Mathematical Foundations Student Solutions Manual

Justin Lariviere

Contents

Chapter 0

Preliminaries

0.1 Sentence Structure

Exercises 0.1.

Common Notation for Sets of Numbers

$\mathbb{N}$: *Natural numbers.*
$\mathbb{Z}$: *Integers.*
$\mathbb{E}$: *Even numbers.*
$\mathbb{O}$: *Odd numbers.*
$\mathbb{Q}$: *Rational numbers.*
$\mathbb{Q}^c$: *Irrational numbers.*
$\mathbb{R}$: *Real numbers.*

Give an example of open sentences $P(x)$ and $Q(x)$ and a universe of discourse U for which the following hold:

1. The statement '$\forall x \in U$, $P(x)$ or $Q(x)$' is true, but the statement '$\forall x \in U$, $P(x)$, or $\forall x \in U$, $Q(x)$' is false.

   Solution.
   For example, take $U = \mathbb{Z}$, $P(x) =$ 'x is even$'$, and $Q(x) =$ 'x is odd$'$, we have that
   '$\forall x \in U$, $P(x)$ or $Q(x)$' is the statement '$\forall x \in \mathbb{Z}$, x is even or x is odd' which is true.
   '$\forall x \in U$, $P(x)$, or $\forall x \in U$, $Q(x)$' is the statement '$\forall x \in \mathbb{Z}$, x is even, or $\forall x \in \mathbb{Z}$, x is odd' which is false. □

2. The statement '$\exists x \in U$, $P(x)$, and $\exists x \in U$, $Q(x)$' is true, but the statement '$\exists x \in U$, $P(x)$ and $Q(x)$' is false.

   Solution.
   For example, take $U = \mathbb{Z}$, $P(x) =$ 'x is even$'$, and $Q(x) =$ 'x is odd$'$, we have that
   '$\exists x \in U$, $P(x)$, and $\exists x \in U$, $Q(x)$' is the statement '$\exists x \in \mathbb{Z}$, x is even, and $\exists x \in \mathbb{Z}$, x is odd' which is true.
   '$\exists x \in U$, $P(x)$ and $\forall x \in U$, $Q(x)$' is the statement '$\exists x \in \mathbb{Z}$, x is even and x is odd' which is false. □

State whether the proposition is true or false.

3. $\exists x \in \mathbb{R}, x < 0$.

Solution.
True. □

4. $\forall x \in \mathbb{N}, 0 \leq x$.

Solution.
True. □

5. $\forall x \in \mathbb{Z}$, if $0 \leq x$, then $x \in \mathbb{N}$.

Solution.
False. □

6. $\forall x \in \mathbb{R}, 0 < x^2$.

Solution.
False. □

7. $\exists x \in \mathbb{R}, x^2 < 0$.

Solution.
False. □

8. $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x = 2y$.

Solution.
True. □

9. $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x = 2y$.

Solution.
False. □

10. $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, y = 2x$.

Solution.
True. □

11. $\forall x \in \mathbb{R}, \exists n \in \mathbb{Z}, n < x$.

Solution.
True. □

12. $\exists x \in \mathbb{R}, \forall n \in \mathbb{Z}, n < x$.

Solution.
False. □

13. $\forall x \in \mathbb{Z}$, if x is odd, then $\forall y \in \mathbb{Z}, x = 2y + 1$.

Solution.
False. □

14. $\forall x, y \in \mathbb{R}$, if $\exists a \in \mathbb{R}, x < a$ and $a \leq z$, then $x < z$.

Solution.
True. □

15. $\forall x \in \mathbb{N}, \exists y \in \mathbb{N}, y < x$.

Solution.
False. □

16. $\forall x \in \mathbb{N}, \exists y \in \mathbb{N}, x < y$.

Solution.
True. □

17. $\exists x \in \mathbb{N}, \forall y \in \mathbb{N}, y \leq x$.

Solution.
False. □

18. $\exists x \in \mathbb{N}, \forall y \in \mathbb{N}, x \leq y$.

Solution.
True. □

19. $\forall x, y \in \mathbb{Z}$, if $\exists a \in \mathbb{Z}, y = ax$, then $\exists b \in \mathbb{Z}, x = by$.

Solution.
False. □

20. $\forall x \in \mathbb{R}$, if $\exists a \in \mathbb{R}$, $ax \leq 0$, then $x \leq 0$.

Solution.
False. □

21. $\forall x \in \mathbb{Z}$, if $x \leq 0$, then $\forall a \in \mathbb{Z}$, $ax \leq 0$.

Solution.
False. □

22. $\forall x \in \mathbb{R}$, if $\forall a \in \mathbb{R}$, $ax \leq 0$, then $\forall b \in \mathbb{R}$, $0 \leq bx$.

Solution.
True. □

23. $\forall a \in \mathbb{R}$, if $\exists x \in \mathbb{R}$, $x \leq xa$, then $1 \leq a$.

Solution.
False. □

24. $\forall a \in \mathbb{R}$, if $\forall x \in \mathbb{R}$, $ax \leq x$, then $a = 1$.

Solution.
True. □

25. $\forall a \in \mathbb{R}$, if $\exists x \in \mathbb{R}$, $ax > 1$, then $\exists y \in \mathbb{R}$, $ay < -1$.

Solution.
True. □

26. $\forall x \in \mathbb{R}$, if $\exists a \in \mathbb{R}$, $ax < 0$, then $\exists b \in \mathbb{R}$, $0 < bx$.

Solution.
True. □

Write the following propositions in symbolic form by identifying all variables and using the appropriate quantifiers.

27. Every natural number is positive

Solution.
$\forall x \in \mathbb{N}, 0 < x$. □

28. The negative of any even integer is even.

Solution.
$\forall x \in \mathbb{Z}$, if x is even, then $-x$ is even.
Alternate Solution. $\forall x \in \mathbb{Z}$, if $\exists a \in \mathbb{Z}, x = 2a$, then $\exists b \in \mathbb{Z}, -x = 2b$. □

29. The negative of any odd integer is odd.

Solution.
$\forall x \in \mathbb{Z}$, if x is odd, then $-x$ is odd.
Alternate Solution. $\forall x \in \mathbb{Z}$, if $\exists a \in \mathbb{Z}, x = 2a + 1$, then $\exists b \in \mathbb{Z}, -x = 2b + 1$. □

30. The sum of any two odd integers is even.

Solution.
$\forall x, y \in \mathbb{Z}$, if x is odd and y is odd, then $x + y$ is even.
Alternate Solution. $\forall x, y \in \mathbb{Z}$, if $\exists a \in \mathbb{Z}, x = 2a + 1$, and $\exists b \in \mathbb{Z}, y = 2b + 1$, then $\exists c \in \mathbb{Z}, x + y = 2c$. □

31. Some integers have even squares.

Solution.
$\exists x \in \mathbb{Z}, x^2$ is even.
Alternate Solution. $\exists x, y \in \mathbb{Z}, x^2 = 2y$. □

32. Not all integers have even squares.

Solution.
$\exists x \in \mathbb{Z}, x^2$ is not even.
Alternate Solution. $\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, x^2 \neq 2y$. □

33. No odd integers have even squares.

Solution.
$\forall x \in \mathbb{Z}$, if x is odd, then x^2 is not even.
Alternate Solution. $\forall x \in \mathbb{Z}$, if $\exists a \in \mathbb{Z}, x = 2a + 1$, then $\forall b \in \mathbb{Z}, x^2 \neq 2b$. □

34. If an integer's square is even, then the integer itself is even.

Solution.
$\forall x \in \mathbb{Z}$, if x^2 is even, then x is even.
Alternate Solution. $\forall x \in \mathbb{Z}$, if $\exists a \in \mathbb{Z}$, $x^2 = 2a$, then $\exists b \in \mathbb{Z}$, $x = 2b$. □

35. Not all integers are even.

Solution.
$\exists x \in \mathbb{Z}$, x is not even.
Alternate Solution. $\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, x \neq 2y$. □

36. No odd integers are even.

Solution.
$\forall x \in \mathbb{Z}$, if x is odd, then x is not even.
Alternate Solution. $\forall x \in \mathbb{Z}$, if $\exists a \in \mathbb{Z}$, $x = 2a + 1$, then $\forall b \in \mathbb{Z}$, $x \neq 2b$. □

37. All integers are either even or odd.

Solution.
$\forall x \in \mathbb{Z}$, x is even or x is odd.
Alternate Solution. $\forall x \in \mathbb{Z}$, ($\exists a \in \mathbb{Z}$, $x = 2a$ or $\exists b \in \mathbb{Z}$, $x = 2b + 1$). □

38. 6 is a multiple of 3.

Solution.
$\exists x \in \mathbb{Z}, 6 = 3x$. □

39. 6 is not a multiple of 5.

Solution.
$\forall x \in \mathbb{Z}, 6 \neq 5x$. □

40. No odd integer is a multiple of an even integer.

Solution.
$\forall x, y \in \mathbb{Z}$, if x is odd and y is even, then x is not a multiple of y.
Alternate Solution. $\forall x, y \in \mathbb{Z}$, if $\exists a \in \mathbb{Z}$, $x = 2a + 1$ and $\exists b \in \mathbb{Z}$, $y = 2b$, then $\forall c \in \mathbb{Z}$, $x \neq yc$. □

41. Some even integers are multiples of odd integers.

Solution.
$\exists x, y \in \mathbb{Z}$, x is even, and y is odd, and x is a multiple of y.
Alternate Solution. $\exists x, y, a, b, c \in \mathbb{Z}$, $x = 2a$, and $y = 2b + 1$, and $x = yc$. □

42. Every real number is smaller than some natural number.

Solution.
$\forall x \in \mathbb{R}, \exists n \in \mathbb{N}, x < n$. □

43. There is no natural number that is larger than every real number.

Solution.
$\forall n \in \mathbb{N}, \exists x \in \mathbb{R}, n \leq x$. □

44. Every element of the interval $(0, 1)$ is smaller than every element of the interval $(1, 2)$.

Solution.
$\forall x \in (0, 1), \forall y \in (1, 2), x < y$. □

45. 1 is the smallest positive integer. (NOTE: 1 is not a variable. It is a constant.)

Solution.
$\forall x \in \mathbb{N}, 1 \leq x$. □

46. There is a smallest natural number.

Solution.
$\exists x \in \mathbb{N}, \forall y \in \mathbb{N}, x \leq y$. □

47. There is no largest natural number.

Solution.
$\forall x \in \mathbb{N}, \exists y \in \mathbb{N}, x < y$. □

48. Between any two distinct real numbers, there is a rational number.

Solution.
$\forall x, y \in \mathbb{R}$, if $x \neq y$, then $\exists z \in \mathbb{Q}$, $x < z < y$ or $y < z < x$. □

49. The equation $y^2 = 4x + 3$ has no integer solutions.

Solution.
$\forall x, y \in \mathbb{Z}, y^2 \neq 4x + 3.$ □

50. There is no real number whose square is negative.

Solution.
$\forall x \in \mathbb{R}, 0 \leq x^2.$ □

51. There is a real number whose square is not positive.

Solution.
$\exists x \in \mathbb{R}, x^2 \leq 0.$ □

52. 0 and 1 are the only real numbers that are equal to their own squares.

Solution.
$\forall x \in \mathbb{R}, x^2 = x$ if and only if $x = 0$ or $x = 1$. □

53. 1 and 7 are the only positive divisors of 7.

Solution.
$\forall x \in \mathbb{N}$, x divides 7 if and only if ($x = 1$ or $x = 7$).
Alternate Solution. $\forall x \in \mathbb{N}$, $(\exists a \in \mathbb{N}, 7 = ax)$ if and only if ($x = 1$ or $x = 7$) □

54. There is no largest real number in the interval $(0, 1)$.

Solution.
$\forall x \in (0, 1), \exists y \in (0, 1), x < y.$ □

55. If a number multiplied by two numbers makes certain numbers, then the numbers so produced have the same ratio as the numbers multiplied. (Here, 'number' should be read as 'natural number.')

Solution.
$\forall a, x, y \in \mathbb{N}, \frac{ax}{ay} = \frac{x}{y}.$ □

56. If a number multiplied by itself makes a cubic number, then it itself is also cubic. (Again, 'number' should be read as 'natural number.')

Solution.
$\forall x \in \mathbb{N}$, if x^2 is cubic, then x is cubic.
Alternate Solution. $\forall x \in \mathbb{N}$, if $\exists a \in \mathbb{N}, x^2 = a^3$, then $\exists b \in \mathbb{N}, x = b^3$. □

0.2 Negation

Exercises 0.2.

Write the negation of each of the following propositions. Determine which is true, the proposition or its negation.

1. $\forall x \in \mathbb{R},\ x \le x$.

   Negation.
   $\exists x \in \mathbb{R},\ x < x$. (The original proposition is true). □

2. $\exists x \in \mathbb{R},\ x^2 < 0$.

   Negation.
   $\forall x \in \mathbb{R},\ 0 \le x^2$. (The negation is true). □

3. $\exists x \in \mathbb{R},\ \forall y \in \mathbb{Z},\ x \le y$.

   Negation.
   $\forall x \in \mathbb{R},\ \exists y \in \mathbb{Z},\ y < x$. (The negation is true). □

4. $\forall x \in \mathbb{Z},\ \exists y \in \mathbb{R},\ x \le y$.

   Negation.
   $\exists x \in \mathbb{Z},\ \forall y \in \mathbb{R},\ y < x$. (The original proposition is true). □

5. $\forall x \in \mathbb{Q},\ \exists y \in \mathbb{Q},\ xy = 1$.

   Negation.
   $\exists x \in \mathbb{Q},\ \forall y \in \mathbb{Q},\ xy \neq 1$. (The negation is true). □

6. $\exists x \in \mathbb{Q},\ \forall y \in \mathbb{Q},\ xy = 1$.

   Negation.
   $\forall x \in \mathbb{Q},\ \exists y \in \mathbb{Q},\ xy \neq 1$. (The negation is true). □

7. $\forall x \in \mathbb{Z}$, $x + (x + 1)$ is odd and $x(x + 1)$ is even.

   Negation.
   $\exists x \in \mathbb{Z}$, $x + (x + 1)$ is not odd or $x(x + 1)$ is not even. (The original proposition is true). □

8. $\forall x, y \in \mathbb{Z}$, x divides y or y divides x.

   Negation.
   $\exists x, y \in \mathbb{Z}$, x does not divide y and y does not divide x. (The negation is true). □

9. $\exists x, y \in \mathbb{R}$, xy is rational, and x or y is irrational.

   Negation.
   $\forall x, y \in \mathbb{R}$, xy is irrational, or (x is rational and y is rational). (The original proposition is true). □

10. $\forall x \in \mathbb{Z}$, $\exists y \in \mathbb{Z}$, y is prime and y divides x.

    Negation.
    $\exists x \in \mathbb{Z}$, $\forall y \in \mathbb{Z}$, y is not prime or y does not divide x. (The negation is true). □

11. $\forall x, y \in \mathbb{Z}$, if $x - y$ is even, then $x + y$ is even.

    Negation.
    $\exists x, y \in \mathbb{Z}$, $x - y$ is even and $x + y$ is not even. (The original proposition is true). □

12. $\forall x, y \in \mathbb{R}$, if x is rational and y is irrational, then $x + y$ is irrational.

    Negation.
    $\exists x, y \in \mathbb{R}$, x is rational, and y is irrational, and $x + y$ is rational. (The original proposition is true). □

13. $\forall x, y \in \mathbb{R}$, if $x > 0$, then $\exists n \in \mathbb{N}$, $y < nx$.

    Negation.
    $\exists x, y \in \mathbb{R}$, $x > 0$ and $\forall n \in \mathbb{N}$, $nx \leq y$. (The original proposition is true). □

14. $\forall x \in \mathbb{R}$, if $x > 0$, then $\exists y \in \mathbb{R}$, $0 < y$ and $y < x$.

    Negation.
    $\exists x \in \mathbb{R}$, $x > 0$ and $\forall y \in \mathbb{R}$, $y \leq 0$ or $x \leq y$. (The original proposition is true). □

15. $\forall x, y \in \mathbb{Q}$, if $x < y$, then $\exists z \in \mathbb{R}$, $z \notin \mathbb{Q}$ and $x < z < y$.

    Negation.
    $\exists x, y \in \mathbb{Q}$, $x < y$ and $\forall z \in \mathbb{R}$, if $z \notin \mathbb{Q}$, then $z \leq x$ or $y \leq z$. (The original proposition is true). □

16. $\forall x \in \mathbb{Z}$, if x is prime, then $\exists y \in \mathbb{Z}$, y is prime and $y > x$.

Negation.
$\exists x \in \mathbb{Z}$, x is prime and $\forall y \in \mathbb{Z}$, if y is prime, then $y \leq x$. (The original proposition is true). □

17. $\forall x \in \mathbb{R}$, if $\exists y \in \mathbb{R}$, $y \neq 0$ and $xy = y$, then $\forall z \in \mathbb{R}$, $xz = z$.

Negation.
$\exists x \in \mathbb{R}$, $\exists y \in \mathbb{R}$, $y \neq 0$ and $xy = y$ and $\exists z \in \mathbb{R}$, $xz \neq z$. (The original proposition is true). □

18. $\forall x \in \mathbb{R}$, if $\forall y \in \mathbb{Z}$, $xy \leq 0$, then $\forall z \in \mathbb{Z}$, $xz \geq 0$.

Negation.
$\exists x \in \mathbb{R}$, $\forall y \in \mathbb{Z}$, $xy \leq 0$ and $\exists z \in \mathbb{Z}$, $xz < 0$. (The original proposition is true). □

19. $\forall x \in \mathbb{Z}$, if $\exists t \in \mathbb{Z}$, $x^2 = 3t$, then $\exists s \in \mathbb{Z}$, $x = 3s$.

Negation.
$\exists x \in \mathbb{Z}$, $\exists t \in \mathbb{Z}$, $x^2 = 3t$, and $\forall s \in \mathbb{Z}$, $x \neq 3s$. (The original proposition is true). □

20. $\forall x, y \in \mathbb{Z}$, if $\exists a \in \mathbb{Z}$, $xy = 6a$, then $\exists b \in \mathbb{Z}$, $x = 6b$ or $\exists c \in \mathbb{Z}$, $y = 6c$.

Negation.
$\exists x, y \in \mathbb{Z}$, $\exists a \in \mathbb{Z}$, $xy = 6a$, and $\forall b \in \mathbb{Z}$, $x \neq 6b$ and $\forall c \in \mathbb{Z}$, $y \neq 6c$. (The negation is true). □

21. $\forall n \in \mathbb{N}$, if $n = 1$ or $n = 7$, then $\forall x \in \mathbb{N}$, $\exists t \in \mathbb{N}$, $7x = nt$.

Negation.
$\exists n \in \mathbb{N}$, $n = 1$ or $n = 7$, and $\exists x \in \mathbb{N}$, $\forall t \in \mathbb{N}$, $7x \neq nt$. (The original proposition is true). □

22. $\forall n \in \mathbb{N}$, if $\forall x \in \mathbb{N}$, $\exists t \in \mathbb{N}$, $7x = nt$ then $n = 1$ or $n = 7$.

Negation.
$\exists n \in \mathbb{N}$, $\forall x \in \mathbb{N}$, $\exists t \in \mathbb{N}$, $7x = nt$, and $n \neq 1$, and $n \neq 7$. (The original proposition is true). □

23. $\forall x \in \mathbb{R}$, if $\forall y \in \mathbb{R}$, if $y > 0$ then $x \leq y$, then $x \leq 0$.

Negation.
$\exists x \in \mathbb{R}$, $0 < x$ and $\forall y \in \mathbb{R}$, if $y > 0$, then $x \leq y$. (The original proposition is true). □

0.3 Logical Operators

Exercises 0.3.

> *".. you should say what you mean," the March Hare went on.*
> *"I do," Alice hastily replied; "at least... at least I mean what I say... that's the same thing, you know."*
> *"Not the same thing a bit!" said the Hatter. "You might just as well say that 'I see what I eat' is the same thing as 'I eat what I see'!"*
> *"You might just as well say," added the March Hare, "that 'I like what I get' is the same thing as 'I get what I like'!"*
> *"You might just as well say," added the Dormouse, who seemed to be talking in his sleep, "that 'I breathe when I sleep' is the same thing as 'I sleep when I breathe'!"*
> *(Alice's Adventures in Wonderland) [?]*

The following exercises are related to logical operators and equivalent propositional forms.

1. Let $x, y \in \mathbb{Z}$. Let P be the Boolean value of the proposition 'x is odd,' and let Q be the Boolean value of the proposition 'y is even.' Using $\wedge$, $\vee$, and $\neg$, write each of the following as a logical operator applied to P and Q:

   (a) x is odd and y is even.

   Solution.
   $P \wedge Q$. □

   (b) x is even and y is odd.

   Solution.
   $\neg P \wedge \neg Q$. □

   (c) Both x and y are even.

   Solution.
   $\neg P \wedge Q$. □

   (d) Neither x nor y is even.

   Solution.
   $P \wedge \neg Q$. □

   (e) At least one of x or y is even.

   Solution.
   $\neg P \vee Q$. □

   (f) At least one of x or y is odd.

   Solution.
   $P \vee \neg Q$. □

(g) At most one of x or y is even.

Solution.
$P \vee \neg Q$. □

(h) At most one of x or y is odd.

Solution.
$\neg P \vee Q$. □

(i) Exactly one of x or y is even.

Solution.
$(P \wedge Q) \vee (\neg P \wedge \neg Q)$. □

(j) Exactly one of x or y is odd.

Solution.
$(P \wedge Q) \vee (\neg P \wedge \neg Q)$. □

2. Given the following pairs of equivalent logical operators:

   (a) $x \Rightarrow (y \Rightarrow z) \equiv (x \wedge y) \Rightarrow z$

   (b) $x \Rightarrow (y \vee z) \equiv (x \wedge \neg y) \Rightarrow z$

   (c) $(x \wedge y) \Rightarrow z \equiv (x \wedge \neg z) \Rightarrow \neg y$

   (d) $(x \vee y) \Rightarrow z \equiv (x \Rightarrow z) \wedge (y \Rightarrow z)$

   (e) $x \Rightarrow (y \wedge z) \equiv (x \Rightarrow y) \wedge (x \Rightarrow z)$

   rewrite the following propositions using the corresponding equivalences above.

   (a) $\forall a, x \in \mathbb{R}$, if $a \neq 0$, then if $ax = a$, then $x = 1$.

   Solution.
   $\forall a, x \in \mathbb{R}$, if $a \neq 0$ and $ax = a$, then $x = 1$. □

   (b) $\forall x, y, z \in \mathbb{R}$, if $xy = 0$, then $y = 0$ or $x = 0$.

   Solution.
   $\forall x, y, z \in \mathbb{R}$, if $xy = 0$ and $y \neq 0$, then $x = 0$. □

   (c) $\forall x, y, z \in \mathbb{Z}$, if $xz = yz$ and $z \neq 0$ then $x = y$.

   Solution.
   $\forall x, y, z \in \mathbb{Z}$, if $xz = yz$ and $x \neq y$, then $z = 0$. □

(d) $\forall x \in \mathbb{Z}$, if x is even or x is odd, then $x^2 - 3x + 1$ is odd.

> *Solution.*
> $\forall x \in \mathbb{Z}$, if x is even, then $x^2 - 3x + 1$ is odd, and if x is odd, then $x^2 - 3x + 1$ is odd. □

(e) $\forall x, y \in \mathbb{Z}$, if xy is odd, then x is odd and y is odd.

> *Solution.*
> $\forall x, y \in \mathbb{Z}$, if xy is odd, then x is odd, and if xy is odd, then y is odd. □

The following exercises are related to the converse and contrapositive of an implication.

3. State the converse and contrapositive of each of the following implications:

   (a) $\forall x \in \mathbb{R}$, if $0 < x$, then $-x < 0$.

   Solution.
   Converse: $\forall x \in \mathbb{R}$, if $-x < 0$, then $0 < x$. *Contrapositive:* $\forall x \in \mathbb{R}$, if $0 \leq -x$, then $x \leq 0$. □

   (b) $\forall x, y, a \in \mathbb{R}$, if $ax = ay$ and $a \neq 0$, then $x = y$.

   Solution.
   Converse: $\forall x, y, a \in \mathbb{R}$, if $x = y$, then $ax = ay$ and $a \neq 0$. *Contrapositive:* $\forall x, y, a \in \mathbb{R}$, if $x \neq y$, then $ax \neq ay$ or $a = 0$. □

   (c) $\forall a \in \mathbb{R}$, if $\forall x \in \mathbb{R}$, $ax = 0$, then $a = 0$.

   Solution.
   Converse: $\forall a \in \mathbb{R}$, if $a = 0$, then $\forall x \in \mathbb{R}$, $ax = 0$. *Contrapositive:* $\forall a \in \mathbb{R}$, if $a \neq 0$, then $\exists x \in \mathbb{R}$, $ax \neq 0$. □

   (d) $\forall a \in \mathbb{R}$, if $\exists x \in \mathbb{R}$, $x \neq 0$ and $ax = 0$, then $a = 0$.

   Solution.
   Converse: $\forall a \in \mathbb{R}$, if $a = 0$, then $\exists x \in \mathbb{R}$, $x \neq 0$ and $ax = 0$. *Contrapositive:* $\forall a \in \mathbb{R}$, if $a \neq 0$, then $\forall x \in \mathbb{R}$, $x = 0$ or $ax \neq 0$. □

   (e) $\forall a, b, x \in \mathbb{R}$, if $a < x$ and $x < b$, then $\exists t \in (0, 1)$, $x = (1 - t)a + tb$.

   Solution.
   Converse: $\forall a, b, x \in \mathbb{R}$, if $\exists t \in (0, 1)$, $x = (1-t)a+tb$, then $a < x$ and $x < b$. *Contrapositive:* $\forall a, b, x \in \mathbb{R}$, if $\forall t \in (0, 1)$, $x \neq (1 - t)a + tb$, then $x \leq a$ or $b \leq x$. □

4. In the passage above, taken from ”Alice’s Adventures in Wonderland” by Lewis Carroll, Alice, the March Hare, the Dormouse, and the Mad Hatter have a conversation about whether or not converse statements are equivalent to one another.

   (a) Rewrite each of the implications in the conversation in the form ‘if P then Q’ and ‘if Q then P’ to see that each pair are in fact converse statements.

   Solution.
   i. “I say what I mean” : “If I mean something, then I say it.”
   ii. “I mean what I say” : “If I say something, then I mean it.”
   iii. “I see what I eat” : “If I eat something, then I see it.”
   iv. “I eat what I see” : “If I see something, then I eat it.”
   v. “I like what I get” : “If I get something, then I like it.”
   vi. “I get what I like” : “If I like something, then I get it.”
   vii. “I breathe when I sleep” : “If I am sleeping, then I am breathing.”
   viii. “I sleep when I breathe” : “If I am breathing, then I am sleeping.”

   □

(b) Write the negation of each statement.

Solution.

i. I mean something, but I do not say it.
ii. I say something, but I do not mean it.
iii. I eat something, but I do not see it.
iv. I see something, but I do not eat it.
v. I get something, but I do not like it.
vi. I like something, but I do not get it.
vii. I am sleeping, but I am not breathing.
viii. I am breathing, but I am not sleeping.

□

(c) Is Alice right when she says "that's the same thing, you know" or is the Hatter right when he says they are "Not the same thing a bit!"?

Solution.
The Hatter is right. □

5. Give, if possible, an example of a **true** implication statement for which:

(a) the converse is true.

Example.
"$\forall x \in \mathbb{R}$, if $0 < x$, then $0 < 2x$" is true, as is its converse:
"$\forall x \in \mathbb{R}$, if $0 < 2x$, then $0 < x$." □

(b) the converse is false.

Example.
"$\forall x \in \mathbb{R}$, if $0 < x$, then $0 < x^2$" is true, but its converse:
"$\forall x \in \mathbb{R}$, if $0 < x^2$, then $0 < x$" is false. □

(c) the contrapositive is true.

Solution.
Since the implication is equivalent to its contrapositive form, any example of a true implication will suffice. For example:
"$\forall x, y \in \mathbb{R}$, if $xy = 0$, then $y = 0$ or $y = 0$" is true, as is its contrapositive form:
"$\forall x, y \in \mathbb{R}$, if $x \neq 0$ and $y \neq 0$, then $xy \neq 0$." □

(d) the contrapositive is false.

Solution.
Since the implication is equivalent to its contrapositive form, there is no such statement. □

6. Give, if possible, an example of a **false** implication statement for which:

   (a) the converse is true.

   Example.
   "$\forall x \in \mathbb{R}$, if $0 < x + 1$, then $0 < x$" is false, but its converse:
   "$\forall x \in \mathbb{R}$, if $0 < x$, then $0 < x + 1$" is true. □

   (b) the converse is false.

   Example.
   "$\forall x \in \mathbb{R}$, if $0 < x$, then $1 < x^2$" is false, as is its converse:
   "$\forall x \in \mathbb{R}$, if $1 < x^2$, then $0 < x$." □

   (c) the contrapositive is true.

   Solution.
   Since the implication is equivalent to its contrapositive form, there is no such statement. □

   (d) the contrapositive is false.

   Solution.
   Since the implication is equivalent to its contrapositive form, any example of a false implication will suffice. For example:
   "$\forall x \in \mathbb{R}$, if $0 < x^2$, then $0 < x$" is false, as is its contrapositive form:
   "$\forall x \in \mathbb{R}$, if $x \leq 0$, then $x^2 \leq 0$." □

7. (a) Show, using a truth table, that for any Boolean values x and y, at least one of $x \Rightarrow y$ or $y \Rightarrow x$ must be true.

   Solution.

x	y	$x \Rightarrow y$	$y \Rightarrow x$	$(x \Rightarrow y) \vee (y \Rightarrow x)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

   □

   (b) Give reasons why the following propositions are both false:

   i. $\forall x \in \mathbb{R}$, if $x < 0$, then $x^2 < 1$.

   Solution.
   For example, $-2 < 0$, by $(-2)^2 = 4 \not< 1$. □

   ii. $\forall x \in \mathbb{R}$, if $x^2 < 1$, then $x < 0$.

   Solution.
   For example, $(\frac{1}{2})^2 = \frac{1}{4} < 1$, but $\frac{1}{2} \not< 0$. □

(c) Explain why part 7b does not contradiction part 7a.

Solution.
The Boolean values of the statements "$x < 0$" and "$x^2 < 1$" depend of the value of x. For a given value of x, say $x = -2$, we have that "$-2 < 0$" is true and "$(-2)^2 < 1$" is false. In this case, the converse implications

"if $-2 < 0$, then $(-2)^2 < 0$" and "if $(-2)^2 < 0$, then $-2 < 0$"

correspond to the second row of the truth table, and are respectively false and true.

Similarly, if we take $\frac{1}{2}$ as the value of x, the statement "$\frac{1}{2} < 0$" is false and the statement "$(\frac{1}{2})^2 < 1$" is true. In this case, the converse implications

"if $\frac{1}{x} < 0$, then $(\frac{1}{2})^2 < 0$" and "if $(\frac{1}{2})^2 < 0$, then $\frac{1}{2} < 0$"

correspond to the third row of the truth table, and are respectively true and false.

Thus, for each particular value of x, we have at least one of the converse implications being true. However, most implication statements in practice are general statements, preceded by a universal ($\forall$) quantifier. In these cases, it is sometimes possible to find one value of the variable x that makes the implication false, and a different value of the variable that makes the converse false. This is the situation in part (b). $\square$

Chapter 1

Structure of a Mathematical Proof

1.1 The Real Numbers

Exercises 1.1.

Notation:
For all $x, y \in \mathbb{R}$, $x - y$ means $x + (-y)$.
For all $x, y \in \mathbb{R}$ with $y \neq 0$, $\frac{x}{y}$ means $x(y^{-1})$.
For all $x, y \in \mathbb{R}$, $x \leq y$ means $x < y$ or $x = y$.
For all $x, y, z \in \mathbb{R}$, $x < y < z$ means $x < y$ and $y < z$.
2 is defined by $2 = 1 + 1$; 3 is defined by $3 = 2 + 1$; 4 is defined by $4 = 3 + 1$; and so on.

Prove the following propositions. At each step indicate the axiom or proposition you have used.

1. $\forall x, y \in \mathbb{R}, -(x + y) = -x - y$.

Proof.
Let $x, y \in \mathbb{R}$.

$$
\begin{aligned}
-(x+y) + ((x+y) + (-x-y)) &= (-(x+y) + (x+y)) + (-x-y) && A2 \\
-(x+y) + ((x+y) + (-y + (-x))) &= 0 + (-x-y) && A1, A4 \\
-(x+y) + ((x + (y + (-y))) + (-x)) &= -x - y && A2, A3 \\
-(x+y) + ((x+0) + (-x)) &= -x - y && A4 \\
-(x+y) + (x + (-x)) &= -x - y && A3 \\
-(x+y) + 0 &= -x - y && A4 \\
-(x+y) &= -x - y && A3
\end{aligned}
$$

Therefore, $\forall x, y \in \mathbb{R}, -(x + y) = -x - y$. □

3. $\forall x \in \mathbb{R} \setminus \{0\}, (x^{-1})^{-1} = x$.

Proof.
Let $x \in \mathbb{R} \setminus \{0\}$.

$$\begin{aligned} (xx^{-1})(x^{-1})^{-1} &= x(x^{-1}(x^{-1})^{-1}) && M2 \\ (1)(x^{-1})^{-1} &= x(1) && M4 \\ (x^{-1})^{-1} &= x && M3 \end{aligned}$$

Therefore, $\forall x \in \mathbb{R} \setminus \{0\}, (x^{-1})^{-1} = x$. □

5. $\forall x, y \in \mathbb{R} \setminus \{0\}, (xy)^{-1} = y^{-1}x^{-1}$.

Proof.
Let $x, y \in \mathbb{R} \setminus \{0\}$.

$$\begin{aligned} (xy)(y^{-1}x^{-1}) &= (x(yy^{-1}))x^{-1} && M2 \\ &= (x(1))x^{-1} && M4 \\ &= xx^{-1} && M3 \\ &= 1 && M4 \end{aligned}$$

Now, since $(xy)(y^{-1}x^{-1}) = 1$, we have $(xy)^{-1}((xy)(y^{-1}x^{-1})) = (xy)^{-1}(1)$.
By M2 and M3, this gives us $((xy)^{-1}(xy))(y^{-1}x^{-1}) = (xy)^{-1}$.
By M4, $1(y^{-1}x^{-1}) = (xy)^{-1}$; and by M3, $y^{-1}x^{-1} = (xy)^{-1}$.
Therefore, $\forall x, y \in \mathbb{R} \setminus \{0\}, (xy)^{-1} = y^{-1}x^{-1}$. □

7. $\forall a, b \in \mathbb{R}, \forall x \in \mathbb{R} \setminus \{0\}, \frac{xa+xb}{x} = a + b$.

Proof.
Let $a, b \in \mathbb{R}$. Let $x \in \mathbb{R} \setminus \{0\}$.

$$\begin{aligned} \frac{xa + xb}{x} &= (xa + xb)(x^{-1}) \\ &= (xa)(x^{-1}) + (xb)(x^{-1}) && DL \\ &= (ax)(x^{-1}) + (bx)(x^{-1}) && M1 \\ &= (a(xx^{-1})) + (b(xx^{-1})) && M2 \\ &= (a1) + (b1) && M4 \\ &= a + b && M3 \end{aligned}$$

Therefore, $\forall a, b \in \mathbb{R}, \forall x \in \mathbb{R} \setminus \{0\}, \frac{xa+xb}{x} = a + b$. □

9. $\forall a, b \in \mathbb{R}, \forall x \in \mathbb{R} \setminus \{0\}, \frac{a}{x} + \frac{b}{x} = \frac{a+b}{x}$.

Proof.
Let $a, b \in \mathbb{R}$. Let $x \in \mathbb{R} \setminus \{0\}$.

$$\begin{aligned} \frac{a}{x} + \frac{b}{x} &= ax^{-1} + bx^{-1} \\ &= (a+b)x^{-1} \qquad DL \\ &= \frac{a+b}{x} \end{aligned}$$

Therefore, $\forall a, b \in \mathbb{R}, \forall x \in \mathbb{R} \setminus \{0\}, \frac{a}{x} + \frac{b}{x} = \frac{a+b}{x}$. □

11. $\forall a, x \in \mathbb{R}, \forall b, y \in \mathbb{R} \setminus \{0\}, \left(\frac{a}{b}\right)\left(\frac{x}{y}\right) = \frac{ax}{by}$.

Proof.
Let $a, x \in \mathbb{R}$. Let $b, y \in \mathbb{R} \setminus \{0\}$.

$$\begin{aligned} \left(\frac{a}{b}\right)\left(\frac{x}{y}\right) &= (a(b^{-1}))(x(y^{-1})) \\ &= a(b^{-1}(x(y^{-1}))) && M2 \\ &= a((x(y^{-1}))b^{-1}) && M1 \\ &= (ax)(y^{-1}b^{-1}) && M2 \\ &= (ax)(by)^{-1} && \text{Exercise 5} \\ &= \frac{ax}{by} \end{aligned}$$

Therefore, $\forall a, x \in \mathbb{R}, \forall b, y \in \mathbb{R} \setminus \{0\}, \left(\frac{a}{b}\right)\left(\frac{x}{y}\right) = \frac{ax}{by}$. □

13. $1 < 3$.

Proof.
Since $0 < 1$ by Proposition 1.1.4, $1 < 1 + 1$ by O3. Now, $0 < 1$ and $1 < 2$, so $0 < 2$ by O2. Finally, since $0 < 2$, we have $0 + 1 < 2 + 1$ by O3. Therefore, $1 < 3$ by A3. □

15. $\forall x \in \mathbb{R}, x < x + 1$.

Proof.
Let $x \in \mathbb{R}$.
Since $0 < 1$ by Proposition 1.1.4, we have $x + 0 < x + 1$ by O3.
Therefore, $x < x + 1$ by A3.
Therefore, $\forall x \in \mathbb{R}, x < x + 1$. □

Prove the following propositions using a direct proof.

17. $\forall x, y, a \in \mathbb{R}$, if $ax = ay$ and $a \neq 0$, then $x = y$.

Proof.
Let $x, y, a \in \mathbb{R}$.
Assume $ax = ay$ and $a \neq 0$.
Since $a \neq 0$, there is a number $a^{-1} \in \mathbb{R}$ with $a^{-1}a = 1$.
Then $a^{-1}ax = a^{-1}ay$, which means $1x = 1y$.
Therefore, $x = y$.
Hence, if $ax = ay$ and $a \neq 0$, then $x = y$.
Therefore, $\forall x, y, a \in \mathbb{R}$, if $ax = ay$ and $a \neq 0$, then $x = y$. □

19. $\forall a, x \in \mathbb{R}$, if $a + x = 0$, then $a = -x$. (That is, $-x$ is the only additive inverse of x).

Proof.
Let $a, x \in \mathbb{R}$.
Assume $a + x = 0$.
Then $(a + x) + (-x) = 0 + (-x)$, which by associativity gives us $a + (x + (-x)) = 0 + (-x)$.
Applying axiom A4, we then have $a + 0 = 0 + (-x)$; hence $a = -x$ by axiom A3.
Therefore, if $a + x = 0$, then $a = -x$.
Therefore, $\forall a, x \in \mathbb{R}$, if $a + x = 0$, then $a = -x$. □

21. $\forall x \in \mathbb{R}$, if $0 < x$, then $0 < x + 1$.

Proof.
Let $x \in \mathbb{R}$.
Assume $0 < x$.
Then $1 < x + 1$ by O3.
Now, $0 < 1$ and $1 < x + 1$, so $0 < x + 1$ by transitivity.
Therefore, if $0 < x$, then $0 < x + 1$.
Therefore, $\forall x \in \mathbb{R}$, if $0 < x$, then $0 < x + 1$. □

23. $\forall x, y \in \mathbb{R}$, $x < y$ if and only if $-y < -x$.

Proof.
Let $x, y \in \mathbb{R}$.
Assume $x < y$.
Then $(-x) + x + (-y) < (-x) + y + (-y)$.
Therefore, $-y < -x$.
Therefore, if $x < y$, then $-y < -x$.
Conversely, assume $-y < -x$.
Then $x + (-y) + y < x + (-x) + y$.
Therefore, $x < y$.
Therefore, if $-y < -x$, then $x < y$.
Therefore, $x < y$ if and only if $-y < -x$.
Therefore, $\forall x, y \in \mathbb{R}$, $x < y$ if and only if $-y < -x$. □

25. $\forall x, y \in \mathbb{R}$, if $x < 0$ and $y < 0$, then $0 < xy$.

Proof.
Let $x, y \in \mathbb{R}$.
Assume $x < 0$ and $y < 0$.
Then $x + (-x) < 0 + (-x)$, and $y + (-y) < 0 + (-y)$.
We then have $0 < -x$ and $0 < -y$.
Multiplying $(-y)$ on both sides of $0 < -x$ gives us $(-y)0 < (-x)(-y)$.
This reduces to $0 < xy$.
Therefore, if $x < 0$ and $y < 0$, then $0 < xy$.
Therefore, $\forall x, y \in \mathbb{R}$, if $x < 0$ and $y < 0$, then $0 < xy$. □

27. $\forall x, y \in \mathbb{R}$, if $x < y$ and $0 < y$, then $2x < 4y$.

Proof.
Let $x, y \in \mathbb{R}$.
Assume $x < y$ and $0 < y$.
Then $2x < 2y$ and $(2)0 < 2y$ by O4.
Now, since $0 < 2y$, we have $2y + 0 < 2y + 2y$; hence $2y < 4y$.
We now have $2x < 2y$ and $2y < 4y$, giving us $2x < 4y$ by transitivity.
Therefore, if $x < y$ and $0 < y$, then $2x < 4y$.
Therefore, $\forall x, y \in \mathbb{R}$, if $x < y$ and $0 < y$, then $2x < 4y$. □

29. $\forall a, b, x, y \in \mathbb{R}$, if $a < b$ and $x < y$, then $a + x < b + y$.

Proof.
Let $a, b, x, y \in \mathbb{R}$.
Assume $a < b$ and $x < y$.
Adding x to both sides of the inequality $a < b$ gives $a + x < b + x$.
Likewise, adding b to both sides of $x < y$ gives $b + x < b + y$.
Since $a + x < b + x$ and $b + x < b + y$, we have $a + x < b + y$ by transitivity.
Therefore, if $a < b$ and $x < y$, then $a + x < b + y$.
Therefore, $\forall a, b, x, y \in \mathbb{R}$, if $a < b$ and $x < y$, then $a + x < b + y$. □

Prove the following propositions using a proof by contraposition.

31. $\forall x, y \in \mathbb{R}$, if $x \leq y$ and $y \leq x$, then $x = y$.

