University of Windsor
Department of Mathematics and Statistics
Mathematical Foundations MATH 1020
Practice Questions for Midterm Test 1

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1. Using a proof by contraposition, prove $\forall x, y \in \mathbb{R}$, if $3y \leq 2x$, then $x \leq 0$ or $y \leq x$.

Proof.
Let $x, y \in \mathbb{R}$.
Assume $0 < x$ and $x < y$.
Since $x < y$ and $0 < 3$, we have $3x < 3y$.
Also, since $0 < 1$, we have $2+0 < 2+1$, which means $2 < 3$. Since $0 < x$, we then have $2x < 3x$.
Now, $2x < 3x$ and $3x < 3y$, which gives us $2x < 3y$.
Therefore, if $0 < x$ and $x < y$, then $2x < 3y$.
Therefore, if $3y \leq 2x$, then $x \leq 0$ or $y \leq x$.
Therefore, $\forall x, y \in \mathbb{R}$, if $3y \leq 2x$, then $x \leq 0$ or $y \leq x$. $\square$

2. Using a proof by contraposition, prove $\forall x, y \in \mathbb{R}$, if $2y \leq x + 1$, then $x \leq 1$ or $y \leq x$.

Proof.
Let $x, y \in \mathbb{R}$.
Assume $1 < x$ and $x < y$.
Since $x < y$ and $0 < 2$, we have $2x < 2y$.
Also, since $1 < x$, we have $x + 1 < x + x$, which means $x + 1 < 2x$.
Now, $x + 1 < 2x$ and $2x < 2y$, which gives us $x + 1 < 2y$.
Therefore, if $1 < x$ and $x < y$, then $x + 1 < 2y$.
Therefore, if $2y \leq x + 1$, then $x \leq 1$ or $y \leq x$.
Therefore, $\forall x, y \in \mathbb{R}$, if $2y \leq x + 1$, then $x \leq 1$ or $y \leq x$. $\square$

3. Using a proof by contradiction, prove $\forall a \in \mathbb{R}$, if $\forall x \in \mathbb{R}$, $ax \neq 1$, then $a = 0$.

Proof.
Let $a \in \mathbb{R}$.
Assume $\forall x \in \mathbb{R}$, $ax \neq 1$ and $a \neq 0$.
Since $a \neq 0$, we have $a^{-1} \in \mathbb{R}$.
Since $a^{-1} \in \mathbb{R}$, we have by our assumption $aa^{-1} \neq 1$.
However, $aa^{-1} = 1$ by axiom M4. This is a contradiction.
Therefore, if $\forall x \in \mathbb{R}$, $ax \neq 1$, then $a = 0$.
Therefore, $\forall a \in \mathbb{R}$, if $\forall x \in \mathbb{R}$, $ax \neq 1$, then $a = 0$. $\square$

4. Using a proof by contradiction, prove $\forall x, y \in \mathbb{R}$, if $xy < x + y$ and $x < y$, then $x < 2$.

Proof.
Let $x, y \in \mathbb{R}$.
Assume $xy < x + y$ and $x < y$ and $2 \leq x$.
Since $x < y$, we have $x + y < y + y$, so $x + y < 2y$.
Since $xy < x + y$ and $x + y < 2y$, we have $xy < 2y$.
Since $0 < 2 \leq x < y$, we have $0 < y$. Then, since $2 \leq x$, we have $2y \leq xy$.
Now, $xy < 2y$ and $2y \leq xy$, which is a contradiction.
Therefore, if $xy < x + y$ and $x < y$, then $x < 2$:
Therefore, $\forall x, y \in \mathbb{R}$, if $xy < x + y$ and $x < y$, then $x < 2$ $\square$

5. Prove $\forall a \in \mathbb{R}, \exists b \in \mathbb{R}, b < a$.

Proof.
Let $a \in \mathbb{R}$.
Let $b = a - 1$.
Since $-1 < 0$, we have $a - 1 < a + 0$, and so $b < a$.
Therefore, $\exists b \in \mathbb{R}, b < a$.
Therefore, $\forall a \in \mathbb{R}, \exists b \in \mathbb{R}, b < a$. $\square$

