) *every idealistic soft n-ary semigroup is soft idempotent*

Definition 4.9 A soft n-ary semigroup (F, A) over S is called a soft regular n-ary semigroup if for each $a \in A, F(a)$ is regular.

The following examples show that if S is a regular n-ary semigroup then soft n-ary semigroup (F, A) over S may not be soft regular and if the soft n-ary semigroup (F, A) over S is soft regular then S may not be regular.

Example 4.10 Consider the regular ternary semigroup in Example 4.4. Let (F, A) be a soft set over S, where $A = S$ and $F : A \to P(S)$ be a set-valued function defined by $F(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}) = S, F(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}) = F(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}) = \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right\}$, $F(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) = F(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}) = \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$. Then (F, A) is soft ternary semigroup over S. But it is not soft regular, because $F(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) = F(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}) = \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$ is not regular ternary subsemigroup of S.

Example 4.11 Let (S, f) be a 4-ary semigroup derived from the semigroup $(S, \cdot)$, where the 4-ary operation defined $f(d, d, d, d) = d$ and $f(x, y, z, u) = a$ where $x, y, z, u \in \{a, b, c\}$. Clearly, (S, f) is not a regular 4-ary semigroup. Let $A = \{\alpha, \beta\}$ be a set of parameters such that $F(\alpha) = \{a\}, F(\beta) = \{a, d\}$. Then (F, A) is a soft regular 4-ary semigroup over (S, f) because $F(\alpha)$ and $F(\beta)$ are regular 4-ary subsemigroups of (S, f).

Remark 4.12 In the above examples, we have demonstrated that the regularity of a n-ary semigroup S does not imply the regularity of a soft n-ary semigroup over S. Also, the regularity of a soft n-ary semigroup over a given n-ary semigroup S does not imply the regularity of the n-ary semigroup. However, we still have the following proposition.

Proposition 4.13 *Let (F, A) be a full soft regular n-ary semigroup over S. Then S is a regular n-ary semigroup.*

Proof. Let (F, A) be a soft regular n-ary semigroup over S. Then $F(\alpha)$ is regular for each $\alpha \in A$. Now let $a \in S$, because (F, A) is a full soft set, we have $S = \bigcup_{\alpha\in A} F(\alpha)$, then there exists $\beta \in A$ such that $a \in F(\beta)$. Since $F(\beta)$ is regular, then exist $x_2, x_3, ..., x_{n-1} \in F(\beta)$ such that $f(a, x_2^{n-1}, a) = a$. Since $x_2, x_3, ..., x_{n-1} \in F(\beta) \subseteq \bigcup_{\alpha\in A} F(\alpha) = S$, hence S is regular. ■

Proposition 4.14 *If S is an idempotent n-ary semigroup, then every soft n-ary semigroup (F, A) over S is soft regular.*

Proof. It is straightforward. ∎

5. Soft congruence relations over n-ary semigroups and homomorphisms

In this section, we give the concept of soft congruence relations over S and introduce quotient n-ary semigroups via soft congruence relations. Some homomorphisms and related properties with respect to soft congruence relations are proposed.

Definition 5.1 A non-null soft set (ρ, A) over $S \times S$ is called a soft congruence relation over S if $\rho(\alpha)$ is a congruence relation on S for all $\alpha \in \mathrm{Supp}(\rho, A)$.

If $\mathrm{Supp}(\rho, A) = \emptyset$, then (ρ, A) is called a null soft congruence relation over S, denoted $\emptyset_A^2$.

Example 5.2 Let (S, f) be an n-ary semigroup in Example 3.6 and $A = \mathbb{N}^+$. Consider the set-valued function $\rho : A \to P(S\times S)$ given by $\rho(\alpha) = \{(x, y) \in S\times S | x \equiv y(mod\, \alpha)\}$ for all $\alpha \in A$. Then $\rho(\alpha)$ is a congruence relation on (S, f). Hence (ρ, A) is a soft congruence relation on (S, f).

Theorem 5.3 *Let (ρ, A) be a soft congruence relation over an n-ary semigroup (S, f) and $S/(\rho, A) = \{[x]_{\rho(\alpha)} | x \in S\}$ where $[x]_{\rho(\alpha)} = \{y \in S | (x, y) \in \rho(\alpha), \alpha \in A\}$. Then for any $\alpha \in A$, $S/(\rho, A)$ is an n-ary semigroup under the n-ary operation defined by*

$$F([x_1]_{\rho(\alpha)}, [x_2]_{\rho(\alpha)}, ..., [x_n]_{\rho(\alpha)}) = [f(x_1^n)]_{\rho(\alpha)}$$

for all $x_1, x_2, ..., x_n \in S$.

Proof. We shall first show that F is well defined. Let $x_1, x_2, ..., x_n, y_1, y_2, ..., y_n \in S$ be such that

$$[x_1]_{\rho(\alpha)} = [y_1]_{\rho(\alpha)}, [x_2]_{\rho(\alpha)} = [y_2]_{\rho(\alpha)}, ..., [x_n]_{\rho(\alpha)} = [y_n]_{\rho(\alpha)}$$

for all $\alpha \in A$. It follows that $(x_1, y_1) \in \rho(\alpha), (x_2, y_2) \in \rho(\alpha), ..., (x_n, y_n) \in \rho(\alpha)$. Since (ρ, A) is a soft congruence relation over (S, f), by Definition 4.1, for any $\alpha \in A$, $\rho(\alpha)$ is a congruence relation on (S, f). Hence we have $(f(x_1^n), f(y_1^n)) \in \rho(\alpha)$. This means $[f(x_1^n)]_{\rho(\alpha)} = [f(y_1^n)]_{\rho(\alpha)}$. Hence F is well defined. $S/(\rho, A)$ is closed under the operation F and F is (i, j)-associative is obvious. Therefore $(S/(\rho, A), F)$ is an n-ary semigroup for all $\alpha \in A$. ∎

Theorem 5.4 *Let (ρ, A) and (σ, B) be two soft congruence relations over an n-ary semigroup S with $(\rho, A)\widetilde{\subseteq}(\sigma, B)$. Then the soft binary relation $(\sigma, B)/(\rho, A)$ over $S/(\rho, A)$, defined by $(\delta, C) = (\sigma, B)/(\rho, A)$ where $C = A \cap B$ and*

$$\delta(\alpha) = \sigma(\alpha)/\rho(\alpha) = \{([x]_{\rho(\alpha)}, [y]_{\rho(\alpha)}) \in S/(\rho, A) \times S/(\rho, A), (x, y) \in \sigma(\alpha)\}$$

for all $\alpha \in C$, is a soft congruence relation over $S/(\rho, A)$ and

$$(S/(\rho, A))/((\sigma, B)/(\rho, A)) \cong S/(\sigma, B).$$

Proof. Since $C = A \cap B = A$, then for any $\alpha \in C$, $\rho(\alpha)$ and $\sigma(\alpha)$ are congruence relations on S, so $\delta(\alpha)$ is an equivalence relation on $S/(\rho, A)$.

Let $a_1, a_2, ..., a_n, b_1, b_2, ..., b_n \in S$, if $(a_1, b_1) \in \sigma(\alpha), ..., (a_n, b_n) \in \sigma(\alpha)$, then we have $(f(a_1^n), f(b_1^n)) \in \sigma(\alpha)$ and $([a_1]_{\rho(\alpha)}, [b_1]_{\rho(\alpha)}) \in \delta(\alpha), ..., ([a_n]_{\rho(\alpha)}, [b_n]_{\rho(\alpha)}) \in \delta(\alpha)$. According to Theorem 5.3,

$$(F([a_1]_{\rho(\alpha)}, ..., [a_n]_{\rho(\alpha)}), F([b_1]_{\rho(\alpha)}, ..., [b_n]_{\rho(\alpha)}) = ([f(a_1^n)]_{\rho(\alpha)}, [f(b_1^n)]_{\rho(\alpha)}) \in \delta(\alpha).$$

This means $\delta(\alpha)$ is a congruence relation on $S/(\rho, A)$ for all $\alpha \in C$. Hence $(\sigma, B)/(\rho, A)$ is a soft congruence relation over $S/(\rho, A)$.

From Theorem 5.3, we know $(S/(\rho, A))/((\sigma, B)/(\rho, A))$ and $S/(\sigma, B)$ are two n-ary semigroups. Define a mapping:

$$h : (S/(\rho, A))/((\sigma, B)/(\rho, A)) \to S/(\sigma, B)$$

by $h([[x]_{\rho(\alpha)}]_{\delta(\alpha)}) = [x]_{\sigma(\alpha)}$ for all $x \in S$ and $\alpha \in A$. If $[[x]_{\rho(\alpha)}]_{\delta(\alpha)} = [[y]_{\rho(\alpha)}]_{\delta(\alpha)}$, then $([x]_{\rho(\alpha)}, [x]_{\rho(\alpha)}) \in \delta(\alpha)$, it follows that $(x, y) \in \sigma(\alpha)$, this means $[x]_{\sigma(\alpha)} = [y]_{\sigma(\alpha)}$. Hence h is well defined.

Let F^* be an n-ary operation of $(S/(\rho, A))/((\sigma, B)/(\rho, A))$. Then we have

$$\begin{aligned}
h(F^*([[x_1]_{\rho(\alpha)}]_{\delta(\alpha)}, ..., [[x_n]_{\rho(\alpha)}]_{\delta(\alpha)}) &= h([F([x_1]_{\rho(\alpha)}, ..., [x_n]_{\rho(\alpha)})]_{\delta(\alpha)}) \\
&= h([[f(x_1^n)]_{\rho(\alpha)}]_{\delta(\alpha)}) \\
&= [f(x_1^n)]_{\sigma(\alpha)} \\
&= F([x_1]_{\sigma(\alpha)}, ..., [x_n]_{\sigma(\alpha)}) \\
&= F(h([[x_1]_{\rho(\alpha)}]_{\delta(\alpha)}), ..., h([[x_n]_{\rho(\alpha)}]_{\delta(\alpha)})).
\end{aligned}$$

This means h is a homomorphism. If $[x]_{\sigma(\alpha)} = [y]_{\sigma(\alpha)}$, then $(x, y) \in \sigma(\alpha)$, so $([x]_{\rho(\alpha)}, [y]_{\rho(\alpha)}) \in \delta(\alpha)$. It follows that $[[x]_{\rho(\alpha)}]_{\delta(\alpha)} = [[x]_{\rho(\alpha)}]_{\delta(\alpha)}$, and h is injective.

Furthermore, for any $[y]_{\sigma(\alpha)} \in S/(\sigma, B)$, there exists $\rho(\alpha) = \sigma(\alpha)$ such that $h([[y]_{\sigma(\alpha)}]_{\delta(\alpha)}) = h([[y]_{\rho(\alpha)}]_{\delta(\alpha)}) = [y]_{\sigma(\alpha)}$ for all $\alpha \in A$. Hence h is surjective. This completes the proof. ∎

Lemma 5.5 *Let $\varphi : (S_1, f) \to (S_2, g)$ be an n-ary semigroup epimorphism.*

(1) *If γ is a congruence relation on S_1, define $\varphi(\gamma)=\{(\varphi(x), \varphi(y)) \in S_2 \times S_2 | (x, y) \in \gamma\}$, then $\varphi(\gamma)$ is a congruence relation on S_2.*

(2) *If θ is a congruence relation on S_2 such that $\varphi^{-1}(\theta) \neq \emptyset$, where $\varphi^{-1}(\theta) = \{(x, y) \in S_1 \times S_1 | (\varphi(x), \varphi(y)) \in \theta\}$, then $\varphi^{-1}(\theta)$ is a congruence relation on S_1.*

Proposition 5.6 *Let $\varphi : (S_1, f) \to (S_2, g)$ be an n-ary semigroup epimorphism.*

(1) *If (ρ, A) is a soft congruence relation over S_1, the image $\varphi(\rho, A)$ of (ρ, A) is denoted by $(\varphi(\rho), A)$, then $(\varphi(\rho), A)$ is a soft congruence relation over S_2, where*

$$\varphi(\rho)(\alpha) = \{(\varphi(x), \varphi(y)) \in S_2 \times S_2 | (x, y) \in \rho(\alpha)\}$$

for all $\alpha \in A$.

(2) *If (σ, B) is a soft congruence relation over S_2 such that $\varphi^{-1}(\sigma, B) \neq \emptyset_B^2$, where $\varphi^{-1}(\sigma, B)$ is the inverse image of (σ, B) is denoted by $(\varphi^{-1}(\sigma), B)$ and*

$$\varphi^{-1}(\sigma)(\beta) = \{(x, y) \in S_1 \times S_1 | (\varphi(x), \varphi(y)) \in \sigma(\beta)\}$$

for all $\beta \in B$. Then $\varphi^{-1}(\sigma, B)$ is a soft congruence relation over S_1.

Theorem 5.7 *Let $\varphi : (S_1, f) \to (S_2, g)$ be an n-ary semigroup epimorphism. If (ρ, A) is a soft congruence relation over S_1, then $S_1/(\rho, A) \cong S_2/\varphi(\rho, A)$.*

Proof. By Theorem 5.3 and Proposition 5.6, $S_1/(\rho, A)$ and $S_2/\varphi(\rho, A)$ are n-ary semigroups. Define a mapping

$$\psi : S_1/(\rho, A) \to S_2/\varphi(\rho, A) \text{ by } \psi([x]_{\rho(\alpha)}) = [\varphi(x)]_{\varphi(\rho)(\alpha)}$$

for all $x \in S_1$ and $\alpha \in A$.

We first show that ψ is well defined. In fact, let $x, x' \in S_1$, if $[x]_{\rho(\alpha)} = [x']_{\rho(\alpha)}$, then $(x.x') \in \rho(\alpha), \alpha \in A$. From Proposition 5.6, we have $(\varphi(x), \varphi(x')) \in \varphi(\rho)(\alpha)$ for all $\alpha \in A$, which implies that $[\varphi(x)]_{\varphi(\rho)(\alpha)} = [\varphi(x')]_{\varphi(\rho)(\alpha)}$. Hence ψ is well defined.

Moreover, ψ is a homomorphism. Let $x_1, x_2, ..., x_n \in S_1, \alpha \in A$, F and F' be two n-ary operations of $S_1/(\rho, A)$ and $S_2/\varphi(\rho, A)$, respectively. Then we have

$$\begin{aligned} \psi(F([x_1]_{\rho(\alpha)}, ..., [x_n]_{\rho(\alpha)}) &= \psi([f(x_1^n)]_{\rho(\alpha)}) \\ &= \varphi(f(x_1^n))_{\varphi(\rho)(\alpha)} \\ &= [g(\varphi(x_1), \varphi(x_2), ..., \varphi(x_n))]_{\varphi(\rho)(\alpha)} \\ &= F'([\varphi(x_1)]_{\varphi(\rho)(\alpha)}, [\varphi(x_2)]_{\varphi(\rho)(\alpha)}, ..., [\varphi(x_n)]_{\varphi(\rho)(\alpha)}). \end{aligned}$$

Hence ψ is a homomorphism.

For any $x, x' \in S_1, \alpha \in A$, if $[\varphi(x)]_{\varphi(\rho)(\alpha)} = [\varphi(x')]_{\varphi(\rho)(\alpha)}$, then $(\varphi(x), \varphi(x')) \in \varphi(\rho)(\alpha)$. By Proposition 5.1, we have $(x, x') \in \rho(\alpha)$, which implies $[x]_{\rho(\alpha)} = [y]_{\rho(\alpha)}$. This means ψ is injective.

Since φ is surjective, then for any $[y]_{\varphi(\rho)(\alpha)} \in S_2/\varphi(\rho, A), y \in S_2, \alpha \in A$, there exists $x_1 \in S_1$ such that $\varphi(x) = y$. So $\psi([x]_{\rho(\alpha)}) = [\varphi(x)]_{\varphi(\rho)(\alpha)} = [y]_{\varphi(\rho)(\alpha)}$, this means that ψ is surjective. Hence ψ is a isomorphism and $S_1/(\rho, A) \cong S_2/\varphi(\rho, A)$. ∎

Theorem 5.8 *Let $\varphi : (S_1, f) \to (S_2, g)$ be an n-ary semigroup epimorphism. If (σ, B) is a soft congruence relation over S_2 and $\varphi^{-1}(\sigma, B) \neq \emptyset_B^2$, then*

$$S_1/\varphi^{-1}(\sigma, B) \cong S_2/(\sigma, B).$$

Proof. It is similar to the proof of Theorem 5.7 and we omit it. ∎

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References

[1] Aktaş, H., Çağman, N., *Soft sets and soft groups*, Inform. Sci., 13 (2007), 2726-2735.

[2] Crombez, G., *0n (n, m)-rings*, Abhand. Math. Sem. Univ. Hamburg, 37(1972) 180-199.

[3] Crombez, G., Timm, J., *On (m, n)-quotient rings*, Abh. Math. Semin. Univ. Hambg., 37 (1972) 200-203.

[4] Davvaz, B., *Approximation in n-ary algebraic systems*, Soft Comput., 12 (2008), 409-418.

[5] Davvaz, B., Dudek, W.A., *Fuzzy n-ary groups as a generalization of Rosenfeld's fuzzy groups*, J. Mult.-Valued Logic Soft Comput., 15 (2009), 451-469.

[6] Dudek, W.A., *Remarks on n-groups*, Demonstratio Math., 13 (1980), 165-181.

[7] Dudek, W.A., *Autodistributive n-groups*, Comment. Math. Ann. Soc. Math. Polon. Prace Mat., 23 (1983), 1-11.

[8] Dudek, W.A., *On (i, j)-associative n-groupoids with the non-empty center*, Ricer. Mat. (Napoli), 35 (1986), 105-111.

[9] Dudek, W.A., *Fuzzification of n-ary groupoids*, Quasigroups Related Systems, 7 (2000), 45-66.

[10] Dudek, W.A., *Idempotents in n-ary semigroups*, Southeast Asian Bull. Math., 25 (2001), 97-104.

[11] Dudek, W.A., Grozdinska, I., *On ideals in regular n-semigroups*, Mat. Bilten Skopje, 4 (1980), 25-44.

[12] Feng, F., Jun, Y.B., Zhao, X.Z., *Soft semirings*, Comput. Math. Appl., 10 (2008), 2621-2628.

[13] Feng, M.I. Ali, M. Shabir, M., *Soft relations applied to semigroups*, Filomat, 7 (2013), 1183-1196.

[14] Feng, F., Li, C.X., Davvaz,B., Ali, M.I., *Soft sets combined with fuzzy sets and rough sets: a tentative approach*, Soft Comput., 14 (2010), 899-911.

[15] Grzymala-Busse, J.W., *Automorphisms of polyadic automata*, J. Assoc. Comput. Mach., 16 (1969), 208-219.

[16] Hila, K., Vougiouklis, T., Naka, K., *On the structure of soft m-ary semihypergroups*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys., 76 (2014), 91-106.

[17] Kasner, R., *An extension of the group concepts*, Bull. Amer. Math. Soc., 10 (1904), 290-291.

[18] Maji, P.K., Biswas, R., Roy, A.R., *Soft set theory*, Comput. Math. Appl., 45 (2003), 555-562.

[19] Molodstov, D., *Soft set theory-first results*, Comput. Math. Appl., 37 (1999), 19-31.

[20] Sioson, F.M., *On regular algebraic systems*, Proc. Japan Acad., 39 (1963), 283-286.

[21] Nambu, Y., *Generalized Hamiltonian mechanics*, Phys. Rev., D7 (1973), 2405-2412.

[22] Vainerman, L., Kerner, R., *On special classes of n-algebras*, J. Math. Phys., 37 (1996), 2553-2565.

[23] Zhan, J., Çağman, N., Singin, A., *Applications of soft union sets to hemirings via SU-h-ideals*, J. Intell. Fuzzy Syst., 26 (2014), 1363-1370.

[24] Zhou, X., Xiang, D., Zhan, J., *A novel study on fuzzy congruences on n-ary semigroups*, U.P.B. Sci. Bull., Series A, 78 (2016), 19-30.

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REPRESENTATIONS OF POLYGROUPS

R. Ameri[1]
School of Mathematics, Statistic and Computer Sciences
University of Tehran
Tehran
Iran
e-mail: rameri@ut.ac.ir

R.A. Borzooei
Department of Mathematics
Shahid Beheshti University
Tehran
Iran
e-mail: borzooei@sbu.ac.ir

K. Ghadimi
Department of Mathematics
Payame Noor University
P.O.Box 19395-3697, Tehran
Iran
e-mail: ghadimi@phd.pnu.ac.ir

Abstract. The purpose of this paper is the study of representation of polygroups based on hyperspaces (hypervector spaces). In this regards we introduce and study representation and weak representation of a given polygroup. In particular, we study irreducible and weak irreducible representation of polygroups and obtain some basic properties of them.

Keywords: polygroup, hypervector space, representation.

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1. Introduction

The theory of algebraic hyperstructures is a well-established branch of classical algebraic theory. Hyperstructure theory was first proposed in 1934 by Marty, who defined hypergroups and began to investigate their properties with applications to groups, rational fractions and algebraic functions [16]. It was later observed

[1]Corresponding author.

that the theory of hyperstructures has many applications in both pure and applied sciences; for example, semi-hypergroups are the simplest algebraic hyperstructures that possess the properties of closure and associativity. The theory of hyperstructures has been widely reviewed (for more see [1]-[7], [12]-[15], [21]).

M.S. Tallini introduced the notion of hyperspaces (hypervector spaces) ([18], [19] and [20]) and studied basic properties of them. R. Ameri and O.R. Dehghan [2] introduced and studied dimension of hyperspaces and, in [17], M. Motameni et.al. studied hypermatrix. In this paper, we consider hyperspaces in the sense of Tallini to give a representation for a given polygroup. In this regards, we give representation and weak representation of a polygroups based on various categories of hyperspaces. Also, we discuss on irreducible and weak irreducible representations of a polygroup. Finally, we investigate the relationship between irreducible and decomposable representations and obtain some results.

2. Preliminaries

In this section we give some notions and results of hypergroups and hyperspaces, which we need to developing our paper.

Definition 2.1. Let H be a set. A map $. : H \times H \longrightarrow P_*(H)$ is called a hyperoperation or join operation, where $P_*(H)$ is the set of all non-empty subsets of H. The join operation is extended to subsets of H in natural way, so that $A.B$ is given by

$$A.B = \bigcup \{a.b : a \in A \textit{ and } b \in B\}.$$

the notations $a.A$ and $A.a$ are used for $\{a\}.A$ and $A.\{a\}$ respectively. Generally, the singleton $\{a\}$ is identified by its element a.

Definition 2.2. [12] A hypergroup is a set H equipped with an associative hyperoperation $. : H \times H \to P_*(H)$ which satisfies the property $x.H = H.x = H$, for all $x \in H$. If the hyperoperation . is associative, then H is called a semihypergroup. In the above definition if $A, B \subseteq H$ and $x \in H$, then we define

$$A.B = \bigcup_{a \in A, b \in B} a.b, \quad x.B = \{x\}.B, \quad \textit{and} \quad A.x = A.\{x\}.$$

A quasicanonical hypergroup, is a special kind of a hypergroup, that first time introduced and studied by Bonansinga and Corsini in [8, 9]. After that this kind of hypergroups studied by Comer [10, 11] as the name of polygroups.

Definition 2.3. [12, 14] A polygroup is a system $\mathcal{P} = \langle P, ., e, ^{-1} \rangle$, where $e \in P$, $^{-1}$ is a unary operation on P, . maps $P \times P$ into nonempty subsets of P, and the following axioms hold for all $x, y, z \in P$:
(P_1) $(x.y).z = x.(y.z)$;
(P_2) $x.e = e.x = x$;
(P_3) $x \in y.z$ implies $y \in x.z^{-1}$ and $z \in y^{-1}.x$.

The following elementary facts about polygroups follow easily from the axioms: $e \in x.x^{-1} \cap x^{-1}.x$, $e^{-1} = e$, $(x^{-1})^{-1} = x$, and $(x.y)^{-1} = y^{-1}.x^{-1}$, where $A^{-1} = \{a^{-1} | a \in A\}$. A polygroup in which every element has order 2 (i.e., $x^{-1} = x$ for all x) is called symmetric. As in group theory it can be shown that a symmetric polygroup is commutative.

The concept of hyperspace, which is a generalization of the concept of ordinary vector space.

Definition 2.4. [18] Let K be a field and $(V, +)$ be an abelian group. We define a hyperspace over K to be the quadrupled $(V, +, \circ, K)$, where $\circ$ is a mapping

$$\circ : K \times V \longrightarrow P_*(V),$$

such that the following conditions hold:
(H_1) $\forall\ a \in K,\ \forall\ x, y \in V,\ a \circ (x + y) \subseteq a \circ x + a \circ y$, right distributive law;
(H_2) $\forall\ a, b \in K,\ \forall\ x \in V,\ (a + b) \circ x \subseteq a \circ x + b \circ x$, left distributive law;
(H_3) $\forall\ a, b \in K,\ \forall\ x \in V,\ a \circ (b \circ x) = (ab) \circ x$, associative law;
(H_4) $\forall\ a \in K,\ \forall\ x \in V,\ a \circ (-x) = (-a) \circ x = -(a \circ x)$;
(H_5) $\forall\ x \in V,\ x \in 1 \circ x$.

Remark 2.5.

(i) In the right hand side of (H_1) the sum is meant in the sense of Frobenius, that is we consider the set of all sums of an element of $a \circ x$ with an element of $a \circ y$. Similarly we have in (H_2).

(ii) We say that $(V, +, \circ, K)$ is anti-left distributive, if

$$\forall\ a, b \in K,\ \forall\ x \in V,\ (a + b) \circ x \supseteq a \circ x + b \circ x,$$

and strongly left distributive, if

$$\forall\ a, b \in K,\ \forall\ x \in V,\ (a + b) \circ x = a \circ x + b \circ x,$$

In a similar way we define the anti-right distributive and strongly right distributive hyperspaces, respectively. V is called strongly distributive if it is both strongly left and strongly right distributive.

(iii) The left hand side of (H_3) means the set-theoretical union of all the sets $a \circ y$, where y runs over the set $b \circ x$, i.e.,

$$a \circ (b \circ x) = \bigcup_{y \in b \circ x} a \circ y.$$

(iv) Let $\Omega_V = 0 \circ 0_V$, where 0_V is the zero of $(V, +)$, In [18] it is shown if V is either strongly right or left distributive, then Ω_V is a subgroup of $(V, +)$.

Example 2.6. [2] Consider abelian group $(\mathbb{R}^2, +)$. Define hyper-compositions:

$$\begin{cases} \circ : \mathbb{R} \times \mathbb{R}^2 \longrightarrow P_*(\mathbb{R}^2) \\ \quad a \circ (x, y) = ax \times \mathbb{R} \end{cases}$$

and

$$\begin{cases} \diamond : \mathbb{R} \times \mathbb{R}^2 \longrightarrow P_*(\mathbb{R}^2) \\ \quad a \diamond (x, y) = \mathbb{R} \times ay. \end{cases}$$

Then $(\mathbb{R}^2, +, \circ, \mathbb{R})$ and $(\mathbb{R}^2, +, \diamond, \mathbb{R})$ are a strongly distributive hyperspaces.

Example 2.7. [2] Let $(V, +, ., K)$ be a classical vector space and P be a subspace of V. Define the hyper-composition:

$$\begin{cases} \circ : K \times V \longrightarrow P_*(V) \\ \quad a \circ x = a.x + P. \end{cases}$$

Then it is easy to verify that $(V, +, \circ, K)$ is a strongly distributive hyperspace.

Example 2.8. [18] In $(\mathbb{R}^2, +)$ define the hyper-composition $\circ$ as follows:

$$\forall a \in \mathbb{R}, \forall x \in \mathbb{R}^2 \ : \ a \circ x = \begin{cases} \text{line } ox & \text{if } x \neq 0_V, \\ \{0_V\} & \text{if } x = 0_V, \end{cases}$$

where $0_V = (0, 0)$. Then $(\mathbb{R}^2, +, \circ, \mathbb{R})$ is a strongly left, but not right distributive hyperspace.

Proposition 2.9. [18] *Every strongly right distributive hyperspace is strongly left distributive hyperspace. Let $(V, +)$ be an abelian group, Ω a subgroup of V and K a field such that $W = V/\Omega$ is a classical vector space over a field K. If $p : V \longrightarrow W$ is the canonical projection of $(V, +)$ onto $(W, +)$ and set:*

$$\begin{cases} \circ : K \times V \longrightarrow P_*(V) \\ \quad a \circ x = p^{-1}(a.p(x)). \end{cases}$$

Then $(V, +, \circ, K)$ is a strongly distributive hyperspace over a field K. Moreover every strongly distributive hyperspace can be obtained in such a way.

Proposition 2.10. [18] *If $(V, +, \circ, K)$ is a left distributive hyperspace, then for all $a \in K$, and $x \in V$:*

(i) *$0 \circ x$ is a subgroup of $(V, +)$;*

(ii) *Ω_V is a subgroup of $(V, +)$;*

(iii) *$a \circ 0_V = \Omega_V = a \circ \Omega_V$;*

(iv) *$\Omega_V \subseteq 0 \circ x$;*

(v) *$x \in 0 \circ x \Longleftrightarrow 1 \circ x = 0 \circ x \Longleftrightarrow a \circ x = 0 \circ x$.*

Remark 2.11. Let $(V, +, \circ, K)$ be a hyperspace and W be a subhyperspace of V. Consider the quotient abelian group $(V/W, +)$. Define the rule:

$$\begin{cases} * : K \times V/W \longrightarrow P_*(V/W) \\ \quad (a, x + W) \longmapsto a \circ x + W. \end{cases}$$

Then it is easy to verify that $(V/W, +, *, K)$ is a hyperspace over a field K and it is called the quotient hyperspace of V over W.

Definition 2.12. [2] Let V be a hyperspace over a field K. A nonempty subset W of V is called a subhyperspace if W is itself a hyperspace with the hyperoperation on V, i.e.,

$$W \neq \emptyset, \;\; W - W \subseteq W, \;\; \forall a \in K, \;\; a \circ W \subseteq W.$$

In this case we write $W \leq V$.

Definition 2.13. [2] Let V be a hyperspace over a field K. If W is a nonempty subset of V, then the linear span of W is defined by

$$\begin{aligned} L(W) &= \{t \in V : t \in \sum_{i=1}^{n} a_i \circ w_i, a_i \in K, w_i \in W, n \in \mathbb{N}\} \\ &= \{t_1 + t_2 + ... + t_n : t_i \in a_i \circ w_i, a_i \in K, w_i \in W, n \in \mathbb{N}\}. \end{aligned}$$

Lemma 2.14. [2] *$L(W)$ is the smallest subhyperspace of V containing W.*

Definition 2.15. [2] Let V be a hyperspace over a field K. A subset W of V is called linearly independent if for every vectors $v_1, v_2, ..., v_n$ in W, $c_1, c_2, ..., c_n \in K$, and $0_V \in c_1 \circ v_1 + ... + c_n \circ v_n$, implies that $c_1 = c_2 = ... = c_n = 0$. A subset W of V is called linearly dependent if it is not linearly independent.

Definition 2.16. [2] Let V be a hyperspace over a field K. A basis for V is a linearly independent subset of V such that span V. We say that V has finite dimensional if it has a finite basis.

Definition 2.17. [2] Let V and W be two hyperspaces over a field K. A mapping $T : V \longrightarrow W$ is called

(i) weak linear transformation iff
$T(x+y) = T(x)+T(y)$ and $T(a\circ x)\cap a\circ T(x) \neq \emptyset$, for all $x, y \in V$, $a \in K$.

(ii) linear transformation iff
$T(x + y) = T(x) + T(y)$ and $T(a \circ x) \subseteq a \circ T(x)$, for all $x, y \in V$, $a \in K$.

(iii) strong linear transformation iff
$T(x + y) = T(x) + T(y)$ and $T(a \circ x) = a \circ T(x)$, for all $x, y \in V$, $a \in K$.

A (resp. weak, strong) linear isomorphism is defined as usual. If $T : V \longrightarrow W$ is a (resp. weak, strong) linear isomorphism, then it is denoted by (resp. $V \cong_w W$, $V \cong_s W$) $V \cong W$.

Definition 2.18. [2] Let V and W be two hyperspaces over a field K and $T : V \longrightarrow W$ be a linear transformation. The kernel and image of T are denoted by $kerT$ and ImT, respectively, are defined by

$$kerT = \{x \in V \mid T(x) \in \Omega_W\},$$

and

$$ImT = \{y \in W \mid y = T(x) \text{ for some } x \in V\}.$$

Proposition 2.19. [2] *Let $T : V \to W$ be a strong linear transformation.*

(i) *If Z is a subhyperspace of V, then the image of Z, $T(Z)$ is a subhyperspace of W. In particular ImT is a subhyperspace of W.*

(ii) *If L be a subhyperspace of W, then the preimage of L, $T^{-1}(L)$ is a subhyperspace of V containing $kerT$.*

Proposition 2.20. [2] *Let V and W be strongly left distributive hyperspaces over a field K, and $T : V \to W$ be a linear transformation. Then $kerT$ is a subhyperspace of V. Moreover, $\Omega_V \subseteq kerT$.*

Proposition 2.21. [2] *Let V and W be strongly left distributive hyperspaces over a field K, and $T : V \to W$ be a linear transformation. Then*

$$V/kerT \cong T(V)/\Omega_W,$$

Moreover if T is onto, then

$$V/kerT \cong W/\Omega_W.$$

Corollary 2.22. [2] *Let V be a strongly left distributive hyperspace over a field K, and let $B = \{x_1, ..., x_n\}$ be a basis for V. Then $V/0 \circ \omega \cong K^n$, where $\omega = \sum_{i=1}^n x_i$.*

Proposition 2.23. *Let $(V, +, \circ, K)$ be a strongly left distributive hyperspace over a field K. If $T : V \longrightarrow V$ be a linear transformation such that $T(0) = 0$, and $(-T)(\alpha) = T(-\alpha)$. We define $l(V) = \{T \mid T \text{ is invertible}\}$ and the operation $+$ as follows:*

$$(T + U)(\alpha) = T(\alpha) + U(\alpha),$$

Also, we define the external composition as

$$(a \circ T)(\alpha) = a \circ T(\alpha).$$

Then $(l(V), +, \circ, K)$ as defined above is a vector space over a field K.

Proof. The external composition $\circ$ is defined as follows:

$$\begin{aligned} \circ : K \times l(V) &\longrightarrow l(V) \\ (a, T) &\longmapsto a \circ T. \end{aligned}$$

First, we show that $(l(V),+)$ is an abelian group. Let T, U, and Z be three linear transformations that belong to $l(V)$. Then

$$\begin{aligned}
(T+(U+Z))(\alpha) &= T(\alpha)+(U+Z)(\alpha)\\
&= T(\alpha)+(U(\alpha)+Z(\alpha))\\
&= (T(\alpha)+U(\alpha))+Z(\alpha),\ \text{since V is hyperspace}\\
&= ((T+U)+Z)(\alpha).
\end{aligned}$$

Also,

$$(T+U)(\alpha)=T(\alpha)+U(\alpha)=U(\alpha)+T(\alpha)=(U+T)(\alpha).$$

Thus associativity and commutativity is hold. We consider the transformation $0:V\longrightarrow 0$ as a 0 for the group and $1_V:V\longrightarrow V$ as the identity. Then there exists a unique inverse $(-T)$ such that

$$\begin{aligned}
(T+(-T))(\alpha) &= T(\alpha)+(-T)(\alpha)\\
&= T(\alpha)+T(-\alpha)\\
&= T(\alpha+(-\alpha))\\
&= T(0)\\
&= 0.
\end{aligned}$$

Therefore, $(l(V),+)$ is an abelian group.

Now, we check that $I(V)$ is a vector space. Let $a,b\in K$ and $T,U\in I(V)$. Then we have

(1) $(a\circ(T+U))(\alpha)=a\circ(T+U)(\alpha)=(a\circ T(\alpha))+(a\circ U(\alpha))$,

(2) $((a+b)\circ T)(\alpha)=(a\circ T(\alpha))+(b\circ T(\alpha))$.

The other conditions will be obtained immediately. Therefore, $(l(V),+,\circ,K)$ is a vector space. ∎

The composition $S\odot T:V\longrightarrow V$ of linear transformations $T:V\longrightarrow V$ and $S:V\longrightarrow V$ is defined as follows:

$$(S\odot T)(x)=ST(x)=S(T(x)).$$

Proposition 2.24. *Let $(V,+,\circ,K)$ be a hyperspace over a field K. Then $(l(V),\odot)$ is a group.*

Proof. First, we show that $l(V)$ is an associative. Let T, U, and Z be three linear transformations that belong to $l(V)$. Then

$$T\odot(U\odot Z))(\alpha)=T(U\odot Z)(\alpha)=T(U(Z(\alpha))),$$

Also,

$$(T\odot U)\odot Z)(\alpha)=(T\odot U)(Z(\alpha))=T(U(Z(\alpha))).$$

Therefore $T\odot(U\odot Z)=(T\odot U)\odot Z$. Also,

$$\exists 1_V\in l(V);\ \forall T\in l(V),\ T\odot 1_V=1_V\odot T=T,$$

and

$$\forall T \in l(V),\ \exists T^{-1} \in l(V);\ T \odot T^{-1} = T^{-1} \odot T = 1_V.$$

Thus $(l(V), \odot)$ is a group. ■

Suppose $\dim V = n$, $K = \mathbb{R}$, $M_n(\mathbb{R}) = \{n \times n$ matrices with entries in $\mathbb{R}\}$, and $GL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid A$ is invertible$\}$. Then $l(V) \cong GL_n(\mathbb{R})$ (for more details see [17]).

Definition 2.25. Let V and W be two hyperspaces over a field K. A multivalued linear transformation (MLT), $T : V \longrightarrow P_*(W)$ is a mapping such that, for all $x, y \in V$ and $a \in K$,

(1) $T(x+y) \subseteq T(x) + T(y)$;

(2) $T(a \circ x) \subseteq a \circ T(x)$;

(3) $T(-a) = -T(a)$.

Remark 2.26.

(i) In Definition 2.25 (1) and (2), if the equality holds, then T is called a strong multivalued linear transformation $(SMLT)$.

(ii) In Definition 2.25, if we use the all conditions of "$=$" and "$\subseteq$" we can find three types of $SMLT$. Here we consider only inclusion and equality cases.

(iii) If T is a MLT, then $0 \in T(x)$. Since $T(x) \neq \emptyset$, so there exists $y \in T(x)$; $0 = y - y \in T(x) - T(x) = T(x) + T(-x) = T(x + (-x)) = T(x - x) = T(0)$.

Definition 2.27. [1] Let V and W be two hyperspaces over a field K and $T : V \longrightarrow P_*(W)$ be a $SMLT$. Then multivalued kernel and multivalued image of T, denoted by $\overline{Ker}T$ and $\overline{Im}T$, respectively, are defined as follows:

$$\overline{Ker}T = \{x \in V \mid 0_W \in T(x)\},$$

and

$$\overline{Im}T = \{y \in W \mid y \in T(x) \text{ for some } x \in V\}.$$

Remark 2.28.

(i) Note that $\overline{Ker}T \neq \emptyset$, by Remark 2.26(iii).

(ii) For hyperspaces V and W over a field K, by $Hom_K(V,W)$ and $Hom_K^s(V,W)$, we mean the set of all MLT and $SMLT$, respectively and sometimes we use morphism instead multivalued linear transformation, respectively.

In the following, we briefly introduced the categories of hyperspaces and study the relationship between monomorphism, epimorphism, isomorphism and monic, epic and iso objects in this category.

Definition 2.29. The category of hyperspaces over a field K denoted by $\mathcal{HV}_K$ is defined as follows:

(1) The objects of $\mathcal{HV}_K$ are all hyperspaces over K;

(2) For the objects V and W of $\mathcal{HV}_K$, the set of all morphisms from V to W denoted by $Hom_K(V,W)$, is the set of all MLT from V to W;

(3) The composition $S \odot T : V \longrightarrow P_*(W)$ of morphisms $T : V \longrightarrow P_*(L)$ and $S : L \longrightarrow P_*(W)$ is defined as follows:

$$S \odot T(x) = ST(x) = \bigcup_{t \in T(x)} S(t).$$

(4) For any object V, the morphism $1_V : V \longrightarrow P_*(V)$, $x \to \{x\}$ is the identity.

Remark 2.30. If in Definition 2.29(2), we replace $Hom_K(V,W)$ by $Hom_K^s(V,W)$, the set of all $SMLT$, then we will obtain a new category, which it denotes by $\mathcal{HV}_K^s$. In fact, $\mathcal{HV}_K^s \preceq \mathcal{HV}_K$ (by $A \preceq B$ we mean A is a subcategory of B). Also, denote the category of all vector spaces over a field K by $\mathcal{V}_K$. Clearly, $\mathcal{V}_K \preceq \mathcal{HV}_K^s$ (for more details see [1]).

Definition 2.31. Let $T : V \longrightarrow P_*(W)$ be a $SMLT$ of hyperspaces. We say that T is weakly injective if

$$\forall\ x, y \in V,\ T(x) \cap T(y) \neq \emptyset \Rightarrow x = y.$$

We say that T is strongly injective if

$$\forall\ x, y \in V,\ T(x) = T(y) \Rightarrow x = y.$$

Remark 2.32. Clearly, every weakly injective morphism is also strongly injective. Note that T is strongly injective, means that T is injective as a function with values in $P_*(W)$.

Proposition 2.33. *Let V and W be strongly left distributive hyperspaces such that $|1 \circ x| = 1,$ for all $x \in V$. If $T : V \longrightarrow P_*(W)$ is monic in $\mathcal{HV}_K^s$, then T is strongly injective.*

Proof. Suppose that $T : V \longrightarrow P_*(W)$ is a monic. Fix $x_1, x_2 \in V$ and let $T(x_1) = T(x_2)$. Define mappings $\hat{x_1}, \hat{x_2} : K \longrightarrow P_*(V)$ by $\hat{x_1}(a) = a \circ x_1$ and $\hat{x_2}(a) = a \circ x_2$ (here K is viewed as a hyperspace). $\hat{x_1}$ and $\hat{x_2}$ are well-defined morphisms of hyperspaces since for all $a, b \in K$, and $i \in \{1, 2\}$, we have

$$\hat{x_i}(a+b) = (a+b) \circ x_i = a \circ x_i + b \circ x_i = \hat{x_i}(a) + \hat{x_i}(b),$$

$$\hat{x_i}(ab) = (ab) \circ x_i = a \circ (b \circ x) = a \circ \hat{x_i}(b),$$

$$\hat{x_i}(-a) = (-a) \circ x_i = -(a \circ x_i) = -\hat{x_i}(a).$$

Moreover, $T\hat{x_1}(a) = T(a \circ x_1) = a \circ T(x_1) = a \circ T(x_2) = T(a \circ x_2) = T\hat{x_2}(a)$, and hence $\hat{x_1} = \hat{x_2}$, since T is monic. In particular $\hat{x_1}(1) = \hat{x_2}(1)$, that $1 \circ x_1 = 1 \circ x_2$, then $x_1 = x_2$. ∎

Proposition 2.34. *In $\mathcal{HV}_K^s$, if $S : V \longrightarrow P_*(W)$ is weakly injective, then it is a monic.*

Proof. Say $S : V \longrightarrow P_*(W)$ is weakly injective and let $T, U : Z \longrightarrow P_*(V)$ be two morphisms such that $ST = SU$. It suffices to show that $T(x) = U(x)$, for all $x \in Z$. Indeed fix $x \in Z$ and $y \in U(x)$. Then $S(y) \subseteq SU(x) = ST(x)$, and therefore for some $z \in T(x), S(y) \cap S(z) \neq \emptyset$. Thus $y = z$ and hence $y \in T(x)$. The other inclusion is proved analogously. ∎

Similarly, we introduce the notions of weakly and strongly surjective. A morphism $T : V \longrightarrow P_*(W)$ of hyperspaces is said to be weakly surjective if for every $y \in W$ there exists $x \in V$ such that $y \in T(x)$ and is strongly surjective, if for every non-empty subset Z of W, there exists $x \in V$ such that $Z = T(x)$.

Remark 2.35. Clearly, every strongly surjective morphism is weakly surjective. But the converse is not true. For example the identity function on every hyperspace is weakly surjective, but is not strongly surjective.

Proposition 2.36. *Let $T : V \longrightarrow P_*(W)$ be a SMLT:*

(i) *If Z is a subhyperspace of V, then $T(Z)$ is also a subhyperspace of W. In particular, $\overline{Im}T$ is a subhyperspace of W.*

(ii) *If L is a subhyperspace of W, then $T^{-1}(L)$ is also a subhyperspace of V containing $\overline{ker}T$, where $T^{-1}(L) = \{x \in V \mid T(x) \subseteq L\}$.*

Proof. Straightforward. ∎

Proposition 2.37. *In $\mathcal{HV}_K^s$, every epic is weakly surjective.*

Proof. Suppose T is an epic. Define morphisms $0, \pi : W \longrightarrow P_*(W/\overline{Im}T)$ by $0(a) = \{\overline{0}\}$ and $\pi(x) = \{\overline{x}\}$. Then:

$$\pi T = \{\overline{0}\} = 0T,$$

so that $\pi = 0$, thus $\pi(x) = 0(x)$ and $x + \overline{Im}T = 0 + \overline{Im}T$, so $x \in \overline{Im}T$ that is T is a weakly surjective. ∎

Proposition 2.38. *In $\mathcal{HV}_K^s$, if $T : V \longrightarrow P_*(W)$ is a strongly surjective, then it is epic.*

Proof. Let $T : V \longrightarrow P_*(W)$ be a strongly surjective and $S, U : W \longrightarrow P_*(Z)$ be two morphisms such that $ST = UT$. For a fixed $y \in W$, suppose $x \in V$ be such that $T(x) = \{y\}$. Then

$$S(y) = S(T(x)) = ST(x) = UT(x) = U(T(x)) = U(y).$$ ∎

Proposition 2.39. *Let V be a hyperspace such that $|1 \circ x| = 1$ for all $x \in V$. Then a morphism T in $\mathcal{HV}_K^s$ is iso if and only if it is a single valued bijective morphism.*

Proof. First, suppose that $T : V \longrightarrow P_*(W)$ is an iso in $\mathcal{HV}_K^s$ and suppose that T is not single valued, that is for some $x \in V$ there exist $y_1, y_2 \in T(a)$ with $y_1 \neq y_2$. Since T is an iso, there exists a morphism $S : W \longrightarrow P_*(V)$ such that $TS = 1_W$ and $ST = 1_V$. In particular, S is an iso and thus it is strongly injective, by Proposition 2.33 Moreover, $S(y_1) = \{x\}$ and $S(y_2) = \{x\}$, so that $y_1 = y_2$ which is a contradiction. Hence T is single valued, strongly injective and weakly surjective by Propositions 2.33 and 2.37, and thus it is a bijective function.

Conversely, suppose that $T : V \longrightarrow P_*(W)$ is a single valued bijective morphism. Define the mapping $S : W \longrightarrow P_*(V)$ by $S(y) = \{x\}$ if and only if $T(x) = \{y\}$. Clearly, $TS = 1_W$ and $ST = 1_V$, and it is easy to check that S is a morphism in $\mathcal{HV}_K^s$. Now, for $y_1, y_2 \in W$, consider $x_1, x_2 \in V$ such that $T(x_1) = y_1$ and $T(x_2) = y_2$. Therefore

$$\begin{aligned} x \in S(y_1 + y_2) &\iff x = S(y_1 + y_2) \\ &\iff y_1 + y_2 = T(x) \\ &\iff T(x) = T(x_1) + T(x_2) = T(x_1 + x_2) \\ &\iff x = x_1 + x_2 \\ &\iff x = S(y_1) + S(y_2). \end{aligned}$$

Thus,

$$S(y_1 + y_2) = S(y_1) + S(y_2).$$

Also,

$$\begin{aligned} x_1 \in S(a \circ y) &\iff x_1 = S(z) \text{ and } z \in a \circ y \\ &\iff z = T(x_1) \text{ and } z \in a \circ y \\ &\iff T(x_1) \in a \circ y = a \circ T(x_2) = T(a \circ x_2) \\ &\iff x_1 \in a \circ x_2 \\ &\iff x_1 \in a \circ S(y). \end{aligned}$$

Therefore S is a morphism in $\mathcal{HV}_K^s$. ■

Proposition 2.40. *In $\mathcal{HV}_K^s$ a morphism $T : V \longrightarrow P_*(W)$ is weakly injective if and only if $\overline{Ker}T = \{0\}$.*

Proof. First, let $x \in \overline{Ker}T$. Then $0 \in T(x)$, also $0 \in T(0)$ by Remark 2.26(iii). Thus $T(0) \cap T(x) \neq \emptyset$ implies $0 = x$. Conversly, let $x, y \in V$ such that $T(x) \cap T(y) \neq \emptyset$. So, there exists $z \in T(x) \cap T(y)$ and

$$0 = z - z \in T(x) - T(y) = T(x - y) \Rightarrow x - y \in \overline{Ker}T = \{0\} \Rightarrow x = y. \quad ■$$

Proposition 2.41. *Let V and W be two left distributive hyperspaces over a field K and $T : V \longrightarrow P_*(W)$ be a $SMLT$. Then $\overline{Ker}T$ is a subhyperspace of V.*

Proof. Let $x, y \in \overline{Ker}T$. Then $0 \in T(x)$ and $0 \in T(y)$, hence $0 = 0 - 0 \in T(x) - T(y) = T(x - y)$ and so $x - y \in \overline{Ker}T$. Also for all $a \in \overline{Ker}T$, $a \circ 0 \subseteq a \circ T(x)$ and by Proposition 2.10, $a \circ 0 = \Omega_W$. Also

$$0 \in \Omega_W \subseteq a \circ T(x) = T(a \circ x).$$

Therefore, $a \circ x \in \overline{Ker}T$. Thus $\overline{Ker}T \leq V$. ■

3. Representation of polygroups

In the sequel, by V we mean a hyperspace over a field K.