Proof.
Let $x, y \in \mathbb{R}$.
Assume $x \neq y$.
By trichotomy, we have either $y < x$ or $x < y$ or $x = y$.
Since $x \neq y$, we have $y < x$ or $x < y$.
Therefore, if $x \neq y$, then $y < x$ or $x < y$.
Therefore, if $x \leq y$ and $y \leq x$, then $x = y$.
Therefore, $\forall x, y \in \mathbb{R}$, if $x \leq y$ and $y \leq x$, then $x = y$. □

33. $\forall x \in \mathbb{R}$, if $x^2 \leq x$, then $x \leq 1$.

Proof.
Let $x \in \mathbb{R}$.
Assume $1 < x$.
Since $0 < 1$ and $1 < x$, we have $0 < x$ by transitivity.
Now, $1 < x$ and $0 < x$, so $1(x) < x(x)$ by O4.
Thus, $x < x^2$.
Therefore, if $1 < x$, then $x < x^2$.
Therefore, if $x^2 \leq x$, then $x \leq 1$.
Therefore, $\forall x \in \mathbb{R}$, if $x^2 \leq x$, then $x \leq 1$. □

35. $\forall x, y \in \mathbb{R}$, if $x^2 \leq y^2$, then $y \leq 0$ or $x \leq y$.

Proof.
Let $x, y \in \mathbb{R}$.
Assume $0 < y$ and $y < x$.
Then $y^2 < xy$ by O4.
Also, since $0 < y$ and $y < x$, we have $0 < x$ by transitivity.
Now, $0 < x$ and $y < x$, which by O4 gives us $xy < x^2$.
We now have $y^2 < xy$ and $xy < y^2$; hence $y^2 < x^2$ by transitivity.
Therefore, if $0 < y$ and $y < x$, then $y^2 < x^2$.
Therefore, if $x^2 \leq y^2$, then $y \leq 0$ or $x \leq y$.
Therefore, $\forall x, y \in \mathbb{R}$, if $x^2 \leq y^2$, then $y \leq 0$ or $x \leq y$. □

37. $\forall x, y \in \mathbb{R}$, if $x^2 - xy \leq xy - y^2$, then $x \leq y$.

Proof.
Let $x, y \in \mathbb{R}$.
    Assume $y < x$.
        Then $0 < x - y$.
        By O4, we have $y(x - y) < x(x - y)$; hence $xy - y^2 < x^2 - xy$.
    Therefore, if $y < x$, then $xy - y^2 < x^2 - xy$.
    Therefore, if $x^2 - xy \leq xy - y^2$, then $x \leq y$.
Therefore, $\forall x, y \in \mathbb{R}$, if $x^2 - xy \leq xy - y^2$, then $x \leq y$. □

Prove the following propositions using a proof by contradiction.

39. $\forall x, y \in \mathbb{R}$, if $x < 0$ and $0 < xy$, then $y < 0$.

Proof.
Let $x, y \in \mathbb{R}$.
Assume $x < 0$ and $0 < xy$ and $0 \leq y$.
Since $x < 0$, we have $x \leq 0$.
Now, $x \leq 0$ and $0 \leq y$, so $xy \leq 0$ by Proposition 1.1.21.
Now, $0 < xy$ and $xy \leq 0$, which is a contradiction.
Therefore, if $x < 0$ and $0 < xy$, then $y < 0$.
Therefore, $\forall x, y \in \mathbb{R}$, if $x < 0$ and $0 < xy$, then $y < 0$. □

41. $\forall x \in \mathbb{R}$, if $x \neq 0$, then $x^{-1} \neq 0$.

Proof.
Let $x \in \mathbb{R}$.
Assume $x \neq 0$ and $x^{-1} = 0$.
Then $xx^{-1} = x(0) = 0$.
However, we also have $xx^{-1} = 1 \neq 0$.
We now have $xx^{-1} = 0$ and $xx^{-1} \neq 0$, which is a contradiction.
Therefore, if $x \neq 0$, then $x^{-1} \neq 0$.
Therefore, $\forall x \in \mathbb{R}$, if $x \neq 0$, then $x^{-1} \neq 0$. □

43. $\forall x, y \in \mathbb{R}$, if $x < y < 0$, then $y^{-1} < x^{-1}$.

Proof.
Let $x, y \in \mathbb{R}$.
Assume $x < y < 0$ and $x^{-1} \leq y^{-1}$. That is, $x < y$ and $y < 0$ and $x^{-1} \leq y^{-1}$.
By transitivity, we have $x < 0$.
Since $x < 0$ and $y < 0$, by exercise 25, we have $0 < xy$.
Now, $x^{-1}xy \leq xy(y^{-1})$; hence $y \leq x$.
We now have the contradiction $x < y$ and $y \leq x$.
Therefore, if $x < y < 0$, then $y^{-1} < x^{-1}$.
Therefore, $\forall x, y \in \mathbb{R}$, if $x < y < 0$, then $y^{-1} < x^{-1}$. □

45. $\forall a, x, y \in \mathbb{R}$, if $ax < ay$ and $y \leq x$, then $a \leq 0$.

Proof.
Let $a, x, y \in \mathbb{R}$.
Assume $ax < ay$ and $y \leq x$ and $0 < a$.
Since $0 < a$, we have $0 < a^{-1}$.
By O4, $a^{-1}ax < a^{-1}ay$; hence $x < y$.
Now, $y \leq x$ and $x < y$, which is a contradiction.
Therefore, if $ax < ay$ and $y \leq x$, then $a \leq 0$.
Therefore, $\forall a, x, y \in \mathbb{R}$, if $ax < ay$ and $y \leq x$, then $a \leq 0$. □

Prove the following propositions.

47. $\exists x \in \mathbb{R}, x < x^2$.

Proof.
Put $x = 2$.
Since $0 < 1$, we have $0 + 1 < 1 + 1$; hence $1 < 2$. Therefore, $2(1) < 2(2)$, which gives us $(2) < (2)^2$.
Thus, $x < x^2$.
Therefore, $\exists x \in \mathbb{R}, x < x^2$. □

49. $\exists x \in \mathbb{R}, x = x^2$.

Proof.
Put $x = 1$.
Since $1 = (1)(1) = (1)^2$, we have $x = x^2$
Therefore, $\exists x \in \mathbb{R}, x = x^2$. □

51. $\forall y \in \mathbb{R}, \exists x \in \mathbb{R}, y = x + 1$.

Proof.
Let $y \in \mathbb{R}$.
Put $x = y - 1$.
Then $y = (y - 1) + 1 = x + 1$.
Therefore, $\exists x \in \mathbb{R}, y = x + 1$.
Therefore, $\forall y \in \mathbb{R}, \exists x \in \mathbb{R}, y = x + 1$. □

53. $\forall y \in \mathbb{R}, \exists x \in \mathbb{R}, y = 5x - 2$.

Proof.
Let $y \in \mathbb{R}$.
Put $x = \frac{1}{5}y + \frac{2}{5}$.
Then $5x - 2 = 5(\frac{1}{5}y + \frac{2}{5}) - 2 = y + 2 - 2 = y$.
Therefore, $\exists x \in \mathbb{R}, y = 5x - 2$.
Therefore, $\forall y \in \mathbb{R}, \exists x \in \mathbb{R}, y = 5x - 2$. □

55. $\forall a, b \in \mathbb{R}$, if $a < b$, then $\exists x \in (0, \infty), a + x = b$.

Proof.
Let $a, b \in \mathbb{R}$.
Assume $a < b$.
Put $x = b - a$.
Since $a < b$, we have $0 < b - a$; hence $0 < x$, which means $x \in (0, \infty)$. Also, $a + x = a + (b - a) = b$.
Therefore, $\exists x \in (0, \infty), a + x = b$.
Therefore, if $a < b$, then $\exists x \in (0, \infty), a + x = b$.
Therefore, $\forall a, b \in \mathbb{R}$, if $a < b$, then $\exists x \in (0, \infty), a + x = b$. □

57. $\forall x \in (0, 1), \exists y \in (0, 1), y < x$.

Proof.
Let $x \in (0, 1)$.
That is, $0 < x$ and $x < 1$.
Put $y = \frac{x}{2}$.
Then $2y = x$, so $0 < 2y$ and $2y < 1$.
By O4, $(\frac{1}{2})0 < (\frac{1}{2})(2y)$; hence $0 < y$.
Since $0 < y$, by O3, $y < 2y$. Therefore, $y < 1$, since $y < 2y$ and $2y < 1$.
We now have $0 < y$ and $y < 1$, which means $y \in (0, 1)$.
Finally, since $y < 2y$, we have $y < x$.
Therefore, $\exists y \in (0, 1), y < x$.
Therefore, $\forall x \in (0, 1), \exists y \in (0, 1), y < x$. □

59. $\forall x, y \in \mathbb{R}$, if $x < y$, then $\exists a \in (0, \infty), x + a < y$.

Proof.
Let $x, y \in \mathbb{R}$.
Assume $x < y$.
Put $a = \frac{y-x}{2}$.
Since $x < y$, we have $0 < y - x$, so $0 < \frac{y-x}{2}$. Thus, $a \in (0, \infty)$.
Also, since $x < y$, we have $x + (x + y - x) < y + (x + y - x)$, which gives us $2x + (y - x) < 2y$.
Therefore, $x + \frac{(y-x)}{2} < y$. In other words, $x + a < y$.
Therefore, $\exists a \in (0, \infty), x + a < y$.
Therefore, if $x < y$, then $\exists a \in (0, \infty), x + a < y$.
Therefore, $\forall x, y \in \mathbb{R}$, if $x < y$, then $\exists a \in (0, \infty), x + a < y$. □

61. $\forall a, b, x \in \mathbb{R}$, if $a < x < b$, then $\exists t \in (0, 1), x = (1 - t)a + tb$.

Proof.
Let $a, b, x \in \mathbb{R}$.
Assume $a < x < b$.
Put $t = \frac{x-a}{b-a}$.
Since $a < x$, we have $0 < x - a$. Likewise, since $a < b$, we have $0 < b - a$.
Therefore, $0 < (b - a)^{-1}$, and hence $0 < \frac{x-a}{b-a}$. That is, $0 < t$.
Further, since $x < b$, we have $x - a < b - a$.
Multiplying by $(b - a)^{-1}$ gives $\frac{x-a}{b-a} < 1$.
That is, $t < 1$, and so $t \in (0, 1)$.
Finally, since $t = \frac{x-a}{b-a}$, we have $t(b - a) = x - a$.
Then $x = a - ta + tb = (1 - t)a + tb$.
Therefore, $\exists t \in (0, 1), x = (1 - t)a + tb$.
Therefore, if $a < x < b$, then $\exists t \in (0, 1), x = (1 - t)a + tb$.
Therefore, $\forall a, b, x \in \mathbb{R}$, if $a < x < b$, then $\exists t \in (0, 1), x = (1 - t)a + tb$. □

63. $\forall x, y \in \mathbb{R}, \exists z \in \mathbb{R}, x < z$ and $y < z$.

Proof.
Let $x, y \in \mathbb{R}$.
Case 1: $x \leq y$.
Put $z = y + 1$.
Since $0 < 1$, we have $y < y + 1$; hence $y < z$. Since $x \leq y$ and $y < z$, we have $x < z$.
Therefore, $\exists z \in \mathbb{R}, x < z$ and $y < z$.
Case 2: $y < x$.
Put $z = x + 1$.
Since $x < x + 1$, we have $x < z$. Since $y < x$ and $x < z$, we have $y < z$.
Therefore, $\exists z \in \mathbb{R}, x < z$ and $y < z$.
Therefore, $\forall x, y \in \mathbb{R}, \exists z \in \mathbb{R}, x < z$ and $y < z$. □

Prove the following propositions.

65. $\forall a \in \mathbb{R}$, if $\forall x \in \mathbb{R}$, $a + x = x$, then $a = 0$. (That is, 0 is the only additive identity).

Proof.
Let $a \in \mathbb{R}$.
  Assume $\forall x \in \mathbb{R}$, $a + x = x$.
    Since $0 \in \mathbb{R}$, we then have $a + 0 = 0$, and hence $a = 0$.
  Therefore, if $\forall x \in \mathbb{R}$, $a + x = x$, then $a = 0$.
Therefore, $\forall a \in \mathbb{R}$, if $\forall x \in \mathbb{R}$, $a + x = x$, then $a = 0$. □

67. $\forall a \in \mathbb{R}$, if $\forall x \in \mathbb{R}$, $ax = a$, then $a = 0$.

Proof.
Let $a \in \mathbb{R}$.
  Assume $\forall x \in \mathbb{R}$, $ax = a$.
    Since $0 \in \mathbb{R}$, we have $a(0) = a$. Therefore, $a = 0$.
  Therefore, if $\forall x \in \mathbb{R}$, $ax = a$, then $a = 0$.
Therefore, $\forall a \in \mathbb{R}$, if $\forall x \in \mathbb{R}$, $ax = a$, then $a = 0$. □

69. $\forall a \in \mathbb{R}$, if $\forall x \in \mathbb{R}$, $ax \leq x$, then $a = 1$.

Proof.
Let $a \in \mathbb{R}$.
  Assume $\forall x \in \mathbb{R}$, $ax \leq x$.
    Since $1 \in \mathbb{R}$, we have $a(1) \leq 1$; hence $a \leq 1$. Since $-1 \in \mathbb{R}$, we have $a(-1) \leq -1$; hence $1 \leq a$.
    We now have $a \leq 1$ and $1 \leq a$; hence $a = 1$.
  Therefore, if $\forall x \in \mathbb{R}$, $ax \leq x$, then $a = 1$.
Therefore, $\forall a \in \mathbb{R}$, if $\forall x \in \mathbb{R}$, $ax \leq x$, then $a = 1$. □

71. $\forall a \in \mathbb{R}$, if $\forall x \in \mathbb{R}$, $ax \leq 0$, then $a = 0$.

Proof.
Let $a \in \mathbb{R}$.
  Assume $\forall x \in \mathbb{R}$, $ax \leq 0$.
    Since $1 \in \mathbb{R}$, we have $a(1) \leq 0$; hence $a \leq 0$. Since $-1 \in \mathbb{R}$, we have $a(-1) \leq 0$; hence $0 \leq a$.
    Now, $a \leq 0$ and $0 \leq a$, so $a = 0$.
  Therefore, if $\forall x \in \mathbb{R}$, $ax \leq 0$, then $a = 0$.
Therefore, $\forall a \in \mathbb{R}$, if $\forall x \in \mathbb{R}$, $ax \leq 0$, then $a = 0$. □

73. $\forall x \in \mathbb{R}$, if $\forall a \in (0, \infty)$, $x \leq a$, then $x \leq 0$.

Proof.
Let $x \in \mathbb{R}$.
Assume $0 < x$.
Put $a = \frac{x}{2}$.
Since $0 < x$, we have $0 < \frac{x}{2}$; hence $0 < a$. This means $a \in (0, \infty)$.
Since $1 < 2$, we have $x < 2x$; hence $\frac{x}{2} < x$. This means $a < x$.
Therefore, $\exists a \in (0, \infty)$, $a < x$.
Therefore, if $0 < x$, then $\exists a \in (0, \infty)$, $a < x$.
Therefore, if $\forall a \in (0, \infty)$, $x \leq a$, then $x \leq 0$.
Therefore, $\forall x \in \mathbb{R}$, if $\forall a \in (0, \infty)$, $x \leq a$, then $x \leq 0$. □

75. $\forall x \in \mathbb{R}$, if $\forall a \in (0, \infty)$, $x \leq a$, then $\forall b \in (0, \infty)$, $x < b$.

Proof.
Let $x \in \mathbb{R}$.
Assume $\forall a \in (0, \infty)$, $x \leq a$.
Let $b \in (0, \infty)$.
Then $0 < b$, so $0 < \frac{b}{2}$. Thus, $\frac{b}{2} \in (0, \infty)$.
Now, since $\forall a \in (0, \infty)$, $x \leq a$, and $\frac{b}{2} \in (0, \infty)$, we have $x \leq \frac{b}{2}$.
Now, $x \leq \frac{b}{2}$ and $\frac{b}{2} < b$, which gives us $x < b$ by transitivity.
Therefore, $\forall b \in (0, \infty)$, $x < b$.
Therefore, if $\forall a \in (0, \infty)$, $x \leq a$, then $\forall b \in (0, \infty)$, $x < b$.
Therefore, $\forall x \in \mathbb{R}$, if $\forall a \in (0, \infty)$, $x \leq a$, then $\forall b \in (0, \infty)$, $x < b$. □

77. $\forall x \in \mathbb{R}$, if $\forall a \in (0, \infty)$, $x < 100a$, then $\forall b \in (0, \infty)$, $x < b$.

Proof.
Let $x \in \mathbb{R}$.
Assume $\forall a \in (0, \infty)$, $x < 100a$.
Let $b \in (0, \infty)$.
Then $0 < b$, so $0 < \frac{b}{100}$. Thus, $\frac{b}{100} \in (0, \infty)$.
Now, since $\forall a \in (0, \infty)$, $x < 100a$, and $\frac{b}{100} \in (0, \infty)$, we have $x < 100\frac{b}{100}$. Thus, $x < b$.
Therefore, $\forall b \in (0, \infty)$, $x < b$.
Therefore, if $\forall a \in (0, \infty)$, $x < 100a$, then $\forall b \in (0, \infty)$, $x < b$.
Therefore, $\forall x \in \mathbb{R}$, if $\forall a \in (0, \infty)$, $x < 100a$, then $\forall b \in (0, \infty)$, $x < b$. □

79. $\forall x, y \in \mathbb{R}$, if $\forall a \in \mathbb{R}$, $x \leq a$ if and only if $y \leq a$, then $x = y$.

Proof.
Let $x, y \in \mathbb{R}$.
Assume $\forall a \in \mathbb{R}$, $x \leq a$ if and only if $y \leq a$.
Since $x \in \mathbb{R}$ and $x \leq x$, we then have $y \leq x$. Likewise, since $y \in \mathbb{R}$ and $y \leq y$, we have $x \leq y$.
Now, since $y \leq x$ and $x \leq y$, we have $x = y$.
Therefore, if $\forall a \in \mathbb{R}$, $x \leq a$ if and only if $y \leq a$, then $x = y$.
Therefore, $\forall x, y \in \mathbb{R}$, if $\forall a \in \mathbb{R}$, $x \leq a$ if and only if $y \leq a$, then $x = y$. □

81. $\forall x, y \in \mathbb{R}$, if $\forall a \in (-\infty, x]$, $a < y$, then $\exists b \in (-\infty, y]$, $x < b$.

Proof.
Let $x, y \in \mathbb{R}$.
    Assume $\forall a \in (-\infty, x]$, $a < y$.
        Since $x \in (-\infty, x]$, we have $x < y$.
        Put $b = y$.
            Then $b \leq y$, so $b \in (-\infty, y]$.
            Also, since $x < y$, we have $x < b$.
        Therefore, $\exists b \in (-\infty, y]$, $x < b$.
    Therefore, if $\forall a \in (-\infty, x]$, $a < y$, then $\exists b \in (-\infty, y]$, $x < b$.
Therefore, $\forall x, y \in \mathbb{R}$, if $\forall a \in (-\infty, x]$, $a < y$, then $\exists b \in (-\infty, y]$, $x < b$. □

Prove the following propositions.

83. $\forall a \in \mathbb{R}$, if $\exists x \in (0, \infty)$, $x < a$, then $0 < a$.

Proof.
Let $a \in \mathbb{R}$.
Assume $\exists x \in (0, \infty)$, $x < a$.
Choose $b \in (0, \infty)$ with $b < a$.
Since $b \in (0, \infty)$, we have $0 < b$.
Now, since $0 < b$ and $b < a$, we have $0 < a$ by transitivity.
Therefore, if $\exists x \in (0, \infty)$, $x < a$, then $0 < a$.
Therefore, $\forall a \in \mathbb{R}$, if $\exists x \in (0, \infty)$, $x < a$, then $0 < a$. □

85. $\forall a \in \mathbb{R}$, if $\exists x \in \mathbb{R}$, $a + x = x$, then $a = 0$.

Proof.

Let $a \in \mathbb{R}$.
Assume $\exists x \in \mathbb{R}$, $a + x = x$.
Choose $x_0 \in \mathbb{R}$ with $a + x_0 = x_0$.
Then $a + x_0 - x_0 = x_0 - x_0$, and hence $a = 0$.
Therefore, if $\exists x \in \mathbb{R}$, $a + x = x$, then $a = 0$.
Therefore, $\forall a \in \mathbb{R}$, if $\exists x \in \mathbb{R}$, $a + x = x$, then $a = 0$. □

87. $\forall a \in \mathbb{R}$, if $\exists x \in \mathbb{R}$, $x \neq 1$ and $ax = a$, then $a = 0$.

Proof.
Let $a \in \mathbb{R}$.
Assume $\exists x \in \mathbb{R}$, $x \neq 1$ and $ax = a$, and $a \neq 0$.
Choose $b \in \mathbb{R}$ with $b \neq 1$ and $ab = a$.
Since $a \neq 0$, a^{-1} is defined. Therefore, $a^{-1}ab = a^{-1}a$.
This gives us $b = 1$, which contradicts $b \neq 1$.
Therefore, if $\exists x \in \mathbb{R}$, $x \neq 1$ and $ax = a$, then $a = 0$.
Therefore, $\forall a \in \mathbb{R}$, if $\exists x \in \mathbb{R}$, $x \neq 1$ and $ax = a$, then $a = 0$. □

89. $\forall a \in \mathbb{R}$, if $\exists x \in \mathbb{R}$, $ax > 1$, then $\exists y \in \mathbb{R}$, $ay < -1$.

Proof.
Let $a \in \mathbb{R}$.
Assume $\exists x \in \mathbb{R}$, $ax > 1$.
Accordingly, choose $t \in \mathbb{R}$ with $at > 1$.
Put $y = -t$.
Since $at > 1$, we have $-at < -1$.
Therefore, $ay < -1$.
Therefore, $\exists y \in \mathbb{R}$, $ay < -1$.
Therefore, if $\exists x \in \mathbb{R}$, $ax > 1$, then $\exists y \in \mathbb{R}$, $ay < -1$.
Therefore, $\forall a \in \mathbb{R}$, if $\exists x \in \mathbb{R}$, $ax > 1$, then $\exists y \in \mathbb{R}$, $ay < -1$. □

91. $\forall x \in \mathbb{R}$, if $\exists a \in (0, \infty)$, $a \le x$, then $\exists b \in (0, \infty)$, $b < x$.

Proof.
Let $x \in \mathbb{R}$.
Assume $\exists a \in (0, \infty)$, $a \le x$.
Choose $t \in (0, \infty)$ with $t \le x$.
Put $b = \frac{t}{2}$.
Since $0 < t$, we have $0 < \frac{t}{2}$; hence $b \in (0, \infty)$.
Also, since $0 < b$, we have $b < 2b$; hence $b < t$.
Now, $b < t$ and $\le x$; so $b < x$.
Therefore, $\exists b \in (0, \infty)$, $b < x$.
Therefore, if $\exists a \in (0, \infty)$, $a \le x$, then $\exists b \in (0, \infty)$, $b < x$.
Therefore, $\forall x \in \mathbb{R}$, if $\exists a \in (0, \infty)$, $a \le x$, then $\exists b \in (0, \infty)$, $b < x$. □

93. $\forall x \in \mathbb{R}$, if $\exists a \in (0, \infty)$, $a < x$, then $\forall b \in (0, \infty)$, $0 < bx$.

Proof.
Let $x \in \mathbb{R}$.
Assume $\exists a \in (0, \infty)$, $a < x$.
Choose such an a. i.e. let $a \in (0, \infty)$ with $a < x$.
Let $b \in (0, \infty)$.
Since $a \in (0, \infty)$, we have $0 < a$.
Now, $0 < a$ and $a < x$, so $0 < x$.
Since $b \in (0, \infty)$, we have $0 < b$. Thus, $0x < bx$; hence $0 < bx$.
Therefore, $\forall b \in (0, \infty)$, $0 < bx$.
Therefore, if $\exists a \in (0, \infty)$, $a < x$, then $\forall b \in (0, \infty)$, $0 < bx$.
Therefore, $\forall x \in \mathbb{R}$, if $\exists a \in (0, \infty)$, $a < x$, then $\forall b \in (0, \infty)$, $0 < bx$. □

95. $\forall x \in \mathbb{R}$, if $\exists a \in (0, \infty)$, $ax < 0$, then $\forall b \in (0, \infty)$, $bx < 0$.

Proof.
Let $x \in \mathbb{R}$.
Assume $\exists a \in (0, \infty)$, $ax < 0$.
Choose such an a. i.e. let $a \in (0, \infty)$ with $ax < 0$.
Let $b \in (0, \infty)$.
We have $0 < b$, $0 < a$, and $ax < 0$.
Since $0 < a$, we have $0 < a^{-1}$; hence $a^{-1}ax < a^{-1}0$.
This gives us $x < 0$, and since $0 < b$, we have $bx < b0$; hence $bx < 0$.
Therefore, $\forall b \in (0, \infty)$, $bx < 0$.
Therefore, if $\exists a \in (0, \infty)$, $ax < 0$, then $\forall b \in (0, \infty)$, $bx < 0$.
Therefore, $\forall x \in \mathbb{R}$, if $\exists a \in (0, \infty)$, $ax < 0$, then $\forall b \in (0, \infty)$, $bx < 0$. □

Prove the following inequalities.

97. $\forall x, y \in \mathbb{R}, 2xy \leq x^2 + y^2$.

Proof.
Let $x, y \in \mathbb{R}$.
By Proposition 1.1.18, we have $0 \leq (x - y)^2$.
Therefore, $0 \leq x^2 - 2xy + y^2$.
Adding $2xy$ to both sides gives us $2xy \leq x^2 + y^2$.
Therefore, $\forall x, y \in \mathbb{R}, 2xy \leq x^2 + y^2$. □

99. $\forall x, y \in \mathbb{R}, 4xy \leq (x + y)^2$.

Proof.
Let $x, y \in \mathbb{R}$.
By Proposition 1.1.18, we have $0 \leq (x - y)^2$.
Therefore, $0 \leq x^2 - 2xy + y^2$.
Adding $4xy$ to both sides gives us $4xy \leq x^2 + 2xy + y^2$.
Thus, $4xy \leq (x + y)^2$.
Therefore, $\forall x, y \in \mathbb{R}, 4xy \leq (x + y)^2$. □

101. $\forall x, y \in \mathbb{R}, 4xy \leq (y + 2x)^2 - y^2$.

Proof.
Let $x, y \in \mathbb{R}$.
By Proposition 1.1.18, we have $0 \leq x^2$.
Therefore, $0 \leq 4x^2$.
Adding $4xy + y^2$ to both sides gives us $4xy + y^2 \leq y^2 + 4xy + 4x^2$.
We now have $4xy + y^2 \leq (y + 2x)^2$; hence $4xy \leq (y + 2x)^2 - y^2$.
Therefore, $\forall x, y \in \mathbb{R}, 4xy \leq (y + 2x)^2 - y^2$. □

Prove the following propositions involving the max **and** min **functions.**

103. $\forall x, y \in \mathbb{R}$, $\min(x, y) \leq x$ and $\min(x, y) \leq y$.

Proof.
Let $x, y \in \mathbb{R}$.
Case 1: $x \leq y$.
In this case, $\min(x, y) = x$; hence $\min(x, y) \leq x$.
Since $\min(x, y) = x$ and $x \leq y$, we have $\min(x, y) \leq y$.
Therefore, $\min(x, y) \leq x$ and $\min(x, y) \leq y$.
Case 2: $y < x$.
In this case, $\min(x, y) = y$. Therefore, $\min(x, y) \leq y$
Moreover, since $\min(x, y) = y$ and $y < x$, we have $\min(x, y) < x$; thus $\min(x, y) \leq x$.
Therefore, we again have $\min(x, y) \leq x$ and $\min(x, y) \leq y$.
Therefore, $\forall x, y \in \mathbb{R}$, $\min(x, y) \leq x$ and $\min(x, y) \leq y$. □

105. $\forall a, x, y \in \mathbb{R}$, if $x \leq y$, then $\max(a, x) \leq \max(a, y)$.

Proof.
Let $a, x, y \in \mathbb{R}$.
Assume $x \leq y$.
Case 1: $a \leq x$.
Then $a \leq y$ by transitivity.
We thus have $\max(a, x) = x$ and $\max(a, y) = y$.
Since $x \leq y$, we have $\max(a, x) \leq \max(a, y)$.
Case 2: $x < a$
Then $\max(a, x) = a$.
Case 2.1: $a \leq y$.
In this case, we have $\max(a, y) = y$.
Since $a \leq y$, we have $\max(a, x) \leq \max(a, y)$.
Case 2.2: $y < a$.
In this case, we have $\max(a, y) = a$.
Now, $max(a, x) = \max(a, y)$, so $\max(a, x) \leq \max(a, y)$.
Therefore, if $x \leq y$, then $\max(a, x) \leq \max(a, y)$.
Therefore, $\forall a, x, y \in \mathbb{R}$, if $x \leq y$, then $\max(a, x) \leq \max(a, y)$. □

107. $\forall a, b, x, y \in \mathbb{R}$, if $x \leq y$ and $a \leq b$, then $\max(a, x) \leq \max(b, y)$.

Proof.
Let $a, b, x, y \in \mathbb{R}$.
Assume $x \leq y$ and $a \leq b$.
By Exercise 105, since $x \leq y$, we have $\max(a, x) \leq \max(a, y)$.
Again, by Exercise 105, since $a \leq b$, we have $\max(a, y) \leq \max(b, y)$.
Now, $\max(a, x) \leq \max(a, y)$ and $\max(a, y) \leq \max(b, y)$, so $\max(a, x) \leq \max(b, y)$ by transitivity.
Therefore, if $x \leq y$ and $a \leq b$, then $\max(a, x) \leq \max(b, y)$.
Therefore, $\forall a, b, x, y \in \mathbb{R}$, if $x \leq y$ and $a \leq b$, then $\max(a, x) \leq \max(b, y)$. □

109. $\forall a, x, y \in \mathbb{R}$, if $\max(a, x) = \max(a, y)$ and $\min(a, x) = \min(a, y)$, then $x = y$.

Proof.
Let $a, x, y \in \mathbb{R}$.
Assume $\max(a, x) = \max(a, y)$ and $\min(a, x) = \min(a, y)$.
Case 1: $a \leq x$ and $a \leq y$.
Then $\max(a, x) = x$ and $\max(a, y) = y$; hence $x = y$.
Case 2: $a \leq x$ and $y < a$.
Then $\max(a, x) = x$ and $\max(a, y) = a$, so $x = a$.
Also, $\min(a, x) = a$ and $\min(a, y) = y$, so $a = y$.
Since $x = a$ and $a = y$, we have $x = y$.
Case 3: $x < a$ and $a \leq y$.
Then $\max(a, x) = a$ and $\max(a, y) = y$, so $a = y$.
$\min(a, x) = x$ and $\min(a, y) = a$, so $x = a$.
Since $x = a$ and $a = y$, we have $x = y$.
Case 4: $x < a$ and $y < a$.
Then $\min(a, x) = x$ and $\min(a, y) = y$; thus $x = y$.
Therefore, if $\max(a, x) = \max(a, y)$ and $\min(a, x) = \min(a, y)$, then $x = y$.
Therefore, $\forall a, x, y \in \mathbb{R}$, if $\max(a, x) = \max(a, y)$ and $\min(a, x) = \min(a, y)$, then $x = y$. □

111. $\forall x, y, z \in \mathbb{R}$, if $x \leq z$, then $\max(x, \min(y, z)) = \min(\max(x, y), z)$.

Proof.
Let $x, y, z \in \mathbb{R}$.
Assume $x \leq z$.
Case 1: $y \leq x$.
Then $y \leq z$ by transitivity.
In this case, $\max(x, \min(y, z)) = \max(x, y) = x$ and $\min(\max(x, y), z) = \min(x, z) = x$.
Therefore, $\max(x, \min(y, z)) = \min(\max(x, y), z)$.
Case 2: $x < y$ and $y \leq z$.
In this case, $\max(x, \min(y, z)) = \max(x, y) = y$ and $\min(\max(x, y), z) = \min(y, z) = y$.
Therefore, $\max(x, \min(y, z)) = \min(\max(x, y), z)$.
Case 3: $x < y$ and $z < y$.
In this case, $\max(x, \min(y, z)) = \max(x, z) = z$ and $\min(\max(x, y), z) = \min(y, z) = z$.
Therefore, $\max(x, \min(y, z)) = \min(\max(x, y), z)$.
Therefore, if $x \leq z$, then $\max(x, \min(y, z)) = \min(\max(x, y), z)$.
Therefore, $\forall x, y, z \in \mathbb{R}$, if $x \leq z$, then $\max(x, \min(y, z)) = \min(\max(x, y), z)$. □

113. $\forall a, b, x \in \mathbb{R}$, if $a < x < b$, then $\max(b - x, x - a) < b - a$.

Proof.
Let $a, b, x \in \mathbb{R}$.
Assume $a < x < b$.
Case 1: $x - a \leq b - x$.
In this case, $\max(b - x, x - a) = b - x$.
Since $a < x$, we have $-x < -a$, and so $b - x < b - a$.
Therefore, $\max(b - x, x - a) < b - a$.
Case 2: $b - x < x - a$.
In this case, $\max(b - x, x - a) = x - a$.
Since $x < b$, we have $x - a < b - a$, and so $\max(b - x, x - a) < b - a$.
Therefore, if $a < x < b$, then $\max(b - x, x - a) < b - a$.
Therefore, $\forall a, b, x \in \mathbb{R}$, if $a < x < b$, then $\max(b - x, x - a) < b - a$. □

115. $\forall x, y, a \in \mathbb{R}$, $\max(x, y) > a$ if and only if $x > a$ or $y > a$.

Proof.
Let $x, y, a \in \mathbb{R}$.
Assume $\max(x, y) > a$.
Case 1: $x \geq y$.
In this case, $\max(x, y) = x$, and hence $x > a$.
Therefore, $x > a$ or $y > a$.
Case 2: $x < y$.
In this case, $\max(x, y) = y$, so $y > a$.
Again, $x > a$ or $y > a$ is true.
Therefore, if $\max(x, y) > a$, then $x > a$ or $y > a$.
Conversely, assume $x > a$ or $y > a$.
Case 1: $x > a$.
Since $\max(x, y) \geq x$ (by Proposition 1.1.22) and $x > a$,
we have $\max(x, y) > a$ by transitivity.
Case 2: $y > a$.
Since $\max(x, y) \geq y$ (by Proposition 1.1.22) and $y > a$,
we have $\max(x, y) > a$ by transitivity.
Therefore, if $x > a$ or $y > a$, then $\max(x, y) > a$.
Therefore, $\forall x, y, a \in \mathbb{R}$, $\max(x, y) > a$ if and only if $x > a$ or $y > a$. □

117. $\forall x, y, a \in \mathbb{R}$, $\max(x, y) < a$ if and only if $x < a$ and $y < a$.

Proof.
Let $x, y, a \in \mathbb{R}$.
  Assume $\max(x, y) < a$.
    Case 1: $y \le x$.
      In this case, $\max(x, y) = x$, which means $x < a$.
      Since $y \le x < a$, we have $y < a$ by transitivity.
      Therefore, $x < a$ and $y < a$.
    Case 2: $x < y$.
      In this case, $\max(x, y) = y$, which means $y < a$.
      Since $x < y < a$, we have $x < a$ by transitivity.
      Therefore, $x < a$ and $y < a$.
  Therefore, if $\max(x, y) < a$, then $x < a$ and $y < a$.
  Conversely, assume $x < a$ and $y < a$.
    Case 1: $y \le x$.
      In this case, $\max(x, y) = x$.
      Since $x < a$, we then have $\max(x, y) < a$.
    Case 2: $x < y$.
      In this case, $\max(x, y) = y$.
      Since $y < a$, we then have $\max(x, y) < a$.
  Therefore, if $x < a$ and $y < a$, then $\max(x, y) < a$.
  Thus, $\max(x, y) < a$ if and only if $x < a$ and $y < a$.
Therefore, $\forall x, y, a \in \mathbb{R}$, $\max(x, y) < a$ if and only if $x < a$ and $y < a$. □

119. $\forall x, y, z \in \mathbb{R}$, if $\forall a \in \mathbb{R}$, $z \le a$ if and only if $x \le a$ and $y \le a$, then $z = \max(x, y)$.

Proof.
Let $x, y, z \in \mathbb{R}$.
  Assume $\forall a \in \mathbb{R}$, $z \le a$ if and only if $x \le a$ and $y \le a$.
    Since $x \le \max(x, y)$ and $y \le \max(x, y)$, we have $z \le \max(x, y)$.
    Also, since $z \le z$, we have $x \le z$ and $y \le z$.
    Case 1: $x \le y$.
      In this case, $\max(x, y) = y$; thus we have $\max(x, y) \le z$.
    Case 2: $y < x$.
      In this case, $\max(x, y) = x$, giving us again $\max(x, y) \le z$.
    We now have $\max(x, y) \le z$ and $z \le \max(x, y)$.
    Therefore, $z = \max(x, y)$
  Therefore, if $\forall a \in \mathbb{R}$, $z \le a$ if and only if $x \le a$ and $y \le a$, then $z = \max(x, y)$.
Therefore, $\forall x, y, z \in \mathbb{R}$, if $\forall a \in \mathbb{R}$, $z \le a$ if and only if $x \le a$ and $y \le a$, then $z = \max(x, y)$. □

Prove the following propositions involving the absolute value function.

121. $\forall x, y \in \mathbb{R}, |xy| = |x||y|$.

Proof.
Let $x, y \in \mathbb{R}$.
Case 1: $0 \leq x$ and $0 \leq y$.
In this case, $|x| = x$ and $|y| = y$.
Since $0 \leq x$ and $0 \leq y$, we have $0 \leq xy$, hence $|xy| = xy = |x||y|$.
Case 2: $0 \leq x$ and $y < 0$.
In this case, $|x| = x$ and $|y| = -y$.
Since $0 \leq x$ and $y < 0$, we have $xy \leq 0$.
Therefore, $|xy| = -xy = x(-y) = |x||y|$.
Case 3: $x < 0$ and $0 \leq y$.
In this case, $|x| = -x$ and $|y| = y$.
Since $x < 0$ and $0 \leq y$, we have $xy \leq 0$.
Therefore, $|xy| = -xy = (-x)y = |x||y|$.
Case 4: $x < 0$ and $y < 0$.
In this case, $|x| = -x$ and $|y| = -y$.
Since $x < 0$ and $y < 0$, we have $xy > 0$.
Therefore, $|xy| = xy = (-x)(-y) = |x||y|$.
Therefore, $\forall x, y \in \mathbb{R}, |xy| = |x||y|$. □

123. $\forall x, y \in \mathbb{R}, ||x| - |y|| \leq |x + y|$.

Proof.
Let $x, y \in \mathbb{R}$.
By the triangle inequality, we have $|(x + y) - y| \leq |x + y| + |-y|$.
Therefore, $|x| \leq |x + y| + |-y|$, and hence $|x| \leq |x + y| + |y|$, since $|-y| = |y|$.
This gives us $|x| - |y| \leq |x + y|$.
Similarly, $|(x + y) + (-x)| \leq |x + y| + |-x|$.
Therefore, $|y| \leq |x + y| + |-x|$, and hence $|y| \leq |x + y| + |x|$, since $|-x| = |x|$.
This gives us $-|x + y| \leq |x| - |y|$.
We now have $-|x + y| \leq |x| - |y| \leq |x + y|$.
Therefore, $||x| - |y|| \leq |x + y|$, by proposition 1.1.27.
Therefore, $\forall x, y \in \mathbb{R}, ||x| - |y|| \leq |x + y|$. □

125. $\forall x, y \in \mathbb{R}$, $y < |x|$ if and only if $x < -y$ or $y < x$.

Proof.
Let $x, y \in \mathbb{R}$.
Assume $y < |x|$.
Case 1: $0 \leq x$.
Then $|x| = x$, so $y < x$.
Therefore, $x < -y$ or $y < x$.
Case 2: $x < 0$.
Then $|x| = -x$, so $y < -x$; which implies $x < -y$.
Again, we have $x < -y$ or $y < x$.
Therefore, if $y < |x|$, then $x < -y$ or $y < x$.
Conversely, assume $x < -y$ or $y < x$.
Case 1: $y < x$.
By Proposition 1.1.26, $x \leq |x|$; hence by transitivity, $y < |x|$.
Case 1: $x < -y$.
Then $y < -x$.
By Proposition 1.1.26, $-x \leq |-x|$, and by Proposition 1.1.25, $|-x| = |x|$.
Combining these, we have $-x \leq |x|$; hence by transitivity, $y < |x|$.
Therefore, if $x < -y$ or $y < x$, then $y < |x|$.
Therefore, $y < |x|$ if and only if $x < -y$ or $y < x$.
Therefore, $\forall x, y \in \mathbb{R}$, $y < |x|$ if and only if $x < -y$ or $y < x$. □

127. $\forall x, y \in \mathbb{R}$, if $-y < x < y$, then $x^2 < y^2$.

Proof.
Let $x, y \in \mathbb{R}$.
Assume $-y < x < y$.
That is, $-y < x$ and $x < y$.
By transitivity, we have $-y < y$; hence, $y + (-y) < y + y$.
This gives us $0 < 2y$, which means $(2^{-1})0 < (2^{-1})2y$; hence $0 < y$.
Case 1: $0 < x$.
In this case, since $x < y$, we have $x^2 < xy$ (multiplying by x).
Likewise, since $x < y$, we have $xy < y^2$ (multiplying by y).
Since $x^2 < xy$ and $xy < y^2$, we have $x^2 < y^2$ by transitivity.
Case 2: $0 = x$.
In this case, since $x = 0$, we have $x^2 = 0$.
Since $0 < y$, we have $0y < y^2$; hence $0 < y^2$.
Therefore, $x^2 < y^2$.
Case 3: $x < 0$.
In this case, since $-y < x$, we have $x^2 < -xy$ (multiplying by x).
Also, since $-y < x$, we have $-y^2 < xy$ (multiplying by y); hence $-xy < y^2$.
Since $x^2 < -xy$ and $-xy < y^2$, we have $x^2 < y^2$ by transitivity.
Therefore, if $-y < x < y$, then $x^2 < y^2$.
Therefore, $\forall x, y \in \mathbb{R}$, if $-y < x < y$, then $x^2 < y^2$. □

129. $\forall x, y \in (0, 1), |x - y| < 1.$

Proof.
Let $x, y \in (0, 1)$.
Then $0 < x$ and $x < 1$ and $0 < y$ and $y < 1$.
Case 1: $y \leq x$.
In this case, $0 \leq x - y$, so $|x - y| = x - y$.
Now, since $x < 1$, $x - y < 1 - y$.
Also, since $0 < y$, $-y < 0$; hence $1 - y < 1$.
Since $x - y < 1 - y$ and $1 - y < 1$, we have $x - y < 1$. Thus, $|x - y| < 1$.
Case 2: $x < y$.
In this case, $x - y < 0$, which means $|x - y| = -(x - y) = y - x$.
Since $y < 1$, we have $y - x < 1 - x$.
Since $0 < x$, we have $-x < 0$; hence $1 - x < 1$.
We now have $y - x < 1 - x$ and $1 - x < 1$, giving us $y - x < 1$. Therefore, $|x - y| < 1$.
Therefore, $\forall x, y \in (0, 1), |x - y| < 1$. □

131. $\forall x \in \mathbb{R}$, if $|x - 1| < 1$, then $x^2 + 3x - 4 < 6$.