6. Prove $\forall a \in \mathbb{R}$, if $a < 1$, then $\exists b \in (0, \infty), a + 2b < 1$.

Proof.
Let $a \in \mathbb{R}$.
Assume $a < 1$.
Let $b = \frac{1-a}{4}$.
Since $a < 1$, we have $a - a < 1 - a$; hence $0 < 1 - a$. Then $\frac{0}{4} < \frac{1-a}{4}$, and so $0 < b$. Thus, $b \in (0, \infty)$.
Also, since $a < 1$, we have $a + 1 < 1 + 1$.
Since $a = 2a - a$, the above inequality can be rewritten as $2a + 1 - a < 2$.
Multiplying by 2^{-1} then gives $a + \frac{1-a}{2} < 1$. Thus, $a + 2b < 1$.
Therefore, $\exists b \in (0, \infty), a + 2b < 1$.
Therefore, if $a < 1$, then $\exists b \in (0, \infty), a + 2b < 1$.
Therefore, $\forall a \in \mathbb{R}$, if $a < 1$, then $\exists b \in (0, \infty), a + 2b < 1$. $\square$

7. Prove $\forall a \in \mathbb{R}$, if $a < 1$, then $\exists b \in \mathbb{R}, 2a < b < 2$.

Proof.
Let $a \in \mathbb{R}$.
Assume $a < 1$.
Let $b = a + 1$.
Since $a < 1$, we have $a + a < a + 1$; hence $2a < a + 1$. Thus, $2a < b$.
Also, since $a < 1$, we have $a + 1 < 1 + 1$; hence $a + 1 < 2$. Thus, $b < 2$.
Therefore, $\exists b \in \mathbb{R}, 2a < b < 2$.
Therefore, if $a < 1$, then $\exists b \in \mathbb{R}, 2a < b < 2$.
Therefore, $\forall a \in \mathbb{R}$, if $a < 1$, then $\exists b \in \mathbb{R}, 2a < b < 2$. $\square$

8. Prove $\forall \varepsilon \in (0, \infty), \exists \delta \in (0, \infty), \forall x \in \mathbb{R}$, if $x - 2 < \delta$, then $3x - 6 < \varepsilon$.

Proof.
Let $\varepsilon \in (0, \infty)$.
Let $\delta = \frac{\varepsilon}{3}$.
Since $0 < \varepsilon$, we have $\frac{0}{3} < \frac{\varepsilon}{3}$, and so $0 < \delta$. Thus, $\delta \in (0, \infty)$.
Let $x \in \mathbb{R}$.
Assume $x - 2 < \delta$.
Then $x - 2 < \frac{\varepsilon}{3}$, and so $3(x - 2) < \varepsilon$. Thus, $3x - 6 < \varepsilon$.
Therefore, if $x - 2 < \delta$, then $3x - 6 < \varepsilon$.
Therefore, $\forall x \in \mathbb{R}$, if $x - 2 < \delta$, then $3x - 6 < \varepsilon$.
Therefore, $\exists \delta \in (0, \infty), \forall x \in \mathbb{R}$, if $x - 2 < \delta$, then $3x - 6 < \varepsilon$.
Therefore, $\forall \varepsilon \in (0, \infty), \exists \delta \in (0, \infty), \forall x \in \mathbb{R}$, if $x - 2 < \delta$, then $3x - 6 < \varepsilon$. $\square$

9. Prove $\forall a \in \mathbb{R}$, if $\forall x \in \mathbb{R}$, $ax = x$, then $a = 1$.

Proof.
Let $a \in \mathbb{R}$.
  Assume $\forall x \in \mathbb{R}$, $ax = x$.
    Since $1 \in \mathbb{R}$, we have $a(1) = 1$.
    Therefore, $a = 1$.
  Therefore, if $\forall x \in \mathbb{R}$, $ax = x$, then $a = 1$.
Therefore, $\forall a \in \mathbb{R}$, if $\forall x \in \mathbb{R}$, $ax = x$, then $a = 1$. $\square$