Definition 3.1. A representation of a polygroup P is a homomorphism $\varphi : P \longrightarrow l(V)$ for some (finite-dimensional) non-zero hyperspace V. The dimension of V is called the degree of φ.

We usually write φ_g for $\varphi(g)$ and $\varphi_g(x)$, or simply $\varphi_g x$, for the action of φ_g on $x \in V$.

Definition 3.2. Two representations $\varphi : P \to l(V)$ and $\psi : P \to l(W)$ are equivalent if there exists an isomorphism $T : V \to W$ such that $\psi_g = T\varphi_g T^{-1}$ for all $g \in P$, i.e., $\psi_g T = T\varphi_g$ for all $g \in P$. In this case, we write $\varphi \sim \psi$. In picture, we have the diagram

$$\begin{tikzcd} V \arrow[r, "\varphi_g"] \arrow[d, "T"'] & V \arrow[d, "T"] \\ W \arrow[r, "\psi_g"'] & W \end{tikzcd}$$

commutes.

Definition 3.3. Let $\varphi : P \to l(V)$ be a representation. A subhyperspace $W \leq V$ is P-invariant if, for all $g \in P$ and $w \in W$, one has $\varphi_g w \in W$.

Definition 3.4. Suppose that representations $\varphi^{(1)} : P \to l(V_1)$ and $\varphi^{(2)} : P \to l(V_2)$ are given. Then their direct sum

$$\varphi^{(1)} \oplus \varphi^{(2)} : P \to l(V_1 \oplus V_2)$$

is given by

$$(\varphi^{(1)} \oplus \varphi^{(2)})_g(v_1, v_2) = (\varphi_g^{(1)}(v_1), \varphi_g^{(2)}(v_2)).$$

Definition 3.5. A representation $\varphi : P \to l(V)$ is said to be irreducible if the only P-invariant subhyperspaces of V are $\{0\}$ and V.

Definition 3.6. Let P be a polygroup. A representation $\varphi : P \to l(V)$ is said to be completely reducible if $V = V_1 \oplus V_2 \oplus ... \oplus V_n$ where the V_i are non-zero P-invariant subhyperspaces and $\varphi\,|_{V_i}$ are irreducible for all $i = 1, ..., n$.

Equivalently, φ is completely reducible if $\varphi \sim \varphi^{(1)} \oplus \varphi^{(2)} \oplus ... \oplus \varphi^{(n)}$ where the $\varphi^{(i)}$ are irreducible representations.

Definition 3.7. We say that φ is decomposable if $V = V_1 \oplus V_2$ with V_1 and V_2 non-zero P-invariant subhyperspaces. Otherwise, V is called indecomposable.

Lemma 3.8. *Let $\varphi : P \to l(V)$ be equivalent to decomposable representation. Then φ is decomposable.*

Proof. Let $\psi : P \to l(W)$ be a decomposable representation with $\psi \sim \varphi$ and $T : V \to W$ a hyperspace isomorphism with $\varphi_g = T^{-1}\psi_g T$. Suppose that W_1 and W_2 are non-zero invariant subhyperspaces of W with $W = W_1 \oplus W_2$. Since T is an equivalence we have

$$\begin{tikzcd} V \arrow[r, "\varphi_g"] \arrow[d, "T"'] & V \arrow[d, "T"] \\ W \arrow[r, "\psi_g"'] & W \end{tikzcd}$$

commutes, i.e., $T\varphi_g = \psi_g T$, for all $g \in P$. Let $V_1 = T^{-1}(W_1)$ and $V_2 = T^{-1}(W_2)$. First we claim $V = V_1 \oplus V_2$, Indeed, if $v \in V_1 \cap V_2$, then $Tv \in W_1 \cap W_2 = \{0\}$ and so $Tv = 0$. But T is injective so this implies $v = 0$. Next, if $v \in V$, then $Tv = w_1 + w_2$ for some $w_1 \in W_1$ and $w_2 \in W_2$. Then $v = T^{-1}w_1 + T^{-1}w_2 \in V_1 + V_2$. Thus $V = V_1 \oplus V_2$.

Next, we show that V_1 and V_2 are P-invariant. If $v \in V_i$ such that $i \in \{1, 2\}$, then $\varphi_g v = T^{-1}\psi_g T v$. But $Tv \in W_i$ implies $\psi_g T v \in W_i$, since W_i is P-invariant. Therefore, we conclude that $\varphi_g v = T^{-1}\psi_g T v \in T^{-1}(W_i) = V_i$, as required. ∎

Similarly, we have the following results, whose proofs we omit.

Lemma 3.9. *Let $\varphi : P \to l(V)$ be equivalent to an irreducible representation. Then φ is irreducible.*

Lemma 3.10. *Let $\varphi : P \to l(V)$ be equivalent to a completely reducible representation. Then φ is completely reducible.*

4. Weak representation of polygroups

Definition 4.1. A weak representation of a polygroup P is a homomorphism $\varphi : P \longrightarrow L(V)$ for some (finite-dimensional) non-zero hyperspace V such that $L(V) = \{T : V \longrightarrow P_*(V) \mid T \text{ is } MLT\}$. The dimension of V is called the degree of φ.

If $T : V \longrightarrow P_*(W)$ is a MLT, then T induced a map $\overline{T} : P_*(V) \longrightarrow P_*(W)$ by $\overline{T}(A) = \bigcup_{a \in A} T(a)$.

Definition 4.2. Two weak representations $\varphi : G \to L(V)$ and $\psi : G \to L(W)$ are equivalent if there exists an iso $T : V \to P_*(W)$ such that $\overline{\psi_g} T = \overline{T}\varphi_g$ for all $g \in P$. In this case, we write $\varphi \sim \psi$. In picture, we have the diagram

$$\begin{tikzcd} V \arrow[r, "\varphi_g"] \arrow[d, "T"'] & P_*(V) \arrow[d, "\overline{T}"] \\ P_*(W) \arrow[r, "\overline{\psi_g}"'] & P_*(W) \end{tikzcd}$$

commutes.

Definition 4.3. Let $\varphi : P \to L(V)$ be a weak representation. A subhyperspace $W \leq V$ is weak P-invariant if, for all $g \in P$ and $w \in W$, one has $\varphi_g w \cap W \neq \emptyset$.

Definition 4.4. Let $\varphi : P \to L(V)$ be a weak representation. A a subhyperspace $W \leq V$ is P-invariant if, for all $g \in P$ and $w \in W$, one has $\varphi_g w \subseteq W$.

Remark 4.5. *Clearly P-invariant subhyperspace is weak P-invariant subhyperspace.*

Definition 4.6. Suppose that weak representations $\varphi^{(1)} : P \to L(V_1)$ and $\varphi^{(2)} : P \to L(V_2)$ are given. Then their direct sum

$$\varphi^{(1)} \oplus \varphi^{(2)} : P \to L(V_1 \oplus V_2)$$

is given by

$$(\varphi_g^{(1)} \oplus \varphi^{(2)})_g(v_1, v_2) = (\varphi_g^{(1)}(v_1), \varphi_g^{(2)}(v_2)).$$

Definition 4.7. A weak representation $\varphi : P \to L(V)$ is said to be weak irreducible if the only weak P-invariant subhyperspaces of V are $\{0\}$ and V.

Definition 4.8. A weak representation $\varphi : P \to L(V)$ is said to be irreducible if the only P-invariant subhyperspaces of V are 0 and V.

Definition 4.9. Let P be a polygroup. A weak representation $\varphi : P \to L(V)$ is said to be completely reducible if $V = V_1 \oplus V_2 \oplus ... \oplus V_n$ where the V_i are non-zero P-invariant subhyperspaces and $\varphi\mid_{V_i}$ are irreducible for all $i = 1, ..., n$.

Equivalently, φ is completely reducible if $\varphi \sim \varphi^{(1)} \oplus \varphi^{(2)} \oplus ... \oplus \varphi^{(n)}$ where the $\varphi^{(i)}$ are irreducible representations.

Definition 4.10. We say that φ is decomposable if $V = V_1 \oplus V_2$ with V_1 and V_2 non-zero P-invariant subhypervector spaces. Otherwise, V is called indecomposable.

Proposition 4.11. *Let V be a strongly left distributive hyperspace such that $|1 \circ x| = 1$, for all $x \in V$, and $\varphi : P \to L(V)$ be equivalent to decomposable representation. Then φ is decomposable.*

Proof. Let $\psi : P \to L(W)$ be a decomposable representation with $\psi \sim \varphi$ and $T : V \to P_*(W)$ a hyperspace iso with $\overline{T}\varphi_g = \overline{\psi_g}T$. Suppose that W_1 and W_2 are non-zero P-invariant subhyperspaces of W with $W = W_1 \oplus W_2$. Since T is an equivalence we have

$$\begin{array}{ccc} V & \xrightarrow{\varphi_g} & P_*(V) \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle \overline{T}} \\ P_*(W) & \xrightarrow[\overline{\psi_g}]{} & P_*(W) \end{array}$$

commutes, i.e., $\overline{T}\varphi_g = \overline{\psi_g}T$, for all $g \in G$. Let $V_1 = T^{-1}(W_1)$ and $V_2 = T^{-1}(W_2)$. First we claim $V = V_1 \oplus V_2$, Indeed, if $v \in V_1 \cap V_2$, then $T(v) \in W_1 \cap W_2 = \{0\}$ and so $Tv = \{0\}$. But T is monic by Proposition 2.33 implies $v = 0$. Next, if

$v \in V$, then for all $w \in Tv$, $w = w_1 + w_2$ for some $w_1 \in W_1$ and $w_2 \in W_2$. Then $v \in T^{-1}(w) = T^{-1}(w_1) + T^{-1}(w_2) \subseteq V_1 + V_2$. Thus $V = V_1 \oplus V_2$.

Next, we show that V_1 and V_2 are P-invariant. If $v \in V_i$ such that $i \in \{1, 2\}$, then $\overline{T}\varphi_g v = \overline{\psi_g} Tv$. But $Tv \subseteq W_i$ implies $\overline{\psi_g} Tv \subseteq W_i$ since W_i is P-invariant. Therefore, we conclude that $\overline{T}\varphi_g v = \overline{\psi_g} Tv \subseteq W_i$, thus $\bigcup_{a \in \varphi_g v} T(a) \subseteq W_i$. Therefore $Ta \subseteq W_i$ and $a \in T^{-1}(W_i) = V_i$, for all $a \in \varphi_g v$. Then $\varphi_g v \subseteq V_i$, as required. ∎

Similarly, we have the following results, whose proofs was omitted.

Proposition 4.12. *Let $\varphi : P \to L(V)$ be equivalent to an irreducible weak representation. Then φ is irreducible.*

Proposition 4.13. *Let $\varphi : P \to L(V)$ be equivalent to a completely reducible weak representation. Then φ is completely reducible.*

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References

[1] Ameri, R., *On categories of hypergroups and hypermodules*, Journal of Discrete Mathematical Science and Cryptography, 6 (2-3) (2003), 121-132.

[2] Ameri, R., Dehghan, O.R., *On dimension of hypervector spaces*, European Journal of Pure and Applied Mathematics, 1 (2) (2008), 32-50.

[3] Ameri, R., Norouzi, M., *On multiplication (m, n)-hypermodules*, European Journal of Combinatorics, 44 (2015), 153-171.

[4] Ameri, R., Norouzi, M., *New fundamental relation of hyperrings*, European Journal of Combinatorics, 34 (2013), 884–891.

[5] Ameri, R., Norouzi, M., *Prime and primary hyperideals in Krasner*, European Journal of Combinatorics, 34 (2013), 379–390.

[6] Ameri, R., Rosenberg, I.G., *Congruences of multialgebras*, Multivalued Logic and Soft Computing, 15 (5-6) (2009), 525-536.

[7] Ameri, R., Zahedi, M.M., *Hyperalgebraic systems*, Italian Journal of Pure and Applied Mathematics, 6 (1999), 21-32.

[8] Bonansinga, P., *Sugli ipergruppi quasicanonici,* Atti Soc. Pelor. Sc. Fis. Mat. Nat., 27 (1981), 9-17.

[9] Bonansinga, P., Corsini, P., *Sugli omomorfismi di semi-ipergruppi e di ipergruppi*, Boll. Un. Mat. Italy, 1-B (1982), 717-727.

[10] Comer, S., *Polygroups derived from cogroups*, J. Algebra, 89 (2) (1984), 397-405.

[11] Comer, S., *Extension of polygroups by polygroups and their representations using color schemes*, Lecture notes in Mathematics, Universal Algebra and Lattice Theory, 1004 (1982) 91-103.

[12] Corsini, P., *Prolegomena of Hypergroup Theory*, Second Edition, Aviani Editor, 1993.

[13] Corsini, P., Leoreanu-Fotea, V., *Applications of Hyperstructure Theory*, Kluwer Academic Publishers, Dordrecht, Hardbound, 2003.

[14] Davvaz, B., *Polygroup Theory and Related Systems*, World Scientific, 2013.

[15] Davvaz, V. Leoreanu-Fotea, V., *Hyperring Theory and Applications*, International Academic Press, USA, 2007.

[16] Marty, F., *Sur une generalization de la notion de groupe*, 8*th* Congress Math. Scandinaves, Stockholm, 1934, 45-49.

[17] Motameni, M., Ameri, R., Sadeghi, R., *Hypermatrix based on Krasner hypervector spaces*, Ratio Mathematica 25 (2013) 77-94.

[18] Scafatti Tallini, M., *Hypervector spaces*, 4*th* AHA, World Scientific, Xanthi, Greece, 1990, 167-174.

[19] Scafatti Tallini, M., *Weak hypervector spaces and norms in such spaces*, Proceedings of the Fifth International Congress on Algebraic Hyperstructures and Applications, Jasi, Rumania, Hadronic Press, 1994, 199-206.

[20] Scafatti Tallini, M., *Dimensions in multivalued algebraic structures*, Italian Journal of Pure and Applied Mathematics, 1 (1997), 51-64.

[21] Vougiouklis, T., *Hyperstructure and Their Representations*, Hadronic Press, Inc., 1994.

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On $H_3(p)$ HANKEL DETERMINANT FOR CERTAIN SUBCLASS OF $p-$VALENT FUNCTIONS

A.A. Amourah

School of Mathematical Sciences
Faculty of Science and Technology
Universiti Kebangsaan Malaysia
Bangi 43600, Selangor D. Ehsan
Malaysia
e-mails: alaammour@yahoo.com

Feras Yousef[1]

Department of Mathematics
Faculty of Science
The University of Jordan
Amman 11942
Jordan
e-mail: fyousef@ju.edu.jo

Tariq Al-Hawary

Department of Applied Science
Ajloun College
Al-Balqa Applied University
Ajloun 26816
Jordan
e-mail: tariq_amh_@yahoo.com

M. Darus

School of Mathematical Sciences
Faculty of Science and Technology
Universiti Kebangsaan Malaysia
Bangi 43600, Selangor D. Ehsan
Malaysia
e-mails: maslina@ukm.edu.my

Abstract. The aim of this paper is to obtain an upper bound to the $H_3(p)$ Hankel determinant for certain subclass of p-valent functions. To do so, we obtain best possible bounds for the functionals $|a_{p+3} - a_{p+1}a_{p+2}|$ and $\left|a_{p+2} - a_{p+1}^2\right|$, then using known upper bound for the functional $\left|a_{p+1}a_{p+3} - a_{p+2}^2\right|$ we obtain the required sharp upper bound to the $H_3(p)$ Hankel determinant.

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[1]Corresponding author

1. Introduction

Let $\mathcal{A}_p$ denote the class of functions f of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, (p \in \mathbb{N} = \{1, 2, 3, ...\}) \tag{1.1}$$

in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let S be the subclass of $\mathcal{A}_1 := \mathcal{A}$, consisting of univalent functions.

The Hankel determinant of f for $q \geq 1$ and $n \geq 1$ was defined by Pommerenke ([17], [16]) as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & \cdots & a_{n+q} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix} \tag{1.2}$$

This determinant has been considered by many authors in the literature [14]. For example, Noor [15] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for functions given by (1.1) with bounded boundary. Ehrenborg [2] studied the Hankel determinant of exponential polynomials. In [7], Janteng et al. studied the Hankel determinant for the classes of starlike and convex functions. Again Janteng et al. discussed the Hankel determinant problem for the classes of starlike functions with respect to symmetric points and convex functions with respect to symmetric points in [5] and for the functions whose derivative has a positive real part in [6]. Also Hankel determinant for various subclasses of $p-$valent functions was investigated by various authors including Krishna and Ramreddy [8] and Hayami and Owa [4].

In this paper, we consider the Hankel determinant in the case of $q = 3$ and $n = p$:

$$H_3(p) = \begin{vmatrix} a_p & a_{p+1} & a_{p+2} \\ a_{p+1} & a_{p+2} & a_{p+3} \\ a_{p+2} & a_{p+3} & a_{p+4} \end{vmatrix}.$$

For $f \in \mathcal{A}_p$, $a_p = 1$, we have

$$H_3(p) = a_{p+2}(a_{p+1}a_{p+3} - a_{p+2}^2) - a_{p+3}(a_{p+3} - a_{p+1}a_{p+2}) + a_{p+4}(a_{p+2} - a_{p+1}^2),$$

and by applying the triangle inequality, we obtain

$$|H_3(p)| \leq |a_{p+2}||a_{p+1}a_{p+3} - a_{p+2}^2| + |a_{p+3}||a_{p+3} - a_{p+1}a_{p+2}| + |a_{p+4}||a_{p+2} - a_{p+1}^2|. \tag{1.3}$$

Incidentally, the sharp upper bound for the functional $\left|a_{p+1}a_{p+3} - a_{p+2}^2\right|$ on the right hand side of the inequality (1.3) for the class of functions which is of our interest in this paper was obtained by Vamshee Krishna and Ramreddy [9]. Thus, in this paper we obtain upper bounds to the functionals $|a_{p+3} - a_{p+1}a_{p+2}|$ and $\left|a_{p+2} - a_{p+1}^2\right|$, then the sharp upper bound on $H_3(p)$ follows as simple corollary.

Definition 1.1 A function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{H}_{p,\alpha}$, if it satisfies the condition

$$\mathrm{Re}\left\{(1-\alpha)\frac{f(z)}{z^p} + \alpha\frac{f'(z)}{pz^{p-1}}\right\} > 0, \quad (0 \le \alpha \le 1), \forall z \in \mathbb{U}. \tag{1.4}$$

For $\alpha = 1$, the class $\mathcal{H}_{p,1}$ reduced to Vamshee Krishna, et al. [10].

In the next section, we state the necessary lemmas, while in Section 3 we present our main results.

2. Preliminaries results

Let Q denote the class of functions

$$q(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ... = 1 + \sum_{k=1}^{\infty} c_k z^k, \tag{2.1}$$

which are analytic in $\mathbb{U}$ and satisfy $\mathrm{Re}\{q(z)\} > 0$ for any $z \in \mathbb{U}$. To prove our main results in the next section, we shall require the following three Lemmas:

Lemma 2.1 ([18], [19]) *Let $q \in Q$. Then $|c_k| \le 2$ for each $k \in \mathbb{N}$. And the inequality is sharp.*

Lemma 2.2 ([3], [11], [12]) *Let $q \in Q$. Then*

$$2c_2 = c_1^2 + (4 - c_1^2)x,$$
$$4c_3 = c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z,$$

for some x and z satisfying $|x| \le 1$, $|z| \le 1$ and $q_1 \in [0, 2]$.

Lemma 2.3 ([1]) *Let $q \in Q$. Then we have the sharp inequalities for any real number σ,*

$$\left|c_2 - \sigma\frac{c_1^2}{2}\right| \le \begin{cases} 2(1-\sigma) & \text{if } \sigma \le 0, \\ 2 & \text{if } 0 \le \sigma \le 2, \\ 2(\sigma - 1) & \text{if } \sigma \ge 0. \end{cases} \tag{2.2}$$

3. Main results

Theorem 3.1 *Let $f(z) \in \mathcal{H}_{p,\alpha}$. Then, for all $j \in \mathbb{N}$, we have the sharp inequalities:*

$$|a_{p+j}| \le \frac{2p}{p + j\alpha}. \tag{3.1}$$

Proof. Since $f \in \mathcal{H}_{p,\alpha}$, then by Definition 1.1 there exists a function $q \in Q$ such that

$$\left\{(1-\alpha)\frac{f(z)}{z^p}+\alpha\frac{f'(z)}{pz^{p-1}}\right\}=p(z)\Rightarrow p(1-\alpha)f(z)+\alpha zf'(z)=pz^p p(z), \tag{3.2}$$

for some $z \in \mathbb{U}$.

Replacing $f(z)$, $f'(z)$ with their equivalent $p-$ valent series expressions, and $p(z)$ with its equivalent series expression in (3.2), we have

$$\begin{aligned} p(1-\alpha)\left[z^p+\sum_{n=p+1}^{\infty}a_nz^n\right]+\alpha z\left(pz^{p-1}+\sum_{n=p+1}^{\infty}na_nz^{n-1}\right) \\ =pz^p\left(1+\sum_{k=1}^{\infty}c_kz^k\right) \end{aligned} \tag{3.3}$$

Simplifying (3.3) implies

$$\begin{aligned} (p+\alpha)a_{p+1}z^{p+1}+(p+2\alpha)a_{p+2}z^{p+2}+(p+3\alpha)a_{p+3}z^{p+3} \\ +(p+4\alpha)a_{p+4}z^{p+4}+...=pc_1z^{p+1}+pc_2z^{p+2}+pc_3z^{p+3}+pc_4z^{p+4}+... \end{aligned} \tag{3.4}$$

Equating coefficients in (3.4) yields

$$a_{p+j}=\frac{pc_j}{p+i\alpha}, \quad \text{for all } j \in \mathbb{N} \tag{3.5}$$

and the result follows by using Lemma 2.2. ∎

Theorem 3.2 *Let $f(z) \in \mathcal{H}_{p,\alpha}$. Then we have the sharp inequalities:*

$$|a_{p+1}a_{p+2}-a_{p+3}| \le \begin{cases} 2 & \text{if} \quad \alpha=0 \\ \dfrac{2p\left(p^2+3\alpha p+6\alpha^2\right)^{\frac{3}{2}}}{3(p+\alpha)(p+2\alpha)(p+3\alpha)\sqrt{6\alpha^2}} & \text{if} \quad 0<\alpha\le 1. \end{cases} \tag{3.6}$$

Proof. By (3.5) (for $j=1,2,$ and 3), we find that

$$|a_{p+1}a_{p+2}-a_{p+3}|=\left|\frac{p^2c_1c_2}{(p+\alpha)(p+2\alpha)}-\frac{pc_3}{(p+3\alpha)}\right|. \tag{3.7}$$

Substituting the values of c_2 and c_3 from Lemma 2.2 in (3.7) and letting $c_1=c$, we have

$$\begin{aligned} |a_{p+1}a_{p+2}-a_{p+3}|=\frac{p}{4(p+\alpha)(p+2\alpha)(p+3\alpha)}\big|(p^2+3\alpha p-2\alpha^2)c^3-4\alpha^2c(4-c^2)x \\ +c(p^2+3\alpha p+2\alpha^2)(4-c^2)x^2-2(p^2+3\alpha p+2\alpha^2)(4-c^2)(1-|x|^2)z\big|. \end{aligned}$$

Without loss of generality, assume that $c = c_1 \in [0, 2]$ (see Lemma 2.1). Then, by applying the triangle inequality with $\delta = |x|$ and noticing that $p^2+3\alpha p-2\alpha^2 \geq 0$, since $0 \leq \alpha \leq 1$ by definition, we get

$$\begin{aligned}|a_{p+1}a_{p+2}-a_{p+3}| \leq \frac{p}{4(p+\alpha)(p+2\alpha)(p+3\alpha)}\big[(p^2+3\alpha p-2\alpha^2)c^3+4\alpha^2 c(4-c^2)\delta \\ +c(p^2+3\alpha p+2\alpha^2)(4-c^2)\delta^2+2(p^2+3\alpha p+2\alpha^2)(4-c^2)(1-\delta^2)\big] \\ = \frac{p}{4(p+\alpha)(p+2\alpha)(p+3\alpha)}\big[2(p^2+3\alpha p+2\alpha^2)(4-c^2)+(p^2+3\alpha p-2\alpha^2)c^3 \\ +4\alpha^2 c(4-c^2)\delta+(p^2+3\alpha p+2\alpha^2)(c-2)(4-c^2)\delta^2\big] \\ := F(\delta),\ \text{where } 0 \leq \delta \leq 1.\end{aligned} \tag{3.8}$$

We then maximize the function $F(\delta)$ on the closed interval $[0, 1]$:

$$F'(\delta) = \frac{p\alpha^2 c(4-c^2)}{(p+\alpha)(p+2\alpha)(p+3\alpha)} + \frac{p(c-2)(4-c^2)}{2(p+3\alpha)}\delta.$$

Note that $F'(\delta) \geq F'(1^-) > 0$ for all $c \in (0, 2)$ and $F'(\delta) \leq 0$ otherwise. Hence, there exists $c^* \in [0, 2]$ such that $F'(\delta) > 0$ for $c \in (c^*, 2]$ and $F'(\delta) \leq 0$ otherwise. Thus, for $c \in (c^*, 2]$ we observe that $F(\delta) \leq F(1)$, that is:

$$|a_{p+1}a_{p+2}-a_{p+3}| \leq \frac{p(p^2+3\alpha p+6\alpha^2)}{(p+\alpha)(p+2\alpha)(p+3\alpha)}c - \frac{2\alpha^2 p}{(p+\alpha)(p+2\alpha)(p+3\alpha)}c^3 := G_1(c). \tag{3.9}$$

If $\alpha = 0$, we have $G_1(c) = c \leq 2$. Otherwise, simplifying the relations (3.8) and (3.9), we get

$$G_1(c) = \frac{p(p^2+3\alpha p+6\alpha^2)}{(p+\alpha)(p+2\alpha)(p+3\alpha)}c - \frac{2\alpha^2 p}{(p+\alpha)(p+2\alpha)(p+3\alpha)}c^3 \tag{3.10}$$

$$G_1'(c) = \frac{p(p^2+3\alpha p+6\alpha^2)}{(p+\alpha)(p+2\alpha)(p+3\alpha)} - \frac{6\alpha^2 p}{(p+\alpha)(p+2\alpha)(p+3\alpha)}c^2 \tag{3.11}$$

$$G_1''(c) = -\frac{12\alpha^2 p}{(p+\alpha)(p+2\alpha)(p+3\alpha)}c \tag{3.12}$$

For an optimum value of $G_1(c)$, consider $G_1'(c) = 0$. From (3.11), we have

$$c^2 = \frac{p^2+3\alpha p+6\alpha^2}{6\alpha^2} \tag{3.13}$$

Using the obtained value of c^2 from (3.13) in (3.12), after simplifying, we get

$$G_1''(c) = -\frac{12\alpha^2 p}{(p+\alpha)(p+2\alpha)(p+3\alpha)}\sqrt{\frac{p^2+3\alpha p+6\alpha^2}{6\alpha^2}} < 0 \text{ for } p \in \mathbb{N}.$$

Therefore, by the second derivative test, $G_1(c)$ has maximum value at c, where c^2 is given by (3.13). Substituting c^2 value in the expression (3.10), upon simplification, the maximum value of $G_1(c)$ at c^2 is obtained as

$$G_1(c)_{\max} = \frac{p}{(p+\alpha)(p+2\alpha)(p+3\alpha)} \frac{2\left(p^2+3\alpha p+6\alpha^2\right)^{\frac{3}{2}}}{3\sqrt{6\alpha^2}}.$$

Upon simplification, we obtain

$$|a_{p+1}a_{p+2} - a_{p+3}| \leq \frac{2p\left(p^2+3\alpha p+6\alpha^2\right)^{\frac{3}{2}}}{3(p+\alpha)(p+2\alpha)(p+3\alpha)\sqrt{6\alpha^2}}.$$

This completes the proof. ∎

Theorem 3.3 *Let $f(z) \in \mathcal{H}_{p,\alpha}$. Then*

$$\left|a_{p+2} - a_{p+1}^2\right| \leq \frac{2p}{p+2\alpha}. \tag{3.14}$$

Proof. Since $f(z) \in \mathcal{H}_{p,\alpha}$, then using equation (3.5), (for $j = 1$ and $j = 2$), we find that

$$\left|a_{p+2} - a_{p+1}^2\right| = \left|\frac{pc_2}{(p+2\alpha)} - \frac{p^2c_1^2}{(p+\alpha)^2}\right| = \frac{p}{p+2\alpha}\left|c_2 - \frac{2p(p+2\alpha)}{(p+\alpha)^2}\left(\frac{c_1^2}{2}\right)\right|.$$

Now, since $0 \leq \sigma = \frac{2p(p+2\alpha)}{(p+\alpha)^2} \leq 2$, then Lemma 2.3 yeilds

$$\left|a_{p+2} - a_{p+1}^2\right| \leq \frac{2p}{p+2\alpha}. \tag{3.15}$$

This completes the proof. ∎

Substituting the above results in (1.3) together with the known inequality $\left|a_{p+1}a_{p+3} - a_{p+2}^2\right| \leq \frac{4p^2}{(p+2\alpha)^2}$ ([9]) and $|a_k| \leq \frac{2}{k}$, where $k \in \{p+1,\ p+2,\ p+3, ...\}$ ([13]), after simplifying, we obtain the sharp inequalities:

Corollary 3.4 *Let $f(z) \in \mathcal{H}_{p,\alpha}$. Then*

$$|H_3(p)| \leq \begin{cases} 4\left[\frac{2p^2}{(p+2)(p+2\alpha)^2} + \frac{p}{(p+4)(p+2\alpha)} + \frac{1}{p+3}\right] & \text{if } \alpha = 0 \\ \frac{2}{p+2\alpha}\left[\frac{4p^2}{(p+2)(p+2\alpha)} + \frac{\sqrt{2}p\left(p^2+3\alpha p+6\alpha^2\right)^{\frac{3}{2}}}{3\sqrt{3\alpha^2}(p+3)(p+\alpha)(p+3\alpha)} + \frac{2p}{p+4}\right] & \text{if } 0 < \alpha \leq 1. \end{cases} \tag{3.16}$$

Remark 3.5 As a final remark, for the choice of $\alpha = 1$, from expressions (3.6) and (3.16), we obtain

$$|a_{p+1}a_{p+2} - a_{p+3}| \leq \frac{\sqrt{2}p\left(p^2+3p+6\right)^{\frac{3}{2}}}{3\sqrt{3}(p+1)(p+2)(p+3)}$$

and

$$|H_3(p)| \leq \frac{2}{p+2}\left[\frac{4p^2}{(p+2)^2} + \frac{\sqrt{2}p\,(p^2+3p+6)^{\frac{3}{2}}}{3\sqrt{3}(p+3)^2(p+1)} + \frac{2p}{p+4}\right],$$

respectively. These inequalities are sharp and coincide with the results obtained by ([10]).

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References

[1] Babalola, K.O., Opoola, T.O., *On the coefficients of a certain class of analytic functions*, Advances in Inequalities for Series, S.S. Dragomir and A. Sofo (Eds.), Nova Science Publishers, 2006, 5-17. (2008), 5-17.

[2] Ehrenborg, R., *The Hankel determinant of exponential polynomials*, American Mathematical Monthly, 2000, 557-560.

[3] Grenander, U., Szegö,G., *Toeplitz Forms and Their Applications*, 2nd edn (New York: Chelsea), 1984.

[4] Hayami, T., Owa, S., *Hankel determinant for $p-$valently starlike and convex functions of orderα*, General Math., 17 (4) (2009), 29-44.

[5] Janteng, A., Halim, S.A., Darus, M., *Hankel determinant for functions starlike and convex with respect to symmetric points*, Journal of Quality Measurement and Analysis, 2 (1) (2006), 37-43.

[6] Janteng, A., Halim, S.A., Darus, M., *Coefficient inequality for a function whose derivative has a positive real part*, Journal of Inequalities in Pure and Applied Mathematics, 7 (2) (2006), 1–5.

[7] Janteng, A., Halim, S.A., Darus, M., *Hankel determinant for starlike and convex functions*, Int. J. Math. Anal., 13 (1) (2007), 619-625.

[8] Krishna, D.V., Ramreddy, T., *Hankel determinant for $p-$valent starlike and convex functions of order α*, Novi Sad J. Math., 42 (2) (2012), 89-96.

[9] Krishna, D.V., Ramreddy, T., *Coefficient inequality for certain subclass of p-valent functions*, Pales. Poly. Univ., 2(2015), 223-228.

[10] Vamshee Krishna, D., Venkateswarlu, B., Ramreddy, T., *Third Hankel determinant for certain subclass of $p-$valent functions*, Complex Variables and Elliptic Equations 60. 9(2015), 1301-1307.

[11] Libera, R.J., Złotkiewicz, E.J., *Early coefficients of the inverse of a regular convex function*, Proceedings of the American Mathematical Society 85. 2(1982), 225-230.

[12] Libera, R.J., Złotkiewicz, E.J., *Coefficient bounds for the inverse of a function with derivative in P*, Proceedings of the American Mathematical Society 87. 2(1983), 251-257.

[13] MacGregor, T.H., *Functions whose derivative has a positive real part*, Transactions of the American Mathematical Society, 104 (3) (1962), 532–537.

[14] Noonan, J.W., Thomas, D.K., *On the second Hankel determinant of areally mean $p-$valent functions*, Transactions of the American Mathematical Society, 223 (1976), 337-346.

[15] Noor, K.I., *Hankel determinant problem for the class of functions with bounded boundary rotation*, Revue Roumaine de Mathématiques Pures et Appliquées, 28 (8) (1983), 731-739.

[16] Pommerenke, C., *On the coefficients and Hankel determinants of univalent functions*, Journal of the London Mathematical Society, 1 (1) (1966), 111-122.

[17] Pommerenke, C., *On the Hankel determinants of univalent functions*, Mathematika, 14 (1) (1967), 108-112.

[18] Pommerenke, C., Jensen, G., *Univalent functions*, Vol. 25, Göttingen: Vandenhoeck und Ruprecht, 1975.

[19] Simon, B., *Orthogonal polynomials on the unit circle*, Part 1, vol. 54 of American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, RI, 2005.

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FILTER THEORY OF PSEUDO HOOP-ALGEBRAS

S.Z. Alavi

Department of Mathematics
Islamic Azad University Central Tehran Branch
Tehran
Iran
e-mail: szalavi@yahoo.com

R.A. Borzooei

M. Aaly Kologani

Department of Mathematics
Shahid Beheshti University
Tehran
Iran
e-mails: borzooei@sbu.ac.ir
mona4011@gmail.com

Abstract. In this paper, by considering the notion of pseudo hoop-algebras, which introduced by G. Georgescu et al. in [11] under the name of residuated integral monoids, and pseudo MV-algebras, pseudo Wajsberg-algebras and pseudo-BL algebras arise as particular cases of them, we introduce the notions of some types of filters ((positive) implicative filters, fantastic filters, associative filters) in pseudo hoop-algebras and to investigate their properties. Several characterizations of (positive) implicative, fantastic and associative filters are derived. Finally, the relations among these filters are investigated.

Keywords: pseudo hoop-algebras, filter, (positive) implicative filter, fantastic filter, associative filter.

1. Introduction

In [13], G. Georgescu, L. Leustean and V. Preoteasa presented pseudo hoops which were originally introduced by Bosbach in ([3], [4]) under the name *residuated integral monoids*. The prefix "pseudo" stands for non-commutative or not necessarily commutative type of algebra. It followed naturally after the introduction of pseudo-MV algebras ([10], [11]), pseudo-Wajsberg algebras ([6], [7]) and pseudo-BL algebras ([12], [9], [8]). All the above are non-commutative generalizations of algebras for many-valued logics. Pseudo-hoops are weaker structures, and pseudo-MV, pseudo-Wajsberg, and pseudo-BL algebras arise as particular cases of them. Pseudo hoops are monoids endowed with orders. Moreover, the orders are canonical (actually inverse canonical) they are given by divisibility relations w.r.t.the monoid operation and the orders have residuals.

Now, in this paper, we study some types of filter (implicative, positive implicative, fantastic, associative filters) and investigate definitions that are equivalent to those and we get the relation between them.

2. Preliminaries

In this section, we recollect some definitions and results which will be used in this paper.

Definition 2.1. [13] A *pseudo hoop* is an algebra $(A, \wedge, \odot, \rightarrow, \rightsquigarrow, 1)$ of type $(2, 2, 2, 2, 0)$ such that, for all $x, y, z \in A$,
(PH-1) $(A, \wedge, 1)$ is a $\wedge$-semilattice,
(PH-2) $(A, \odot, 1)$ is a monoid with unit 1,
(PH-3) $x \odot y \leq z$ if and only if $x \leq y \rightarrow z$ if and only if $y \leq x \rightsquigarrow z$,
(PH-4) $(x \rightarrow y) \odot x = x \odot (x \rightsquigarrow y)$. (Divisibility condition)

A pseudo hoop can be thought of as an algebra $(A, \odot, \rightarrow, \rightsquigarrow, 1)$, where for all $x, y \in A$, $x \wedge y = x \odot (x \rightsquigarrow y)$. Then, it is easy to see that $(A, \wedge, 1)$ is a $\wedge$-semilattice.

On Pseudo hoop A, we define $x \leq y$ if and only if $x \rightarrow y = 1$ (this is equivalent to $x \rightsquigarrow y = 1$). Then $\leq$ is a partial order relation on A. If $\odot$ is commutative (equivalent $\rightarrow = \rightsquigarrow$), A is said to be a *hoop*. We say that a Pseudo hoop A is *bounded* if it has an element $0 \in A$ such that $0 \leq x$, for all $x \in A$. We let $x^0 = 1, x^n = x^{n-1} \odot x$, for any $n \in \mathbb{N}$. We define two unary operations, $x^- = x \rightarrow 0$ and $x^\sim = x \rightsquigarrow 0$, for all $x \in A$. If $(x^-)^\sim = (x^\sim)^- = x$, for all $x \in A$, then the bounded Pseudo hoop A is said to have the *pseudo double negation property*, (PDN) for short. (See [13])

Definition 2.2. [14] A *lattice-ordered group* or *ℓ-group* is a group which is also a lattice that satisfies the identities $x(y \wedge z)t = xyt \wedge xzt$ and $x(y \vee z)t = xyt \vee xzt$. Throughout we write $x \leq y$ instead of $x \vee y = y$ or $x \wedge y = x$, and ℓ-group as an abbreviation for lattice-ordered group.

Example 2.3. [13] Let $(G, +, -, 0, \wedge, \vee)$ be an arbitrary ℓ-group. For an arbitrary element $0 \leq u \in G$ define the following operations, on the set $[0, u]$,

$$x \odot y = (x - u + y) \vee 0, \quad x \rightarrow y = (y - x + u) \wedge u \quad and \quad x \rightsquigarrow y = (u - x + y) \wedge u$$

for any $x, y \in [0, u]$. By routine calculation, we can see that $([0, u], \odot, \rightarrow, \rightsquigarrow, u)$ is a bounded Pseudo hoop.

Example 2.4. [13] Let $G = (G, +, -, 0, \vee, \wedge)$ be an arbitrary ℓ-group and $N(G)$ be the negative cone of G, that is $N(G) = a \in G \mid a \leq 0$. On $N(G)$ we define the following operations:

$$a \odot b = a + b, \quad a \rightarrow b = (b - a) \wedge 0 \quad and \quad a \rightsquigarrow b = (-a + b) \wedge 0$$

Then $N(G) = (N(G), \odot, \to, \rightsquigarrow, 0)$ is an unbounded pseudo hoop. The following proposition provide some properties of pseudo hoops.

The following proposition provide some properties of pseudo hoops.

Proposition 2.5. [3],[4] *Let $(A, \odot, \to, \rightsquigarrow, 1)$ be a Pseudo hoop. Then the following conditions hold, for all $x, y, z, a \in A$,*

(i) $x \leq y$ *if and only if* $x \to y = 1$ *if and only if* $x \rightsquigarrow y = 1$,

(ii) $x \rightsquigarrow (y \to z) = y \to (x \rightsquigarrow z)$, $x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z)$,

(iii) $((x \to y) \rightsquigarrow y) \to y = x \to y$, $((x \rightsquigarrow y) \to y) \rightsquigarrow y = x \rightsquigarrow y$,

(iv) $x \leq y \to x$, $x \leq y \rightsquigarrow x$,

(v) $1 \to x = x$, $1 \rightsquigarrow x = x$,

(vi) $x \to 1 = 1$, $x \rightsquigarrow 1 = 1$,

(vii) $x \odot (x \rightsquigarrow y) \leq y$, $(x \to y) \odot x \leq y$,

(viii) $x \leq (x \to y) \rightsquigarrow y$, $x \leq (x \rightsquigarrow y) \to y$,

(ix) $x \to y \leq (y \to z) \rightsquigarrow (x \to z)$, $x \rightsquigarrow y \leq (y \rightsquigarrow z) \to (x \rightsquigarrow z)$,

(x) $(y \to z) \odot (x \to y) \leq x \to z$, $(x \rightsquigarrow y) \odot (y \rightsquigarrow z) \leq x \rightsquigarrow z$,

(xi) $x \leq y$ *implies* $x \odot z \leq y \odot z$, $z \odot x \leq z \odot y$,

(xii) $x \leq y$ *implies* $y \to z \leq x \to z$, $y \rightsquigarrow z \leq x \rightsquigarrow z$,

(xiii) $x \leq y$ *implies* $z \to x \leq z \to y$, $z \rightsquigarrow x \leq z \rightsquigarrow y$,

(xiv) $z \to x \leq (y \to z) \to (y \to x)$, $z \rightsquigarrow x \leq (y \rightsquigarrow z) \rightsquigarrow (y \rightsquigarrow x)$.

Theorem 2.6. [13] *Let $A = (A, \odot, \to, \rightsquigarrow, 1)$ be a Pseudo hoop. Then the following identities hold, for all $x, y, z \in A$,*

(i) $x \to x = x \rightsquigarrow x = 1$,

(ii) $(x \odot y) \to z = x \to (y \to z)$,

(iii) $(x \odot y) \rightsquigarrow z = y \rightsquigarrow (x \rightsquigarrow z)$.

Proposition 2.7. [3],[4] *Let A be a bounded Pseudo hoop. Then $x^- \leq x \to y$ and $x^\sim \leq x \rightsquigarrow y$, for all $x, y \in A$.*

Proposition 2.8. [13] *Let A be a Pseudo hoop and for any $x, y \in A$, we define, $x \sqcup_1 y = ((x \to y) \rightsquigarrow y) \wedge ((y \to x) \rightsquigarrow x)$ and $x \sqcup_2 y = ((x \rightsquigarrow y) \to y) \wedge ((y \rightsquigarrow x) \to x)$. Then the following conditions are equivalent:*

(i) $\sqcup_1$ *and* $\sqcup_2$ *are associative,*

(ii) $x \leq y$ *implies* $x \sqcup z \leq y \sqcup z$, *for all* $x, y, z \in A$ *and* $\sqcup \in \{\sqcup_1, \sqcup_2\}$,

(iii) $x \sqcup (y \wedge z) \leq (x \sqcup y) \wedge (x \sqcup y)$, *for all* $x, y, z \in A$ *and* $\sqcup \in \{\sqcup_1, \sqcup_2\}$,

(iv) $\sqcup_1$ *and* $\sqcup_2$ *are the pseudo join operation on* A.

Definition 2.9. [13] A Pseudo hoop A is called a $\sqcup$-*pseudo hoop*, if $\sqcup$ is a pseudo join operation on A, where $\sqcup \in \{\sqcup_1, \sqcup_2\}$

Proposition 2.10. [13] *Let A be a $\sqcup$-pseudo hoop, for $\sqcup \in \{\sqcup_1, \sqcup_2\}$. Then $(x \sqcup y) \rightsquigarrow z = (x \rightsquigarrow z) \wedge (y \rightsquigarrow z)$, for all $x, y, z \in A$.*

Definition 2.11. [13] Let A be a Pseudo hoop. A non-empty subset F of A is a *filter* if it satisfies,
(F1) $x, y \in F$ implies $x \odot y \in F$,
(F2) $x \in F$ and $x \leq y$ imply $y \in F$, for any $x, y \in A$.

A filter F of A is *proper* if and only if $F \neq A$.

Proposition 2.12. [13] *Let A be a Pseudo hoop and F be a non-empty subset of A such that $1 \in F$. Then the following statements are equivalent, for any $x, y, z \in A$,*

(i) *F is a filter,*
(ii) *if $x, x \to y \in F$, then $y \in F$,*
(iii) *if $x, x \rightsquigarrow y \in F$, then $y \in F$.*

Definition 2.13. [13] Let A be a Pseudo hoop and F be a filter F. Then F is called *normal* if $x \to y \in F$ if and only if $x \rightsquigarrow y \in F$.

Note. From now one in this paper, we let $A = (A, \odot, \to, \rightsquigarrow, 1)$ be a pseudo hoop, unless otherwise state.

3. (Positive) Implicative filters

In this section, we introduce the notions of implicative and positive implicative filters in pseudo hoops and investigate some properties of them and the relation between them.

Definition 3.1. Let F be a non-empty subset of A. Then F is called an *implicative filter* of A if,
(IF1) $1 \in F$,
(IF2) $x \to ((y \to z) \rightsquigarrow y) \in F$ and $x \in F$ imply $y \in F$, for any $x, y, z \in A$,
(IF3) $x \rightsquigarrow ((y \rightsquigarrow z) \to y) \in F$ and $x \in F$ imply $y \in F$, for any $x, y, z \in A$.

Example 3.2. Let $A = \{0, a, b, c, d, 1\}$. We define $\odot$, $\to$ and $\rightsquigarrow$ on A as follows:

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	d	0	d	a
b	0	d	c	c	0	b
c	0	0	c	c	0	c
d	0	d	0	0	0	d
1	0	a	b	c	d	1

$\to=\rightsquigarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	b	c	b	1
b	d	a	1	b	a	1
c	a	a	1	1	a	1
d	b	1	1	b	1	1
1	0	a	b	c	d	1

Routine calculations show that A is a pseudo hoop and $F = \{b, c, 1\}$ is an implicative filter.

Proposition 3.3. *Every implicative filter of A is a filter.*

Proof. Let F be an implicative filter and $x, x \to y \in F$, for $x, y \in A$. By Proposition 2.5(vi) and (v), $x \to ((y \to 1) \rightsquigarrow y) = x \to y \in F$. Since $x \in F$ and F is an implicative filter, by (IF2), $y \in F$. Therefore, F is a filter. ∎

Remark 3.4. The converse of Proposition 3.3 may not be true. In Example 3.2, $F = \{a, 1\}$ is a filter, but is not an implicative filter of A, because, $1 \to ((b \to c) \rightsquigarrow b) = 1 \in F$, but $b \notin F$.

Theorem 3.5. *Let A be bounded and F be a filter of A. Then, for all $x, y \in A$, the following conditions are equivalent,*

(i) *F is an implicative filter,*

(ii) *if $(x \to y) \rightsquigarrow x \in F$, then $x \in F$ and if $(x \rightsquigarrow y) \to x \in F$, then $x \in F$,*

(iii) *$((x \rightsquigarrow y) \to x) \to x \in F$ and $((x \to y) \rightsquigarrow x) \rightsquigarrow x \in F$,*

(iv) *$(x^- \rightsquigarrow x) \rightsquigarrow x \in F$ and $(x^\sim \to x) \to x \in F$,*

(v) *if $(z^- \odot x) \rightsquigarrow y \in F$ and $y \rightsquigarrow z \in F$, then $x \rightsquigarrow z \in F$ and if $(x \odot z^\sim) \to y \in F$ and $y \to z \in F$, then $z \to x \in F$,*

(vi) *if F is a normal filter and $(y^- \odot x) \rightsquigarrow y \in F$, then $x \rightsquigarrow y \in F$ and if $(x \odot y^\sim) \to y \in F$, then $y \to x \in F$.*

Proof. (i)⇒(ii) Let F be an implicative filter and $(x \to y) \rightsquigarrow x \in F$. By Proposition 2.5(v), $1 \to ((x \to y) \rightsquigarrow x) = (x \to y) \rightsquigarrow x \in F$. Since $1 \in F$ and F is an implicative filter, by (IF2), $x \in F$. The proof of other case is similar.