Proof.
Let $x \in \mathbb{R}$.
Assume $|x - 1| < 1$.
Then $-1 < x - 1 < 1$.
Adding 5 to both sides of $-1 < x - 1$ gives $4 < x + 4$.
Since $0 < 4$ and $4 < x + 4$, we have $0 < x + 4$ by transitivity.
Therefore, since $x - 1 < 1$, we have $(x + 4)(x - 1) < (x + 4)(1)$.
That is, $x^2 + 3x - 4 < x + 4$.
Now, adding 5 to both sides of $x - 1 < 1$ gives $x + 4 < 6$.
Since $x^2 + 3x - 4 < x + 4$ and $x + 4 < 6$, we have $x^2 + 3x - 4 < 6$ by transitivity.
Therefore, if $|x - 1| < 1$ then $x^2 + 3x - 4 < 6$.
Therefore, $\forall x \in \mathbb{R}$, if $|x - 1| < 1$ then $x^2 + 3x - 4 < 6$. □

133. $\forall x, y \in \mathbb{R}$, if $\forall a \in (0, \infty), |x - y| \leq a$, then $x = y$.

Proof.
Let $x, y \in \mathbb{R}$.
Assume $x \neq y$.
Then $|x - y| \neq 0$.
Since $|x - y| \geq 0$ and $|x - y| \neq 0$, we have $|x - y| > 0$.
Put $a = \frac{1}{2}|x - y|$.
Since $a = \frac{1}{2}|x - y|$, we have $a < |x - y|$.
Since $|x - y| > 0$, we have $a > 0$, and so $a \in (0, \infty)$.
Therefore, $\exists a \in (0, \infty), a < |x - y|$.
Therefore, if $x \neq y$, then $\exists a \in (0, \infty), a < |x - y|$.
Therefore, if $\forall a \in (0, \infty), |x - y| \leq a$, then $x = y$.
Therefore, $\forall x, y \in \mathbb{R}$, if $\forall a \in (0, \infty), |x - y| \leq a$, then $x = y$. □

Prove the following propositions.

135. $\forall x, y \in \mathbb{R}$, $\max(x, y) = \frac{1}{2}(|x - y| + x + y)$.

Proof.
Let $x, y \in \mathbb{R}$.
Case 1: $y \leq x$.
In this case, $\max(x, y) = x$. Also, in this case $0 \leq x - y$, so $|x - y| = x - y$.
Now, $\max(x, y) = x = \frac{1}{2}(x + x) = \frac{1}{2}(x - y + x + y)) = \frac{1}{2}(|x - y| + x + y)$.
Case 2: $x < y$.
In this case, $\max(x, y) = y$. Also, in this case $x - y < 0$, so $|x - y| = y - x$.
$\max(x, y) = y = \frac{1}{2}(y + y) = \frac{1}{2}(y - x + x + y) = \frac{1}{2}(|x - y| + x + y)$.
Therefore, $\max(x, y) = \frac{1}{2}(|x - y| + x + y)$.
Therefore, $\forall x, y \in \mathbb{R}$, $\max(x, y) = \frac{1}{2}(|x - y| + x + y)$. $\square$

137. $\forall x, y \in \mathbb{R}$, if $\min(x, y) < |x - y|$, then $2\min(x, y) < \max(x, y)$.

Proof.
Let $x, y \in \mathbb{R}$.
Assume $\min(x, y) < |x - y|$.
Case 1: $y \leq x$.
Then $\min(x, y) = y$ and and $\max(x, y) = x$.
Further, we have $0 \leq x - y$, so $|x - y| = x - y$.
We then have $y < x - y$; hence $2y < x$, which means $2\min(x, y) < \max(x, y)$.
Case 2: $x < y$.
In this case, $\min(x, y) = x$ and $\max(x, y) = y$.
Also, $x - y < 0$, so $|x - y| = -(x - y) = y - x$.
Since $\min(x, y) < |x - y|$, we have $x < y - x$, giving us $2x < y$; hence $2\min(x, y) < \max(x, y)$.
Therefore, if $\min(x, y) < |x - y|$, then $2\min(x, y) < \max(x, y)$.
Therefore, $\forall x, y \in \mathbb{R}$, if $\min(x, y) < |x - y|$, then $2\min(x, y) < \max(x, y)$. $\square$

139. $\forall x, y \in \mathbb{R}$, if $x > 0$ and $y > 0$, then $|x - y| < \max(x, y)$.

Proof.
Let $x, y \in \mathbb{R}$.
Assume $x > 0$ and $y > 0$.
Case 1: $x \geq y$.
In this case, $x - y \geq 0$, and hence $|x - y| = x - y$.
Also, in this case, $\max(x, y) = x$.
Since $y > 0$, we have $-y < 0$, and hence $x - y < x$.
Therefore, $|x - y| < \max(x, y)$.
Case 2: $x < y$.
In this case, $x - y < 0$, which means $|x - y| = y - x$.
Also, $\max(x, y) = y$.
Since $x > 0$, we have $-x < 0$, and hence $y - x < y$.
Therefore, $|x - y| < \max(x, y)$.
Therefore, if $x > 0$ and $y > 0$, then $|x - y| < \max(x, y)$.
Therefore, $\forall x, y \in \mathbb{R}$, if $x > 0$ and $y > 0$, then $|x - y| < \max(x, y)$. $\square$

Prove the following propositions using the Archimdedean property.

141. $\forall x, y \in \mathbb{R}$, if $0 < x$, then $\exists n \in \mathbb{N}$, $y \leq nx$.

Proof.
Let $x, y \in \mathbb{R}$,
  Assume $0 < x$.
    By the Archimedean property, $\exists n \in \mathbb{N}$, $y < nx$.
    Choose such an n.
      Since $y < nx$, we have $y \leq nx$.
    Therefore, $\exists n \in \mathbb{N}$, $y \leq nx$.
  Therefore, if $0 < x$, then $\exists n \in \mathbb{N}$, $y \leq nx$.
Therefore, $\forall x, y \in \mathbb{R}$, if $0 < x$, then $\exists n \in \mathbb{N}$, $y \leq nx$. □

143. $\forall x, y \in \mathbb{R}$, $\exists n \in \mathbb{N}$, $y \leq x + n$.

Proof.
Let $x, y \in \mathbb{R}$.
  By the Archimedean property, $\exists n \in \mathbb{N}$, $y - x < n(1)$.
  Choose such an n.
    Then $y - x + x < x + n$; hence $y < x + n$.
    Therefore, $y \leq x + n$.
  Therefore, $\exists n \in \mathbb{N}$, $y \leq x + n$.
Therefore, $\forall x, y \in \mathbb{R}$, $\exists n \in \mathbb{N}$, $y \leq x + n$. □

145. $\forall x \in \mathbb{R}$, if $\forall n \in \mathbb{N}$, $x \leq 3 + \frac{1}{n}$, then $x \leq 3$.

Proof.
Let $x \in \mathbb{R}$.
  Assume $3 < x$.
    The $0 < x - 3$
    By the Archimedean property, $\exists n \in \mathbb{N}$, $1 < n(x - 3)$.
    Choose such an n.
      Then $\frac{1}{n} < x - 3$; hence $3 + \frac{1}{n} < x$.
    Therefore, $\exists n \in \mathbb{N}$, $3 + \frac{1}{n} < x$.
  Therefore, if $3 < x$, then $\exists n \in \mathbb{N}$, $3 + \frac{1}{n} < x$.
  Therefore, if $\forall n \in \mathbb{N}$, $x \leq 3 + \frac{1}{n}$, then $x \leq 3$.
Therefore, $\forall x \in \mathbb{R}$, if $\forall n \in \mathbb{N}$, $x \leq 3 + \frac{1}{n}$, then $x \leq 3$. □

147. $\forall x \in \mathbb{R}$, if $\forall n \in \mathbb{N}$, $3 - \frac{1}{n} \leq x$, then $3 \leq x$.

Proof.
Let $x \in \mathbb{R}$.
  Assume $x < 3$.
    The $0 < 3 - x$
    Applying the Archimedean property, choose $n \in \mathbb{N}$ with $1 < n(3 - x)$.
      Then $\frac{1}{n} < 3 - x$; hence $x < 3 - \frac{1}{n}$.
    Therefore, $\exists n \in \mathbb{N}$, $x < 3 - \frac{1}{n}$.
  Therefore, if $x < 3$, then $\exists n \in \mathbb{N}$, $x < 3 - \frac{1}{n}$.
  Therefore, if $\forall n \in \mathbb{N}$, $3 - \frac{1}{n} \leq x$, then $3 \leq x$.
Therefore, $\forall x \in \mathbb{R}$, if $\forall n \in \mathbb{N}$, $3 - \frac{1}{n} \leq x$, then $3 \leq x$. □

149. $\forall a, x, y \in \mathbb{R}$, if $\forall n \in \mathbb{N}$, $x + an \leq y$, then $a \leq 0$.

Proof.
Let $a, x, y \in \mathbb{R}$.
  Assume $0 < a$.
    By the Archimedean property, $\exists n \in \mathbb{N}$, $y - x < an$.
    Choose such an n.
      Since $y - x < an$, we have $y < x + an$.
    Therefore, $\exists n \in \mathbb{N}$, $y < x + an$.
  Therefore, if $0 < a$, then $\exists n \in \mathbb{N}$, $y < x + an$.
  Therefore, if $\forall n \in \mathbb{N}$, $x + an \leq y$, then $a \leq 0$.
Therefore, $\forall a, x, y \in \mathbb{R}$, if $\forall n \in \mathbb{N}$, $x + an \leq y$, then $a \leq 0$. □

151. $\forall x, y \in \mathbb{R}$, if $\exists b \in \mathbb{R}$, $\forall n \in \mathbb{N}$, $|x - y| < \frac{b}{n}$, then $x = y$.

Proof.
Let $x, y \in \mathbb{R}$.
  Assume $x \neq y$.
    Then $x - y \neq 0$, and hence $|x - y| \neq 0$.
    Since $|x - y| \geq 0$, we then have $|x - y| > 0$.
    Let $b \in \mathbb{R}$.
      By the Archimedean property, $\exists n \in \mathbb{N}$, $b < n|x - y|$.
        For such an n, we have $\frac{b}{n} < |x - y|$; hence $\frac{b}{n} < |x - y|$.
      Therefore, $\exists n \in \mathbb{N}$, $\frac{b}{n} \leq |x - y|$.
    Therefore, $\forall b \in \mathbb{R}$, $\exists n \in \mathbb{N}$, $\frac{b}{n} \leq |x - y|$.
  Therefore, if $x \neq y$, then $\forall b \in \mathbb{R}$, $\exists n \in \mathbb{N}$, $\frac{b}{n} \leq |x - y|$.
  Therefore, if $\exists b \in \mathbb{R}$, $\forall n \in \mathbb{N}$, $|x - y| < \frac{b}{n}$, then $x = y$.
Therefore, $\forall x, y \in \mathbb{R}$, if $\exists b \in \mathbb{R}$, $\forall n \in \mathbb{N}$, $|x - y| < \frac{b}{n}$, then $x = y$. □

153. $\forall x, \varepsilon \in \mathbb{R}$, if $\varepsilon > 0$, then $\exists n \in \mathbb{N}$, $\frac{x}{n} < \varepsilon$.

Proof.
Let $x, \varepsilon \in \mathbb{R}$.
Assume $\varepsilon > 0$.
Applying the Archimedean property, choose $n \in \mathbb{N}$ with $x < n\varepsilon$.
Then $\frac{x}{n} < \varepsilon$.
Therefore, $\exists n \in \mathbb{N}$, $\frac{x}{n} < \varepsilon$.
Therefore, if $\varepsilon > 0$, then $\exists n \in \mathbb{N}$, $\frac{x}{n} < \varepsilon$.
Therefore, $\forall x, \varepsilon \in \mathbb{R}$, if $\varepsilon > 0$, then $\exists n \in \mathbb{N}$, $\frac{x}{n} < \varepsilon$. □

155. $\forall a, x, y \in \mathbb{R}$, if $x < y$, then $\exists n \in \mathbb{N}$, $x(n + a) < y(n - a)$.

Proof.
Let $a, x, y \in \mathbb{R}$.
Assume $x < y$.
Then $0 < y - x$.
By the Archimedean property, $\exists n \in \mathbb{N}$, $xa + ya < (y - x)n$.
Choose such an n.
Then $xa + ya < yn - xn$; $xn + xa < yn - ya$; $x(n + a) < y(n - a)$.
Therefore, $\exists n \in \mathbb{N}$, $x(n + a) < y(n - a)$.
Therefore, if $x < y$, then $\exists n \in \mathbb{N}$, $x(n + a) < y(n - a)$.
Therefore, $\forall a, x, y \in \mathbb{R}$, if $x < y$, then $\exists n \in \mathbb{N}$, $x(n + a) < y(n - a)$. □

157. $\forall x \in \mathbb{R}$, $x < 10$ if and only if $\exists n \in \mathbb{N}$, $x + \frac{1}{n} \leq 10$.

Proof.
Let $x \in \mathbb{R}$.
Assume $x < 10$.
Then $0 < 10 - x$.
Applying the Archimedean property, choose $n \in \mathbb{N}$ with $1 < (10 - x)n$.
Then $1 \leq (10 - x)n$, giving us $\frac{1}{n} \leq 10 - x$; hence $x + \frac{1}{n} < 10$.
Therefore, $\exists n \in \mathbb{N}$, $x + \frac{1}{n} \leq 10$.
Therefore, if $x < 10$, then $\exists n \in \mathbb{N}$, $x + \frac{1}{n} \leq 10$.
Conversely, assume $\exists n \in \mathbb{N}$, $x + \frac{1}{n} \leq 10$.
Choose such an n. i.e. let $n \in \mathbb{N}$ with $x + \frac{1}{n} \leq 10$.
Since $n \in \mathbb{N}$, we have $0 < n$; hence $0 < \frac{1}{n}$, giving us $10 < 10 + \frac{1}{n}$.
Now, $x + \frac{1}{n} \leq 10$ and $10 < 10 + \frac{1}{n}$, giving us $x + \frac{1}{n} < 10 + \frac{1}{n}$. Thus, $x < 10$.
Therefore, if $\exists n \in \mathbb{N}$, $x + \frac{1}{n} \leq 10$, then $x < 10$.
Therefore, $x < 10$ if and only if $\exists n \in \mathbb{N}$, $x + \frac{1}{n} \leq 10$.
Therefore, $\forall x \in \mathbb{R}$, $x < 10$ if and only if $\exists n \in \mathbb{N}$, $x + \frac{1}{n} \leq 10$. □

1.2 The Integers

Exercises 1.2.

Prove the following propositions using the Well-Ordering Property.

1. $\forall x \in \mathbb{R}$, if $0 < x$, then $\exists n \in \mathbb{N}$, $n - 1 < x \leq n$.

Proof.
Let $x \in \mathbb{R}$ and assume $0 < x$.
Let $S = \{k \in \mathbb{Z} \mid x \leq< k\}$.
By the Archimedean property, we have $\exists m \in \mathbb{N}$, $x < m$.
For such an m, we have $m \in S$, and hence $S \neq \emptyset$.
Further, for any $k \in S$, we have $0 < x \leq k$, and hence $0 < k$.
Thus, S is bounded below by 0.
Now, $S \neq \emptyset$ and S is bounded below.
By the well-ordering property, choose n to be the smallest element of S.
Since $n \in S$, $x \leq n$.
Further, since $n - 1 < n$, we have $n - 1 \notin S$.
Therefore, $n - 1 < x$.
Therefore, $\exists n \in \mathbb{N}$, $n - 1 < x \leq n$.
Therefore, $\forall x \in \mathbb{R}$, if $0 < x$, then $\exists n \in \mathbb{N}$, $n - 1 < x \leq n$. □

3. Let $S \subseteq \mathbb{N}$. Let $A = \{x \in \mathbb{R} \mid \frac{1}{x} \in S\}$. If $S \neq \emptyset$, then A has a largest element.

Proof.
Assume $S \neq \emptyset$.
Since S is non-empty, and bounded below by 0,
by the well-ordering property, S has a smallest element.
Choose $n \in S$ to be the smallest element of S.
Put $a = \frac{1}{n}$. We claim that a is the largest element of A.
Indeed, since $\frac{1}{a} = n$, we have $\frac{1}{a} \in S$ and hence $a \in A$.
Now, let $x \in A$.
Then $\frac{1}{x} \in S$, and so $n \leq \frac{1}{x}$.
Therefore, $x \leq \frac{1}{n}$, since $n > 0$ and $x > 0$.
In other words, $x \leq a$.
Therefore, $\forall x \in A$, $x \leq a$.
Thus, a is the largest element of S.
Therefore, if $S \neq \emptyset$, then A has a largest element. □

5. Let $a \in \mathbb{N}$. If $a \neq 1$, then the set $S = \{x \in \mathbb{N} \mid x \text{ divides } a \text{ and } x \neq a\}$ has a largest element.

Proof.
Let $a \in \mathbb{N}$.
Assume $a \neq 1$, and let $S = \{x \in \mathbb{N} \mid x \text{ divides } a \text{ and } x \neq a\}$.
  Since 1 divides a and $1 \neq a$, we have $1 \in S$.
  Therefore, $S \neq \emptyset$.
  Let $x \in S$.
    Then x divides a, and hence $x \leq a$.
  Therefore, $\forall x \in S$, $x \leq a$. That is, a is an upper bound of S.
  Since $S \neq \emptyset$ and S is bounded above, S has a largest element.
Therefore, if $a \neq 1$, then S has a largest element. □

7. $\forall x \in \mathbb{N}$, $3^x \geq 1 + 2^x$.

Proof.
Suppose $\exists x \in \mathbb{N}$, $3^x < 1 + 2^x$.
  Let $S = \{x \in \mathbb{N} \mid 3^x < 1 + 2^x\}$.
  By our assumption $S \neq \emptyset$ and so by the Well-Ordering Property, S has a smallest element.
  Let n be the smallest element of S.
  Note that since $3^1 = 1 + 2^1$, we have $1 \notin S$, so $n \neq 1$; hence $n - 1 \in \mathbb{N}$.
  Now, since $n - 1 \notin S$, we must have $1 + 2^{n-1} \leq 3^{n-1}$.
  Since $1 < 2$, we have $1 + 2^n < 2 + 2^n$; hence $1 + 2^n < 2(1 + 2^{n-1})$. Call this inequality A.
  Since $2 < 3$, we have $2(1 + 2^{n-1}) < 3(1 + 2^{n-1})$. Call this inequality B.
  Since $1 + 2^{n-1} \leq 3^{n-1}$, we have $3(1 + 2^{n-1}) \leq 3^n$. Call this inequality C.
  Now, from inequalities A, B, and C, we have $1 + 2^n < 3^n$, which contradicts $n \in S$.
Therefore, $\forall x \in \mathbb{N}$, $3^x \geq 1 + 2^x$. □

9. $\forall x \in \mathbb{Z}$, if x is odd, then $\forall n \in \mathbb{N}$, x^n is odd.

Proof.
Let $x \in \mathbb{Z}$ and assume x is odd.
  Suppose $\exists n \in \mathbb{N}$, x^n is even.
    Let $S = \{n \in \mathbb{N} \mid x^n \text{ is even}\}$.
    By our assumption $S \neq \emptyset$ and so by the Well-Ordering Property, S has a smallest element.
    Let n be the smallest element of S.
    Since x^1 is odd, $1 \notin S$, so $n \neq 1$; hence $n - 1 \in \mathbb{N}$.
    Now, $n \in S$, meaning x^n is even, and $n - 1 \notin S$, meaning x^{n-1} is odd.
    Choose $a, b \in \mathbb{Z}$ with $x^n = 2a$ and $x^{n-1} = 2b + 1$.
    Put $c = a - bx$.
      $2a = x^n = xx^{n-1} = x(2b + 1) = 2bx + x$; thus $x = 2a - 2bx = 2c$.
    Therefore, x is even, which is a contradiction.
  Therefore, $\forall n \in \mathbb{N}$, x^n is odd.
Therefore, $\forall x \in \mathbb{Z}$, if x is odd, then $\forall n \in \mathbb{N}$, x^n is odd. □

11. $\forall x \in \mathbb{R}$, if $\exists n \in \mathbb{N}$, $x^n < 0$, then $x < 0$.

Proof.
Let $x \in \mathbb{R}$.
Suppose $\exists n \in \mathbb{N}$, $x^n < 0$ and $0 \leq x$.
Let $S = \{n \in \mathbb{N} \mid x^n < 0\}$.
By our assumption $S \neq \emptyset$ and so by the Well-Ordering Property, S has a smallest element.
Let n be the smallest element of S.
Since $0 \leq x^1$, $1 \notin S$, so $n \neq 1$; hence $n - 1 \in \mathbb{N}$.
Now, $n - 1 \notin S$, which implies $0 \leq x^{n-1}$.
Since $0 \leq x$ and $0 \leq x^{n-1}$, we have $0 \leq xx^{n-1}$; hence $0 \leq x^n$.
This is a contradiction, since $n \in S$.
Therefore, if $\exists n \in \mathbb{N}$, $x^n < 0$, then $x < 0$.
Therefore, $\forall x \in \mathbb{R}$, if $\exists n \in \mathbb{N}$, $x^n < 0$, then $x < 0$. □

13. $\forall x \in \mathbb{R}$, if $1 < x$, then $\forall n \in \mathbb{N}$, $1 < x^n$.

Proof.
Let $x \in \mathbb{R}$.
Suppose $1 < x$ and $\exists n \in \mathbb{N}$, $x^n \leq 1$.
Let $S = \{n \in \mathbb{N} \mid x^n \leq 1\}$.
By our assumption $S \neq \emptyset$ and so by the Well-Ordering Property, S has a smallest element.
Let n be the smallest element of S.
Since $1 < x^1$, $1 \notin S$, so $n \neq 1$; hence $n - 1 \in \mathbb{N}$.
Now, $n - 1 \notin S$, which implies $1 < x^{n-1}$.
Since $1 < x$, we have $0 < x$ by transitivity.
Therefore, $x < xx^{n-1}$, giving us $1 < x < x^n \leq 1$.
By transitivity, $1 < 1$, which is a contradiction.
Therefore, if $1 < x$, then $\forall n \in \mathbb{N}$, $1 < x^n$.
Therefore, $\forall x \in \mathbb{R}$, if $1 < x$, then $\forall n \in \mathbb{N}$, $1 < x^n$. □

15. $\forall x \in \mathbb{R}$, if $0 < x < 1$, then $\forall n \in \mathbb{N}$, $x^n < 1$.

Proof.
Let $x \in \mathbb{R}$.
Suppose $0 < x < 1$ and $\exists n \in \mathbb{N}$, $1 \leq x^n$.
Let $S = \{n \in \mathbb{N} \mid 1 \leq x^n\}$.
By our assumption $S \neq \emptyset$ and so by the Well-Ordering Property, S has a smallest element.
Let n be the smallest element of S.
Since $x^1 < 1$, $1 \notin S$, so $n \neq 1$; hence $n - 1 \in \mathbb{N}$.
Now, $n - 1 \notin S$, which implies $x^{n-1} < 1$.
Since $0 < x$, we have $xx^{n-1} < x$.
This gives us $1 \leq x^n < x < 1$.
By transitivity, $1 < 1$, which is a contradiction.
Therefore, if $0 < x < 1$, then $\forall n \in \mathbb{N}$, $x^n < 1$.
Therefore, $\forall x \in \mathbb{R}$, if $0 < x < 1$, then $\forall n \in \mathbb{N}$, $x^n < 1$. □

17. $\forall m, n \in \mathbb{N}$, if $m < n$, then $2^m < 2^n$.

Proof.
Let $m \in \mathbb{N}$.
Suppose $\exists n \in \mathbb{N}$, $m < n$ and $2^n \le 2^m$.
Let $S = \{n \in \mathbb{N} \mid m < n \text{ and } 2^n \le 2^m\}$.
By our assumption $S \neq \emptyset$ and so by the Well-Ordering Property, S has a smallest element.
Let n be the smallest element of S.
Since $1 \le m < n$, we have that $n \neq 1$, and so $n - 1 \in \mathbb{N}$.
Since $\frac{1}{2} < 1$, we have $2^n \left(\frac{1}{2}\right) < 2^n(1)$; hence $2^{n-1} < 2^n$. Also, $2^n \le 2^m$, and so $2^{n-1} \le 2^m$ by transitivity.
Since $n - 1 \notin S$, we must not have $m < n - 1$. However, since $m < n$, we have $m \le n - 1$.
This means we must have $m = n - 1$. However, since $2^{n-1} < 2^n$, this means $2^m < 2^n$.
This is a contradiction, since we also have $2^n \le 2^m$.
Therefore, $\forall n \in \mathbb{N}$, if $m < n$, then $2^m < 2^n$.
Therefore, $\forall m, n \in \mathbb{N}$, if $m < n$, then $2^m < 2^n$. □

19. $\forall x \in \mathbb{R}$, if $0 < x < 1$, then $\forall m, n \in \mathbb{N}$, if $m < n$, then $x^n < x^m$.

Proof.
Let $x \in \mathbb{R}$, and assume $0 < x < 1$.
Let $m \in \mathbb{N}$, and suppose $\exists n \in \mathbb{N}$, $m < n$ and $x^m \le x^n$.
Let $S = \{n \in \mathbb{N} \mid m < n \text{ and } x^m \le x^n\}$.
By our assumption $S \neq \emptyset$ and so by the Well-Ordering Property, S has a smallest element.
Let n be the smallest element of S.
Since $1 \le m < n$, we have that $n \neq 1$, and so $n - 1 \in \mathbb{N}$.
Since $0 < x < 1$, we have $x^{n-1}x < x^{n-1}(1)$; hence $x^n < x^{n-1}$.
Also, $x^m \le x^n$, and so $x^m \le x^{n-1}$ by transitivity.
Since $n - 1 \notin S$, we must not have $m < n - 1$. However, since $m < n$, we have $m \le n - 1$.
This means we must have $m = n - 1$. However, since $x^n < x^{n-1}$, this means $x^n < x^m$.
This, with the fact that $x^m \le x^n$, gives us a contradiction.
Therefore, $\forall m, n \in \mathbb{N}$, if $m < n$, then $x^n < x^m$.
Therefore, $\forall x \in \mathbb{R}$, if $0 < x < 1$, then $\forall m, n \in \mathbb{N}$, if $m < n$, then $x^n < x^m$. □

Prove the following propositions using theorem 1.2.3 or its corollary.

21. $\forall x \in \mathbb{Z}$, if $x < 0$, then $x \le -1$.

Proof.
Let $x \in \mathbb{Z}$.
Assume $x < 0$.
Then $0 < -x$.
By Theorem 1.2.3, we then have $1 \le -x$.
Therefore, $x \le -1$.
Therefore, if $x < 0$, then $x \le -1$.
Therefore, $\forall x \in \mathbb{Z}$, if $x < 0$, then $x \le -1$. □

23. $\forall x \in \mathbb{N}$, if x divides 2, then $x = 1$ or $x = 2$.

Proof.
Let $x \in \mathbb{N}$.
Assume x divides 2.
Let $a \in \mathbb{Z}$ with $2 = ax$.
Since $0 < x$ and $0 < 2$, we have $0 < a$; hence $1 \le a$ by Theorem 1.2.3.
Now, $x \le ax$, so $x \le 2$.
Since $0 < x$, we have $1 \le x$ by Theorem 1.2.3.
Case 1: $1 = x$.
In this case, we have the desired result: $x = 1$ or $x = 2$.
Case 2: $1 < x$.
Then $0 < x - 1$; hence $1 \le x - 1$ by Theorem 1.2.3; hence $2 \le x$.
We now have $2 \le x$ and $x \le 2$, so $x = 2$.
Therefore, it is again true that $x = 1$ or $x = 2$.
Therefore, if x divides 2, then $x = 1$ or $x = 2$.
Therefore, $\forall x \in \mathbb{N}$, if x divides 2, then $x = 1$ or $x = 2$. □

25. $\forall x \in \mathbb{R}$, $\forall m, n \in \mathbb{Z}$, if $n \le x < n + 1$ and $m \le x < m + 1$, then $m = n$.

Proof.
Let $x \in \mathbb{R}$ and let $m, n \in \mathbb{Z}$.
Assume $n \le x < n + 1$ and $m \le x < m + 1$
Since $n \le x$ and $x < m + 1$, we have $n < m + 1$ by transitivity.
Likewise, since $m \le x$ and $x < n + 1$, we have $m < n + 1$.
Now, since $n < m + 1$, we have $0 < m - n + 1$; hence $1 \le m - n + 1$ by Theorem 1.2.3.
This gives us $0 \le m - n$; which means $n \le m$.
Similarly, since $m < n + 1$, we have $m \le n$.
Therefore, $m = n$.
Therefore, if $n \le x < n + 1$ and $m \le x < m + 1$, then $m = n$.
Therefore, $\forall x \in \mathbb{R}$, $\forall m, n \in \mathbb{Z}$, if $n \le x < n + 1$ and $m \le x < m + 1$, then $m = n$. □

Prove the following propositions.

27. $\forall x \in \mathbb{Z}$, if x^2 is odd then x is odd.

Proof.
Let $x \in \mathbb{Z}$.
Assume x is not odd.
Then x is even.
Choose $a \in \mathbb{Z}$ with $x = 2a$.
Put $b = 2a^2$.
$x^2 = (2a)^2 = 4a^2 = 2(2a^2) = 2b$.
Therefore, x^2 is even; hence x^2 is not odd.
Therefore, if x is not odd, then x^2 is not odd.
Therefore, if x^2 is odd, then x is odd.
Therefore, $\forall x \in \mathbb{Z}$, if x^2 is odd then x is odd. □

29. $\forall x, y \in \mathbb{Z}$, if x is even and $x + y$ is even, then y is even.

Proof.
Let $x, y \in \mathbb{Z}$.
Assume x is even and $x + y$ is even.
Then $\exists k \in \mathbb{Z}, x = 2k$ and $\exists k \in \mathbb{Z}, x + y = 2k$.
Let $a \in \mathbb{Z}$ with $x = 2a$ and let $b \in \mathbb{Z}$ with $x + y = 2b$.
Put $c = b - a$.
$y = x + y - x = (2b) - (2a) = 2b - 2a = 2(b - a) = 2c$.
Therefore, $\exists c \in \mathbb{Z}, y = 2c$.
Thus, y is even.
Therefore, if x is even and $x + y$ is even, then y is even.
Therefore, $\forall x, y \in \mathbb{Z}$, if x is even and $x + y$ is even, then y is even. □

31. $\forall x, y \in \mathbb{Z}$, if x is odd and $x + y$ is even, then y is odd.

Proof.
Let $x, y \in \mathbb{Z}$.
Assume x is odd and $x + y$ is even.
Then $\exists k \in \mathbb{Z}, x = 2k + 1$ and $\exists k \in \mathbb{Z}, x + y = 2k$.
Let $a \in \mathbb{Z}$ with $x = 2a + 1$ and let $b \in \mathbb{Z}$ with $x + y = 2b$.
Put $c = b - a - 1$.
$y = x + y - x = (2b) - (2a + 1) = 2b - 2a - 1 = 2b - 2a - 2 + 1 = 2(b - a - 1) + 1 = 2c + 1$.
Therefore, $\exists c \in \mathbb{Z}, y = 2c + 1$.
Thus, y is odd.
Therefore, if x is odd and $x + y$ is even, then y is odd.
Therefore, $\forall x, y \in \mathbb{Z}$, if x is odd and $x + y$ is even, then y is odd. □

33. $\forall x, y \in \mathbb{Z}$, if x is odd and xy is odd, then y is odd.

Proof.
Let $x, y \in \mathbb{Z}$.
Assume x is odd and xy is odd.
Then $\exists t \in \mathbb{Z}, x = 2t + 1$ and $\exists t \in \mathbb{Z}, xy = 2t + 1$.
Let $a \in \mathbb{Z}$ with $x = 2a + 1$ and let $b \in \mathbb{Z}$ with $xy = 2b + 1$.
Put $c = b - ay$.
Since $x = 2a + 1$ and $xy = 2b + 1$, we have $(2a + 1)y = 2b + 1$.
This gives us $2ay + y = 2b + 1$; hence $y = 2b - 2ay + 1 = 2(b - ay) + 1 = 2c + 1$.
Therefore, $\exists c \in \mathbb{Z}, y = 2c + 1$.
This shows that y is odd.
Therefore, if x is odd and xy is odd, then y is odd.
Therefore, $\forall x, y \in \mathbb{Z}$, if x is odd and xy is odd, then y is odd. □

35. $\forall x, y \in \mathbb{Z}$, if xy is even, then x is even or y is even.

Proof.
Let $x, y \in \mathbb{Z}$.
Assume x is not even and y is not even.
Then x is odd and y is odd.
Let $a \in \mathbb{Z}$ with $x = 2a + 1$ and let $b \in \mathbb{Z}$ with $y = 2b + 1$
Put $c = 2ab + a + b$.
$xy = (2a + 1)(2b + 1) = 4ab + 2a + 2b + 1 = 2(2ab + a + b) + 1 = 2c + 1$.
Therefore, $\exists c \in \mathbb{Z}, xy = 2c + 1$.
This means xy is odd; thus xy is not even.
Therefore, if x is not even and y is not even, then xy is not even.
Therefore, if xy is even, then x is even or y is even.
Therefore, $\forall x, y \in \mathbb{Z}$, if xy is even, then x is even or y is even. □

37. $\forall x, y, z \in \mathbb{Z}$, if $x - y$ is even and $y - z$ is even, then $x - z$ is even.

Proof.
Let $x, y, z \in \mathbb{Z}$.
Assume $x - y$ is even and $y - z$ is even.
Then $\exists q \in \mathbb{Z}, x - y = 2q$ and $\exists q \in \mathbb{Z}, y - z = 2q$.
Choose $a \in \mathbb{Z}$ with $x - y = 2a$, and choose $b \in \mathbb{Z}$ with $y - z = 2b$.
Put $c = a + b$.
$x - z = x - y + y - z = 2a + 2b = 2(a + b) = 2c$.
Therefore, $\exists c \in \mathbb{Z}, x - z = 2c$.
That is, $x - z$ is even.
Therefore, if $x - y$ is even and $y - z$ is even, then $x - z$ is even.
Therefore, $\forall x, y, z \in \mathbb{Z}$, if $x - y$ is even and $y - z$ is even, then $x - z$ is even. □

39. $\forall x, y, z \in \mathbb{Z}$, if x divides y and y divides z, then x divides z.

Proof.
Let $x, y, z \in \mathbb{Z}$.
Assume x divides y and y divides z.
That is, $\exists q \in \mathbb{Z}$, $y = xq$ and $\exists q \in \mathbb{Z}$, $z = yq$.
Choose $a, b \in \mathbb{Z}$ with $y = xa$ and $z = yb$.
Put $c = ab$.
Then $z = yb = (xa)b = xc$.
Therefore, $\exists c \in \mathbb{Z}$, $z = xc$.
Hence, x divides z.
Therefore, if x divides y and y divides z, then x divides z.
Therefore, $\forall x, y, z \in \mathbb{Z}$, if x divides y and y divides z, then x divides z. □

41. $\forall x \in \mathbb{Z}$, if 3 divides x, then 3 divides $9 - x$.

Proof.
Let $x \in \mathbb{Z}$.
Assume 3 divides x.
Then, $\exists k \in \mathbb{Z}$, $x = 3k$. Choose such a k.
Then, $9 - x = 9 - 3k = 3(3 - k)$.
Putting $n = 3 - k$ gives us $9 - x = 3n$.
so, $\exists n \in \mathbb{Z}$, $9 - x = 3n$.
Therefore, 3 divides $9 - x$.
Therefore, if 3 divides x, then 3 divides $9 - x$.
Therefore, $\forall x \in \mathbb{Z}$, if 3 divides x, then 3 divides $9 - x$. □

43. $\forall x, y \in \mathbb{Z}$, if 5 divides $11x + 6y$, then 5 divides $x + y$.

Proof.

Let $x, y \in \mathbb{Z}$.
Assume 5 divides $11x + 6y$.
Choose $a \in \mathbb{Z}$ with $11x + 6y = 5a$.
$x + y = 11x + 6y - 10x - 5y = 5a - 5(2x + y)$.
Put $b = a - 2x - y$.
Then $x + y = 5b$.
Therefore, $\exists b \in \mathbb{Z}$, $x + y = 5b$.
That is, 5 divides $x + y$.
Therefore, if 5 divides $11x + 6y$, then 5 divides $x + y$.
Therefore, $\forall x, y \in \mathbb{Z}$, if 5 divides $11x + 6y$, then 5 divides $x + y$. □

45. $\forall x, y \in \mathbb{Z}$, if x divides y, then x divides $|y|$.

Proof.
Let $x, y \in \mathbb{Z}$.
Assume x divides y.
Let $t \in \mathbb{Z}$ with $y = xt$.
Case 1: $0 \leq y$.
In this case, $|y| = y$, and since x divides y, we have that x divides $|y|$.
Case 2: $y < 0$.
In this case, $|y| = -y$.
Put $s = -t$.
Since $y = xt$, we have $-y = x(-t)$; hence $|y| = xs$.
Therefore, $\exists s \in \mathbb{Z}, |y| = xs$.
This shows that x divides $|y|$.
Therefore, if x divides y, then x divides $|y|$.
Therefore, $\forall x, y \in \mathbb{Z}$, if x divides y, then x divides $|y|$. □

47. $\forall x, y \in \mathbb{Z}$, if $|x|$ divides y, then x divides y.

Proof.
Let $x, y \in \mathbb{Z}$.
Assume $|x|$ divides y.
Let $a \in \mathbb{Z}$ with $y = a|x|$.
Case 1: $0 \leq x$.
In this case, $|x| = x$, and since $|x|$ divides y, we have that x divides y.
Case 2: $x < 0$.
In this case, $|x| = -x$, and so $y = -ax$.
Put $b = -a$.
Then $y = bx$
Therefore, $\exists b \in \mathbb{Z}, y = bx$; thus x divides y.
Therefore, if $|x|$ divides y, then x divides y.
Therefore, $\forall x, y \in \mathbb{Z}$, if $|x|$ divides y, then x divides y. □

49. $\forall x \in \mathbb{Z}$, if 3 divides x and 2 divides x, then 6 divides x.

Proof.
Let $x \in \mathbb{Z}$.
Assume 3 divides x and 2 divides x.
Choose $a, b \in \mathbb{Z}$ with $x = 3a$ and $x = 2b$.
Put $c = b - a$.
$x = 3x - 2x = 3(2b) - 2(3a) = 6(b - a) = 6c$.
Therefore, $\exists c \in \mathbb{Z}, x = 6c$.
That is, 6 divides x.
Therefore, if 3 divides x and 2 divides x, then 6 divides x.
Therefore, $\forall x \in \mathbb{Z}$, if 3 divides x and 2 divides x, then 6 divides x. □

51. $\forall x \in \mathbb{Z}$, if 30 divides x, then 5 divides x and 6 divides x.

Proof.
Let $x \in \mathbb{Z}$.
Assume 30 divides x.
Choose $c \in \mathbb{Z}$ with $x = 30c$.
Put $a = 6c$.
$x = 30c = 5(6c) = 5a$.
Therefore, $\exists a \in \mathbb{Z}, x = 5a$.
That is, 5 divides x.
Next, put $b = 5c$.
$x = 30c = 6(5c) = 6b$.
Therefore, $\exists b \in \mathbb{Z}, x = 6b$.
Thus, 6 divides x.
Therefore, if 30 divides x, then 5 divides x and 6 divides x.
Therefore, $\forall x \in \mathbb{Z}$, if 30 divides x, then 5 divides x and 6 divides x. $\square$

Prove the following propositions about greatest common divisors and least common multiples.

53. $\forall x, y \in \mathbb{Z}$, if $x \neq 0$ and $y \neq 0$, then $\exists f \in \mathbb{Z}$, $f = \operatorname{lcm}(x, y)$.

Proof.
Let $x, y \in \mathbb{Z}$.
Assume $x \neq 0$ and $y \neq 0$.
Let $S = \{n \in \mathbb{Z} \mid x$ divides n and y divides n and $0 < n\}$
To prove $S \neq \emptyset$, we claim that $x^2y^2 \in S$.
Indeed, putting $a = xy^2$ gives us $x^2y^2 = xa$; hence x divides x^2y^2.
Likewise, putting $b = x^2y$ gives us $x^2y^2 = yb$; hence y divides x^2y^2.
Finally, since $x \neq 0$ and $y \neq 0$, we have $x^2y^2 \neq 0$; thus $0 < x^2y^2$.
Now, since x divides x^2y^2 and y divides x^2y^2 and $0 < x^2y^2$, we have $x^2y^2 \in S$.
Therefore, $S \neq \emptyset$.
By the well-ordering property, S has a smallest element.
Let $f \in S$ be the smallest element of S.
Since $f \in S$, we have that x divides f and y divides f and $0 < f$.
It only remains to show $\forall a \in \mathbb{Z}$, if x divides a and y divides a and $a \neq 0$, then $f \leq |a|$.
To this end, let $a \in \mathbb{Z}$ and assume x divides a and y divides a and $a \neq 0$.
Note, since x divides a, we have that x divides $|a|$.
Likewise, since y divides a, we have that y divides $|a|$.
Since $a \neq 0$, we have $0 < |a|$. Thus, $|a| \in S$.
Since f is the smallest element of S, we have $f \leq |a|$.
Therefore, $\forall a \in \mathbb{Z}$, if x divides a and y divides a and $a \neq 0$, then $f \leq |a|$.
Therefore, $f = \operatorname{lcm}(x, y)$.
Therefore, $\exists f \in \mathbb{Z}$, $f = \operatorname{lcm}(x, y)$.
Therefore, if $x \neq 0$ and $y \neq 0$ then $\exists f \in \mathbb{Z}$, $f = \operatorname{lcm}(x, y)$.
Therefore, $\forall x, y \in \mathbb{Z}$, if $x \neq 0$ and $y \neq 0$ then $\exists f \in \mathbb{Z}$, $f = \operatorname{lcm}(x, y)$. □

55. $\forall a, x, y \in \mathbb{Z}$, if x divides y, then $\gcd(a, x)$ divides $\gcd(a, y)$.