10. Prove $\forall x \in \mathbb{R}$, if $\forall a \in \mathbb{R}$, $ax = a$, then $2x^2 + x = 2x + 1$.

Proof.
Let $x \in \mathbb{R}$.
  Assume $\forall a \in \mathbb{R}$, $ax = a$.
    Since $2x + 1 \in \mathbb{R}$, we have $(2x + 1)x = 2x + 1$.
    Then, $2x^2 + x = 2x + 1$.
  Therefore, if $\forall a \in \mathbb{R}$, $ax = a$, then $2x^2 + x = 2x + 1$.
Therefore, $\forall x \in \mathbb{R}$, if $\forall a \in \mathbb{R}$, $ax = a$, then $2x^2 + x = 2x + 1$. $\square$

11. Prove $\forall a \in \mathbb{R}$, if $\exists x \in (0, \infty)$, $x < a$, then $\forall x \in (-\infty, 0)$, $x < a$.

Proof.
Let $a \in \mathbb{R}$.
  Assume $\exists x \in (0, \infty)$, $x < a$.
    Let $b \in (0, \infty)$ with $b < a$. Then $0 < b$.
    Let $x \in (-\infty, 0)$.
      Then $x < 0$ and $0 < b$, and $b < a$. By transitivity, $x < a$.
    Therefore, $\forall x \in (-\infty, 0)$, $x < a$.
  Therefore, if $\exists x \in (0, \infty)$, $x < a$, then $\forall x \in (-\infty, 0)$, $x < a$.
Therefore, $\forall a \in \mathbb{R}$, if $\exists x \in (0, \infty)$, $x < a$, then $\forall x \in (-\infty, 0)$, $x < a$. $\square$

12. Prove $\forall x, y \in \mathbb{R}$, if $\forall a \in (-\infty, x]$, $a < y$, then $\exists b \in (0, \infty)$, $x + b \leq y$.

Proof.
Let $x, y \in \mathbb{R}$.
  Assume $\forall a \in (-\infty, x]$, $a < y$.
    Since $x \in (-\infty, x]$, we have $x < y$.
    Let $b = y - x$.
      Since $x < y$, we have $0 < y - x$, and so $0 < b$. This means $b \in (0, \infty)$.
      Also, since $b = y - x$, we have $x + b = y$, which implies $x + b \leq y$.
    Therefore, $\exists b \in (0, \infty)$, $x + b \leq y$.
  Therefore, $\forall a \in (-\infty, x]$, $a < y$, then $\exists b \in (0, \infty)$, $x + b \leq y$.
Therefore, $\forall x, y \in \mathbb{R}$, if $\forall a \in (-\infty, x]$, $a < y$, then $\exists b \in (0, \infty)$, $x + b \leq y$. $\square$

13. Prove $\forall a \in \mathbb{R}$, if $\exists x \in (0,\infty)$, $\forall y \in (0,\infty)$, $a + x \leq y$, then $\exists z \in (-\infty, 0)$, $a < z$.