(ii)⇒(i) Suppose that (ii) holds. Let $x \to ((y \to z) \rightsquigarrow y) \in F$ and $x \in F$. Since F is a filter, by Proposition 2.12(ii), $(y \to z) \rightsquigarrow y \in F$. Now, by (ii), $y \in F$. Therefore, F is an implicative filter.

(iii)⇒(i) Let $x \to ((y \to z) \rightsquigarrow y) \in F$ and $x \in F$. Since F is a filter, by Proposition 2.12(ii), $(y \to z) \rightsquigarrow y \in F$ and so by (iii), $((y \to z) \rightsquigarrow y) \rightsquigarrow y \in F$. Thus, by Proposition 2.12(iii), $y \in F$. Therefore, F is an implicative filter.

(iii)⇒(iv) Since A is bounded, it is enough to take $y = 0$ in (iii). Then $((x \rightsquigarrow 0) \to x) \to x \in F$. Hence, $(x^\sim \to x) \to x \in F$, for all $x \in A$. The proof of other case is similar.

(iv)⇒(iii) Since A is bounded, for any $y \in A$, $0 \le y$. By Proposition 2.5(xiii), for any $x \in A$, $x \to 0 \le x \to y$, and so $x^- \le x \to y$. Now, by Proposition 2.5(xii), $(x \to y) \rightsquigarrow x \le x^- \rightsquigarrow x$, then $(x^- \rightsquigarrow x) \rightsquigarrow x \le ((x \to y) \rightsquigarrow x) \rightsquigarrow x$. Since F is a filter and by assumption $(x^- \rightsquigarrow x) \rightsquigarrow x \in F$, by (F2), $((x \to y) \rightsquigarrow x) \rightsquigarrow x \in F$, for all $x, y \in A$. The proof of other case is similar.

(v)⇒(vi) Suppose that F is a normal filter and $(y^- \odot x) \rightsquigarrow y \in F$, for $x, y \in A$. Then by Theorem 2.6(iii) and (i), $x \rightsquigarrow (y^- \rightsquigarrow y) \in F$ and $y \rightsquigarrow y = 1 \in F$. Then by (v), $x \rightsquigarrow y \in F$. The proof of other case is similar.

(vi)⇒(v) Suppose that $(x \odot z^\sim) \to y \in F$ and $y \to z \in F$. Since F is a filter, by (F1), $(y \to z) \odot ((x \odot z^\sim) \to y) \in F$. By Proposition 2.5(x),

$(y \to z) \odot ((x \odot z^{\sim}) \to y) \leq (x \odot z^{\sim}) \to z$. Since $(y \to z) \odot ((x \odot z^{\sim}) \to y) \in F$, by (F2), $(x \odot z^{\sim}) \to z \in F$. Hence, by (vi), $z \to x \in F$. The proof of other case is similar.

(vi)$\Rightarrow$(iv) By Theorem 2.6(iii),(i), $(x^{-} \odot (x^{-} \rightsquigarrow x)) \rightsquigarrow x = 1 \in F$. Then by (vi), $(x^{-} \rightsquigarrow x) \rightsquigarrow x \in F$. Therefore, (iv) holds.

(i)$\Rightarrow$(vi) Let F be an implicative filter. By Proposition 2.5(iv), $y \leq x \to y$. Then by Proposition 2.5(xii), $(x \to y) \to 0 \leq y \to 0$, thus by Proposition 2.5(xii), $(y^{-} \rightsquigarrow y) \leq (x \to y)^{-} \rightsquigarrow y$. Now, by Proposition 2.5(xiii), $x \rightsquigarrow (y^{-} \rightsquigarrow y) \leq x \rightsquigarrow ((x \to y)^{-} \rightsquigarrow y)$. Hence, by Theorem 2.6(iii), $(y^{-} \odot x) \rightsquigarrow y \leq ((x \to y)^{-} \odot x) \rightsquigarrow y$. Since $(y^{-} \odot x) \rightsquigarrow y \in F$, by (F2), $((x \to y)^{-} \odot x) \rightsquigarrow y \in F$. Then $x \rightsquigarrow ((x \to y)^{-} \rightsquigarrow y) \in F$. Since F is a normal filter, we get $x \to ((x \to y)^{-} \rightsquigarrow y) \in F$. By Proposition 2.5(ii), $(x \to y)^{-} \rightsquigarrow (x \to y) \in F$, thus by Proposition 2.5(v), $1 \to ((x \to y)^{-} \rightsquigarrow (x \to y)) \in F$. Since F is an implicative filter and $1 \in F$, by (IF2), $x \to y \in F$, and so $x \rightsquigarrow y \in F$. ∎

Corollary 3.6. *Let A be bounded and F be an implicative filter of A. Then $x^{\sim -} \to x \in F$ and $x^{-\sim} \rightsquigarrow x \in F$, for all $x \in A$.*

Proof. Let F be an implicative filter of A. Since A is a bounded Pseudo hoop, we have $0 \leq x$, for all $x \in A$. Then by Proposition 2.5(xiii), $x^{-} \rightsquigarrow 0 \leq x^{-} \rightsquigarrow x$, and so by Proposition 2.5(i), $x^{-\sim} \rightsquigarrow (x^{-} \rightsquigarrow x) = 1$. Thus, by (IF1), $x^{-\sim} \rightsquigarrow (x^{-} \rightsquigarrow x) = 1 \in F$ and by (PH-3) $(x^{-} \odot x^{-\sim}) \rightsquigarrow x \in F$. Hence, by Theorem 3.5(vi), $x^{-\sim} \rightsquigarrow x \in F$. By the similar way, we get $x^{\sim -} \to x \in F$. ∎

Proposition 3.7. *Let A be bounded and F and G be two filters of A such that $F \subseteq G$. If F is a normal and implicative filter, then G is an implicative filter, too.*

Proof. Let F and G be two filters of bounded Pseudo hoop A such that $F \subseteq G$ and F be an implicative filter. By Theorem 3.5, it suffices to prove that (vi) holds. Suppose that $(y^{-} \odot x) \rightsquigarrow y \in G$, for $x, y \in A$. Let $u = (y^{-} \odot x) \rightsquigarrow y$. By Theorem 2.6(i), $u \to u = 1$, and so $u \to ((y^{-} \odot x) \rightsquigarrow y) = 1$, then by Proposition 2.5(ii), $(y^{-} \odot x) \rightsquigarrow (u \to y) = 1$. Since F is a filter and $1 \in F$, then $(y^{-} \odot x) \rightsquigarrow (u \to y) \in F$. By Proposition 2.5(iv), $y \leq u \to y$, thus, by Proposition 2.5(xii), $(u \to y) \to 0 \leq y \to 0$, then by Proposition 2.5(xi), $(u \to y)^{-} \odot x \leq y^{-} \odot x$. Again, by Proposition 2.5(xii), $(y^{-} \odot x) \rightsquigarrow (u \to y) \leq ((u \to y)^{-} \odot x) \rightsquigarrow (u \to y)$. Since $(y^{-} \odot x) \rightsquigarrow (u \to y) \in F$, by (F2), $((u \to y)^{-} \odot x) \rightsquigarrow (u \to y) \in F$. Since F is an implicative filter, by Theorem 3.5(vi), $x \rightsquigarrow (u \to y) \in F$. Then by Proposition 2.5(ii), $u \to (x \rightsquigarrow y) \in F$, and so $u \to (x \rightsquigarrow y) \in G$. Since $u \in G$, by Proposition 2.12(ii), $x \rightsquigarrow y \in G$. Therefore, G is an implicative filter. ∎

Theorem 3.8. *Let A be a bounded $\sqcup$-pseudo hoop and F be a filter of A. If $x \sqcup x^{-} \in F$, for any $x \in A$ and $\sqcup \in \{\sqcup_1, \sqcup_2\}$, then F is an implicative filter.*

Proof. Let $x \sqcup x^{-} \in F$. Then by Proposition 2.10, $(x \sqcup x^{-}) \rightsquigarrow x = (x \rightsquigarrow x) \wedge (x^{-} \rightsquigarrow x)$. Hence by Theorem 2.6(i), $(x \sqcup x^{-}) \rightsquigarrow x = x^{-} \rightsquigarrow x$. Also, by Proposition 2.7, $x^{-} \leq x \to y$. Then $(x \to y) \rightsquigarrow x \leq x^{-} \rightsquigarrow x$. Since $(x \to y)$

$\rightsquigarrow x \in F$, by (F2), $x^- \rightsquigarrow x \in F$. Since $(x \sqcup x^-) \rightsquigarrow x = x^- \rightsquigarrow x$, we have $(x \sqcup x^-) \rightsquigarrow x \in F$. Moreover, $x \sqcup x^- \in F$, thus, by Proposition 2.12(iii), $x \in F$. Then by Theorem 3.5(ii), F is an implicative filter. The proof of other case is similar. Therefore, F is an implicative filter. ∎

Definition 3.9. A non-empty subset F of A is called a *positive implicative filter* of A if,
(PIF1) $1 \in F$,
(PIF2) $x \to (y \to z) \in F$ and $x \rightsquigarrow y \in F$ imply $x \to z \in F$, for any $x, y, z \in A$,
(PIF3) $x \rightsquigarrow (y \rightsquigarrow z) \in F$ and $x \to y \in F$ imply $x \rightsquigarrow z \in F$, for any $x, y, z \in A$.

Example 3.10. Let $A = \{0, a, b, c, 1\}$. We define $\odot$, $\to$ and $\rightsquigarrow$ on A as follows:

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	b	a	b
c	0	a	a	c	c
1	0	a	b	c	1

$\to = \rightsquigarrow$	0	a	b	c	1
0	0	1	1	1	1
a	0	1	1	1	1
b	0	c	1	c	1
c	0	b	b	1	1
1	1	a	b	c	1

Routine calculations show that A is a pseudo hoop and $F = \{b, 1\}$ is a positive implicative filter.

Proposition 3.11. *Every positive implicative filter is a filter.*

Proof. Suppose that F is a positive implicative filter. Then by Proposition 2.12(iii), it is enough to prove that if $x, x \rightsquigarrow y \in F$, then $y \in F$, for any $x, y \in A$. By Proposition 2.5(v), $1 \rightsquigarrow (x \rightsquigarrow y) \in F$ and $1 \to x = x \in F$. Since F is a positive implicative filter, we have $1 \rightsquigarrow y = y \in F$. Therefore, F is a filter. ∎

Proposition 3.12. *Let F be a positive implicative filter of A. Then, for all $x, y \in A$ the following statements are hold:*

(i) *if $x \to (x \to y) \in F$, then $x \to y \in F$,*

(ii) *if $x \rightsquigarrow (x \rightsquigarrow y) \in F$, then $x \rightsquigarrow y \in F$,*

(iii) $x \to x^2 \in F$,

(iv) $x \rightsquigarrow x^2 \in F$.

Proof. We prove (ii) and (iii), the proofs of (i) and (iv) are similar.

(ii) Since F is a positive implicative filter, by Proposition 3.11, F is a filter. Now, let $x \rightsquigarrow (x \rightsquigarrow y) \in F$, for $x, y \in A$. Then, by Theorem 2.6(i), $x \to x = 1 \in F$, thus, by (PIF3), $x \rightsquigarrow y \in F$.

(iii) Since F is a filter, by Theorem 2.6(i), for any $x \in A$, $(x \odot x) \to (x \odot x) = 1 \in F$. Then by Theorem 2.6(iii), $x \to (x \to (x \odot x)) \in F$ and $x \rightsquigarrow x \in F$. Hence, by (PIF2), $x \to (x \odot x) \in F$. ∎

Theorem 3.13. *Let A be a chain and F be a positive implicative filter of A. Then F is an implicative filter if and only if $(x \to y) \rightsquigarrow y \in F$ implies $(y \to x) \rightsquigarrow x \in F$, for any $x, y \in A$.*

Proof. ($\Leftarrow$) Let $(x \to y) \rightsquigarrow x \in F$, for $x, y \in A$. Since A is a chain, $x \leq y$ or $y \leq x$. If $x \leq y$, then $x \to y = 1$. Thus, by Proposition 2.5(v), $x = 1 \rightsquigarrow x = (x \to y) \rightsquigarrow x \in F$. Now, let $y \leq x$. Since $x \leq (x \to y) \rightsquigarrow y$, by Proposition 2.5(xiii), $(x \to y) \rightsquigarrow x \leq (x \to y) \rightsquigarrow ((x \to y) \rightsquigarrow y)$. By Proposition 3.11, F is a filter and $(x \to y) \rightsquigarrow x \in F$. Then $(x \to y) \rightsquigarrow ((x \to y) \rightsquigarrow y) \in F$. Since $(x \to y) \rightsquigarrow (x \to y) = 1 \in F$ and F is a positive implicative filter, by (PIF3), $(x \to y) \rightsquigarrow y \in F$. By assumption, $(y \to x) \rightsquigarrow x \in F$. Since $y \leq x$, we have $y \to x = 1$. Then $x = 1 \rightsquigarrow x = (y \to x) \rightsquigarrow x \in F$. By the similar way, if $(x \rightsquigarrow y) \to x \in F$, then $x \in F$. Hence, by Theorem 3.5(ii), F is an implicative filter of A.

($\Rightarrow$) Let F be an implicative filter and $(x \to y) \rightsquigarrow y \in F$, for $x, y \in A$. By Proposition 2.5(viii), $y \leq (y \to x) \rightsquigarrow x$. Then by Proposition 2.5(xiii), $(x \to y) \rightsquigarrow y \leq (x \to y) \rightsquigarrow ((y \to x) \rightsquigarrow x)$. Since $(x \to y) \rightsquigarrow y \in F$, by (F2), $(x \to y) \rightsquigarrow ((y \to x) \rightsquigarrow x) \in F$. Also, since $x \leq (y \to x) \rightsquigarrow x$, we have $((y \to x) \rightsquigarrow x) \to y \leq x \to y$. Hence, $(x \to y) \rightsquigarrow ((y \to x) \rightsquigarrow x) \leq (((y \to x) \rightsquigarrow x) \to y) \rightsquigarrow ((y \to x) \rightsquigarrow x)$, by Proposition 3.11 and (F2), $(((y \to x) \rightsquigarrow x) \to y) \rightsquigarrow ((y \to x) \rightsquigarrow x) \in F$. Since F is an implicative filter, by Theorem 3.5(ii), $(y \to x) \rightsquigarrow x \in F$. ∎

Theorem 3.14. *Let F be a positive implicative normal filter of A such that $(x \to y) \rightsquigarrow y \in F$ implies $(y \to x) \rightsquigarrow x \in F$, for any $x, y \in A$. Then F is an implicative filter.*

Proof. Let $(x \to y) \rightsquigarrow x \in F$, for any $x, y \in A$. By Proposition 2.5(viii), $x \leq (x \to y) \rightsquigarrow y$. Then by Proposition 2.5(xiii), $(x \to y) \rightsquigarrow x \leq (x \to y) \rightsquigarrow ((x \to y) \rightsquigarrow y)$. Since $(x \to y) \rightsquigarrow x \in F$, by Proposition 3.11, F is a filter and so we have, $(x \to y) \rightsquigarrow ((x \to y) \rightsquigarrow y) \in F$. Since F is a positive implicative filter, by Theorem 3.12(ii), $(x \to y) \rightsquigarrow y \in F$, and so by assumption, $(y \to x) \rightsquigarrow x \in F$. Also, by Proposition 2.5(iv), $y \leq x \to y$. Then, by Proposition 2.5(xii), $(x \to y) \rightsquigarrow x \leq y \rightsquigarrow x$, thus by (F2), $y \rightsquigarrow x \in F$. Since F is a normal filter, $y \to x \in F$. Then, by Proposition 2.12(ii), $x \in F$. By the similar way, if $(x \rightsquigarrow y) \to x \in F$, we get $x \in F$. Therefore, F is an implicative filter. ∎

Proposition 3.15. *Let F be a normal filter of A and*

(i) *if $x \rightsquigarrow (x \rightsquigarrow y) \in F$, then $x \rightsquigarrow y \in F$, for any $x, y \in A$.*

(ii) *if $x \to (x \to y) \in F$, then $x \to y \in F$, for any $x, y \in A$.*

Then F is a positive implicative filter of A.

Proof. Suppose that (i) holds. Let $x \rightsquigarrow (y \rightsquigarrow z) \in F$ and $x \to y \in F$, for $x, y \in A$. Since F is a normal filter and $x \rightsquigarrow (y \rightsquigarrow z) \in F$, we have $x \to (y \rightsquigarrow z) \in F$. By Proposition 2.5(ii), $y \rightsquigarrow (x \to z) \in F$. Since F is a normal filter, we have $y \to (x \to z) \in F$ and $x \to y \in F$. By (F1), $(y \to (x \to z)) \odot (x \to y) \in F$. Then by Proposition 2.5(x), $(y \to (x \to z)) \odot (x \to y) \leq x \to (x \to z)$, then by (F2), $x \to (x \to z) \in F$. Also, by (ii), $x \to z \in F$. Since F is a normal filter, we get $x \rightsquigarrow z \in F$. Then we have (PIF3). The proof of (PIF2) is similar. Therefore, F is a positive implicative filter. ∎

Definition 3.16. Let A be bounded and $x \in A$. If there exists a smallest positive integer number $n \in \mathbb{N}$ such that $x^n = 0$, then we say that the order of x is n and we denote by $ord(x) = n$. We say $ord(x) = \infty$, if no such n exists.

Example 3.17. In Example 3.2, $ord(a) = ord(b) = ord(c) = ord(1) = \infty$ and $ord(d) = 2$.

Proposition 3.18. *Let A be bounded such that for any $x \in A$, $ord(x) = 2$. If F is a positive implicative filter, then $x^-, x^\sim \in F$.*

Proof. By Theorem 3.12, The proof is clear. ∎

Theorem 3.19. *Let F and G be two filters of A such that $F \subseteq G$. If F is a positive implicative filter of A, then G is a positive implicative filter of A, too.*

Proof. Let F and G be two filters of Pseudo hoop A such that $F \subseteq G$ and for any $x, y \in A$, $x \rightsquigarrow (x \rightsquigarrow y) \in G$. Let $u = x \rightsquigarrow (x \rightsquigarrow y) \in G$. Then by Theorem 2.6(iii), $u = (x \odot x) \rightsquigarrow y \in G$ and by Theorem 2.6(i), $u \to ((x \odot x) \rightsquigarrow y) = 1 \in F$. Thus, by Proposition 2.5(ii), $(x \odot x) \rightsquigarrow (u \to y) \in F$. Since F is a positive implicative filter and $x \to x = 1 \in F$, by (PIF3), $x \rightsquigarrow (u \to y) \in F$. By Proposition 2.5(ii), $u \to (x \rightsquigarrow y) \in F$. Since $F \subseteq G$, $u \in G$, and by Proposition 2.12(ii), $x \rightsquigarrow y \in G$. ∎

Theorem 3.20. *Let A be bounded and F be an implicative filter of A.*

(i) *If $x^{-\sim} \in F$ or $x^{\sim -} \in F$, for any $x \in A$, then F is a positive implicative filter,*

(ii) *If for any $x \in A$, $x^- \rightsquigarrow x \in F$ or $x^\sim \to x \in F$, then F is a positive implicative filter.*

Proof. (i) Let F be an implicative filter and $x^{-\sim} \in F$, for any $x \in A$. Suppose that $x \to (y \to z) \in F$ and $x \rightsquigarrow y \in F$, for $x, y \in A$. Since $z^{-\sim} \in F$, we have $(z \to 0) \rightsquigarrow 0 \in F$. So by Theorem 3.13, $(0 \to z) \rightsquigarrow z \in F$. Then by Proposition 2.5(i), (v), $1 \to z = z \in F$. And by similar way $x \in F$. Thus, $x \to z \in F$. Therefore, F is a positive implicative filter. The proof of other case is similar.

(ii) Let F be an implicative filter of A and $x^- \rightsquigarrow x \in F$, for any $x \in A$. Suppose that, $x \to (y \to z) \in F$ and $x \rightsquigarrow y \in F$, for $x, y \in A$. Since F is an implicative filter, by Theorem 3.5(iv), $(x^- \rightsquigarrow x) \rightsquigarrow x \in F$, for any $x \in A$. Since $x^- \rightsquigarrow x \in F$, by Proposition 2.12(ii), $x \in F$. Since $x \rightsquigarrow y \in F$ and $x \in F$, we get $y \in F$. Now, y, $y \to z \in F$, then $z \in F$. Since, $z \leq x \to z$, we get $x \to z \in F$. Therefore, F is a positive implicative filter. ∎

4. Fantastic and associative filters

In this section, we introduce the notions of fantastic and associative filters in pseudo hoops and investigate their properties of them and we study the relation between them.

Definition 4.1. Let F be a non-empty subset of A. Then F is called a *fantastic filter* A if it satisfies the following properties,
(FF1) $1 \in F$,
(FF2) $z \to (x \to y) \in F$ and $z \in F$ imply $((y \to x) \rightsquigarrow x) \to y \in F$, for any $x, y, z \in A$,
(FF3) $z \rightsquigarrow (x \rightsquigarrow y) \in F$ and $z \in F$ imply $((y \rightsquigarrow x) \to x) \rightsquigarrow y \in F$, for any $x, y, z \in A$.

Example 4.2. According to Example 2.3, $([0, u], \odot, \to, \rightsquigarrow, u)$ is a bounded Pseudo hoop. By [13], Let K be a normal convex ℓ-group of G and $F{=}\{a \in [o, u] \mid u{-}a \in k\}$ is a normal filter of $[0, u]$. suppose that $z \to (x \to y) \in F$ and $z \in F$ then $((y \to x) \rightsquigarrow x) \to y = (((x-y+u) \wedge u) \rightsquigarrow x) \to y = ((u-((x-y+u) \wedge u)+x) \wedge u) \to y = (((u{-}u{+}y{-}x{+}x) \vee (u{-}u{+}x)) \wedge u) \to y = (y \vee x) \to y = (y{-}(y \vee x){+}u) \wedge u = u \in F$. thus we have (FF2). The similar way (FF3), holds. Therefore, F is a fantastic filter.

Proposition 4.3. *Every fantastic filter of A is a filter.*

Proof. Let F be a fantastic filter and $x, x \to y \in F$. By Proposition 2.5(v), $x \to y = x \to (1 \to y) \in F$. Since $x \in F$ and F is a fantastic filter, by (FF2), $((y \to 1) \rightsquigarrow 1) \to y \in F$. Then, by Proposition 2.5(v) and (vi), $y \in F$. Therefore, F is a filter. ∎

Proposition 4.4. *Let F be a filter of A. Then F is a fantastic filter if and only if $x \to y \in F$ implies $((y \to x) \rightsquigarrow x) \to y \in F$, for any $x, y \in A$.*

Proof. ($\Leftarrow$) Let F be a fantastic filter and $y \to x \in F$, for any $x, y \in A$. Then by Proposition 2.5(v), $y \to x = 1 \to (y \to x) \in F$. Since F is a filter, $1 \in F$, and by (FF2), $((y \to x) \rightsquigarrow x) \to y \in F$.

($\Rightarrow$) Since F is a filter, (FF1) holds. Now, let $z \to (y \to x) \in F$ and $z \in F$, for any $x, y \in A$. Since F is a filter, by Proposition 2.12(ii), $y \to x \in F$. Then, by assumption, $((x \to y) \rightsquigarrow y) \to x \in F$. Therefore, F is a fantastic filter. ∎

Proposition 4.5. *Let F and G be two filters of A such that $F \subseteq G$. If F is a fantastic filter, then G is a fantastic filter, too.*

Proof. Let F and G be two filters of A such that $F \subseteq G$. Suppose that $y \to x \in G$, for any $x, y \in A$. By Theorem 2.6(i), $(y \to x) \rightsquigarrow (y \to x) = 1$. Then by Proposition 2.5(ii), $y \to ((y \to x) \rightsquigarrow x) = 1$ and so $y \to ((y \to x) \rightsquigarrow x) = 1 \in F$. Since F is a fantastic filter and $y \to ((y \to x) \rightsquigarrow x) \in F$, we have $((((y \to x) \rightsquigarrow x) \to y) \rightsquigarrow y) \to ((y \to x) \rightsquigarrow x) = (y \to x) \rightsquigarrow (((((y \to x) \rightsquigarrow x) \to y) \rightsquigarrow y) \to x) \in F$. Since $F \subseteq G$, $(y \to x) \rightsquigarrow (((((y \to x) \rightsquigarrow x) \to y) \rightsquigarrow y) \to x) \in G$.

Also, since $y \to x \in G$, by Proposition 2.12(ii), $(((((y \to x) \rightsquigarrow x) \to y) \rightsquigarrow y) \to x) \in G$. Let $\alpha = (((((y \to x) \rightsquigarrow x) \to y) \rightsquigarrow y) \to x)$. Now, by Proposition 2.5(xii),(vi),(ii), and Theorem 2.6(i), we get

$$\begin{aligned} \alpha \rightsquigarrow (((x \to y) \rightsquigarrow y) \to x) &\geq ((x \to y) \rightsquigarrow y) \to ((((y \to x) \rightsquigarrow x) \to y) \rightsquigarrow y) \\ &\geq (((y \to x) \rightsquigarrow x) \to y) \rightsquigarrow (x \to y) \\ &\geq x \to ((y \to x) \rightsquigarrow x) \\ &= (y \to x) \rightsquigarrow (x \to x) \\ &= 1 \end{aligned}$$

By assumption, G is a filter, then $1 \in G$ and by (F2), $\alpha \rightsquigarrow (((x \to y) \rightsquigarrow y) \to x) \in G$. Since $\alpha \in G$, by Proposition 2.12(ii), $((x \to y) \rightsquigarrow y) \to x \in G$. Therefore, G is a fantastic filter. ∎

Theorem 4.6. *Let A be bounded. If F is an implicative filter, then F is a fantastic filter.*

Proof. Let F be an implicative filter and $y \rightsquigarrow x \in F$. By Proposition 2.7, $x^- \leq x \to y$. Then $(x \to y) \rightsquigarrow y \leq x^- \rightsquigarrow y$ and so $((x \to y) \rightsquigarrow y) \rightsquigarrow (x^- \rightsquigarrow y) = 1$. By Proposition 3.3, F is a filter, and so $((x \to y) \rightsquigarrow y) \rightsquigarrow (x^- \rightsquigarrow y) \in F$, then by Theorem 2.6(iii), $(x^- \odot ((x \to y) \rightsquigarrow y)) \rightsquigarrow y \in F$. Also, by assumption, $y \rightsquigarrow x \in F$. Since F is an implicative filter, by Theorem 3.5(v), $((x \rightsquigarrow y) \rightsquigarrow y) \rightsquigarrow x \in F$. Therefore, F is a fantastic filter. ∎

Theorem 4.7. *Let F be a fantastic and positive implicative filter of A. Then F is an implicative filter.*

Proof. Let F be a fantastic and positive implicative filter and $(x \to y) \rightsquigarrow x \in F$, for $x, y \in A$. Since F is a fantastic filter, by (FF2), $((x \to (x \to y)) \rightsquigarrow (x \to y)) \to x \in F$. Then by Theorem 2.6(ii), $((x^2 \to y) \rightsquigarrow (x \to y)) \to x \in F$. By Proposition 2.5(ix), $x \to x^2 \leq (x^2 \to y) \rightsquigarrow (x \to y)$. Since F is a positive implicative filter, by Theorem 3.12(iii), $x \to x^2 \in F$, and so by (F2), $(x^2 \to y) \rightsquigarrow (x \to y) \in F$. Since $((x^2 \to y) \rightsquigarrow (x \to y)) \to x \in F$, then by Proposition 2.12(ii), $x \in F$. Hence, by Theorem 3.5(ii), F is an implicative filter. ∎

Definition 4.8. Let F be a non-empty subset of A. Then F is called an *associative filter* of A if, for any $x, y, z \in A$;
(AF1) $1 \in F$,
(AF2) $x \to (y \to z) \in F, x \rightsquigarrow y \in F$ imply $z \in F$,
(AF3) $x \rightsquigarrow (y \rightsquigarrow z) \in F, x \rightsquigarrow y \in F$ imply $z \in F$.

Example 4.9. Let $A = \{0, a, b, c, d, 1\}$. We define $\odot, \to$ and $\rightsquigarrow$ on A as follows:

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	b	a	d	b	a
b	0	a	b	0	0	b
c	0	d	0	c	d	c
d	0	b	0	d	0	d
1	0	a	b	c	c	1

$\to = \rightsquigarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	a	c	c	1
b	c	1	1	c	c	1
c	b	a	b	1	a	1
d	a	1	a	1	1	1
1	0	a	b	c	d	1

Then, A with these operations is a pseudo hoop. Let $F = \{1, a, b\}$. Then routine calculations show that F is an associative filter of A.

Theorem 4.10. *Let F be an associative filter of A. Then*

(i) *F is a filter,*
(ii) *F is an implicative filter,*
(iii) *F is a positive implicative filter,*
(iv) *F is a fantastic filter, if A is bounded.*

Proof. (i) Suppose that F is an associative filter and $x, x \to y \in F$, for $x, y \in A$. Then by Proposition 2.5(v), $1 \to (x \to y) = x \to y \in F$ and $1 \rightsquigarrow x = x \in F$. Since F is an associative filter, we have $y \in F$. Therefore, F is a filter.

(ii) Let F be an associative filter and $(x \to y) \rightsquigarrow x \in F$. Then by Proposition 2.5(v),(iii), $(x \to y) \rightsquigarrow (1 \rightsquigarrow x) \in F$ and $(x \to y) \to 1 = 1 \in F$. Since F is associative filter, by (AF3), $x \in F$. Therefore, F is an implicative filter.

(iii) Let F be an associative filter such that $x \to (y \to z) \in F$ and $x \rightsquigarrow y \in F$. Then by (AF2), $z \in F$. Since F is a filter and by Proposition 2.5(iv), $z \leq x \to z$, and so $x \to z \in F$. Therefore, F is a Positive implicative filter.

(iv) By (ii) and 4.6, the proof is clear. ∎

Example 4.11. (i) Let G be a Pseudo hoop similar as Example 2.3. We show that $\{u\}$ is a filter of $G[u]$. $u \in \{u\}$ so (F1), holds. If $x, y \in \{u\}$, then $x \odot y = u \odot u = (u - u + u) \vee 0 = u \in \{u\}$. This is clear that (F2), holds, and so $\{u\}$ is a filter. Let $x = z \neq u$ and $y = u$. We obtained, $z \to (u \to z) = u \in \{u\}, z \rightsquigarrow u = u \in \{u\}$. Since $\{u\}$ is an associative filter, by (AF2), $z \in \{u\}$, which is a contradiction. Therefore, F is not an associative filter.

(ii) Let A be a Pseudo hoop similar as Example 3.2. Then $F = \{b, c, 1\}$ is an implicative filter but it is not an associative filter, because, $a \to (b \to a)$ $= a \to a = 1 \in F$ and $a \to b = b \in F$, but $a \notin F$.

(iii) In Example 3.2, $F = \{b, c, 1\}$, is a positive implicative filter, but it is not an associative filter.

(iv) In Example 3.2, $F = \{b, c, 1\}$, is a fantastic filter, but it is not an associative filter.

Proposition 4.12. *Let A be bounded and F be an associative filter of A. Then $x^-, x^\sim \in F$, for any $x \in A$.*

Proof. Let F be an associative filter of A. By Proposition 2.5(iv),(v), $(x \to 0) \to (1 \to (x \to 0)) = 1 \in F$ and $(x \to 0) \rightsquigarrow 1 = 1 \in F$. Since F is an associative filter, by (AF2), $x^- \in F$. By similar way, we can see that $x^\sim \in F$. ∎

Theorem 4.13. *Let A be bounded and F be an implicative filter of A. If one of following conditions holds, then F is an associative filter.*

(i) *if $x^- \rightsquigarrow x \in F$, for any $x \in A$.*
(ii) *if $x^\sim \to x \in F$, for any $x \in A$.*

Proof. (i) Let F be an implicative filter and $x \to (y \to z) \in F$ and $x \rightsquigarrow y \in F$. Since F is an implicative filter, by Theorem 3.5(iv), $(x^- \rightsquigarrow x) \rightsquigarrow x \in F$, for any $x \in A$. Since $x^- \rightsquigarrow x \in F$, then by Proposition 2.12(iii), $x \in F$. Since $x \to (y \to z) \in F$ and $x \rightsquigarrow y \in F$, we get $y \to z \in F$, and $y \in F$, then $z \in F$. Therefore, F is an associative filter.

(ii) The proof of (ii) is similar. ∎

Corollary 4.14. *Let A is bounded. If F is a fantastic and positive implicative filter of A and $x^- \rightsquigarrow x \in F$ or $x^\sim \to x \in F$, for $x \in A$, then F is an associative filter.*

Proof. By Theorems 4.7 and 4.13, the proof is clear. ∎

Theorem 4.15. *Let A be bounded and F be an implicative filter of A. If one of following conditions holds, then F is an associative filter.*

(i) *if $x^{-\sim} \in F$, for any $x \in A$,*

(ii) *if $x^{\sim -} \in F$, for any $x \in A$.*

Proof. (i) Let F be an implicative filter and $x^{-\sim} \in F$, for any $x \in A$. Suppose that $x \to (y \to z) \in F$ and $x \rightsquigarrow y \in F$. Since $z^{-\sim} \in F$, we have $(z \to 0) \rightsquigarrow 0 \in F$. So by Theorem 3.13, $(0 \to z) \rightsquigarrow z \in F$. Then by Proposition 2.5(i),(v), $1 \rightsquigarrow z = z \in F$. Therefore, (AF2) holds. The proof of (AF3), is similar.

(ii) Let F be an implicative filter and $x \to (y \to z) \in F$ and $x \rightsquigarrow y \in F$. Since F is an implicative filter, by Corollary 3.6, $x^{\sim -} \to x \in F$. Since $x^{\sim -} \in F$, then by Proposition 2.12(ii), $x \in F$. Now, since $x \to (y \to z) \in F$ and $x \in F$, we get $y \to z$. Since $y \to z \in F$ and $y \in F$, by Proposition 2.12(ii), $z \in F$. Therefore, (AF2) holds. By similar way (AF3), holds. ∎

Proposition 4.16. *Let A be bounded and F and G be two filters A such that $F \subseteq G$ and $x^- \rightsquigarrow x \in F$, for any $x \in A$. If F is an associative filter, then G is an associative filter, too.*

Proof. By Proposition 3.7 and Theorems 4.10 and 4.13 the proof is clear. ∎

5. Conclusions and future works

The aim of this paper is to introduce the notions of (positive)implicative, fantastic and associative filters in Pseudo hoops and to investigate their properties. Several characterizations of these filters are derived but in this paper we try to investigate the relation between them. I want to examine quotient algebra associated with the filters in the future.

References

[1] Borzooei, R.A., Khosravi Shoar, S., Ameri, R., *Some types of filters in MTL-algebras*, Fuzzy Sets and Systems, 187 (2012), 92-102.

[2] Borumand Saeid, A., Rezaei, A., Borzooei, R.A., *Some types of filters in BE-algebras*, Mathematics in Computer Science, 7 (3) (2013), 341-352.

[3] Bosbach, B., *Komplementäre Halbgruppen, Axiomatik und Arithmetik*, Fundamenta Mathematicae, 3 (64) (1969), 257-287.

[4] Bosbach, B., *Komplementäre Halbgruppen, Kongruenzen und Quotienten*, Fundamenta Mathematicae, 1 (69) (1970), 1-14.

[5] Büchi, J.R., Owens, T.M., *Complemented monoids and hoops*, unpublished manuscript, 1975.

[6] Ceterchi, R., *On Algebras with Implication, Categorically Equivalent to Pseudo-MV Algebras*, The Proceedings of the Fourth International Simposium on Economic Imformatics, Bucharest, Romania, 1999, 911-916.

[7] Ceterchi, R., *Pseudo-Wajsberg algebras*, Multiple-Valued Logic, 6 (2001), 67-88.

[8] Di Nola, A., Georgescu, G., Iorgulescu, A., *Pseudo-BL algebras: Part II*, Multiple-Valued Logic, 8 (5-6) (2002), 717-750.

[9] Di Nola, A., Georgescu, G., Iorgulescu, A., *Pseudo-BL algebras: Part I*, Multiple-Valued Logic, 8 (5-6) (2002), 673-716.

[10] Georgescu, G., Iorgulescu, A., *Pseudo-MV algebras: a non-commutative extension of MV algebras*, The Proceedings of the Fourth International Symposium on Economic Informatics, Bucharest, Romania, 1999, 961-968.

[11] Georgescu, G., Iorgulescu, A., *Pseudo MV-algebras*, Multiple-Valued Logic, 6 (2001), 95-135.

[12] Georgescu, G., Iorgulescu, A., *Pseudo-BL algebras: a non-commutative extension of BL-algebras*, Fifth International Conference FSTA, 2000, 90-92.

[13] Georgescu, G., Leustean, L., Preoteasa, V., *Pseudo-hoops*, Journal of Multiple-Valued logic and Soft Computing, 11 (1-2) (2005), 153-184.

[14] Glass, A.M. ., *Sublattice subgroups of finitely presented lattice-ordered groups*, Journal of Algebra, 301 (2) (2006), 509-530.

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WEAK CLOSURE OPERATIONS WITH SPECIAL TYPES IN LOWER BCK-SEMILATTICES

Dapeng Yu

Hashem Bordbar[1]

Faculty of Mathematics
Statistics and Computer Science
Shahid Bahonar University
Kerman
Iran
e-mail: bordbar.amirh@gmail.com

Mohammad Mehdi Zahedi

Department of Mathematics
Graduate University of Advanced Technology
Mahan-Kerman
Iran
e-mail: zahedi_mm@kgut.ac.ir

Young Bae Jun

Department of Mathematics Education (and RINS)
Gyeongsang National University
Jinju 52828
Korea
e-mail: skywine@gmail.com

Abstract. The notions of (strong) quasi prime mapping on the set of all ideals, t-type weak closure operation, and tender (resp., naive, sheer, feeble tender) weak closure operation are introduced, and their relations and properties are investigated. Conditions for a weak closure operation to be of t-type are provided. Given a weak closure operation, conditions for the new weak closure operation to be of t-type and to be a naive (sheer, feeble tender) weak closure operation are considered. We show that the new weak closure operation is the smallest tender weak closure operation containing the given weak closure operation.

Keywords: (strong) quasi prime mapping, t-type weak closure operation, naive (sheer, tender, feeble tender) weak closure operation.

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[1]Corresponding author

1. Introduction

In [4], Bordbar et al. introduced a weak closure operation, which is more general form than closure operation, on ideals of BCK-algebras. Bordbar and Zahedi [2], [3] studied a finite type closure operations and semi-prime closure operations on BCK-algebras. Regarding weak closure operation "cl", they defined another weak closure operation "cl_t" in [1].

In this paper, we introduce the notions of (strong) quasi prime mapping on the set of all ideals, t-type weak closure operation, and tender (resp., naive, sheer, feeble tender) weak closure operation, and investigates their relations and properties. We provide conditions for a weak closure operation to be of t-type.

We consider conditions for "cl_t" to be a t-type weak closure operation.

We discuss conditions for "cl_t" to be a naive (sheer, feeble tender) weak closure operation.

We show that "cl_t" is the smallest tender weak closure operation containing the weak closure operation "cl".

2. Preliminaries

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki and was extensively investigated by several researchers.

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a *BCI-algebra* if it satisfies the following conditions

(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,

(II) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,

(III) $(\forall x \in X)\ (x * x = 0)$,

(IV) $(\forall x, y \in X)\ (x * y = 0,\ y * x = 0 \ \Rightarrow\ x = y)$.

If a BCI-algebra X satisfies the following identity

(V) $(\forall x \in X)\ (0 * x = 0)$,

then X is called a *BCK-algebra*.

A BCK-algebra X is called a *lower BCK-semilattice* (see [8]) if X is a lower semilattice with respect to the BCK-order.

A subset A of a BCK/BCI-algebra X is called an *ideal* of X (see [8]) if it satisfies

$$0 \in A, \tag{2.1}$$

$$(\forall x \in X)\,(\forall y \in A)\,(x * y \in A \ \Rightarrow\ x \in A)\,. \tag{2.2}$$

Note that every ideal A of a BCK/BCI-algebra X satisfies the following implication (see [8]).

$$(\forall x, y \in X)\,(x \leq y,\ y \in A \ \Rightarrow\ x \in A)\,. \tag{2.3}$$

For any subset A of X, the ideal generated by A is defined to be the intersection of all ideals of X containing A, and it is denoted by $\langle A\rangle$. If A is finite, then we say that $\langle A\rangle$ is *finitely generated ideal* of X (see [8]).

Let $\mathcal{I}(X)$ and $\mathcal{I}_f(X)$ be the set of all ideals of X and the set of all finitely generated ideals of X, respectively.

We refer the reader to the books [7], [8] for further information regarding BCK/BCI-algebras.

3. t-type weak closure operations

In what follows, let X be a lower BCK-semilattice unless otherwise specified.

Definition 3.1. [4] An element x of X is called a *zeromeet element* of X if the condition

$$(\exists\, y \in X \setminus \{0\})\,(x \wedge y = 0)$$

is valid. Otherwise, x is called a *non-zeromeet element* of X.

Denote by $Z(X)$ the set of all zeromeet elements of X, that is,

$$Z(X) = \{x \in X \mid x \wedge y = 0 \text{ for some nonzero element } y \in X\}.$$

Obviously, $0 \in Z(X)$ and if X has the greatest element 1, then $1 \in X \setminus Z(X)$.

Lemma 3.2. [4] *For any $x, y \in X$, if $x, y \notin Z(X)$, then $x \wedge y \notin Z(X)$, that is, the set $X \setminus Z(X)$ is closed under the operation $\wedge$.*

Definition 3.3. [6] For any nonempty subsets A and B of X, we denote

$$A \wedge B := \langle\{a \wedge b \mid a \in A, b \in B\}\rangle$$

which is called the *meet ideal* of X generated by A and B. In this case, we say that the operation "$\wedge$" is a *meet operation.* If $A = \{a\}$, then $\{a\} \wedge B$ is denoted by $a \wedge B$. Also, if $B = \{b\}$, then $A \wedge \{b\}$ is denoted by $A \wedge b$.

Definition 3.4. [5] For any nonempty subsets A and B of X, we define a set

$$(A :_\wedge B) := \{x \in X \mid x \wedge B \subseteq A\}$$

which is called the *relative annihilator* of B with respect to A.

For a nonempty subset B of X, consider the following condition:

$$(\forall x, y \in X)(\forall b \in B)\,((x \wedge b) * (y \wedge b) \leq (x * y) \wedge b)\,. \tag{3.1}$$

Lemma 3.5. [5] *If A and B are ideals of X, then the relative annihilator $(A :_\wedge B)$ of B with respect to A is an ideal of X.*

Lemma 3.6. [5] *If A is an ideal of X, then $(A :_\wedge X) = A$ and $(A :_\wedge A) = X$.*

Definition 3.7. [4] A mapping $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ is called a *weak closure operation* on $\mathcal{I}(X)$ if the following conditions are valid.

$$(\forall A \in \mathcal{I}(X))\,(A \subseteq cl(A))\,, \tag{3.2}$$

$$(\forall A, B \in \mathcal{I}(X))\,(A \subseteq B \;\Rightarrow\; cl(A) \subseteq cl(B))\,. \tag{3.3}$$

If a weak closure operation $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ satisfies the condition

$$(\forall A \in \mathcal{I}(X))\,(cl(cl(A)) = cl(A))\,, \tag{3.4}$$

then we say that "cl" is a closure operation on $\mathcal{I}(X)$ (see [2]). In what follows, we use A^{cl} instead of $cl(A)$.

For any mapping $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ and every ideal A of X, let

$$K := \cup\{((a \wedge A)^{cl} :_\wedge \langle b\rangle) \mid a, b \in X \setminus Z(X)\}. \tag{3.5}$$

Then the mapping

$$cl^* : \mathcal{I}(X) \to \mathcal{I}(X),\; A \mapsto \langle K\rangle \tag{3.6}$$

is not a weak closure operation on $\mathcal{I}(X)$ as seen in the following example.

Example 3.8. Consider a lower BCK-semilattice $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	4	4	4	0

There are six ideals: $A_0 = \{0\}$, $A_1 = \{0, 1\}$, $A_2 = \{0, 2\}$, $A_3 = \{0, 1, 2\}$, $A_4 = \{0, 1, 2, 3\}$ and $A_5 = X$.

Define a mapping $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ by $A_0^{cl} = A_1$, $A_1^{cl} = A_0$, $A_2^{cl} = A_1$, $A_3^{cl} = A_1$, $A_4^{cl} = A_2$ and $A_5^{cl} = A_3$. Then "cl" is not a weak closure operation on $\mathcal{I}(X)$ because $A_3 \nsubseteq A_1 = A_3^{cl}$.

Note that $Z(X) = \{0, 1, 2\}$. For non-zeromeet elements $3, 4$ of X, we have
$((3 \wedge A_2)^{cl} :_\wedge \langle 3\rangle) = (A_2^{cl} :_\wedge A_4) = (A_1 :_\wedge A_4) = A_1$,
$((3 \wedge A_2)^{cl} :_\wedge \langle 4\rangle) = (A_2^{cl} :_\wedge A_5) = (A_1 :_\wedge A_5) = A_1$,
$((4 \wedge A_2)^{cl} :_\wedge \langle 3\rangle) = (A_2^{cl} :_\wedge A_4) = (A_1 :_\wedge A_4) = A_1$,
$((4 \wedge A_2)^{cl} :_\wedge \langle 4\rangle) = (A_2^{cl} :_\wedge A_5) = (A_1 :_\wedge A_5) = A_1$.

It follows that

$$A_2^{cl^*} = \langle \cup\{((a \wedge A_2)^{cl} :_\wedge \langle b\rangle) \mid a, b \in X \setminus Z(X)\}\rangle = \langle A_1\rangle = A_1 \nsupseteq A_2$$

which shows that "cl^*" is not a weak closure operation on $\mathcal{I}(X)$.

If "cl" is a weak closure operation on $\mathcal{I}(X)$, then K in (3.5) is an ideal of X containing A^{cl} (see [1, Theorem 3.28]).

Assume that X has the greatest element 1. For a weak closure operation "cl" on $\mathcal{I}(X)$, we define a new function

$$cl_t : \mathcal{I}(X) \to \mathcal{I}(X),\ A \mapsto \cup\{((a \wedge A)^{cl} :_{\wedge} \langle b \rangle) \mid a, b \in X \setminus Z(X)\}. \tag{3.7}$$

Then "cl_t" is also a weak closure operation on $\mathcal{I}(X)$ (see [1, Theorem 3.29]).

We investigate relations between "cl" and "cl_t". The following example shows that they are not equal, that is, there exists $A \in \mathcal{I}(X)$ such that $A^{cl} \neq A^{cl_t}$.

Example 3.9. Consider the lower BCK-semilattice $X = \{0, 1, 2, 3, 4\}$ which is given in Example 3.8. Note that the element 4 is the greatest element of X and we have 6 ideals, $A_0 = \{0\}$, $A_1 = \{0, 1\}$, $A_2 = \{0, 2\}$, $A_3 = \{0, 1, 2\}$, $A_4 = \{0, 1, 2, 3\}$ and $A_5 = X$.

Define a mapping $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ by $A_0^{cl} = A_1$, $A_1^{cl} = A_3$, $A_2^{cl} = A_3$, $A_3^{cl} = A_4$, $A_4^{cl} = A_4$ and $A_5^{cl} = A_5$. Then "cl" is a weak closure operation on $\mathcal{I}(X)$.