Proof.
Let $a, x, y \in \mathbb{Z}$.
Assume x divides y.
Choose $k \in \mathbb{Z}$ with $y = xk$.
Since $\gcd(a, x)$ divides x, we can also choose $m \in \mathbb{Z}$ with $x = \gcd(a, x)m$.
Then $y = xk = \gcd(a, x)mk$. This implies that $\gcd(a, x)$ divides y.
Now, since $\gcd(a, x)$ divides a and $\gcd(a, x)$ divides y,
we have by Corollary 1.2.12 $\gcd(a, x)$ divides $\gcd(a, y)$.
Therefore, if x divides y, then $\gcd(a, x)$ divides $\gcd(a, y)$.
Therefore, $\forall a, x, y \in \mathbb{N}$, if x divides y, then $\gcd(a, x)$ divides $\gcd(a, y)$. □

57. $\forall a, b, x, y \in \mathbb{Z}$, if x divides y and a divides b, then $\gcd(a, x)$ divides $\gcd(b, y)$.

Proof.
Let $a, b, x, y \in \mathbb{Z}$.
Assume x divides y and a divides b.
Since x divides y, from exercise 55 we have that $\gcd(a, x)$ divides $\gcd(a, y)$.
Likewise, since a divides b, we have that $\gcd(a, y)$ divides $\gcd(b, y)$.
Now, since $\gcd(a, x)$ divides $\gcd(a, y)$ and $\gcd(a, y)$ divides $\gcd(b, y)$,
we have that $\gcd(a, x)$ divides $\gcd(b, y)$ by exercise 39.
Therefore, if x divides y and a divides b, then $\gcd(a, x)$ divides $\gcd(b, y)$.
Therefore, $\forall a, b, x, y \in \mathbb{Z}$, if x divides y and a divides b, then $\gcd(a, x)$ divides $\gcd(b, y)$. □

59. $\forall m, n \in \mathbb{Z}$, if $m, n \neq 0$ and $\gcd(m, n) = 1$, then $\forall x \in \mathbb{Z}, \exists u, v \in \mathbb{Z}, x = mu + nv$.

Proof.
Let $m, n \in \mathbb{Z}$.
Assume $m, n \neq 0$ and $\gcd(m, n) = 1$.
Choose $s, t \in \mathbb{Z}$ with $1 = sm + tn$.
Let $x \in \mathbb{Z}$.
Put $u = xs$ and $v = xt$.
$x = x(1) = x(sm + tn) = xsm + xtn = mu + nv$.
Therefore, $\exists u, v \in \mathbb{Z}, x = mu + nv$.
Therefore, $\forall x \in \mathbb{Z}, \exists u, v \in \mathbb{Z}, x = mu + nv$.
So, if $\gcd(m, n) = 1$ then $\forall x \in \mathbb{Z}, \exists u, v \in \mathbb{Z}, x = mu + nv$.
Thus, $\forall m, n \in \mathbb{Z}$, if $\gcd(m, n) = 1$ then $\forall x \in \mathbb{Z}, \exists u, v \in \mathbb{Z}, x = mu + nv$. □

61. $\forall x, y, a \in \mathbb{Z}$, if $x, y \neq 0$ and x divides a and y divides a and $\gcd(x, y) = 1$, then xy divides a.

Proof.
Let $x, y, a \in \mathbb{Z}$.
Assume x divides a and y divides a and $\gcd(x, y) = 1$.
Choose $p \in \mathbb{Z}$ with $a = xp$.
Choose $q \in \mathbb{Z}$ with $a = yq$.
Choose $s, t \in \mathbb{Z}$ with $1 = sx + ty$.
Put $k = qs + tp$.
$a = a(1) = a(sx + ty) = asx + aty = (yq)sx + (xp)ty = xy(qs + tp) = xyk$.
Therefore, $\exists k \in \mathbb{Z}, a = xyk$.
That is, xy divides a.
Therefore, if x divides a and y divides a and $\gcd(x, y) = 1$, then xy divides a.
Thus, $\forall x, y, a \in \mathbb{Z}$, if x divides a and y divides a and $\gcd(x, y) = 1$, then xy divides a. □

Prove the following propositions about rational and irrational numbers.

63. $\forall x, y \in \mathbb{R}$, if x is rational and y is rational, then $x + y$ is rational.

Proof.
Let $x, y \in \mathbb{R}$.
Assume x is rational and y is rational.
Then, by the definition, we choose
$m, n \in \mathbb{Z}$, with $m = nx$ and $n \neq 0$ and $p, q \in \mathbb{Z}$ with $p = qx$ and $q \neq 0$.
Put $a = mq + pn$ and $b = nq$.
Then $nq \neq 0$, since $n \neq 0$ and $q \neq 0$.
Moreover, $a = mq + pn = nxq + qyn = nq(x + y) = b(x + y)$.
Hence $\exists a, b \in \mathbb{Z}$, $a = b(x + y)$ and $b \neq 0$.
So, $x + y$ is rational.
Therefore, if x is rational and y is rational, then $x + y$ is rational.
Therefore, $\forall x, y \in \mathbb{R}$, if x is rational and y is rational, then $x + y$ is rational. □

65. $\forall x, y \in \mathbb{R}$, if $x \neq 0$ and xy is rational and y is irrational, then x is irrational.

Proof.
Let $x, y \in \mathbb{R}$.
Assume $x \neq 0$ and xy is rational and y is irrational and x is rational.
Choose $m, n \in \mathbb{Z}$ with $mxy = n$ and $m \neq 0$,
and choose $p, q \in \mathbb{Z}$ with $px = q$ and $p \neq 0$.
Now, since $x \neq 0$ and $p \neq 0$, we have $q \neq 0$.
Further, since $mxy = n$, we have $mpxy = np$, and hence $mqy = np$.
Put $s = mq$ and $t = np$.
Then $sy = t$.
Further, since $m \neq 0$ and $q \neq 0$, we have $s \neq 0$.
Therefore, $\exists s, t \in \mathbb{Z}$, $sy = t$ and $s \neq 0$.
That is, y is rational.
This gives us the contradiction y is rational and y is irrational.
Therefore, if $x \neq 0$ and xy is rational and y is irrational, then x is irrational.
Therefore, $\forall x, y \in \mathbb{R}$, if $x \neq 0$ and xy is rational and y is irrational, then x is irrational. □

67. $\exists x, y \in \mathbb{R}$, x is irrational and y is irrational and xy is rational.

Proof.
Put $x = \sqrt{2}$ and $y = \sqrt{2}$.
Then $xy = 2$.
Now, $2 = (1)xy$ and $1 \neq 0$; so we have $\exists a, b \in \mathbb{Z}$, $a = bxy$ and $b \neq 0$.
Therefore, xy is rational.
Also, since it was shown that $\sqrt{2}$ is irrational, we have x is irrational and y is irrational.
Therefore, $\exists x, y \in \mathbb{R}$, x is irrational and y is irrational and xy is rational. □

Prove the following propositions. They are analogous to lemma 1.2.14 and proposition 1.2.15.

69. $\forall x \in \mathbb{Z}$, if 3 divides x^2, then 3 divides x.

Proof.
Let $x \in \mathbb{Z}$.
Assume 3 does not divide x.
By the division algorithm, choose $q, r \in \mathbb{Z}$ with $x = 3q + r$ and $0 \leq r < 3$.
Since 3 does not divides x, we have $r \neq 0$.
Therefore, $0 < r < 3$, which means $r = 1$ or $r = 2$.
Case 1: $r = 1$.
Then $x = 3q + 1$, which means $x^2 = 9q^2 + 6q + 1$.
Put $p = 3q^2 + 2q$.
$x^2 = 9q^2 + 6q + 1 = 3(3q^2 + 2q) + 1 = 3p + 1$.
Therefore, $\exists p \in \mathbb{Z}, x^2 = 3p + 1$.
By the uniqueness of the quotient and remainder in the division algorithm, we have that 3 does not divides x^2.
Case 2: $r = 2$.
In this case, $x = 3q + 2$, so $x^2 = 9q^2 + 12q + 4$.
Put $p = 3q^2 + 4q + 1$.
$x^2 = 9q^2 + 12q + 4 = 9q^2 + 12q + 3 + 1 = 3(3q^2 + 4q + 1) + 1 = 3p + 1$.
Therefore, $\exists p \in \mathbb{Z}, x^2 = 3p + 1$.
Again, by the uniqueness of the quotient and remainder in the division algorithm, 3 does not divides x^2.
Therefore, if 3 does not divide x, then 3 does not divides x^2.
Therefore, if 3 divides x^2, then 3 divides x.
Therefore, $\forall x \in \mathbb{Z}$, if 3 divides x^2, then 3 divides x. □

Using the Well-Ordering Property, prove the following forms of the Principle of Mathematical Induction.

71. Let $a \in \mathbb{Z}$, and let $A \subseteq \mathbb{Z}$. If $a \in A$ and $\forall n \in \mathbb{Z}$, if $n \in A$ then $n - 1 \in A$, then $\mathbb{Z}_{\leq a} \subseteq A$.

Proof.
Let $a \in \mathbb{Z}$ and let $A \subseteq \mathbb{Z}$.
Assume $a \in A$ and $\forall n \in \mathbb{Z}$, if $n \in A$ then $n - 1 \in A$.
Further, suppose $\mathbb{Z}_{\leq a} \nsubseteq A$. That is, $\exists x \in \mathbb{Z}_{\leq a}$, $x \notin A$.
Let $S = \{x \in \mathbb{Z}_{\leq a} \mid x \notin A\}$.
By our assumption $S \neq \emptyset$. Also, since $S \subseteq \mathbb{Z}_{\leq a}$, S is bounded above by a.
By the Well-Ordering Property, S has a largest element.
Let n be the largest element of S.
Then $n \leq a$ and $n \notin A$.
Since $a \in A$, we have $n \neq a$; hence $n < a$. This gives us $n + 1 \leq a$.
Since n is the largest element of S, $n + 1 \notin S$; thus $n + 1 \in A$.
Now, since $n + 1 \in A$, we have $(n + 1) - 1 \in A$.
This means $n \in A$, which is a contradiction.
Therefore, $\mathbb{Z}_{\leq a} \subseteq A$.
Therefore, if $a \in A$ and $\forall n \in \mathbb{Z}$, if $n \in A$ then $n - 1 \in A$, then $\mathbb{Z}_{\leq a} \subseteq A$. □

73. Let $A \subseteq \mathbb{Z}$. If $A \neq \emptyset$ and $\forall n \in \mathbb{Z}$, if $n \in A$ then $n + 1 \in A$ and $n - 1 \in A$, then $A = \mathbb{Z}$.

Proof.
Let $A \subseteq \mathbb{Z}$.
Assume $A \neq \emptyset$ and $\forall n \in \mathbb{Z}$, if $n \in A$ then $n + 1 \in A$ and $n - 1 \in A$.
Further, suppose $A \neq \mathbb{Z}$.
Since $A \neq \emptyset$, we can choose an element $a \in A$. Also, since $A \neq \mathbb{Z}$, we can choose an integer $b \notin A$.
Case 1: $b < a$.
In this case, let $S = \{x \in \mathbb{Z} \mid x < a \text{ and } x \notin A\}$.
Then S is bounded above by a, and since $b \in S$, $S \neq \emptyset$.
Using the Well-Ordering Property, we can choose n to be the largest element of S.
Then $n + 1 \notin S$, from which it follows that $n + 1 \in A$.
Now, since $n + 1 \in A$, we have $(n + 1) - 1 \in A$; hence $n \in A$.
This is a contradiction, since $n \in S$.
Case 2: $a < b$.
In this case, let $S = \{x \in \mathbb{Z} \mid a < x \text{ and } x \notin A\}$.
Then S is bounded below by a, and since $b \in S$, $S \neq \emptyset$.
Using the Well-Ordering Property, we can choose n to be the smallest element of S.
Then $n - 1 \notin S$, from which it follows that $n - 1 \in A$.
Now, since $n - 1 \in A$, we have $(n - 1) + 1 \in A$; hence $n \in A$.
This is a contradiction, since $n \in S$.
Therefore, $A = \mathbb{Z}$.
Therefore, if $A \neq \emptyset$ and $\forall n \in \mathbb{Z}$, if $n \in A$ then $n + 1 \in A$ and $n - 1 \in A$, then $A = \mathbb{Z}$. □

Prove the following propositions using the Principle of Mathematical Induction.

75. $\forall x \in \mathbb{N}$, 5 divides $8^x + 2(3^{x-1})$.

Proof.
Let $A = \{x \in \mathbb{N} \mid 5 \text{ divides } 8^x + 2(3^{x-1})\}$.
Putting $q = 2$ gives us $8^1 + 2(3^{1-1}) = 10 = 5(2) = 5q$.
Therefore, $\exists q \in \mathbb{Z}, 8^1 + 2(3^{1-1}) = 5q$. Thus, 5 divides $8^1 + 2(3^{1-1})$, which means $1 \in A$.
Let $n \in \mathbb{N}$, and assume $n \in A$.
Then 5 divides $8^n + 2(3^{n-1})$. Accordingly, choose $p \in \mathbb{Z}$ with $8^n + 2(3^{n-1}) = 5p$.
Put $k = 8^n + 3p$

$$\begin{aligned}
8^{n+1} + 2(3^{n+1-1}) &= 8(8^n) + 3(2)(3^{n-1}) \\
&= (5+3)(8^n) + 3(2)(3^{n-1}) \\
&= 5(8^n) + 3(8^n) + 3(2)(3^{n-1}) \\
&= 5(8^n) + 3(8^n + 2(3^{n-1})) \\
&= 5(8^n) + 3(5p) \\
&= 5(8^n + 3p) \\
&= 5k
\end{aligned}$$

Therefore, $\exists k \in \mathbb{Z}, 8^{n+1} + 2(3^{n+1-1}) = 5k$, and so $n + 1 \in A$.
Therefore, $\forall n \in \mathbb{N}$, if $n \in A$, then $n + 1 \in A$.
By the Principle of Mathematical Induction $\mathbb{N} \subseteq A$.
Therefore, $\forall x \in \mathbb{N}$, 5 divides $8^x + 2(3^{x-1})$. $\square$

77. $\forall x, y \in \mathbb{Z}, \forall m \in \mathbb{N}$, $x - y$ divides $x^m - y^m$.

Proof.
Let $x, y \in \mathbb{Z}$. Let $A = \{m \in \mathbb{N} \mid x - y \text{ divides } x^m - y^m\}$.
Since $x^1 - y^1 = x - y = (x - y)(1)$, we have (putting $q = 1$) that $\exists q \in \mathbb{Z}\ x^1 - y^1 = (x - y)q$.
This means $x - y$ divides $x^1 - y^1$. Thus, $1 \in A$.
Next, let $n \in \mathbb{N}$ and assume $n \in A$.
Then $x - y$ divides $x^n - y^n$. We can therefore choose $a \in \mathbb{Z}$ with $y^n - y^n = a(x - y)$.
Put $b = xa + y^n$.

$$\begin{aligned}
x^{n+1} - y^{n+1} &= x^n x - y^n y \\
&= x^n x - y^n x + y^n x - y^n y \\
&= x(x^n - y^n) + y^n(x - y) \\
&= xa(x - y) + y^n(x - y) \\
&= (xa + y^n)(x - y) \\
&= b(x - y)
\end{aligned}$$

Therefore, $\exists b \in \mathbb{Z}, x^{n+1} - y^{n+1} = b(x - y)$, which means $x - y$ divides $x^{n+1} - y^{n+1}$. Thus, $n + 1 \in A$.
Therefore, $\forall n \in \mathbb{N}$, if $n \in A$, then $n + 1 \in A$.
By the Principle of Mathematical Induction, $\mathbb{N} \subseteq A$.
Therefore, $\forall m \in \mathbb{N}$, $x - y$ divides $x^m - y^m$.
Therefore, $\forall x, y \in \mathbb{Z}, \forall m \in \mathbb{N}$, $x - y$ divides $x^m - y^m$. $\square$

79. $\forall x, y, a, b \in \mathbb{Z}$, if $x - y$ divides $a + b$, then $\forall k \in \mathbb{N}$, $x - y$ divides $ax^k + by^k$.

Proof.
Let $x, y, a, b \in \mathbb{Z}$ and assume $x - y$ divides $a + b$.
Let $A = \{n \in \mathbb{N} \mid x - y \text{ divides } ax^n + by^n\}$.
Since $x - y$ divides $a + b$, choose $s \in \mathbb{Z}$ with $a + b = (x - y)s$.
Put $t = sx - b$.

$$\begin{aligned} ax^1 + by^1 &= ax + by \\ &= ax + bx - bx + by \\ &= (a + b)x - b(x - y) \\ &= (x - y)sx - b(x - y) \\ &= (x - y)(sx - b) = (x - y)t \end{aligned}$$

Therefore, $\exists t \in \mathbb{Z}$, $ax^1 + by^1 = (x - y)t$. Thus, $x - y$ divides $ax^1 + by^1$. Giving us $1 \in A$.
Let $n \in \mathbb{N}$ and assume $n \in A$.
Then $x - y$ divides $ax^n + by^n$. Choose $q \in \mathbb{Z}$ with $ax^n + by^n = (x - y)q$.
Put $p = xq - by^n$.

$$\begin{aligned} ax^{n+1} + bx^{n+1} &= axx^n + byy^n \\ &= axx^n + bxy^n - bxy^n + byy^n \\ &= x(ax^n + by^n) - by^n(x - y) \\ &= x(x - y)q - by^n(x - y) = (x - y)p \end{aligned}$$

Therefore, $x - y$ divides $ax^{n+1} + bx^{n+1}$; hence $n + 1 \in A$.
Therefore, $\forall n \in \mathbb{N}$, if $n \in A$, then $n + 1 \in A$.
By the Principle of Mathematical Induction, $\mathbb{N} \subseteq A$.
Therefore, $\forall x, y, a, b \in \mathbb{Z}$, if $x - y$ divides $a + b$, then $\forall k \in \mathbb{N}$, $x - y$ divides $ax^k + by^k$. □

81. $\forall a, b \in \mathbb{R}$, if $a \geq 0$ and $b \geq 0$, then $\forall x \in \mathbb{N}$, $(a + b)^x \geq a^x + b^x$.

Proof.
Let $a, b \in \mathbb{R}$ and assume $a \geq 0$ and $b \geq 0$.
Let $A = \{x \in \mathbb{N} \mid (a + b)^x \geq a^x + b^x\}$.
Since $(a + b)^1 = a + b = a^1 + b^1$, we have $(a + b)^1 \geq a^1 + b^1$. Hence, $1 \in A$.
Let $n \in \mathbb{N}$, and assume $n \in A$.
Then $(a + b)^n \geq a^n + b^n$.
Since $a \geq 0$ and $b \geq 0$, we have $a + b \geq 0$.
Now, $a + b \geq 0$ and $(a + b)^n \geq a^n + b^n$, so $(a + b)^{n+1} \geq (a + b)(a^n + b^n)$.
Next, since $a \geq 0$ and $b \geq 0$, we have $ba^n + ab^n \geq 0$.
Adding $a^{n+1} + b^{n+1}$ to both sides gives $a^{n+1} + ba^n + ab^n + b^{n+1} \geq a^{n+1} + b^{n+1}$.
That is, $(a + b)(a^n + b^n) \geq a^{n+1} + b^{n+1}$.
We now have $(a + b)^{n+1} \geq (a + b)(a^n + b^n)$ and $(a + b)(a^n + b^n) \geq a^{n+1} + b^{n+1}$.
Thus, by transitivity, $(a + b)^{n+1} \geq a^{n+1} + b^{n+1}$. Hence, $n + 1 \in A$.
Therefore, $\forall n \in \mathbb{N}$, if $n \in A$, then $n + 1 \in A$.
By the Principle of Mathematical Induction $\mathbb{N} \subseteq A$.
Therefore, $\forall x \in \mathbb{N}$, $(a + b)^x \geq a^x + b^x$.
Therefore, $\forall a, b \in \mathbb{R}$, if $a \geq 0$ and $b \geq 0$, then $\forall x \in \mathbb{N}$, $(a + b)^x \geq a^x + b^x$. □

83. $\forall x \in \mathbb{Z}$, if $\exists n \in \mathbb{N}$, x^n is odd, then x is odd.

Proof.
Let $x \in \mathbb{Z}$.
Assume x is not odd.
Then x is even, so we may let $a \in \mathbb{Z}$ with $x = 2a$.
Let $A = \{k \in \mathbb{N} \mid x^k$ is not odd$\}$.
Since $x^1 = x$ and x is even, we have that x^1 is not odd.
Thus, $1 \in A$.
Next, let $n \in \mathbb{N}$ and assume $n \in A$.
Then x^n is not odd, which means x^n is even.
Accordingly, let $b \in \mathbb{Z}$ with $x^n = 2b$.
Put $c = 2ab$.
Now, $x^{n+1} = x^n x = (2b)(2a) = 2(2ab) = 2c$.
Therefore, $\exists c \in \mathbb{Z}$, $x^{n+1} = 2c$.
This means x^{n+1} is even, and hence x^{n+1} is not odd.
Thus, $n + 1 \in A$.
Therefore, $\forall n \in \mathbb{N}$, if $n \in A$, then $n + 1 \in A$.
By the Principle of Mathematical Induction, $\mathbb{N} \subseteq A$.
Therefore, $\forall n \in \mathbb{N}$, x^n is not odd.
Therefore, if x is not odd, then $\forall n \in \mathbb{N}$, x^n is not odd.
Therefore, if $\exists n \in \mathbb{N}$, x^n is odd, then x is odd.
Therefore, $\forall x \in \mathbb{Z}$, if $\exists n \in \mathbb{N}$, x^n is odd, then x is odd. □

85. $\forall x \in \mathbb{R}$, if $x > 1$, then $\forall n \in \mathbb{N}$, $x^n > 1$.

Proof.
Let $x \in \mathbb{R}$.
Assume $x > 1$.
Let $A = \{n \in \mathbb{N} \mid x^n > 1\}$.
Since $x^1 > 1$, we have $1 \in A$.
Let $n \in \mathbb{N}$ and assume $n \in A$.
Hence $x^n > 1$.
Since $0 < 1$ and $1 < x$, we have $0 < x$, so which means now, $x^n(x) > 1(x)$.
Now, $x^{n+1} > x$ and $x > 1$, so $x^{n+1} > 1$ by transitivity. Thus, $n + 1 \in A$.
Therefore, $\forall n \in \mathbb{N}$, if $n \in A$ then $n + 1 \in A$.
By the Principle of Mathematical Induction, $\mathbb{N} \subseteq A$.
Therefore, $\forall n \in \mathbb{N}$, $x^n > 1$.
Therefore, if $x > 1$, then $\forall n \in \mathbb{N}$, $x^n > 1$.
Therefore, $\forall x \in \mathbb{R}$, if $x > 1$ then $\forall n \in \mathbb{N}$, $x^n > 1$. □

87. $\forall x \in \mathbb{R}$, if $\exists n \in \mathbb{N}$, $x^n < x$, then $x < 1$.

Proof.
Let $x \in \mathbb{R}$.
  Assume $1 \leq x$.
    let $A = \{k \in \mathbb{N} \mid x \leq x^k\}$.
      Since $x = x^1$, we have $x \leq x^1$. Thus, $1 \in A$.
      Let $n \in \mathbb{N}$ and assume $n \in A$.
        Then $x \leq x^n$.
        First, since $1 \leq x$ and $x \leq x^n$, we have $1 \leq x^n$.
        Also, since $0 \leq 1$ and $1 \leq x$, we have $0 \leq x$.
        Now, since $1 \leq x^n$ and $0 \leq x$, we have $1(x) \leq x^n(x)$; hence $x \leq x^{n+1}$.
        Therefore, $n + 1 \in A$.
      Therefore, $\forall n \in \mathbb{N}$, if $n \in A$, then $n + 1 \in A$.
    By the Principle of Mathematical Induction, $\mathbb{N} \subseteq A$.
    Therefore, $\forall n \in \mathbb{N}$, $x \leq x^n$.
  Therefore, if $1 \leq x$, then $\forall n \in \mathbb{N}$, $x \leq x^n$.
  Therefore, if $\exists n \in \mathbb{N}$, $x^n < x$, then $x < 1$.
Therefore, $\forall x \in \mathbb{R}$, if $\exists n \in \mathbb{N}$, $x^n < x$, then $x < 1$. □

89. $\forall x, y \in \mathbb{R}$, if $0 < x < y$, then $\forall n \in \mathbb{N}$, $x^n < y^n$.

Proof.
Let $x, y \in \mathbb{R}$.
  Assume $0 < x < y$.
    Let $A = \{n \in \mathbb{N} \mid x^n < y^n\}$.
    Since $x < y$, we have $x^1 < y^1$, and hence $1 \in A$.
    Let $n \in \mathbb{N}$, and assume $n \in A$.
      Then $x^n < y^n$.
      Since $0 < x$ and $x^n < y^n$, we have $x^{n+1} < xy^n$.
      Since $0 < y^n$ and $x < y$, we have $xy^n < y^{n+1}$.
      By transitivity, $x^{n+1} < y^{n+1}$.
      Thus, $n + 1 \in A$.
    Therefore, $\forall n \in \mathbb{N}$, if $n \in A$, then $n + 1 \in A$.
    By the Principle of Mathematical Induction $\mathbb{N} \subseteq A$.
    Therefore, $\forall n \in \mathbb{N}$, $x^n < y^n$.
  Therefore, if $0 < x < y$, then $\forall n \in \mathbb{N}$, $x^n < y^n$.
Therefore, $\forall x, y \in \mathbb{R}$, if $0 < x < y$, then $\forall n \in \mathbb{N}$, $x^n < y^n$. □

91. $\forall m, n \in \mathbb{N}$, if $m < n$, then $\left(\frac{1}{2}\right)^n < \left(\frac{1}{2}\right)^m$.

Proof.
Let $m \in \mathbb{N}$.
Let $A = \left\{n \in \mathbb{N} \mid \text{if } m < n, \text{ then } \left(\frac{1}{2}\right)^n < \left(\frac{1}{2}\right)^m\right\}$.
The statement "if $m < 1$, then $\left(\frac{1}{2}\right)^1 < \left(\frac{1}{2}\right)^m$" is vacuously true, since $1 \leq m$. Thus, $1 \in A$.
Let $n \in \mathbb{N}$, and assume $n \in A$.
Assume $m < n + 1$. Then $m \leq n$.
First note that since $1 < 2$, we have $1\left(\frac{1}{2}\right)^{n+1} < 2\left(\frac{1}{2}\right)^{n+1}$, and so $\left(\frac{1}{2}\right)^{n+1} < \left(\frac{1}{2}\right)^n$.
With this in mind, and since we know $m \leq n$, we will consider the two cases $m = n$, and $m < n$.
Case 1: $m = n$.
Since $\left(\frac{1}{2}\right)^{n+1} < \left(\frac{1}{2}\right)^n$ and $m = n$, we have $\left(\frac{1}{2}\right)^{n+1} < \left(\frac{1}{2}\right)^m$.
Case 2: $m < n$.
In this case, since $n \in A$, we have $\left(\frac{1}{2}\right)^n < \left(\frac{1}{2}\right)^m$.
With $\left(\frac{1}{2}\right)^{n+1} < \left(\frac{1}{2}\right)^n$, we then have $\left(\frac{1}{2}\right)^{n+1} < \left(\frac{1}{2}\right)^m$, by transitivity.
Therefore, if $m < n + 1$, then $\left(\frac{1}{2}\right)^{n+1} < \left(\frac{1}{2}\right)^m$. Thus, $n + 1 \in A$.
Therefore, $\forall n \in \mathbb{N}$, if $n \in A$, then $n + 1 \in A$.
By the PMI, $\mathbb{N} \subseteq A$, and so $\forall n \in \mathbb{N}$, if $m < n$, then $\left(\frac{1}{2}\right)^n < \left(\frac{1}{2}\right)^m$.
Since m is arbitrary, we have $\forall m, n \in \mathbb{N}$, if $m < n$, then $\left(\frac{1}{2}\right)^n < \left(\frac{1}{2}\right)^m$. □

93. $\forall x \in \mathbb{R}$, if $1 < x$, then $\forall m, n \in \mathbb{N}$, if $m < n$, then $x^m < x^n$.

Proof.
Let $x \in \mathbb{R}$ and assume $1 < x$.
Let $m \in \mathbb{N}$.
Let $A = \{n \in \mathbb{N} \mid \text{if } m < n, \text{ then } x^m < x^n\}$.
The statement "if $m < 1$, then $x^m < x^n$" is vacuously true, since $1 \leq m$. Thus, $1 \in A$.
Let $n \in \mathbb{N}$, and assume $n \in A$.
Assume $m < n + 1$.
First note that since $1 < x$, we have $1(x^n) < x(x^n)$, and so $x^n < x^{n+1}$.
Now, since $m < n + 1$, we have $m \leq n$.
Case 1: $m = n$.
Since $x^n < x^{n+1}$ and $m = n$, we have $x^m < x^{n+1}$.
Case 2: $m < n$.
In this case, since $n \in A$, we have $x^m < x^n$, and so $x^m < x^{n+1}$ by transitivity.
Therefore, if $m < n + 1$, then $x^m < x^{n+1}$. Thus, $n + 1 \in A$.
Therefore, $\forall n \in \mathbb{N}$, if $n \in A$, then $n + 1 \in A$.
By the PMI, $\mathbb{N} \subseteq A$, and so $\forall n \in \mathbb{N}$, if $m < n$, then $x^m < x^n$.
Since m is arbitrary, we have $\forall m, n \in \mathbb{N}$, if $m < n$, then $x^m < x^n$.
Therefore, $\forall x \in \mathbb{R}$, if $1 < x$, then $\forall m, n \in \mathbb{N}$, if $m < n$, then $x^m < x^n$. □

95. Let $a \in \mathbb{R}$ with $0 < a < 1$. Then, $\forall n \in \mathbb{N}$, $\frac{a^{n+1}-1}{n+1} > \frac{a^n-1}{n}$.

Proof.
Let $a \in \mathbb{R}$ with $0 < a < 1$.
Let $A = \{n \in \mathbb{N} \mid \frac{a^{n+1}-1}{n+1} > \frac{a^n-1}{n}\}$.
Since $a < 1$, we have $a - 1 < 0$. This gives us $0 < (a-1)^2$.
Now, $0 < a^2 - 2a + 1$; $2a < a^2 + 1$; $2a - 2 < a^2 - 1$; $a - 1 < \frac{a^2-1}{2}$.
This prove $\frac{a^2-1}{2} > \frac{a^1-1}{1}$. Thus, $1 \in A$.
Let $n \in \mathbb{N}$ and assume $n \in A$.
Then $\frac{a^n-1}{n} < \frac{a^{n+1}-1}{n+1}$.
First, since $n^2 + 2 < n^2 + 2n + 1$, we have $n(n+2) < (n+1)^2$; thus $\frac{n}{n+1} < \frac{n+1}{n+2}$.
Therefore, $(\frac{a^n-1}{n})(\frac{n}{n+1}) < (\frac{a^{n+1}-1}{n+1})(\frac{n+1}{n+2})$, giving us $\frac{a^n-1}{n+1} < \frac{a^{n+1}-1}{n+2}$.
Next, since $0 < a$, we have $a(\frac{a^n-1}{n+1}) < a(\frac{a^{n+1}-1}{n+2})$, giving us $\frac{a^{n+1}-a}{n+1} < \frac{a^{n+2}-a}{n+2}$.
Now, since $n + 1 < n + 2$, we have $\frac{1}{n+2} < \frac{1}{n+1}$.
Since $a < 1$, we have $a - 1 < 0$; hence $\frac{a-1}{n+1} < \frac{a-1}{n+2}$.
Adding this last inequality to $\frac{a^{n+1}-a}{n+1} < \frac{a^{n+2}-a}{n+2}$ gives us:
$\frac{a^{n+1}-a}{n+1} + \frac{a-1}{n+1} < \frac{a^{n+2}-a}{n+2} + \frac{a-1}{n+2}$, which is $\frac{a^{n+1}-1}{n+1} < \frac{a^{n+2}-1}{n+2}$.
Thus, $n + 1 \in A$.
Therefore, $\forall n \in \mathbb{N}$, if $n \in A$, then $n + 1 \in A$.
By the Principle of Mathematical Induction $\mathbb{N} \subseteq A$.
Therefore, $\forall n \in \mathbb{N}$, $\frac{a^{n+1}-1}{n+1} > \frac{a^n-1}{n}$. □

97. Every finite set of real numbers has a minimum element.

Proof.
We will prove $\forall n \in \mathbb{N}$, if $S \subseteq \mathbb{R}$ is a set with n elements, then S has a minimum element.
Let $A = \{n \in \mathbb{N} \mid$ if $S \subseteq \mathbb{R}$ is a set with n elements, then S has a minimum element.$\}$.
Suppose $S = \{a\}$ is a set with 1 element.
Then since $\forall x \in S$, $x = a$, we have $\forall x \in S$, $a \leq x$. Thus, S has a minimum element.
Therefore, $1 \in A$.
Let $n \in \mathbb{N}$ and assume $n \in A$.
Now, let S be a set with $n + 1$ elements and choose an element $b \in S$.
The set $V = S \setminus \{b\}$ has n elements, and since $n \in A$, V must have a smallest element.
Let c be the smallest element of V.
Case 1: $c \leq b$.
Let $x \in S$.
Then either $x = b$ or $x \in V$.
In either case, $c \leq x$.
Therefore, $\forall x \in S$, $c \leq x$. Thus, S has a smallest element.
Case 2: $b < c$.
Let $x \in S$.
Again, either $x = b$ or $x \in V$.
If $x = b$, then $b \leq x$.
If $x \in V$, then $c \leq x$ and so by transitivity, $b \leq x$.
Therefore, $\forall x \in S$, $b \leq x$. Thus, S has a smallest element.
Therefore, $n + 1 \in A$.
Therefore, $\forall n \in \mathbb{N}$, if $n \in A$, then $n + 1 \in A$.
By the Principle of Mathematical Induction $\mathbb{N} \subseteq A$.
Therefore, $\forall n \in \mathbb{N}$, if S is a set with n elements, then S has a minimum element. □

99. Let $S \subseteq \mathbb{R}$. If S is closed under addition in the sense that $\forall x, y \in S$, $x + y \in S$, then $\forall a \in S$, $\forall n \in \mathbb{N}$, $na \in S$.

Proof.
Let $S \subseteq \mathbb{R}$ and assume S is closed under addition.
  Let $a \in S$..
    Let $A = \{n \in \mathbb{N} \mid na \in S\}$.
    Since $a(1) = a \in S$, we have $1 \in A$.
    Let $n \in A$.
      $na \in S$.
      Since $na \in S$ and $a \in S$, and S is closed under addition, we have $na + a \in S$.
      Therefore, $(n + 1)a \in S$, and hence $n + 1 \in A$.
    Therefore, if $n \in A$, then $n + 1 \in A$.
    By the PMI, $\mathbb{N} \subseteq A$, and hence $\forall n \in \mathbb{N}$, $na \in S$.
  Therefore, $\forall a \in S$, $\forall n \in \mathbb{N}$, $na \in S$.
Therefore, if S is closed under addition, then $\forall a \in S$, $\forall n \in \mathbb{N}$, $na \in S$. □

Prove the following propositions about inductive sets.

101. The set of rational numbers $\mathbb{Q}$ is an inductive subset of $\mathbb{R}$.

Proof.
Let $x \in \mathbb{R}$.
Assume $x \in \mathbb{Q}$.
Choose $a, b \in \mathbb{Z}$ with $a = bx$ and $b \neq 0$.
Put $p = a + b$.
Then $p = a + b = bx + b = b(x + 1)$.
Therefore, $\exists p, b \in \mathbb{Z}$, $p = b(x + 1)$ and $b \neq 0$.
Thus, $x + 1 \in \mathbb{Q}$.
Therefore, if $x \in \mathbb{Q}$, then $x + 1 \in \mathbb{Q}$.
Therefore, $\mathbb{Q}$ is inductive. □

103. M_0 (the smallest inductive set containing 0) is closed under addition.

Proof.
Let $y \in M_0$.
Let $A = \{x \in \mathbb{R} \mid x + y \in M_0\}$.
Since $0 + y = y \in M_0$, we have $0 \in A$.
Let $x \in \mathbb{R}$ and assume $x \in A$.
Then $x + y \in M_0$.
Since M_0 is inductive, we have $x + y + 1 \in M_0$.
This gives us $(x + 1) + y \in M_0$; hence $x + 1 \in A$.
Therefore, A is inductive.
Since $0 \in A$ and A is inductive, we have $M_0 \subseteq A$ (by definition of M_0).
Therefore, $\forall x \in M_0$, $x \in A$. This means $\forall x \in M_0$, $x + y \in M_0$.
Since $y \in M_0$ was arbitrary, we have $\forall x, y \in M_0$, $x + y \in M_0$.
Thus, M_0 is closed under addition. □

105. Let $a \in \mathbb{R}$, and let M_a be the smallest inductive set containing a. $\forall x, y \in M_a$, if $x < y$, then $y - x \in \mathbb{N}$.

Proof.
Let $x \in M_a$.
Let $A = \{z \in \mathbb{R} \mid z \leq x \text{ or } z - x \in \mathbb{N}\}$.
By Proposition 1.2.24, we have $a \leq x$, since $x \in M_a$. Thus, $a \in A$.
Let $z \in A$.
Then either $z \leq x$ or $z - x \in \mathbb{N}$.
Case 1: $z \leq x$.
if $z = x$, then $(z + 1) - x = (x + 1) - x = 1 \in \mathbb{N}$; hence $z + 1 \in A$.
if $z < x$, then $z + 1 \leq x$; hence $z + 1 \in A$.
Case 2: $z - x \in \mathbb{N}$.
Then $z - x + 1 \in \mathbb{N}$, which means $(z + 1) - x \in \mathbb{N}$. Thus, $z + 1 \in A$.
Therefore, if $z \in A$, then $z + 1 \in A$; hence A is inductive.
Since $a \in A$ and A is inductive, we have $M_a \subseteq A$.
Now, let $y \in M_a$ and assume $x < y$.
Since $y \in M_a$, we have $y \in A$.
Since $x < y$, we have $y - x \in \mathbb{N}$.
Therefore, $\forall y \in M_a$, if $x < y$, then $y - x \in \mathbb{N}$.
Therefore, $\forall x, y \in M_a$, if $x < y$, then $y - x \in \mathbb{N}$. □

Prove the following propositions using the recursive definition of exponents.

107. $\forall x \in \mathbb{R} \setminus \{0\}, \forall n \in \mathbb{N}, (x^{-1})^n = (x^n)^{-1}$.

Proof.
Let $x \in \mathbb{R} \setminus \{0\}$.
Let $A = \{n \in \mathbb{N} \mid (x^{-1})^n = (x^n)^{-1}\}$.
$(x^{-1})^1 = x^{-1} = (x^1)^{-1}$; hence $1 \in A$.
Let $n \in \mathbb{N}$ and assume $n \in A$.
Then $(x^{-1})^n = (x^n)^{-1}$.
Now, $(x^{n+1})^{-1} = (x^n x)^{-1} = (x^n)^{-1} x^{-1} = (x^{-1})^n x^{-1} = (x^{-1})^{n+1}$.
Thus, $n + 1 \in A$.
Therefore, $\forall n \in \mathbb{N}$, if $n \in A$, then $n + 1 \in A$.
By the PMI, $\mathbb{N} \subseteq A$.
Therefore, $\forall n \in \mathbb{N}, (x^{-1})^n = (x^n)^{-1}$.
Therefore, $\forall x \in \mathbb{R} \setminus \{0\}, \forall n \in \mathbb{N}, (x^{-1})^n = (x^n)^{-1}$. □

109. $\forall x \in \mathbb{R}, \forall n, m \in \mathbb{N}, (x^n)^m = x^{nm}$.

Proof.
Let $x \in \mathbb{R}$.
Let $n \in \mathbb{N}$, and let $A = \{m \in \mathbb{N} \mid (x^n)^m = x^{nm}\}$.
$(x^n)^1 = x^n = x^{n1}$; hence $1 \in A$.
Let $k \in \mathbb{N}$.
Assume $k \in A$.
Then $(x^n)^k = x^{nk}$.
Now, $(x^n)^{k+1} = (x^n)^k x^n = x^{nk} x^n = x^{nk+n} = x^{n(k+1)}$.
Thus, $k + 1 \in A$.
Therefore, if $k \in A$, then $k + 1 \in A$.
By the PMI, $\mathbb{N} \subseteq A$.
Therefore, $\forall m \in \mathbb{N}, (x^n)^m = x^{nm}$.
Therefore, $\forall n, m \in \mathbb{N}, (x^n)^m = x^{nm}$.
Therefore, $\forall x \in \mathbb{R}, \forall n, m \in \mathbb{N}, (x^n)^m = x^{nm}$. □

Prove the following properties of series.

111. $\forall a \in \mathbb{R}, \forall n \in \mathbb{N}, \displaystyle\sum_{k=1}^{n} a = na.$

Proof.
Let $a \in \mathbb{R}$.
Let $A = \{x \in \mathbb{N} \mid \sum_{k=1}^{x} a = xa\}$.
$\sum_{k=1}^{1} a = a = (1)a$, and hence $1 \in A$.
Let $n \in A$.
Then $\sum_{k=1}^{n} a = na$.
$\sum_{k=1}^{n+1} a = \left(\sum_{k=1}^{n} a\right) + a = na + a = (n+1)a$.
Hence, $n + 1 \in A$.
Therefore, if $n \in A$, then $n + 1 \in A$. By the PMI, $\mathbb{N} \subseteq A$, which means $\forall n \in \mathbb{N}, \sum_{k=1}^{n} a = na$.
Therefore, $\forall a \in \mathbb{R}, \forall n \in \mathbb{N}, \sum_{k=1}^{n} a = na$. □

113. For sequences of real numbers $(a_k)_{k\in\mathbb{N}}$ and $(b_k)_{k\in\mathbb{N}}$, $\forall n \in \mathbb{N}, \displaystyle\sum_{k=1}^{n}(a_k + b_k) = \left(\sum_{k=1}^{n} a_k\right) + \left(\sum_{k=1}^{n} b_k\right)$.

Proof.
Let $A = \{x \in \mathbb{N} \mid \sum_{k=1}^{x}(a_k + b_k) = (\sum_{k=1}^{x} a_k) + (\sum_{k=1}^{x} b_k)\}$.
$\sum_{k=1}^{1}(a_k + b_k) = a_1 + b_1 = (\sum_{k=1}^{1} a_k) + (\sum_{k=1}^{1} b_k)$, and so $1 \in A$.
Let $n \in A$.
Then $\sum_{k=1}^{n}(a_k + b_k) = (\sum_{k=1}^{n} a_k) + (\sum_{k=1}^{n} b_k)$.

$$\begin{aligned}\sum_{k=1}^{n+1}(a_k + b_k) &= \left(\sum_{k=1}^{n}(a_k + b_k)\right) + a_{n+1} + b_{n+1} = \left(\sum_{k=1}^{n} a_k\right) + \left(\sum_{k=1}^{n} b_k\right) + a_{n+1} + b_{n+1} \\ &= \left(\sum_{k=1}^{n} a_k\right) + a_{n+1} + \left(\sum_{k=1}^{n} b_k\right) + b_{n+1} = \left(\sum_{k=1}^{n+1} a_k\right) + \left(\sum_{k=1}^{n+1} b_k\right)\end{aligned}$$

Hence, $n + 1 \in A$.
Therefore, if $n \in A$, then $n + 1 \in A$.
By the PMI, $\mathbb{N} \subseteq A$, and so $\forall n \in \mathbb{N}, \sum_{k=1}^{n}(a_k + b_k) = \left(\sum_{k=1}^{n} a_k\right) + \left(\sum_{k=1}^{n} b_k\right)$. □

115. For a sequence of real numbers $(a_k)_{k\in\mathbb{N}}$, $\forall m, n \in \mathbb{N}$, if $m < n$, then $\displaystyle\sum_{k=1}^{m} a_k + \sum_{k=m+1}^{n} a_k = \sum_{k=1}^{n} a_k$.