Proof.
Let $a \in \mathbb{R}$.
Assume $\exists x \in (0,\infty)$, $\forall y \in (0,\infty)$, $a + x \leq y$.
Let $x \in (0,\infty)$ such that $\forall y \in (0,\infty)$, $a + x \leq y$.
Since $\frac{x}{2} \in (0,\infty)$, we have $a + x \leq \frac{x}{2}$, and so $a \leq -\frac{x}{2}$.
Let $z = -\frac{x}{3}$.
Since $x \in (0,\infty)$, we have $0 < x$, and so $-x < 0$. Then $-\frac{x}{3} < 0$, so $z \in (-\infty, 0)$.
Also, since $-x < 0$, and $2 < 3$, we have $-3x < -2x$. Then $-\frac{3x}{6} < -\frac{2x}{6}$; hence $-\frac{x}{2} < z$.
Now, $a \leq -\frac{x}{2}$ and $-\frac{x}{2} < z$, so $a < z$.
Therefore, $\exists z \in (-\infty, 0)$, $a < z$.
Therefore, if $\exists x \in (0,\infty)$, $\forall y \in (0,\infty)$, $a + x \leq y$, then $\exists z \in (-\infty, 0)$, $a < z$.
Therefore, $\forall a \in \mathbb{R}$, if $\exists x \in (0,\infty)$, $\forall y \in (0,\infty)$, $a + x \leq y$, then $\exists z \in (-\infty, 0)$, $a < z$. $\square$

14. Prove $\forall a \in \mathbb{R}$, if $\exists x \in (0,\infty)$, $\forall y \in (0,\infty)$, $a + x \leq y$, then $\forall z \in (0,\infty)$, $a < z$.

Proof.
Let $a \in \mathbb{R}$.
Assume $\exists x \in (0,\infty)$, $\forall y \in (0,\infty)$, $a + x \leq y$.
Let $x \in (0,\infty)$ such that $\forall y \in (0,\infty)$, $a + x \leq y$.
Let $z \in (0,\infty)$.
Since $z \in (0,\infty)$ and $\forall y \in (0,\infty)$, $a + x \leq y$, we have $a + x \leq z$. Thus, $a \leq z - x$.
Also, since $0 < x$, we have $-x < 0$, and so $z - x < z$.
Now, $a \leq z - x$ and $z - x < z$, which gives us $a < z$.
Therefore, $\forall z \in (0,\infty)$, $a < z$.
Therefore, if $\exists x \in (0,\infty)$, $\forall y \in (0,\infty)$, $a + x \leq y$, then $\forall z \in (0,\infty)$, $a < z$.
Therefore, $\forall a \in \mathbb{R}$, if $\exists x \in (0,\infty)$, $\forall y \in (0,\infty)$, $a + x \leq y$, then $\forall z \in (0,\infty)$, $a < z$. $\square$

15. Prove $\forall x, y \in \mathbb{B}$, if $y \Rightarrow \neg x$, then $x \wedge \neg y = x$.

Proof.
Let $x, y \in \mathbb{B}$.
Assume $y \Rightarrow \neg x$.
Then $y \vee \neg x = \neg x$. (Using that $a \Rightarrow b$ is equivalent to $a \vee b = b$).
Negating both sides, we have $\neg(y \vee \neg x) = \neg\neg x$.
By DeMorgan's law, this is $\neg y \wedge x = x$, and so $x \wedge \neg y = x$.
Therefore, if $y \Rightarrow \neg x$, then $x \wedge \neg y = x$.
Therefore, $\forall x, y \in \mathbb{B}$, if $y \Rightarrow \neg x$, then $x \wedge \neg y = x$. $\square$

16. Prove $\forall a, x, y \in \mathbb{B}$, if $(\neg x) \wedge a \Rightarrow y$, then $(\neg y) \wedge a \Rightarrow x$.

Proof.
Let $a, x, y \in \mathbb{B}$.
Assume $(\neg x) \wedge a \Rightarrow y$.
Then $\neg y \Rightarrow \neg(\neg x \wedge a)$, which gives us $\neg y \Rightarrow x \vee \neg a$.
By order preservation, $(\neg y) \wedge a \Rightarrow (x \vee \neg a) \wedge a$.
Distributing, $(\neg y) \wedge a \Rightarrow (x \wedge a) \vee (\neg a \wedge a)$, which is $(\neg y) \wedge a \Rightarrow (x \wedge a) \vee F$. Thus, $(\neg y) \wedge a \Rightarrow (x \wedge a)$.
Since consistency also gives us $x \wedge a \Rightarrow x$, by transitivity, we have $(\neg y) \wedge a \Rightarrow x$.
Therefore, if $(\neg x) \wedge a \Rightarrow y$, then $(\neg y) \wedge a \Rightarrow x$.
Therefore, $\forall a, x, y \in \mathbb{B}$, if $(\neg x) \wedge a \Rightarrow y$, then $(\neg y) \wedge a \Rightarrow x$. $\square$

17. Prove $\forall x, y, z \in \mathbb{B}$, if $x \Rightarrow y$ and $\neg y \Rightarrow z$, then $x \vee \neg z \Rightarrow y$.