Note that $Z(X) = \{0, 1, 2\}$. For non-zeromeet element 3 of X, we have

$$((3 \wedge A_3)^{cl} :_{\wedge} \langle 3 \rangle) = (A_3^{cl} :_{\wedge} \{0, 1, 2, 3\}) = (A_4 :_{\wedge} A_4) = X.$$

Thus $A_3^{cl_t} = \cup\{((a \wedge A_3)^{cl} :_{\wedge} \langle b \rangle) \mid a, b \in X \setminus Z(X)\} = X$. Therefore

$$A_3^{cl} = A_4 \neq X = A_3^{cl_t}.$$

Proposition 3.10. *Assume that X has the greatest element 1. If "cl" is a weak closure operation on $\mathcal{I}(X)$, then "cl" is contained in "cl_t", that is, $A^{cl} \subseteq A^{cl_t}$ for all $A \in \mathcal{I}(X)$.*

Proof. Suppose that $x \in A^{cl}$. Since $1 \wedge A = A$ and $\langle 1 \rangle = X$, we have

$$A^{cl} = ((1 \wedge A)^{cl} :_{\wedge} \langle 1 \rangle) \subseteq A^{cl_t}.$$

by Lemma 3.6. Therefore, $x \in A^{cl_t}$ and $A^{cl} \subseteq A^{cl_t}$ for all $A \in \mathcal{I}(X)$. ∎

Definition 3.11. Assume that X has the greatest element 1. A weak closure operation "cl" on $\mathcal{I}(X)$ is said to be of *t-type* if the following assertion is valid.

$$(\forall A \in \mathcal{I}(X)) \left(A^{cl} = A^{cl_t}\right). \tag{3.8}$$

Example 3.12. Consider a lower BCK-semilattice $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	2	1	0	3
4	4	4	4	4	0

The element 4 is the greatest element of X and we have 5 ideals: $A_0 = \{0\}$, $A_1 = \{0,1\}$, $A_2 = \{0,2\}$, $A_3 = \{0,1,2,3\}$ and $A_4 = X$.

Define a mapping $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ by $A_0^{cl} = A_0$, $A_1^{cl} = A_1$, $A_2^{cl} = A_2$, $A_3^{cl} = A_4$ and $A_4^{cl} = A_4$. Then "cl" is a weak closure operation on $\mathcal{I}(X)$.

Note that $Z(X) = \{0,1,2\}$. For non-zeromeet elements 3 and 4 of X, we have $\langle 3\rangle = A_3$ and $\langle 4\rangle = A_4$. Also,

$((3 \wedge A_0)^{cl} :_\wedge \langle 3\rangle) = (A_0^{cl} :_\wedge A_3) = (A_0 :_\wedge A_3) = A_0$.
$((3 \wedge A_0)^{cl} :_\wedge \langle 4\rangle) = (A_0^{cl} :_\wedge A_4) = (A_0 :_\wedge A_4) = A_0$.
$((4 \wedge A_0)^{cl} :_\wedge \langle 3\rangle) = (A_0^{cl} :_\wedge A_3) = (A_0 :_\wedge A_3) = A_0$.
$((4 \wedge A_0)^{cl} :_\wedge \langle 4\rangle) = (A_0^{cl} :_\wedge A_4) = (A_0 :_\wedge A_4) = A_0$.

Hence $A_0^{cl_t} = A_0^{cl}$. Similarly

$((3 \wedge A_1)^{cl} :_\wedge \langle 3\rangle) = (A_1^{cl} :_\wedge A_3) = (A_1 :_\wedge A_3) = A_1$.
$((3 \wedge A_1)^{cl} :_\wedge \langle 4\rangle) = (A_1^{cl} :_\wedge A_4) = (A_1 :_\wedge A_4) = A_1$.
$((4 \wedge A_1)^{cl} :_\wedge \langle 3\rangle) = (A_1^{cl} :_\wedge A_3) = (A_1 :_\wedge A_3) = A_1$.
$((4 \wedge A_1)^{cl} :_\wedge \langle 4\rangle) = (A_1^{cl} :_\wedge A_4) = (A_1 :_\wedge A_4) = A_1$.

Thus $A_1^{cl_t} = A_1^{cl}$. By the similar way, we have

$$A_i^{cl_t} = A_i^{cl},\ i = \{2,3,4\}.$$

Therefore "cl" is a t-type weak closure operation on $\mathcal{I}(X)$.

Given a weak closure operation "cl" on $\mathcal{I}(X)$, we discuss conditions for "cl" to be of t-type.

Theorem 3.13. *Assume that X has the greatest element 1. If the greatest element 1 is the only non-zeromeet element of X, then every weak closure operation on $\mathcal{I}(X)$ is of t-type.*

Proof. Let "cl" be a weak closure operation on $\mathcal{I}(X)$. For any $A \in \mathcal{I}(X)$, we have $1 \wedge A = A$ and $\langle 1\rangle = X$. It follows from Lemma 3.6 that

$$\begin{aligned} A^{cl_t} &= \cup\{((a \wedge A)^{cl} :_\wedge \langle b\rangle) \mid a, b \in X \setminus Z(X)\} \\ &= ((1 \wedge A)^{cl} :_\wedge \langle 1\rangle) = (A^{cl} :_\wedge X) = A^{cl}. \end{aligned}$$

Therefore "cl" is a t-type weak closure operation on $\mathcal{I}(X)$. ■

Definition 3.14. A mapping $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ is said to be

- *quasi-prime* if it satisfies:

$$(3.9) \qquad (\forall a \in X \setminus Z(X))\,(\forall A \in \mathcal{I}(X))\,\left(a \wedge A^{cl} \subseteq (a \wedge A)^{cl}\right).$$

- *strong quasi-prime* if it satisfies:

$$(3.10) \qquad (\forall a \in X \setminus Z(X))\,(\forall A \in \mathcal{I}(X))\,\left(a \wedge A^{cl} = (a \wedge A)^{cl}\right).$$

Example 3.15. Consider a lower BCK-semilattice $X = \{0,1,2,3,4\}(B_{5-1-2})$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	0	0
3	3	3	3	0	3
4	4	4	4	4	0

There are five ideals: $A_0 = \{0\}$, $A_1 = \{0,1,2\}$, $A_2 = \{0,1,2,3\}$, $A_3 = \{0,1,2,4\}$ and $A_4 = X$.

Define a mapping $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ by $A_0^{cl} = A_0$, $A_1^{cl} = A_2$, $A_2^{cl} = A_3$, $A_3^{cl} = A_4$ and $A_4^{cl} = A_4$. It is routine to verify that "cl" is a quasi-prime mapping. But it is not a weak closure operation on $\mathcal{I}(X)$ since $A_2 \nsubseteq A_3 = A_2^{cl}$.

Lemma 3.16. *Every ideal A of X satisfies the following assertion.*

$$(\forall a,b,z \in X)\,(a \wedge b \in A \;\Rightarrow\; a \wedge \langle b \wedge z\rangle \subseteq A)\,. \tag{3.11}$$

Proof. Let $p \in \langle b \wedge z\rangle$. Then $p * (b \wedge z)^n = 0$ for some $n \in \mathbb{N}$. Since $b \wedge z \leq b$, we have $(b \wedge z)^n \leq b$, which implies that

$$p * b \leq p * (b \wedge z)^n = 0.$$

Hence $p * b = 0$, that is, $p \leq b$. It follows that

$$a \wedge p \leq a \wedge b \in A$$

and so that $a \wedge p \in A$. Therefore $a \wedge \langle b \wedge z\rangle \subseteq A$. ∎

Theorem 3.17. *Assume that X has the greatest element 1. If "cl" is a quasi-prime weak closure operation on $\mathcal{I}(X)$, then "cl_t" is a t-type weak closure operation on $\mathcal{I}(X)$.*

Proof. Note that "cl_t" is a weak closure operation on $\mathcal{I}(X)$. Let $x \in A^{cl_t}$. Then $x \in ((a \wedge A)^{cl} :_\wedge \langle b\rangle)$, and so $x \wedge \langle b\rangle \subseteq (a \wedge A)^{cl}$ for some $a, b \in X \setminus Z(X)$ by (3.7). It follows from the quasi-primeness of "cl" that

$$x \wedge c \wedge z \in (a \wedge A)^{cl} \wedge z = z \wedge (a \wedge A)^{cl} \subseteq (z \wedge a \wedge A)^{cl} = (z \wedge (a \wedge A))^{cl}$$

for all $c, z \in X \setminus Z(X)$. Thus $x \wedge z \in ((z \wedge (a \wedge A))^{cl} :_\wedge c)$, and so

$$x \wedge z \in ((z \wedge (a \wedge A))^{cl} :_\wedge \langle b\rangle) \subseteq (a \wedge A)^{cl_t}. \tag{3.12}$$

Now suppose that $w \in X \setminus Z(X)$. Then $z \wedge w \in X \setminus Z(X)$ by Lemma 3.2. Using Lemma 3.16 and (3.12) induces $x \wedge \langle z \wedge w\rangle \subseteq (a \wedge A)^{cl_t}$, and thus

$$x \in ((a \wedge A)^{cl_t} :_\wedge \langle z \wedge w\rangle) \subseteq \cup\{((a \wedge A)^{cl_t} :_\wedge \langle b\rangle) \mid a, b \in X \setminus Z(X)\}.$$

Conversely, suppose that $x \in A^{(cl_t)_t}$. Then $x \in ((a \wedge A)^{cl_t} :_\wedge \langle b\rangle)$ for some $a, b \in X \setminus Z(X)$. Then $x \wedge z \in (a \wedge A)^{cl_t}$ for all $z \in \langle b\rangle$. It follows from (3.7) that there exist $p, q \in X \setminus Z(X)$ such that

$$x \wedge \langle b\rangle \subseteq ((p \wedge (a \wedge A))^{cl} :_\wedge \langle q\rangle).$$

Thus $x \wedge \langle b \wedge q\rangle \subseteq x \wedge \langle b\rangle \wedge \langle q\rangle \subseteq (p \wedge (a \wedge A))^{cl}$, which implies that

$$x \in ((p \wedge a \wedge A)^{cl} :_\wedge \langle b \wedge q\rangle)$$

Since $p \wedge a$ and $b \wedge q$ are elements of $X \setminus Z(X)$ by Lemma 3.2, we conclude that $x \in A^{cl_t}$. Consequently, "cl_t" is a t-type weak closure operation on $\mathcal{I}(X)$. ∎

Corollary 3.18. *Assume that X has the greatest element 1. If "cl" is a strong quasi-prime weak closure operation on $\mathcal{I}(X)$, then "cl_t" is a t-type weak closure operation on $\mathcal{I}(X)$.*

Definition 3.19. A weak closure operation "*cl*" on $\mathcal{I}(X)$ is said to be

• *tender* if for any $A \in \mathcal{I}(X)$ and any non-zeromeet elements a and b of X, the equality

$$((a \wedge A)^{cl} :_\wedge \langle b\rangle) = A^{cl} \tag{3.13}$$

is valid,

• *feeble tender* if for any $A \in \mathcal{I}(X)$ and any non-zeromeet element a of X, the equality

$$((a \wedge A)^{cl} :_\wedge \langle a\rangle) = A^{cl} \tag{3.14}$$

is valid,

• *naive* if for any $A \in \mathcal{I}(X)$ there exist non-zeromeet elements a and b of X such that the equality (3.13) is valid.

• *sheer* if for any $A \in \mathcal{I}(X)$ there exists non-zeromeet element a of X such that the equality (3.14) is valid.

Example 3.20. Consider a lower BCK-semilattice $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	4	4	4	0

X has 6 ideals: $A_0 = \{0\}$, $A_1 = \{0, 1\}$, $A_2 = \{0, 2\}$, $A_3 = \{0, 1, 2\}$, $A_4 = \{0, 1, 2, 3\}$ and $A_5 = X$.

Define a mapping $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ by $A_0^{cl} = A_0$, $A_1^{cl} = A_3$, $A_2^{cl} = A_3$, $A_3^{cl} = A_3$, $A_4^{cl} = X$ and $A_5^{cl} = X$. Then "cl" is a weak closure operation on $\mathcal{I}(X)$.

Note that $Z(X) = \{0, 1, 2\}$. For non-zeromeet elements 3 and 4 of X, we have $\langle 3\rangle = A_4$ and $\langle 4\rangle = X$. Also,

$((3 \wedge A_0)^{cl} :_\wedge \langle 3\rangle) = (A_0^{cl} :_\wedge A_4) = (A_0 :_\wedge A_4) = A_0 = A_0^{cl}$.
$((4 \wedge A_0)^{cl} :_\wedge \langle 4\rangle) = (A_0^{cl} :_\wedge A_5) = (A_0 :_\wedge A_5) = A_0 = A_0^{cl}$.
$((3 \wedge A_1)^{cl} :_\wedge \langle 3\rangle) = (A_1^{cl} :_\wedge A_4) = (A_3 :_\wedge A_4) = A_3 = A_1^{cl}$.
$((4 \wedge A_1)^{cl} :_\wedge \langle 4\rangle) = (A_1^{cl} :_\wedge A_5) = (A_3 :_\wedge A_5) = A_3 = A_1^{cl}$.
$((3 \wedge A_2)^{cl} :_\wedge \langle 3\rangle) = (A_2^{cl} :_\wedge A_4) = (A_3 :_\wedge A_4) = A_3 = A_2^{cl}$.
$((4 \wedge A_2)^{cl} :_\wedge \langle 4\rangle) = (A_2^{cl} :_\wedge A_5) = (A_3 :_\wedge A_5) = A_3 = A_2^{cl}$.
$((3 \wedge A_3)^{cl} :_\wedge \langle 3\rangle) = (A_3^{cl} :_\wedge A_4) = (A_3 :_\wedge A_4) = A_3 = A_3^{cl}$.
$((4 \wedge A_3)^{cl} :_\wedge \langle 4\rangle) = (A_3^{cl} :_\wedge A_5) = (A_3 :_\wedge A_5) = A_3 = A_3^{cl}$.
$((3 \wedge A_4)^{cl} :_\wedge \langle 3\rangle) = (A_4^{cl} :_\wedge A_4) = (A_5 :_\wedge A_4) = A_5 = A_4^{cl}$.
$((4 \wedge A_4)^{cl} :_\wedge \langle 4\rangle) = (A_4^{cl} :_\wedge A_5) = (A_5 :_\wedge A_5) = A_5 = A_4^{cl}$.
$((3 \wedge A_5)^{cl} :_\wedge \langle 3\rangle) = (A_4^{cl} :_\wedge A_4) = (A_5 :_\wedge A_4) = A_5 = A_5^{cl}$.
$((4 \wedge A_5)^{cl} :_\wedge \langle 4\rangle) = (A_5^{cl} :_\wedge A_5) = (A_5 :_\wedge A_5) = A_5 = A_5^{cl}$.

Thus "cl" is a (feeble) tender weak closure operation on $\mathcal{I}(X)$.

Obviously, every tender weak closure operation is a naive weak closure operation, but the converse is not true in general as seen in the following example.

Example 3.21. Consider a lower BCK-semilattice $X = \{0, 1, 2, 3\}$ with the following Cayley table.

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	1	0	1
3	3	3	3	0

We have 3 ideals of X, and they are $A_0 = \{0\}$, $A_1 = \{0, 1, 2\}$ and $A_2 = X$.

Define a mapping $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ by $A_0^{cl} = A_1^{cl} = A_1$ and $A_2^{cl} = A_2$.

We can easily check that "cl" is a naive weak closure operation on $\mathcal{I}(X)$. But, it is not a tender weak closure operation on $\mathcal{I}(X)$. In fact, we know that there are two non-zeromeet elements 2 and 3. Thus

$$((3 \wedge A_1)^{cl} :_\wedge \langle 2\rangle) = (A_1^{cl} :_\wedge \langle 2\rangle) = (A_1 :_\wedge A_1) = X \neq A_1 = A_1^{cl}.$$

Obviously, every feeble tender weak closure operation is a sheer weak closure operation, but the converse is not true in general as seen in the following example.

Example 3.22. Consider a lower BCK-semilattice $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	2	0	0	0
3	3	3	3	0	3
4	4	4	4	4	0

There are six ideals: $A_0 = \{0\}$, $A_1 = \{0, 1\}$, $A_2 = \{0, 1, 2\}$, $A_3 = \{0, 1, 2, 3\}$, $A_4 = \{0, 1, 2, 4\}$ and $A_5 = X$. Define a mapping $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ by $A_0^{cl} = A_0$, $A_1^{cl} = A_4$, $A_2^{cl} = A_4$, $A_3^{cl} = X$, $A_4^{cl} = X$ and $A_5^{cl} = X$. Then "cl" is a weak closure operation on $\mathcal{I}(X)$.

Note that $Z(X) = \{0\}$. For non-zeromeet elements 3 and 4 of X, we have $\langle 3\rangle = A_3$ and $\langle 4\rangle = A_4$. Also,

$((3 \wedge A_0)^{cl} :_\wedge \langle 3\rangle) = (A_0^{cl} :_\wedge A_3) = (A_0 :_\wedge A_3) = A_0 = A_0^{cl}$.
$((3 \wedge A_1)^{cl} :_\wedge \langle 3\rangle) = (A_1^{cl} :_\wedge A_3) = (A_4 :_\wedge A_3) = A_4 = A_1^{cl}$.
$((3 \wedge A_2)^{cl} :_\wedge \langle 3\rangle) = (A_2^{cl} :_\wedge A_3) = (A_4 :_\wedge A_3) = A_4 = A_2^{cl}$.
$((4 \wedge A_3)^{cl} :_\wedge \langle 4\rangle) = (A_2^{cl} :_\wedge A_4) = (A_4 :_\wedge A_4) = X = A_3^{cl}$.
$((4 \wedge A_4)^{cl} :_\wedge \langle 4\rangle) = (A_4^{cl} :_\wedge A_4) = (X :_\wedge A_4) = X = A_4^{cl}$.
$((4 \wedge A_5)^{cl} :_\wedge \langle 4\rangle) = (A_4^{cl} :_\wedge A_4) = (X :_\wedge A_4) = X = A_5^{cl}$.

Thus "cl" is a sheer weak closure operation. But it is not feeble tender since

$$((3 \wedge A_4)^{cl} :_\wedge \langle 3\rangle) = (A_2^{cl} :_\wedge A_3) = (A_4 :_\wedge A_3) = A_4 \neq X = A_4^{cl}.$$

Theorem 3.23. *Assume that X has the greatest element 1. If "cl" is a quasi-prime weak closure operation on $\mathcal{I}(X)$, then "cl_t" is a naive weak closure operation on $\mathcal{I}(X)$.*

Proof. Note that "cl_t" is a weak closure operation on $\mathcal{I}(X)$. Suppose that A is an ideal of X and $x \in A^{cl_t}$. Then there exist $p, q \in X \setminus Z(X)$ such that $x \in ((p \wedge A)^{cl} :_\wedge \langle q\rangle)$. So $x \wedge \langle q\rangle \subseteq (p \wedge A)^{cl}$. Let $a \in X \setminus Z(X)$. Then

$$a \wedge x \wedge \langle q\rangle \subseteq a \wedge (p \wedge A)^{cl} \subseteq (a \wedge p \wedge A)^{cl}$$

by the quasi-primeness of "cl", and thus

$$x \wedge a \in ((p \wedge (a \wedge A))^{cl} :_\wedge \langle q\rangle) \subseteq (a \wedge A)^{cl_t}.$$

It follows from Lemma 3.16 that

$$x \wedge \langle a \wedge b\rangle \subseteq (a \wedge A)^{cl_t}$$

for $b \in X \setminus Z(X)$. Therefore $x \in ((a \wedge A)^{cl_t} :_\wedge \langle a \wedge b\rangle)$ which means that there exist non-zeromeet elements s, t such that $x \in ((s \wedge A)^{cl_t} :_\wedge \langle t\rangle)$.

Conversely, let $x \in ((a \wedge A)^{cl_t} :_\wedge \langle b\rangle)$ for some $a, b \in X \setminus Z(X)$. Then $x \wedge \langle b\rangle \subseteq (a \wedge A)^{cl_t}$, and so there exist $p, q \in X \setminus Z(X)$ such that

$$x \wedge \langle b\rangle \subseteq ((p \wedge (a \wedge A))^{cl} :_\wedge \langle q\rangle).$$

Thus $x \wedge \langle q \wedge b\rangle \subseteq x \wedge \langle q\rangle \wedge \langle b\rangle \subseteq (p \wedge (a \wedge A))^{cl}$, which means that

$$x \in (((p \wedge a) \wedge A)^{cl} :_\wedge \langle q \wedge b\rangle) \subseteq A^{cl_t}.$$

Consequently, "cl_t" is a naive weak closure operation on $\mathcal{I}(X)$. ∎

Corollary 3.24. *Assume that X has the greatest element 1. If "cl" is a strong quasi-prime weak closure operation on $\mathcal{I}(X)$, then "cl_t" is a naive weak closure operation on $\mathcal{I}(X)$.*

Lemma 3.25. [4] *Assume that X has the greatest element 1. If "cl" is a tender weak closure operation on $\mathcal{I}(X)$, then so is the function "cl_t" in* (3.7).

Theorem 3.26. *Assume that X has the greatest element 1. If "cl" is a tender weak closure operation on $\mathcal{I}(X)$, then "cl_t" is the smallest tender weak closure operation on $\mathcal{I}(X)$ such that "cl" is contained in "cl_t", that is, $A^{cl} \subseteq A^{cl_t}$ for all $A \in \mathcal{I}(X)$.*

Proof. By using Proposition 3.10 and Lemma 3.25, "cl_t" is a tender weak closure operation which contains "cl". Now suppose that "cl_1" is a tender weak closure operation which contains "cl". For any $A \in \mathcal{I}(X)$, if $x \in A^{cl_t}$, then $x \in ((p \wedge A)^{cl} :_\wedge \langle q\rangle)$ for some $p, q \in X \setminus Z(X)$. Since $A^{cl} \subseteq A^{cl_1}$ and "cl_1" is a tender weak closure operation, we have

$$x \in ((p \wedge A)^{cl_1} :_\wedge \langle q\rangle) = A^{cl_1},$$

which shows that $A^{cl_t} \subseteq A^{cl_1}$. ∎

Theorem 3.27. *Assume that X has the greatest element 1. If "cl" is a quasi-prime weak closure operation on $\mathcal{I}(X)$, then "cl_t" is a feeble tender weak closure operation on $\mathcal{I}(X)$.*

Proof. Note that "cl_t" is a weak closure operation on $\mathcal{I}(X)$. Suppose that A is an ideal of X and $x \in A^{cl_t}$. Then there exist $p, q \in X \setminus Z(X)$ such that $x \in ((p \wedge A)^{cl} :_\wedge \langle q\rangle)$. So $x \wedge \langle q\rangle \subseteq (p \wedge A)^{cl}$. Let $a \in X \setminus Z(X)$ be an arbitrary element. Using the quasi-primeness of "cl" implies

$$a \wedge x \wedge \langle q\rangle \subseteq a \wedge (p \wedge A)^{cl} \subseteq (a \wedge p \wedge A)^{cl}.$$

Thus $x \wedge a \in ((p \wedge (a \wedge A))^{cl} :_\wedge \langle q\rangle) \subseteq (a \wedge A)^{cl_t}$. It follows from Lemma 3.16 that

$$x \wedge \langle a\rangle = x \wedge \langle a \wedge a\rangle \subseteq (a \wedge A)^{cl_t}$$

and so that $x \in ((a \wedge A)^{cl_t} :_\wedge \langle a\rangle)$ for all $a \in X \setminus Z(X)$

Conversely, let $x \in ((a \wedge A)^{cl_t} :_\wedge \langle a\rangle)$ for $a \in X \setminus Z(X)$. Then $x \wedge z \in (a \wedge A)^{cl_t}$ for every element $z \in \langle a\rangle$. So there exist $p, q \in X \setminus Z(X)$ such that

$$x \wedge z \in ((p \wedge (a \wedge A))^{cl} :_\wedge \langle q\rangle).$$

Hence $x \wedge \langle a\rangle \subseteq ((p \wedge (a \wedge A))^{cl} :_\wedge \langle q\rangle)$, and so $x \wedge \langle q \wedge a\rangle \subseteq x \wedge \langle q\rangle \wedge \langle a\rangle \subseteq (p \wedge (a \wedge A))^{cl}$. Therefore

$$x \in (((p \wedge a) \wedge A)^{cl} :_\wedge \langle q \wedge a\rangle) \subseteq A^{cl_t}.$$

Consequently, "cl_t" is a feeble tender weak closure operation on $\mathcal{I}(X)$. ∎

Corollary 3.28. *Assume that X has the greatest element 1. If "cl" is a quasi-prime weak closure operation on $\mathcal{I}(X)$, then "cl_t" is a sheer weak closure operation on $\mathcal{I}(X)$.*

Corollary 3.29. *Assume that X has the greatest element 1. If "cl" is a strong quasi-prime weak closure operation on $\mathcal{I}(X)$, then "cl_t" is a feeble tender weak closure operation on $\mathcal{I}(X)$ and so a sheer weak closure operation on $\mathcal{I}(X)$.*

References

[1] Bordbar, H., Jun, Y.B., Novak, M., *Tender and naive weak closure operations on lower BCK-semilattices,* Math. Slovaca (submitted).

[2] Bordbar, H., Zahedi, M.M., *A finite type of closure operations on BCK-algebra,* Appl. Math. Inf. Sci. Lett., 4 (2) (2016), 1–9.

[3] Bordbar, H., Zahedi, M.M., *Semi-prime closure operations on BCK-algebra,* Commun. Korean Math. Soc., 30 (5) (2015), 385–402.

[4] Bordbar, H., Zahedi, M.M., Ahn, S.S., Jun, Y.B., *Weak closure operations on ideals of BCK-algebras,* J. Comput. Anal. Appl. (in press).

[5] Bordbar, H., Zahedi, M.M., Jun, Y.B., *Relative annihilators in lower BCK-semilattices,* Demonstratio Mathematica (submitted).

[6] Bordbar, H., Ahn, S.S., Zahedi, M.M., Jun, Y.B., *Semiring structures based on meet and plus ideals in lower BCK-semilattices,* J. Comput. Anal. Appl. (submitted).

[7] Huang, Y., *BCI-algebra,* Science Press, Beijing 2006.

[8] Meng, J., Jun, Y.B., BCK-algebras, Kyung Moon Sa Co., Seoul 1994.

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ON A GROUP OF THE FORM 2^{10}:($U_5(2)$:2)

Ayoub B.M. Basheer[1, 2]
Jamshid Moori[3]

Department of Mathematical Sciences
North-West University (Mafikeng)
P.O.Box X2046, Mmabatho 2735
South Africa
e-mails: ayoubbasheer@gmail.com or ayoub.basheer@nwu.ac.za
jamshid.moori@nwu.ac.za

Abstract. The full automorphism group $U_5(2)$:2 of the special unitary group $U_5(2)$ has a 10-dimensional absolutely irreducible module over $GF(2)$. Hence a split extension of the form $\overline{G} = 2^{10}$:($U_5(2)$:2) does exist. In this paper we first determine the conjugacy classes of $\overline{G}$ using the coset analysis technique. The structures of the inertia factor groups were determined. These are the groups $U_5(2)$:2, 2_{-}^{1+6}:((3^{1+2}:8):2) and $O_5(2)$:2. We then determine the Fischer matrices and apply the Clifford-Fischer theory to compute the ordinary character table of $\overline{G}$. The Fischer matrices $\mathcal{F}_i$ of $\overline{G}$ are all $\mathbb{Z}$-valued, with sizes range between 1 and 5. The full character table of $\overline{G}$, which is 109×109 $\mathbb{C}$-valued matrix is available in the PhD Thesis [1] of the first author, which could be accessed online.

Keywords: Group extensions, unitary group, extra-special p-group, character table, inertia groups, Fischer matrices.
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1. Introduction

Let $U = U_5(2)$ be the special unitary group consisting of 5×5 matrices over $\mathbb{F}_4$ that preserves a non-singular Hermitian form. The outer automorphism of U is 2 (see the $\mathbb{ATLAS}$ [8]) and thus the full automorphism group of U is a group of the form $U_5(2)$:2. We denote this group by G and we note that $|G| =$ 27 371 520. By the electronic Atlas of Wilson [18], we observe that G has a

[1]Corresponding author.
[2]The first author is currently a postdoctoral fellow at the North-West University, Mafikeng campus.
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10-dimensional absolutely irreducible module over $\mathbb{F}_2$, which is the 5-dimentional Hermitian $\mathbb{F}_4$-vector space involved in the definition of $\overline{G}$. Hence a split extension of the form $\overline{G} := 2^{10}{:}(U_5(2){:}2)$ does exist. In this paper our main aims are to fully study this group, to determine its inertia factor groups (and their respective ordinary character tables) and to compute the Fischer matrices. It will turn out that the character table of $\overline{G}$ is a 109×109 complex matrix and it is partitioned into three blocks corresponding to the three inertia factor groups $H_1 = U_5(2){:}2$, $H_2 = 2_{-}^{1+6}{:}((3^{1+2}{:}8){:}2)$ and $H_3 = O_5(2){:}2$ (see Section 3).

Clifford-Fischer Theory provides much more interesting information on the group and on the character table, in particular the character table produced by Clifford-Fischer Theory is in a special format that could not be achieved by direct computations using GAP [10] or Magma [7]. Also providing examples of applications of Clifford-Fischer Theory to both split and non-split extensions is sensible choice, since each group requires individual approach. The readers (particulary young researchers) will highly benefit from the theoretical background required for these computations. GAP and Magma are computational tools and would not replace good powerful and theoretical arguments.

For the notation used in this paper and the description of Clifford-Fischer theory technique, we follow [1], [2], [3], [4], [5], [6].

Using the 10-dimensional matrices over $\mathbb{F}_2$ that generate $G = U_5(2){:}2$, given at the electronic ATLAS of Wilson, together with GAP, we were able to construct the group $\overline{G}$ inside $PSL(11,2)$. The following two elements $\overline{g}_1$ and $\overline{g}_2$ generate $\overline{G}$.

$$\overline{g}_1 = \begin{pmatrix} 1&0&1&1&0&0&0&1&0&0&0\\ 0&1&1&1&0&0&1&1&1&0&0\\ 1&1&1&1&1&1&1&0&0&1&0\\ 1&0&0&1&0&0&1&0&0&0&0\\ 1&1&1&0&1&1&1&1&1&0&0\\ 1&1&1&1&1&0&0&1&1&0&0\\ 1&1&0&0&1&1&1&1&0&0&0\\ 0&0&1&1&0&0&0&1&1&1&0\\ 0&0&1&0&0&0&1&0&1&0&0\\ 1&1&1&1&0&0&0&1&0&0&0\\ 0&1&1&0&1&0&0&0&0&0&1 \end{pmatrix}, \quad \overline{g}_2 = \begin{pmatrix} 1&1&1&1&0&1&1&0&1&0&0\\ 0&1&1&0&0&1&1&1&1&0&0\\ 1&0&1&1&0&0&1&1&1&1&0\\ 0&0&1&0&0&0&1&1&0&0&0\\ 0&0&1&1&0&1&0&0&1&0&0\\ 1&0&1&1&1&1&1&1&0&1&0\\ 0&0&1&1&1&1&1&0&0&0&0\\ 0&1&0&0&0&1&1&0&1&1&0\\ 1&1&0&0&0&1&1&0&0&0&0\\ 1&0&0&0&0&0&1&1&1&1&0\\ 1&0&1&0&0&1&0&0&1&0&1 \end{pmatrix},$$

with $o(\overline{g}_1) = 24,\ o(\overline{g}_2) = 8$ and $o(\overline{g}_1\overline{g}_2) = 20$.

Now having the group $\overline{G}$ constructed in GAP, it is easy to obtain all its normal subgroups. In fact $\overline{G}$ has only two non-trivial proper normal subgroups, one is of order 1024 and the other is of order 14 014 218 240. The normal subgroup of order 1024 is an elementary abelian 2-group and thus is isomorphic to N. Generators $n_1, n_2, \cdots, n_{10}$ of N, in terms of 11-dimensional matrices over $\mathbb{F}_2$ are given in Basheer [1].

In Magma or GAP one can check for the complements of $N = \langle n_1, n_2, \cdots, n_{10}\rangle$ in $\overline{G} = \langle \overline{g}_1, \overline{g}_2\rangle$, where here we obtained only one complement G. The following two elements g_1 and g_2 generate the complement G of N in $\overline{G}$. Note that G is a subgroup of $\overline{G}$ isomorphic to the quotient $\overline{G}/N \cong U_5(2){:}2$ and together with N creates the split extension $\overline{G}$ in consideration.

$$g_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix},$$

with $o(g_1) = 16,\ o(g_2) = 16$ and $o(g_1g_2) = 12$.

For the notations used in this paper and the description of Clifford-Fischer theory technique, we follow Basheer [1] and Basheer and Moori [2], [3].

2. Conjugacy Classes of $\overline{G} = 2^{10}{:}(U_5(2){:}2)$

In this section we use the method of the coset analysis technique (see Basheer [1], Basheer and Moori [2], [3], [5] or Moori [14] and [15] for more details)as we are interested to organize the classes of $\overline{G}$ corresponding to the classes of G. We list the conjugacy classes of $\overline{G}$ in Table 1, where in this table:

- g_i is the i^{th} conjugacy class of G as listed in Table 11.14 of [1].
- g_{ij} is a representative of a conjugacy class of $\overline{G}$ correspond to the class g_i of G.
- k_i is the number of orbits $Q_{i1}, Q_{i2}, \cdots, Q_{ik_i}$ on the action of N on the coset $N\overline{g}_i = Ng_i = 2^{10}g_i$. In particular, the action of N on the identity coset N produces 1024 orbits each consists of singleton. Thus $k_1 = 1024$.
- f_{ij} is the number of orbits fused together under the action of $C_G(g_i)$ on $Q_{i1}, Q_{i2}, \cdots, Q_{ik_i}$. In particular, the action of $C_G(g_1) = G$ on the orbits $Q_{11}, Q_{12}, \cdots, Q_{1k_1}$ affords three orbits of lengths 1, 495 and 528. Thus $f_{11} = 1$, $f_{12} = 495$ and $f_{13} = 528$.
- m_{ij} are weights attached to each class of $\overline{G}$ that will be used later in computing the Fischer matrices of $\overline{G}$. These weights are computed through the formula

$$m_{ij} = [N_{\overline{G}}(N\overline{g}_i) : C_{\overline{G}}(g_{ij})] = |N|\frac{|C_G(g_i)|}{|C_{\overline{G}}(g_{ij})|}. \tag{1}$$

Table 1: The conjugacy classes of $\overline{G} = 2^{10}{:}(U_5(2){:}2)$

$[g_i]_G$	k_i	f_{ij}	m_{ij}	$[g_{ij}]_{\overline{G}}$	$o(g_{ij})$	$\lvert[g_{ij}]_{\overline{G}}\rvert$	$\lvert C_{\overline{G}}(g_{ij})\rvert$
$g_1 = 1A$	$k_1 = 1024$	$f_{11} = 1$	$m_{11} = 1$	g_{11}	1	1	28028436480
		$f_{12} = 495$	$m_{12} = 495$	g_{12}	2	495	56623104
		$f_{13} = 528$	$m_{13} = 528$	g_{13}	2	528	53084160
$g_2 = 2A$	$k_2 = 256$	$f_{21} = 1$	$m_{21} = 4$	g_{21}	2	660	42467328
		$f_{22} = 27$	$m_{22} = 108$	g_{22}	2	17820	1572864
		$f_{23} = 36$	$m_{23} = 144$	g_{23}	2	23760	1179648
		$f_{24} = 192$	$m_{24} = 768$	g_{24}	4	126720	221184
$g_3 = 2B$	$k_3 = 64$	$f_{31} = 1$	$m_{31} = 16$	g_{31}	2	47520	589824
		$f_{32} = 3$	$m_{32} = 48$	g_{32}	2	142560	196608
		$f_{33} = 24$	$m_{33} = 384$	g_{33}	4	1140480	24576
		$f_{34} = 36$	$m_{34} = 576$	g_{34}	4	1710720	16384
$g_4 = 2C$	$k_4 = 32$	$f_{41} = 1$	$m_{41} = 32$	g_{41}	2	608256	46080
		$f_{42} = 1$	$m_{42} = 32$	g_{42}	4	608256	46080
		$f_{43} = 15$	$m_{43} = 480$	g_{43}	4	9123840	3072
		$f_{44} = 15$	$m_{44} = 480$	g_{44}	4	9123840	3072
$g_5 = 3A$	$k_5 = 1$	$f_{51} = 1$	$m_{51} = 1024$	g_{51}	3	360448	77760
$g_6 = 3B$	$k_6 = 64$	$f_{61} = 1$	$m_{61} = 16$	g_{61}	3	112640	248832
		$f_{62} = 27$	$m_{62} = 432$	g_{62}	6	3041280	9216
		$f_{63} = 36$	$m_{63} = 576$	g_{63}	6	4055040	6912
$g_7 = 3C$	$k_7 = 16$	$f_{71} = 1$	$m_{71} = 64$	g_{71}	3	4055040	6912
		$f_{72} = 6$	$m_{72} = 384$	g_{72}	6	24330240	1152
		$f_{73} = 9$	$m_{73} = 576$	g_{73}	6	36495360	768
$g_8 = 3D$	$k_8 = 4$	$f_{81} = 1$	$m_{81} = 256$	g_{81}	3	10813440	2592
		$f_{82} = 3$	$m_{82} = 768$	g_{82}	6	32440320	864
$g_9 = 4A$	$k_9 = 64$	$f_{91} = 1$	$m_{91} = 16$	g_{91}	4	190080	147456
		$f_{92} = 3$	$m_{92} = 3$	g_{92}	4	570240	49152
		$f_{93} = 6$	$m_{93} = 96$	g_{93}	4	1140480	24576
		$f_{94} = 6$	$m_{94} = 96$	g_{94}	4	1140480	24576
		$f_{95} = 48$	$m_{95} = 768$	g_{95}	4	9123840	3072
$g_{10} = 4B$	$k_{10} = 16$	$f_{10,1} = 1$	$m_{10,1} = 64$	$g_{10,1}$	4	2880960	12288
		$f_{10,2} = 1$	$m_{10,2} = 64$	$g_{10,2}$	4	2880960	12288
		$f_{10,3} = 2$	$m_{10,3} = 128$	$g_{10,3}$	4	4561920	6144
		$f_{10,4} = 12$	$m_{10,4} = 768$	$g_{10,4}$	4	27371520	1024
$g_{11} = 4C$	$k_{11} = 16$	$f_{11,1} = 1$	$m_{11,1} = 64$	$g_{11,1}$	4	9123840	3072
		$f_{11,2} = 3$	$m_{11,2} = 192$	$g_{11,2}$	4	27371520	1024
		$f_{11,3} = 12$	$m_{11,3} = 768$	$g_{11,3}$	8	109486080	256
$g_{12} = 4D$	$k_{12} = 8$	$f_{12,1} = 1$	$m_{12,1} = 128$	$g_{12,1}$	4	36495360	768
		$f_{12,2} = 1$	$m_{12,2} = 128$	$g_{12,2}$	4	36495360	768
		$f_{12,3} = 3$	$m_{12,3} = 384$	$g_{12,3}$	8	109486080	256
		$f_{12,4} = 3$	$m_{12,4} = 384$	$g_{12,4}$	8	109486080	256
$g_{13} = 5A$	$k_{13} = 4$	$f_{13,1} = 1$	$m_{13,1} = 256$	$g_{13,1}$	5	233570304	120
		$f_{13,2} = 3$	$m_{13,2} = 768$	$g_{13,2}$	10	700710912	40
$g_{14} = 6A$	$k_{14} = 1$	$f_{14,1} = 1$	$m_{14,1} = 1024$	$g_{14,1}$	6	16220160	1728
$g_{15} = 6B$	$k_{15} = 4$	$f_{15,1} = 1$	$m_{15,1} = 256$	$g_{15,1}$	6	5406720	5184
		$f_{15,2} = 3$	$m_{15,2} = 768$	$g_{15,2}$	12	16220160	1728
$g_{16} = 6C$	$k_{16} = 16$	$f_{16,1} = 1$	$m_{16,1} = 64$	$g_{16,1}$	6	4055040	6912
		$f_{16,2} = 3$	$m_{16,2} = 192$	$g_{16,2}$	6	12165120	2304
		$f_{16,3} = 12$	$m_{16,3} = 768$	$g_{16,3}$	12	48660480	576
$g_{17} = 6D$	$k_{17} = 16$	$f_{17,1} = 1$	$m_{17,1} = 64$	$g_{17,1}$	6	4055040	6912
		$f_{17,2} = 6$	$m_{17,2} = 384$	$g_{17,2}$	6	24330240	1152
		$f_{17,3} = 9$	$m_{17,3} = 576$	$g_{17,3}$	6	36495360	768
$g_{18} = 6E$	$k_{18} = 1$	$f_{18,1} = 1$	$m_{18,1} = 1024$	$g_{18,1}$	6	97320960	288
$g_{19} = 6F$	$k_{19} = 4$	$f_{19,1} = 1$	$m_{19,1} = 256$	$g_{19,1}$	6	48660480	576
		$f_{19,2} = 3$	$m_{19,2} = 768$	$g_{19,2}$	12	145981440	192
$g_{20} = 6G$	$k_{20} = 4$	$f_{20,1} = 1$	$m_{20,1} = 256$	$g_{20,1}$	6	64880640	432
		$f_{20,2} = 3$	$m_{20,2} = 768$	$g_{20,2}$	6	194641920	144
$g_{21} = 6H$	$k_{21} = 4$	$f_{21,1} = 1$	$m_{21,1} = 256$	$g_{21,1}$	6	97320960	288
		$f_{21,2} = 3$	$m_{21,2} = 768$	$g_{21,2}$	6	291962880	96
$g_{22} = 6I$	$k_{22} = 2$	$f_{22,1} = 1$	$m_{22,1} = 512$	$g_{22,1}$	6	389283840	72
		$f_{22,2} = 1$	$m_{22,2} = 512$	$g_{22,2}$	12	389283840	72
$g_{23} = 6J$	$k_{23} = 8$	$f_{23,1} = 1$	$m_{23,1} = 128$	$g_{23,1}$	6	97320960	288
		$f_{23,2} = 1$	$m_{23,2} = 128$	$g_{23,2}$	12	97320960	288
		$f_{23,3} = 3$	$m_{23,3} = 384$	$g_{23,3}$	12	291962880	96
		$f_{23,4} = 3$	$m_{23,4} = 384$	$g_{23,4}$	12	291962880	96

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Table 1 (continued from previous page)

$[g_i]_G$	k_i	f_{ij}	m_{ij}	$[g_{ij}]_{\overline{G}}$	$o(g_{ij})$	$\lvert[g_{ij}]_{\overline{G}}\rvert$	$\lvert C_{\overline{G}}(g_{ij})\rvert$
$g_{24} = 8A$	$k_{24} = 8$	$f_{24,1} = 1$	$m_{24,1} = 128$	$g_{24,1}$	8	18247680	1536
		$f_{24,2} = 3$	$m_{24,2} = 384$	$g_{24,2}$	8	54743040	512
		$f_{24,3} = 4$	$m_{24,3} = 512$	$g_{24,3}$	8	72990720	384
$g_{25} = 8B$	$k_{25} = 8$	$f_{25,1} = 1$	$m_{25,1} = 128$	$g_{25,1}$	8	18247680	1536
		$f_{25,2} = 3$	$m_{25,2} = 384$	$g_{25,2}$	8	54743040	512
		$f_{25,1} = 4$	$m_{25,3} = 512$	$g_{25,3}$	8	72990720	384
$g_{26} = 8C$	$k_{26} = 8$	$f_{26,1} = 1$	$m_{26,1} = 128$	$g_{26,1}$	8	109486080	256
		$f_{26,2} = 1$	$m_{26,2} = 128$	$g_{26,2}$	8	109486080	256
		$f_{26,3} = 2$	$m_{26,3} = 256$	$g_{26,3}$	8	218972160	128
		$f_{26,4} = 2$	$m_{26,4} = 256$	$g_{26,4}$	8	218972160	128
		$f_{26,5} = 2$	$m_{26,5} = 256$	$g_{26,5}$	8	218972160	128
$g_{27} = 8D$	$k_{27} = 4$	$f_{27,1} = 1$	$m_{27,1} = 256$	$g_{27,1}$	8	218972160	128
		$f_{27,2} = 1$	$m_{27,2} = 256$	$g_{27,2}$	8	218972160	128
		$f_{27,3} = 2$	$m_{27,3} = 512$	$g_{27,3}$	8	437944320	64
$g_{28} = 9A$	$k_{28} = 1$	$f_{28,1} = 1$	$m_{28,1} = 1024$	$g_{28,1}$	9	519045120	54
$g_{29} = 9B$	$k_{29} = 4$	$f_{29,1} = 1$	$m_{29,1} = 256$	$g_{29,1}$	9	259522560	108
		$f_{29,2} = 3$	$m_{29,2} = 768$	$g_{29,2}$	18	778567680	36
$g_{30} = 10A$	$k_{30} = 2$	$f_{30,1} = 1$	$m_{30,1} = 512$	$g_{30,1}$	10	1401421824	20
		$f_{30,2} = 1$	$m_{30,2} = 512$	$g_{30,2}$	20	1401421824	20
$g_{31} = 11A$	$k_{31} = 1$	$f_{31,1} = 1$	$m_{31,1} = 1024$	$g_{31,1}$	11	2548039680	11
$g_{32} = 12A$	$k_{32} = 1$	$f_{32,1} = 1$	$m_{32,1} = 1024$	$g_{32,1}$	12	194641920	144
$g_{33} = 12B$	$k_{33} = 4$	$f_{33,1} = 1$	$m_{33,1} = 256$	$g_{33,1}$	12	389283840	288
		$f_{33,2} = 3$	$m_{33,2} = 768$	$g_{33,2}$	12	291962880	96
$g_{34} = 12C$	$k_{34} = 16$	$f_{34,1} = 1$	$m_{34,1} = 64$	$g_{34,1}$	12	24330240	1152
		$f_{34,2} = 3$	$m_{34,2} = 192$	$g_{34,2}$	12	72990720	384
		$f_{34,3} = 3$	$m_{34,3} = 192$	$g_{34,3}$	12	72990720	384
		$f_{34,4} = 3$	$m_{34,4} = 192$	$g_{34,4}$	12	72990720	384
		$f_{34,5} = 6$	$m_{34,5} = 384$	$g_{34,5}$	12	145981440	192
$g_{35} = 12D$	$k_{35} = 1$	$f_{35,1} = 1$	$m_{35,1} = 1024$	$g_{35,1}$	12	1167851520	24
$g_{36} = 12E$	$k_{36} = 4$	$f_{36,1} = 1$	$m_{36,1} = 256$	$g_{36,1}$	12	291962880	96
		$f_{36,2} = 1$	$m_{36,2} = 256$	$g_{36,2}$	12	291962880	96
		$f_{36,3} = 1$	$m_{36,3} = 256$	$g_{36,3}$	12	291962880	96
		$f_{36,4} = 1$	$m_{36,4} = 256$	$g_{36,4}$	12	291962880	96
$g_{37} = 12F$	$k_{37} = 2$	$f_{37,1} = 1$	$m_{37,1} = 512$	$g_{37,1}$	12	1167851520	24
		$f_{37,2} = 1$	$m_{37,2} = 512$	$g_{37,2}$	12	1167851520	24
$g_{38} = 15A$	$k_{38} = 1$	$f_{38,1} = 1$	$m_{38,1} = 1024$	$g_{38,1}$	15	1868562432	15
$g_{39} = 16A$	$k_{39} = 2$	$f_{39,1} = 1$	$m_{39,1} = 512$	$g_{39,1}$	16	875888640	32
		$f_{39,2} = 1$	$m_{39,2} = 512$	$g_{39,2}$	16	875888640	32
$g_{40} = 16B$	$k_{40} = 2$	$f_{40,1} = 1$	$m_{40,1} = 512$	$g_{40,1}$	16	875888640	32
		$f_{40,2} = 1$	$m_{40,2} = 512$	$g_{40,2}$	16	875888640	32
$g_{41} = 18A$	$k_{41} = 1$	$f_{41,1} = 1$	$m_{41,1} = 1024$	$g_{41,1}$	18	1557135360	18
$g_{42} = 24A$	$k_{42} = 2$	$f_{42,1} = 1$	$m_{42,1} = 512$	$g_{42,1}$	24	583925760	48
		$f_{42,2} = 1$	$m_{42,2} = 512$	$g_{42,2}$	24	583925760	48
$g_{43} = 24B$	$k_{43} = 2$	$f_{43,1} = 1$	$m_{43,1} = 512$	$g_{43,1}$	24	583925760	48
		$f_{43,2} = 1$	$m_{43,2} = 512$	$g_{43,2}$	24	583925760	48

3. Inertia Factor Groups of $\overline{G} = 2^{10}$:($U_5(2)$:2)

We have seen in Section 2 that the action of $\overline{G} = 2^{10}$:($U_5(2)$:2) on $N = 2^{10}$ yielded three orbits of lengths 1, 495 and 528. By a theorem of Brauer (see Lemma 4.5.2 of [9]), it follows that the action of $\overline{G}$ on $\mathrm{Irr}(N)$ will also produce three orbits. Since $N = 2^{10}$ is a vector space, the action of $\overline{G}$ on $\mathrm{Irr}(2^{10})$ can be viewed as the action of $\overline{G}$ on N^*, where N^* is the dual space of N. In fact we act the group generated by the transposed matrices of the matrix generators of $\overline{G}$ on N. We have found that the action of $\overline{G}$ on $\mathrm{Irr}(N)$ is self-dual to the action of $\overline{G}$ on N, that is the orbit lengths of $\overline{G}$ on $\mathrm{Irr}(N)$ are 1, 495 and 528. This can also be deduced from the associated geometry of classical groups (for example see Liebeck [12]), that is the action of $SU_n(q^2)$ on its natural module $\mathbb{V} = \mathbb{F}_{q^2}^n$ yields three orbits of lengths 1 (consisting of the zero vector) and

$$\begin{cases} (q^n-1)(q^{n-1}+1),\ q^{n-1}(q-1)(q^n-1) & \text{if } n \text{ is even,} \\ (q^n+1)(q^{n-1}-1),\ q^{n-1}(q-1)(q^n+1) & \text{if } n \text{ is odd.} \end{cases}$$

From Grove [11] we know that the number of isotropic vectors together with the zero vector of $\mathbb{V}$ is given by $q^{2n-1} + (-1)^n(q^n - q^{n-1})$. This distinguishes the orbits of isotropic and non-isotropic vectors in each case of n. In the following, we determine the structures of the inertia factor groups of $\overline{G}$.