Proof.
Let $m \in \mathbb{N}$ and let $A = \{n \in \mathbb{N} \mid \text{if } m < n, \text{ then } \sum_{k=1}^{m} a_k + \sum_{k=m+1}^{n} a_k = \sum_{k=1}^{n} a_k\}$.
Since $1 \leq m$, we have $1 \in A$.
Let $n \in \mathbb{N}$ and assume $n \in A$.
Assume $m < n + 1$. From this, we have $m \leq n$.
Case 1: $n = m$.
Then $\sum_{k=1}^{n+1} a_k = (\sum_{k=1}^{n} a_k) + a_{n+1} = \sum_{k=1}^{n} a_k + \sum_{k=n+1}^{n+1} a_k = \sum_{k=1}^{m} a_k + \sum_{k=m+1}^{n+1} a_k$
Case 2: $m < n$.
In this case, $\sum_{k=1}^{m} a_k + \sum_{k=m+1}^{n} a_k = \sum_{k=1}^{n} a_k$.
Then $\sum_{k=1}^{n+1} a_k = (\sum_{k=1}^{n} a_k) + a_{n+1} = (\sum_{k=1}^{m} a_k) + (\sum_{k=m+1}^{n} a_k) + a_{n+1} = \sum_{k=1}^{m} a_k + \sum_{k=m+1}^{n+1} a_k$.
Therefore, $n + 1 \in A$.
Therefore, $\forall n \in \mathbb{N}$, if $n \in A$, then $n + 1 \in A$.
By the Principle of Mathematical Induction, $\mathbb{N} \subseteq A$.
Therefore, $\forall m, n \in \mathbb{N}$, if $m < n$, then $\sum_{k=1}^{m} a_k + \sum_{k=m+1}^{n} a_k = \sum_{k=1}^{n} a_k$. □

117. For a sequence of real numbers $(a_k)_{k\in\mathbb{N}}$, $\forall n \in \mathbb{N}$, $\displaystyle\sum_{k=1}^{n}(a_{k+1} - a_k) = a_{n+1} - a_1$.

(Such a series is called a **telescoping sum**)

Proof.
Let $A = \{x \in \mathbb{N} \mid \sum_{k=1}^{x}(a_{k+1} - a_k) = a_{x+1} - a_1\}$.
$\sum_{k=1}^{1}(a_{k+1} - a_k) = a_{1+1} - a_1$, and so $1 \in A$.
Let $n \in A$.
Then $\sum_{k=1}^{n}(a_{k+1} - a_k) = a_{n+1} - a_1$.
$\sum_{k=1}^{n+1}(a_{k+1} - a_k) = \left(\sum_{k=1}^{n}(a_{k+1} - a_k)\right) + a_{n+2} - a_{n+1} = a_{n+1} - a_1 + a_{n+2} - a_{n+1} = a_{n+2} - a_1$.
Hence, $n + 1 \in A$.
Therefore, if $n \in A$, then $n + 1 \in A$.
By the PMI, $\mathbb{N} \subseteq A$, and so $\forall n \in \mathbb{N}$, $\sum_{k=1}^{n}(a_{k+1} - a_k) = a_{n+1} - a_1$. □

119. For a sequence of integers $(a_k)_{k\in\mathbb{N}}$, if $\forall k \in \mathbb{N}$, a_k is even, then $\forall n \in \mathbb{N}$, $\displaystyle\sum_{k=1}^{n} a_k$ is even.

Proof.
Assume $\forall k \in \mathbb{N}$, a_k is even.
Let $A = \{x \in \mathbb{N} \mid \sum_{k=1}^{x} a_k \text{ is even}\}$.
Since $1 \in \mathbb{N}$ and $\forall k \in \mathbb{N}$, a_k is even, we have that a_1 is even.
Further, since $\sum_{k=1}^{1} a_k = a_1$ and a_1 is even, we have that $\sum_{k=1}^{1} a_k$ is even.
Thus, $1 \in A$.
Let $n \in \mathbb{N}$ and assume $n \in A$.
Then $\sum_{k=1}^{n} a_k$ is even. Accordingly, choose $q \in \mathbb{Z}$ with $\sum_{k=1}^{n} a_k = 2q$.
Also, since $n + 1 \in \mathbb{N}$ and $\forall k \in \mathbb{N}$, a_k is even, we have that a_{n+1} is even.
Thus, we may choose $p \in \mathbb{Z}$ with $a_{n+1} = 2p$.
Putting $s = q + p$, we have:
$\sum_{k=1}^{n+1} a_k = (\sum_{k=1}^{n} a_k) + a_{n+1} = 2q + 2p = 2s$.
Therefore, $\exists s \in \mathbb{Z}$, $\sum_{k=1}^{n+1} a_k = 2s$, which means $\sum_{k=1}^{n+1} a_k$ is even.
Thus, $n + 1 \in A$.
Therefore, $\forall n \in \mathbb{N}$, if $n \in A$, then $n + 1 \in A$.
By the PMI, $\mathbb{N} \subseteq A$.
Therefore, $\forall n \in \mathbb{N}$, $\sum_{k=1}^{n} a_k$ is even.
Therefore, if $\forall k \in \mathbb{N}$, a_k is even, then $\forall n \in \mathbb{N}$, $\sum_{k=1}^{n} a_k$ is even. □

Prove the following propositions.

121. $\forall n \in \mathbb{N}, \displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$

Proof.
Let $A = \{x \in \mathbb{N} \mid \sum_{k=1}^{x} k^2 = \frac{x(x+1)(2x+1)}{6}\}$.
$\sum_{k=1}^{1} k^2 = 1^2 = 1 = \frac{(1)(2)(3)}{6} = \frac{1(1+1)(2(1)+1)}{6}$. Thus, $1 \in A$.
Let $n \in \mathbb{N}$ and assume $n \in A$.
Then $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.
$\sum_{k=1}^{n+1} k^2 = \left(\sum_{k=1}^{n} k^2\right) + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2$
$= \frac{(n+1)(n(2n+1)+6(n+1))}{6} = \frac{(n+1)(2n^2+7n+6)}{6} = \frac{(n+1)(n+2)(2n+3)}{6} = \frac{(n+1)(n+1+1)(2(n+1)+1)}{6}$
Thus, $n+1 \in A$.
Therefore, $\forall n \in \mathbb{N}$, if $n \in A$, then $n+1 \in A$.
By the PMI, $\mathbb{N} \subseteq A$. Therefore, $\forall n \in \mathbb{N}, \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$. □

123. $\forall n \in \mathbb{N}, \displaystyle\sum_{k=1}^{n} k^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}.$

Proof.
Let $A = \{x \in \mathbb{N} \mid \sum_{k=1}^{n} k^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}\}$.
$\sum_{k=1}^{1} k^4 = 1^4 = 1 = \frac{(1)(2)(3)(5)}{30} = \frac{(1)(1+1)(2(1)+1)(3(1)^2+3(1)-1)}{30}$. Thus, $1 \in A$.
Let $n \in \mathbb{N}$ and assume $n \in A$.
Then $\sum_{k=1}^{n} k^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$.
$\sum_{k=1}^{n+1} k^4 = \left(\sum_{k=1}^{n} k^4\right) + (n+1)^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} + (n+1)^4 = \frac{(n+1)(n(2n+1)(3n^2+3n-1)+30(n+1)^3)}{30}$
$= \frac{(n+1)(6n^4++39n^3+91n^2+89n+30)}{30} = \frac{(n+1)(n+2)(2n+3)(3n^2+9n+5)}{30} = \frac{(n+1)(n+1+1)(2(n+1)+1)(3(n+1)^2+3(n+1)-1)}{6}$
Thus, $n+1 \in A$.
Therefore, $\forall n \in \mathbb{N}$, if $n \in A$, then $n+1 \in A$.
By the PMI, $\mathbb{N} \subseteq A$. Therefore, $\forall n \in \mathbb{N}, \sum_{k=1}^{n} k^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$. □

125. $\forall n \in \mathbb{N}, \displaystyle\sum_{k=1}^{n} \frac{k}{(k+1)!} = 1 - \frac{1}{(n+1)!}.$

Proof.
Let $A = \{x \in \mathbb{N} \mid \sum_{k=1}^{n} \frac{k}{(k+1)!} = 1 - \frac{1}{(n+1)!}\}$.
$\sum_{k=1}^{1} \frac{k}{(k+1)!} = \frac{1}{2!} = \frac{1}{2(1!)} = \frac{1}{2} = 1 - \frac{1}{2} = 1 - \frac{1}{2!}$. Thus, $1 \in A$.
Let $n \in \mathbb{N}$ and assume $n \in A$.
Then $\sum_{k=1}^{n} \frac{k}{(k+1)!} = 1 - \frac{1}{(n+1)!}$.
$\sum_{k=1}^{n+1} \frac{k}{(k+1)!} = (\sum_{k=1}^{n} \frac{k}{(k+1)!}) + \frac{n+1}{(n+2)!} = 1 - \frac{1}{(n+1)!} + \frac{n+1}{(n+2)!} = 1 - \frac{n+2}{(n+2)(n+1)!} + \frac{n+1}{(n+2)!} = 1 - \frac{1}{(n+2)!}$.
Thus, $n+1 \in A$.
Therefore, $\forall n \in \mathbb{N}$, if $n \in A$, then $n+1 \in A$.
By the PMI, $\mathbb{N} \subseteq A$. Therefore, $\forall n \in \mathbb{N}, \sum_{k=1}^{n} \frac{k}{(k+1)!} = 1 - \frac{1}{(n+1)!}$. □

For each of the following recursively defined functions, guess an explicit formula for $f(x)$ and prove that your formula is true for all $x \in \mathbb{N}$.

127. $f : \mathbb{N} \to \mathbb{N}$ given by: $f(1) = 2$ and for each $n \in \mathbb{N}$, $f(n+1) = f(n) + 2$.

Proof. Guess: $\forall x \in \mathbb{N}$, $f(x) = 2x$.
Let $A = \{x \in \mathbb{N} \mid f(x) = 2x\}$.
Since $f(1) = 2 = 2(1)$, we have $1 \in A$.
Let $n \in \mathbb{N}$ and assume $n \in A$.
Then $f(n) = 2n$.
Now, $f(n+1) = f(n) + 2 = 2n + 2 = 2(n+1)$.
Therefore, $n + 1 \in A$.
Therefore, $\forall n \in \mathbb{N}$, if $n \in A$, then $n + 1 \in A$.
By the PMI, $\mathbb{N} \subseteq A$.
Therefore, $\forall x \in \mathbb{N}$, $f(x) = 2x$. □

129. $f : \mathbb{N} \to \mathbb{N}$ given by: $f(1) = 1$, $f(2) = 4$ and for each $n \geq 2$, $f(n+1) = 2(f(n) + 1) - f(n-1)$.

Proof. Guess: $\forall x \in \mathbb{N}$, $f(x) = x^2$.
Let $S = \{x \in \mathbb{N} \mid f(x) = x^2\}$.
Since $f(1) = 1 = 1^2$, we have $1 \in S$.
Let $n \in N$, and assume $\{1, \ldots, n\} \subseteq S$.
Case 1: $n = 1$.
Then $f(n+1) = f(2) = 4 = 2^2 = (n+1)^2$.
Thus, $n + 1 \in S$.
Case 2: $n \geq 2$.
Then $n \in S$ and $n - 1 \in S$.
Therefore, $f(n) = n^2$ and $f(n-1) = (n-1)^2$.
Now, $f(n+1) = 2(f(n)+1) - f(n-1) = 2(n^2+1) - (n-1)^2 = 2n^2 + 2 - n^2 + 2n - 1 = n^2 + 2n + 1 = (n+1)^2$.
Therefore, $n + 1 \in S$.
Thus, $\forall n \in \mathbb{N}$, if $\{1, \ldots, n\} \subseteq S$, then $n + 1 \in S$.
By the principle of complete induction, $\mathbb{N} \subseteq S$.
Therefore, $\forall x \in \mathbb{N}$, $f(x) = x^2$. □

Chapter 2

Sets

2.1 Relations and Operations

Exercises 2.1.

Let A, B, C, and D be sets (assume the elements of these sets belong to a common universe of discourse U). Prove the following propositions.

1. $U \setminus A = A^c$.

Proof.
Let $x \in U$.
Assume $x \in U \setminus A$.
Then $x \in U$ and $x \notin A$. Since $x \notin A$, we have $x \in A^c$.
Therefore, if $x \in U \setminus A$, then $x \in A^c$.
Therefore, $U \setminus A \subseteq A^c$.
Conversely, let $x \in U$ and assume $x \in A^c$.
Then $x \notin A$, giving us $x \in U$ and $x \notin A$. This means, $x \in U \setminus A$.
Therefore, $\forall x \in U$, if $x \in A^c$, then $x \in U \setminus A$.
Therefore, $A^c \subseteq U \setminus A$.
Thus, $U \setminus A = A^c$. □

3. (a) If $A \subseteq B$, then $A \cap B = A$.

Proof.
Assume $A \subseteq B$.
Let $x \in U$ and assume $x \in A \cap B$.
Then $x \in A$ and $x \in B$.
In particular, $x \in A$.
Therefore, $\forall x \in U$, if $x \in A \cap B$, then $x \in A$. This means $A \cap B \subseteq A$.
Conversely, let $x \in U$ and assume $x \in A$.
Since $x \in A$ and $A \subseteq B$, we have $x \in B$.
Now, $x \in A$ and $x \in B$, giving us $x \in A \cap B$.
Therefore, $\forall x \in U$, if $x \in A$, then $x \in A \cap B$. This means $A \subseteq A \cap B$.
Thus, $A \cap B = A$.
Therefore, if $A \subseteq B$, then $A \cap B = A$. □

(c) If $A \cup B = B$, then $A \setminus B = \emptyset$.

Proof.
Assume $A \cup B = B$ and $A \setminus B \neq \emptyset$.
Since $A \setminus B \neq \emptyset$, $\exists x \in U$, $x \in A \setminus B$. Choose such an x.
Now, since $x \in A \setminus B$, we have $x \in A$ and $x \notin B$.
Since $x \in A$ and $A \subseteq A \cup B$, we have $x \in A \cup B$.
Since $x \in A \cup B$ and $A \cup B = B$, we have $x \in B$.
We now have the contradiction $x \in B$ and $x \notin B$.
Therefore, if $A \cup B = B$, then $A \setminus B = \emptyset$. □

4. (a) If $A \cap B = \emptyset$, then $A \subseteq B^c$.

Proof.
Assume $A \cap B = \emptyset$.
Let $x \in U$.
Suppose $x \in A$ and $x \notin B^c$.
Since $x \notin B^c$, we have $x \in B$.
Now, since $x \in A$ and $x \in B$, we have $x \in A \cap B$.
Since $A \cap B = \emptyset$, this means $x \in \emptyset$, which is a contradiction.
Therefore, if $x \in A$, then $x \in B^c$.
Therefore, $A \subseteq B^c$.
Therefore, if $A \cap B = \emptyset$, then $A \subseteq B^c$. □

(c) If $(A \cup B) \setminus B = A$, then $B \subseteq A^c$.

Proof.
Assume $(A \cup B) \setminus B = A$.
Let $x \in U$.
Assume $x \notin A^c$.
Since $x \notin A^c$, we have $x \in A$.
Since $x \in A$ and $(A \cup B) \setminus B = A$, we have $x \in (A \cup B) \setminus B$.
This means $x \in A \cup B$ and $x \notin B$. In particular, $x \notin B$.
This proves, if $x \notin A^c$, then $x \notin B$.
Therefore, if $x \in B$, then $x \in A^c$.
Therefore, $B \subseteq A^c$.
Therefore, If $(A \cup B) \setminus B = A$, then $B \subseteq A^c$. □

(e) If $A \setminus B = A$, then $B \setminus A = B$.

Proof.
Assume $A \setminus B = A$.
Let $x \in U$, and assume $x \in B \setminus A$.
Then $x \in B$ and $x \notin A$. In particular, $x \in B$.
Therefore, $\forall x \in U$, if $x \in B \setminus A$, then $x \in B$. That is, $B \setminus A \subseteq B$.
Conversely, let $x \in U$ and suppose $x \in B$ and $x \notin B \setminus A$.
$x \notin B \setminus A$ means either $x \notin B$ or $x \in A$.
Since $x \in B$, it must be the case that $x \in A$.
Now, since $x \in A$ and $A = A \setminus B$, we have $x \in A \setminus B$.
Then $x \notin B$, which is a contradiction, since by assumption $x \in B$.
Therefore, $\forall x \in U$, if $x \in B$, then $x \in B \setminus A$. That is, $B \subseteq B \setminus A$, and thus $B \setminus A = B$.
Therefore, if $A \setminus B = A$, then $B \setminus A = B$. □

5. $(A \setminus B) \setminus C = (A \setminus C) \setminus (B \setminus C)$.

Proof.
Let $x \in (A \setminus B) \setminus C$.
Then $x \in A \setminus B$ and $x \notin C$. This means $x \in A$ and $x \notin B$ and $x \notin C$.
Since $x \in A$ and $x \notin C$, we have $x \in A \setminus C$.
Since $x \notin B$, we have $\neg(x \in B$ and $x \notin C)$, and hence $x \notin B \setminus C$.
In summary, $x \in A \setminus C$ and $x \notin B \setminus C$; that is, $x \in (A \setminus C) \setminus (B \setminus C)$.
Therefore, $(A \setminus B) \setminus C \subseteq (A \setminus C) \setminus (B \setminus C)$.
Conversely, let $x \in (A \setminus C) \setminus (B \setminus C)$.
Then $x \in A \setminus C$ and $x \notin B \setminus C$. This means $x \in A$ and $x \notin C$, and either $x \notin B$ or $x \in C$.
Since $x \notin C$, it must be the case that $x \notin B$.
Therefore, $x \in A$ and $x \notin B$. Hence, $x \in A \setminus B$.
Now, since $x \in A \setminus B$ and $x \notin C$, we have $x \in (A \setminus B) \setminus C$.
Therefore, $(A \setminus C) \setminus (B \setminus C) \subseteq (A \setminus B) \setminus C$. Thus, $(A \setminus B) \setminus C = (A \setminus C) \setminus (B \setminus C)$. □

7. $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

Proof.
Let $x \in A \setminus (B \cup C)$.
Then $x \in A$ and $x \notin B \cup C$. This means $x \in A$ and $x \notin B$ and $x \notin C$.
Since $x \in A$ and $x \notin B$, we have $x \in A \setminus B$.
Since $x \in A$ and $x \notin C$, we have $x \in A \setminus C$.
Now, $x \in A \setminus B$ and $x \in A \setminus C$, so $x \in (A \setminus B) \cap (A \setminus C)$.
Therefore, $A \setminus (B \cup C) \subseteq (A \setminus B) \cap (A \setminus C)$.
Conversely, let $x \in (A \setminus B) \cap (A \setminus C)$.
Then $x \in A \setminus B$ and $x \in A \setminus C$. This means $x \in A$ and $x \notin B$ and $x \in A$ and $x \notin C$.
Since $x \notin B$ and $x \notin C$, We have $x \notin B \cup C$.
Therefore, $x \in A$ and $x \notin B \cup C$. Hence, $x \in A \setminus (B \cup C)$.
Therefore, $(A \setminus B) \cap (A \setminus C) \subseteq A \setminus (B \cup C)$, and hence $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$. □

9. $A = (A \setminus B) \cup (A \cap B)$.

Proof.
Let $x \in A$.
Case 1: $x \in B$.
In this case, $x \in A$ and $x \in B$, so $x \in A \cap B$.
Therefore, $x \in (A \setminus B) \cup (A \cap B)$.
Case 2: $x \notin B$.
In this case, $x \in A$ and $x \notin B$, so $x \in A \setminus B$.
Therefore, $x \in (A \setminus B) \cup (A \cap B)$.
Therefore, $A \subseteq (A \setminus B) \cup (A \cap B)$.
Conversely, let $x \in (A \setminus B) \cup (A \cap B)$.
Then $x \in A \setminus B$ or $x \in A \cap B$.
Case 1: $x \in A \setminus B$.
Then $x \in A$ and $x \notin B$. In particular, $x \in A$.
Case 2: $x \in A \cap B$.
Then $x \in A$ and $x \in B$. In particular, $x \in A$.
Therefore, $(A \setminus B) \cup (A \cap B) \subseteq A$. Thus, $A = (A \setminus B) \cup (A \cap B)$. □

11. If $A \subseteq B$, then $A \cup C \subseteq B \cup C$.

Proof.
Assume $A \subseteq B$.
Let $x \in U$ and assume $x \in A \cup C$.
Then $x \in A$ or $x \in C$.
Case 1: $x \in A$.
Since $A \subseteq B$, we then have $x \in B$.
This proves $x \in B$ or $x \in C$, which means $x \in B \cup C$.
Case 2: $x \in C$.
In this case, we again have $x \in B$ or $x \in C$, so $x \in B \cup C$.
Therefore, $\forall x \in U$, if $x \in A \cup C$, then $x \in B \cup C$.
Thus, $A \cup C \subseteq B \cup C$.
Therefore, if $A \subseteq B$, then $A \cup C \subseteq B \cup C$. □

13. if $A \subseteq B$ then $C \setminus B \subseteq C \setminus A$.

Proof.
Assume $A \subseteq B$.
Let $x \in U$.
Suppose $x \in C \setminus B$ and $x \notin C \setminus A$.
Since $x \in C \setminus B$, we have $x \in C$ and $x \notin B$.
Since $x \notin C \setminus A$, we have $x \notin C$ or $x \in A$.
Since we know $x \in C$, it must be the case that $x \in A$.
Now, since $x \in A$ and $A \subseteq B$, we have $x \in B$.
We now have the contradiction $x \in B$ and $x \notin B$.
Therefore, if $x \in C \setminus B$, then $x \in C \setminus A$.
Therefore, $C \setminus B \subseteq C \setminus A$.
Therefore, if $A \subseteq B$ then $C \setminus B \subseteq C \setminus A$. □

15. If $A \subseteq B$ and $C \subseteq D$, then $A \setminus D \subseteq B \setminus C$.

Proof.
Assume $A \subseteq B$ and $C \subseteq D$.
Let $x \in A \setminus D$.
Then $x \in A$ and $x \notin D$.
Since $x \in A$ and $A \subseteq B$, we have $x \in B$.
Suppose $x \in C$.
Then $x \in D$, since $C \subseteq D$.
This is a contradiction, since $x \notin D$.
Therefore, $x \notin C$.
Now, $x \in B$ and $x \notin C$, which means $x \in B \setminus C$.
Therefore, if $x \in A \setminus D$, then $x \in B \setminus C$.
Thus, $A \setminus D \subseteq B \setminus C$.
Therefore, if $A \subseteq B$ and $C \subseteq D$, then $A \setminus D \subseteq B \setminus C$. □

16. (a) If $A \cap E \subseteq B \cap E$ for all sets E, then $A \subseteq B$.

Proof.
Assume $A \cap E \subseteq B \cap E$ for all sets E.
Since U is a set, we then have $A \cap U \subseteq B \cap U$.
Since $A \cap U = A$ and $B \cap U = B$, this means $A \subseteq B$.
Therefore, if $A \cap E \subseteq B \cap E$ for all sets E, then $A \subseteq B$. □

(c) If $A \setminus E \subseteq B \setminus E$ for all sets E, then $A \subseteq B$.

Proof.
Assume $A \setminus E \subseteq B \setminus E$ for all sets E.
Since $\emptyset$ is a set, we have $A \setminus \emptyset \subseteq B \setminus \emptyset$.
Since $A \setminus \emptyset = A \cap \emptyset^c = A \cap U = A$ and similarly, $B \setminus \emptyset = B$, we then have $A \subseteq B$.
Therefore, if $A \setminus E \subseteq B \setminus E$ for all sets E, then $A \subseteq B$. □

17. (a) If $A \cap E = \emptyset$ for all sets E, then $A = \emptyset$.

Proof.
Assume $A \cap E = \emptyset$ for all sets E.
Since A is a set, we have $A \cap A = \emptyset$.
Since $A \cap A = A$, this gives us $A = \emptyset$.
Therefore, if $A \cap E = \emptyset$ for all sets E, then $A = \emptyset$. □

(c) If $A \cup E = E$ for all sets E, then $A = \emptyset$.

Proof.
Assume $A \cup E = E$ for all sets E.
Then $A \cup \emptyset = \emptyset$.
Since $A \cup \emptyset = A$, we then have $A = \emptyset$.
Therefore, if $A \cup E = E$ for all sets E, then $A = \emptyset$. □

18. (a) If $A \cup E = U$ for all sets E, then $A = U$.

Proof.

Assume $A \cup E = U$ for all sets E.
Since $\emptyset$ is a set, we have $A \cup \emptyset = U$.
Since $A \cup \emptyset = A$, this means $A = U$.
Therefore, if $A \cup E = U$ for all sets E, then $A = U$. □

(c) If $A \cap E = E$ for all sets E, then $A = U$.

Proof.
Assume $A \cap E = E$ for all sets E.
Since U is a set, we have $A \cap U = U$.
Since $A \cap U = A$, we then have $A = U$.
Therefore, if $A \cap E = E$ for all sets E, then $A = U$. □

19. If $A \cap B = \emptyset$ and $A \cup B = U$, then $A = B^c$.

Proof.
Assume $A \cap B = \emptyset$ and $A \cup B = U$.
Let $x \in U$ and suppose $x \in A$ and $x \notin B^c$.
Since $x \notin B^c$, we have $x \in B$.
Now, $x \in A$ and $x \in B$, which means $x \in A \cap B$, which is a contradiction, since $A \cap B = \emptyset$.
Therefore, $\forall x \in U$, if $x \in A$, then $x \in B^c$. Therefore, $A \subseteq B^c$.
Conversely, let $x \in U$ and assume $x \in B^c$.
Since $x \in U$ and $A \cup B = U$, we have $x \in A \cup B$. This means $x \in A$ or $x \in B$.
However, since $x \in B^c$, we have $x \notin B$.
Therefore, it must be the case that $x \in A$.
Therefore, $\forall x \in U$, if $x \in B^c$, then $x \in A$. Thus, $B^c \subseteq A$.
We now have $A = B^c$.
Therefore, if $A \cap B = \emptyset$ and $A \cup B = U$, then $A = B^c$. □

21. If $U \setminus A \subseteq A$, then $A = U$.

Proof.
Assume $U \setminus A \subseteq A$ and assume $A \neq U$.
Since $A \subseteq U$, it must be the case that $U \nsubseteq A$.
That is, $\exists x \in U$, $x \notin A$. Choose such an x.
Then $x \in U$ and $x \notin A$, which means $x \in U \setminus A$.
Since $U \setminus A \subseteq A$, we then have $x \in A$.
We now have the contradiction $x \in A$ and $x \notin A$.
Therefore, if $U \setminus A \subseteq A$, then $A = U$. □

23. If $C \subseteq A \cup B$, then $C \setminus A \subseteq B$.

Proof.
Assume $C \subseteq A \cup B$.
Let $x \in U$ and assume $x \in C \setminus A$.
Then $x \in C$ and $x \notin A$.
Since $x \in C$ and $C \subseteq A \cup B$, we have $x \in A \cup B$.
Now, $x \in A$ or $x \in B$, but since $x \notin A$, it must be the case that $x \in B$.
Therefore, $\forall x \in U$, if $x \in C \setminus A$, then $x \in B$. That is, $C \setminus A \subseteq B$.
Therefore, if $C \subseteq A \cup B$, then $C \setminus A \subseteq B$. □

25. If $A \cap B \cap C = \emptyset$, then $C \subseteq A^c \cup B^c$.

Proof.
Assume $A \cap B \cap C = \emptyset$.
Let $x \in U$ and suppose $x \in C$ and $x \notin A^c \cup B^c$.
Since $x \notin A^c \cup B^c$, we have $x \notin A^c$ and $x \notin B^c$. This means $x \in A$ and $x \in B$.
We now have $x \in A$ and $x \in B$ and $x \in C$, which means $x \in A \cap B \cap C$.
Since $A \cap B \cap C = \emptyset$, we have $x \in \emptyset$, which is a contradiction.
Therefore, $\forall x \in U$, if $x \in C$, then $x \in A^c \cup B^c$. This means $C \subseteq A^c \cup B^c$.
Therefore, if $A \cap B \cap C = \emptyset$, then $C \subseteq A^c \cup B^c$. □

Let $(A_k)_{k\in\mathbb{N}}$ be a sequence of sets, and let B be a set. Prove the following propositions.

27. $\forall m, n \in \mathbb{N}$, if $m \le n$, then $A_m \subseteq \displaystyle\bigcup_{k=1}^{n} A_k$.

Proof.
Let $m \in \mathbb{N}$.
Let $S = \{n \in \mathbb{N} \mid \text{if } m \le n, \text{ then } A_m \subseteq \bigcup_{k=1}^{n} A_k\}$.
Assume $m \le 1$.
Since $m \in \mathbb{N}$, we then have $m = 1$.
Since $\bigcup_{k=1}^{1} A_k = A_1$, we have $A_1 \subseteq \bigcup_{k=1}^{1} A_k$;
hence, $A_m \subseteq \bigcup_{k=1}^{1} A_k$.
Therefore, if $m \le 1$, then $A_m \subseteq \bigcup_{k=1}^{1} A_k$.
Thus, $1 \in S$.
Next, let $n \in \mathbb{N}$ and assume $n \in S$.
Assume $m \le n + 1$.
Case 1: $m = n + 1$.
In this case, $A_m = A_{n+1}$.
Since $A_{n+1} \subseteq (\bigcup_{k=1}^{n} A_k) \cup A_{n+1}$, we have $A_{n+1} \subseteq \bigcup_{k=1}^{n+1} A_k$.
Thus, $A_m \subseteq \bigcup_{k=1}^{n+1} A_k$.
Case 2: $m < n + 1$.
In this case, we have $m \le n$, and since $n \in S$, this means $A_m \subseteq \bigcup_{k=1}^{n} A_k$.
Therefore, $A_m \cup A_{n+1} \subseteq (\bigcup_{k=1}^{n} A_k) \cup A_{n+1}$.
Since we also have $A_m \subseteq A_m \cup A_{n+1}$, this gives $A_m \subseteq \bigcup_{k=1}^{n+1} A_k$ by transitivity.
Therefore, if $m \le n + 1$, then $A_m \subseteq \bigcup_{k=1}^{n+1} A_k$.
That is, $n + 1 \in S$.
Therefore, $\forall n \in \mathbb{N}$, if $n \in S$, then $n + 1 \in S$.
Therefore, $\mathbb{N} \subseteq S$ by the Principle of Mathematical Induction.

Therefore, $\forall m, n \in \mathbb{N}$, if $m \le n$, then $A_m \subseteq \displaystyle\bigcup_{k=1}^{n} A_k$. □

29. $\forall n \in \mathbb{N}$, $B \cup \displaystyle\bigcap_{k=1}^{n} A_k = \bigcap_{k=1}^{n}(B \cup A_k)$.

Proof.
Let $S = \{n \in \mathbb{N} \mid B \cup \bigcap_{k=1}^{n} A_k = \bigcap_{k=1}^{n}(B \cup A_k)\}$.
$B \cup \bigcap_{k=1}^{1} A_k = B \cup A_1 = \bigcap_{k=1}^{1}(B \cup A_k)$; hence $1 \in S$.
Let $n \in \mathbb{N}$ and assume $n \in S$.
Then $B \cup \bigcap_{k=1}^{n} A_k = \bigcap_{k=1}^{n}(B \cup A_k)$.
By the distributive law, $B \cup (\bigcap_{k=1}^{n} A_k \cap A_{n+1}) = (B \cup \bigcap_{k=1}^{n} A_k) \cap (B \cup A_{n+1})$.
This gives us $B \cup (\bigcap_{k=1}^{n+1} A_k) = (\bigcap_{k=1}^{n}(B \cup A_k)) \cap (B \cup A_{n+1})$.
Thus, $B \cup \bigcap_{k=1}^{n+1} A_k = \bigcap_{k=1}^{n+1}(B \cup A_k)$.
Therefore, $n + 1 \in S$.
Therefore, $\forall n \in \mathbb{N}$, if $n \in S$, then $n + 1 \in S$.
By the PMI, $\mathbb{N} \subseteq S$.
Therefore, $\forall n \in \mathbb{N}$, $B \cup \bigcap_{k=1}^{n} A_k = \bigcap_{k=1}^{n}(B \cup A_k)$. □

31. $\forall n \in \mathbb{N}, B \setminus \bigcup_{k=1}^{n} A_k = \bigcap_{k=1}^{n} (B \setminus A_k)$.

Proof.
Let $S = \{n \in \mathbb{N} \mid B \setminus \bigcup_{k=1}^{n} A_k = \bigcap_{k=1}^{n} (B \setminus A_k)\}$.
$B \setminus \bigcup_{k=1}^{1} A_k = B \setminus A_1 = \bigcap_{k=1}^{1} (B \setminus A_k)$; hence $1 \in S$.
Let $n \in \mathbb{N}$ and assume $n \in S$.
Then $B \setminus \bigcup_{k=1}^{n} A_k = \bigcap_{k=1}^{n} (B \setminus A_k)$.
$B \setminus ((\bigcup_{k=1}^{n} A_k) \cup A_{n+1}) = (B \setminus (\bigcup_{k=1}^{n} A_k)) \cap (B \setminus A_{n+1})$ by exercise 7.
This gives us $B \setminus (\bigcup_{k=1}^{n+1} A_k) = (\bigcap_{k=1}^{n} (B \setminus A_k)) \cap (B \setminus A_{n+1}) = \bigcap_{k=1}^{n+1} (B \setminus A_k)$.
Therefore, $n + 1 \in S$.
Therefore, $\forall n \in \mathbb{N}$, if $n \in S$, then $n + 1 \in S$.
By the PMI, $\mathbb{N} \subseteq S$.
Therefore, $\forall n \in \mathbb{N}, B \setminus \bigcup_{k=1}^{n} A_k = \bigcap_{k=1}^{n} (B \setminus A_k)$. □

33. $\forall n \in \mathbb{N}, \left(\bigcup_{k=1}^{n} A_k\right)^c = \bigcap_{k=1}^{n} (A_k)^c$.

Proof.
Let $S = \{x \in \mathbb{N} \mid (\bigcup_{k=1}^{x} A_k)^c = \bigcap_{k=1}^{x} (A_k)^c\}$.
$(\bigcup_{k=1}^{1} A_k)^c = A_1^c = \bigcap_{k=1}^{1} A_k^c$; hence $1 \in S$.
Let $n \in \mathbb{N}$, and assume $n \in S$.
Then $(\bigcup_{k=1}^{n} A_k)^c = \bigcap_{k=1}^{n} (A_k)^c$.
Now, $\left(\bigcup_{k=1}^{n+1} A_k\right)^c = \left(\left(\bigcup_{k=1}^{n} A_k\right) \cup A_{n+1}\right)^c = \left(\bigcup_{k=1}^{n} A_k\right)^c \cap A_{n+1}^c = \left(\bigcap_{k=1}^{n} A_k^c\right) \cap A_{n+1}^c = \bigcap_{k=1}^{n+1} (A_k)^c$.
Therefore, $n + 1 \in S$.
Therefore, $\forall n \in \mathbb{N}$, if $n \in S$, then $n + 1 \in S$.
By the PMI, $\mathbb{N} \subseteq S$.
Therefore, $\forall n \in \mathbb{N}, \left(\bigcup_{k=1}^{n} A_k\right)^c = \bigcap_{k=1}^{n} (A_k)^c$. □

35. If $B \subseteq A_1$, then $\forall n \in \mathbb{N}, B \subseteq \bigcup_{k=1}^{n} A_k$.

Proof.
Assume $B \subseteq A_1$.
Let $S = \{n \in \mathbb{N} \mid B \subseteq \bigcup_{k=1}^{n} A_k\}$.
Since $\bigcup_{k=1}^{1} A_k = A_1$ and $B \subseteq A_1$, we have that $B \subseteq \bigcup_{k=1}^{1} A_k$.
Thus, $1 \in S$.
Let $n \in \mathbb{N}$ and assume $n \in S$.
Then $B \subseteq \bigcup_{k=1}^{n} A_k$.
Now, since $\bigcup_{k=1}^{n} A_k \subseteq (\bigcup_{k=1}^{n} A_k) \cup A_{n+1}$, we have $\bigcup_{k=1}^{n} A_k \subseteq \bigcup_{k=1}^{n+1} A_k$.
Since $B \subseteq \bigcup_{k=1}^{n} A_k$ and $\bigcup_{k=1}^{n} A_k \subseteq \bigcup_{k=1}^{n+1} A_k$, we have $B \subseteq \bigcup_{k=1}^{n+1} A_k$ by transitivity.
Thus, $n + 1 \in S$.
Therefore, $\forall n \in \mathbb{N}$, if $n \in S$, then $n + 1 \in S$.
By the PMI, $\mathbb{N} \subseteq S$.
Therefore, $\forall n \in \mathbb{N}, B \subseteq \bigcup_{k=1}^{n} A_k$.
Therefore, if $B \subseteq A_1$, then $\forall n \in \mathbb{N}, B \subseteq \bigcup_{k=1}^{n} A_k$. □

37. If $\forall m \in \mathbb{N}$, $B \subseteq A_m$, then $\forall n \in \mathbb{N}$, $B \subseteq \bigcap_{k=1}^{n} A_k$.

Proof.
Assume $\forall m \in \mathbb{N}$, $B \subseteq A_m$.
Let $S = \{n \in \mathbb{N} \mid B \subseteq \bigcap_{k=1}^{n} A_k\}$.
Since $B \subseteq A_1$ and $\bigcap_{k=1}^{1} A_k = A_1$, we have $B \subseteq \bigcap_{k=1}^{1} A_k$. Thus, $1 \in S$.
Let $n \in \mathbb{N}$ and assume $n \in S$.
Then $B \subseteq \bigcap_{k=1}^{n} A_k$.
Let $x \in U$ and assume $x \in B$.
Since $B \subseteq \bigcap_{k=1}^{n} A_k$, we have $x \in \bigcap_{k=1}^{n} A_k$.
Since $\forall m \in \mathbb{N}$, $B \subseteq A_m$, we have that $B \subseteq A_{n+1}$. Since $x \in B$, we then have $x \in A_{n+1}$.
Now, $x \in \bigcap_{k=1}^{n} A_k$ and $x \in A_{n+1}$, so $x \in (\bigcap_{k=1}^{n} A_k) \cap A_{n+1}$. Thus, $x \in \bigcap_{k=1}^{n+1} A_k$.
Therefore, $\forall x \in U$, if $x \in B$, then $x \in \bigcap_{k=1}^{n+1} A_k$.
Thus, $B \subseteq \bigcap_{k=1}^{n+1} A_k$, which proves $n + 1 \in S$.
Therefore, $\forall n \in \mathbb{N}$, if $n \in S$, then $n + 1 \in S$. Thus, by the PMI, $\mathbb{N} \subseteq S$.
Therefore, $\forall n \in \mathbb{N}$, $B \subseteq \bigcap_{k=1}^{n} A_k$.
Therefore, if $\forall m \in \mathbb{N}$, $B \subseteq A_m$, then $\forall n \in \mathbb{N}$, $B \subseteq \bigcap_{k=1}^{n} A_k$. □

39. If $\forall k \in \mathbb{N}$, $A_{k+1} \subseteq A_k$, then $\forall m, n \in \mathbb{N}$, if $m \leq n$, then $A_n \subseteq A_m$.

Proof.
Assume $\forall k \in \mathbb{N}$, $A_{k+1} \subseteq A_k$.
Let $m \in \mathbb{N}$, and let $S = \{n \in \mathbb{N} \mid \text{if } m \leq n, \text{ then } A_n \subseteq A_m\}$.
Assume $m \leq 1$.
Then $m = 1$, and so $A_1 = A_m$; hence $A_1 \subseteq A_m$.
Therefore, if $m \leq 1$, then $A_1 \subseteq A_m$. Thus, $1 \in S$.
Let $n \in \mathbb{N}$ and assume $n \in S$.
Assume $m \leq n + 1$.
Case 1: $m = n + 1$.
In this case, we have $A_{n+1} = A_m$, and hence $A_{n+1} \subseteq A_m$.
Case 2: $m < n + 1$.
In this case, we have $m \leq n$, and since $n \in S$, this implies $A_n \subseteq A_m$.
Also, since $\forall k \in \mathbb{N}$, $A_{k+1} \subseteq A_k$, we have $A_{n+1} \subseteq A_n$.
Therefore, $A_{n+1} \subseteq A_m$ by transitivity.
Therefore, if $m \leq n + 1$, then $A_{n+1} \subseteq A_m$. Thus, $n + 1 \in S$.
Therefore, $\forall n \in \mathbb{N}$, if $n \in S$, then $n + 1 \in S$. Thus, by the PMI, $\mathbb{N} \subseteq S$.
Therefore, $\forall m, n \in \mathbb{N}$, if $m \leq n$, then $A_n \subseteq A_m$.
Therefore, if $\forall m \in \mathbb{N}$, $A_{m+1} \subseteq A_m$, then $\forall m, n \in \mathbb{N}$, if $m \leq n$, then $A_n \subseteq A_m$. □

41. If $\forall m \in \mathbb{N}$, $A_{m+1} \subseteq A_m$, then $\forall n \in \mathbb{N}$, $\displaystyle\bigcap_{k=1}^{n} A_k = A_n$.