Proof.
Let $x, y, z \in \mathbb{B}$.
Assume $x \Rightarrow y$ and $\neg y \Rightarrow z$.
Since $\neg y \Rightarrow z$, we have $\neg z \Rightarrow y$. Then $y \vee \neg z \Rightarrow y \vee y$. Thus, $y \vee \neg z \Rightarrow y$.
Also, since $x \Rightarrow y$, we have $x \vee \neg z \Rightarrow y \vee \neg z$.
Now, $x \vee \neg z \Rightarrow y \vee \neg z$ and $y \vee \neg z \Rightarrow y$. By transitivity, $x \vee \neg z \Rightarrow y$.
Therefore, if $x \Rightarrow y$ and $\neg y \Rightarrow z$, then $x \vee \neg z \Rightarrow y$.
Therefore, $\forall x, y, z \in \mathbb{B}$, if $x \Rightarrow y$ and $\neg y \Rightarrow z$, then $x \vee \neg z \Rightarrow y$.
$\square$

18. Prove $\forall x \in \mathbb{R}$, if $|x - y| = 0$, then $x = y$.

Proof.
Let $x, y \in \mathbb{R}$.
Assume $|x - y| = 0$ and $x \neq y$.
Since $x \neq y$, we have either $x < y$ or $y < x$.
Case 1: $x < y$.
In this case, $x - y < 0$, and so $|x - y| = -(x - y) = y - x$.
Since $|x - y| = 0$, we then have $y - x = 0$, and so $x = y$. This is a contradiction.
Case 2: $y < x$.
In this case, $0 < x - y$, and so $|x - y| = x - y$.
Since $|x - y| = 0$, we then have $x - y = 0$, and so $x = y$. This is a contradiction.
Therefore, if $|x - y| = 0$, then $x = y$.
Therefore, $\forall x \in \mathbb{R}$, if $|x - y| = 0$, then $x = y$. $\square$

19. Prove $\forall x, y \in \mathbb{R}$, $|x - y| = \max(x, y) - \min(x, y)$.

Proof.
Let $x, y \in \mathbb{R}$.
Case 1: $y \leq x$.
In this case, $0 \leq x - y$, and so $|x - y| = x - y$. Also, $\max(x, y) = x$ and $\min(x, y) = y$.
This gives us $|x - y| = x - y = \max(x, y) - \min(x, y)$.
Case 2: $x < y$.
In this case, $x - y < 0$, and so $|x - y| = -(x - y) = y - x$. Also, $\max(x, y) = y$ and $\min(x, y) = x$.
We then have $|x - y| = y - x = \max(x, y) - \min(x, y)$.
Therefore, $\forall x, y \in \mathbb{R}$, $|x - y| = \max(x, y) - \min(x, y)$. $\square$

20. Prove $\forall x \in \mathbb{R}$, $|x| = \max(x, -x)$.

Proof.
Let $x \in \mathbb{R}$.
Case 1: $0 \leq x$.
In this case, $|x| = x$.
Also, $-x \leq 0$ and $0 \leq x$, which gives us $-x \leq x$.
This means $\max(x, -x) = x$. Therefore, $|x| = \max(x, -x)$.
Case 2: $x < 0$.
In this case, $|x| = -x$.
Since $x < 0$, we have $0 < -x$, and so $x < -x$ by transitivity.
This means $\max(x, -x) = -x = |x|$.
Therefore, $\forall x \in \mathbb{R}$, $|x| = \max(x, -x)$. $\square$