Let $H_1,\ H_2$ and H_3 be the respective inertia factor groups of representatives of characters from the previous orbits of $\overline{G}$ on $\mathrm{Irr}(N)$. We notice that these inertia factors have indices 1, 495 and 528 respectively in $U_5(2){:}2$. Clearly $H_1 = U_5(2){:}2$ and the character table of this group is given as Table 11.14 of Basheer [1]. By looking at the $\mathbb{ATLAS}$, the group $U_5(2){:}2$ has 7 conjugacy classes of maximal subgroups. Let $M[1], M[2], \cdots, M[7]$ be representatives of these classes of maximal subgroups. That is $M[1] = U_5(2)$, $M[2] = 2_-^{1+6}{:}3_-^{1+2}{:}2S_4$, $M[3] = (3 \times U_4(2)){:}2$, $M[4] = 2^{4+4}{:}(3 \times A_5){:}2$, $M[5] = 3^4{:}(2 \times S_5)$, $M[6] = S_3 \times 3_+^{1+2}{:}2S_4$ and $M[7] = PSL(2,11){:}2$. Note that these maximal subgroups have indices 2, 165, 176, 297, 1408, 3520 and 20736 respectively in G. By considering the indices of H_2 and H_3 in G, we infer that H_2 must be an index 3 subgroup of $M[2]$, while H_3 is either an index 264 subgroup of $M[1]$ or of index 3 in $M[3]$. However the possibility $H_3 \leq M[1]$ is not feasible as we can see from the $\mathbb{ATLAS}$ that $U_5(2)$ does not contain a subgroup of index that is a divisor of 264. This leaves us with the other possibility that H_3 is an index 3 subgroup of $M[3]$. As subgroups of $G = \langle g_1, g_2 \rangle$, the group $M[2]$ is generated by α_1 and α_2, while the group $M[3]$ is generated by β_1 and β_2, where

$$\alpha_1 = \begin{pmatrix} 0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&1&0&1&1&1&1&1&0 \\ 1&1&1&0&0&0&0&1&0&0&0 \\ 1&1&0&1&1&1&0&0&0&0&0 \\ 0&1&0&0&1&0&0&1&1&0&0 \\ 1&1&1&1&0&1&1&0&1&0&0 \\ 1&1&0&0&1&1&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&1&0 \\ 0&1&1&0&1&0&0&1&1&0&0 \\ 0&1&1&1&1&1&1&1&0&0&0 \\ 1&0&0&1&1&0&1&1&1&0&1 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0&1&1&0&0&1&0&0&1&0&0 \\ 0&1&1&1&0&0&1&1&0&1&0 \\ 1&1&1&0&0&1&0&1&1&1&0 \\ 0&0&0&1&1&1&0&1&1&0&0 \\ 1&1&1&1&1&0&0&1&1&0&0 \\ 0&1&0&0&1&1&0&0&1&0&0 \\ 0&1&0&0&1&1&0&1&1&1&0 \\ 1&1&0&1&1&0&0&1&0&0&0 \\ 1&0&0&0&1&1&1&0&1&1&0 \\ 0&1&1&0&1&1&1&0&0&0&0 \\ 1&0&1&0&0&1&1&0&0&1&1 \end{pmatrix},$$

$$\beta_1 = \begin{pmatrix} 1&0&0&0&0&1&0&1&1&1&0 \\ 1&0&1&0&0&0&1&1&0&0&0 \\ 0&1&0&1&0&1&0&0&0&0&0 \\ 0&0&0&0&1&0&1&0&1&0&0 \\ 0&0&1&0&0&1&1&1&0&1&0 \\ 0&0&1&0&0&1&0&0&0&0&0 \\ 0&1&1&1&0&1&1&1&1&0&0 \\ 0&0&0&0&1&1&1&1&0&0&0 \\ 1&0&0&1&0&1&0&1&1&0&0 \\ 1&1&1&0&0&0&1&0&0&1&0 \\ 0&1&1&0&0&0&0&1&0&1&1 \end{pmatrix}, \quad \beta_2 = \begin{pmatrix} 0&1&0&1&1&1&0&0&0&1&0 \\ 0&1&1&0&1&1&0&1&1&0&0 \\ 1&1&1&0&1&0&1&0&0&1&0 \\ 1&1&1&0&1&0&0&1&1&0&0 \\ 1&1&0&0&1&1&0&0&1&0&0 \\ 1&1&1&0&0&0&1&0&0&1&0 \\ 1&1&0&0&0&0&0&1&1&0&0 \\ 0&0&1&0&0&0&1&0&0&1&0 \\ 0&1&1&0&0&1&1&1&0&1&0 \\ 1&0&0&0&0&0&0&0&0&1&0 \\ 1&0&1&1&0&1&0&1&0&1&1 \end{pmatrix}.$$

Using GAP we were able to locate all the maximal subgroups of $M[2]$ and $M[3]$. We list brief information about these subgroups in Table 2.

Table 2: Some information on the maximal subgroups of $M[2]$ and $M[3]$

Maximal Subgroups of $M[2]$ & $M[3]$	$M[ij]$	$\lvert M[ij]\rvert$	$[M[i] : M[ij]]$	$\lvert \mathrm{Irr}(M[ij])\rvert$
$M[2] = 2_{-}^{1+6}{:}3_{-}^{1+2}{:}2S_4$	$M[21]$	82944	2	64
	$M[22]$	55296	3	41
	$M[23]$	41472	4	55
	$M[24]$	18432	9	50
	$M[25]$	2592	64	36
$M[3] = (3 \times U_4(2)){:}2$	$M[31]$	77760	2	60
	$M[32]$	51840	3	25
	$M[33]$	5760	27	30
	$M[34]$	4320	36	33
	$M[35]$	3888	40	39
	$M[36]$	3888	40	42
	$M[37]$	3456	45	48

From Table 2 we can see that the two groups H_2 and H_3 are in the conjugacy classes of maximal subgroups containing $M[22]$ and $M[32]$ respectively.

As subgroups of $G = \langle g_1, g_2\rangle$, the group $M[22]$ is generated by μ_1 and μ_2, while the group $M[32]$ is generated by ζ_1 and ζ_2, where

$$\mu_1 = \begin{pmatrix} 0&1&0&1&0&1&0&1&1&1&0\\ 0&1&0&1&0&1&1&0&1&1&0\\ 1&1&0&0&0&1&0&0&0&0&0\\ 0&0&0&1&1&1&0&0&0&1&0\\ 1&1&0&1&1&1&0&0&0&0&0\\ 1&1&1&0&0&0&1&1&1&1&0\\ 0&1&0&0&0&1&0&0&1&0&0\\ 0&1&0&1&1&0&1&1&0&1&0\\ 0&1&0&1&0&0&0&0&1&1&0\\ 0&1&1&1&1&1&1&1&0&1&0\\ 0&0&0&1&0&0&0&0&1&1&1 \end{pmatrix}, \quad \mu_2 = \begin{pmatrix} 0&1&1&0&1&0&0&1&1&0&0\\ 1&1&1&0&0&1&1&1&0&0&0\\ 1&1&0&0&0&0&0&1&1&0&0\\ 0&1&1&0&0&1&1&1&0&1&0\\ 0&0&1&0&0&0&0&0&1&0&0\\ 0&1&1&0&0&0&0&1&1&0&0\\ 0&0&0&1&0&0&1&0&1&0&0\\ 0&0&1&0&0&1&0&0&0&0&0\\ 1&0&1&1&0&1&1&1&0&0&0\\ 0&0&1&1&1&1&0&1&1&0&0\\ 1&0&1&1&0&0&0&0&0&0&1 \end{pmatrix},$$

with $o(\mu_1) = 24,\ o(\mu_2) = 16$ and $o(\mu_1\mu_2) = 12$,

$$\zeta_1 = \begin{pmatrix} 1&1&1&0&0&0&1&0&0&1&0\\ 0&1&0&1&1&1&0&0&1&1&0\\ 0&0&1&1&0&0&1&0&0&0&0\\ 0&0&0&1&1&0&1&1&1&1&0\\ 1&1&1&1&0&1&0&0&1&0&0\\ 1&1&1&0&1&1&1&1&1&0&0\\ 0&0&1&1&1&0&0&0&0&0&0\\ 1&0&0&0&1&1&0&1&1&1&0\\ 1&0&0&0&0&1&1&1&1&0&0\\ 0&0&1&1&0&0&0&1&1&0&0\\ 0&0&0&0&1&0&1&1&1&1&1 \end{pmatrix}, \quad \zeta_2 = \begin{pmatrix} 1&1&0&0&0&0&0&0&1&1&0\\ 1&0&0&0&0&1&1&1&1&1&0\\ 0&1&0&0&0&1&0&1&1&0&0\\ 0&0&1&0&1&1&1&0&0&1&0\\ 1&1&1&1&0&1&0&1&0&1&0\\ 0&0&0&0&0&0&0&0&1&1&0\\ 1&1&1&1&1&0&0&1&1&1&0\\ 0&1&1&1&1&1&0&1&1&1&0\\ 0&1&1&0&0&0&1&0&1&1&0\\ 0&0&0&1&1&1&0&1&1&1&0\\ 1&1&1&0&0&1&1&0&0&1&1 \end{pmatrix},$$

with $o(\zeta_1) = 3,\ o(\zeta_2) = 4$ and $o(\zeta_1\zeta_2) = 10$.

Remark 1. With the aid of GAP we were able to determine the structures of all the normal subgroups of H_2 and H_3 and the corresponding quotient groups. In fact we have found that $H_2 \cong 2_{-}^{1+6}{:}((3^{1+2}{:}8){:}2)$, while the group $H_3 = M[32] \cong O_5(3){:}2 \cong U_4(2){:}2 \cong Sp(4,3){:}2$.

We recall that knowledge of the appropriate character tables of inertia factor groups is crucial in calculating the full character table of any group extension. Since in our extensions $\overline{G}$, the normal subgroup N is abelian and the extension splits, it follows by applications of Mackey's Theorem (see for example Theorem 3.3.4 of Whitley [17]), that every character of N is extendible to an ordinary character of its respective inertia group $\overline{H}_k$. Thus all the character tables of the inertia factor groups that we will use to construct the character tables of $\overline{G}$ are the ordinary ones. The character table of $H_2 = 2_{-}^{1+6}{:}((3^{1+2}{:}8){:}2)$ is not in a library of GAP. One can use the generators μ_1 and μ_2 of H_2 to generate H_2 inside Magma [7] or GAP and then obtain its character table. In fact the character table of H_2 appears as Table 11.13 of [1]. The character table of H_3 is stored in GAP, or one can use any set of generators of $U_4(2){:}2$ in the forms of 6, 8 or 14-dimensional $\mathbb{F}_2$-representations supplied at [18] to generate this group in either Magma or GAP and then obtain its character table. Alternatively one can use the generators ζ_1 and ζ_2 to find the character table of H_3. Note that from Table 1 the group $\overline{G}$ has 109 conjugacy classes. By the description of Section 3 of [2] the 109 irreducible characters of $\overline{G}$ are distributed into three blocks of characters correspond to the inertia factor groups. That is $|\mathrm{Irr}(\overline{G})| = |\mathrm{Irr}(H_1)| + |\mathrm{Irr}(H_2)| + |\mathrm{Irr}(H_3)|$. From Tables 11.14, 11.15 and 11.16 of [1], we can see that $|\mathrm{Irr}(H_1)| + |\mathrm{Irr}(H_2)| + |\mathrm{Irr}(H_3)| = 43 + 41 + 25 = 109 = |\mathrm{Irr}(\overline{G})|$.

In Tables 3 and 4 we list respectively the fusions of classes of H_2 and H_3 into classes of G.

Table 3: The fusion of classes of H_2 into classes of G

Class of H_2	$\hookrightarrow$	Class of $U_5(2){:}2$	Class of H_2	$\hookrightarrow$	Class of $U_5(2){:}2$
$1a = g_{121}$		$1A$	$8a = g_{24,21}$		$8A$
$2a = g_{221}$		$2A$	$8b = g_{25,21}$		$8B$
$2b = g_{222}$		$2A$	$8c = g_{26,21}$		$8C$
$2c = g_{321}$		$2B$	$8d = g_{27,21}$		$8D$
$2d = g_{322}$		$2B$	$8e = g_{27,22}$		$8D$
$2e = g_{421}$		$2C$	$8f = g_{26,22}$		$8C$
$3a = g_{721}$		$3C$	$8g = g_{24,22}$		$8A$
$3b = g_{621}$		$3B$	$8h = g_{25,22}$		$8B$
$4a = g_{921}$		$4A$	$8i = g_{26,23}$		$8C$
$4b = g_{10,21}$		$4B$	$12a = g_{33,21}$		$12B$
$4c = g_{11,21}$		$4C$	$12b = g_{34,21}$		$12C$
$4d = g_{12,21}$		$4D$	$12c = g_{36,21}$		$12E$
$4e = g_{922}$		$4A$	$12d = g_{34,22}$		$12C$
$4f = g_{10,22}$		$4B$	$12e = g_{36,22}$		$12E$
$4g = g_{923}$		$4A$	$12f = g_{36,23}$		$12E$
$4h = g_{10,23}$		$4B$	$12g = g_{34,23}$		$12C$
$6a = g_{15,21}$		$6B$	$16a = g_{39,21}$		$16A$
$6b = g_{16,21}$		$6C$	$16b = g_{40,21}$		$16B$
$6c = g_{19,21}$		$6F$	$24a = g_{42,21}$		$24A$
$6d = g_{17,21}$		$6D$	$24b = g_{43,21}$		$24B$
$6e = g_{23,21}$		$6J$			

Table 4: The fusion of classes of H_3 into classes of G

Class of H_3	$\hookrightarrow$	Class of $U_5(2){:}2$	Class of H_3	$\hookrightarrow$	Class of $U_5(2){:}2$
$1a = g_{131}$		$1A$	$6a = g_{14,31}$		$6D$
$2a = g_{431}$		$2C$	$6b = g_{20,31}$		$6G$
$2b = g_{231}$		$2A$	$6c = g_{23,31}$		$6J$
$2c = g_{331}$		$2B$	$6d = g_{16,31}$		$6C$
$2d = g_{432}$		$2C$	$6e = g_{22,31}$		$6I$
$3a = g_{731}$		$3C$	$6f = g_{21,31}$		$6H$
$3b = g_{831}$		$3D$	$6g = g_{23,32}$		$6J$
$3c = g_{631}$		$3B$	$8a = g_{26,31}$		$8C$
$4a = g_{12,31}$		$4D$	$9a = g_{29,31}$		$9B$
$4b = g_{933}$		$4A$	$10a = g_{30,31}$		$10A$
$4c = g_{12,32}$		$4D$	$12a = g_{37,31}$		$12F$
$4d = g_{11,31}$		$4C$	$12b = g_{34,31}$		$12C$
$5a = g_{13,31}$		$5A$			

4. Fischer matrices of $\overline{G} = 2^{10}{:}(U_5(2){:}2)$

We recall from [1, 2] that we label the top and bottom of the columns of the Fischer matrix $\mathcal{F}_i$, corresponding to g_i, by the sizes of the centralizers of $g_{ij},\ 1 \leq j \leq c(g_i)$ in $\overline{G}$ and m_{ij} respectively. In Table 1 we supplied $|C_{\overline{G}}(g_{ij})|$ and $m_{ij},\ 1 \leq i \leq 43$, $1 \leq j \leq c(g_i)$. Also the fusions of the classes of H_2 and H_3 into classes of G are given in Tables 3 and 4 respectively. Since the size of the Fischer matrix $\mathcal{F}_i$ is $c(g_i)$, it follows from Table 1 that the sizes of the Fischer matrices of $\overline{G}$ range between 1 and 5 for every $i \in \{1, 2, ..., 43\}$.

We have used the arithmetical properties of the Fischer matrices, given in Proposition 3.6 of [2], to calculate some of the entries of these matrices and also to build an algebraic system of equations. For example since the extension splits and $N = 2^{10}$ is abelian, then every coset $N\overline{g}_i$ (or just Ng_i) is a split coset (see Schiffer [16]) and it results that $a_{i1}^{(k,m)} = \frac{|C_{U_5(2):2}(g_i)|}{|C_{H_k}(g_{ikm})|}$, $\forall i \in \{1, 2, ..., 43\}$. With the help of the symbolic mathematical package Maxima [13], we were able to solve these systems of equations and hence we have computed all the Fischer matrices of $\overline{G}$, which we list below.

$\mathcal{F}_1$

g_1		g_{11}	g_{12}	g_{13}
$o(g_{1j})$		1	2	2
$\|C_{\overline{G}}(g_{1j})\|$		28028436480	56623104	53084160
(k,m)	$\|C_{H_k}(g_{1km})\|$			
(1, 1)	27371520	1	1	1
(2, 1)	55296	495	15	−17
(3, 1)	51840	528	−16	16
m_{1j}		1	495	528

$\mathcal{F}_2$

g_2		g_{21}	g_{22}	g_{23}	g_{24}
$o(g_{2j})$		2	2	2	4
$\|C_{\overline{G}}(g_{2j})\|$		42467328	1572864	1179648	221184
(k,m)	$\|C_{H_k}(g_{2km})\|$				
(1, 1)	165888	1	1	1	1
(2, 1)	55296	3	3	3	−1
(2, 2)	1536	108	−20	12	0
(3, 1)	1152	144	16	−16	0
m_{2j}		4	108	144	768

$\mathcal{F}_3$

g_3		g_{31}	g_{32}	g_{33}	g_{34}
$o(g_{3j})$		2	2	4	4
$\|C_{\overline{G}}(g_{3j})\|$		589824	196608	24576	16384
(k,m)	$\|C_{H_k}(g_{3km})\|$				
(1, 1)	9216	1	1	1	1
(2, 1)	1536	6	6	2	−2
(2, 2)	1024	9	9	−3	1
(3, 1)	192	48	−16	0	0
m_{3j}		16	48	384	576

$\mathcal{F}_4$

g_4		g_{41}	g_{42}	g_{43}	g_{44}
$o(g_{4j})$		2	4	4	4
$\|C_{\overline{G}}(g_{4j})\|$		46080	46080	3072	3072
(k,m)	$\|C_{H_k}(g_{4km})\|$				
(1, 1)	1440	1	1	1	1
(2, 1)	96	15	15	−1	−1
(3, 1)	1440	1	−1	1	−1
(3, 2)	96	15	−15	−1	1
m_{4j}		32	32	480	480

$\mathcal{F}_5$

g_5		g_{51}
$o(g_{5j})$		3
$\|C_{\overline{G}}(g_{5j})\|$		77760
(k,m)	$\|C_{H_k}(g_{5km})\|$	
(1, 1)	77760	1
m_{5j}		1024

$\mathcal{F}_6$

g_6		g_{61}	g_{62}	g_{63}
$o(g_{6j})$		3	6	6
$\|C_{\overline{G}}(g_{6j})\|$		248832	9216	6912
(k,m)	$\|C_{H_k}(g_{6km})\|$			
(1, 1)	3888	1	1	1
(2, 1)	144	27	3	−5
(3, 1)	108	36	−4	4
m_{6j}		16	432	576

$\mathcal{F}_7$

g_7		g_{71}	g_{72}	g_{73}
$o(g_{7j})$		3	6	6
$\|C_{\overline{G}}(g_{7j})\|$		6912	1152	768
(k,m)	$\|C_{H_k}(g_{7km})\|$			
(1, 1)	432	1	1	1
(2, 1)	48	9	−3	1
(3, 1)	72	6	2	−2
m_{7j}		64	384	576

$\mathcal{F}_8$

g_8		g_{81}	g_{82}
$o(g_{8j})$		3	6
$\lvert C_{\overline{G}}(g_{8j})\rvert$		2592	864
(k,m)	$\lvert C_{H_k}(g_{8km})\rvert$		
$(1,1)$	648	1	1
$(3,1)$	216	3	-1
m_{8j}		256	768

$\mathcal{F}_9$

g_9		g_{91}	g_{92}	g_{93}	g_{94}	g_{95}
$o(g_{9j})$		4	4	4	4	4
$\lvert C_{\overline{G}}(g_{9j})\rvert$		147456	49152	24576	24576	3072
(k,m)	$\lvert C_{H_k}(g_{9km})\rvert$					
$(1,1)$	2304	1	1	1	1	1
$(2,1)$	768	3	3	3	3	-1
$(2,2)$	192	12	12	-4	-4	0
$(2,3)$	96	24	-8	-8	8	0
$(3,1)$	96	24	-8	8	-8	0
m_{9j}		16	48	96	96	768

$\mathcal{F}_{10}$

g_{10}		$g_{10,1}$	$g_{10,2}$	$g_{10,3}$	$g_{10,4}$
$o(g_{10j})$		4	4	4	4
$\lvert C_{\overline{G}}(g_{10j})\rvert$		12288	12288	6144	1024
(k,m)	$\lvert C_{H_k}(g_{10km})\rvert$				
$(1,1)$	768	1	1	1	1
$(2,1)$	256	3	3	3	-1
$(2,2)$	192	4	4	-4	0
$(2,3)$	96	8	-8	0	0
m_{10j}		64	64	128	768

$\mathcal{F}_{11}$

g_{11}		$g_{11,1}$	$g_{11,2}$	$g_{11,3}$
$o(g_{11j})$		4	4	8
$\lvert C_{\overline{G}}(g_{11j})\rvert$		3072	1024	256
(k,m)	$\lvert C_{H_k}(g_{11km})\rvert$			
$(1,1)$	192	1	1	1
$(2,1)$	64	3	3	-1
$(3,1)$	16	12	-4	0
m_{11j}		64	192	768

$\mathcal{F}_{12}$

g_{12}		$g_{12,1}$	$g_{12,2}$	$g_{12,3}$	$g_{12,4}$
$o(g_{12j})$		4	4	8	8
$\lvert C_{\overline{G}}(g_{12j})\rvert$		768	768	256	256
(k,m)	$\lvert C_{H_k}(g_{12km})\rvert$				
$(1,1)$	96	1	1	1	1
$(2,1)$	32	3	3	-1	-1
$(3,1)$	96	1	-1	1	-1
$(3,2)$	32	3	-3	-1	1
m_{12j}		128	128	384	384

$\mathcal{F}_{13}$

g_{13}		$g_{13,1}$	$g_{13,2}$
$o(g_{13j})$		5	10
$\lvert C_{\overline{G}}(g_{13j})\rvert$		120	40
(k,m)	$\lvert C_{H_k}(g_{13km})\rvert$		
$(1,1)$	30	1	1
$(3,1)$	10	3	-1
m_{13j}		256	768

$\mathcal{F}_{14}$

g_{14}		$g_{14,1}$
$o(g_{14j})$		6
$\lvert C_{\overline{G}}(g_{14j})\rvert$		1728
(k,m)	$\lvert C_{H_k}(g_{14km})\rvert$	
$(1,1)$	1728	1
m_{14j}		1024

$\mathcal{F}_{15}$

g_{15}		$g_{15,1}$	$g_{15,2}$
$o(g_{15j})$		6	12
$\lvert C_{\overline{G}}(g_{15j})\rvert$		5184	1728
(k,m)	$\lvert C_{H_k}(g_{15km})\rvert$		
$(1,1)$	1296	1	1
$(2,1)$	432	3	-1
m_{15j}		256	768

$\mathcal{F}_{16}$

g_{16}		$g_{16,1}$	$g_{16,2}$	$g_{16,3}$
$o(g_{16j})$		6	6	12
$\lvert C_{\overline{G}}(g_{16j})\rvert$		6912	2304	576
(k,m)	$\lvert C_{H_k}(g_{16km})\rvert$			
$(1,1)$	432	1	1	1
$(2,1)$	144	3	3	-1
$(3,1)$	36	12	-4	0
m_{16j}		64	192	768

$\mathcal{F}_{17}$

g_{17}		$g_{17,1}$	$g_{17,2}$	$g_{17,3}$
$o(g_{17j})$		6	6	6
$\lvert C_{\overline{G}}(g_{17j})\rvert$		6912	1152	768
(k,m)	$\lvert C_{H_k}(g_{17km})\rvert$			
$(1,1)$	432	1	1	1
$(2,1)$	48	9	-3	1
$(3,1)$	72	6	2	-2
m_{17j}		64	384	576

$\mathcal{F}_{18}$

g_{18}		$g_{18,1}$
$o(g_{18j})$		6
$\lvert C_{\overline{G}}(g_{18j})\rvert$		288
(k,m)	$\lvert C_{H_k}(g_{18km})\rvert$	
$(1,1)$	288	1
m_{18j}		1024

$\mathcal{F}_{19}$

g_{19}		$g_{19,1}$	$g_{19,2}$
$o(g_{19j})$		6	12
$\lvert C_{\overline{G}}(g_{19j})\rvert$		576	192
(k,m)	$\lvert C_{H_k}(g_{19km})\rvert$		
$(1,1)$	144	1	1
$(2,1)$	48	3	-1
m_{19j}		256	768

$\mathcal{F}_{20}$

g_{20}		$g_{20,1}$	$g_{20,2}$
$o(g_{20j})$		6	6
$\lvert C_{\overline{G}}(g_{20j})\rvert$		432	144
(k,m)	$\lvert C_{H_k}(g_{20km})\rvert$		
$(1,1)$	108	1	1
$(3,1)$	36	3	-1
m_{20j}		256	768

$\mathcal{F}_{21}$

g_{21}		$g_{21,1}$	$g_{21,2}$
$o(g_{21j})$		6	6
$\lvert C_{\overline{G}}(g_{21j})\rvert$		288	96
(k,m)	$\lvert C_{H_k}(g_{21km})\rvert$		
$(1,1)$	72	1	1
$(3,1)$	24	3	-1
m_{21j}		256	768

$\mathcal{F}_{22}$

g_{22}		$g_{22,1}$	$g_{22,2}$
$o(g_{22j})$		6	12
$\lvert C_{\overline{G}}(g_{22j})\rvert$		72	72
(k,m)	$\lvert C_{H_k}(g_{22km})\rvert$		
$(1,1)$	36	1	1
$(3,1)$	36	1	-1
m_{22j}		512	512

$\mathcal{F}_{23}$

g_{23}		$g_{23,1}$	$g_{23,2}$	$g_{23,3}$	$g_{23,4}$
$o(g_{23j})$		6	12	12	12
$\lvert C_{\overline{G}}(g_{23j})\rvert$		288	288	96	96
(k,m)	$\lvert C_{H_k}(g_{23km})\rvert$				
$(1,1)$	36	1	1	1	1
$(2,1)$	12	3	3	-1	-1
$(3,1)$	36	1	-1	1	-1
$(3,2)$	12	3	-3	-1	1
m_{23j}		128	128	384	384

$\mathcal{F}_{24}$

g_{24}		$g_{24,1}$	$g_{24,2}$	$g_{24,3}$
$o(g_{24j})$		8	8	8
$\lvert C_{\overline{G}}(g_{24j})\rvert$		1536	512	384
(k,m)	$\lvert C_{H_k}(g_{24km})\rvert$			
$(1,1)$	192	1	1	1
$(2,1)$	192	1	1	-1
$(2,2)$	32	6	-2	0
m_{24j}		128	384	512

$\mathcal{F}_{25}$

g_{25}		$g_{25,1}$	$g_{25,2}$	$g_{25,3}$
$o(g_{25j})$		8	8	8
$\lvert C_{\overline{G}}(g_{25j})\rvert$		1536	512	384
(k,m)	$\lvert C_{H_k}(g_{25km})\rvert$			
$(1,1)$	192	1	1	1
$(2,1)$	192	1	1	-1
$(2,2)$	32	6	-2	0
m_{25j}		128	384	512

$\mathcal{F}_{26}$

g_{26}		$g_{26,1}$	$g_{26,2}$	$g_{26,3}$	$g_{26,4}$	$g_{26,5}$
$o(g_{26j})$		8	8	8	8	8
$\lvert C_{\overline{G}}(g_{26j})\rvert$		256	256	128	128	128
(k,m)	$\lvert C_{H_k}(g_{26km})\rvert$					
$(1,1)$	32	1	1	1	1	1
$(2,1)$	32	1	1	-1	1	-1
$(2,2)$	32	1	1	1	-1	-1
$(2,3)$	32	1	1	-1	-1	1
$(3,1)$	8	4	-4	0	0	0
m_{26j}		128	128	256	256	256

$\mathcal{F}_{27}$

g_{27}		$g_{27,1}$	$g_{27,2}$	$g_{27,3}$
$o(g_{27j})$		8	8	8
$\lvert C_{\overline{G}}(g_{27j})\rvert$		128	128	64
(k,m)	$\lvert C_{H_k}(g_{27km})\rvert$			
$(1,1)$	32	1	1	1
$(2,1)$	32	1	1	-1
$(2,2)$	16	2	-2	0
m_{27j}		256	256	512

$\mathcal{F}_{28}$

g_{28}		$g_{28,1}$
$o(g_{28j})$		9
$\lvert C_{\overline{G}}(g_{28j})\rvert$		54
(k,m)	$\lvert C_{H_k}(g_{28km})\rvert$	
$(1,1)$	54	1
m_{28j}		1024

$\mathcal{F}_{29}$

g_{29}		$g_{29,1}$	$g_{29,2}$
$o(g_{29j})$		9	18
$\lvert C_{\overline{G}}(g_{29j})\rvert$		108	36
(k,m)	$\lvert C_{H_k}(g_{29km})\rvert$		
$(1,1)$	27	1	1
$(3,1)$	9	3	-1
m_{29j}		256	768

$\mathcal{F}_{30}$

g_{30}		$g_{30,1}$	$g_{30,2}$
$o(g_{30j})$		10	20
$\lvert C_{\overline{G}}(g_{30j})\rvert$		20	20
(k,m)	$\lvert C_{H_k}(g_{30km})\rvert$		
$(1,1)$	10	1	1
$(3,1)$	10	1	-1
m_{30j}		512	512

$\mathcal{F}_{31}$

g_{31}		$g_{31,1}$
$o(g_{31j})$		11
$\lvert C_{\overline{G}}(g_{31j})\rvert$		11
(k,m)	$\lvert C_{H_k}(g_{31km})\rvert$	
$(1,1)$	11	1
m_{31j}		1024

$\mathcal{F}_{32}$

g_{32}		$g_{32,1}$
$o(g_{32j})$		12
$\lvert C_{\overline{G}}(g_{32j})\rvert$		144
(k,m)	$\lvert C_{H_k}(g_{32km})\rvert$	
$(1,1)$	144	1
m_{32j}		1024

$\mathcal{F}_{33}$

g_{33}		$g_{33,1}$	$g_{33,2}$
$o(g_{33j})$		12	12
$\lvert C_{\overline{G}}(g_{33j})\rvert$		288	96
(k,m)	$\lvert C_{H_k}(g_{33km})\rvert$		
$(1,1)$	72	1	1
$(2,1)$	24	3	-1
m_{33j}		256	768

$\mathcal{F}_{34}$

g_{34}		$g_{34,1}$	$g_{34,2}$	$g_{34,3}$	$g_{34,4}$	$g_{34,5}$
$o(g_{34j})$		12	12	12	12	12
$\lvert C_{\overline{G}}(g_{34j})\rvert$		1152	384	384	384	192
(k,m)	$\lvert C_{H_k}(g_{34km})\rvert$					
$(1,1)$	72	1	1	1	1	1
$(2,1)$	24	3	-1	-1	3	-1
$(2,2)$	24	3	-1	3	-1	-1
$(2,3)$	24	3	3	-1	-1	-1
$(3,1)$	12	6	-2	-2	-2	2
m_{34j}		64	192	192	192	384

$\mathcal{F}_{35}$

g_{35}		$g_{35,1}$
$o(g_{35j})$		12
$\lvert C_{\overline{G}}(g_{35j})\rvert$		24
(k,m)	$\lvert C_{H_k}(g_{35km})\rvert$	
$(1,1)$	24	1
m_{35j}		1024

$\mathcal{F}_{36}$

g_{36}		$g_{36,1}$	$g_{36,2}$	$g_{36,3}$	$g_{36,4}$
$o(g_{36j})$		12	12	12	12
$\lvert C_{\overline{G}}(g_{36j})\rvert$		96	96	96	96
(k,m)	$\lvert C_{H_k}(g_{36km})\rvert$				
$(1,1)$	24	1	1	1	1
$(2,1)$	24	1	-1	1	-1
$(2,2)$	24	1	1	-1	-1
$(2,3)$	24	1	-1	-1	1
m_{36j}		256	256	256	256

$\mathcal{F}_{37}$

g_{37}		$g_{37,1}$	$g_{37,2}$
$o(g_{37j})$		12	12
$\lvert C_{\overline{G}}(g_{37j})\rvert$		24	24
(k,m)	$\lvert C_{H_k}(g_{37km})\rvert$		
$(1,1)$	12	1	1
$(3,1)$	12	1	-1
m_{37j}		512	512

$\mathcal{F}_{38}$

g_{38}		$g_{38,1}$
$o(g_{38j})$		15
$\lvert C_{\overline{G}}(g_{38j})\rvert$		15
(k,m)	$\lvert C_{H_k}(g_{38km})\rvert$	
$(1,1)$	15	1
m_{38j}		1024

$\mathcal{F}_{39}$

g_{39}		$g_{39,1}$	$g_{39,2}$
$o(g_{39j})$		16	16
$\lvert C_{\overline{G}}(g_{39j})\rvert$		32	32
(k,m)	$\lvert C_{H_k}(g_{39km})\rvert$		
$(1,1)$	16	1	1
$(2,1)$	16	1	-1
m_{39j}		512	512

$\mathcal{F}_{40}$

g_{40}		$g_{40,1}$	$g_{40,2}$
$o(g_{40j})$		16	16
$\lvert C_{\overline{G}}(g_{40j})\rvert$		32	32
(k,m)	$\lvert C_{H_k}(g_{40km})\rvert$		
$(1,1)$	16	1	1
$(2,1)$	16	1	-1
m_{40j}		512	512

$\mathcal{F}_{41}$

g_{41}		$g_{41,1}$
$o(g_{41j})$		18
$\lvert C_{\overline{G}}(g_{41j})\rvert$		18
(k,m)	$\lvert C_{H_k}(g_{41km})\rvert$	
$(1,1)$	18	1
m_{41j}		1024

$\mathcal{F}_{42}$

g_{42}		$g_{42,1}$	$g_{42,2}$
$o(g_{42j})$		24	24
$\lvert C_{\overline{G}}(g_{42j})\rvert$		48	48
(k,m)	$\lvert C_{H_k}(g_{42km})\rvert$		
$(1,1)$	24	1	1
$(2,1)$	24	1	-1
m_{42j}		512	512

$\mathcal{F}_{43}$

g_{43}		$g_{43,1}$	$g_{43,2}$
$o(g_{43j})$		24	24
$\lvert C_{\overline{G}}(g_{43j})\rvert$		48	48
(k,m)	$\lvert C_{H_k}(g_{43km})\rvert$		
$(1,1)$	24	1	1
$(2,1)$	24	1	-1
m_{43j}		512	512

5. The character table of $\overline{G} = 2^{10}{:}(U_5(2){:}2)$

From Sections 2, 3, 4 and the Appendix of Basheer [1], we have

- the conjugacy classes of $\overline{G}$ (Table 1),
- the fusions of classes of the inertia factors H_2 and H_3 into classes of G (Tables 3 and 4 respectively),
- the character tables of the inertia factors H_1, H_2 and H_3 (Tables 11.14, 11.15 and 11.16 of [1] respectively),
- the Fischer matrices of $\overline{G}$ (see Section).

By [1] or [2], it follows that the full character table of $\overline{G}$ can be constructed easily. One can apply similar arguments used in [2, 3] to obtain the character table of $\overline{G}$, which is a 109×109 $\mathbb{C}$-valued matrix, partitioned into 129 parts $\mathcal{K}_{ik}\mathcal{F}_{ik}$, where $1 \le i \le 43,\ 1 \le k \le 3$. The full character table of $\overline{G}$, in the format of Clifford-Fischer theory appears as Table 11.17 of [1].

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References

[1] Basheer, A.B.M., *Clifford-Fischer Theory Applied to Certain Groups Associated with Symplectic, Unitary and Thompson Groups*, PhD Thesis, University of KwaZulu-Natal, Pietermaitzburg, 2012.

[2] Basheer, A.B.M., Moori, J., *Fischer matrices of Dempwolff group* $2^{5\cdot}GL(5,2)$, International Journal of Group Theory, 1 (4) (2012), 43-63.

[3] Basheer, A.B.M., Moori, J., *On the non-split extension group* $2^{6\cdot}Sp(6,2)$, Bulletin of the Iranian Mathematical Society, 39 (6) (2013), 1189-1212.

[4] Basheer, A.B.M., Moori, J., *On the non-split extension* $2^{2n\cdot}Sp(2n,2)$, Bulletin of the Iranian Mathematical Society, 41 No. 2 (2015), 499-518.

[5] Basheer, A.B.M., Moori, J., *On a maximal subgroup of the Thompson simple group*, Mathematical Communications, 20 (2015), 201 - 218.

[6] Basheer, A.B.M., Moori, J., *A survey on Clifford-Fischer theory*, London Mathematical Society Lecture Notes Series, 422, published by Cambridge University Press, 2015, 160-172.

[7] Bosma, W., Cannon, J.J., *Handbook of Magma Functions*, Department of Mathematics, University of Sydeny, November 1994.

[8] Conway, J.H., Curtis, R.T., Norton, S.P., Parker, R.A., Wilson, R.A., *Atlas of Finite Groups*, Clarendon Press, Oxford, 1985.

[9] Gorenstein, D., *Finite Groups*, Harper and Row Publishers, New York, 1968.

[10] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.4.10*; 2007. (http://www.gap-system.org)

[11] Grove, L.C., *Classical Groups and Geometric Algebra*, American Mathematical Society, Graduate Text in Mathematics, 39 2002.

[12] Liebeck, M.W., *The affine permutation groups of rank three*, Proceedings of the London Mathematical Society, 54 (1987), 477-516.

[13] Maxima, a Computer Algebra System. Version 5.18.1; 2009. (http://maxima.sourceforge.net)

[14] Moori, J., *On the Groups* G^+ *and* $\overline{G}$ *of the form* $2^{10}{:}M_{22}$ *and* $2^{10}{:}\overline{M}_{22}$, PhD Thesis, University of Birmingham, 1975.

[15] Moori, J., *On certain groups associated with the smallest Fischer group*, J. London Math. Soc., 2 (1981), 61-67.

[16] Schiffer, U., *Cliffordmatrizen*, Diplomarbeit, Lehrstul D Fur Matematik, RWTH, Aachen, 1995.

[17] Whitely, N.S., *Fischer Matrices and Character Tables of Group Extensions*, MSc Thesis, University of Natal, Pietermaritzburg, 1993.

[18] Wilson, R.A. et al., *Atlas of finite group representations*, (http://brauer.maths.qmul.ac.uk/Atlas/v3/)

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ORTHOGONAL-BASED HYBRID BLOCK METHOD FOR SOLVING GENERAL SECOND ORDER INITIAL VALUE PROBLEMS

E.O. Adeyefa

Department of Mathematics
Federal University Oye-Ekiti
P.M.B. 373, Oye-Ekiti
Ekiti State
Nigeria
e-mail: adeoluman@yahoo.com

Abstract. The direct integration of general second order initial value problems is considered in this paper. We employ a new class of orthogonal polynomials constructed as basis function to develop One Step Hybrid Block Method (OSHBM) adopting collocation technique. We present the recursive formula of the class of polynomials constructed and give analysis of the basic properties of OSHBM as findings show that the method is accurate and convergent.

Keywords: orthogonal polynomials, algorithm, block method, collocation, interpolation, zero-stable.

AMS Mathematical Subject Classification: 65L05, 65L06.

1. Introduction

In many area of physical problems such as in science, engineering and management, second order differential equations of the form

$$y''(x) = f(x, y(x), y'(x)) \quad y(x_0) = y_0, \;\; y'(x_0) = y_0^{'}, p \leq x \leq q \tag{1}$$

arise frequently. The difficulties encounter in solving such problems has led to development of numerical methods. The solution of (1) has been discussed by various researchers among them are Lambert (1973,1991), Norsett (1989), Sirisena (1999), Gear (1971). However, experience has shown in Lie and Norsett (1989), Fatunla (1988, 1991, 1994), Hairer and Wanner (1993), Lambert (1973, 1991), Brugnano and Trigiante (1998), Onumanyi et al. (1999, 2008) and Jator (2007) that to derive these methods, polynomials play a vital role. Notable among the well-known polynomials are the orthogonal polynomials. Orthogonal polynomial sequence is a family of polynomials such that any two different polynomials in the sequence are orthogonal to each other under some inner product. The first orthogonal polynomials were the Legendre polynomials. Then came the Chebyshev polynomials, the general Jacobi polynomials, the Hermite and the Laguerre polynomials. All these classical orthogonal polynomials play an important role in many applied problems.

Asymptotic formulae for orthogonal polynomials were first discovered by G. Szego, Szego (1975). Lanczos, C. (1938) introduced Chebyshev polynomials as trial function. Several researchers have employed these polynomials as trial functions to formulate algorithms (see Shampine and Watts (1969), Tanner (1979), Dahlquist (1979), Jator (2007), Awoyemi (1991)).
In this work, we shall employ a non-negative weight function to construct a new class of orthogonal polynomials which will serve as trial functions to formulate numerical algorithms for the solution of initial value problems.

2. Construction of orthogonal basis function

We define the orthogonal polynomial of the first kind of degree n over the interval [-1, 1] with respect to weight function $w(x) = (x^2 - 1)^2$ as

$$q_r(x) = \sum_{r=0}^{n} C_r^{(n)} x^r \tag{2a}$$

The following requirements are considered:

$$< q_m(x),\ q_n(x) >= 0,\ m = 0, 1, 2, ..., n-1 \tag{2b}$$

For the purpose of constructing the basis function, we adopt the approach discussed extensively in Adeyefa and Adeniyi (2015) and use additional property (the normalization)

$$q_n(1) = 1 \tag{2c}$$

Using (2b) and (2c), equation (2a) yields

$$\begin{cases} q_0(x) = 1 \\ q_1(x) = x \\ q_2(x) = \frac{1}{6}(7x^2 - 1) \\ q_3(x) = \frac{1}{2}(3x^3 - x) \\ q_4(x) = \frac{1}{16}(33x^4 - 18x^2 + 1) \\ q_5(x) = \frac{1}{48}(143x^5 - 110x^3 + 15x) \\ q_6(x) = \frac{1}{32}(143x^6 - 143x^4 + 33x^2 - 1) \\ q_7(x) = \frac{1}{32}(221x^7 - 273x^5 + 91x^3 - 7x) \\ q_8(x) = \frac{1}{384}(4199x^8 - 6188x^6 + 2730x^4 - 364x^2 + 7) \\ q_9(x) = \frac{1}{128}(2261x^9 - 3876x^7 + 2142x^5 - 420x^3 + 21x) \\ q_{10}(x) = \frac{1}{256}(7429x^{10} - 14535x^8 + 9690x^6 - 2550x^4 + 225x^2 - 3) \end{cases} \tag{3}$$

In the spirit of Golub and Fischer (1992), equation (3) must satisfy three-term recurrence relation

$$c_j p(t) = (t - a_j)p_{j-1}(t) - b_j p_{j-2}(t), j = 1, 2, ..., p_{-1}(t) = 0,\ p_0(t) \equiv p_0$$

where
b_j, $c_j > 0$ for $j \geq 1$ (b_1 is arbitrary).
$c_j p(t) = (n+5)P_{n+1}(x),\ (t-a_j)p_{j-1}(t) = (2n+5)xP_n(x),\ b_j p_{j-2}(t) = nP_{n-1}(x)$, $n = 1, 2, ...$
The recursive formula for these orthogonal polynomials is therefore given as

$$P_{n+1}(x) = \frac{1}{n+5}\left[(2n+5)xP_n(x) - nP_{n-1}(x)\right],\ n \geq 1,\ P_0(x) = 1,\ P_1(x) = x$$

This relation, along with the two polynomials P_0(x) and P_1(x), allows the new set of polynomials to be generated recursively.
In what immediately follows, we shall develop an algorithm to integrate second order differential equations where polynomials q_n(x) shall be employed as basis function. Thereafter, the analysis of the method for convergence and implementation of the method through some test problems shall be presented. Finally, conclusion shall be made.