Proof.
Assume $\forall m \in \mathbb{N}$, $A_{m+1} \subseteq A_m$.
Let $S = \{n \in \mathbb{N} \mid \bigcap_{k=1}^{n} A_k = A_n\}$.
Since $\bigcap_{k=1}^{1} A_k = A_1$, we have $1 \in S$.
Let $n \in S$.
Then $\bigcap_{k=1}^{n} A_k = A_n$.
Since $(\bigcap_{k=1}^{n} A_k) \cap A_{n+1} \subseteq A_{n+1}$, we have $\bigcap_{k=1}^{n+1} A_k \subseteq A_{n+1}$.
Conversely, let $x \in A_{n+1}$.
Then $x \in A_n$, since $A_{n+1} \subseteq A_n$.
Therefore, $x \in \bigcap_{k=1}^{n} A_k$, since $\bigcap_{k=1}^{n} A_k = A_n$.
We now have $x \in \bigcap_{k=1}^{n} A_k$ and $x \in A_{n+1}$.
Therefore, $x \in (\bigcap_{k=1}^{n} A_k) \cap A_{n+1} = \bigcap_{k=1}^{n+1} A_k$.
Therefore, $A_{n+1} \subseteq \bigcap_{k=1}^{n+1} A_k$.
We now have, $\bigcap_{k=1}^{n+1} A_k = A_{n+1}$, and hence $n+1 \in S$.
Therefore, if $n \in S$, then $n+1 \in S$.
By the PMI, $S = \mathbb{N}$, which means $\forall n \in \mathbb{N}$, $\bigcap_{k=1}^{n} A_k = A_n$.
Therefore, if $\forall m \in \mathbb{N}$, $A_{m+1} \subseteq A_m$, then $\forall n \in \mathbb{N}$, $\bigcap_{k=1}^{n} A_k = A_n$. □

43. If $\forall m \in \mathbb{N}$, $A_{m+1} \subseteq A_m$, then $\forall n \in \mathbb{N}$, $\displaystyle\bigcup_{k=1}^{n} A_k = A_1$.

Proof.
Assume $\forall m \in \mathbb{N}$, $A_{m+1} \subseteq A_m$.
Let $S = \{n \in \mathbb{N} \mid \bigcup_{k=1}^{n} A_k = A_1\}$.
Since $\bigcup_{k=1}^{1} A_k = A_1$, we have $1 \in S$.
Let $n \in \mathbb{N}$ and assume $n \in S$.
Then $\bigcup_{k=1}^{n} A_k = A_1$.
Since $\forall m \in \mathbb{N}$, $A_{m+1} \subseteq A_m$, and since $1 \leq n+1$, we have $A_{n+1} \subseteq A_1$ by exercise 39.
Now, $\bigcup_{k=1}^{n+1} A_k = (\bigcup_{k=1}^{n} A_k) \cup A_{n+1} = A_1 \cup A_{n+1}$.
Since $A_{n+1} \subseteq A_1$, we have $A_1 \cup A_{n+1} \subseteq A_1 \cup A_1$; hence $\bigcup_{k=1}^{n+1} A_k \subseteq A_1$.
Conversely, since $\bigcup_{k=1}^{n} A_k \subseteq (\bigcup_{k=1}^{n} A_k) \cup A_{n+1}$, we have $A_1 \subseteq \bigcup_{k=1}^{n+1} A_k$.
Therefore, $\bigcup_{k=1}^{n+1} A_k = A_1$, and hence $n+1 \in S$.
Therefore, $\forall n \in \mathbb{N}$, if $n \in S$, then $n+1 \in S$. Thus, by the PMI, $\mathbb{N} \subseteq S$.
Therefore, $\forall n \in \mathbb{N}$, $\bigcup_{k=1}^{n} A_k = A_1$.
Therefore, if $\forall m \in \mathbb{N}$, $A_{m+1} \subseteq A_m$, then $\forall n \in \mathbb{N}$, $\bigcup_{k=1}^{n} A_k = A_1$. □

Prove the following propositions.

45. $(-2, 1] \cup [0, 3) = (-2, 3)$.

Proof.
Let $x \in \mathbb{R}$ and assume $x \in (-2, 1] \cup [0, 3)$.
That is, $x \in (-2, 1]$ or $x \in [0, 3)$.
Case 1: $x \in (-2, 1]$.
In this case, we have $-2 < x$ and $x \leq 1$.
Since $x \leq 1$ and $1 < 3$, we have by transitivity $x < 3$.
Now, $-2 < x$ and $x < 3$, which means $x \in (-2, 3)$.
Case 2: $x \in [0, 3)$.
In this case, we have $0 \leq x$ and $x < 3$.
Since $-2 < 0$ and $0 \leq x$, we have by transitivity $-2 < x$.
Since $-2 < x$ and $x < 3$, we have $x \in (-2, 3)$.
Therefore, $\forall x \in \mathbb{R}$, if $x \in (-2, 1] \cup [0, 3)$ then $x \in (-2, 3)$. Thus, $(-2, 1] \cup [0, 3) \subseteq (-2, 3)$.
Let $x \in \mathbb{R}$ and assume $x \in (-2, 3)$.
That is, $-2 < x$ and $x < 3$.
Case 1: $x \leq 1$.
In this case, we have $-2 < x$ and $x \leq 1$; hence $x \in (-2, 1]$.
So, $x \in (-2, 1] \cup [0, 3)$.
Case 2: $1 < x$.
Since $0 < 1$ and $1 < x$, we have $0 < x$; hence $0 \leq x$.
We now have $0 \leq x$ and $x < 3$, which means $x \in [0, 3)$.
So, $x \in (-2, 1] \cup [0, 3)$.
Therefore, $\forall x \in \mathbb{R}$, if $x \in (-2, 3)$ then $x \in (-2, 1] \cup [0, 3)$. This means $(-2, 3) \subseteq (-2, 1] \cup [0, 3)$.
Thus, $(-2, 1] \cup [0, 3) = (-2, 3)$. □

47. $(-2, 1] \setminus [0, 3) = (-2, 0)$.

Proof.
Let $x \in \mathbb{R}$ and assume $x \in (-2, 1] \setminus [0, 3)$.
Then $x \in (-2, 1]$ and $x \notin [0, 3)$. That is, $-2 < x$ and $x \leq 1$, and either $x < 0$ or $3 \leq x$.
Since $x \leq 1$ and $1 < 3$, we have $x < 3$.
Thus, it is not the case that $3 \leq x$, which means it must be the case that $x < 0$.
Now, $-2 < x$ and $x < 0$; hence $x \in (-2, 0)$.
Therefore, $\forall x \in \mathbb{R}$, if $x \in (-2, 1] \setminus [0, 3)$, then $x \in (-2, 0)$. This means $(-2, 1] \setminus [0, 3) \subseteq (-2, 0)$.
Conversely, let $x \in \mathbb{R}$, and assume $x \in (-2, 0)$.
Then $-2 < x$ and $x < 0$.
Since $x < 0$ and $0 < 1$, we have $x < 1$; hence $x \leq 1$.
Now, $-2 < x$ and $x \leq 1$, which shows $x \in (-2, 1]$.
Next, suppose $x \in [0, 3)$.
Then $0 \leq x$ and $x < 3$.
Now, $0 \leq x$ and $x < 0$, which is a contradiction.
Therefore, $x \notin [0, 3)$.
Since $x \in (-2, 1]$ and $x \notin [0, 3)$, we have $x \in (-2, 1] \setminus [0, 3)$.
Therefore, $\forall x \in \mathbb{R}$, if $x \in (-2, 0)$, then $x \in (-2, 1] \setminus [0, 3)$.
Therefore, $(-2, 0) \subseteq (-2, 1] \setminus [0, 3)$, and hence $(-2, 1] \setminus [0, 3) = (-2, 0)$. □

49. $(-\infty, 3) \setminus (-2, 1] = (-\infty, -2] \cup (1, 3)$.

Proof.

Let $x \in \mathbb{R}$ and assume $x \in (-\infty, 3) \setminus (-2, 1]$.
Then $x \in (-\infty, 3)$ and $x \notin (-2, 1]$. This means $x < 3$ and either $x \leq -2$ or $1 < x$.
Case 1: $x \leq -2$.
Then $x \in (-\infty, -2]$; hence $x \in (-\infty, -2] \cup (1, 3)$ (since $(-\infty, -2] \subseteq (-\infty, -2] \cup (1, 3)$).
Case 2: $1 < x$.
We then have $1 < x$ and $x < 3$; hence $x \in (1, 3)$.
Since $(1, 3) \subseteq (-\infty, -2] \cup (1, 3)$, we then have $x \in (-\infty, -2] \cup (1, 3)$.
Therefore, $\forall x \in \mathbb{R}$, if $x \in (-\infty, 3) \setminus (-2, 1]$, then $x \in (-\infty, -2] \cup (1, 3)$. Thus, $(-\infty, 3) \setminus (-2, 1] \subseteq (-\infty, -2] \cup (1, 3)$.
Let $x \in \mathbb{R}$ and assume $x \in (-\infty, -2] \cup (1, 3)$.
Then $x \in (-\infty, -2]$ or $x \in (1, 3)$.
Case 1: $x \in (-\infty, -2]$.
In this case, $x \leq -2$.
Since $x \leq -2$, it is not the case that $-2 < x \leq 1$; hence $x \notin (-2, 1]$.
Also, since $x \leq -2$ and $-2 < 3$, we have $x < 3$; hence $x \in (-\infty, 3)$.
Now, $x \in (-\infty, 3)$ and $x \notin (-2, 1]$, which means $x \in (-\infty, 3) \setminus (-2, 1]$.
Case 2: $x \in (1, 3)$.
In this case, we have $1 < x$ and $x < 3$.
Since $x < 3$, we have $x \in (-\infty, 3)$.
Also, since $1 < x$, it is not the case that $-2 < x \leq 1$; hence $x \notin (-2, 1]$.
We now have $x \in (-\infty, 3)$ and $x \notin (-2, 1]$, which again means $x \in (-\infty, 3) \setminus (-2, 1]$.
Therefore, $\forall x \in \mathbb{R}$, if $x \in (-\infty, -2] \cup (1, 3)$, then $x \in (-\infty, 3) \setminus (-2, 1]$. This means $(-\infty, -2] \cup (1, 3) \subseteq (-\infty, 3) \setminus (-2, 1]$.
Therefore, $(-\infty, 3) \setminus (-2, 1] = (-\infty, -2] \cup (1, 3)$. □

51. $\langle 5 \rangle \cap \langle 6 \rangle = \langle 30 \rangle$.

Proof.
Let $x \in \langle 5 \rangle \cap \langle 6 \rangle$.
Then $x \in \langle 5 \rangle$ and $x \in \langle 6 \rangle$.
Choose $a, b \in \mathbb{Z}$ such that $x = 5a$ and $x = 6b$.
Put $q = a - b$.
Then $x = 6x - 5x = 6(5a) - 5(6b) = 30a - 30b = 30(a - b) = 30q$
Therefore, $\exists q \in \mathbb{Z}, x = 30q$; hence $x \in \langle 30 \rangle$.
Therefore, $\langle 5 \rangle \cap \langle 6 \rangle \subseteq \langle 30 \rangle$.
Conversely, let $x \in \langle 30 \rangle$.
Choose $c \in \mathbb{Z}$ such that $x = 30c$.
Put $a = 6c$ and $b = 5c$.
Then $x = 5(6c) = 5a$, and $x = 6(5c) = 6b$.
Therefore, $\exists a \in \mathbb{Z}, x = 5a$, and $\exists b \in \mathbb{Z}, x = 6b$.
Thus, $x \in \langle 5 \rangle$ and $x \in \langle 6 \rangle$; hence $x \in \langle 5 \rangle \cap \langle 6 \rangle$.
Therefore, $\langle 30 \rangle \subseteq \langle 5 \rangle \cap \langle 6 \rangle$.
Therefore, $\langle 5 \rangle \cap \langle 6 \rangle = \langle 30 \rangle$. □

53. $\forall a, b \in \mathbb{Z}$, if $\gcd(a, b) = 1$, then $\langle a\rangle \cap \langle b\rangle = \langle ab\rangle$.

Proof.
Let $a, b \in \mathbb{Z}$ and assume $\gcd(a, b) = 1$.
Choose $s, t \in \mathbb{Z}$ with $as + bt = 1$.
Let $x \in \mathbb{Z}$ and assume $x \in \langle a\rangle \cap \langle b\rangle$.
Then $x \in \langle a\rangle$ and $x \in \langle b\rangle$, so we can choose $m, n \in \mathbb{Z}$ with $x = am$ and $x = bn$.
Now, $x = x(1) = x(as + bt) = asx + btx = asbn + btam = ab(sn + tm)$.
Putting $k = sn + tm$ gives us $x = abk$; hence $x \in \langle ab\rangle$.
Therefore, $\langle a\rangle \cap \langle b\rangle \subseteq \langle ab\rangle$.
Conversely, let $x \in \mathbb{Z}$ and assume $x \in \langle ab\rangle$.
Choose $u \in \mathbb{Z}$ with $x = abu$.
Put $v = bu$ and $w = au$. Then $x = av$ and $x = bw$.
Therefore, $x \in \langle a\rangle$ and $x \in \langle b\rangle$. Thus, $x \in \langle a\rangle \cap \langle b\rangle$.
Therefore, $\langle ab\rangle \subseteq \langle a\rangle \cap \langle b\rangle$, and so $\langle a\rangle \cap \langle b\rangle = \langle ab\rangle$.
Therefore, $\forall a, b \in \mathbb{Z}$, if $\gcd(a, b) = 1$, then $\langle a\rangle \cap \langle b\rangle = \langle ab\rangle$. □

55. Let $A = \{x \in \mathbb{Z} \mid \exists t \in \mathbb{Z}, x = 15t + 7\}$, $B = \{x \in \mathbb{Z} \mid \exists s \in \mathbb{Z}, x = 3s + 1\}$, and $C = \{x \in \mathbb{Z} \mid \exists r \in \mathbb{Z}, x = 5r + 2\}$. Then $A = B \cap C$.

Proof.
Let $x \in \mathbb{Z}$ and assume $x \in B \cap C$.
Then, $x \in B$ and $x \in C$, which means $\exists s \in \mathbb{Z}, x = 3s + 1$ and $\exists r \in \mathbb{Z}, x = 5r + 2$.
Choose such s and r, and put $t = 2r - s$.
$x = 6x - 5x = 6(5r + 2) - 5(3s + 1) = 30r + 12 - 15s - 5 = 15(2r - s) + 7 = 15t + 7$.
Therefore, $\exists t \in \mathbb{Z}, x = 15t + 7$.
Hence, $x \in A$.
Therefore, $B \cap C \subseteq A$.
Let $x \in \mathbb{Z}$ and assume $x \in A$.
That is, $\exists t \in \mathbb{Z}, x = 15t + 7$. Choose such a t.
Put $s = 5t + 2$.
$x = 15t + 7 = 15t + 6 + 1 = 3(5t + 2) + 1 = 3s + 1$.
Therefore, $\exists s \in \mathbb{Z}, x = 3s + 1$. Hence $x \in B$.
Put $r = 3t + 1$.
$x = 15t + 7 = 15t + 5 + 2 = 5(3t + 1) + 2 = 5r + 2$.
Therefore, $\exists r \in \mathbb{Z}, x = 5r + 2$. Hence $x \in C$.
We now have $x \in B$ and $x \in C$, so $x \in B \cap C$.
Therefore, $A \subseteq B \cap C$. Thus, $A = B \cap C$. □

57. $\forall n \in \mathbb{N}, \displaystyle\bigcup_{k=1}^{n}(0,k] = (0,n]$.

Proof.
Let $S = \{n \in \mathbb{N} \mid \bigcup_{k=1}^{n}(0,k] = (0,n]\}$.
$\bigcup_{k=1}^{1}(0,k] = (0,1]$, and so $1 \in S$.
Let $n \in S$.
Then $\bigcup_{k=1}^{n}(0,k] = (0,n]$.
Let $x \in \bigcup_{k=1}^{n+1}(0,k]$.
Then $x \in (\bigcup_{k=1}^{n}(0,k]) \cup (0,n+1]$. This means $x \in \bigcup_{k=1}^{n}(0,k]$ or $x \in (0,n+1]$.
Since $\bigcup_{k=1}^{n}(0,k] = (0,n]$, we have $x \in (0,n]$ or $x \in (0,n+1]$.
In case $x \in (0,n]$, we have $0 < x \leq n \leq n+1$, and hence $x \in (0,n+1]$.
In case $x \in (0,n+1]$, we again have $x \in (0,n+1]$.
Therefore, $\bigcup_{k=1}^{n+1}(0,k] \subseteq (0,n+1]$.
Next, let $x \in (0,n+1]$.
Then $x \in (\bigcup_{k=1}^{n}(0,k]) \cup (0,n+1]$, since $(0,n+1] \subseteq (\bigcup_{k=1}^{n}(0,k]) \cup (0,n+1]$.
This means, $x \in \bigcup_{k=1}^{n+1}(0,k]$.
Thus, $(0,n+1] \subseteq \bigcup_{k=1}^{n+1}(0,k]$.
Now, $\bigcup_{k=1}^{n+1}(0,k] = (0,n+1]$, and so $n+1 \in S$.
Therefore, if $n \in S$, then $n+1 \in S$. By the principle of mathematical induction, $\mathbb{N} \subseteq S$.
Therefore, $\forall n \in \mathbb{N}, \bigcup_{k=1}^{n}(0,k] = (0,n]$. □

2.2 Real Intervals

Exercises 2.2.

Prove the following propositions.

1. For every subset A of $\mathbb{R}$, A is bounded above if and only if $\exists k \in \mathbb{N}, \forall x \in A, x \leq k$.

Proof.
Let A be a subset of $\mathbb{R}$.
Assume A is bounded above. That is, $\exists a \in \mathbb{R}, \forall x \in A, x \leq a$.
Choose $a \in \mathbb{R}$ for which $\forall x \in A, x \leq a$.
Also, applying the Archimedean property, choose $k \in \mathbb{N}$ with $a \leq k$.
Let $x \in A$.
Then $x \leq a$ and $a \leq k$, so $x \leq k$.
Therefore, $\forall x \in A, x \leq k$.
Therefore, $\exists k \in \mathbb{N}, \forall x \in A, x \leq k$.
Therefore, if A is bounded above, then $\exists k \in \mathbb{N}, \forall x \in A, x \leq k$.
Conversely, assume $\exists k \in \mathbb{N}, \forall x \in A, x \leq k$.
Choosing $a \in \mathbb{N}$ with $\forall x \in A, x \leq a$, we have that since $\mathbb{N} \subseteq \mathbb{R}$, $a \in \mathbb{R}$.
Therefore, $\exists a \in \mathbb{R}, \forall x \in A, x \leq a$. Thus, A is bounded above.
Therefore, if $\exists k \in \mathbb{N}, \forall x \in A, x \leq k$, then A is bounded above.
Therefore, for every subset A of $\mathbb{R}$, A is bounded above if and only if $\exists k \in \mathbb{N}, \forall x \in A, x \leq k$. □

3. $\forall a, b \in \mathbb{R}, (a, b) \subseteq (a, b] \subset [a, b]$.

Proof.
Let $a, b \in \mathbb{R}$.
Let $x \in \mathbb{R}$ and assume $x \in (a, b)$.
This means $a < x$ and $b < x$, and since $b < x$, we have $b \leq x$.
Now, $a < x$ and $b \leq x$, so $x \in (a, b]$.
Therefore, $(a, b) \subseteq (a, b]$.
Next, let $x \in \mathbb{R}$ and assume $x \in (a, b]$.
This means $a < x$ and $x \leq b$. Since $a < x$, we have $a \leq x$.
Now, $a \leq x$ and $x \leq b$, so $x \in [a, b]$.
Therefore, $(a, b] \subseteq [a, b]$.
We now have $(a, b) \subseteq (a, b]$ and $(a, b] \subseteq [a, b]$.
Therefore, $\forall a, b \in \mathbb{R}, (a, b) \subseteq (a, b] \subset [a, b]$. □

5. $\forall a, b \in \mathbb{R}$, if $b \leq a$, then $(a, b] = \emptyset$. (So the empty set is an interval.)

Proof.
Let $a, b \in \mathbb{R}$ and assume $b \leq a$ and $(a, b] \neq \emptyset$.
Since $(a, b] \neq \emptyset$, we can choose an element $x \in (a, b]$.
For this x, we have $a < x$ and $x \leq b$; hence $a < b$ by transitivity.
Now, $a < b$ and $b \leq a$, which is a contradiction.
Therefore, $\forall a, b \in \mathbb{R}$, if $b \leq a$, then $(a, b] = \emptyset$. □

7. $\forall a, b \in \mathbb{R}$, if $a = b$, then $[a, b] = \{a\}$.

Proof.
Let $a, b \in \mathbb{R}$ and assume $a = b$.
Let $x \in [a, b]$.
Then $a \leq x$ and $x \leq b$. Since $x \leq b$ and $a = b$, we have $x \leq a$.
Now $a \leq x$ and $x \leq a$, so $x = a$; hence $x \in \{a\}$.
Therefore, $[a, b] \subseteq \{a\}$.
Conversely, let $x \in \{a\}$. This means $x = a$; hence $x = b$.
Since $x = a$, we have $a \leq x$, and since $x = b$, we have $x \leq b$. Thus, $x \in [a, b]$.
Therefore, $\{a\} \subseteq [a, b]$. We now have $[a, b] = \{a\}$.
Therefore, $\forall a, b \in \mathbb{R}$, if $a = b$, then $[a, b] = \{a\}$. □

9. $\forall x, y \in \mathbb{R}$, if $(-\infty, x) = (-\infty, y)$, then $x = y$.

Proof.
Let $x, y \in \mathbb{R}$ and assume $(-\infty, x) = (-\infty, y)$ and $x \neq y$.
Since $x \neq y$, we can assume without loss of generality that $x < y$.
Then $x \in (-\infty, y)$ and since $(-\infty, x) = (-\infty, y)$, we have $x \in (-\infty, x)$.
This means $x < x$, which is a contradiction.
Therefore, $\forall x, y \in \mathbb{R}$, if $(-\infty, x) = (-\infty, y)$, then $x = y$. □

11. $\forall a, b, x, y \in \mathbb{R}$, if $a \leq x$ and $y \leq b$, then $(x, y) \subseteq (a, b)$.

Proof.
Let $a, b, x, y \in \mathbb{R}$ and assume $a \leq x$ and $y \leq b$.
Let $t \in \mathbb{R}$ and assume $t \in (x, y)$.
Then $x < t$ and $t < y$.
Since $a \leq x$ and $x < t$, we have $a < t$ by transitivity.
Likewise, since $t < y$ and $y \leq b$, we have $t < b$.
We now have $a < t$ and $t < b$. Therefore, $t \in (a, b)$.
Hence, $(x, y) \subseteq (a, b)$.
Therefore, for all $a, b, x, y \in \mathbb{R}$, if $a \leq x$ and $y \leq b$, then $(x, y) \subseteq (a, b)$. □

13. $\forall a, b \in \mathbb{R}_{\geq 0}$, if $[0, a) \subseteq [0, b)$, then $a \leq b$.

Proof.
Let $a, b \in \mathbb{R}_{\geq 0}$. i.e. $0 \leq a$ and $0 \leq b$.
Assume $[0, a) \subseteq [0, b)$ and $b < a$.
Since $0 \leq b$ and $b < a$, we have $b \in [0, a)$.
Since $b \in [0, a)$ and $[0, a) \subseteq [0, b)$, we then have $b \in [0, b)$.
This means $0 \leq b$ and $b < b$, which is a contradiction.
Therefore, if $[0, a) \subseteq [0, b)$, then $a \leq b$.
Therefore, $\forall a, b \in \mathbb{R}_{\geq 0}$, if $[0, a) \subseteq [0, b)$, then $a \leq b$. □

15. $\forall a, b \in \mathbb{R}_{\geq 0}$, if $[0, a] = [0, b]$, then $a = b$.

Proof.
Let $a, b \in \mathbb{R}_{\geq 0}$. i.e. $0 \leq a$ and $0 \leq b$.
Assume $[0, a] = [0, b]$.
Since $0 \leq a$ and $a \leq a$, we have $a \in [0, a]$. This implies $a \in [0, b]$, and so $a \leq b$.
Likewise, since $b \in [0, b]$, we have $b \in [0, a]$, and so $b \leq a$.
Therefore, $a = b$, since $a \leq b$ and $b \leq a$.
Therefore, if $[0, a] = [0, b]$ if and only if $a = b$.
Therefore, $\forall a, b \in \mathbb{R}_{\geq 0}$, if $[0, a] = [0, b]$, then $a = b$. □

17. $\forall a \in \mathbb{R}$, if $(0, \infty) \subseteq [a, \infty)$, then $(0, \infty) \subseteq (a, \infty)$.

Proof.
Let $a \in \mathbb{R}$ and assume $(0, \infty) \subseteq [a, \infty)$.
Let $x \in (0, \infty)$, and suppose $x \notin (a, \infty)$.
This means $0 < x$ and $x \leq a$.
Also, since $(0, \infty) \subseteq [a, \infty)$, we have $x \in [a, \infty)$, and so $a \leq x$.
Now, since $a \leq x$ and $x \leq a$, we have $x = a$. Now, since $0 < x$, we must have $0 < a$.
This means $0 < \frac{a}{2}$, and so $\frac{a}{2} \in (0, \infty)$, which implies $\frac{a}{2} \in [a, \infty)$.
We now have $a \leq \frac{a}{2}$, which gives us $2a \leq a$.
Subtracting a from both sides gives $a \leq 0$, which is a contradiction, since $0 < a$.
Therefore, if $x \in (0, \infty)$, then $x \in (a, \infty)$.
This means, $(0, \infty) \subseteq (a, \infty)$.
Therefore, $\forall a \in \mathbb{R}$, if $(0, \infty) \subseteq [a, \infty)$, then $(0, \infty) \subseteq (a, \infty)$. □

Prove the following propositions about unions, intersections, and complements of intervals.

19. $\forall a, b, x, y \in \mathbb{R}$, if $x \in (a, b)$ and $b \in (x, y)$, then $(a, b) \cap (x, y) = (x, b)$.

Proof.
Let $a, b, x, y \in \mathbb{R}$ and assume $x \in (a, b)$ and $b \in (x, y)$.
This gives us the following three inequalities: $a < x$, $x < b$, and $b < y$.
Let $t \in (a, b) \cap (x, y)$.
Then $t \in (a, b)$ and $t \in (x, y)$, which means $a < t$, $t < b$, $x < t$, and $t < y$.
Since $x < t$ and $t < b$, we have $t \in (x, b)$.
Therefore, $(a, b) \cap (x, y) \subseteq (x, b)$.
Conversely, let $t \in (x, b)$.
Then $x < t$ and $t < b$.
Now, $a < x$ and $x < t$, so $a < t$. This gives us $a < t$ and $t < b$, which means $t \in (a, b)$.
Also, $t < b$ and $b < y$, so $t < y$. This gives us $x < t$ and $t < y$; hence $t \in (x, y)$.
Since $t \in (a, b)$ and $t \in (x, y)$, we have $t \in (a, b) \cap (x, y)$.
Therefore, $(x, b) \subseteq (a, b) \cap (x, y)$.
Therefore, $\forall a, b, x, y \in \mathbb{R}$, if $x \in (a, b)$ and $b \in (x, y)$, then $(a, b) \cap (x, y) = (x, b)$. □

21. $\forall a, b, c, d \in \mathbb{R}$, $(a, b) \cap (c, d) = (e, f)$, where $e = \max(a, c)$ and $f = \min(b, d)$.

Proof.
Let $a, b, c, d \in \mathbb{R}$, and let $e = \max(a, c)$ and $f = \min(b, d)$.
Let $x \in (a, b) \cap (c, d)$.
Then $x \in (a, b)$ and $x \in (c, d)$, which means $a < x$, $x < b$, $c < x$, and $x < d$.
Since $a < x$ and $c < x$, we have $\max(a, c) < x$; hence $e < x$.
Since $x < b$ and $x < d$, we have $x < \min(b, d)$; hence $x < f$. Thus, $x \in (e, f)$.
Therefore, $(a, b) \cap (c, d) \subseteq (e, f)$.
Conversely, let $x \in (e, f)$. i.e. $e < x$ and $x < f$.
Since $e = \max(a, c)$, we have $a \leq e$ and $c \leq e$.
Since $a \leq e$ and $e < x$, we have $a < x$. Similarly, since $c \leq e$ and $e < x$, we have $c < x$.
Likewise, since $f = \min(b, d)$, we have $f \leq b$ and $f \leq d$.
Since $x < f$ and $x \leq b$, we have $x < b$. Similarly, since $x < f$ and $f \leq d$, we have $x < d$.
We have thus shown the four inequalities: $a < x$, $x < b$, $c < x$, and $x < d$.
$a < x$ and $x < b$ give us $x \in (a, b)$, and $c < x$ and $x < d$ give us $x \in (c, d)$. Thus, $x \in (a, b) \cap (c, d)$.
Therefore, $(e, f) \subseteq (a, b) \cap (c, d)$, and so $(a, b) \cap (c, d) = (e, f)$.
Therefore, $\forall a, b, c, d \in \mathbb{R}$, $(a, b) \cap (c, d) = (e, f)$, where $e = \max(a, c)$ and $f = \min(b, d)$. □

23. $\forall a, b \in \mathbb{R}$, $(a, \infty) \cap (b, \infty) = (c, \infty)$, where $c = \max(a, b)$.

Proof.
Let $a, b \in \mathbb{R}$, and let $c = \max(a, b)$.
Let $x \in (a, \infty) \cap (b, \infty)$.
Then $x \in (a, \infty)$ and $x \in (b, \infty)$, which means $a < x$ and $b < x$.
Since $a < x$ and $b < x$, we have $\max(a, b) < x$. This means $c < x$, and so $x \in (c, \infty)$.
Therefore, $(a, \infty) \cap (b, \infty) \subseteq (c, \infty)$.
Conversely, let $x \in (c, \infty)$. i.e. $c < x$.
Since $c = \max(a, b)$, we have $a \leq c$ and $b \leq c$.
Now, $a \leq c$ and $c < x$, which implies $a < x$. This gives us $x \in (a, \infty)$.
Similarly, since $b \leq c$ and $c < x$, we have $b < x$, and so $x \in (b, \infty)$.
Now, $x \in (a, \infty)$ and $x \in (b, \infty)$, which means $x \in (a, \infty) \cap (b, \infty)$.
Therefore, $(c, \infty) \subseteq (a, \infty) \cap (b, \infty)$, and so $(a, \infty) \cap (b, \infty) = (c, \infty)$.
Therefore, $\forall a, b \in \mathbb{R}$, $(a, \infty) \cap (b, \infty) = (c, \infty)$, where $c = \max(a, b)$. □

25. $\forall a \in \mathbb{R}$, $\mathbb{R} \setminus (a, \infty) = (-\infty, a]$.

Proof.
Let $a \in \mathbb{R}$.
Let $x \in \mathbb{R}$ and assume $\mathbb{R} \setminus (a, \infty)$.
Then $x \notin (a, \infty)$, which means $x \leq a$. Therefore, $x \in (-\infty, a]$.
Therefore, $\mathbb{R} \setminus (a, \infty) \setminus (-\infty, a]$.
Conversely, let $x \in \mathbb{R}$ and assume $x \in (-\infty, a]$.
Then $x \leq a$, which means $a \not< x$, so $x \notin (a, \infty)$.
Now, $x \in \mathbb{R}$ and $x \notin (a, \infty)$, which gives us $x \in \mathbb{R} \setminus (a, \infty)$.
Therefore, $(-\infty, a] \subseteq \mathbb{R} \setminus (a, \infty)$.
Therefore, $\forall a \in \mathbb{R}$, $\mathbb{R} \setminus (a, \infty) = (-\infty, a]$. □

27. $\forall a, b, c \in \mathbb{R}$, $[a, b] \cap (-\infty, c] = [a, u]$, where $u = \min(b, c)$.

Proof.
Let $a, b, c \in \mathbb{R}$, and let $u = \min(b, c)$.
Let $x \in [a, b] \cap (-\infty, c]$.
Then $x \in [a, b]$ and $x \in (-\infty, c]$, which means $a \leq x$, $x \leq b$, and $x \leq c$.
Since $x \leq b$ and $x \leq c$, we have $x \leq \min(b, c)$; hence $x \leq u$.
Now, $a \leq x$ and $x \leq u$, which gives us $x \in [a, u]$.
Therefore, $[a, b] \cap (-\infty, c] \subseteq [a, u]$.
Conversely, let $x \in [a, u]$. i.e. $a \leq x$ and $x \leq u$.
Since $u = \min(b, c)$, we have $u \leq b$ and $u \leq c$.
Since $x \leq u$ and $u \leq b$, we have $x \leq b$. Now, $a \leq x \leq b$, which means $x \in [a, b]$.
Similarly, since $x \leq u$ and $u \leq c$, we have $x \leq c$; hence $x \in (-\infty, c]$.
We now have $x \in [a, b]$ and $x \in (-\infty, c]$, which gives us $x \in [a, b] \cap (-\infty, c]$.
Therefore, $[a, u] \subseteq [a, b] \cap (-\infty, c]$, and hence $[a, b] \cap (-\infty, c] = [a, u]$.
Therefore, $\forall a, b, c \in \mathbb{R}$, $[a, b] \cap (-\infty, c] = [a, u]$, where $u = \min(b, c)$. □

29. $\forall a, b, c, d \in \mathbb{R}$, $[a, b] \setminus (c, d) = [a, u] \cup [v, b]$, where $u = \min(b, c)$ and $v = \max(a, d)$.

Proof.
Let $a, b, c, d \in \mathbb{R}$ and let $u = \min(b, c)$ and $v = \max(a, d)$.
Let $x \in [a, b] \setminus (c, d)$.
Then $x \in [a, b]$, and $x \notin (c, d)$. This means $a \leq x$ and $x \leq b$, and either $x \leq c$ or $d \leq x$.
Case 1: $x \leq c$.
Since $x \leq b$ and $x \leq c$, we have $x \leq \min(b, c)$. Hence, $x \leq u$.
Now, $a \leq x$ and $x \leq u$, so $x \in [a, u]$. Hence $x \in [a, u] \cup [v, b]$.
Case 2: $d \leq x$.
Since $a \leq x$ and $d \leq x$, we have $\max(a, d) \leq x$. That is, $v \leq x$.
Now, $v \leq x$ and $x \leq b$, which means $x \in [v, b]$. So, again $x \in [a, u] \cup [v, b]$.
Therefore, if $x \in [a, b] \setminus (c, d)$, then $x \in [a, u] \cup [v, b]$. Thus, $[a, b] \setminus (c, d) \subseteq [a, u] \cup [v, b]$.
Conversely, let $x \in [a, u] \cup [v, b]$. This means $x \in [a, u]$ or $x \in [v, b]$.
Case 1: $x \in [a, u]$. i.e. $a \leq x$ and $x \leq u$.
Since $x \leq u$ and $u \leq b$, we have $x \leq b$.
Now, $a \leq x$ and $x \leq b$, so $x \in [a, b]$.
Further, since $x \leq u$ and $u \leq c$, we have $x \leq c$. Thus, $x \notin (c, d)$.
We now have, $x \in [a, b] \setminus (c, d)$.
Case 2: $x \in [v, b]$. i.e. $v \leq x$ and $x \leq b$.
Since $a \leq v$ and $v \leq x$, we have $a \leq x$.
Now, $a \leq x$ and $x \leq b$, so $x \in [a, b]$.
Also, since $d \leq v$ and $v \leq x$, we have $d \leq x$.
Therefore, $x \notin (c, d)$, and hence $x \in [a, b] \setminus (c, d)$.
Therefore, if $x \in [a, u] \cup [v, b]$, then $x \in [a, b] \setminus (c, d)$. Thus, $[a, u] \cup [v, b] \subseteq [a, b] \setminus (c, d)$.
Therefore, $[a, b] \setminus (c, d) = [a, u] \cup [v, b]$.
Therefore, $\forall a, b, c, d \in \mathbb{R}$, $[a, b] \setminus (c, d) = [a, u] \cup [v, b]$, where $u = \min(b, c)$ and $v = \max(a, d)$. $\square$

31. $\forall a, b, c \in \mathbb{R}$, if $(-\infty, a) \cap (b, c) \neq \emptyset$, then $(-\infty, a) \cup (b, c) = (-\infty, u)$, where $u = \max(a, c)$.

Proof.
Let $a, b, c \in \mathbb{R}$, and let $u = \max(a, c)$.
Assume $(-\infty, a) \cap (b, c) \neq \emptyset$, and accordingly, choose an element $t \in (-\infty, a) \cap (b, c)$.
We then have $t \in (-\infty, a)$ and $t \in (b, c)$, meaning $t < a$, $b < t$, and $t < c$. Thus, $b < a$.
Let $x \in (-\infty, a) \cup (b, c)$. This means $x \in (-\infty, a)$ or $x \in (b, c)$.
Case 1: $x \in (-\infty, a)$. i.e. $x < a$.
Since $u = \max(a, c)$, we have $a \leq u$, and so by transitivity, $x < u$. Thus, $x \in (-\infty, u)$.
Case 2: $x \in (b, c)$. i.e. $b < x$ and $x < c$.
Since $u = \max(a, c)$, we have $c \leq u$, and since $x < c$, we then have $x < u$. Thus, $x \in (-\infty, u)$.
Therefore, $(-\infty, a) \cup (b, c) \subseteq (-\infty, u)$.
Conversely, let $x \in (-\infty, u)$. This means $x < u$.
Case 1: $x < a$.
In this case, $x \in (-\infty, a)$, and so $x \in (-\infty, a) \cup (b, c)$.
Case 2: $a \leq x$.
In this case, $a \leq x$ and $x < u$, which gives us $a < u$. This means $u \neq a$, and so $u = c$.
Now, since $x < u$ and $u = c$, we have $x < c$. Also, since $b < a$ and $a \leq x$, we have $b < x$.
Thus, $b < x$ and $x < c$, which means $x \in (b, c)$, which implies $x \in (-\infty, a) \cup (b, c)$.
Therefore, $(-\infty, u) \subseteq (-\infty, a) \cup (b, c)$. Thus, $(-\infty, a) \cup (b, c) = (-\infty, u)$.
Therefore, if $(-\infty, a) \cap (b, c) \neq \emptyset$, then $(-\infty, a) \cup (b, c) = (-\infty, u)$, where $u = \max(a, c)$.
Therefore, $\forall a, b, c \in \mathbb{R}$, if $(-\infty, a) \cap (b, c) \neq \emptyset$, then $(-\infty, a) \cup (b, c) = (-\infty, u)$, where $u = \max(a, c)$. $\square$

33. $\forall a \in \mathbb{R}$, if $J \subseteq \mathbb{R}$ with $(a, \infty) \subseteq J \subseteq [a, \infty)$, then $J = (a, \infty)$ or $J = [a, \infty)$.

Proof.
Let $a \in \mathbb{R}$ and assume $J \subseteq \mathbb{R}$ with $(a, \infty) \subseteq J \subseteq [a, \infty)$.
Case 1: $a \in J$.
Let $x \in [a, \infty)$. This means $a \leq x$.
If $a = x$, then since $a \in J$, we have $x \in J$.
Otherwise, we have $a < x$, and so $x \in (a, \infty)$. Since $(a, \infty) \subseteq J$, we again have $x \in J$.
Therefore, $[a, \infty) \subseteq J$, and since we also have $J \subseteq [a, \infty)$, this proves $J = [a, \infty)$.
Case 2: $a \notin J$.
Let $x \in J$.
Then $x \in [a, \infty)$, since $J \subseteq [a, \infty)$. This means $a \leq x$.
Since $x \in J$ and $a \notin J$, we have $x \neq a$; hence $a < x$. Thus, $x \in (a, \infty)$.
Therefore, $J \subseteq (a, \infty)$, and since $(a, \infty) \subseteq J$, we have $J = (a, \infty)$.
Therefore, in either case, we have $J = (a, \infty)$ or $J = [a, \infty)$.
Therefore, $\forall a \in \mathbb{R}$, if $J \subseteq \mathbb{R}$ with $(a, \infty) \subseteq J \subseteq [a, \infty)$, then $J = (a, \infty)$ or $J = [a, \infty)$. □

Prove the following propositions using theorems 2.2.9 and 2.2.11.

35. Let I be an interval of $\mathbb{R}$. For all $a \in \mathbb{R}$, if $a \notin I$ and $\exists b \in I, a < b$, then $\forall x \in I, a < x$.

Proof.
Let $a \in \mathbb{R}$, and assume $a \notin I$ and $\exists b \in I, a < b$.
Choose $y \in I$ with $a < y$.
Let $x \in I$, and assume $x \leq a$.
Since $x \in I$ and $a \notin I$, we have $x \neq a$; hence $x < a$.
We now have $x \in I$ and $y \in I$, and $x < a < y$.
By theorem 2.2.9, we then have $a \in I$, which is a contradiction.
Therefore, $\forall x \in i, a < x$.
Therefore, for all $a \in \mathbb{R}$, if $a \notin I$ and $\exists b \in I, a < b$, then $\forall x \in I, a < x$. □

37. The intersection of any two intervals is an interval.

Proof.
Let I and J be intervals.
Let $x, y \in I \cap J$, let $z \in \mathbb{R}$, and assume $x < z < y$.
Then $x \in I$, $x \in J$, $y \in I$, and $y \in J$.
Since $x, y \in I$ and $x < z < y$, we have $z \in I$ by theorem 2.2.9.
Similarly, since $x, y \in J$ and $x < z < y$, we have $z \in J$ by theorem 2.2.9.
Now, $z \in I$ and $z \in J$, which means $z \in I \cap J$.
Therefore, $\forall x, y \in I \cap J, \forall z \in \mathbb{R}, x < z < y$ implies $z \in I \cap J$.
By theorem 2.2.11, $I \cap J$ is an interval.
Therefore, if I and J are intervals, then $I \cap J$ is an interval. □

Prove the following propositions characterizing bounded intervals.

39. $\forall a, b \in \mathbb{R}$, if $a < b$, then $(a, b) = \{x \in \mathbb{R} \mid \exists t \in (0,1),\ x = (1-t)a + tb\}$.

Proof.
Let $a, b \in \mathbb{R}$ and assume $a < b$.
Let $x \in (a, b)$. This means $a < x < b$.
Put $t = \frac{x-a}{b-a}$.
Since $a < x$, we have $0 < x - a$. Likewise, since $a < b$, we have $0 < b - a$.
Therefore, $0 < \frac{x-a}{b-a}$; hence, $0 < t$.
Further, since $x < b$, we have $x - a < b - a$, and so $\frac{x-a}{b-a} < 1$.
Thus, $t < 1$, which means $t \in (0, 1)$.
Finally, since $t = \frac{x-a}{b-a}$, we have $t(b - a) = x - a$, which give us $x = a - ta + tb = (1 - t)a + tb$.
Therefore, $\exists t \in (0, 1)$, $x = (1 - t)a + tb$.
Therefore, $(a, b) \subseteq \{x \in \mathbb{R} \mid \exists t \in (0,1),\ x = (1-t)a + tb\}$.
Conversely, let $x \in \{x \in \mathbb{R} \mid \exists t \in (0,1),\ x = (1-t)a + tb\}$.
Choose $t \in (0, 1)$ with $x = (1 - t)a + tb$.
Since $a < b$ and $0 < t$, we have $at < bt$.
Adding $(1 - t)a$ to both sides gives $(1 - t)a + at < (1 - t)a + bt$. That is, $a < x$.
Next, since $t < 1$, we have $0 < 1 - t$, and so since $a < b$, we have $a(1 - t) < b(1 - t)$.
Adding bt to both sides gives $a(1 - t) + bt < b(1 - t) + bt$. That is, $x < b$.
Now, $a < x < b$, so $x \in (a, b)$.
Therefore, $\{x \in \mathbb{R} \mid \exists t \in (0,1),\ x = (1-t)a + tb\} \subseteq (a, b)$. Thus, $(a, b) = \{x \in \mathbb{R} \mid \exists t \in (0,1),\ x = (1-t)a + tb\}$.
Therefore, $\forall a, b \in \mathbb{R}$, if $a < b$, then $(a, b) = \{x \in \mathbb{R} \mid \exists t \in (0,1),\ x = (1-t)a + tb\}$. □

41. $\forall a, b \in \mathbb{R}$, if $a \neq b$, then $(a, b) \cup (b, a) = \{x \in \mathbb{R} \mid \exists t \in (0,1),\ x = (1-t)a + tb\}$.