21. Prove $\forall x \in \mathbb{R}$, if $x < 1$, then $\exists n \in \mathbb{N}$, $x + \frac{2}{n} < 1$.

Proof.
Let $x \in \mathbb{R}$.
Assume $x < 1$.
Then $0 < 1 - x$.
By the Archimedean Property, $\exists n \in \mathbb{N}$, $2 < (1 - x)n$.
Let $n \in \mathbb{N}$ with $2 < (1 - x)n$.
Then $\frac{2}{n} < 1 - x$, and so $x + \frac{2}{n} < 1$.
Therefore, $\exists n \in \mathbb{N}$, $x + \frac{2}{n} < 1$.
Therefore, if $x < 1$, then $\exists n \in \mathbb{N}$, $x + \frac{2}{n} < 1$.
Therefore, $\forall x \in \mathbb{R}$, if $x < 1$, then $\exists n \in \mathbb{N}$, $x + \frac{2}{n} < 1$. $\square$

22. Prove $\forall x \in \mathbb{R}$, if $2 < x$, then $\exists n \in \mathbb{N}$, $2 < x - \frac{3}{n}$.

Proof.
Let $x \in \mathbb{R}$.
Assume $2 < x$.
Then $0 < x - 2$.
By the Archimedean Property, $\exists n \in \mathbb{N}$, $3 < (x - 2)n$.
Let $n \in \mathbb{N}$ with $3 < (x - 2)n$.
Then $\frac{3}{n} < x - 2$, and so $2 < x - \frac{3}{n}$.
Therefore, $\exists n \in \mathbb{N}$, $2 < x - \frac{3}{n}$.
Therefore, if $2 < x$, then $\exists n \in \mathbb{N}$, $2 < x - \frac{3}{n}$.
Therefore, $\forall x \in \mathbb{R}$, if $2 < x$, then $\exists n \in \mathbb{N}$, $2 < x - \frac{3}{n}$. $\square$

23. Prove $\forall x \in \mathbb{R}$, if $1 < x$, then $\exists n \in \mathbb{N}$, $\frac{n+1}{n} < x$.

Proof.
Let $x \in \mathbb{R}$.
Assume $1 < x$, and so $0 < x - 1$.
By the Archimedean property, $\exists n \in \mathbb{N}$, $1 < (x - 1)n$.
Let $n \in \mathbb{N}$ with $1 < (x - 1)n$.
We then have $1 < xn - n$; $n + 1 < xn$; $\frac{n+1}{n} < x$.
Therefore, $\exists n \in \mathbb{N}$, $\frac{n+1}{n} < x$.
Therefore, if $1 < x$, then $\exists n \in \mathbb{N}$, $\frac{n+1}{n} < x$.
Therefore, $\forall x \in \mathbb{R}$, if $1 < x$, then $\exists n \in \mathbb{N}$, $\frac{n+1}{n} < x$. $\square$

24. Prove $\forall x \in \mathbb{R}$, if $3 < x$, then $\exists n \in \mathbb{N}$, $\frac{3n+4}{n} < x$.

Proof.
Let $x \in \mathbb{R}$.
Assume $3 < x$.
Then $0 < x - 3$.
By the Archimedean property, $\exists n \in \mathbb{N}$, $4 < (x - 3)n$.
Let $n \in \mathbb{N}$ with $4 < (x - 3)n$.
Now, $4 < xn - 3n$; $3n + 4 < xn$; $\frac{3n+4}{n} < x$.
Therefore, $\exists n \in \mathbb{N}$, $\frac{3n+4}{n} < x$.
Therefore, if $3 < x$, then $\exists n \in \mathbb{N}$, $\frac{3n+4}{n} < x$.
Therefore, $\forall x \in \mathbb{R}$, if $3 < x$, then $\exists n \in \mathbb{N}$, $\frac{3n+4}{n} < x$. $\square$