3. Materials and methods

3.1. Development of the method

In this section, our aim is to derive a continuous hybrid scheme which shall serve as direct integrator to second order initial value problems (IVPs) of the form (1). To make this happen, we shall seek an approximant

$$y(x) = \sum_{r=0}^{s+k-1} a_r q_r(x) \tag{4}$$

to obtain the solution of second order initial value problems in ordinary differential equations. Transforming q_n(x) to interval [0, 1], we have $x = \frac{2X-2x_n-ph}{ph}$, where p varies as the method to be developed. In this case, p = 1, s and k in (4) are points of interpolation and collocation respectively. The procedure involves interpolating (4) at points s = $\frac{1}{4}, \frac{1}{2}$ and collocating the second derivative of (4) at points k = 0, $\frac{1}{4}, \frac{1}{3}, \frac{1}{2}$, 1. The a_r, r = 0(1)6 from the resulting system of equations are obtained as

$$\left\{\begin{aligned}
a_0 &= y_{n+\frac{1}{2}} - \tfrac{h^2}{1386}f_n + \tfrac{h^2}{77}f_{n+\frac{1}{4}} - \tfrac{81h^2}{4928}f_{n+\frac{1}{3}} + \tfrac{241h^2}{11088}f_{n+\frac{1}{2}} + \tfrac{13h^2}{44352}f_{n+1}\\
a_1 &= -2y_{n+\frac{1}{4}} + 2y_{n+\frac{1}{2}} - \tfrac{1703h^2}{506880}f_n + \tfrac{119h^2}{2376}f_{n+\frac{1}{4}} - \tfrac{8037h^2}{112640}f_{n+\frac{1}{3}}\\
&\qquad\qquad + \tfrac{10867h^2}{126720}f_{n+\frac{1}{2}}\tfrac{379h^2}{276480}f_{n+1}\\
a_2 &= -\tfrac{51h^2}{8008}f_n + \tfrac{120h^2}{1001}f_{n+\frac{1}{4}} - \tfrac{1215h^2}{8008}f_{n+\frac{1}{3}} + \tfrac{285h^2}{2002}f_{n+\frac{1}{2}} + \tfrac{3h^2}{1001}f_{n+1}\\
a_3 &= -\tfrac{h^2}{117}f_n + \tfrac{40h^2}{351}f_{n+\frac{1}{4}} - \tfrac{45h^2}{208}f_{n+\frac{1}{3}} + \tfrac{25h^2}{234}f_{n+\frac{1}{2}} + \tfrac{23h^2}{5616}f_{n+1}\\
a_4 &= -\tfrac{2h^2}{495}f_n + \tfrac{16h^2}{165}f_{n+\frac{1}{4}} - \tfrac{27h^2}{220}f_{n+\frac{1}{3}} + \tfrac{13h^2}{495}f_{n+\frac{1}{2}} + \tfrac{7h^2}{1980}f_{n+1}\\
a_5 &= -\tfrac{3h^2}{2860}f_n - \tfrac{16h^2}{715}f_{n+\frac{1}{4}} + \tfrac{243h^2}{5720}f_{n+\frac{1}{3}} - \tfrac{3h^2}{143}f_{n+\frac{1}{2}} + \tfrac{1h^2}{520}f_{n+1}\\
a_6 &= \tfrac{2h^2}{715}f_n - \tfrac{64h^2}{2145}f_{n+\frac{1}{4}} + \tfrac{27h^2}{715}f_{n+\frac{1}{3}} - \tfrac{8h^2}{715}f_{n+\frac{1}{2}} + \tfrac{h^2}{2145}f_{n+1}
\end{aligned}\right. \tag{5}$$

Substituting (5) into (4) yields the continuous implicit method

$$sy(x) = \alpha_{\frac{1}{4}}(x)y_{n+\frac{1}{4}} + \alpha_{\frac{1}{2}}(x)y_{n+\frac{1}{2}} + h^2(\beta_k(x)f_{n+k}),\ \ k = 0, \tfrac{1}{4}, \tfrac{1}{3}, \tfrac{1}{2}, 1 \tag{6}$$

Evaluating equation (6) at x = x_{n+m}, m=0, $\frac{1}{3}$, 1 yields the discrete equations

$$\begin{aligned}
&\text{(7a)} \quad y_{n+1} = -2y_{n+\frac{1}{4}} + 3y_{n+\frac{1}{2}} - \tfrac{109h^2}{5120}f_n + \tfrac{41h^2}{120}f_{n+\frac{1}{4}} - \tfrac{5103h^2}{10240}f_{n+\frac{1}{3}} + \tfrac{449h^2}{1280}f_{n+\frac{1}{2}} \\
&\qquad\qquad + \tfrac{451h^2}{30720}f_{n+1} \\
&\text{(7b)} \quad y_{n+\frac{1}{3}} = \tfrac{2}{3}y_{n+\frac{1}{4}} + \tfrac{1}{3}y_{n+\frac{1}{2}} - \tfrac{137h^2}{3732480}f_n + \tfrac{53h^2}{87480}f_{n+\frac{1}{4}} - \tfrac{1817h^2}{276480}f_{n+\frac{1}{3}} - \tfrac{883h^2}{933120}f_{n+\frac{1}{2}} \\
&\qquad\qquad + \tfrac{103h^2}{22394880}f_{n+1} \\
&\text{(7c)} \quad y_n = 2y_{n+\frac{1}{4}} - y_{n+\frac{1}{2}} + \tfrac{71h^2}{15360}f_n + \tfrac{7h^2}{120}f_{n+\frac{1}{4}} - \tfrac{81h^2}{10240}f_{n+\frac{1}{3}} + \tfrac{29h^2}{3840}f_{n+\frac{1}{2}} - \tfrac{h^2}{10240}f_{n+1}
\end{aligned}$$

To develop the block method from the continuous scheme, we adopt the general block formula proposed in Shampine and Watts (1969) in the normalized form given as

$$A^{(0)}Y_m = ey_m + h^{\mu-\lambda}df(y_m) + h^{\mu-\lambda}bF(y_m) \tag{8}$$

Evaluating the first derivative of (6) at x = x_{n+j}, j = 0, $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{2}$, 1 to obtain the first derivative equations (FDE). Substituting the resulting equations FDE and equation (7) into (8) and solving simultaneously gives a block formulae represented as

$$\left\{\begin{aligned}
y_{n+\frac{1}{4}} &= \tfrac{147}{10240}h^2f_n + \tfrac{7}{144}h^2f_{n+\frac{1}{4}} - \tfrac{783}{20480}h^2f_{n+\frac{1}{3}} + \tfrac{17}{2560}h^2f_{n+\frac{1}{2}} \\
&\quad - \tfrac{23}{184320}h^2f_{n+1} + \tfrac{1}{4}hy' + y_n \\
y_{n+\frac{1}{3}} &= \tfrac{301}{14580}h^2f_n + \tfrac{928}{10935}h^2f_{n+\frac{1}{4}} - \tfrac{13}{216}h^2f_{n+\frac{1}{3}} + \tfrac{38}{3645}h^2f_{n+\frac{1}{2}} \\
&\quad - \tfrac{17}{14580}h^2f_{n+1} + \tfrac{1}{3}hy' + y_n \\
y_{n+\frac{1}{2}} &= \tfrac{1}{30}h^2f_n + \tfrac{7}{45}h^2f_{n+\frac{1}{4}} - \tfrac{27}{320}h^2f_{n+\frac{1}{3}} + \tfrac{1}{48}h^2f_{n+\frac{1}{2}} + \tfrac{1}{2880}h^2f_{n+1} \\
&\quad + \tfrac{1}{2}hy' + y_n \\
y_{n+1} &= \tfrac{1}{20}h^2f_n + \tfrac{32}{45}h^2f_{n+\frac{1}{4}} - \tfrac{27}{40}h^2f_{n+\frac{1}{3}} + \tfrac{2}{5}h^2f_{n+\frac{1}{2}} + \tfrac{1}{72}h^2f_{n+1} + hy' + y_n \\
y'_{n+\frac{1}{4}} &= \tfrac{581}{7680}hf_n + \tfrac{49}{120}hf_{n+\frac{1}{4}} - \tfrac{1431}{5120}hf_{n+\frac{1}{3}} + \tfrac{89}{1920}hf_{n+\frac{1}{2}} - \tfrac{13}{15360}hf_{n+1} + y'_n \\
y'_{n+\frac{1}{3}} &= \tfrac{61}{810}hf_n + \tfrac{544}{1215}hf_{n+\frac{1}{4}} - \tfrac{7}{30}hf_{n+\frac{1}{3}} + \tfrac{2}{45}hf_{n+\frac{1}{2}} - \tfrac{1}{1215}hf_{n+1} + y'_n \\
y'_{n+\frac{1}{2}} &= \tfrac{37}{480}hf_n + \tfrac{2}{5}hf_{n+\frac{1}{4}} - \tfrac{27}{320}hf_{n+\frac{1}{3}} + \tfrac{13}{120}hf_{n+\frac{1}{2}} - \tfrac{1}{960}hf_{n+1} + y'_n \\
y'_{n+1} &= -\tfrac{1}{30}hf_n + \tfrac{32}{15}hf_{n+\frac{1}{4}} - \tfrac{27}{10}hf_{n+\frac{1}{3}} + \tfrac{22}{15}hf_{n+\frac{1}{2}} + \tfrac{2}{15}hf_{n+1} + y'_n
\end{aligned}\right. \tag{9}$$

Equation (9) is our desired block method of which its basic properties shall be discussed in the next section.

3.2. Implementation of the method

Equation (7) and FDE were solved simultaneously to obtain y_{n+j} and y'_{n+j} using the block (8) through maple code. Block equation (9) is applied directly to (1) without requiring any predictor to obtain numerical values for y_{n+j}, y'_{n+j} and f_{n+j}. The necessary starting value is obtained from the last values y_{n+1} and y'_{n+1} of the previous block whose loss of accuracy do not affect subsequent points, thus the order of the algorithm is preserved.

4. Analysis of the method

4.1. Order and error constant

Following Henrici (1962), the approach adopted in Fatunla (1991, 1994) and Lambert (1973), we define the local truncation error associated with equation (9) by the difference operator

$$L[y(x):h]=\sum_{j=0}^{k}\left[\alpha_j y(x_n+jh)-h^2\beta_j f(x_n+jh)\right] \tag{10}$$

where $y(x)$ is an arbitrary function, continuously differentiable on [a, b].

Expanding (10) in Taylor series about the point x, we obtain the expression

$$L[y(x);h]=C_0y(x)+C_1hy'(x)+C_2h^2y''(x)+\ldots+C_{p+3}h^{p+3}y^{p+3}(x)$$

where the $C_0, C_1, C_2, C_p, \ldots$ are obtained as

$$C_0=\sum_{j=0}^{k}\alpha_j,\quad C_1=\sum_{j=1}^{k}j\alpha_j,\quad C_2=\frac{1}{2!}\sum_{j=1}^{k}j^2\alpha_j,$$
$$C_q=\frac{1}{q!}\left[\sum_{j=1}^{k}j^q\alpha_j-q(q-1)(q-2)\sum_{j=1}^{k}\beta_j j^{q-3}\right]$$

According to Lambert (1973), equations (7) and (9) are of order p if

$$C_0=C_1=C_2=\ldots C_p=C_{p+1}=0 \text{ and } C_{p+2}\neq 0$$

The $C_{p+2}\neq 0$ is called the error constant and $C_{p+2}h^{p+2}y^{p+2}(x_n)$ is the principal local truncation error at the pointx_n.

Thus, equations (7) and (9) are all of order 5 with the error constants

$$C_{p+2}=\left[\tfrac{-1}{196608}\ \tfrac{329}{6449725440}\ \tfrac{1}{5898240}\right]^T$$

and

$$C_{p+2}=\left[-\tfrac{1}{241920}\ \tfrac{1}{1548288}\ \tfrac{13}{35271936}\ \tfrac{59}{247726080}\ \tfrac{7}{4423680}\ \tfrac{13}{8398080}\ \tfrac{1}{552960}\ -\tfrac{1}{34560}\right]^T$$

respectively.

4.2. Zero stability of the method

According to Lambert (1973), a linear multistep method is said to be zero-stable if no root of the first characteristic polynomial $\rho(R)$ has modulus greater than one and if every root of modulus one has multiplicity not greater than the order of the differential equation.

To analyze the zero-stability of the method, we present (9) in vector notation form of column vectors $e = (e_1 \dots e_r)^T$, $d = (d_1 \dots d_r)^T$, $y_m = (y_{n+1} \dots y_{n+r})^T$, $F(y_m) = (f_{n+1} \dots f_{n+r})^T$ and matrices $A = (a_{ij})$, $B = (b_{ij})$.

Thus, equation (9) forms the block formula

$$A^0 y_m = hBF(y_m) + A^1 y_n + hdf_n \tag{11a}$$

where h is a fixed mesh size within a block.

The first characteristic polynomial of the hybrid block method (9a) is given by

$$\rho(R) = \det(RA^0 - A^1) \tag{11b}$$

where

$$A^0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n+\frac{1}{4}} \\ y_{n+\frac{1}{3}} \\ y_{n+\frac{1}{2}} \\ y_{n+1} \\ y'_{n+\frac{1}{4}} \\ y'_{n+\frac{1}{3}} \\ y'_{n+\frac{1}{2}} \\ y'_{n+1} \end{pmatrix},$$

$$A^1 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n-\frac{1}{4}} \\ y_{n-\frac{1}{3}} \\ y_{n-\frac{1}{2}} \\ y_n \\ y'_{n-\frac{1}{4}} \\ y'_{n-\frac{1}{3}} \\ y'_{n-\frac{1}{2}} \\ y'_n \end{pmatrix},$$

$$B = \begin{pmatrix} \frac{7}{144} & \frac{-783}{20480} & \frac{17}{2560} & \frac{-23}{184320} & 0 & 0 & 0 & 0 \\ \frac{928}{10935} & \frac{-13}{216} & \frac{38}{3645} & \frac{-17}{14580} & 0 & 0 & 0 & 0 \\ \frac{7}{45} & \frac{-27}{320} & \frac{1}{48} & \frac{1}{2880} & 0 & 0 & 0 & 0 \\ \frac{32}{45} & \frac{-27}{40} & \frac{2}{5} & \frac{1}{72} & 0 & 0 & 0 & 0 \\ \frac{49}{120} & \frac{-1431}{5120} & \frac{89}{1920} & \frac{-13}{15360} & 0 & 0 & 0 & 0 \\ \frac{544}{1215} & \frac{-7}{30} & \frac{2}{45} & \frac{-1}{1215} & 0 & 0 & 0 & 0 \\ \frac{2}{5} & \frac{-27}{320} & \frac{13}{120} & \frac{-1}{960} & 0 & 0 & 0 & 0 \\ \frac{32}{15} & \frac{-27}{10} & \frac{22}{15} & \frac{2}{15} & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} f_{n+\frac{1}{4}} \\ f_{n+\frac{1}{3}} \\ f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n-\frac{1}{4}} \\ f_{n-\frac{1}{3}} \\ f_{n-\frac{1}{2}} \\ f_{n-1} \end{pmatrix},$$

$$d = \begin{pmatrix} 0 & 0 & 0 & \frac{147}{10240} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{301}{14580} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{30} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{20} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{581}{7680} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{61}{810} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{37}{480} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-1}{30} & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} f_{n-\frac{1}{4}} \\ f_{n-\frac{1}{3}} \\ f_{n-\frac{1}{2}} \\ f_n \\ f_{n-1} \\ f_{n-2} \\ f_{n-3} \\ f_{n-4} \end{pmatrix}$$

Substituting A^0 and A^1 in (11b), we obtain $\rho(R) = R^6(R-1)^2$ which implies that $R_1 = ... = R_6 = 0$, $R_7 = R_8 = 1$.

According to Fatunla (1988 , 1991), the our block method equation are zero-stable since from $\rho(R) = 0$ satisfies $|R_j| \leq 1$, $j = 1$ and for those roots with $|R_j| = 1$, the multiplicity does not exceed two.

4.3. Region of absolute stability of the main methods

For the region of absolute stability, the following definitions are considered.

Given the stability polynomial

$$\pi\left(z, \overline{h}\right) = \rho(z) - \overline{h}\sigma(z) = 0 \tag{12}$$

where $\overline{h} = h^2\lambda^2$ and $\lambda = \frac{df}{dy}$ are assumed constants.

The scheme (7) is said to be absolutely stable if for a given $\overline{h}$ all the roots z_s of (12) satisfy $|z_s| < 1$, $s = 1, 2, ...n$, where $\overline{h} = \lambda h$.

Definition 1.1. The region $\Re$ of the complex $\overline{h}$-plane such that the roots of $\pi\left(z, \overline{h}\right) = 0$ lies within the unit circle whenever $\overline{h}$ lies in the interior of the region is called the region of absolute stability.

Remark. Let $\Re$ be the boundary of the region$\Re$. Since the roots of the stability polynomial are continuous functions of $\overline{h}$, $\overline{h}$ will lie on $\Re$ when one of the roots of the $\pi\left(z, \overline{h}\right) = 0$ lies on the boundary of the unit circle. Thus we define (12) in terms of Euler's number, $\exp i\theta$, as follows;

$$\pi(\exp(i\theta), h) = \rho(\exp(i\theta) - \overline{h}\sigma(\exp(i\theta)) = 0 \tag{13}$$

So that, the locus of the boundary $\Re$ is given by

$$\overline{h}(\theta) = \frac{\rho(e^{i\theta})}{\sigma(e^{i\theta})} \tag{14}$$

From (7a), the boundary of the region of absolute stability is

$$\overline{h}(\theta) = \frac{2\cos\frac{1}{4}\theta - 3\cos\frac{1}{2}\theta - 3i\sin\frac{1}{2}\theta + 2i\sin\frac{1}{4}\theta + \cos\theta + i\sin\theta}{\begin{pmatrix} \frac{41}{120}i\sin\frac{1}{4}\theta - \frac{5103}{10240}i\sin\frac{1}{3}\theta + \frac{449}{1280}i\sin\frac{1}{2}\theta + \frac{451}{30720}i\sin\theta - \frac{109}{5120} \\ +\frac{41}{120}\cos\frac{1}{4}\theta - \frac{5103}{10240}\cos\frac{1}{3}\theta + \frac{449}{1280}\cos\frac{1}{2}\theta + \frac{451}{30720}\cos\theta \end{pmatrix}}$$

Definition 1.2. According to Widlund (1967), a numerical method is said to be A(α)-stable, $\alpha \in (0, \frac{\pi}{2})$, if its region of absolute stability contains the infinite wedge $W_\alpha = \{h\lambda \,|\, - \ \alpha < \pi - \arg h\lambda \ < \ \alpha\}$.

The A(α)-stability property is shown in Figure 1.

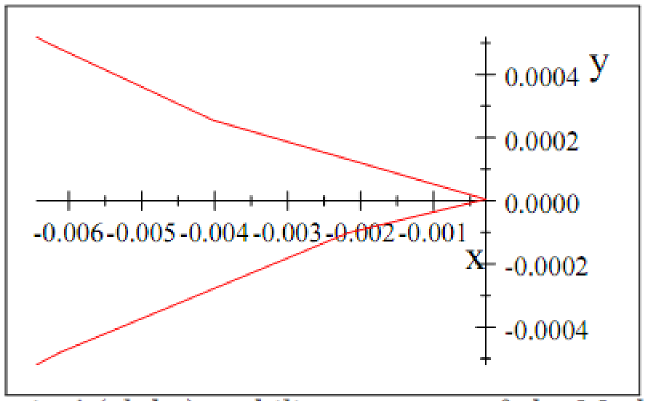

Fig. 1: A(alpha)-stability property of the Method

4.4. Consistency of the method

According to Lambert (1973), a linear multistep method is said to be consistent if it has order at least one. Owing to this definition, equations (7) and (9) are consistent.

4.5. Convergency of the Method

According to the theorem of Dahlquist, the necessary and sufficient condition for a LMM to be convergent is to be consistent and zero stable. Since the method satisfies the two conditions hence its convergence.

4.6. Numerical experiment

Problem 1. We consider here the highly stiff initial value problem

$$y'' = -1001y' - 1000y, y(0) = 1, \; y'(0) = -1, \; h = 0.05$$

Exact Solution: $y(x) = \exp(-x)$.

Problem 2. Here we implement OSHBM using linear initial value problem

$$y" = -\frac{6}{x}y' - \frac{4}{x^2}y, y(1) = 1, \; y'(1) = 1, \; h = \frac{0.1}{32}$$

Exact Solution: $y(x) = \frac{5}{3x} - \frac{2}{3x^4}$.

Problem 3. The Vanderpol's Oscillator Problem

$$y'' = 2\cos x - \cos^3 x - y' - y - y^2y', y(0) = 0, \; y'(0) = 1, \; h = 0.1$$

whose exact solution is y(x) = sinx is considered as our third test problem.

Problem 4. We consider the second order system equations

$$y_1'' = -4t^2 y_1 - \frac{2y_2}{\sqrt{y_1^2+y_2^2}}, y_1\left(\sqrt{\tfrac{\pi}{2}}\right) = 0, y_1'\left(\sqrt{\tfrac{\pi}{2}}\right) = -2\sqrt{\tfrac{\pi}{2}}$$
$$y_2'' = -4t^2 y_2 + \frac{2y_1}{\sqrt{y_1^2+y_2^2}}, y_2\left(\sqrt{\tfrac{\pi}{2}}\right) = 1, y_2'\left(\sqrt{\tfrac{\pi}{2}}\right) = 0, \quad \sqrt{\tfrac{\pi}{2}} \le t \le 10$$

with exact solution given by $y_1(t) = \cos(t^2), y_2(t) = \sin(t^2)$, see Sommeijer (1993).

Table 1. Numerical Results of Problem 1

X	Error in OSHBM h=0.05, p=5	Error in OSHBM h=0.1, p=5	Error in [2], h=0.1, p=4
0.1	3.23959e-15	3.9996926e-13	2.05e-11
0.2	1.794055e-14	2.75748143e-12	4.39e-11
0.3	6.909555e-14	1.37032147e-11	6.55e-11
0.4	2.3715412e-13	6.272985728e-11	8.38e-11
0.5	7.8100585e-13	2.8095960466e-10	9.86e-11
0.6	2.5337002e-12	1.25130028034e-09	1.10e-10
0.7	8.17574399e-12	5.5650421949e-09	1.19e-10
0.8	2.6332157e-11	2.474161011314e-08	1.24e-10
0.9	8.475545968e-11	1.0998987818295e-07	1.28e-10
1.0	2.7274422458e-10	4.8895562799261e-07	1.30e-10

Table 2. Numerical Results of Problem 2

X	Error in OSHBM h=0.1/32, p=5	Error in OSHBM h=0.1, p=5	Error in [5], h=0.1/32, p=6
0.003125	4.6e-18	1.001124239044e-07	3.8354 E-05
0.00625	4.16e-17	6.413822486422e-07	7.5004E-05
0.009375	1.09e-16	1.136405e-06	1.0592 E-04
0.0125	2.053e-16	1.49966e-06	1.35476 E-04
0.015625	3.288e-16	1.7427e-06	1.55567E-04
0.01875	4.782e-16	1.894407e-06	1.86372E-04
0.021875	6.521e-16	1.980796e-06	1.96055E-04
0.025	8.491e-16	2.021807e-06	2.21045E-04
0.028125	1.0679e-15	2.03174e-06	2.05628E-04
0.03125	1.3073e-15	2.0206e-06	2.77908E-04

Table 3. Numerical Results of Problem 3

X	Exact	OSHBM	Error
0.1	0.09983341664682815231	0.09983341664641143268	4.16719627e-13
0.2	0.19866933079506121546	0.19866933079151260797	3.54860749e-12
0.3	0.29552020666133957511	0.29552020665229235391	9.0472212e-12
0.4	0.38941834230865049167	0.38941834229214808125	1.650241042e-11
0.5	0.47942553860420300027	0.47942553857875939095	2.544360932e-11
0.6	0.56464247339503535720	0.56464247335967945648	3.535590072e-11
0.7	0.64421768723769105367	0.64421768719198266396	4.570838971e-11
0.8	0.71735609089952276163	0.71735609084353295021	5.598981142e-11
0.9	0.78332690962748338846	0.78332690956173874562	6.574464284e-11
1.0	0.84147098480789650665	0.84147098473329359276	7.460291389e-11

Table 4. Numerical Results of Problem 4

Method	p	M=400	M = 800	M = 1600	M = 3200	6400
N4	4	0.6	1.8	3.0	4.2	5.4
H8	8	0.3	2.6	5.2	7.6	10.0
BG8	8	0.9	3.1	5.6	8.0	10.4
OSHBM	5	5.2	6..7	8.2	9.7	11.0

5. Discussion of results

Problem 1 is a highly stiff problem and has been solved by Adeniran and Ogundare (2015) with a method of order 4. The numerical solutions are shown in Table 1. The OSHBM compared favourably well with the Adeniran and Ogundare method though the OSHBM is of order p=5. In Table 2, the solutions of problem 2 is presented as comparison of our order 5 OSHBM is made with order 6 method of Badmus and Yahaya. The superiority of the method has been established numerically. Table 3 shows the desirability of the method as we compare the solution of Problem 3 with the analytical method. Table 4 presents the solutions of Problem 4 which has been integrated in the interval $[\sqrt{\frac{\pi}{2}}, 10]$. This problem has also been solved by Sommeijer (1993) using N4 method of order four, the eighth-order, eight-stage RKN (H8) method constructed by Hairer (1977) and the eight-order, nine-stage method of order 8 (BG8) constructed by Beentjes and Gerritsen (1976). The results of these methods are compared with the OSHBM of order 5 in Table 4. The superiority of OSHBM has been established numerically as it performs better than those in Sommeijer (1993) in terms of accuracy (larger CD values) and efficiency (smaller NFEs).

6. Conclusion

Formulation of initial value problem solver has been developed using a new class orthogonal polynomials with recursive formula. Four test problems have been considered to show the efficiency and accuracy of the method. Tables 1, 2, 3 and 4 display the accuracy and comparison of the numerical results of the OSHBM with existing methods. The method is desirability as its superiority has been established by the numerical results. With little extension, the approach adopted in this paper is viable for the solution of higher order initial value problems of ordinary differential equations.

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References

[1] Adeyefa, E.O., Adeniyi, R.B., *Construction of Orthogonal Basis Function and formulation of continuous hybrid schemes for the solution of third order ODEs*, Journal of the Nigerian Association of Mathematical Physics, 29 (2015), 21-28.

[2] Adeniran, A.O., Ogundare, B.S., *An efficient hybrid numerical scheme for solving general second order initial value problems (IVPs)*, International Journal of Applied Mathematical Research, 4 (2) (2015), 411-419.

[3] Awoyemi, D.O., *A class of Continuous Linear Multistep Method for general second order initial value problems in ordinary differential equations*, International Journal of Computer Mathematics, 72 (1991), 29-37.

[4] Awoyemi, D.O., *A class of continuous method for general second order initial value problems in ordinary differential equations*, Int. J. Comput. Math., 72 (1999), 29-37.

[5] Badmus, A.M., Yahaya, Y.A., *An Accurate Uniform Order 6 Block Method for Direct Solution of General Second Order Ordinary Differential Equations*, The Pacific Journal of Science and Technology, 10 (2) (2009), 248-254.

[6] Beentjes, P.A., Gerritsen, W.J., *Higher order Runge-Kutta methods for the numerical solution of second order differential equations without first derivative*, Report NW 34/76, Centre for Mathematics and Computer Science, Amsterdam, 1976.

[7] Brown, R.L., *Some characteristics of implicitly multi-derivative integration formulas*, SIAM. J. Numer. Anal., 14 (1977), 982-983.

[8] Brugnano, L., Trigiante, D., *Solving Differential Problems by Multitep Initial and Boundary Value Methods*, Gordon and Breach Science Publishers, Amsterdam, 1998.

[9] Dahlquist, G., *Some properties of linear multistep and one leg method for ordinary differential equations. Department of computer science*, Royal institute of technology, Stockholm, 1979.

[10] Ehigie, J.O., Okunuga, S.A., Sofoluwe, A.B., *On generalized 2-step continuous linear multistep methods of hybrid type for the integration of second order ordinary differential equations*, Archives of Applied Science Research, 2 (6) (2010), 362–372.

[11] Fatunla, S.O., *Numerical Methods for initial value problems for ordinary differential equations*, U.S.A Academy Press, Boston, 1988, p.295.

[12] Fatunla, S.O., *Block method for second order initial value problem (IVP)*, International Journal of Computer Mathematics, 41 (1991), 55-63.

[13] Fatunla, S.O., *A class of block methods for second order IVPs*, Int. J. Comput. Math., 55 (1994), 119-133.

[14] Gear, C.W., *The automatic integration of ordinary differential equations*, Comm. ACM., 14 (1971), 176-179.

[15] Gear, C.W., *Simultaneous numerical solution of differential-algebraic equations*, IEEE Transactions in Circuit Theory, 18 (1971), 89-95.

[16] Golub, G.H., Fischer B., *How to generate unknown polynomials out of known orthogonal polynomials*, Journal of Computational and Applied Mathematics, 43 (1992), 99-115.

[17] Jator, S.N., *A sixth order Linear Multistep Method for direct solution of* $y^{''} = f(x, y, y^{'})$, International Journal of Pure and Applied Mathematics, 40 (1) (2007), 407-472.

[18] Hairer, E., Nørsett, S.P., Wanner, G., *Solving Ordinary Differential Equations* I, Nonstiff Problems, Springer-Verlag, Berlin, Heidelberg, 1993.

[19] Hairer, E., *Méthodes de Nystrøm pour l'équation différentielle* $y'' = f(x, y)$, Numer. Math. 25 (1977), 283–300.

[20] Henrici, P., *Discrete variable methods for ODEs*, John Wiley, New York, 1962.

[21] Ismail, F., Yap, L.K., Mohamad, O., *Explicit and implicit 3-point block methods for solving special second order ordinary differential equations directly,* International Journal of Mathematical Analysis, 3 (5) (2009), 239–254.

[22] Lambert, J.D., *Numerical Methods for ordinary differential systems* New York John Willey and Sons, 1991, p. 293.

[23] Lambert, J.D., *Computational methods for ordinary differential equations*, John Wiley, New York, 1973.

[24] Lanczos, C., *Trigonometric interpolation of empirical and analytical functions*, J. Math. Physics, 17 (1938), 123-199.

[25] Lie, I., Norselt, S.P., *Super convergence for multistep collocation*, Math. Comp., 52 (1989), 65-79.

[26] Milne W.E., *Numerical Solution of Differential Equations*, John Wiley & Sons, New York, NY, USA, 1953.

[27] Onumanyi, P., Sirisena, U.W., Jator, S.N., *Continuous finite difference approximations for solving differential equations*, Int. J. Comput. Math., 72 (1999), 15–27.

[28] Onumanyi, P., Fatokun, J., Adejo, B.O., *Accurate Numerical Differentiation by continuous integration for ordinary differential equations*, J. Mathe. Soc., 27 (2008), 69-90.

[29] Ramos, H., Mehta, S., Vigo-Aguiar, J., *A unified approach for the development of k-step block Falkner-type methods for solving general second-order initial-value problems in ODEs*, J. of Computational and Applied Mathematics. Article in Press, 2016

[30] Rosser, J.B., *A Runge-Kutta for all seasons*, SIAM Review, 9 (1967), 417-452.

[31] Shampine, L.F., Watts, H.A., *Block implicit one-step methods*, Journal of Computer Maths., 23 (1969), 731-740.

[32] Shampine, L.F., Watts, H.A., *Block implicit one-step methods*, Mathematics of Computation, 23 (1959), 731-740.

[33] Sirisena U.W., *An accurate implementation of the Butcher hybrid formula for the IVP in ODEs*, Nig. J. of Mathematics and Applications, 12 (1999), 199-206.

[34] Sommeijer, B.P., *Explicit high-order Runge–Kutta–Nystrm methods for parallel computers*, Appl. Numer. Math., 13 (1993), 221-240.

[35] Suleiman, M.B., *Solving higher order ODEs directly by the direct integration method*, Appl. Math. Comput., 33 (1989), 197–219.

[36] Szego, G., *Orthogonal polynomials*, Amer. Math. Soc. Colloquium publications, 1975.

[37] Tanner, L.H., *The spreading of Silicone Oil Drops on Horizontal Surfaces*, J. Phys. Appl. Phys., 12 (1979), 1473-1484.

[38] Widlund, O.B., *A note on unconditionally stable linear multistep methods*, BIT, 7 (1967), 65-70.

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A NOVEL VIEW OF ROUGH SOFT SEMIGROUPS BASED ON FUZZY IDEALS

Qiumei Wang
Jianming Zhan[1]

Department of Mathematics
Hubei University for Nationalities
Enshi, 445000
P.R. China

Abstract. By using a special t-level relation $U(\mu,t) = \{(x,y) \in S \times S | (\mu(x) \wedge \mu(y)) \vee Id_S(x,y) \geq t\}$ based on a fuzzy ideal μ of a semigroup S, which is a congruence relation, we study the roughness of soft semigroups under this special ideal of S, such as rough soft subsemigroups, rough soft ideals and rough soft prime ideals.

Keywords: fuzzy ideal; rough soft set; rough soft (prime, bi) ideal.

1. Introduction

Pawlak firstly proposed rough set theory as a new way to solve imprecision, inconsistent and incomplete problems in [14]. The core of rough set theory is a pair of lower and upper approximations operators induced from an approximation space. Just because rough sets are widely used in many areas such as knowledge discovery, machine learning, data analysis, approximate classification, conflict analysis, and so on. After rough sets were introduced, more and more scholars paid their attention to rough sets. Just like the study of rough sets in [2], [17] and [19]. With the studing of rough sets, some researchers turned their attention to the roughness of algebraic systems, such as in [9], Kuroki showed us rough ideals in semigroups. Followed by this, the roughness of τ-subsemigroups and ideals in τ -semigroups were discussed by Jun in [7]. In [3], making use of an ideal of a ring, the author introduced rough ideals and rough subrings and considered the ring as a universal set. Especially in [17], the roughness of semigroups were researched and Zhan et al. in [18] studied rough soft rings.

[1]Corresponding author. Email address: zhanjianming@hotmail.com

Molodtsov firstly put forward soft sets in 1999 in [11], which were mainly used to solve the uncertainty and vagueness problems. Since Molodtsov in [11] proposed soft sets, the applications of soft sets were widely used, such as smoothness of functions, game theory, operations research, Rlemann-integration, Perron integration, probability, theory of measurement and so on. As a new mathematical branch, soft sets were researched by more and more researchers. With the help of rough sets of Pawlak [14], Maji et al. [13], defined a parameter set over soft sets, and gave the decision making problems of soft sets at the same time. Kong et al. [10] presented a new definition of the parameter reduction. Maji et al. in [12] gave a theoretical study on soft sets. Since then, the study of soft sets theories begun to build over algebraic structure increasingly in recent years. And then in [1], Aktaş and Çağman firstly put up the definitions of soft groups. They also showed us the relations between soft sets and fuzzy sets ang rough sets. Jun in [8], introduced the soft BCK/BCI-algebras and soft subalgebras. Jun in [7] put forward roughness of gamma-subsemigroups/ideals in gamma-subsemigroups. In [5], Feng et al. gave a study of soft semirings over soft sets theory, and then obtained some relevant conclusions. And in [4], Feng et al. introduced soft relations over semigroups, Shabir et al. researched soft ternary semigroups in [15]. As we all know, soft set is a new way to deal with decision-making problems.

In [11] and [14], as we all know, rough sets and soft sets are all the tools which can solve uncertainty and vagueness problems. And we know, a soft set (F, A) can be regarded as a parameterized family of subsets of universe U, which gives an approximation (soft) description of the objects in U. As pointed in [11], for any parameter $\epsilon \in A$, the subset $F(\epsilon) \subseteq U$ may be considered as the set of ϵ-approximation elements in the soft set (F, A). As a set, soft sets also have the rough approximations. So we realized that we can combine the rough sets and soft sets and research their properties. Just like in [6], Feng et al. told us a new tentative approach that applied fuzzy sets and rough sets to soft sets. And in the following researches, Zhan in [20] showed us that the roughness of soft hemirings and obtained the rough soft hemirings based on equivalence relation ρ, and got some related properties and conclusions. It is no doubt that an equivalence relation is essential in any rough sets, so in Zhan et al. in [21] made use of a strong h-ideal as the equivalence relation and got the rough soft hemirings, especially gave us the applications of rough soft hemirings in decision making. It is clearly that the t-level relation $U(\mu, t)$ of a fuzzy ideal μ is a congruence relation.

As a special algebraic structure, a semigroup has many good properties. Since 1950, the studies of finite semigroups are especially important in theoretical computer science, just because there is a natural connection between finite semigroups and finite automaton. Also we can combine semigroups with rough sets or soft sets, just like in [17], Xiao et al. gave us rough prime ideals and rough fuzzy prime ideals in semigroups. And in [22], Zhan et al. made use of the fuzzy ideal as the congruence relation and defined the rough n-ary semigroups, and then got some conclusions and properties. Recently, Wang and Zhan [16] defined a t-level relation $U(\mu, t)$ of a fuzzy ideal μ in a semigroups S and proved that it is a congruence relation.

In this paper, we study the rough soft semigroups in four parts. Firstly, we recall some basic notions which are needed in this paper in Section 2. And then we define the rough soft semigroups over fuzzy ideals in Section 3. Finally, we research the rough structure of soft semigroups over fuzzy ideals and study some relative properties in Section 4.

2. Preliminaries

In this section, some basic notions such as semigroups, rough sets, soft sets and so on, are briefly described.

2.1. Semigroups

A semigroup S is a nonempty set with a binary operation "$\cdot$" such that

(i) $a \cdot b \in S$, for all $a, b \in S$;

(ii) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$, for all $a, b, c \in S$.

A subset A of S is called a subsemigroup if A is closed under "$\cdot$"; A is called a left (right) ideal if A is closed under "$\cdot$" and $SA \subseteq A(AS \subseteq A)$; If A is not only a left ideal but also is a right ideal, then it is an ideal; A ideal A is called a prime ideal if $a \cdot b \in A$ implies $a \in A$ or $b \in A$, for all $a, b \in A$; A nonempty subsemigroup T of S is called a bi-ideal if $TST \subseteq T$.

A fuzzy set μ of S is called a fuzzy subsemigroup if it satisfies $\mu(x \cdot y) \geq \mu(x) \wedge \mu(y)$; μ is called a fuzzy left (right) ideal if it satisfies $(1)\mu(x \cdot y) \geq \mu(x) \wedge \mu(y)$ $(2)\mu(x \cdot y) \geq \mu(y)$ $(\mu(x \cdot y) \geq \mu(x))$, for all $x, y \in S$; μ is called a fuzzy ideal if it is not only a fuzzy left ideal but also is a fuzzy right ideal.

2.2. Soft sets

Definition 2.1 [11] A pair (F, A) is called a soft set over U, where $A \subseteq E$ and $F : A \to P(U)$ is a set-valued mapping.

Definition 2.2 [21] Let $\mathfrak{S} = (F, A)$ and $\mathfrak{T} = (G, B)$ be two soft sets over a common semigroup. Then the "multiplication" of $\mathfrak{S}$ and $\mathfrak{T}$, denoted by $\mathfrak{S} \cdot \mathfrak{T} = (F, A) \cdot (G, B) = (H, A \times B)$, where $H(x, y) = F(x) \cdot G(y)$ for all $(x, y) \in A \times B$.

Definition 2.3 Let (F, A) be a soft set over S. Then

(1) (F, A) is called a soft semigroup over S if $F(x)$ is a semigroup of S for all $x \in \mathrm{Supp}(F, A)$.

(2) (F, A) is called an idealistic soft semigroup if $F(x)$ is an ideal of S for all $x \in \mathrm{Supp}(F, A)$.

(3) (F, A) is called a prime idealistic soft semigroup if $F(x)$ is a prime ideal of S for all $x \in \mathrm{Supp}(F, A)$.

(4) (F, A) is called a bi-idealistic soft semigroup if $F(x)$ is a bi-ideal of S for all $x \in \mathrm{Supp}(F, A)$.

2.3. Rough sets

Definition 2.4 [14] Let ρ be an equivalence relation on the universe U, (U, ρ) be a Pawlak approximation space. A subset $A \subseteq U$ is called definable if $\underline{\rho}(A) = \overline{\rho}(A)$, otherwise, U is a rough set, where

$$\underline{\rho}(A) = \{x \in U : [x]_\rho \subseteq A\},$$

and

$$\overline{\rho}(A) = \{x \in U : [x]_\rho \cap A \neq \emptyset\}.$$

Definition 2.5 [6] Let (U, ρ) be a Pawlak approximation space and $\mathfrak{S} = (F, A)$ a soft set over U. The lower and upper rough approximations of $\mathfrak{S} = (F, A)$ w.r.t. (U, ρ) are denoted by $\underline{\rho}(\mathfrak{S}) = (\underline{F}, A)$, and $\overline{\rho}(\mathfrak{S}) = (\overline{F}, A)$, which are soft sets over U with

$$\underline{F}(x) = \underline{\rho}(F(x)) = \{y \in U | [y]_\rho \subseteq F(x)\},$$

and

$$\overline{F}(x) = \overline{\rho}(F(x)) = \{y \in U | [y]_\rho \cap F(x) \neq \emptyset\},$$

for all $x \in A$.

If $\underline{\rho}(\mathfrak{S}) = \overline{\rho}(\mathfrak{S})$, $\mathfrak{S}$ is called definable; otherwise $\mathfrak{S}$ is called a rough soft set.

3. Rough soft semigroups based on fuzzy ideals

In [16], we proved that $U(\mu, t)$ is a congruence relation in a semigroup S if μ is a fuzzy ideal of S. By means of this point, we study rough soft semigroups in this section. And in this paper, let S be a semigroup in the following.

Definition 3.1 *[16]* Let μ be a fuzzy ideal of S. For each $t \in [0, 1]$, the set $U(\mu, t) = \{(x, y) \in S \times S | (\mu(x) \wedge \mu(y)) \vee Id_S(x, y) \geq t\}$ is called a t-level relation of μ.

Lemma 3.2 [16] *Let μ and ν be two fuzzy ideals of S such that $\mu \subseteq \nu$ and $t \in [0, 1]$. Then $[x]_{(\mu,t)} \subseteq [x]_{(\nu,t)}$ for all $x \in S$.*

For any fuzzy ideal μ, $\mu(0) \geq \mu(x)$ and $\mu(0) \leq 1$, so when $t \in [0, \mu(0)]$, $U(\mu, t)$ is a congruence relation on S (see [16]). We say x is congruent to y model μ, written $x \equiv_t y(mod\mu)$. If for elements $x, y \in S$, $t \in [0, 1]$, $(\mu(x) \wedge \mu(y)) \vee Ids(x, y) \geq t$, let $[x]_{(\mu,t)}$ as the equivalence class of x. However, $U(\mu, t)$ is not a complete congruence relation. We can only obtain a conclusion as follows.

Lemma 3.3 [16] *Let μ be a fuzzy ideal of S, and $t \in [0,1]$, then $[x]_{(\mu,t)} \cdot [y]_{(\mu,t)} \subseteq [xy]_{(\mu,t)}$.*

Example 3.4 Let $S = \{a, b, c, d\}$ be a semigroup with the following "$\cdot$" table.

$\cdot$	a	b	c	d
a	a	b	b	d
b	b	b	b	d
c	b	b	b	d
d	d	d	d	d

Let $\mu = \frac{0.3}{a} + \frac{0.5}{b} + \frac{0.1}{c} + \frac{0.8}{d}$ be the fuzzy set of S and $t = 0.4$, it is easy to prove μ is a fuzzy ideal of S. Thus $U(\mu; 0.4) = \{(a,a), (b,b), (c,c), (d,d), (b,d)\}$, so $[a]_{(\mu,0.4)} = \{a\}$, $[b]_{(\mu,0.4)} = \{b, d\}$, $[c]_{(\mu,0.4)} = \{c\}$, $[d]_{(\mu,0.4)} = \{b, d\}$. $[a]_{(\mu,0.4)} \cdot [c]_{(\mu,0.4)} = \{b\}$. Since $a \cdot c = b$, so $[ac]_{(\mu,0.4)} = \{b, d\}$. Obviously, $[a]_{(\mu,0.4)} \cdot [c]_{(\mu,0.4)} \subseteq [a \cdot c]_{(\mu,0.4)}$.

Definition 3.5 $U(\mu; t)$ is called a complete congruence relation if it satisfies: For any elements $x, y \in S$, $[x]_{(\mu,t)} \cdot [y]_{(\mu,t)} = [xy]_{(\mu,t)}$.

Example 3.6 Let $S = \{0, a, b, c\}$ be a semigroup with the following "$\cdot$" table.

$\cdot$	0	a	b	c
0	0	a	b	c
a	a	a	b	c
b	b	b	b	c
c	c	c	c	b

Assume that $\mu = \frac{0.1}{0} + \frac{0.4}{a} + \frac{0.7}{b} + \frac{0.7}{c}$ is a fuzzy set of S and $t = 0.7$, obviously, μ is a fuzzy ideal of S. Thus $U(\mu; 0.7) = \{(0,0), (a,a), (b,b), (c,c), (b,c)\}$, so $[0]_{(\mu,0.7)} = \{0\}$, $[a]_{(\mu,0.7)} = \{a\}$, $[b]_{(\mu,0.7)} = \{b, c\}$, $[c]_{(\mu,0.7)} = \{b, c\}$. Obviously, we can easily check $U(\mu; t)$ is a complete congruence relation.

Let μ be a fuzzy ideal of S, $t \in [0,1]$. Thus $U(\mu; t)$ is a congruence relation. Therefore, when $U = S$ and ρ is the above equivalence relation, then we use (S, μ, t) instead of approximation space (U, ρ).

Definition 3.7 Let (S, μ, t) be a Pawlak approximation space and $\mathfrak{S} = (F, A)$ be a soft set over S. The lower and upper rough approximations of $\mathfrak{S} = (F, A)$ with respect to (S, μ, t) are denoted by: $\underline{U}(\mu, t, \mathfrak{S}) = (\underline{F}_\mu, A)$ and $\overline{U}(\mu, t, \mathfrak{S}) = (\overline{F}_\mu, A)$, which are soft sets over S with $\underline{F}_\mu(x) = \underline{U}(\mu, t, F(x)) = \{y \in S | [y]_{(\mu,t)} \subseteq F(x)\}$ and $\overline{F}_\mu(x) = \overline{U}(\mu, t, F(x)) = \{y \in S | [y]_{(\mu,t)} \cap F(x) \neq \emptyset\}$, for all $x \in A$.

(i) $\underline{U}(\mu, t, \mathfrak{S}) = \overline{U}(\mu, t, \mathfrak{S})$, $\mathfrak{S}$ is called definable.

(ii) $\underline{U}(\mu, t, \mathfrak{S}) \neq \overline{U}(\mu, t, \mathfrak{S})$, $\underline{U}(\mu, t, \mathfrak{S})(\overline{U}(\mu, t, \mathfrak{S}))$ is called a lower (upper) rough soft set. Moreover, $\mathfrak{S}$ is called a rough soft set.

Example 3.8 Based on Example 3.4, then we have $[a]_{(\mu,0.4)} = \{a\}$, $[b]_{(\mu,0.4)} = \{b,d\}$, $[c]_{(\mu,0.4)} = \{c\}$, $[d]_{(\mu,0.4)} = \{b,d\}$. Define a soft set $\mathfrak{S} = (F,A)$ over S. Let $A = \{x_1, x_2\}$, where $F(x_1) = \{a,b\}$, $F(x_2) = \{c,d\}$, obviously, $\underline{F}_\mu(x_1) = \{a\}$, $\underline{F}_\mu(x_2) = \{c\}$, $\overline{F}_\mu(x_1) = \{a,b,d\}$, $\overline{F}_\mu(x_2) = \{a,b,d\}$. Thus, $\mathfrak{S}$ is a rough soft set.

The following two theorems are straightforward and we omit the proofs.

Theorem 3.9 *Let μ be a fuzzy ideal of S and $\mathfrak{S} = (F,A)$ be a soft set over S. Then we have:*

(1) $\underline{U}(\mu,t,\mathfrak{S}) \subseteq \mathfrak{S} \subseteq \overline{U}(\mu,t,\mathfrak{S})$.

(2) $\underline{U}(\mu,t,\underline{U}(\mu,t,\mathfrak{S})) = \underline{U}(\mu,t,\mathfrak{S})$.

(3) $\overline{U}(\mu,t,\overline{U}(\mu,t,\mathfrak{S})) = \overline{U}(\mu,t,\mathfrak{S})$.

(4) $\overline{U}(\mu,t,\underline{U}(\mu,t,\mathfrak{S})) = \underline{U}(\mu,t,\mathfrak{S})$.

(5) $\underline{U}(\mu,t,\overline{U}(\mu,t,\mathfrak{S})) = \overline{U}(\mu,t,\mathfrak{S})$.

(6) $\underline{U}(\mu,t,\mathfrak{S}) = (\overline{U}(\mu,t,\mathfrak{S})^r)^r$.

(7) $\overline{U}(\mu,t,\mathfrak{S}) = (\underline{U}(\mu,t,\mathfrak{S})^r)^r$.

Theorem 3.10 *Let μ be a fuzzy ideal of S and $\mathfrak{S} = (F,A)$ and $\mathfrak{T} = (G,B)$ be soft sets over S. Then we have:*

(1) $\underline{U}(\mu,t,\mathfrak{S} \Cap \mathfrak{T}) = \underline{U}(\mu,t,\mathfrak{S}) \Cap \underline{U}(\mu,t,\mathfrak{T})$.

(2) $\underline{U}(\mu,t,\mathfrak{S} \sqcap_\varepsilon \mathfrak{T}) = \underline{U}(\mu,t,\mathfrak{S}) \sqcap_\varepsilon \underline{U}(\mu,t,\mathfrak{T})$.

(3) $\overline{U}(\mu,t,\mathfrak{S} \Cap \mathfrak{T}) \subseteq \overline{U}(\mu,t,\mathfrak{S}) \Cap \overline{U}(\mu,t,\mathfrak{T})$.

(4) $\overline{U}(\mu,t,\mathfrak{S} \sqcap_\varepsilon \mathfrak{T}) \subseteq \overline{U}(\mu,t,\mathfrak{S}) \sqcap_\varepsilon \overline{U}(\mu,t,\mathfrak{T})$.

(5) $\underline{U}(\mu,t,\mathfrak{S} \cup_R \mathfrak{T}) \supseteq \underline{U}(\mu,t,\mathfrak{S}) \cup_R \underline{U}(\mu,t,\mathfrak{T})$.

(6) $\underline{U}(\mu,t,\mathfrak{S} \tilde{\cup} \mathfrak{T}) \supseteq \underline{U}(\mu,t,\mathfrak{S}) \tilde{\cup} \underline{U}(\mu,t,\mathfrak{T})$.

(7) $\overline{U}(\mu,t,\mathfrak{S} \cup_R \mathfrak{T}) = \overline{U}(\mu,t,\mathfrak{S}) \cup_R \overline{U}(\mu,t,\mathfrak{T})$.

(8) $\overline{U}(\mu,t,\mathfrak{S} \tilde{\cup} \mathfrak{T}) = \overline{U}(\mu,t,\mathfrak{S}) \tilde{\cup} \overline{U}(\mu,t,\mathfrak{T})$.

(9) $\mathfrak{S} \subseteq \mathfrak{T} \Rightarrow \underline{U}(\mu,t,\mathfrak{S}) \subseteq \underline{U}(\mu,t,\mathfrak{T})$, $\overline{U}(\mu,t,\mathfrak{S}) \subseteq \overline{U}(\mu,t,\mathfrak{T})$.