Proof.
Let $a, b \in \mathbb{R}$ and assume $a \neq b$.
Case 1: $a < b$.
Suppose $(b, a) \neq \emptyset$. Accordingly, choose an element $x \in (b, a)$.
Then $b < x$ and $x < a$, which gives us $b < a$. This is a contradiction.
This proves $(b, a) = \emptyset$, and hence $(a, b) \cup (b, a) = (a, b) \cup \emptyset = (a, b)$.
By exercise 39, since $a < b$, we have $(a, b) = \{x \in \mathbb{R} \mid \exists t \in (0,1),\ x = (1-t)a + tb\}$.
Therefore, $(a, b) \cup (b, a) = \{x \in \mathbb{R} \mid \exists t \in (0,1),\ x = (1-t)a + tb\}$.
Case 2: $b < a$.
Similar to the case above, in this case we have $(a, b) = \emptyset$, and hence $(a, b) \cup (b, a) = (b, a)$.
From exercise 39, we have $(b, a) = \{x \in \mathbb{R} \mid \exists s \in (0,1),\ x = (1-s)b + sa\}$.
Let $x \in (a, b) \cup (b, a)$. Then $x \in (b, a)$, so $\exists s \in (0, 1)$, $x = (1 - s)b + sa$.
Choose $s \in (0, 1)$, with $x = (1 - s)b + sa$, and put $t = 1 - s$.
Since $0 < s$, $1 - s < 1$; and so $t < 1$. Since $s < 1$, $0 < 1 - s$, and so $0 < t$. Thus, $t \in (0, 1)$.
Since $x = (1 - s)b + sa$, we have $x = tb + (1 - t)a$. Therefore, $\exists t \in (0, 1)$, $x = (1 - t)a + tb$.
Therefore, $(a, b) \cup (b, a) \subseteq \{x \in \mathbb{R} \mid \exists t \in (0,1),\ x = (1-t)a + tb\}$.
Conversely, let $x \in \{x \in \mathbb{R} \mid \exists t \in (0,1),\ x = (1-t)a + tb\}$.
Choose $t \in (0, 1)$, $x = (1 - t)a + tb$, and put $s = 1 - t$.
Similar to above, since $t \in (0, 1)$, we have $s \in (0, 1)$.
Also, since $x = (1 - t)a + tb$, we have $x = sa + (1 - s)b$. Thus, $\exists s \in (0, 1)$, $x = (1 - s)b + sa$.
Therefore, $x \in (b, a)$, and so $x \in (a, b) \cup (b, a)$.
Therefore, $(a, b) \cup (b, a) = \{x \in \mathbb{R} \mid \exists t \in (0,1),\ x = (1-t)a + tb\}$.
Therefore, $\forall a, b \in \mathbb{R}$, if $a \neq b$, then $(a, b) \cup (b, a) = \{x \in \mathbb{R} \mid \exists t \in (0,1),\ x = (1-t)a + tb\}$. □

43. $\forall a, b \in \mathbb{R}$, if $a < b$, then $(a, b] = \{x \in \mathbb{R} \mid \exists t \in (0, 1],\ x = (1 - t)a + tb\}$.

Proof.
Let $a, b \in \mathbb{R}$ and assume $a < b$.
Let $x \in (a, b]$. This means $a < x \leq b$.
Put $t = \frac{x-a}{b-a}$.
Since $a < x$, we have $0 < x - a$. Likewise, since $a < b$, we have $0 < b - a$.
Therefore, $0 < \frac{x-a}{b-a}$; hence, $0 < t$.
Further, since $x \leq b$, we have $x - a \leq b - a$, and so $\frac{x-a}{b-a} \leq 1$.
Thus, $t \leq 1$, which means $t \in (0, 1]$.
Finally, since $t = \frac{x-a}{b-a}$, we have $t(b - a) = x - a$, which give us $x = a - ta + tb = (1 - t)a + tb$.
Therefore, $\exists t \in (0, 1]$, $x = (1 - t)a + tb$.
Therefore, $(a, b) \subseteq \{x \in \mathbb{R} \mid \exists t \in (0, 1],\ x = (1 - t)a + tb\}$.
Conversely, let $x \in \{x \in \mathbb{R} \mid \exists t \in (0, 1],\ x = (1 - t)a + tb\}$.
Choose $t \in (0, 1]$ with $x = (1 - t)a + tb$.
Since $a < b$ and $0 < t$, we have $at < bt$.
Adding $(1 - t)a$ to both sides gives $(1 - t)a + at < (1 - t)a + bt$. That is, $a < x$.
Next, since $t \leq 1$, we have $0 \leq 1 - t$, and so since $a < b$, we have $a(1 - t) \leq b(1 - t)$.
Adding bt to both sides gives $a(1 - t) + bt \leq b(1 - t) + bt$. That is, $x \leq b$.
Now, $a < x \leq b$, so $x \in (a, b]$.
Therefore, $\{x \in \mathbb{R} \mid \exists t \in (0, 1],\ x = (1-t)a+tb\} \subseteq (a, b]$. Thus, $(a, b] = \{x \in \mathbb{R} \mid \exists t \in (0, 1],\ x = (1-t)a+tb\}$.
Therefore, $\forall a, b \in \mathbb{R}$, if $a < b$, then $(a, b] = \{x \in \mathbb{R} \mid \exists t \in (0, 1],\ x = (1 - t)a + tb\}$. □

2.3 Ideals of the Integers

Exercises 2.3.

Prove the following propositions.

1. $\forall a, x, y \in \mathbb{Z}$, if $x \in \langle a \rangle$ and $y \in \langle a \rangle$, then $x + y \in \langle a \rangle$.

Proof.
Let $a, x, y \in \mathbb{Z}$ and assume $x \in \langle a \rangle$ and $y \in \langle a \rangle$.
Choose $s, t \in \mathbb{Z}$ with $x = as$ and $y = at$.
Putting $q = s + t$ gives us $x + y = as + at = a(s + t) = aq$. Thus, $x + y \in \langle a \rangle$.
Therefore, $\forall a, x, y \in \mathbb{Z}$, if $x \in \langle a \rangle$ and $y \in \langle a \rangle$, then $x + y \in \langle a \rangle$. □

3. $\forall a, x \in \mathbb{Z}$, if $x \in \langle a \rangle$, then $\forall t \in \mathbb{Z}$, $xt \in \langle a \rangle$.

Proof.
Let $a, x \in \mathbb{Z}$ and assume $x \in \langle a \rangle$.
Choose $k \in \mathbb{Z}$ with $x = ak$. Let $t \in \mathbb{Z}$.
Putting $m = kt$ gives us $xt = akt = am$. Thus, $xt \in \langle a \rangle$.
Therefore, $\forall t \in \mathbb{Z}$, $xt \in \langle a \rangle$.
Therefore, $\forall a, x \in \mathbb{Z}$, if $x \in \langle a \rangle$, then $\forall t \in \mathbb{Z}$, $xt \in \langle a \rangle$. □

5. $\langle 4 \rangle \cap \langle 6 \rangle = \langle 12 \rangle$.

Proof.
Let $x \in \langle 4 \rangle \cap \langle 6 \rangle$.
Then $x \in \langle 4 \rangle$ and $x \in \langle 6 \rangle$, so we can choose $a, b \in \mathbb{Z}$ with $x = 4a$ and $x = 6b$.
Putting $c = a - b$ gives us $x = 3x - 2x = 3(4a) - 2(6b) = 12a - 12b = 12(a - b) = 12c$. Thus, $x \in \langle 12 \rangle$.
Therefore, $\langle 4 \rangle \cap \langle 6 \rangle \subseteq \langle 12 \rangle$.
Conversely, let $x \in \langle 12 \rangle$, and choose $k \in \mathbb{Z}$ with $x = 12k$.
Putting $m = 3k$ gives $x = 4(3k) = 4m$; hence $x \in \langle 4 \rangle$. Putting $n = 2k$ gives $x = 6(2k) = 6n$; hence $x \in \langle 6 \rangle$.
Now, $x \in \langle 4 \rangle$ and $x \in \langle 6 \rangle$, which means $x \in \langle 4 \rangle \cap \langle 6 \rangle$.
Therefore, $\langle 12 \rangle \subseteq \langle 4 \rangle \cap \langle 6 \rangle$. Thus, $\langle 4 \rangle \cap \langle 6 \rangle = \langle 12 \rangle$. □

7. $\forall a \in \mathbb{Z}$, $\langle a \rangle + \langle a \rangle = \langle a \rangle$.

Proof.
Let $a \in \mathbb{Z}$.
Let $x \in \langle a \rangle + \langle a \rangle$.
Choose $s, t \in \mathbb{Z}$ with $x = as + at$.
Putting $q = s + t$ gives us $x = as + at = a(s + t) = aq$; hence $x \in \langle a \rangle$.
Therefore, $\langle a \rangle + \langle a \rangle \subseteq \langle a \rangle$.
Conversely, let $x \in \langle a \rangle$ and choose $k \in \mathbb{Z}$ with $x = ak$.
Putting $m = 0$ gives us $x = ak + am$; hence $x \in \langle a \rangle + \langle a \rangle$.
Therefore, $\langle a \rangle \subseteq \langle a \rangle + \langle a \rangle$.
Therefore, $\forall a \in \mathbb{Z}$, $\langle a \rangle + \langle a \rangle = \langle a \rangle$. □

9. $\forall a, b \in \mathbb{Z}$, if $\langle a\rangle + \langle b\rangle = \langle b\rangle$, then $\langle a\rangle \subseteq \langle b\rangle$.

Proof.
Let $a, b \in \mathbb{Z}$ and assume $\langle a\rangle + \langle b\rangle = \langle b\rangle$.
Let $x \in \langle a\rangle$.
Choose $k \in \mathbb{Z}$ with $x = ak$.
Putting $m = 0$ gives us $x = ak + bm$; hence $x \in \langle a\rangle + \langle b\rangle$.
Therefore, $x \in \langle b\rangle$, since $\langle a\rangle + \langle b\rangle = \langle b\rangle$.
Therefore, $\langle a\rangle \subseteq \langle b\rangle$.
Therefore, $\forall a, b \in \mathbb{Z}$, if $\langle a\rangle + \langle b\rangle = \langle b\rangle$, then $\langle a\rangle \subseteq \langle b\rangle$. □

11. $\forall a, b \in \mathbb{Z}$, $\langle a\rangle \cap (\langle a\rangle + \langle b\rangle) = \langle a\rangle$.

Proof.
Let $a, b \in \mathbb{Z}$.
By Proposition 2.1.5, we have $\langle a\rangle \cap (\langle a\rangle + \langle b\rangle) \subseteq \langle a\rangle$.
Conversely, let $x \in \langle a\rangle$.
Choose $k \in \mathbb{Z}$ with $x = ak$.
Putting $m = 0$ gives us $x = ak + bm$; hence $x \in \langle a\rangle + \langle b\rangle$.
Now, $x \in \langle a\rangle$ and $x \in \langle a\rangle + \langle b\rangle$, so $x \in \langle a\rangle \cap (\langle a\rangle + \langle b\rangle)$.
Therefore, $\langle a\rangle \subseteq \langle a\rangle \cap (\langle a\rangle + \langle b\rangle)$.
Therefore, $\forall a, b \in \mathbb{Z}$, $\langle a\rangle \cap (\langle a\rangle + \langle b\rangle) = \langle a\rangle$. □

13. $\forall a, b, c \in \mathbb{Z}$, if $\langle c\rangle \subseteq \langle a\rangle$, then $\langle a\rangle \cap (\langle b\rangle + \langle c\rangle) = (\langle a\rangle \cap \langle b\rangle) + \langle c\rangle$.

Proof.
let $a, b, c \in \mathbb{Z}$ and assume $\langle c\rangle \subseteq \langle a\rangle$.
Let $x \in \langle a\rangle \cap (\langle b\rangle + \langle c\rangle)$.
Then $x \in \langle a\rangle$ and $x \in \langle b\rangle + \langle c\rangle$. Choose $k, m, n \in \mathbb{Z}$ with $x = ak$ and $x = bm + cn$.
Since $cn \in \langle c\rangle$, we have $cn \in \langle a\rangle$. Choose $s \in \mathbb{Z}$ with $cn = as$.
Now, $bm = x - cn = ak - asn = a(k - sn) \in \langle a\rangle$. Also, $bm \in \langle b\rangle$. Therefore, $bm \in \langle a\rangle \cap \langle b\rangle$.
Now, $x = bm + cn$ and since $bm \in \langle a\rangle \cap \langle b\rangle$ and $cn \in \langle c\rangle$, we have $x \in (\langle a\rangle \cap \langle b\rangle) + \langle c\rangle$.
Therefore, $\langle a\rangle \cap (\langle b\rangle + \langle c\rangle) \subseteq (\langle a\rangle \cap \langle b\rangle) + \langle c\rangle$.
Conversely, let $x \in (\langle a\rangle \cap \langle b\rangle) + \langle c\rangle$.
Choose $u \in \langle a\rangle \cap \langle b\rangle$ and $v \in \langle c\rangle$ with $x = u + v$. Then, $u \in \langle a\rangle$ and $u \in \langle b\rangle$.
Since $v \in \langle c\rangle$ and $\langle c\rangle \subseteq \langle a\rangle$, we have $v \in \langle a\rangle$.
Now, $u \in \langle a\rangle$ and $v \in \langle a\rangle$, so $x = u + v \in \langle a\rangle$.
Also, since $u \in \langle b\rangle$ and $v \in \langle c\rangle$, we have $x = u + v \in \langle b\rangle + \langle c\rangle$.
Now, $x \in \langle a\rangle$ and $x \in \langle b\rangle + \langle c\rangle$, so $x \in \langle a\rangle \cap (\langle b\rangle + \langle c\rangle)$.
Therefore, $(\langle a\rangle \cap \langle b\rangle) + \langle c\rangle \subseteq \langle a\rangle \cap (\langle b\rangle + \langle c\rangle)$.
Thus, $\langle a\rangle \cap (\langle b\rangle + \langle c\rangle) = (\langle a\rangle \cap \langle b\rangle) + \langle c\rangle$.
Therefore, $\forall a, b, c \in \mathbb{Z}$, if $\langle c\rangle \subseteq \langle a\rangle$, then $\langle a\rangle \cap (\langle b\rangle + \langle c\rangle) = (\langle a\rangle \cap \langle b\rangle) + \langle c\rangle$. □

15. $\forall a, x, y \in \mathbb{Z}$, if $\langle x \rangle \subseteq \langle y \rangle$, then $\langle a \rangle + \langle x \rangle \subseteq \langle a \rangle + \langle y \rangle$.

Proof.
Let $a, x, y \in \mathbb{Z}$ and assume $\langle x \rangle \subseteq \langle y \rangle$.
Let $w \in \langle a \rangle + \langle x \rangle$.
Choose $s, t \in \mathbb{Z}$ with $w = as + xt$.
Since $xt \in \langle x \rangle$ and $\langle x \rangle \subseteq \langle y \rangle$, we have $xt \in \langle y \rangle$. Choose $q \in \mathbb{Z}$ with $xt = yq$.
Then $w = as + xt = as + yq$. Thus, $w \in \langle a \rangle + \langle y \rangle$.
Therefore, $\langle a \rangle + \langle x \rangle \subseteq \langle a \rangle + \langle y \rangle$.
Therefore, $\forall a, x, y \in \mathbb{Z}$, if $\langle x \rangle \subseteq \langle y \rangle$, then $\langle a \rangle + \langle x \rangle \subseteq \langle a \rangle + \langle y \rangle$. □

17. $\forall x, y \in \mathbb{Z} \setminus \{0\}$, $\gcd(x, y) = 1$ if and only if $\langle x \rangle + \langle y \rangle = \mathbb{Z}$.

Proof.
Let $x, y \in \mathbb{Z} \setminus \{0\}$.
Assume $\gcd(x, y) = 1$ and choose $s, t \in \mathbb{Z}$ with $1 = sx + ty$.
Let $w \in \mathbb{Z}$.
Then $w = w(1) = wsx + wty$; hence $w \in \langle x \rangle + \langle y \rangle$.
Therefore, $\mathbb{Z} \subseteq \langle x \rangle + \langle y \rangle$.
Since we also have $\langle x \rangle + \langle y \rangle \subseteq \mathbb{Z}$, this proves $\langle x \rangle + \langle y \rangle = \mathbb{Z}$.
Therefore, if $\gcd(x, y) = 1$, then $\langle x \rangle + \langle y \rangle = \mathbb{Z}$.
Conversely, assume $\langle x \rangle + \langle y \rangle = \mathbb{Z}$.
Since $1 \in \mathbb{Z}$, we have $1 \in \langle x \rangle + \langle y \rangle$. We can therefore choose $m, n \in \mathbb{Z}$ with $1 = mx + ny$.
Therefore, $\gcd(x, y) = 1$.
Therefore, if $\langle x \rangle + \langle y \rangle = \mathbb{Z}$, then $\gcd(x, y) = 1$.
Therefore, $\forall x, y \in \mathbb{Z} \setminus \{0\}$, $\gcd(x, y) = 1$ if and only if $\langle x \rangle + \langle y \rangle = \mathbb{Z}$. □

19. $\forall a, x, y \in \mathbb{Z} \setminus \{0\}$, if $xy \in \langle a \rangle$ and $\gcd(a, x) = 1$, then $y \in \langle a \rangle$.

Proof.
Let $a, x, y \in \mathbb{Z} \setminus \{0\}$, and assume $xy \in \langle a \rangle$ and $\gcd(a, x) = 1$.
Choose $k \in \mathbb{Z}$ with $xy = ak$, and choose $s, t \in \mathbb{Z}$ with $1 = as + xt$.
Then $y = y(1) = asy + xyt = asy + akt = a(sy + kt)$.
Putting $m = sy + kt$ gives us $y = am$. Thus, $y \in \langle a \rangle$.
Therefore, $\forall a, x, y \in \mathbb{Z} \setminus \{0\}$, if $xy \in \langle a \rangle$ and $\gcd(a, x) = 1$, then $y \in \langle a \rangle$. □

21. $\forall a \in \mathbb{Z}$, if a is prime, then $\forall x \in \mathbb{Z}$, $\langle a \rangle + \langle x \rangle = \mathbb{Z}$ or $\langle a \rangle + \langle x \rangle = \langle a \rangle$.

Proof.
Let $a \in \mathbb{Z}$ and assume a is prime.
Let $x \in \mathbb{Z}$.
By proposition 2.3.11, we can choose $b \in \mathbb{Z}$ with $\langle a \rangle + \langle x \rangle = \langle b \rangle$.
Since $\langle a \rangle \subseteq \langle a \rangle + \langle x \rangle$, we have $\langle a \rangle \subseteq \langle b \rangle$.
Since a is prime, by proposition 2.3.14, we have $\langle b \rangle = \mathbb{Z}$ or $\langle b \rangle = \langle a \rangle$.
Thus, $\langle a \rangle + \langle x \rangle = \mathbb{Z}$ or $\langle a \rangle + \langle x \rangle = \langle a \rangle$.
Therefore, $\forall x \in \mathbb{Z}$, $\langle a \rangle + \langle x \rangle = \mathbb{Z}$ or $\langle a \rangle + \langle x \rangle = \langle a \rangle$.
Therefore, $\forall a \in \mathbb{Z}$, if a is prime, then $\forall x \in \mathbb{Z}$, $\langle a \rangle + \langle x \rangle = \mathbb{Z}$ or $\langle a \rangle + \langle x \rangle = \langle a \rangle$. □

23. (Euclid's Lemma) $\forall a \in \mathbb{Z}$, if a is prime, then $\forall x, y \in \mathbb{Z}$, if $xy \in \langle a \rangle$, then $x \in \langle a \rangle$ or $y \in \langle a \rangle$. (Hint: Combine the result of exercise 21, with those of exercises 9 and 19.)

Proof.
Let $a \in \mathbb{Z}$ and assume a is prime.
Let $x, y \in \mathbb{Z}$ and assume $xy \in \langle a \rangle$.
By exercise 21, $\langle a \rangle + \langle x \rangle = \mathbb{Z}$ or $\langle a \rangle + \langle x \rangle = \langle a \rangle$.
Case 1: $\langle a \rangle + \langle x \rangle = \mathbb{Z}$.
By exercise 17, $\gcd(a, x) = 1$, and so by exercise 19, $y \in \langle a \rangle$.
Case 2: $\langle a \rangle + \langle x \rangle = \langle a \rangle$.
By exercise 9, we then have $\langle x \rangle \subseteq \langle a \rangle$.
Since $x \in \langle x \rangle$, we then have $x \in \langle a \rangle$.
In either case, we have $x \in \langle a \rangle$ or $y \in \langle a \rangle$.
Therefore, $\forall x, y \in \mathbb{Z}$, if $xy \in \langle a \rangle$, then $x \in \langle a \rangle$ or $y \in \langle a \rangle$.
Therefore, $\forall a \in \mathbb{Z}$, if a is prime, then $\forall x, y \in \mathbb{Z}$, if $xy \in \langle a \rangle$, then $x \in \langle a \rangle$ or $y \in \langle a \rangle$. □

Let $(I_k)_{k\in\mathbb{N}}$ be a sequence of ideals. That is, for each $k \in \mathbb{N}$, $\exists a_k \in \mathbb{Z}$, $I_k = \langle a_k \rangle$. Prove the following propositions.

25. $\forall x \in \mathbb{Z}$, if $\forall k \in \mathbb{N}$, $a_k \in \langle x \rangle$, then $\forall n \in \mathbb{N}$, $\displaystyle\sum_{k=1}^{n} a_k \in \langle x \rangle$.

Proof.
Let $x \in \mathbb{Z}$ and assume $\forall k \in \mathbb{N}$, $a_k \in \langle x \rangle$.
Let $A = \{n \in \mathbb{N} \mid \sum_{k=1}^{n} a_k \in \langle x \rangle\}$.
Since $\sum_{k=1}^{1} a_k = a_1 \in \langle x \rangle$, we have $1 \in A$.
Let $n \in \mathbb{N}$ and assume $n \in A$.
Then $\sum_{k=1}^{n} a_k \in \langle x \rangle$. Choose $t \in \mathbb{Z}$ with $\sum_{k=1}^{n} a_k = xt$.
Also, since $a_{n+1} \in \langle x \rangle$, we can choose $s \in \mathbb{Z}$ with $a_{n+1} = xs$.
Putting $q = t + s$ give us $\sum_{k=1}^{n+1} a_k = (\sum_{k=1}^{n} a_k) + a_{n+1} = xt + xs = x(t + s) = xq$.
Therefore, $\sum_{k=1}^{n+1} a_k \in \langle x \rangle$, and so $n + 1 \in A$.
Therefore, $\forall n \in \mathbb{N}$, if $n \in A$, then $n + 1 \in A$. By the PMI, $\mathbb{N} \subseteq A$.
Therefore, $\forall n \in \mathbb{N}$, $\displaystyle\sum_{k=1}^{n} a_k \in \langle x \rangle$.
Therefore, $\forall x \in \mathbb{Z}$, if $\forall k \in \mathbb{N}$, $a_k \in \langle x \rangle$, then $\forall n \in \mathbb{N}$, $\displaystyle\sum_{k=1}^{n} a_k \in \langle x \rangle$. □

2.4 Families of Sets

Exercises 2.4.

Let A, B, and C be sets whose elements belong to a common universe of discourse U. Prove the following propositions.

1. If $\mathcal{P}(A) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$, then $A \subseteq B$.

Proof.
Assume $\mathcal{P}(A) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$.
Since $A \subseteq A$, we have $A \in \mathcal{P}(A)$.
Since $\mathcal{P}(A) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$, we then have $A \in \mathcal{P}(A) \cap \mathcal{P}(B)$.
In particular $A \in \mathcal{P}(B)$, which means $A \subseteq B$.
Therefore, if $\mathcal{P}(A) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$, then $A \subseteq B$. □

3. (a) $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$.

Proof.
Let $S \in \mathcal{P}(A) \cup \mathcal{P}(B)$.
Then $S \in \mathcal{P}(A)$ or $S \in \mathcal{P}(B)$, which means $S \subseteq A$ or $S \subseteq B$.
Case 1: $S \subseteq A$.
Since $A \subseteq A \cup B$, we have $S \subseteq A \cup B$ by transitivity.
Thus, $S \in \mathcal{P}(A \cup B)$.
Case 2: $S \subseteq B$.
Since $B \subseteq A \cup B$, we then have $S \subseteq A \cup B$, and so $S \in \mathcal{P}(A \cup B)$.
Therefore, if $S \in \mathcal{P}(A) \cup \mathcal{P}(B)$, then $S \in \mathcal{P}(A \cup B)$.
Therefore, $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$. □

(c) if $\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$, then $A \subseteq B$ or $B \subseteq A$.

Proof.
Assume $\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$.
Since $A \cup B \subseteq A \cup B$, we have $A \cup B \in \mathcal{P}(A \cup B)$.
This implies, $A \cup B \in \mathcal{P}(A) \cup \mathcal{P}(B)$, which means $A \cup B \in \mathcal{P}(A)$ or $A \cup B \in \mathcal{P}(B)$.
Case 1: $A \cup B \in \mathcal{P}(A)$.
In this case, $A \cup B \subseteq A$.
Now, since $B \subseteq A \cup B$ and $A \cup B \subseteq A$, we have $B \subseteq A$.
Case 2: $A \cup B \in \mathcal{P}(B)$.
In this case, $A \cup B \subseteq B$.
Since $A \subseteq A \cup B$ and $A \cup B \subseteq B$, we have $A \subseteq B$.
Therefore, $A \subseteq B$ or $B \subseteq A$.
Therefore, if $\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$, then $A \subseteq B$ or $B \subseteq A$. □

4. (a) There are no sets A and B for which $\mathcal{P}(A \setminus B) \subseteq \mathcal{P}(A) \setminus \mathcal{P}(B)$.

Proof.
Let A and B be sets.
Since $\emptyset \subseteq A \setminus B$, we have $\emptyset \in \mathcal{P}(A \setminus B)$.
Since $\emptyset \subseteq B$, we have $\emptyset \in \mathcal{P}(B)$. Thus, $\emptyset \notin \mathcal{P}(A) \setminus \mathcal{P}(B)$.
Therefore, $\mathcal{P}(A \setminus B) \nsubseteq \mathcal{P}(A) \setminus \mathcal{P}(B)$.
Therefore, for all sets A and B, $\mathcal{P}(A \setminus B) \nsubseteq \mathcal{P}(A) \setminus \mathcal{P}(B)$. □

(c) if $A \cap B = \emptyset$, then $\mathcal{P}(A) \setminus \mathcal{P}(B) \subseteq \mathcal{P}(A \setminus B)$.

Proof.
Assume $A \cap B = \emptyset$.
Let $S \in \mathcal{P}(A) \setminus \mathcal{P}(B)$ and suppose $S \notin \mathcal{P}(A \setminus B)$.
Then $S \in \mathcal{P}(A)$ and $S \notin \mathcal{P}(B)$ and $S \notin \mathcal{P}(A \setminus B)$, which means $S \subseteq A$ and $S \nsubseteq B$ and $S \nsubseteq A \setminus B$.
Since $S \nsubseteq A \setminus B$, we have that $\exists x \in U$, $x \in S$ and $x \notin A \setminus B$. Choose such an x.
Then $x \in S$ and either $x \notin A$ or $x \in B$.
However, since $x \in S$ and $S \subseteq A$, we have $x \in A$.
This means it cannot be the case that $x \notin A$, and so it must be the case that $x \in B$.
Now, $x \in A$ and $x \in B$, which gives us $x \in A \cap B$, and hence $x \in \emptyset$, which is a contradiction.
Therefore, if $S \in \mathcal{P}(A) \setminus \mathcal{P}(B)$, then $S \in \mathcal{P}(A \setminus B)$.
That is, $\mathcal{P}(A) \setminus \mathcal{P}(B) \subseteq \mathcal{P}(A \setminus B)$.
Therefore, if $A \cap B = \emptyset$, then $\mathcal{P}(A) \setminus \mathcal{P}(B) \subseteq \mathcal{P}(A \setminus B)$. □

5. If $(\mathcal{P}(A))^c \subseteq \mathcal{P}(A^c)$, then $A = U$ or $A = \emptyset$.

Proof.
Suppose $(\mathcal{P}(A))^c \subseteq \mathcal{P}(A^c)$ and $A \neq U$ and $A \neq \emptyset$.
Since $A \neq U$ and $A \subseteq U$, we have $U \nsubseteq A$. This means $U \notin \mathcal{P}(A)$, and so $U \in (\mathcal{P}(A))^c$.
Since $(\mathcal{P}(A))^c \subseteq \mathcal{P}(A^c)$, this implies $U \in \mathcal{P}(A^c)$, meaning $U \subseteq A^c$.
Since $A \neq \emptyset$, we can choose an element $y \in U$ with $y \in A$.
Since $U \subseteq A^c$, we then have $y \in A^c$, which means $y \notin A$. This is a contradiction.
Therefore, if $(\mathcal{P}(A))^c \subseteq \mathcal{P}(A^c)$, then $A = U$ or $A = \emptyset$. □

7. Let $\mathcal{S} = \{A, B\}$. Then $\bigcap\limits_{S \in \mathcal{S}} S = A \cap B$.

Proof.
Let $x \in \bigcap_{S \in \mathcal{S}} S$.
This means $\forall S \in \mathcal{S}$, $x \in S$.
Since $A \in \mathcal{S}$, we have $x \in A$. Likewise, since $B \in \mathcal{S}$, we have $x \in B$.
We now, have $x \in A \cap B$.
Therefore, $\bigcap_{S \in \mathcal{S}} S \subseteq A \cap B$.
Conversely, let $x \in A \cap B$. This means $x \in A$ and $x \in B$.
Let $S \in \mathcal{S}$.
Then $S \in \{A, B\}$, and so $S = A$ or $S = B$.
In the case $S = A$, we have $x \in S$ since $x \in A$.
Likewise, in the case $S = B$, we have $x \in S$ since $x \in B$.
Therefore, $\forall S \in \mathcal{S}$, $x \in S$. This means $x \in \bigcap_{S \in \mathcal{S}} S$.
Therefore, $A \cap B \subseteq \bigcap_{S \in \mathcal{S}} S$. Thus, $\bigcap_{S \in \mathcal{S}} S = A \cap B$. □

Let $\mathcal{A} = \{S_k \mid k \in I\}$ be a family of sets with index set $I \neq \emptyset$, and let B be a set. Prove the following propositions.

9. $B \cap \bigcup_{k \in I} S_k = \bigcup_{k \in I}(B \cap S_k)$.

Proof.
Let $x \in B \cap \bigcup_{k \in I} S_k$.
Then $x \in B$ and $x \in \bigcup_{k \in I} S_k$.
This means $\exists k \in I, x \in S_k$. Choose $m \in I$ with $x \in S_m$.
We now have $x \in B$ and $x \in S_m$; hence $x \in B \cap S_m$.
Therefore, $\exists k \in I, x \in B \cap S_k$. Thus, $x \in \bigcup_{k \in I}(B \cap S_k)$.
Therefore, $B \cap \bigcup_{k \in I} S_k \subseteq \bigcup_{k \in I}(B \cap S_k)$.
Conversely, let $x \in \bigcup_{k \in I}(B \cap S_k)$.
This means $\exists k \in I, x \in B \cap S_k$. Choose $n \in I$ with $x \in B \cap S_n$.
Now, $x \in B \cap S_n$ means $x \in B$ and $x \in S_n$. In particular, $x \in B$.
Also, since $x \in S_n$, we have $\exists k \in I, x \in S_k$, which means $x \in \bigcup_{k \in I} S_k$.
We now have $x \in B$ and $x \in \bigcup_{k \in I} S_k$. Thus, $x \in B \cap \bigcup_{k \in I} S_k$.
Therefore, $\bigcup_{k \in I}(B \cap S_k) \subseteq B \cap \bigcup_{k \in I} S_k$, and so $B \cap \bigcup_{k \in I} S_k = \bigcup_{k \in I}(B \cap S_k)$. □

11. $B \setminus \bigcup_{k \in I} S_k = \bigcap_{k \in I}(B \setminus S_k)$.

Proof.
Let $x \in B \setminus \bigcup_{k \in I} S_k$
Then $x \in B$ and $x \notin \bigcup_{k \in I} S_k$. $x \notin \bigcup_{k \in I} S_k$ means $\forall k \in I, x \notin S_k$.
Let $k \in I$.
Then $x \notin S_k$, and since $x \in B$, we have $x \in B \setminus S_k$.
Therefore, $\forall k \in I, x \in B \setminus S_k$, which means $x \in \bigcap_{k \in I}(B \setminus S_k)$.
Therefore, $B \setminus \bigcup_{k \in I} S_k \subseteq \bigcap_{k \in I}(B \setminus S_k)$.
Conversely, let $x \in \bigcap_{k \in I}(B \setminus S_k)$.
Since $I \neq \emptyset$, choose $m \in I$.
Then $x \in B \setminus S_m$, which means $x \in B$ and $x \notin S_m$. In particular, $x \in B$.
Suppose $x \in \bigcup_{k \in I} S_k$. This means $\exists k \in I, x \in S_k$. Choose $n \in I$ with $x \in S_n$.
Since $n \in I$ and $x \in \bigcap_{k \in I}(B \setminus S_k)$, we must have $x \in B \setminus S_n$.
This gives us $x \in B$ and $x \notin S_n$, which is a contradiction, since $x \in S_n$.
Therefore, $x \notin \bigcup_{k \in I} S_k$.
We now have $x \in B$ and $x \notin \bigcup_{k \in I} S_k$. This means $x \in B \setminus \bigcup_{k \in I} S_k$.
Therefore, $\bigcap_{k \in I}(B \setminus S_k) \subseteq B \setminus \bigcup_{k \in I} S_k$. Thus, $B \setminus \bigcup_{k \in I} S_k = \bigcap_{k \in I}(B \setminus S_k)$. □

13. $\left(\bigcup_{k\in I} S_k\right)^c = \bigcap_{k\in I}(S_k)^c$.

Proof.
Let $x \in (\bigcup_{k\in I} S_k)^c$.
Then $x \notin \bigcup_{k\in I} S_k$.
That is, $\forall k \in I$, $x \notin S_k$.
Therefore, $\forall k \in I$, $x \in S_k^c$.
This means $x \in \bigcap_{k\in I}(S_k)^c$.
Therefore, $(\bigcup_{k\in I} S_k)^c \subseteq \bigcap_{k\in I}(S_k)^c$.
Conversely, let $x \in \bigcap_{k\in I}(S_k)^c$.
This means $\forall k \in I$, $x \in S_k^c$.
In other words, $\forall k \in I$, $x \notin S_k$.
This means $x \notin \bigcup_{k\in I} S_k$.
Hence, $x \in (\bigcup_{k\in I} S_k)^c$.
Therefore, $\bigcap_{k\in I}(S_k)^c \subseteq (\bigcup_{k\in I} S_k)^c$.
Therefore, $(\bigcup_{k\in I} S_k)^c = \bigcap_{k\in I}(S_k)^c$. □

15. If $\bigcup_{k\in I} S_k = \emptyset$, then $\forall m \in I$, $S_m = \emptyset$.

Proof.
Suppose $\bigcup_{k\in I} S_k = \emptyset$ and $\exists m \in I$, $S_m \neq \emptyset$.
Choose $m \in I$ with $S_m \neq \emptyset$, and since $S_m \neq \emptyset$, choose an element $x \in S_m$.
Now, since $x \in S_m$ and $m \in I$, we have $\exists k \in i$, $x \in S_k$. Thus, $x \in \bigcup_{k\in I} S_k$.
This gives us $x \in \emptyset$, which is a contradiction.
Therefore, if $\bigcup_{k\in I} S_k = \emptyset$, then $\forall m \in I$, $S_m = \emptyset$. □

17. $\forall J \in \mathcal{P}(I)$, $\bigcup_{k\in J} S_k \subseteq \bigcup_{k\in I} S_k$.

Proof.
Let $J \in \mathcal{P}(I)$. This means $J \subseteq I$.
Let $x \in \bigcup_{k\in J} S_k$.
Then, $\exists k \in J$, $x \in S_k$. Choose $k \in J$ with $x \in A_k$.
Since $k \in J$ and $J \subseteq I$, we have $k \in I$.
Therefore, $\exists k \in I$, $x \in S_k$. That is, $x \in \bigcup_{k\in I} S_k$.
Therefore, if $x \in \bigcup_{k\in J} S_k$, then $x \in \bigcup_{k\in I} S_k$. Hence, $\bigcup_{k\in J} S_k \subseteq \bigcup_{k\in I} S_k$.
Therefore, $\forall J \in \mathcal{P}(I)$, $\bigcup_{k\in J} S_k \subseteq \bigcup_{k\in I} S_k$. □

19. For all $n \in \mathbb{N}$, if $I = \{k \in \mathbb{N} \mid k \leq n\}$, then $\displaystyle\bigcap_{k \in I} S_k = \bigcap_{k=1}^{n} S_k$.

Proof.
Let $A = \{n \in \mathbb{N} \mid \text{if } I = \{k \in \mathbb{N} \mid k \leq n\}, \text{ then } \bigcap_{k \in I} S_k = \bigcap_{k=1}^{n} S_k\}$.
Assume $I = \{k \in \mathbb{N} \mid k \leq 1\}$.
Let $x \in \bigcap_{k \in I} S_k$.
Then $\forall k \in I$, $x \in S_k$. In particular, since $1 \in I$, we have $x \in S_1$, and hence $x \in \bigcap_{k=1}^{1} S_k$.
Therefore, $\bigcap_{k \in I} S_k \subseteq \bigcap_{k=1}^{1} S_k$.
Conversely, let $x \in \bigcap_{k=1}^{1} S_k$. That is, $x \in S_1$.
Let $k \in I$.
Then $k \in \mathbb{N}$ and $k \leq 1$, so $k = 1$. Since $x \in S_1$, we then have $x \in S_k$.
Therefore, $\forall k \in I$, $x \in S_k$, which means $x \in \bigcap_{k \in I} S_k$.
Therefore, $\bigcap_{k=1}^{1} S_k \subseteq \bigcap_{k \in I} S_k$. Thus, $\bigcap_{k \in I} S_k = \bigcap_{k=1}^{1} S_k$.
This proves, if $I = \{k \in \mathbb{N} \mid k \leq 1\}$, then $\bigcap_{k \in I} S_k = \bigcap_{k=1}^{1} S_k$. Thus, $1 \in A$.
Now, let $n \in \mathbb{N}$ and assume $n \in A$.
Let $J = \{k \in \mathbb{N} \mid k \leq n\}$, and let $I = \{k \in \mathbb{N} \mid k \leq n + 1\}$.
Since $\forall k \in \mathbb{N}$, if $k \leq n$, then $k \leq n + 1$, we have $J \subseteq I$. Therefore, $\bigcap_{k \in I} S_k \subseteq \bigcap_{k \in J} S_k$.
Further, since $n \in S$, we have $\bigcap_{k \in J} S_k = \bigcap_{k=1}^{n} S_k$.
Let $x \in \bigcap_{k \in I} S_k$.
Since $n + 1 \in I$, we have $x \in S_{n+1}$.
Further, since $\bigcap_{k \in I} S_k \subseteq \bigcap_{k \in J} S_k$, we have $x \in \bigcap_{k \in J} S_k$. Hence, $x \in \bigcap_{k=1}^{n} S_k$.
Now, $x \in \bigcap_{k=1}^{n} S_k$ and $x \in S_{n+1}$. That is, $x \in (\bigcap_{k=1}^{n} S_k) \cap S_{n+1}$, which means $x \in \bigcap_{k=1}^{n+1} S_k$.
Therefore, $\bigcap_{k \in I} S_k \subseteq \bigcap_{k=1}^{n+1} S_k$.
Conversely, let $x \in \bigcap_{k=1}^{n+1} S_k$.
Then $x \in (\bigcap_{k=1}^{n} S_k) \cap S_{n+1}$, so $x \in \bigcap_{k \in J} S_k$ and $x \in S_{n+1}$.
Let $k \in I$. This means $k \leq n + 1$.
In then case $k = n + 1$, we have $x \in S_k$, since $x \in S_{n+1}$.
In case $k < n + 1$, we have $k \leq n$, and hence $k \in J$. In this case, $x \in S_k$, since $x \in \bigcap_{k \in J} S_k$.
Therefore, $\forall k \in I$, $x \in S_k$. Hence, $x \in \bigcap_{k \in I} S_k$.
Therefore, $\bigcap_{k=1}^{n+1} S_k \subseteq \bigcap_{k \in I} S_k$. Thus, $\bigcap_{k \in I} S_k = \bigcap_{k=1}^{n+1} S_k$.
Therefore, if $I = \{k \in \mathbb{N} \mid k \leq n + 1\}$, then $\bigcap_{k \in I} S_k = \bigcap_{k=1}^{n+1} S_k$. Thus, $n + 1 \in A$.
Therefore, $\forall n \in \mathbb{N}$, if $n \in A$, then $n + 1 \in A$, and so by the PMI, we have $\mathbb{N} \subseteq A$.
Therefore, $\forall n \in \mathbb{N}$, if $I = \{k \in \mathbb{N} \mid k \leq n\}$, then $\bigcap_{k \in I} S_k = \bigcap_{k=1}^{n} S_k$. $\square$

21. If $\forall k \in I$, $S_k \subseteq \mathbb{R}$ is an interval, then $\displaystyle\bigcap_{k \in I} S_k$ is an interval. (Hint: Use theorems 2.2.9 and 2.2.11).