Proposition 3.11 *Let $\mathfrak{S} = (F, A)$ and $\mathfrak{T} = (G, B)$ be two non-null soft sets over S, μ be a fuzzy ideal of S. Then*

$$\overline{U}(\mu, t, \mathfrak{S}) \cdot \overline{U}(\mu, t, \mathfrak{T}) \subseteq \overline{U}(\mu, t, \mathfrak{S} \cdot \mathfrak{T}).$$

Proof. For any $a \in Supp(F, A)$, $b \in Supp(G, B)$. Let $m \in \overline{U}(\mu, t, F(x)) \cdot \overline{U}(\mu, t, G(y))$. There exists $a_i \in \overline{U}(\mu, t, F(x))$, $b_i \in \overline{U}(\mu, t, G(y))$ such that $m = a_ib_i$, so $[a_i]_{(\mu,t)} \cap F(x) \neq \emptyset$ and $[b_i]_{(\mu,t)} \cap G(y) \neq \emptyset$. Now let $x_i \in [a_i]_{(\mu,t)} \cap F(x)$, $y_i \in [b_i]_{(\mu,t)} \cap G(y)$, then $(x_i, a_i) \in U(\mu, t)$ and $(y_i, b_i) \in U(\mu, t)$. As $U(\mu, t)$ is a congruence relation, so $(x_iy_i, a_ib_i) \in U(\mu, t)$. Since $x_iy_i \in F(x) \cdot G(y)$, Therefore $[a_ib_i]_{(\mu,t)} \cap F(x) \cdot G(y) \neq \emptyset$, that is $[m]_{(\mu,t)} \cap F(x) \cdot G(y) \neq \emptyset$. This means $m \in \overline{U}(\mu, t, F(x)G(y))$ and $\overline{U}(\mu, t, \mathfrak{S}) \cdot \overline{U}(\mu, t, \mathfrak{T}) \subseteq \overline{U}(\mu, t, \mathfrak{S} \cdot \mathfrak{T})$. ∎

The following example shows that the converse of Proposition 3.11 is not true.

Example 3.12 Let $S = \{a, b, c, d\}$ be a semigroup with the following "·" table.

$\cdot$	a	b	c	d
a	a	a	a	d
b	a	b	a	d
c	a	a	c	d
d	d	d	d	d

Let $\mu = \frac{0.5}{a} + \frac{0.3}{b} + \frac{0.1}{c} + \frac{0.8}{d}$ be a fuzzy set of S, μ is a fuzzy ideal of S. Let $t = 0.3$, then $[a]_{(\mu,0.3)} = [b]_{(\mu,0.3)} = [d]_{(\mu,0.3)} = \{a, b, d\}$, $[c]_{(\mu,0.3)} = \{c\}$. Obviously, $U(\mu; t)$ is a congruence relation. Then we define two soft sets $\mathfrak{S} = (F, A)$ and $\mathfrak{T} = (G, B)$ over S, where $F(e_1) = \{a, c\}$, $F(e_2) = \{b, c\}$ and $G(e_3) = \{a, b, d\}$. Then $\overline{U}(\mu, t, F(e_1)) = \{a, b, c, d\}$, $\overline{U}(\mu, t, F(e_2)) = \{a, b, c, d\}$, $\overline{U}(\mu, t, G(e_3)) = \{a, b, d\}$. Thus $\overline{U}(\mu, t, F(e_1)) \cdot \overline{U}(\mu, t, G(e_3)) = \{a, b, d\}$, $\overline{U}(\mu, t, F(e_2)) \cdot \overline{U}(\mu, t, G(e_3)) = \{a, b, d\}$. On the other hand, $\overline{U}(\mu, t, F(e_1) \cdot G(e_3)) = \{a, b, d\}$, $\overline{U}(\mu, t, F(e_2) \cdot G(e_3)) = \{a, b, c, d\}$. Therefore, obviously, $\overline{U}(\mu, t, F(e_2)) \cdot \overline{U}(\mu, t, G(e_3)) \subseteq \overline{U}(\mu, t, F(e_2) \cdot G(e_3))$. Hence, $\overline{U}(\mu, t, \mathfrak{S}) \cdot \overline{U}(\mu, t, \mathfrak{T}) \subsetneqq \overline{U}(\mu, t, \mathfrak{S} \cdot \mathfrak{T})$.

Proposition 3.13 *Let $\mathfrak{S} = (F, A)$ and $\mathfrak{T} = (G, B)$ be two non-null soft sets over S, μ be a fuzzy ideal of S, $U(\mu, t)$ be a complete congruence relation. Then*

$$\overline{U}(\mu, t, \mathfrak{S}) \cdot \overline{U}(\mu, t, \mathfrak{T}) = \overline{U}(\mu, t, \mathfrak{S} \cdot \mathfrak{T}).$$

Proof. Firstly, from Proposition 3.11, we have $\overline{U}(\mu, t, \mathfrak{S}) \cdot \overline{U}(\mu, t, \mathfrak{T}) \subseteq \overline{U}(\mu, t, \mathfrak{S} \cdot \mathfrak{T})$. So we only prove $\overline{U}(\mu, t, \mathfrak{S}) \cdot \overline{U}(\mu, t, \mathfrak{T}) \supseteq \overline{U}(\mu, t, \mathfrak{S} \cdot \mathfrak{T})$. For any $a \in Supp(F, A)$, $b \in Supp(G, B)$. Let $m \in \overline{U}(\mu, t, \mathfrak{S} \cdot \mathfrak{T})$, then $[m]_{(\mu,t)} \cap F(x)G(y) \neq \emptyset$, that is, there exists $n \in [m]_{(\mu,t)}$ and $n \in F(x) \cdot G(y)$ such that $n = a_ib_i$, for $a_i \in F(x), b_i \in G(y)$ and $m \in [n]_{(\mu,t)} = [a_ib_i]_{(\mu,t)} = [a_i]_{(\mu,t)} \cdot [b_i]_{(\mu,t)}$. Since $U(\mu, t)$ is a complete congruence relation, then there exists $x_i \in [a_i]_{(\mu,t)}$ and $y_i \in [b_i]_{(\mu,t)}$ such that $m = x_iy_i$, hence $a_i \in [x_i]_{(\mu,t)} \cap F(x)$ and $b_i \in [x_i]_{(\mu,t)} \cap G(y)$, therefore $m \in \overline{U}(F(x)) \cdot \overline{U}(G(y))$, so $\overline{U}(\mu, t, \mathfrak{S}) \cdot \overline{U}(\mu, t, \mathfrak{T}) = \overline{U}(\mu, t, \mathfrak{S} \cdot \mathfrak{T})$. ∎

Example 3.14 Let $S = \{a, b, c, d\}$ be a semigroup with the following "$\cdot$" table:

$\cdot$	a	b	c	d
a	a	a	a	d
b	a	b	a	d
c	a	a	c	d
d	d	d	d	d

Assume that $\mu = \frac{0.6}{a} + \frac{0.4}{b} + \frac{0.1}{c} + \frac{0.9}{d}$ is a fuzzy set of S and $t = 0.5$, μ is a fuzzy ideal of S, then we have $U(\mu; 0.5) = \{(a, a), (b, b), (c, c), (d, d), (a, d)\}$, so we have $[a]_{(\mu,0.5)} = [d]_{(\mu,0.5)} = \{a, d\}$, $[b]_{(\mu,0.5)} = \{b\}$, $[c]_{(\mu,0.5)} = \{c\}$. Obviously, $U(\mu; t)$ is a complete congruence relation.

Let $\mathfrak{S} = (F, A)$ and $\mathfrak{T} = (G, B)$ be two soft sets over S, where $A = \{e_1, e_2\}, B = \{e_3\}$, $G(e_3) = \{b, c, d\}$. And $F(e_1) = \{a, b, c\}$, $F(e_2) = \{b, d\}$ and $G(e_3) = \{b, c, d\}$. Then $\overline{U}(\mu, t, F(e_1)) = \{a, b, c, d\}$, $\overline{U}(\mu, t, F(e_2)) = \{a, b, d\}$, $\overline{U}(\mu, t, G(e_3)) = \{a, b, c, d\}$. Thus $\overline{U}(\mu, t, F(e_1)) \cdot \overline{U}(\mu, t, G(e_3)) = \{a, b, c, d\}$, $\overline{U}(\mu, t, F(e_2)) \cdot \overline{U}(\mu, t, G(e_3)) = \{a, b, d\}$. $\overline{U}(\mu, t, F(e_1) \cdot G(e_3)) = \{a, b, c, d\}$, $\overline{U}(\mu, t, F(e_2) \cdot G(e_3)) = \{a, b, d\}$. So $\overline{U}(\mu, t, F(e_1)) \cdot \overline{U}(\mu, t, G(e_3)) = \overline{U}(\mu, t, F(e_1) \cdot G(e_3)$, $\overline{U}(\mu, t, F(e_2)) \cdot \overline{U}(\mu, t, G(e_3)) = \overline{U}(\mu, t, F(e_2) \cdot G(e_3))$. Therefore, we know Proposition 3.13 is proper.

Proposition 3.15 *Let $\mathfrak{S} = (F, A)$ and $\mathfrak{T} = (G, B)$ be two non-null soft sets over S, μ be a fuzzy ideal of S and $U(\mu, t)$ be a complete congruence relation. Then*

$$\underline{U}(\mu, t, \mathfrak{S}) \cdot \underline{U}(\mu, t, \mathfrak{T}) \subseteq \underline{U}(\mu, t, \mathfrak{S} \cdot \mathfrak{T}).$$

Proof. For any $a \in Supp(F, A)$, $b \in Supp(G, B)$. Let $m \in \underline{U}(\mu, t, \mathfrak{S}) \cdot \underline{U}(\mu, t, \mathfrak{T})$, then $m = a_i b_i$ where $a_i \in \underline{U}(\mu, t, F(x))$ and $b_i \in \underline{U}(\mu, t, G(y))$, that is, $[a_i]_{(\mu,t)} \subseteq F(x)$ and $[b_i]_{(\mu,t)} \subseteq G(y)$. So $[m]_{(\mu,t)} = [a_i b_i]_{(\mu,t)} = [a_i]_{(\mu,t)} \cdot [b_i]_{(\mu,t)} \subseteq F(x) \cdot G(y)$, therefore $m \in \underline{U}(\mu, t, F(x) \cdot G(y))$, that implies, $\underline{U}(\mu, t, \mathfrak{S}) \cdot \underline{U}(\mu, t, \mathfrak{T}) \subseteq \underline{U}(\mu, t, \mathfrak{S} \cdot \mathfrak{T})$. ∎

Example 3.16 In Example 3.4, let $\mathfrak{S} = (F, A)$ and $\mathfrak{T} = (G, B)$ be two soft sets over S, where $F(e_1) = \{a, b, c\}$, $F(e_2) = \{b, d\}$, $G(e_3) = \{b, c, d\}$, thus $\underline{U}(\mu, t, F(e_1)) = \{b, c\}$, $\underline{U}(\mu, t, F(e_2)) = \{b\}$, $\underline{U}(\mu, t, G(e_3)) = \{b, c\}$, so $\underline{U}(\mu, t, F(e_1)) \cdot \underline{U}(\mu, t, G(e_3)) = \{a, b, c\}$, $\underline{U}(\mu, t, F(e_2)) \cdot \underline{U}(\mu, t, G(e_3)) = \{a, b\}$, $\underline{U}(\mu, t, F(e_1) \cdot G(e_3)) = \{a, b, c, d\}$, $\underline{U}(\mu, t, F(e_2) \cdot G(e_3)) = \{a, b, d\}$, so $\underline{U}(\mu, t, \mathfrak{S}) \cdot \underline{U}(\mu, t, \mathfrak{T}) \subsetneqq \underline{U}(\mu, t, \mathfrak{S} \cdot \mathfrak{T})$.

4. Characterizations of rough soft semigroups based on $U(\mu, t)$

In this section, we study the characterizations of rough soft semigroups based on $U(\mu, t)$.

Proposition 4.1 *Let μ and ν be two fuzzy ideals of S and $t \in [0, 1]$. If $\mathfrak{S} = (F, A)$ is a non-null soft set over S, then*

$$\overline{U}(\mu \cap \nu, t, \mathfrak{S}) \subseteq \overline{U}(\mu, t, \mathfrak{S}) \Cap \overline{U}(\nu, t, \mathfrak{S}).$$

Proof. For all $x \in \mathrm{Supp}(F,A)$, let $a \in \overline{F}_{\mu\cap\nu}(x)$, then we have $[a]_{(\mu\cap\nu,t)} \cap F(x) \neq \emptyset$. So there exists $m \in [a]_{(\mu\cap\nu,t)} \cap F(x)$, it follows that $(a,m) \in U(\mu\cap\nu,t)$ and $m \in F(x)$. This means $((\mu\cap\nu)(a) \wedge (\mu\cap\nu)(m)) \vee Id_S(a,m) \geq t$ and $m \in F(x)$. If $a = m$, then $a \in F(x)$ and $a \in [a]_{(\mu,t)}$. So $a \in [a]_{(\mu,t)} \cap F(x)$, that is $a \in \overline{F}_\mu(x)$. In similar way, we have $a \in \overline{F}_\nu(x)$. Hence $a \in \overline{F}_\mu(x) \cap \overline{F}_\nu(x)$. If $a \neq m$, then $(\mu\cap\nu)(a) \wedge (\mu\cap\nu)(m) \geq t$ and $m \in F(x)$, it implies $\mu(a) \wedge \mu(m) \geq t$ and $\nu(a) \wedge \nu(m) \geq t$. Thus, $(a,m) \in U(\mu,t), (a,m) \in U(\nu,t)$ and $m \in F(x)$. Hence $m \in [a]_{(\mu,t)} \cap F(x)$ and $m \in [a]_{(\nu,t)} \cap F(x)$. This means $[a]_{(\mu,t)} \cap F(x) \neq \emptyset$ and $[a]_{(\nu,t)} \cap F(x) \neq \emptyset$, it implies $a \in \overline{F}_\mu(x)$ and $a \in \overline{F}_\nu(x)$. So $a \in \overline{F}_\mu(x) \cap \overline{F}_\nu(x)$. Hence $\overline{F}_{(\mu\cap\nu)}(x) \subseteq \overline{F}_\mu(x) \cap \overline{F}_\nu(x)$. Therefore $\overline{U}(\mu\cap\nu,t,\mathfrak{S}) \subseteq \overline{U}(\mu,t,\mathfrak{S}) \Cap \overline{U}(\nu,t,\mathfrak{S})$. This completes the proof. ∎

Proposition 4.2 *Let μ and ν be two fuzzy ideals of S and $t \in [0,1]$. If $\mathfrak{S} = (F,A)$ is a non-null soft set over S, then*

$$\underline{U}(\mu,t,\mathfrak{S}) \Cap \underline{U}(\nu,t,\mathfrak{S}) \subseteq \underline{U}(\mu\cap\nu,t,\mathfrak{S}).$$

Proof. For all $x \in \mathrm{Supp}(F,A)$, let $a \in \underline{F}_\mu(x) \cap \underline{F}_\nu(x)$, we have $a \in \underline{F}_\mu(x)$ and $a \in \underline{F}_\nu(x)$, it implies $[a]_{(\mu,t)} \subseteq F(x)$ and $[a]_{(\nu,t)} \subseteq F(x)$. By Lemma 3.2, $[a]_{(\mu\cap\nu,t)} \subseteq [a]_{(\mu,t)} \subseteq F(x)$, this means $a \in \underline{F}_{\mu\cap\nu}(x)$. Hence $\underline{U}(\mu,t,\mathfrak{S}) \Cap \underline{U}(\nu,t,\mathfrak{S}) \subseteq \underline{U}(\mu\cap\nu,t,\mathfrak{S})$. ∎

Example 4.3 Let $S = \{a,b,c,d\}$ be a semigroup in Example 3.12. Let $\mathfrak{S} = (F,A)$ be a soft set and μ and ν be fuzzy ideals over S, where $F(e_1) = \{b,c\}$, $F(e_2) = \{a,b\}$, $F(e_3) = \{c,d\}$, $\mu = \frac{0.7}{a} + \frac{0.3}{b} + \frac{0.5}{c} + \frac{0.8}{d}$, $\nu = \frac{0.5}{a} + \frac{0.5}{b} + \frac{0.4}{c} + \frac{0.9}{d}$, $t = 0.5$, $\mu\cap\nu = \frac{0.5}{a} + \frac{0.3}{b} + \frac{0.4}{c} + \frac{0.8}{d}$, then we have $U(\mu,0.5) = \{\{a,a\},\{b,b\},\{c,c\},\{d,d\},\{a,c\},\{a,d\},\{c,d\}\}$, so $[a]_{(\mu,0.5)} = [c]_{(\mu,0.5)} = [d]_{(\mu,0.5)} = \{a,c,d\}$, $[b]_{(\mu,0.5)} = \{b\}$. $U(\nu,0.5) = \{\{a,a\},\{b,b\},\{c,c\},\{d,d\},\{a,b\},\{a,d\},\{b,d\}\}$ and $[a]_{(\mu,0.5)} = [b]_{(\mu,0.5)} = [d]_{(\mu,0.5)} = \{a,b,d\}$, $[c]_{(\mu,0.5)} = \{c\}$, $U(\mu\cap\nu,0.5) = \{\{a,a\},\{b,b\},\{c,c\},\{d,d\},\{a,d\}\}$ and $[a]_{(\mu,0.5)} = [d]_{(\mu,0.5)} = \{a,d\}$, $[b]_{(\mu,0.5)} = \{b\}$, $[c]_{(\mu,0.5)} = \{c\}$, thus $\overline{U}(\mu,t,F(e_1)) = \{a,b,c,d\}, \overline{U}(\mu,t,F(e_2)) = \{a,b,c,d\}$, $\overline{U}(\mu,t,F(e_3)) = \{a,c,d\}$ and $\underline{U}(\mu,t,F(e_1)) = \{b\}, \underline{U}(\mu,t,F(e_2)) = \{b\}$, $\underline{U}(\mu,t,F(e_3)) = \emptyset$, $\underline{U}(\nu,t,F(e_1)) = \{c\}, \underline{U}(\nu,t,F(e_2)) = \emptyset, \underline{U}(\nu,t,F(e_3)) = \{c\}$ and $\overline{U}(\nu,t,F(e_1)) = \{a,b,c,d\}, \overline{U}(\nu,t,F(e_2)) = \{a,b,d\}, \overline{U}(\nu,t,F(e_3)) = \{a,b,c,d\}$, $\overline{U}(\mu\cap\nu,t,F(e_1)) = \{b,c\}, \overline{U}(\mu\cap\nu,t,F(e_2)) = \{a,b,d\}, \overline{U}(\mu\cap\nu,t,F(e_3)) = \{a,c,d\}$ and $\underline{U}(\mu\cap\nu,t,F(e_1)) = \{b,c\}, \underline{U}(\mu\cap\nu,t,F(e_2)) = \{b\}, \underline{U}(\mu\cap\nu,t,F(e_3)) = \{c\}$, so, obviously, we obtain $\overline{U}(\mu\cap\nu,t,F(e_1)) \subsetneqq \underline{U}(\mu,t,F(e_1)) \Cap \overline{U}(\nu,t,F(e_1))$ and $\underline{U}(\mu,t,F(e_3)) \Cap \underline{U}(\nu,t,F(e_3)) \subsetneqq \underline{U}(\mu\cap\nu,t,F(e_3))$. Therefore, we cannot use " $='$ " to replace " $\subseteq$ " in Propositions 4.1 and 4.2.

Definition 4.4 Let (S,μ,t) be a Pawlak approximation space and $\mathfrak{S} = (F,A)$ a soft set over S.

(1) Then $\underline{U}(\mu,t,\mathfrak{S})(\overline{U}(\mu,t,\mathfrak{S}))$ is called a lower (upper) rough soft semigroup (resp., ideal, prime ideal, bi-ideal) of S, if $\underline{F}_\mu(x)(\overline{F}_\mu(x))$ is a subsemigroup (resp., ideal, prime ideal, bi-ideal) of S, for all $x \in A$.

(2) Moreover, $\mathfrak{S}$ is called a rough soft semigroup (resp., rough soft ideal, rough soft prime ideal, bi-ideal) of S, if $\underline{F}_\mu(x)$ and $\overline{F}_\mu(x)$ are subsemigroups (resp., ideals, prime ideals, bi-ideal) of S for all $x \in A$.

Theorem 4.5 *Let (S, μ, t) be a Pawlak approximation space and $\mathfrak{S} = (F, A)$ a soft semigroup over S. Then $\mathfrak{S} = (F, A)$ is an upper rough soft semigroup of S.*

Proof. For any $a \in A$, let $x_1 \in \overline{F}_\mu(a), x_2 \in \overline{F}_\mu(a)$, then $[x_1]_{(\mu,t)} \cap F(a) \neq \emptyset$, $[x_2]_{(\mu,t)} \cap F(a) \neq \emptyset$. This means there exist $b_1 \in [x_1]_{(\mu,t)} \cap F(a)$, $b_2 \in [x_2]_{(\mu,t)} \cap F(a)$. Thus $b_1 \in [x_1]_{(\mu,t)}$, $b_1 \in F(a), b_2 \in [x_2]_{(\mu,t)}$, $b_2 \in F(a)$ and $(x_1, b_1) \in U(\mu, t)$, $(x_2, b_2) \in U(\mu, t)$. Since $U(\mu, t)$ is a congruence relation on S, we have (x_1x_2, b_1b_2) $\in U(\mu, t)$, it implies $[x_1x_2]_{(\mu,t)}{=}[b_1b_2]_{(\mu,t)}$. Since $\mathfrak{S} = (F, A)$ is a soft semigroup over S, so $F(a)$ is an subsemigroup of S, so $b_1b_2 \in F(a)$. That is $[b_1b_2]_{(\mu,t)} \cap F(a) \neq \emptyset$. Hence $[x_1x_2]_{(\mu,t)} \cap F(a) \neq \emptyset$ and $x_1x_2 \in \overline{F}_\mu(a)$. Then $\overline{F}_\mu(a)$ is a subsemigroup of S. Therefore $\mathfrak{S} = (F, A)$ is a upper rough soft semigroup over S. ∎

Theorem 4.6 *Let (S, μ, t) be a Pawlak approximation space and $\mathfrak{S} = (F, A)$ a soft semigroup over S. If $U(\mu, t)$ is a complete congruence relation on S and $\underline{U}(\mu, t, \mathfrak{S})$ is non-null, then $\mathfrak{S} = (F, A)$ is a lower rough soft semigroup of S.*

Proof. For any $a \in A$, since $\underline{U}(\mu, t, \mathfrak{S})$ is non-null, then $\underline{F}_\mu(a) \neq \emptyset$. Let $x_1 \in \underline{F}_\mu(a), x_2 \in \underline{F}_\mu(a)$, then $[x_1]_{(\mu,t)} \subseteq F(a), [x_2]_{(\mu,t)} \subseteq F(a)$. Since $U(\mu, t)$ is a complete congruence relation and $\mathfrak{S} = (F, A)$ is a soft semigroup over S, we have $[x_1 \cdot x_2]_{(\mu,t)} = [x_1]_{(\mu,t)} \cdot [x_2]_{(\mu,t)} \subseteq F(a) \cdot F(a) \subseteq F(a)$, so $x_1 \cdot x_2 \in \underline{F}_\mu(a)$, Hence $\underline{F}_\mu(a)$ is an subsemigroup of S. Therefore $\underline{U}(\mu, t, \mathfrak{S})$ is a soft semigroup of S. ∎

Theorem 4.7 *Let (S, μ, t) be a Pawlak approximation space and $\mathfrak{S} = (F, A)$ an idealistic soft semigroup over S. Then $\mathfrak{S} = (F, A)$ is an upper rough soft ideal of S.*

Proof. For any $a \in A, x \in \overline{F}_\mu(a), m \in S$, since $\mathfrak{S} = (F, A)$ an idealistic soft semigroup over S, according to Theorem 4.5, we can obtain that $\overline{U}(\mu, t, \mathfrak{S})$ is a soft semigroup of S. In the following, we only prove the properties of left (right). $x \in \overline{F}_\mu(a)$, so we have $[x]_{(\mu,t)} \cap F(a) \neq \emptyset$. Now let $n \in [x]_{(\mu,t)} \cap F(a)$, then we have $(n, x) \in U(\mu, t)$, and $n \in F(a)$, Since $U(\mu, t)$ is a congruence relation on S, so $(mn, mx) \in U(\mu, t)$, we have $[mn]_{(\mu,t)} = [mx]_{(\mu,t)}$. For $n \in F(a), m \in S$, and $F(a)$ is a ideal of S, then we have $mn \in F(a)$, so $[mx]_{(\mu,t)} \cap F(a) \neq \emptyset$,so $mx \in \overline{F}_\mu(a)$. Therefore, $\overline{F}_\mu(a)$ is a left ideal, we can prove $\overline{F}_\mu(a)$ in the similar way. On balance, $\overline{F}_\mu(a)$ is an ideal of S. Then $\mathfrak{S} = (F, A)$ is an upper rough soft ideal of S. ∎

Theorem 4.8 *Let (S, μ, t) be a Pawlak approximation space and $\mathfrak{S} = (F, A)$ an idealistic soft semigroup over S. If $U(\mu, t)$ is a complete congruence relation on S and $\underline{U}(\mu, t, \mathfrak{S})$ is non-null, then $\mathfrak{S} = (F, A)$ is a lower rough soft ideal of S.*

Proof. For any $a \in A$, since $\mathfrak{S} = (F, A)$ an idealistic soft semigroup over S, then $F(a)$ is an ideal of S. Since $\underline{U}(\mu, t, \mathfrak{S})$ is non-null, then $\underline{F}_\mu(a) \neq \emptyset$. Let

$m \in \underline{F}_\mu(a), x \in S$, then $[m]_{(\mu,t)} \subseteq F(a)$. Since $U(\mu,t)$ is a complete congruence relation on S, then $[mx]_{(\mu,t)} = [m]_{(\mu,t)} \cdot [n]_{(\mu,t)} \subseteq F(a) \cdot S \subseteq F(a)$, and so $mx \in \underline{F}_\mu(a)$. Similarly, we can prove that $xm \in \underline{F}_\mu(a)$. This means $\underline{F}_\mu(a)$ is an ideal of S. Hence $\underline{U}(\mu,t,\mathfrak{S})$ is a soft ideal of S. ∎

Theorem 4.9 *Let (S,μ,t) be a Pawlak approximation space and $\mathfrak{S} = (F,A)$ a prime idealistic soft semigroup over S. If $U(\mu,t)$ is a complete congruence relation over S, then $\mathfrak{S} = (F,A)$ is an upper rough soft prime ideal of S.*

Proof. Since $\mathfrak{S} = (F,A)$ is prime idealistic soft semigroup over S, then for all $a \in A$, $F(a)$ is a prime ideal of S. By Theorem 4.7, we know $\overline{F}_\mu(a)$ is an ideal of S. Let $ab \in \overline{F}_\mu(a)$ for some $a,b \in S$, since $U(\mu,t)$ is a complete congruence relation on S, then we have $[a]_{(\mu,t)} \cdot [b]_{(\mu,t)} \cap F(a) = [ab]_{(\mu,t)}) \cap F(a) \neq \emptyset$. So there exist $x_1 \in [a]_{(\mu,t)}, x_2 \in [b]_{(\mu,t)}$ such that $x_1 \cdot x_2 \in F(a)$. Since $F(a)$ is a prime ideal of S, then we have $x_1 \in F(a)$ or $x_2 \in F(a)$. Thus $[a]_{(\mu,t)} \cap F(a) \neq \emptyset$ or $[b]_{(\mu,t)} \cap F(a) \neq \emptyset$, and so $a \in \overline{F}_\mu(a)$ or $b \in \overline{F}_\mu(a)$. Hence $\overline{F}_\mu(a)$ is a prime ideal of S. Thus $\mathfrak{S} = (F,A)$ is an upper rough soft prime ideal of S. ∎

Example 4.10 Let $S = \{a,b,c,d,e\}$ be a semigroup and "$\cdot$" a operation defined by the table:

$\cdot$	a	b	c	d	e
a	b	b	d	d	d
b	b	b	d	d	d
c	d	d	c	d	c
d	d	d	d	d	d
e	d	d	c	d	c

Let $\mu = \frac{0.4}{a} + \frac{0.7}{b} + \frac{0.5}{c} + \frac{0.9}{d} + \frac{0.1}{e}$ be a fuzzy set of S, μ is a fuzzy ideal of S. Let $t = 0.5$, then $U(\mu;0.5) = \{(a,a),(b,b),(c,c),(d,d),(e,e),(b,c),(b,d)\}$, so $[a]_{(\mu,0.5)} = \{a\}$, $[b]_{(\mu,0.5)} = [c]_{(\mu,0.5)} = [d]_{(\mu,0.5)} = \{b,c,d\}$, $[e]_{(\mu,0.5)} = \{e\}$. And let $\mathfrak{S} = (F,A)$ be a soft set over S, $A = \{x_1,x_2\}$, $F(x_1) = \{b,e\}$, $F(x_2) = \{a,b,e\}$. Then we have $\underline{F}_\mu(x_1) = \{e\}, \underline{F}_\mu(x_2) = \{a,e\}$, $\overline{F}_\mu(x_1) = \{b,c,d,e\}, \overline{F}_\mu(x_2) = \{a,b,c,d,e\}$. Obviously, we have $\underline{F}_\mu(x_2), \overline{F}_\mu(x_1), \overline{F}_\mu(x_2)$ are ideals of S, but $\underline{F}_\mu(x_1)$ is not a ideal of S. Therefore, we say $\mathfrak{S}$ is a upper rough ideal of S, but it is not a lower rough ideal of S.

Theorem 4.11 *Let (S,μ,t) be a Pawlak approximation space and $\mathfrak{S} = (F,A)$ a prime idealistic soft semigroup over S. If $U(\mu,t)$ is a complete congruence relation on S and $\underline{U}(\mu,t,\mathfrak{S})$ is non-null, then $\mathfrak{S} = (F,A)$ is a lower rough soft prime ideal of S.*

Proof. Since $\mathfrak{S} = (F,A)$ is prime idealistic soft semigroup over S, then for all $a \in A$, $F(a)$ is a prime ideal of S. By Theorem 4.6, we know $\underline{F}_\mu(a)$ is an ideal of S. Let $x_1 \cdot x_2 \in \underline{F}_\mu(a)$ for some $x_1, x_2 \in S$, since $U(\mu,t)$ is a congruence relation over S, then we have $[x_1]_{(\mu,t)} \cdot [x_2]_{(\mu,t)} = [x_1 \cdot x_2]_{(\mu,t)} \subseteq F(a)$. We suppose that $\underline{F}_\mu(a)$ is not a prime ideal of S, then there exist $x_1, x_2 \in S$ such that $x_1 \cdot x_2 \in \underline{F}_\mu(a)$ but $x_1 \notin \underline{F}_\mu(a), x_2 \notin \underline{F}_\mu(a)$. Thus $[x_1]_{(\mu,t)} \nsubseteq F(a), [x_2]_{(\mu,t)} \nsubseteq F(a)$, then exist $x_1' \in [x_1]_{(\mu,t)}, x_1' \notin F(a), x_2' \in [x_2]_{(\mu,t)}, x_2' \notin F(a)$. Thus $x_1' \cdot x_2' \in [x_1]_{(\mu,t)} \cdot [x_2]_{(\mu,t)} \subseteq F(a)$. Since $F(a)$ is a prime ideal of S, we have $x_1' \in F(a)$ or $x_2' \in F(a)$. It contradicts the supposition. Hence $\mathfrak{S} = (F, A)$ is a lower rough soft prime ideal of S. ∎

Theorem 4.12 *Let (S, μ, t) be a Pawlak approximation space and $\mathfrak{S} = (F, A)$ a bi-idealistic soft semigroup over S. If $U(\mu, t)$ is a congruence relation over S, then $\overline{U}(\mu, t, \mathfrak{S})$ is a soft bi-ideal of S.*

Proof. Since $\mathfrak{S} = (F, A)$ is bi-idealistic soft semigroup over S, then for all $a \in A$, $F(a)$ is a bi-ideal of S, that is $F(a)$ is a subsemigroup of S and $F(a) \cdot S \cdot F(a) \subseteq F(a)$. By Theorem 4.5, we know $\overline{F}_\mu(a)$ is a subsemigroup of S. Let $y \in \overline{F}_\mu(a) \cdot S \cdot \overline{F}_\mu(a)$, there exists $m \in \overline{F}_\mu(a), s \in S, n \in \overline{F}_\mu(a)$ such that $y = msn$, let $k \in [m]_{(\mu,t)} \cap F(a) \neq \emptyset$ and $l \in [n]_{(\mu,t)} \cap F(a) \neq \emptyset$, then we have $[k, m] \in U(\mu, t), [l, n] \in U(\mu, t)$, $[ksl, msn] \in U(\mu, t)$, so $ksl \in [msn]_{(\mu,t)}$ and $ksl \in F(a) \cdot S \cdot F(a) \subseteq F(a)$, therefore $[y]_{(\mu,t)} = [msn]_{(\mu,t)} \cap F(a) \neq \emptyset$, $y \in \overline{F}_\mu(a)$. We have $\overline{F}_\mu(a) \cdot S \cdot \overline{F}_\mu(a) \subseteq \overline{F}_\mu(a)$. Then $\overline{U}(\mu, t, \mathfrak{S})$ is a soft bi-ideal of S. ∎

Theorem 4.13 *Let (S, μ, t) be a Pawlak approximation space and $\mathfrak{S} = (F, A)$ a bi-idealistic soft semigroup over S. If $U(\mu, t)$ is a complete congruence relation over S and $\underline{U}(\mu, t, \mathfrak{S}) \neq \emptyset$. $\mathfrak{S} = (F, A)$ is a lower rough soft bi-ideal of S.*

Proof. Since $\mathfrak{S} = (F, A)$ is bi-idealistic soft semigroup over S, then for all $a \in A$, $F(a)$ is a bi-ideal of S, that is $F(a)$ is a subsemigroup of S and $F(a) \cdot S \cdot F(a) \subseteq F(a)$. By Theorem 4.6, we know $\underline{F}_\mu(a)$ is a subsemigroup of S. Let $y \in \underline{F}_\mu(a) \cdot S \cdot \underline{F}_\mu(a)$, there exists $m \in \underline{F}_\mu(a), s \in S, n \in \underline{F}_\mu(a)$ such that $y = msn$, let $[m]_{(\mu,t)} \subseteq F(a)$ and $[n]_{(\mu,t)} \subseteq F(a)$, since $\mathfrak{S} = (F, A)$ a bi-idealistic soft semigroup over S and $U(\mu, t)$ is a complete congruence relation on S, then we have $[m]_{(\mu,t)} \cdot s \cdot [n]_{(\mu,t)} \subseteq [m]_{(\mu,t)} \cdot [s]_{(\mu,t)} \cdot [n]_{(\mu,t)} = [msn]_{(\mu,t)} \subseteq F(a) \cdot S \cdot F(a) \subseteq F(a)$, so $[y]_{(\mu,t)} \subseteq F(a)$, therefore $y \in \underline{F}_\mu(a)$. Therefore, $\mathfrak{S} = (F, A)$ is a lower rough soft bi-ideal of S. ∎

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References

[1] Aktaş, H., Çağman, N., *Soft sets and soft groups*, Inform. Sci., 177 (2007), 2726-2735.

[2] Ali, M.I., Davvaz, B., Shabir, M., *Some properties of generalized rough sets*, Inform. Sci., 224 (2013), 170-179.

[3] Davvaz, B., *Roughness in rings*, Inform. Sci., 164 (2004) 147-163.

[4] Feng, F., Ali, M.I., Shabir, M., *Soft relations applied to semigroups*, Filomat., 27 (2013), 1183-1196.

[5] Feng, F., Jun, Y.B., Zhao, X., *Soft semirings*, Comput. Math. Appl., 56 (2008), 2621-2628.

[6] Feng, F., Li, Y., *Soft subsets and soft product operations*, Inform. Sci., 232 (2013), 44-57.

[7] Jun, Y.B., *Roughness of gamma-subsemigroups/ideals in gamma-subsemigroups*, Bull. Korean Math. Soc., 40 (2003), 531-536.

[8] Jun, Y.B., *Soft BCK/BCI-algebras*, Comput. Math. Appl., 56 (2008), 1408-1413.

[9] Kuroki, N., *Rough ideals in semigroups*, Inform. Sci., 100 (1997), 139-163.

[10] Kim, C.S., Kang, J.G., Kim, J.S., *Uni-soft (quasi) ideals of semigroups*, Appl. Math. Sci., 7 (2013), 2455-2468.

[11] Molodtsov, D., *Soft set theory – first results*, Comput. Math. Appl., 37 (1999), 19-31.

[12] Maji, P.K., Biswas, R., Roy, A.R., *Soft set theory*, Comput. Math. Appl., 45 (2003), 555-562.

[13] Maji, P.K., Roy, A.R., Biswas, R., *An application of soft sets in a decision making problem*, Comput. Math. Appl., 44 (2002), 1077-1083.

[14] Pawlak, Z., *Rough sets*, Int. J. Inf. Comp. Sci., 11 (1982), 341-356.

[15] Shabir, M., Ali, A., *On soft ternary semigroups*, Ann. Fuzzy Math. Inform., 3 (2012), 39-59.

[16] Wang, Q., Zhan, J., *Rough semigroups and rough fuzzy semigroups based on fuzzy ideals*, Open Math., 2017, in press.

[17] Xiao, Q., Zhang, Z., *Rough prime ideals and rough fuzzy prime ideals in semigroups*, Inform. Sci., 176 (2006), 725-733.

[18] Zhan, J., Davvaz, B., *A kind of new rough set: Rough soft sets and rough soft rings*, J. Intell. Fuzzy Syst., 30 (2016), 475-483.

[19] Zhang, X., Dai, J., Yu, Y., *On the union and intersection operations of rough sets based on various approximation spaces*, Inform. Sci., 292 (2015), 214-229.

[20] Zhan, J., Liu, Q., Davvaz, B., *A new rough set theory: rough soft hemirings*, J. Intell. Fuzzy Syst., 28 (2015), 1687-1697.

[21] ZHAN, J., LIU, Q., ZHU, W., *Another approach to soft hemirings and corresponding decision making*, Soft Comput, DOI: 10.1007/s00500-016-2058-5.

[22] ZHAN, J., ZHOU, X., XIANG, D., *Roughness in n-ary semigroups based on fuzzy ideals*, J. Intell. Fuzzy Syst., 30 (2016), 2833-2841.

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SOLVING A CLASS OF BOUNDARY VALUE PROBLEMS IN STRUCTURAL ENGINEERING AND FLUID MECHANICS USING HOMOTOPY PERTURBATION AND ADOMIAN DECOMPOSITION METHODS

H. Hosseinzadeh[1]
Department of Mathematics
University of Mazandaran
P.O. Box 47416-95447, Babolsar
Iran

H. Jafari
Department of Mathematics
University of Mazandaran
P.O. Box 47416-95447, Babolsar
Iran
Department of Mathematical Sciences
University of South Africa
P.O. Box 392, UNISA 0003
South Africa

M.R. Gholami
Department of Mathematics
University of Mazandaran
P.O. Box 47416-95447, Babolsar
Iran

D.D. Ganji
Department of Mathematical Engineering
Babol University of Technology
P.O. Box 484, Babol
Islamic Republic of Iran

Abstract. In this article, the performance of two analytical methods known as the homotopy perturbation method (HPM) and Adomian decomposition method (ADM) on solving linear and nonlinear boundary value problems structural engineering and fluid mechanics are compared. In order to compare these mathematical models, various problems in inelastic and viscoelastic flows, deformation of beams, and plate deflection theory are chosen. In addition, the results of these two methods are compared with exact solutions to evaluate the precision and accuracy of these numerical methods.

Keywords: deformation of elastic beams, plate deflection theory, homotopy perturbation method, Adomian's decomposition method, boundary-value problems, exact solution.

[1]Corresponding author: E-mail: hosseinzadeh.umz@gmail.com

1. Introduction

Recently, several researches and works are devoted to develop analytical methods for solving nonlinear problems. In addition, many scientists try to compare the existed methods to evaluate the efficiency of them. This paper focuses on the analytical approximate solution of fourth-order equations when nonlinear boundary conditions contains third-order derivatives. Equation (1.1) presents The the general type of the equation when a integer n, $n \geq 2$, is fixed and positive. Thus, differential equation of order $2n$ is as follows:

$$y^{(2n)} + f(x, y) = 0 \tag{1.1}$$

The following equations define the boundary conditions

$$y^{(2j)}(a) = A_2 j, y^{(2j)}(b) = B_{2j}, j = 0, \cdots, n-1, \tag{1.2}$$

where $-\infty < a \leq x \leq b < \infty$, A_{2j}, B_{2j}, $j = 0, \cdots, n-1$ are finite constants.

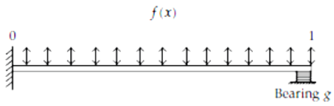

Figure 1: Beam on elastic bearing

In this modeling, it is assumed that the value of y is adequately different and a unique solution of (1.1) existed. One of main issue in this type of problem is plate-deflection theory. It is also crucial in fluid mechanics for modeling viscoelastic and inelastic flows [1], [2]. Usmani [1], [2] presented sixth order methods for solving linear differential equation $y^{(4)} + P(x)y = q(x)$ when the boundary conditions is as follows: $y(a) = A_0, y''(a) = A_2,\ y(b) = B_0,\ y''(b) = B_2$. According to the method described in [1], five diagonal linear systems are obtained and they contains p', p'', q', q'' at a and b, while the method described in [2] constrains nine diagonal linear systems.

In the other work, iterative solutions were proposed by Ma and Silva [3] for modeling beams (1.1) as elastic foundations. According to the classical beam theory, they found that if $u = u(x)$ represents the pattern of the deformed beam, the bending moment satisfies the following relation $M = -EIu''$, where E is the Young modulus of elasticity and I is the inertial moment. Since a load $f = f(x)$, induces the deformation, a free-body diagram, leads following $f = -v'$ and $v = M' = -EIu''$, where v represents the shear force and u is an elastic beam with length $L = 1$, with clamped at its left side ($x = 0$). The load f applied along its length and this causes deformations in the beam (Figure 1), Maand Silva [3] presented the following equation for this boundary value problem by assuming $EI = 1$:

$$u^{iv}(x) = f(x, u(x)),\ 0 < x < 1, \tag{1.3}$$

the boundary conditions are as below:

$$u(0) = u'(0) = 0, \tag{1.4}$$

$$u''(1) = 0,\ u'''(1) = g(u(1)), \tag{1.5}$$

where $f \in C([0,1] \times \mathbb{R})$ and $g \in C(\mathbb{R})$ are real functions. The boundary conditions are obtained through real modelling assumptions. $u'''(1)$ is obtained from the shear force at $x = 1$, and the second boundary condition equation (1.5) shows that the vertical force is equal to $g(u(1))$. Indeed, it presents a nonlinear relation between the vertical force and the displacement $u(1)$. In addition, this assumption $u''(1) = 0$ expresses that no bending moment occurs at $x = 1$, and the beam is assumed at rest on the bearing g.

Ma and Silva [3] presented solutions for this nonlinear equation by means of iterative procedures and explained that the accuracy of results is highly proportional to the integration method used in the iterative process.

Recently, nonlinear sciences have extensively developed, and several methods were proposed to solve differential nonlinear equations such as boundary value problems (BVPS). Among various approaches, four methods are sophisticated and robust: the perturbation parameter method (PPM), homotopy perturbation method (HPM), Adomian decomposition method (ADM) and the variational iteration method (VIM) [4]–[15]. In this research, the abilities and accuracy of two perturbation methods, homotopy perturbation method and Adomian decomposition method, for solving different forms of equation (1.1) with various boundary conditions are compared.

2. The methods

2.1. Basic idea of homotopy perturbation method

Homotopy perturbation method (HPM) is summarized by the following function:

$$A(u) - f(r) = 0,\ r \in \Omega \tag{2.1}$$

with the following boundary conditions of:

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0,\ r \in \Omega, \tag{2.2}$$

where A, B, $f(r)$ and Ω are a general differential operator, a boundary operator, a known analytical function and the boundary of the domain Ω, respectively. In this method, it is assumed that the operator A contains two parts: a linear part L and a non-linear part $N(u)$. Therefore, equation (2.1) rewritten as follows:

$$L(u) + N(u) - f(r) = 0, \tag{2.3}$$

By applying the homotopy technique, we have the following relation:

$$H(\nu, p) = (1-p)[L(\nu) - L(u_0)] + p[A(\nu) - f(r)] = 0,\ p \in [0,1],\ r \in \Omega, \tag{2.4}$$

or

$$H(\nu, p) = L(\nu) - L(u_0) + pL(u_0) + p[A(\nu) - f(r)] = 0, \tag{2.5}$$

where $p \in [0, 1]$ is an embedding parameter, while u_0 is an initial estimation of equation (2.1), to satisfy the boundary conditions. Therefore, equations (2.4) and (2.5) are transferred to the followings relations:

$$H(\nu, 0) = L(\nu) - L(u_0) = 0, \tag{2.6}$$
$$H(\nu, 1) = A(\nu) - f(r) = 0, \tag{2.7}$$

As the p value is changed from zero to unity, $\nu(r, p)$ is transformed from u_0 to $u(r)$. In the topology, this is known as a deformation, and $L(\nu)L(u_0)$ and $A(\nu)f(r)$ are recognized as Homotopy. According to the HPM method, it is assumed that the parameter p is a small parameter in the initial steps, and the solutions of equations (2.4) and (2.5) is a power series in p:

$$\nu = \nu_0 + p^1\nu_1 + p^2\nu_2 + \cdots, \tag{2.8}$$

when $p = 1$, it changes in the approximate solution of equation (2.8) to:

$$u = \lim_{p \to 1} \nu = \nu_0 + \nu_1 + \nu_2 + \cdots \tag{2.9}$$

In recent decades, the perturbation method and the homotopy method are combined and called the HPM. This method removes the disadvantages of the conventional perturbation methods while it contains all of their advantages. The series (2.9) is convergent for most of physical problems. However, the convergent rate highly related to the nonlinear operator $A(\nu)$. Furthermore, He (He, 1999b) proposed the following suggestions:

- The second derivative term of $N(\nu)$ must be small in comparison with ν because the parameter may be relatively large, i.e., $p \to 1$.
- The norm of $L^{-1}\frac{\partial N}{\partial \nu}$ must be lesser than one to elevate the converges of series.

2.2. Basic idea of Adomian decomposition method

The Adomian decomposition method transfers a general nonlinear equations in the following form:

$$Lu + Ru + Nu = g. \tag{2.10}$$

In this equation, the highest order derivative term is assigned to L which is assumed to be easily invertible, R is the linear differential operator of terms with less order than L, Nu refers the nonlinear terms and g is known as the source term. In the next step, the inverse operator L^1 is applied to the both sides of equation (2.10), and we obtain following relation with the given conditions:

$$u = f(x) - L^{-1}(Ru) - L^{-1}(Nu), \tag{2.11}$$

where the function $f(x)$ represents the terms obtained from the integration of the source term $g(x)$ with given boundary conditions.

In nonlinear differential equations, it is supposed that the nonlinear operator $N(u) = F(u)$ could be represented by an infinite series of the so-called Adomian polynomials

$$F(u) = \sum_{m=0}^{\infty} A_m. \tag{2.12}$$

The polynomials series (A_m) are produced for all types of nonlinearity terms so that A_0 is only proportional to u_0, A_1 depends on u_0 and u_1, and so on. According to the Adomian polynomials introduced above, it is found that the summation of subscripts of the component of u for each term of A_m is equal to n. The Adomian method presents a following series for the solution $u(x)$

$$u = \sum_{m=0}^{\infty} u_m. \tag{2.13}$$

In addition, the infinite series of a Taylor expansion about u_0 is proposed for $F(u)$ as follows:

$$F(u) = F(u_0) + F'(u_0)(u - u_0) + F''(u_0)\frac{(u-u_0)}{2!} + F'''(u_0)\frac{(u-u_0)^2}{3!} + \cdots \tag{2.14}$$

Then, equation (2.13) is reformed as $uu_0 = u_1 + u_2 + u_3 + \cdots$, and replaced in equation (2.15). In the next step, two definitions of $F(u)$ presented in equation (2.14) and equation (2.12) become equals, results formulas for the Adomian polynomials in the form of

(2.15)
$$F(u) = A_1 + A_2 + \cdots = F(u_0) + F'(u_0)(u_1 + u_2 + \cdots) + F''(u_0)\frac{(u_1 + u_2 +)^2}{2!} + \cdots$$

Then, it is supposed that each terms in equation (2.15) is equivalent with Adomians polynomials A_0, A_1, A_2, A_3 and A_4 as follows:

$$A_0 = F(u_0) \tag{2.16}$$

$$A_1 = u_1 F'(u_0) \tag{2.17}$$

$$A_2 = u_2 F'(u_0) + \frac{1}{2!}u_1^2 F''(u_0) \tag{2.18}$$

$$A_3 = u_3 F'(u_0) + u_1 u_2 F''(u_0) + \frac{1}{3!}u_1^3 F'''(u_0) \tag{2.19}$$

$$A_4 = u_4 F'(u_0) + (\frac{1}{2!}u_2^2 + u_1 u_3)F''(u_0) + \frac{1}{2!}u_1^2 u_2 F'''(u_0) + \frac{1}{4!}u_1^4 F^{(iv)}(u_0). \tag{2.20}$$

3. Application of methods

In this section, various problems are solved with these two approaches and the result are compared with exact solution.

3.1. Application of HPM

Example 3.1. Equation (3.1) is chosen as our first linear boundary value problem:

$$u^{(4)}(x) = u(x) + 4e^x \tag{3.1}$$

with following boundary conditions

$$u(0) = 1,\ u'(0) = 2,\ u(1) = 2e,\ u'(1) = 3e. \tag{3.2}$$

It is clear that the exact solution of this problem is

$$u(x) = (1 + x)e^x. \tag{3.3}$$

In order to applied HPM for solving equation (3.1), the following processes are considered after the linear and nonlinear portions of the equation are separated. Thus, following equation is obtained by Homotopy method.

$$H(x, p) = (1 - p)\left(\frac{d^4}{dx^4}u(x) - \frac{d^4}{dx^4}u_0(x)\right) + p\left(\frac{d^4}{dx^4}u(x) - u(x) - 4e^x\right). \tag{3.4}$$

Then, $\nu = \nu_0 + p\nu_1 + \cdots$, is replaced in to Equation (2.2) and the following equivalents are obtained:

$$p^1 : \frac{d^4}{dx^4}u_1(x) + \frac{d^4}{dx^4}u_0(x) - u_0(x) - 4e^x = 0, \tag{3.5}$$

$$p^2 : \frac{d^4}{dx^4}u_2(x) - u_1(x) = 0, \tag{3.6}$$

$$p^3 : \frac{d^4}{dx^4}u_3(x) - u_2(x) = 0, \tag{3.7}$$

$$p^4 : \frac{d^4}{dx^4}u_4(x) - u_3(x) = 0, \tag{3.8}$$

In order to solve this equation with HPM, we proposed an arbitrary initial approximation:

$$u_0(x) = ax^3 + bx^2 + cx + d. \tag{3.9}$$

By applying equation (3.9) in equations (3.5)-(3.8), we obtain the following:

$$\begin{aligned}
u_1(x) &= \frac{1}{840}ax^7 + \frac{1}{360}bx^6 + \frac{1}{120}cx^5 + 4e^x + \frac{1}{24}dx^4,\\
u_2(x) &= \frac{1}{6652800}ax^{11} + \frac{1}{1814400}bx^{10} + \frac{1}{362880}cx^9 + 4e^x + \frac{1}{40320}dx^8,\\
u_3(x) &= \frac{1}{217945728000}ax^{15} + \frac{1}{43589145600}bx^{14} + \frac{1}{6227020800}cx^{13}\\
&\quad +4e^x + \frac{1}{479001600}dx^{12},\\
u_4(x) &= \frac{1}{20274183401472000}ax^{19} + \frac{1}{3201186852864000}bx^{18}\\
&\quad +\frac{1}{355687428096000}cx^{17} + 4e^x + \frac{1}{20922789888000}dx^{16}.
\end{aligned}$$

According to this approach, all of the components were determined. In addition, HPM results:

$$u(x) = \lim_{p\to 1} \nu(t) = \nu_0(x) + \nu_1(x) + \cdots \tag{3.10}$$

Then, the values of $u_0(x), u_1(x), u_2(x), u_3(x)$ and $u_4(x)$ are replaced to equation (3.10) and this yields:

$$\begin{aligned} u(x) = {} & ax^3 + bx^2 + cx + d + \frac{1}{840}ax^7 + \frac{1}{360}bx^6 + \frac{1}{120}cx^5 + 16e^x \\ & + \frac{1}{24}dx^4 + \frac{1}{6652800}ax^{11} + \frac{1}{1814400}bx^{10} + \frac{1}{362880}cx^9 + \frac{1}{40320}dx^8 \\ & + \frac{1}{217945728000}ax^{15} + \frac{1}{43589145600}bx^{14} + \frac{1}{6227020800}cx^{13} \\ & + \frac{1}{479001600}dx^{12} + \frac{1}{20274183401472000}ax^{19} + \frac{1}{3201186852864000}bx^{18} \\ & + \frac{1}{355687428096000}cx^{17} + \frac{1}{20922789888000}dx^{16}. \end{aligned} \tag{3.11}$$

As the boundary conditions (3.2) applied into $u(x)$, all of coefficients are determined:

$$a = -1.537,\ b = -6.756,\ c = -14.,\ d = -15. \tag{3.12}$$

Finally, the following approximate solution is obtained:

$$\begin{aligned} u(x) = {} & -1.537x^3 - 6.756x^2 - 14x - 15 - 0.0018298x^7 - 0.018767x^6 \\ & -0.11667x^5 + 16.e^x - 0.62500x^4 - 2.310310^{-7}x^11 - 0.0000037236x^{10} \\ & -0.000038580x^9 - 0.00037202x^8 - 7.052210^{-12}x^{15} - 1.5499x^210^{-10}x^{14} \\ & -2.248310^{-9}x^{13} - 3.131510^{-8}x^12 - 7.581110^{-17}x^{19} - 2.110410^{-15}x^{18} \\ & -3.936010^{-14}x^{17} - 7.169210^{-13}x^{16}, \end{aligned}$$

Example 3.2. Equation (3.13) is another example of linear boundary value problem:

$$u^{(4)}(x) = u(x) + u''(x) + (x-3)e^x, \tag{3.13}$$

with these boundary conditions:

$$u(0) = 1,\ u'(0) = 0,\ u(1) = 0,\ u'(1) = -e. \tag{3.14}$$

Equation (3.15) is the exact solution for this problem

$$u(x) = (1-x)e^x, \tag{3.15}$$

As mentioned in the previous example, the linear and nonlinear parts of the equation is initially separated to solve equation (3.13) by means of HPM. Hence, the following equation is formed:

$$H(x,p) = (1-p)(u^{(4)}(x) - u_0^{(4)}(x)) + p(u^{(4)}(x) - u(x) - u^{(2)}(x) - (x-3)e^x). \tag{3.16}$$

Then, this $\nu = \nu_0 + p\nu_1 + \cdots$ substituted into equation (3.16) and we obtain the following:

(3.17) $p^0 : 0,$

(3.18) $p^1 : \left(-e^x x + \frac{d^4}{dx^4}u_1(x) + \frac{d^4}{dx^4}u_0(x) - u_0(x) - \left(\frac{d^2}{dx^2}u_0(x)\right) + 3e^x\right) = 0,$

The following polynomial is chosen as an arbitrary initial approximation:

$$u_0(x) = ax^3 + bx^2 + cx + d. \tag{3.19}$$

By solving equations (3.18) and (3.19), we obtain the following:

$$u_1(x) = e^x x - 7e^x + \frac{1}{840}ax^7 + \frac{1}{360}bx^6 + \frac{1}{120}cx^5 + \frac{1}{20}ax^5 + \frac{1}{24}(d+2b)x^4. \tag{3.20}$$

In the same way, the rest of components were determined by using the Maple package. Hence, we found that:

$$u(x) = \lim_{p\to 1} \nu(t) = \nu_0(x) + \nu_1(x) + \cdots \tag{3.21}$$

In addition, the values of $u_0(x)$, $u_1(x)$ obtained from equations (3.19) and (3.20) is substituted in equation(3.21) and this yields:

$$\begin{aligned} u(x) &= ax^3 + bx^2 + cx + d + e^x x - 7e^x \\ &\quad + \frac{1}{840}ax^7 + \frac{1}{360}bx^6 + \frac{1}{12}cx^5 + \frac{1}{20}ax^5 + \frac{1}{24}(d+2b)x^4. \end{aligned} \tag{3.22}$$

As the boundary conditions (3.14) are applied into $u(x)$, all of the coefficients are defined:

$$a = \frac{7730702}{323149} - \frac{2950080}{323149}e,\ b = -\frac{11761596}{323149} + \frac{4640400}{323149}e,\ c = 6,\ d = 8 \tag{3.23}$$

Thus, the final results are obtained as follows:

$$\begin{aligned} u(x) = &\left(\frac{7730702}{323149} - \frac{2950080}{323149}e\right)x^3 + \left(-\frac{11761596}{323149} + \frac{4640400}{323149}e\right)x^2 \\ &+6x + 8 + e^x x - 7e^x + \frac{1}{840}\left(\frac{7730702}{323149} - \frac{2950080}{323149}e\right)x^7 \\ &+\frac{1}{360}\left(-\frac{11761596}{323149} + \frac{4640400}{323149}e\right)x^6 + \frac{1}{20}x^5 \\ &+\frac{1}{20}\left(\frac{7730702}{323149} - \frac{2950080}{323149}e\right)x^5 + \frac{1}{24}\left(-\frac{20938000}{323149} + \frac{9280800}{323149}e\right)x^4, \end{aligned}$$

3.2. Application of ADM

Example 3.3. In this example, equation (3.1) are solved by means of ADM. Hence, the equation is reformed as

$$u(x) = L_{4x}u - 4e^x, \tag{3.24}$$

where $L_t = \dfrac{\partial}{\partial t},\ L_{2t} = \dfrac{\partial^2}{\partial t^2},\ L_{3t} = \dfrac{\partial^3}{\partial t^3}, \cdots$

According to the decomposition method, the approximate solution has the form

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \tag{3.25}$$

Thus,

$$u(x) = L_{4x}\left(\sum_{n=0}^{\infty} u_n(x)\right) - 4e^x, \tag{3.26}$$

so, we find:

$$u_1(x) = L_{4x}(u_0) - 4e^x = -4e^x, \tag{3.27}$$

As the values of $u_0(x)$, $u_1(x)$ obtained from equations (3.9) and (3.27) substituted into equation (3.25), it yields the following:

$$u(x) = ax^3 + bx^2 + cx + d - 4e^x. \tag{3.28}$$

Then, the boundary conditions (2.1) applied into $u(x)$, and the following coefficients are obtained:

$$a = 16 - 5e,\ b = -27 + 11e,\ c = 6,\ d = 5 \tag{3.29}$$

Thus, the solution $u(x)$ is determined:

$$u(x) = (16 - 5e)x^3 + (-27 + 11e)x^2 + 6x + 5 - 4e^x, \tag{3.30}$$

Example 3.4. Here, equation (3.13) is solved by means of ADM. Hence, the equation is presented as

$$u(x) = L_{4x}u - L_{2x}u - (x-3)e^x, \tag{3.31}$$

where $L_t = \dfrac{\partial}{\partial t},\ L_{2t} = \dfrac{\partial^2}{\partial t^2},\ L_{3t} = \dfrac{\partial^3}{\partial t^3}, \cdots$

In the decomposition method, the approximate result is as follows:

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \tag{3.32}$$

Thus,

$$u(x) = L_{4x}\left(\sum_{n=0}^{\infty} u_n(x)\right) - L_{2x}\left(\sum_{n=0}^{\infty} u_n(x)\right) - (x-3)e^x. \tag{3.33}$$

So, we find the following:

$$u_1(x) = L_{4x}(u_0) - L_{2x}(u_0) - (x-3)e^x = -6ax - 2b - (x-3)e^x, \tag{3.34}$$
$$u_2(x) = L_{4x}(u_1) - L_{2x}(u_1) - (x-3)e^x = -2e^x - (x-3)e^x. \tag{3.35}$$

Furthermore, the values of $u_0(x), u_1(x), u_2(x)$ obtained from equations (3.19), (3.34) and (3.35) applied into equation (3.32) lead to the following:

$$u(x) = ax^3 + bx^2 + cx + d - 6ax - 2b - 2(x-3)e^x - 2e^x, \tag{3.36}$$

As the boundary conditions (3.14) applied into $u(x)$ and the coefficients are obtained; the following solution $u(x)$ is resulted:

$$u(x) = (-8+3e)x^3+(13-5e)x^2+(-50+18e)x-3-6(-8+3e)x-2(x-3)e^x-2e^x.$$

Tables (1) and (2) compare the approximate solutions with exact solution.

Table 1: Comparison between HPM&ADM with exact solution for equation (3.1)

x	HPM	ADM	Exact solution	Error(HPM)	Error(ADM)
0	1.0000000000	1.0000000000	1.0000000000	0.0000000000	0.0000000000
0.1	1.2135895620	1.2107359200	1.2156880100	0.0020984480	0.0049520900
0.2	1.4589237340	1.4497016990	1.4656833100	0.0067595760	0.0159816110
0.3	1.7429453950	1.7266957310	1.7548164500	0.0118710550	0.0281207190
0.4	2.0727440300	2.0510270400	2.0885545770	0.0158105470	0.0375275370
0.5	2.4555841900	2.4314638000	2.4730819060	0.0174977160	0.0416181060
0.6	2.8989385100	2.8761764650	2.9153900800	0.0164515700	0.0392136150
0.7	3.4105262700	3.3926748900	3.4233796020	0.0128533320	0.0307047120
0.8	3.9983578000	3.9877388780	4.0059736700	0.0076158700	0.0182347920
0.9	4.6707856300	4.6673413760	4.6732459110	0.0024602810	0.0059045350
1	5.4365636600	5.4365636569	5.4365636569	−0.0000000031	0.0000000000

Table 2: Comparison between HPM&ADM with exact solution for equation (3.13)

x	HPM	ADM	Exact solution	Error(HPM)	Error(ADM)
0	1.0000000000	1.0000000000	1.0000000000	0.0000000000	0.0000000000
0.1	0.9998588457	0.9938902421	0.9946538262	−0.0052050195	0.0007635841
0.2	0.9937091960	0.9746323273	0.9771222064	−0.0165869896	0.0024898791
0.3	0.9737227060	0.9404739520	0.9449011656	−0.0288215404	0.0044272136
0.4	0.9330798160	0.8891236830	0.8950948188	−0.0379849972	0.0059711358
0.5	0.8659651150	0.8176672130	0.8243606355	−0.0416044795	0.0066934225
0.6	0.7675654620	0.7224719740	0.7288475200	−0.0387179420	0.0063755460
0.7	0.6340703500	0.5990785600	0.6041258121	−0.0299445379	0.0050472521
0.8	0.4626739700	0.4420772650	0.4451081856	−0.0175657844	0.0030309206
0.9	0.2515788300	0.2449677980	0.2459603111	−0.0056185189	0.0009925131
1	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.0000000000

Moreover, Figures 1 and 2 illustrate these values and show a remarkable agreement between these methods. Of course, the accuracy of solutions could be significantly enhanced.

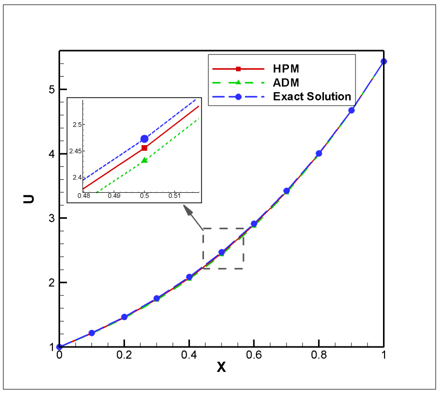

Fig. 1 Comparison between different solutions for equation (3.1)

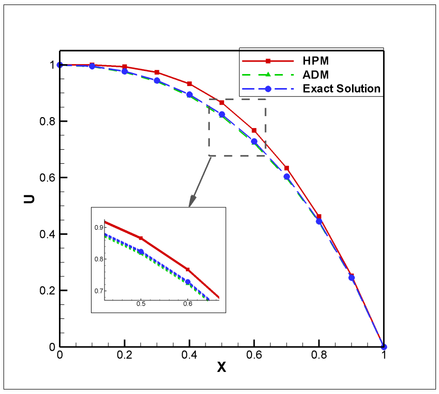

Fig. 2 Comparison between different solutions for equation (3.13)

4. Conclusion

In this study, the results of homotopy perturbation method (HPM) and Adomian decomposition method (ADM) are compared and it is found that both approaches are remarkably effective for solving boundary value problems. In these methods, a fourth-order differential equation was solved for specific engineering applications to prove their effectiveness. In this research, boundary conditions are chosen base on real physical condition of the problem. Also, the results and error of each method are compared with exact solutions to evaluate the accuracy of both approaches. Our findings show that the HPM and ADM are recommended for solving partial differential equations with minimum calculation process.

References

[1] Usmani, R.A., *On the numerical integration of a boundary value problem involving a fourth order linear differential equation,* BIT, 17 (2) (1977), 227-234.

[2] Usmani, R.A., *An $O(h^6)$ finite difference analogue for the solution of some differential equations occurring in plate deflection theory,* Journal of the Institute of Mathematics and Its Applications, 20 (3) (1977), 331-333.

[3] Ma, T.F., da Silva, J., *Iterative solutions for a beam equation with nonlinear boundary conditions of third order,* Applied Mathematics and Computation, 159 (1) (2004)1, 11-18.

[4] He, J.H., *New interpretation of homotopy perturbation method,* International Journal of Modern Physics B, 20 (18) (2006), 2561-2568.

[5] Ganji, D.D., Sadighi, A., *Application of He's homotopy-perturbation method to nonlinear coupled systems of reaction-diffusion equations,* International Journal of Nonlinear Sciences and Numerical Simulation, 7 (4) (2006), 411-418.

[6] Rafei, M., Ganji, D.D., *Explicit solutions of Helmholtz equation and fifth-order KdV equation using homotopy-perturbation method,* International Journal of Nonlinear Sciences and Numerical Simulation, 7 (3) (2006), 321-328.

[7] Jafari, H., Ghasempoor, S., Khalique, C.M., *A Comparison between Adomian's Polynomials and He's Polynomials for Nonlinear Functional Equations*, Mathematical Problems in Engineering 2013, (2013).

[8] Momani, S.M., Odibat, Z., *Numerical comparison of methods for solving linear differential equations of fractional order,* Chaos, Solitons& Fractals, 31 (5) (2007), 1248-1255.

[9] Jafari, H., Zabihi, M., Salehpoor, E., *Application of variational iteration method for modified CamassaHolm and DegasperisProcesi equations*, Numerical Methods for Partial Differential Equations 26 (5) (2010), 1033-1039.

[10] Bildik, N., Konuralp, A., *The use of variational iteration method, differential transform method and Adomian decomposition method for solving different types of nonlinear partial differential equations,* International Journal of Nonlinear Sciences and Numerical Simulation, 7 (1) (2006), 65-70.

[11] He, J.H., *Variational iteration methoda kind of nonlinear analytical technique: some examples,* International Journal of Non-Linear Mechanics, 34 (4) (1999), 699-708.

[12] Jafari, H., *Numerical solution of time-fractional KleinGordon equation by using the decomposition methods*, Journal of Computational and Nonlinear Dynamics 11 (4) (2016), 041015.

[13] Barari, A., Omidvar, M., Ganji, D.D., Poor, A.T., *An Approximate Solution for Boundary Value Problems in Structural Engineering and Fluid Mechanics*, Hindawi Publishing Corporation, 2008.

[14] Choobasti, A.J., Barari, A., Farrokhzad, F., Ganji, D.D., *Analytical investigation of fourth-order boundary value problem in deformation of beams and plate deflection theory*, J. Applied Sci., 8 (11) (2008), 2148-2152.

[15] Kachapi, S.H., Ganji, D.D., *Nonlinear Equations: Analytical Methods and Applications*, Springer, 2012.

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APPROXIMATE SOLUTIONS TO THE GENERALIZED TIME-FRACTIONAL ITO SYSTEM

H.M. Jaradat[1]
Department of Mathematics
Al al-Bayt University
Jordan
Department of Mathematics and Applied Sciences
Dhofar University, Salalah
Oman

Imad Jaradat
Marwan Alquran
Department of Mathematics and Statistics
Jordan University of Science and Technology
Irbid 22110
Jordan

M.M.M. Jaradat
Zead Mustafa
Department of Mathematics, Statistics and Physics
Qatar University, Doha
Qatar

Khedr M. Abohassan
Ra'ed Abdelkarim
Department of Mathematics and Applied Sciences
Dhofar University, Salalah
Oman

Abstract. In this work, we consider the time-fractional version of the well-known integrable Ito system to study the effect of its fractional index "memory index". A modified approach of a relatively new method called residual power series (RPS) is applied to construct an analytical solution for the fractional system. For purpose of comparison, we derive one of the classical Ito system solitary wave solution using Tanh method.

Keywords: time-fractional Ito model, Tanh method, residual power series method.

[1]Corresponding author. E-mail: husseinjaradat@yahoo.com

1. Introduction

The literature has recently been enriched by significant studies on the theory of fractional calculus and its applications. Systematic development on the mathematical structures and physical interpretations of different types was the start point in this field [1]-[7]. A progress on better understanding of the realistic of fractional calculus urged researchers to seek solutions of fractional differential equations (FDEs) by conducting well-posed numerical and analytical schemes. For example, the adomian decomposition method (ADM) [8], [9], the modified (G'/G)-expansion method [10], the differential transform method [11], [12], the homotopy perturbation method [13], the variational iteration method [14], [15], the finite difference method [16], the finite element method [17], [18], the simplified bilinear method [19]-[22] and other methods [23]-[26]. Recently, new trends in the field of fractional nonlinear equations has been approached by scholars by considering fractional versions of well-known nonlinear evolution systems connecting their findings with "Memory-index" feature may such models possess [38], [39]. In this work, we proceed with this trend and we will explore the time-fractional integrable Ito system [27] that reads

$$\begin{aligned} u_t &= v_x \\ v_t &= -2v_{xxx} - 6(uv)_x. \end{aligned} \tag{1.1}$$

Tam, Hu and Wang [28] implemented the Hirota bilinear method and obtained soliton solutions of (1.1). Also, they studied the integrability of this system in the sense of existence of Lax pairs, infinitely many conservations laws and N-soliton solutions.

The study on searching for exact or approximate solutions to partial differential equations enable us to understand physical models better. Therefore, the purpose of this paper is twofold. First, Finding soliton solutions of system (1.1) by using the tanh method [29]-[34]. Second, we use a modified implementation of the RPS method [35]-[43] to find approximate solution to a fractional version of system (1.1) which is

$$\begin{aligned} D_t^{\alpha} u &= v_x \\ D_t^{\alpha} v &= -2v_{xxx} - 6(uv)_x, \end{aligned} \tag{1.2}$$

where α represent the fractional derivative in Caputo sense and also is regarded as the memory index of this model.

2. Survey of the Tanh method

One of the most efficient solitary wave methods is the Tanh method. This technique is based on the assumption that the traveling wave solutions can be expressed in terms of the tanh function [29]-[34]. We therefore introduce a new independent variable

$$Y = \tanh(\mu\zeta). \tag{2.1}$$

Then, the solution can be introduced as a finite power series in Y in the form:

$$u(\mu\zeta) = S(Y) = \sum_{i=0}^{M} a_i Y^i. \tag{2.2}$$

The index M is a positive integer, in most cases, that will be determined by using a balance procedure, where by comparing the behavior of Y^i in the highest derivative against its counterpart within the nonlinear terms. Once M is determined, the coefficients a_i can be determined by setting the coefficients of powers of Y in the resulting equation to zero.

3. Solitary wave solutions to the Ito system

The wave variable $\zeta = x - ct$ carries (1.1) into the ordinary differential system

$$\begin{aligned} -cu' &= v' \\ -cv' &= -2v''' - 6(uv)'. \end{aligned} \tag{3.1}$$

Integrating the above differential equations (DEs) with respect to ζ and considering the constant of integration to be zero, we reach to the following single DE

$$-\frac{c}{2}u + u'' + 3(u^2) = 0. \tag{3.2}$$

Now, by means of Tanh method, u can be assumed as

$$u(\zeta) = \sum_{i=0}^{M} a_i \tanh^i(\mu\zeta). \tag{3.3}$$

Balancing u'' with u^2 in (3.3) gives $M + 2 = 2M$. Thus, $M = 2$.

Now, substituting (3.3) in (3.2) and collecting the coefficients of $\tanh^i$, we obtain a system of algebraic equations for $a_0,\ a_1,\ \ldots\ a_M$ and $\mu,\ \zeta$. By solving this system, one of the obtained solutions is read as:

$$\begin{aligned} u(x,t) &= \frac{2\mu^2}{3} - 2\mu^2 \tanh^2(\mu(x + 8\mu^2 t)), \\ v(x,t) &= 8\mu^2 \left(\frac{2\mu^2}{3} - 2\mu^2 \tanh^2(\mu(x + 8\mu^2 t)) \right). \end{aligned} \tag{3.4}$$

4. Analytical solution of the fractional Ito system

Consider the time-fractional Ito system

$$\begin{aligned} D_t^{\alpha} u &= v_x \\ D_t^{\alpha} v &= -2v_{xxx} - 6(uv)_x. \end{aligned} \tag{4.1}$$

subject to the initial conditions:

$$\begin{aligned} u(x,0) &= f(x), \\ v(x,0) &= g(x). \end{aligned} \tag{4.2}$$

Now we write the solution of Eqs. (4.1-4.2) as a fractional power series about $t = 0$

$$\begin{aligned} u(x,t) &= \sum_{n=0}^{\infty} a_n(x) \, \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}, \\ v(x,t) &= \sum_{n=0}^{\infty} b_n(x) \, \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}. \end{aligned} \tag{4.3}$$

Provided that $0 < \alpha \leq 1$. For the RPS purposes, we let $u_j(x,t),\ v_j(x,t)$ to denote the j-th truncated series of $u(x,t),\ v(x,t)$, respectively, i.e.

$$\begin{aligned} u_j(x,t) &= \sum_{n=0}^{j} a_n(x) \, \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}, \\ v_j(x,t) &= \sum_{n=0}^{j} b_n(x) \, \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}. \end{aligned} \tag{4.4}$$

Applying the conditions given in (4.2), the 0-th RPS approximate solutions of $u(x,t),\ v(x,t)$ are

$$\begin{aligned} u_0(x,t) &= a_0(x) = u(x,0) = f(x) \\ v_0(x,t) &= b_0(x) = v(x,0) = g(x). \end{aligned} \tag{4.5}$$

Therefore, Eqs. (4.4) can be written as

$$\begin{aligned} u_j(x,t) &= f(x) + \sum_{n=1}^{j} a_n(x) \, \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}, \\ v_j(x,t) &= g(x) + \sum_{n=1}^{j} b_n(x) \, \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}, \end{aligned} \tag{4.6}$$

where $j = 1, 2, 3, \ldots$.

Now, we define the residual functions, $Res_u,\ Res_v$, for equations (4.1)

$$\begin{aligned} Res_u(x,t) &= D_t^{\alpha} u - v_x \\ Res_v(x,t) &= D_t^{\alpha} v + 2v_{xxx} + 6(uv)_x, \end{aligned} \tag{4.7}$$

and therefore, the j-th residual functions, $Res_{u,j},\ Res_{v,j}$ are

$$\begin{aligned} Res_{u,j}(x,t) &= D_t^{\alpha} u_j - \frac{\partial v_j}{\partial x} \\ Res_{v,j}(x,t) &= D_t^{\alpha} v_j + 2\frac{\partial^3 v_j}{\partial x^3} + 6\frac{\partial (u_j v_j)}{\partial x}. \end{aligned} \tag{4.8}$$

To determine the coefficients $a_n(x),\ b_n(x):\ n=1,2,3,...,j$ in Eqs. (4.6), we solve the system

$$\begin{aligned} D_t^{(j-1)\alpha} Res_{u,j}(x,0) &= 0,\\ D_t^{(j-1)\alpha} Res_{v,j}(x,0) &= 0, \end{aligned} \tag{4.9}$$

Now, we are ready to formulate the following steps.

Step 1. To determine $a_1(x),\ b_1(x)$, we consider $(j=1)$ in (4.8)

$$\begin{aligned} Res_{u,1} &= D_t^{\alpha} u_1 - \frac{\partial v_1}{\partial x}\\ Res_{v,1} &= D_t^{\alpha} v_1 + 2\frac{\partial^3 v_1}{\partial x^3} + 6\frac{\partial (u_1 v_1)}{\partial x}. \end{aligned} \tag{4.10}$$

But, $u_1(x,t)=f(x)+a_1(x)\frac{t^\alpha}{\Gamma(1+\alpha)}$ and $v_1(x,t)=g(x)+b_1(x)\frac{t^\alpha}{\Gamma(1+\alpha)}$. Therefore,

$$\begin{aligned} Res_{u,1}(x,t) &= a_1(x) - \left(g'(x)+b_1'(x)\frac{t^\alpha}{\Gamma(1+\alpha)}\right),\\ Res_{v,1}(x,t) &= b_1(x) + 2\left(g'''(x)+b_1'''(x)\tfrac{t^\alpha}{\Gamma(1+\alpha)}\right)\\ &+\ 6\left(f(x)+a_1(x)\frac{t^\alpha}{\Gamma(1+\alpha)}\right)\left(g'(x)+b_1'(x)\frac{t^\alpha}{\Gamma(1+\alpha)}\right)\\ &+\ 6\left(f'(x)+a_1'(x)\frac{t^\alpha}{\Gamma(1+\alpha)}\right)\left(g(x)+b_1(x)\frac{t^\alpha}{\Gamma(1+\alpha)}\right). \end{aligned} \tag{4.11}$$

Using equations (4.9), we have $Res_{u,1}(x,0)=0,\ Res_{v,1}(x,0)=0$. Therefore,

$$\begin{aligned} a_1(x) &= g'(x),\\ b_1(x) &= -2g'''(x) - 6(f(x)g(x))'. \end{aligned} \tag{4.12}$$

Step 2. To determine $a_2(x),\ b_2(x)$, we consider $(j=2)$ in (4.8), where $u_2(x,t)=f(x)+a_1(x)\frac{t^\alpha}{\Gamma(1+\alpha)}+a_2(x)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}$ and $v_1(x,t)=g(x)+b_1(x)\frac{t^\alpha}{\Gamma(1+\alpha)}+b_2(x)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}$. Then, we solve the corresponding algebraic equations $D_t^\alpha Res_{u,2}(x,0)=0$ and $D_t^\alpha Res_{v,2}(x,0)=0$. Equivalently, this step can be done easily by considering the coefficient of the variable t^α in each of the functions $D_t^\alpha Res_{u,2}(x,t)$ and $D_t^\alpha Res_{v,2}(x,t)$. Then we multiply each obtained coefficient by the factor $\Gamma(1+\alpha)$ and set to zero to find the unknown required functions $a_2(x),\ b_2(x)$. The reason of adopting this approach the fact that by Caputo derivative, $D_t^\alpha\left(t^\alpha\right)=\Gamma(1+\alpha)$ and $D_t^\alpha\left(t^b\right)\downarrow_{t=0}=0,\ b>\alpha$. Using this argument, leads to the following result formulas.

$$\begin{aligned} a_2(x) &= b_1'(x),\\ b_2(x) &= -2b_1'''(x) - 6\left(f(x)b_1(x)\right)' - 6\left(a_1(x)g(x)\right)'. \end{aligned} \tag{4.13}$$

Step 3. To find the functions $a_3(x),\ b_3(x)$, we find the coefficient of the variable $t^{2\alpha}$ in the resulting expansion of $D_t^{2\alpha}Res_{u,3}(x,t)$, $D_t^{2\alpha}Res_{v,3}(x,t)$ and multiply it by the factor $\Gamma(1+2\alpha)$ and set to zero. Provided that $D_t^{2\alpha}\left(t^{2\alpha}\right)=\Gamma(1+2\alpha)$ and $D_t^{2\alpha}\left(t^{b}\right)\downarrow_{t=0}=0,\ b>2\alpha$. Therefore,

$$\begin{aligned} a_3(x) &= b_2'(x) \\ b_3(x) &= -2b_2'''(x)-6\left(f(x)b_2(x)\right)'-6\left(a_2(x)g(x)\right)' \\ &\quad -6\frac{\Gamma(1+2\alpha)}{\Gamma^2(1+\alpha)}\left(a_1(x)b_1(x)\right)' \end{aligned} \tag{4.14}$$

Finally, following the above routine, we reach at,

$$\begin{aligned} a_4(x) &= b_3'(x) \\ b_4(x) &= -2b_3'''(x)-6\left(f(x)b_3(x)\right)'-6\left(a_3(x)g(x)\right)' \\ &\quad -\frac{6\ \Gamma(1+3\alpha)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)}\left[\left(b_1(x)a_2(x)\right)'+\left(b_2(x)a_1(x)\right)'\right]. \end{aligned} \tag{4.15}$$

5. Discussions and concluding remarks

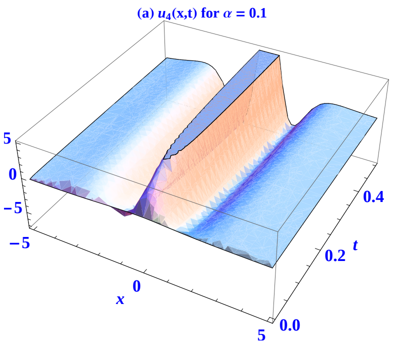
In this section, we study the nature of the solution of the fractional Ito system as the fractional derivative parameter varies from 0 to 1. We solve the system subject to the initial conditions

$$\begin{aligned} u(x,0) &= f(x)=\frac{1}{6}-\frac{1}{2}\tanh^2\left(\frac{1}{2}x\right) \\ v(x,0) &= g(x)=\frac{1}{3}-\tanh^2\left(\frac{1}{2}x\right). \end{aligned} \tag{5.1}$$

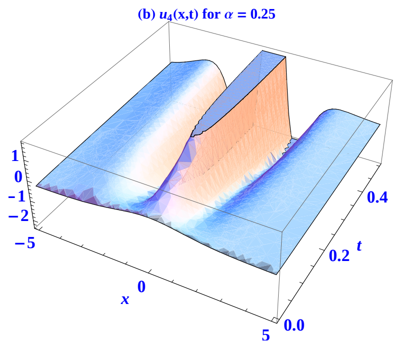
It is to be noted that the exact solution of this system when $\alpha=1$ is (see Section 3, when $\mu=\frac{1}{2}$):

$$\begin{aligned} u(x,t) &= \frac{1}{6}-\frac{1}{2}\tanh^2\left(\frac{x}{2}+t\right) \\ v(x,t) &= \frac{1}{3}-\tanh^2\left(\frac{x}{2}+t\right). \end{aligned} \tag{5.2}$$

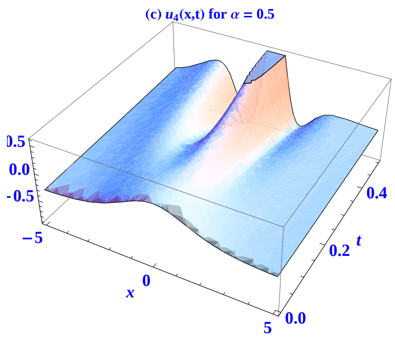
Figures 1 and 2, represent the effect of the fractional derivative parameter α on the chaotic behavior of the obtained solution. It is been observed that as the memory index "The fractional order" getting smaller as to 0, the RPS solutions bifurcate and has a wave-like shape. On the other hand, when the fractional derivative reaches 1, the RPS solution is a complete wave.



(a) $u_4(x, t, \alpha = 0.10)$

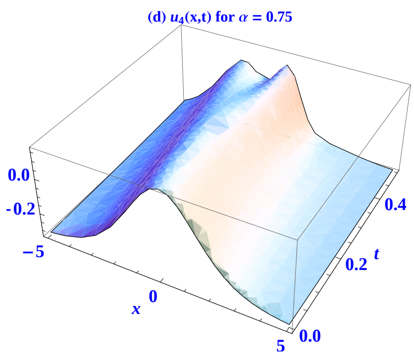


(b) $u_4(x, t, \alpha = 0.25)$

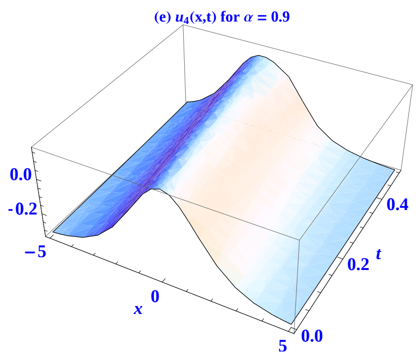


(c) $u_4(x, t, \alpha = 0.50)$

Figure 1: The 4-th RPS approximate solution.



(d) $u_4(x, t, \alpha = 0.75)$



(e) $u_4(x, t, \alpha = 0.90)$.

Figure 2: The 4-th RPS approximate solution.

In order to illustrate that the obtained RPS solution is efficient and accurate, we first present the comparison of the numerical solution using the present method "4-th RPS solution" and the exact solution given in (5.2) when $\alpha = 1$, see Figure 3.

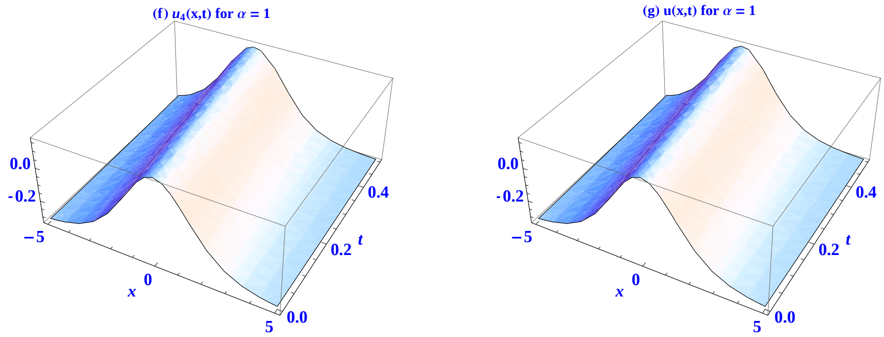


Figure 3: The approximate and exact solutions, respectively, when $-5 < x < 5$ and $0 < t < 0.5$ and $\alpha = 1$.

To validate the accuracy of the numerical scheme, we give explicit values of the variables x, t and compute the absolute error between the exact solution and the 5-th RPS solution when $\alpha = 1$, see Table 1.

$x\|t$	0.1	0.15	0.2	0.25	0.3
-4	2.47×10^{-9}	2.92×10^{-8}	1.71×10^{-7}	6.76×10^{-7}	2.09×10^{-6}
-2	1.14×10^{-8}	1.47×10^{-8}	9.35×10^{-7}	3.97×10^{-6}	1.31×10^{-5}
0	1.88×10^{-7}	2.13×10^{-6}	1.18×10^{-5}	4.46×10^{-5}	1.32×10^{-5}
2	5.93×10^{-9}	5.39×10^{-8}	2.32×10^{-7}	6.27×10^{-7}	1.15×10^{-6}
4	2.12×10^{-9}	2.33×10^{-8}	1.26×10^{-7}	4.65×10^{-7}	1.33×10^{-6}

Table 1: Absolute error of the RPS approximate solution against the exact solution when $\alpha = 1$

In conclusion, the Residual power series method has been successfully applied to find an analytical solution to the fractional Ito system. Comparison of the result obtained by the present method when $\alpha = 1$ with the exact solution obtained by Tanh method evidences that the numerical results are harmonious and can be easily improved by adding new terms of the power series approximations. Applications, concluding remarks and some theory regards the Residual power series method can be found in ([40]-[42]) and ([35]-[39]).

References

[1] Oldham, K.B., Spanier, J., *The Fractional Calculus*, Academic Press, New York, USA, 1974.

[2] Samko, S.G., Kilbas, A.A., Marichev, O.I., *Fractional Integrals and Derivatives*, Gordon and Breach Science, Yverdon, Switzerland, 1993.

[3] Miller, K.S., Ross, B., *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons, New York, USA, 1993.

[4] Podlubny, I., *Fractional Differential Equations*, Academic Press, San Diego, USA, 1999, 198.

[5] Hilfer, R., *Applications of Fractional Calculus in Physics*, World Scientific Publishing, River Edge, USA, 2000.

[6] West, B.J., Bologna, M., Grigolini, P., *Physics of Fractal Operators*, Springer, New York, USA, 2003.

[7] Kilbas, A., Srivastava, H.M., Trujillo, J.J., *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, Elsevier Science, Amsterdam, the Netherlands, 2006, 204.

[8] Hu, Y., Luo, Y, Lu, Z., *Analytical solution of the linear fractional differential equation by Adomian decomposition method*, J. Comput. Appl. Math., 215 (1) (2008), 220-229.

[9] El-Sayed A.M.A., Gaber, M., *The adomian decomposition method for solving partial differential equations of fractal order in finite domains*, Phys. Lett., A. 359 (20) (2006), 175-182.

[10] Katatbeh, Q., Abu-Irwaq, I., Alquran, M., Qawasmeh, A., *Solitary wave solutions to time-fractional coupled degenerate hamiltonian equations*, International Journal of Pure and Applied Mathematics, 93 (3) (2014), 377-387.

[11] Momani, S., Odibat, Z., Erturk, V.S., *Generalized differential transform method for solving a space and time-fractional diffusion-wave equation*, Phys. Lett., A. 370 (5-6) (2007), 379-387.

[12] Odibat, Z., Momani, S., *A generalized differential transform method for linear partial differential equations of fractional order*, Appl. Math. Lett., 21 (2) (2008), 194-199.

[13] He, J.H., *Homotopy perturbation technique*, Comput. Methods Appl. Mech. Eng., 178 (3-4) (1999), 257-262.

[14] Odibat, Z., Momani, S., *The variational iteration method: An efficient scheme for handling fractional partial differential equations in fluid mechanics*, Comput. Math. Appl., 58(11-12) (2009), 2199-2208.

[15] Inc, M., *The approximate and exact solutions of the space- and time-fractional Burgers equations with initial conditions by variational iteration method*, J. Math. Anal. Appl., 345 (1) (2008), 476-484.

[16] Li, C., Chen, A., Ye, J., *Numerical approaches to fractional calculus and fractional ordinary differential equation*, J. Comput. Phys., 230 (9) (2011), 3352-3368.

[17] Gao, G.H., Sun, Z.Z., Zhang, Y.N., *A finite difference scheme for fractional sub-diffusion equations on an unbounded domain using artificial boundary conditions*, J. Comput. Phys., 231 (7) (2012), 2865-2879.

[18] Deng, W., *Finite element method for the space and time fractional Fokker-Planck equation*, SIAM Journal on Numerical Analysis, 47 (1) (2008), 204-226.

[19] H.M. Jaradat, H.M., Awawdeh, F., Al-Shara', S., Alquran, M., Momani, S., *Controllable dynamical behaviors and the analysis of fractal burgers hierarchy with the full effects of inhomogeneities of media*, Romanian Journal of Physics, 60 (3-4) (2015), 324-343.

[20] Jaradat, H.M., *New solitary wave and multiple soliton solutions for the time-space fractional boussinesq equation*, Italian journal of pure and applied mathematics, 36 (2016), 367-376.

[21] Alquran, M., Jaradat, H.M., Al-Shara', S., Awawdeh, F., *A New Simplified Bilinear Method for the N-Soliton Solutions for a Generalized FmKdV Equation with Time-Dependent Variable Coefficients*, International Journal of Nonlinear Sciences and Numerical Simulation, 16 (6) (2015), 259-269.

[22] Jaradat, H.M., *Dynamic behavior of traveling wave solutions for a class for the time-space coupled fractional kdV system with time-dependent coefficients*, Italian Journal of Pure and Aplied Mathematics, 36 (2016), 945-958.

[23] F. Awawdeh, F., Al-Shara', S., Jaradat, H.M., Alomari, A.K., Alshorman, R., *Symbolic computation on soliton solutions for variable coefficient quantum Zakharov-Kuznetsov equation in magnetized dense plasmas*, International Journal of Nonlinear Sciences and Numerical Simulation, 15 (1) (2014), 35-45.

[24] O. Alsayyed, O., Jaradat, H.M., Jaradat, M.M.M., Mustafa, Z., Shatat, F., *Multi-soliton solutions of the BBM equation arisen in shallow water*, Journal of Nonlinear Sciences and Applications, 9 (4) (2016), 1807-1814.

[25] F. Awawdeh, F., Al-Shara', S., Jaradat, H.M., Alomari, A.K., Alshorman, R., *Symbolic computation on soliton solutions for variable coefficient quantum Zakharov-Kuznetsov equation in magnetized dense plasmas*, International Journal of Nonlinear Sciences and Numerical Simulation, 15 (1) (2014), 35-45.

[26] Jaradat, H.M., Al-Shara', S., Awawdeh, F., Alquran, M., *Variable coefficient equations of the Kadomtsev-Petviashvili hierarchy: multiple soliton solutions and singular multiple soliton solutions*, Phys. Scr., vol. 85 (2012).

[27] Ito, M., *An extension of nonlinear evolution equations of the KdV (mKdV) type of higher orders*, J. Phys. Soc. Jpn., 49 (1980), 771-8.

[28] Tam, H.W., Hu, X.B., Wang, D.J., *Two integrable systems*, J. Phys. Soc. Jpn., 68 (1999), 369-79.

[29] Malfliet, W., Hereman, W., *The tanh method. I: Exact solutions of nonlinear evolution and wave equations*, Phys. Scripta., 54 (1996), 563-568.

[30] Alquran, M., Al-Khaled, K., *Sinc and solitary wave solutions to the generalized Benjamin-Bona-Mahony-Burgers equations*, Physica Scripta, 83, (2011), 065010 (6 pp).

[31] Alquran, M., Al-Khaled, K., *The tanh and sine-cosine methods for higher order equations of Korteweg-de Vries type*, Physica Scripta, Volume 84, (2011) 025010 (4 pp).

[32] Alquran, M., Al-Khaled, K., *Mathematical methods for a reliable treatment of the* $(2+1)$*-dimensional Zoomeron equation*, Mathematical Sciences, 6 (12), (2012).

[33] Wazwaz, A.M., *The extended Tanh method for Abundant solitary wave solutions of nonlinear wave equations*, Applied Mathematics and Computation, 187 (2007), 1131-1142.

[34] Wazwaz, A.M., *The tanh method for travelling wave solutions to the Zhiber-Shabat equation and other related equations*, Comm. Nonlinear Sci. Numer Simul., 13 (2008), 584-92.

[35] Alquran, M., Al-Khaled, K., Chattopadhyay, J., *Analytical solutions of fractional population diffusion model: Residual power series*, Nonlinear Studies, 22 (1) (2015), 31-39.

[36] Alquran, M., *Analytical solutions of fractional foam drainage equation by residual power series method*, Mathematical Sciences, 8 (4) (2014), 153-160.

[37] Alquran, M., *Analytical solution of time-fractional two-component evolutionary system of order 2 by residual power series method*, Journal of Applied Analysis and Computation, 5 (4) (2015), 589-599.

[38] Alquran, M., Al-Khaled, K., Sarda, T., Chattopadhyay, J., *Revisited Fisher's equation in a new outlook: A fractional derivative approach*, Physica A: Statistical Mechanics and its Applications, 438 (2015), 81-93.

[39] Jaradat, H.M., Al-Shara, S., Khan, Q.J.A., Alquran, M., Al-Khaled, K., *Analytical Solution of Time-Fractional Drinfeld-Sokolov-Wilson System Using Residual Power Series Method*, IAENG International Journal of Applied Mathematics, 46 (1) (2016), 64-70.

[40] Abu Arqub, O., *Series solution of fuzzy differential equations under strongly generalized differentiability*, Journal of Advanced Research in Applied Mathematics, 5 (2013), 31-52.

[41] Abu Arqub, O., El-Ajou, A., Bataineh, A., Hashim, I., *A representation of the exact solution of generalized Lane Emden equations using a new analytical method*, Abstract and Applied Analysis, (2013), Article ID 378593, 10 pp.

[42] El-Ajou, A., Abu Arqub, O., Momani, S., *Approximate analytical solution of the nonlinear fractional KdV-Burgers equation: a new iterative algorithm*, Journal of Computational Physics, 293 (2015), 81-95.

[43] El-Ajou, A., Abu Arqub, O., Al Zhour, Z., Momani, S., *New results on fractional power series: theories and applications*, Entropy, 15 (2013), 5305-5323.

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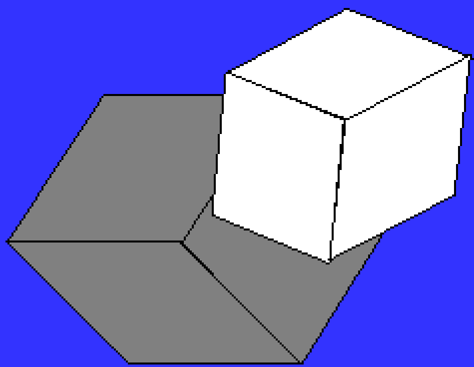

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