Proof.
Assume $\forall k \in I$, $S_k \subseteq \mathbb{R}$ is an interval.
Let $x, y \in \bigcap_{k \in I} S_k$, let $z \in \mathbb{R}$, and assume $x < z < y$.
Let $k \in I$.
Since $x, y \in \bigcap_{k \in I} S_k$, we have $x, y \in S_k$.
Since S_k is an interval and $x < z < y$, we have $z \in S_k$ by theorem 2.2.9.
Therefore, $\forall k \in I$, $z \in S_k$. This means $z \in \bigcap_{k \in I} S_k$.
Therefore, $\forall x, y \in \bigcap_{k \in I} S_k$, $\forall z \in \mathbb{R}$, if $x < z < y$, then $z \in \bigcap_{k \in I} S_k$.
Therefore, $\bigcap_{k \in I} S_k$ is an interval by theorem 2.2.11.
Therefore, if $\forall k \in I$, $S_k \subseteq \mathbb{R}$ is an interval, then $\bigcap_{k \in I} S_k$ is an interval. $\square$

23. Let $(S_k)_{k\in\mathbb{N}}$ be a sequence of ideals of $\mathbb{Z}$. If $\forall k \in \mathbb{N}$, $S_k \subseteq S_{k+1}$, then $\displaystyle\bigcup_{k\in\mathbb{N}} S_k$ is an ideal.

Proof.
Let $(S_k)_{k\in\mathbb{N}}$ be a sequence of ideals of $\mathbb{Z}$, and assume $\forall k \in \mathbb{N}$, $S_k \subseteq S_{k+1}$.
Since $0 \in S_1$, we have that $\exists k \in \mathbb{N}$, $0 \in S_k$. Thus, $0 \in \bigcup_{k\in\mathbb{N}} S_k$, which proves $\bigcup_{k\in\mathbb{N}} S_k \neq \emptyset$.
Next, let $x, y \in \bigcup_{k\in\mathbb{N}} S_k$.
Choose $m, n \in \mathbb{N}$ with $x \in S_m$ and $y \in S_n$.
Case 1: $m \leq n$.
Since $\forall k \in \mathbb{N}$, $S_k \subseteq S_{k+1}$, we have $S_m \subseteq S_n$ by exercise 2.1.40.
Now, since $x \in S_m$, we have $x \in S_n$, and so $x, y \in S_n$.
Since S_n is an ideal, we have $x + y \in S_n$ by exercise 1.
Therefore, $\exists k \in \mathbb{N}$, $x + y \in S_k$. Thus, $x + y \in \bigcup_{k\in\mathbb{N}} S_k$.
Case 2: $n \leq m$.
Similar to the previous case, we have $S_n \subseteq S_m$ by exercise 2.1.40, and so $y \in S_m$.
Now, since $x, y \in S_m$, and S_m is an ideal, we have $x + y \in S_m$ by exercise 1.
Therefore, $\exists k \in \mathbb{N}$, $x + y \in S_k$, and so $x + y \in \bigcup_{k\in\mathbb{N}} S_k$.
Therefore, $\forall x, y \in \bigcup_{k\in\mathbb{N}} S_k$, $x + y \in \bigcup_{k\in\mathbb{N}} S_k$.
Finally, let $x \in \bigcup_{k\in\mathbb{N}} S_k$, and choose $k \in \mathbb{N}$ with $x \in S_k$.
Since S_k is an ideal, we have $-x \in S_k$.
Therefore, $\exists k \in \mathbb{N}$, $-x \in S_k$, and hence $-x \in \bigcup_{k\in\mathbb{N}} S_k$.
Therefore, $\forall x \in \bigcup_{k\in\mathbb{N}} S_k$, $-x \in \bigcup_{k\in\mathbb{N}} S_k$.
By theorem 2.3.7, $\bigcup_{k\in\mathbb{N}} S_k$ is an ideal.
Therefore, if $\forall k \in \mathbb{N}$, $S_k \subseteq S_{k+1}$, then $\displaystyle\bigcup_{k\in\mathbb{N}} S_k$ is an ideal. □

Prove the following propositions.

25. $\bigcup_{k\in\mathbb{Z}} \langle k \rangle = \mathbb{Z}$.

Proof.

$\bigcup_{k\in\mathbb{Z}}\langle k\rangle \subseteq \mathbb{Z}$, since $\mathbb{Z}$ is the universe of discourse.
Conversely, let $x \in \mathbb{Z}$.
Putting $k = 1$ gives us $x = x(1) = xk$, and so $x \in \langle k\rangle$.
Therefore, $\exists k \in \mathbb{Z}, x \in \langle k\rangle$. This means $x \in \bigcup_{k\in\mathbb{Z}}\langle k\rangle$.
Therefore, $\mathbb{Z} \subseteq \bigcup_{k\in\mathbb{Z}}\langle k\rangle$. Thus, $\bigcup_{k\in\mathbb{Z}}\langle k\rangle = \mathbb{Z}$. □

27. $\bigcap_{n\in\mathbb{N}} [n, \infty) = \emptyset$.

Proof.
Suppose $\bigcap_{n\in\mathbb{N}}[n,\infty) \neq \emptyset$.
We can then choose a real number $x \in \bigcap_{n\in\mathbb{N}}[n,\infty)$.
By the Archimedean property, we can choose $n \in \mathbb{N}$ with $x < n$.
Since $n \in \mathbb{N}$ and $x \in \bigcap_{n\in\mathbb{N}}[n,\infty)$, we have $x \in [n,\infty)$. This means $n \leq x$.
Now, $x < n$ and $n \leq x$, which is a contradiction.
Therefore, $\bigcap_{n\in\mathbb{N}}[n,\infty) = \emptyset$. □

29. $\bigcup_{n\in\mathbb{N}} [0, n) = [0, \infty)$.

Proof.

Let $x \in \bigcup_{n\in\mathbb{N}}[0,n)$.
Then $\exists n \in \mathbb{N}, x \in [0,n)$. Choose such an n.
Then $x \in [0,n)$, which means $0 \leq x < n$. In particular, $0 \leq x$; hence $x \in [0,\infty)$.
Therefore, $\bigcup_{n\in\mathbb{N}}[0,n) \subseteq [0,\infty)$.
Conversely, let $x \in [0,\infty)$. i.e. $0 \leq x$.
By the Archimedean property, choose $n \in \mathbb{N}$ with $x < n$.
Now, $0 \leq x$ and $x < n$, which gives us $x \in [0,n)$.
Therefore, $\exists n \in \mathbb{N}, x \in [0,n)$, which means $x \in \bigcup_{n\in\mathbb{N}}[0,n)$.
Therefore, $[0,\infty) \subseteq \bigcup_{n\in\mathbb{N}}[0,n)$. Thus, $\bigcup_{n\in\mathbb{N}}[0,n) = [0,\infty)$. □

31. $\bigcap_{x\in\mathbb{R}} (-\infty, x) = \emptyset$.

Proof.
Suppose $\bigcap_{x\in\mathbb{R}}(-\infty,x) \neq \emptyset$.
Accordingly, choose a real number $y \in \bigcap_{x\in\mathbb{R}}(-\infty,x)$.
Since $y \in \mathbb{R}$, we must then have $y \in (-\infty,y)$; hence $y < y$. This is a contradiction.
Therefore, $\bigcap_{x\in\mathbb{R}}(-\infty,x) = \emptyset$. □

33. $\displaystyle\bigcup_{n\in\mathbb{N}}[1,3n) = [1,\infty)$.

Proof.
Let $x \in \mathbb{R}$ and assume $x \in \bigcup_{n\in\mathbb{N}}[1,3n)$.
This means $\exists n \in \mathbb{N}, x \in [1,3n)$. Choose such an n.
Since $x \in [1,3n)$, we have $1 \leq x$ and $x < 3n$.
Since $1 \leq x$, we have $x \in [1,\infty)$.
Therefore, $\bigcup_{n\in\mathbb{N}}[1,3n) \subseteq [1,\infty)$.
Conversely, let $x \in \mathbb{R}$ and assume $x \in [1,\infty)$. i.e. $1 \leq x$.
By the Archimedean property, since $0 < 3$, $\exists n \in \mathbb{N}, x < 3n$. Choose such an n.
Now, $1 \leq x$ and $x < 3n$, so $x \in [1,3n)$.
Therefore, $\exists n \in \mathbb{N}, x \in [1,3n)$. This means $x \in \bigcup_{n\in\mathbb{N}}[1,3n)$.
Therefore, $[1,\infty) \subseteq \bigcup_{n\in\mathbb{N}}[1,3n)$. Thus, $\bigcup_{n\in\mathbb{N}}[1,3n) = [1,\infty)$. □

35. $\displaystyle\bigcap_{n\in\mathbb{N}}\left(-\infty,\frac{1}{n}\right] = (-\infty,0]$.

Proof.
Let $x \in \mathbb{R}$ and assume $x \in \bigcap_{n\in\mathbb{N}}(-\infty,\frac{1}{n}]$ and $x \notin (-\infty,0]$.
Since $x \notin (-\infty,0]$, we have $0 < x$. By the Archimedean property, we can choose $n \in \mathbb{N}$ with $\frac{1}{n} < x$.
However, since $n \in \mathbb{N}$ and $x \in \bigcap_{n\in\mathbb{N}}(-\infty,\frac{1}{n}]$, we have $x \in (-\infty,\frac{1}{n}]$. Thus, $x \leq \frac{1}{n}$.
We now have the contradiction $\frac{1}{n} < x$ and $x \leq \frac{1}{n}$.
Therefore, $\bigcap_{n\in\mathbb{N}}(-\infty,\frac{1}{n}] \subseteq (-\infty,0]$.
Conversely, let $x \in \mathbb{R}$ and assume $x \in (-\infty,0]$. i.e. $x \leq 0$.
Let $n \in \mathbb{N}$.
Then $0 < n$, and so $0 < \frac{1}{n}$.
By transitivity, we then have $x \leq \frac{1}{n}$, which means $x \in (-\infty,\frac{1}{n}]$.
Therefore, $\forall n \in \mathbb{N}, x \in (-\infty,\frac{1}{n}]$. Thus, $x \in \bigcap_{n\in\mathbb{N}}(-\infty,\frac{1}{n}]$.
Therefore, $(-\infty,0] \subseteq \bigcap_{n\in\mathbb{N}}(-\infty,\frac{1}{n}]$. Thus, $\bigcap_{n\in\mathbb{N}}(-\infty,\frac{1}{n}] = (-\infty,0]$. □

37. $\displaystyle\bigcup_{a\in(-\infty,1)}[0,2+a) = [0,3)$.

Proof.
Let $x \in \mathbb{R}$ and assume $x \in \bigcup_{a\in(-\infty,1)}[0,2+a)$.
This means $\exists a \in (-\infty,1), x \in [0,2+a)$. Choose such an a.
Then $a \in (-\infty,1)$ and $x \in [0,2+a)$, which means $a < 1$, $0 \leq x$ and $x < 2+a$.
Since $a < 1$, we have $2+a < 3$, and since $x < 2+a$, we have $x < 3$ by transitivity.
Now, $0 \leq x$ and $x < 3$ gives us $x \in [0,3)$.
Therefore, $\bigcup_{a\in(-\infty,1)}[0,2+a) \subseteq [0,3)$.
Conversely, let $x \in \mathbb{R}$ and assume $x \in [0,3)$. i.e. $0 \leq x$ and $x < 3$.
Put $a = \frac{x-1}{2}$.
Since $x < 3$, we have $x-1 < 2$; hence $a < 1$. This means $a \in (-\infty,1)$.
Also, since $x < 3$, adding $x-4$ to both sides gives us $2x-4 < x-1$.
Dividing both sides by 2 then gives us $x-2 < a$; hence $x < 2+a$. This proves $x \in [0,2+a)$.
Therefore, $\exists a \in (-\infty,1), x \in [0,2+a)$, and so $x \in \bigcup_{a\in(-\infty,1)}[0,2+a)$.
Therefore, $[0,3) \subseteq \bigcup_{a\in(-\infty,1)}[0,2+a)$. Thus, $\bigcup_{a\in(-\infty,1)}[0,2+a) = [0,3)$. □

39. $\displaystyle\bigcap_{a\in(-\infty,1)} [0, 2-a] = [0, 1]$.

Proof.
Let $x \in \mathbb{R}$ and assume $x \in \bigcap_{a\in(-\infty,1)}[0, 2-a]$.
Since $0 \in (-\infty, 1)$, we have $x \in [0, 2-0]$; hence $0 \le x$ and $x \le 2$.
Suppose $1 < x$.
Take $a = \frac{3-x}{2}$.
Since $1 < x$, we have $3 - x < 3 - 1$; hence $\frac{3-x}{2} < 1$. This shows, $a < 1$; hence $a \in (-\infty, 1)$.
Since $x \in \bigcap_{a\in(-\infty,1)}[0, 2-a]$, we then have $x \in [0, 2-a]$; hence $x \le 2 - a$.
Therefore, $x \le 2 - \frac{3-x}{2}$, giving us $2x \le 4 - (3 - x)$. This simplifies to $x \le 1$, which is a contradiction.
Therefore, $x \le 1$. Since $0 \le x$ and $x \le 1$, we have $x \in [0, 1]$.
Therefore, $\bigcap_{a\in(-\infty,1)}[0, 2-a] \subseteq [0, 1]$.
Conversely, let $x \in \mathbb{R}$ and assume $x \in [0, 1]$. i.e. $0 \le x$ and $x \le 1$.
Let $a \in (-\infty, 1)$. i.e. $a < 1$.
Since $a < 1$, we have $2 - 1 < 2 - a$; hence $1 < a$.
Since $x \le 1$ and $1 < 2 - a$, we have $x \le 2 - a$.
Now, $0 \le x$ and $x \le 2 - a$, which means $x \in [0, 2-a]$.
Therefore, $\forall a \in (-\infty, 1)$, $x \in [0, 2-a]$, which means $x \in \bigcap_{a\in(-\infty,1)}[0, 2-a]$.
Therefore, $[0, 1] \subseteq \bigcap_{a\in(-\infty,1)}[0, 2-a]$. Thus, $\bigcap_{a\in(-\infty,1)}[0, 2-a] = [0, 1]$. □

41. $\displaystyle\bigcup_{a\in(0,\infty)} [a, 2] = (0, 2]$.

Proof.
Let $x \in \mathbb{R}$ and assume $x \in \bigcup_{a\in(0,\infty)}[a, 2]$.
This means $\exists a \in (0, \infty)$, $x \in [a, 2]$. Choose such an a.
Then $a \in (0, \infty)$ and $x \in [a, 2]$, which means $0 < a$, $a \le x$ and $x \le 2$.
Since $0 < a$ and $a \le x$, we have $0 < x$ by transitivity. Now $0 < x$ and $x \le 2$, so $x \in (0, 2]$.
Therefore, $\bigcup_{a\in(0,\infty)}[a, 2] \subseteq (0, 2]$.
Conversely, let $x \in \mathbb{R}$ and assume $x \in (0, 2]$. i.e. $0 < x$ and $x \le 2$.
Put $a = x$.
Since $0 < x$, we have $0 < a$; hence $a \in (0, \infty)$.
Also, since $x = a$, we have $a \le x$, and since we also have $x \le 2$, this gives us $x \in [a, 2]$.
Therefore, $\exists a \in (0, \infty)$, $x \in [a, 2]$, and so $x \in \bigcup_{a\in(0,\infty)}[a, 2]$.
Therefore, $(0, 2] \subseteq \bigcup_{a\in(0,\infty)}[a, 2]$. Thus, $\bigcup_{a\in(0,\infty)}[a, 2] = (0, 2]$. □

43. $\displaystyle\bigcap_{a\in(0,\infty)} (1-a, 2] = [1, 2]$.

Proof.
Let $x \in \mathbb{R}$ and assume $x \in \bigcap_{a\in(0,\infty)}(1-a, 2]$ and $x \notin [1, 2]$. Then $x < 1$ or $2 < x$.
Since $1 \in (0, \infty)$, we have $x \in (1-1, 2]$; hence $x \leq 2$. This means it must be the case that $x < 1$.
Take $a = 1-x$. Since $x < 1$, we have $0 < 1-x$, which means $a \in (0, \infty)$.
This implies $x \in (1-a, 2]$; hence $1-a < x$. But, $1-a = x$, so this is a contradiction.
Therefore, $\bigcap_{a\in(0,\infty)}(1-a, 2] \subseteq [1, 2]$.
Conversely, let $x \in \mathbb{R}$ and assume $x \in [1, 2]$. i.e. $1 \leq x$ and $x \leq 2$.
Let $a \in (0, \infty)$. i.e. $0 < a$.
Since $0 < a$, we have $1-a < 1$. Since $1 \leq x$, this gives us $1-a < x$. Thus, $x \in (1-a, 2]$.
Therefore, $\forall a \in (0, \infty)$, $x \in (1-a, 2]$, which means $x \in \bigcap_{a\in(0,\infty)}(1-a, 2]$.
Therefore, $[1, 2] \subseteq \bigcap_{a\in(0,\infty)}(1-a, 2]$. Thus, $\bigcap_{a\in(0,\infty)}(1-a, 2] = [1, 2]$. □

45. $\displaystyle\bigcup_{a\in(-\infty,1)} (1-a, 2+a) = (0, 3)$.

Proof.
Let $x \in \mathbb{R}$ and assume $x \in \bigcup_{a\in(-\infty,1)}(1-a, 2+a)$.
Choose $a \in (-\infty, 1)$ with $x \in (1-a, 2+a)$. Then $a < 1$, $1-a < x$, and $x < 2+a$.
Since $a < 1$, we have $0 < 1-a$; hence $0 < x$ by transitivity.
Also, since $a < 1$, we have $2+a < 3$; hence $x < 3$ by transitivity. Therefore, $x \in (0, 3)$.
Therefore, $\bigcup_{a\in(-\infty,1)}(1-a, 2+a) \subseteq (0, 3)$
Conversely, let $x \in \mathbb{R}$ and assume $x \in (0, 3)$. i.e. $0 < x$ and $x < 3$.
Put $a = \max(\frac{2-x}{2}, \frac{x-1}{2})$.
Since $0 < x$, $2-x < 2$, so $\frac{2-x}{2} < 1$. Since $x < 3$, $x-1 < 2$, so $\frac{x-1}{2} < 1$. Thus, $a < 1$, so $a \in (-\infty, 1)$.
Now, since $\frac{2-x}{2} \leq a$, we have $2-x \leq 2a$; hence $1-a \leq \frac{x}{2}$. Therefore, $1-a < x$.
Also, since $\frac{x-1}{2} \leq a$, we have $x \leq 1+2a$.
Adding $x \leq 1+2a$ and $x < 3$ gives $2x < 4+2a$; hence $x < 2+a$. Thus, $x \in (1-a, 2+a)$.
Therefore, $\exists a \in (-\infty, 1)$, $x \in (1-a, 2+a)$, and so $x \in \bigcup_{a\in(-\infty,1)}(1-a, 2+a)$.
Therefore, $(0, 3) \subseteq \bigcup_{a\in(-\infty,1)}(1-a, 2+a)$. Thus, $\bigcup_{a\in(-\infty,1)}(1-a, 2+a) = (0, 3)$. □

47. $\displaystyle\bigcap_{a\in(-\infty,1)} [a, 2-a] = \{1\}$.

Proof.
Let $x \in \bigcap_{a\in(-\infty,1)}[a, 2-a]$ and suppose $x \notin \{1\}$. i.e. $x \neq 1$.
Case 1: $x < 1$.
Taking $a = \frac{x+1}{2}$ gives us $x < a$ and $a < 1$. Then $a \in (-\infty, 1)$, so $x \in [a, 2-a]$, which contradicts $x < a$.
Case 2: $1 < x$. In this case, $2-x < 1$.
Take $b = \frac{2-x+1}{2}$. Then $2-x < b < 1$, so $b \in (-\infty, 1)$. Then $x \in [b, 2-b]$, which contradicts $2-x < b$.
Therefore, $\bigcap_{a\in(-\infty,1)}[a, 2-a] \subseteq \{1\}$.
Conversely, let $x \in \{1\}$. i.e. $x = 1$.
Let $a \in (-\infty, 1)$. This means $a < 1$; hence $a \leq 1$.
Since $a \leq 1$, we have $a \leq x$. Since $a \leq 1$, $1 \leq 2-a$; hence $x \leq 2-a$. Therefore, $x \in [a, 2-a]$.
Therefore, $\forall a \in (-\infty, 1)$, $x \in [a, 2-a]$. This means $x \in \bigcap_{a\in(-\infty,1)}[a, 2-a]$.
Therefore, $\{1\} \subseteq \bigcap_{a\in(-\infty,1)}[a, 2-a]$, and so $\bigcap_{a\in(-\infty,1)}[a, 2-a] = \{1\}$. □

49. $\displaystyle\bigcup_{a\in(0,1)}[a, 2+a) = (0, 3)$.

Proof.
Let $x \in \bigcup_{a\in(0,1)}[a, 2+a)$.
Then $\exists a \in (0, 1)$, $x \in [a, 2+a)$. Choose such an a.
For this a, we have $0 < a$, $a < 1$, $a \leq x$, and $x < 2+a$.
Since $0 < a$ and $a \leq x$, we have $0 < x$. Since $a < 1$, $2+a < 3$; hence $x < 3$ by transitivity.
Now, $0 < x$ and $x < 3$, which means $x \in (0, 3)$.
Therefore, $\bigcup_{a\in(0,1)}[a, 2+a) \subseteq (0, 3)$.
Conversely, let $x \in (0, 3)$. i.e. $0 < x$ and $x < 3$.
Case 1: $x \leq 1$.
Put $a = \frac{x}{2}$.
Since $0 < x$, we have $0 < a$, and since $x \leq 1$, we have $a \leq \frac{1}{2}$; hence $a < 1$. Thus, $a \in (0, 1)$.
Also, since $0 < x$, we have $\frac{x}{2} < x$; hence $a < x$. This implies $a \leq x$.
Since $0 < a$ and $1 < 2$, we have $1 < 2+a$. Since $x \leq 1$, we have $x < 2+a$. Thus, $x \in [a, 2+a)$.
Therefore, $\exists a \in (0, 1)$, $x \in [a, 2+a)$. This means $x \in \bigcup_{a\in(0,1)}[a, 2+a)$.
Case 2: $1 < x$.
Put $a = \frac{x-1}{2}$.
Since $1 < x$, we have $0 < x-1$; hence $0 < a$. Since $x < 3$, we have $x-1 < 2$; hence $a < 1$.
Thus, $a \in (0, 1)$.
Also, since $a < 1$ and $1 < x$, we have $a < x$. This implies $a \leq x$.
Since $x < 3$, we have $2x < 3+x$, so $2x < 4+(x-1)$. Dividing by 2 gives $x < 2+a$. Thus, $x \in [a, 2+a)$.
Therefore, $\exists a \in (0, 1)$, $x \in [a, 2+a)$. Hence, $x \in \bigcup_{a\in(0,1)}[a, 2+a)$.
Therefore, $(0, 3) \subseteq \bigcup_{a\in(0,1)}[a, 2+a)$. Thus, $\bigcup_{a\in(0,1)}[a, 2+a) = (0, 3)$. □

51. $\displaystyle\bigcap_{a\in(0,1)}(1-a, 2-a] = \{1\}$.

Proof.
Let $x \in \bigcap_{a\in(0,1)}(1-a, 2-a]$ and suppose $x \notin \{1\}$. i.e. $x \neq 1$.
First, since $\frac{1}{2} \in (0, 1)$, we have $x \in (\frac{1}{2}, \frac{3}{2}]$. Therefore, $\frac{1}{2} < x$ and $x \leq \frac{3}{2}$.
This implies $0 < x$ and $x < 2$ by transitivity, since $0 < \frac{1}{2}$ and $\frac{3}{2} < 2$.
Case 1: $x < 1$.
Taking $a = 1-x$, we have since $x < 1$, $0 < a$. Since $0 < x$, we have $1-x < 1$, and so $a < 1$.
This proves $a \in (0, 1)$. It follows that $x \in (1-a, 2-a]$. In particular, $1-a < x$.
However, $1-a = x$, so this is a contradiction.
Case 2: $1 < x$.
Take $b = \frac{3-x}{2}$. Since $x < 2$, $x < 3$; hence $0 < 3-x$, so $0 < b$. Also, since $1 < x$, $3-x < 2$, so $b < 1$.
Therefore, $b \in (0, 1)$, which implies $x \in (1-b, 2-b]$. In particular, $x \leq 2-b$, and so $2x \leq 4-2b$.
This can be written as $2x \leq 4-(3-x)$, which implies $x \leq 1$. This is a contradiction, since $1 < x$.
Therefore, $\bigcap_{a\in(0,1)}(1-a, 2-a] \subseteq \{1\}$.
Conversely, let $x \in \{1\}$. i.e. $x = 1$.
Let $a \in (0, 1)$. i.e. $0 < a$ and $a < 1$.
Since $0 < a$, we have $1-a < 1$; hence $1-a < x$. Since $a < 1$, we have $1 < 2-a$; hence $x < 2-a$.
We now have $1-a < x$ and $x \leq 2-a$. Therefore, $x \in (1-a, 2-a]$.
Therefore, $\forall a \in (0, 1)$, $x \in (1-a, 2-a]$. That is, $x \in \bigcap_{a\in(0,1)}(1-a, 2-a]$.
Therefore, $\{1\} \subseteq \bigcap_{a\in(0,1)}(1-a, 2-a]$. Thus, $\bigcap_{a\in(0,1)}(1-a, 2-a] = \{1\}$. □

53. $\displaystyle\bigcup_{n\in\mathbb{N}}\left[0, 2+\frac{1}{n}\right) = [0,3)$.

Proof.
Let $x \in \bigcup_{n\in\mathbb{N}}[0, 2+\frac{1}{n})$.
Then $\exists n \in \mathbb{N}$, $x \in [0, 2+\frac{1}{n})$. Choosing such an n, we have $0 \leq x$ and $x < 2+\frac{1}{n}$.
Since $n \in \mathbb{N}$, we have $1 \leq n$, and so $\frac{1}{n} \leq 1$. This gives us $2+\frac{1}{n} \leq 3$; hence $x < 3$. Thus, $x \in [0,3)$.
Therefore, $\bigcup_{n\in\mathbb{N}}\left[0, 2+\frac{1}{n}\right) \subseteq [0,3)$.
Conversely, let $x \in [0,3)$. i.e. $0 \leq x$ and $x < 3$.
Put $m = 1$.
$2+\frac{1}{m} = 2+1 = 3$, and since $x < 3$, we have $x < 2+\frac{1}{m}$. Thus, $x \in [0, 2+\frac{1}{m})$.
Therefore, $\exists m \in \mathbb{N}$, $x \in [0, 2+\frac{1}{m})$. Thus, $x \in \bigcup_{n\in\mathbb{N}}[0, 2+\frac{1}{n})$.
Therefore, $[0,3) \subseteq \bigcup_{n\in\mathbb{N}}[0, 2+\frac{1}{n})$, and so $\bigcup_{n\in\mathbb{N}}[0, 2+\frac{1}{n}) = [0,3)$. □

55. $\displaystyle\bigcap_{n\in\mathbb{N}}\left[0, 2-\frac{1}{n}\right] = [0,1]$.

Proof.
Let $x \in \bigcap_{n\in\mathbb{N}}[0, 2-\frac{1}{n}]$. This means $\forall n \in \mathbb{N}$, $x \in [0, 2-\frac{1}{n}]$.
Since $1 \in \mathbb{N}$, we then have $x \in [0, 2-\frac{1}{1}]$; hence $x \in [0,1]$.
Therefore, $\bigcap_{n\in\mathbb{N}}[0, 2-\frac{1}{n}] \subseteq [0,1]$.
Conversely, let $x \in [0,1]$. i.e. $0 \leq x$ and $x \leq 1$.
Let $n \in \mathbb{N}$.
Then $1 \leq n$, so $\frac{1}{n} \leq 1$. Then $1 \leq 2-\frac{1}{n}$, which implies $x \leq 2-\frac{1}{n}$. Thus, $x \in [0, 2-\frac{1}{n}]$.
Therefore, $\forall n \in \mathbb{N}$, $x \in [0, 2-\frac{1}{n}]$, which means $x \in \bigcap_{n\in\mathbb{N}}[0, 2-\frac{1}{n}]$.
Therefore, $[0,1] \subseteq \bigcap_{n\in\mathbb{N}}[0, 2-\frac{1}{n}]$. Thus, $\bigcap_{n\in\mathbb{N}}[0, 2-\frac{1}{n}] = [0,1]$. □

57. $\displaystyle\bigcup_{n\in\mathbb{N}}\left[\frac{1}{n}, 2\right] = (0,2]$.

Proof.
Let $x \in \bigcup_{n\in\mathbb{N}}[\frac{1}{n}, 2]$.
This means $\exists n \in \mathbb{N}$, $x \in [\frac{1}{n}, 2]$. Choosing such an n, we have $\frac{1}{n} \leq x$ and $x \leq 2$.
Since $0 < n$, we have $0 < \frac{1}{n}$, and so by transitivity, $0 < x$. Since also, $x \leq 2$, we have $x \in (0,2]$.
Therefore, $\bigcup_{n\in\mathbb{N}}[\frac{1}{n}, 2] \subseteq (0,2]$.
Conversely, let $x \in (0,2]$. Then $0 < x$ and $x \leq 2$.
Since $0 < x$, we have choose $n \in \mathbb{N}$ with $\frac{1}{n} < x$, by the Archimedean property.
Then $\frac{1}{n} \leq x$ and $x \leq 2$, so $x \in [\frac{1}{n}, 2]$.
Therefore, $\exists n \in \mathbb{N}$, $x \in [\frac{1}{n}, 2]$. Thus, $x \in \bigcup_{n\in\mathbb{N}}[\frac{1}{n}, 2]$.
Therefore, $(0,2] \subseteq \bigcup_{n\in\mathbb{N}}[\frac{1}{n}, 2]$, and hence $\bigcup_{n\in\mathbb{N}}[\frac{1}{n}, 2] = (0,2]$. □

59. $\displaystyle\bigcap_{n\in\mathbb{N}}\left(1-\frac{1}{n},2\right] = [1,2]$.

Proof.
Let $x \in \bigcap_{n\in\mathbb{N}}(1-\frac{1}{n},2]$ and suppose $x \notin [1,2]$. Then either $x < 1$ or $2 < x$.
Since $1 \in \mathbb{N}$, we have $x \in (1-\frac{1}{1},2]$. In particular, $x \leq 2$. It must then be the case that $x < 1$.
Since $x < 1$, we have $0 < 1-x$. By the Archimedean property, choose $n \in \mathbb{N}$ with $\frac{1}{n} < 1-x$.
Since $n \in \mathbb{N}$ and $x \in \bigcap_{n\in\mathbb{N}}(1-\frac{1}{n},2]$, we have $x \in (1-\frac{1}{n},2]$, and so $1-\frac{1}{n} < x$.
However, since $\frac{1}{n} < 1-x$, we have $x < 1-\frac{1}{n}$, which is a contradiction.
Therefore, $\bigcap_{n\in\mathbb{N}}(1-\frac{1}{n},2] \subseteq [1,2]$.
Conversely, let $x \in [1,2]$. i.e. $1 \leq x$ and $x \leq 2$.
Let $n \in \mathbb{N}$.
Since $0 < n$, we have $0 < \frac{1}{n}$, and so $1-\frac{1}{n} < 1$. By transitivity, we then have $1-\frac{1}{n} < x$.
Now, $1-\frac{1}{n} < x$ and $x \leq 2$, which means $x \in (1-\frac{1}{n},2]$.
Therefore, $\forall n \in \mathbb{N}, x \in (1-\frac{1}{n},2]$. Thus, $x \in \bigcap_{n\in\mathbb{N}}(1-\frac{1}{n},2]$.
Therefore, $[1,2] \subseteq \bigcap_{n\in\mathbb{N}}(1-\frac{1}{n},2]$. Thus, $\bigcap_{n\in\mathbb{N}}(1-\frac{1}{n},2] = [1,2]$. □

61. $\displaystyle\bigcup_{n\in\mathbb{N}}\left(1-\frac{1}{n},2+\frac{1}{n}\right) = (0,3)$.

Proof.
Let $x \in \bigcup_{n\in\mathbb{N}}(1-\frac{1}{n},2+\frac{1}{n})$.
This means $\exists n \in \mathbb{N}, x \in (1-\frac{1}{n},2+\frac{1}{n})$. Choose such an n. Then $1-\frac{1}{n} < x$ and $x < 2+\frac{1}{n}$.
Since $1 \leq n$, we have $\frac{1}{n} \leq 1$; hence $0 \leq 1-\frac{1}{n}$ and $2+\frac{1}{n} \leq 3$. By transitivity, $0 < x$ and $x < 3$. Thus, $x \in (0,3)$.
Therefore, $(1-\frac{1}{n},2+\frac{1}{n}) \subseteq (0,3)$.
Conversely, let $x \in (0,3)$. i.e. $0 < x$ and $x < 3$.
Putting $n = 1$ gives us $1-\frac{1}{n} = 0$ and $2+\frac{1}{n} = 3$. Thus, $1-\frac{1}{n} < x$ and $x < 2+\frac{1}{n}$. Hence, $x \in (1-\frac{1}{n},2+\frac{1}{n})$.
Therefore, $\exists n \in \mathbb{N}, x \in (1-\frac{1}{n},2+\frac{1}{n})$. This means $x \in \bigcup_{n\in\mathbb{N}}(1-\frac{1}{n},2+\frac{1}{n})$.
Therefore, $(0,3) \subseteq \bigcup_{n\in\mathbb{N}}(1-\frac{1}{n},2+\frac{1}{n})$. Thus, $\bigcup_{n\in\mathbb{N}}(1-\frac{1}{n},2+\frac{1}{n}) = (0,3)$. □

63. $\displaystyle\bigcap_{n\in\mathbb{N}}\left[\frac{1}{n},2-\frac{1}{n}\right] = \{1\}$.

Proof.
Let $x \in \bigcap_{n\in\mathbb{N}}[\frac{1}{n},2-\frac{1}{n}]$. This means $\forall n \in \mathbb{N}, x \in [\frac{1}{n},2-\frac{1}{n}]$.
Since $1 \in \mathbb{N}$, we have $x \in [\frac{1}{1},2-\frac{1}{1}]$.
This means $x \in [1,1]$, so $1 \leq x$ and $x \leq 1$. Thus, $x = 1$, and hence $x \in \{1\}$.
Therefore, $\bigcap_{n\in\mathbb{N}}[\frac{1}{n},2-\frac{1}{n}] \subseteq \{1\}$.
Conversely, let $x \in \{1\}$. i.e. $x = 1$.
Let $n \in \mathbb{N}$.
Since $1 \leq n$, we have $\frac{1}{n} \leq 1$; hence $\frac{1}{n} \leq x$.
Also, since $\frac{1}{n} \leq 1$, we have $2-1 \leq 2-\frac{1}{n}$. Thus, $x \leq 2-\frac{1}{n}$, and so $x \in [\frac{1}{n},2-\frac{1}{n}]$.
Therefore, $\forall n \in \mathbb{N}, x \in [\frac{1}{n},2-\frac{1}{n}]$. Thus, $x \in \bigcap_{n\in\mathbb{N}}[\frac{1}{n},2-\frac{1}{n}]$.
Therefore, $\{1\} \subseteq \bigcap_{n\in\mathbb{N}}[\frac{1}{n},2-\frac{1}{n}]$, and so $\bigcap_{n\in\mathbb{N}}[\frac{1}{n},2-\frac{1}{n}] = \{1\}$. □

65. $\displaystyle\bigcup_{n\in\mathbb{N}}\left[\frac{1}{n}, 2+\frac{1}{n}\right) = (0,3)$.

Proof.
Let $x \in \bigcup_{n\in\mathbb{N}}[\frac{1}{n}, 2+\frac{1}{n})$.
Then $\exists n \in \mathbb{N}$, $x \in [\frac{1}{n}, 2+\frac{1}{n})$. Choosing such an n, we have $\frac{1}{n} \leq x$ and $x \leq 2+\frac{1}{n}$.
Since $1 \leq n$, $\frac{1}{n} \leq 1$, and so $2+\frac{1}{n} \leq 3$. By transitivity, $x < 3$.
Since $0 < n$, we have $0 < \frac{1}{n}$. By transitivity, $0 < x$. Now, $0 < x < 3$, and so $x \in (0,3)$.
Therefore, $\bigcup_{n\in\mathbb{N}}[\frac{1}{n}, 2+\frac{1}{n}) \subseteq (0,3)$. Conversely, $x \in (0,3)$. i.e. $0 < x$ and $x < 3$.
Case 1: $1 \leq x$.
Putting $n = 1$, we have $\frac{1}{n} = 1$ and $2+\frac{1}{n} = 3$. Therefore, $\frac{1}{n} \leq x$ and $x < 2+\frac{1}{n}$, and so $x \in [\frac{1}{n}, 2+\frac{1}{n})$.
Therefore, $\exists n \in \mathbb{N}$, $x \in [\frac{1}{n}, 2+\frac{1}{n})$. Thus, $x \in \bigcup_{n\in\mathbb{N}}[\frac{1}{n}, 2+\frac{1}{n})$.
Case 2: $x < 1$.
Since $0 < x$, by the Archimedean property, we can choose $m \in \mathbb{N}$ with $\frac{1}{m} \leq x$.
Since $0 < m$, we have $0 < \frac{1}{m}$, and so $2 < 2+\frac{1}{m}$. Since $1 < 2$, we have $1 < 2+\frac{1}{m}$.
We then have $x < 2+\frac{1}{m}$ by transitivity. Thus, $x \in [\frac{1}{m}, 2+\frac{1}{m})$.
Therefore, $\exists m \in \mathbb{N}$, $x \in [\frac{1}{m}, 2+\frac{1}{m})$. This means $x \in \bigcup_{n\in\mathbb{N}}[\frac{1}{n}, 2+\frac{1}{n})$.
Therefore, $(0,3) \subseteq \bigcup_{n\in\mathbb{N}}[\frac{1}{n}, 2+\frac{1}{n})$. Thus, $\bigcup_{n\in\mathbb{N}}[\frac{1}{n}, 2+\frac{1}{n}) = (0,3)$. $\square$

67. $\displaystyle\bigcap_{n\in\mathbb{N}}\left(1-\frac{1}{n}, 2-\frac{1}{n}\right] = \{1\}$.

Proof.
Let $x \in \bigcap_{n\in\mathbb{N}}(1-\frac{1}{n}, 2-\frac{1}{n}]$ and suppose $x \notin \{1\}$. i.e. $x \neq 1$.
Case 1: $x < 1$.
Then $0 < 1-x$. By the Archimedean property, choose $n \in \mathbb{N}$ with $\frac{1}{n} < 1-x$. Then $x < 1-\frac{1}{n}$.
But, $n \in \mathbb{N}$ and $x \in \bigcap_{n\in\mathbb{N}}(1-\frac{1}{n}, 2-\frac{1}{n}]$, so $x \in (1-\frac{1}{n}, 2-\frac{1}{n}]$. Thus, $1-\frac{1}{n} < x$; a contradiction.
Case 2: $1 < x$.
Since $1 \in \mathbb{N}$ and $x \in \bigcap_{n\in\mathbb{N}}(1-\frac{1}{n}, 2-\frac{1}{n}]$, we have $x \in (1-\frac{1}{1}, 2-\frac{1}{1}]$; hence $x \leq 1$. This is a contradiction.
Therefore, $\bigcap_{n\in\mathbb{N}}(1-\frac{1}{n}, 2-\frac{1}{n}] \subseteq \{1\}$.
Conversely, let $x \in \{1\}$. That is, $x = 1$.
Let $n \in \mathbb{N}$.
Since $0 < n$, we have $0 < \frac{1}{n}$, and so $1-\frac{1}{n} < 1$. Thus, $1-\frac{1}{n} < x$.
Since $1 \leq n$, we have $\frac{1}{n} \leq 1$, and so $2-1 \leq 2-\frac{1}{n}$. This proves $x \leq 2-\frac{1}{n}$.
Now, $1-\frac{1}{n} < x$ and $x \leq 2-\frac{1}{n}$, which gives us $x \in (1-\frac{1}{n}, 2-\frac{1}{n}]$.
Therefore, $\forall n \in \mathbb{N}$, $x \in (1-\frac{1}{n}, 2-\frac{1}{n}]$. This means $x \in \bigcap_{n\in\mathbb{N}}(1-\frac{1}{n}, 2-\frac{1}{n}]$.
Therefore, $\{1\} \subseteq \bigcap_{n\in\mathbb{N}}(1-\frac{1}{n}, 2-\frac{1}{n}]$. Thus, $\bigcap_{n\in\mathbb{N}}(1-\frac{1}{n}, 2-\frac{1}{n}] = \{1\}$. $\square$

69. $\bigcup_{k\in\langle 6\rangle}\langle k\rangle = \langle 6\rangle$.

Proof.
Let $x \in \bigcup_{k\in\langle 6\rangle}\langle k\rangle$.
Then $\exists k \in \langle 6\rangle, x \in \langle k\rangle$. Choose such a k.
Since $k \in \langle 6\rangle$ and $x \in \langle k\rangle$, choose $a, b \in \mathbb{Z}$ with $k = 6a$ and $x = kb$.
Then $x = 6ab$. Putting $c = ab$, we have $x = 6c$; hence $x \in \langle 6\rangle$.
Therefore, $\bigcup_{k\in\langle 6\rangle}\langle k\rangle \subseteq \langle 6\rangle$.
Conversely, let $x \in \langle 6\rangle$.
Putting $k = 6$, we have $k \in \langle 6\rangle$ and $x \in \langle k\rangle$.
Therefore, $\exists k \in \langle 6\rangle, x \in \langle k\rangle$. This means $x \in \bigcup_{k\in\langle 6\rangle}\langle k\rangle$.
Therefore, $\langle 6\rangle \subseteq \bigcup_{k\in\langle 6\rangle}\langle k\rangle$. Thus, $\bigcup_{k\in\langle 6\rangle}\langle k\rangle = \langle 6\rangle$. □

71. $\bigcap_{k\in\mathbb{Z}}(-\infty, k] \cup [k+1, \infty) = \mathbb{Z}$.

Proof.
Let $x \in \bigcap_{k\in\mathbb{Z}}(-\infty, k] \cup [k+1, \infty)$.
According to proposition 1.2.5, we can choose $n \in \mathbb{Z}$ with $n - 1 \leq x < n$.
Since $n - 1 \in \mathbb{Z}$ and $x \in \bigcap_{k\in\mathbb{Z}}(-\infty, k] \cup [k+1, \infty)$, we have $x \in (-\infty, n-1] \cup [n, \infty)$.
This gives us $x \leq n - 1$ or $n \leq x$. Since $x < n$, it must be the case that $x \leq n - 1$.
Now, $n - 1 \leq x$ and $x \leq n - 1$, which gives us $x = n - 1$. Since $n - 1 \in \mathbb{Z}$, we have $x \in \mathbb{Z}$.
Therefore, $\bigcap_{k\in\mathbb{Z}}(-\infty, k] \cup [k+1, \infty) \subseteq \mathbb{Z}$.
Conversely, let $x \in \mathbb{Z}$.
Let $k \in \mathbb{Z}$.
Case 1: $x \leq k$.
In this case, $x \in (-\infty, k]$, and so $x \in (-\infty, k] \cup [k+1, \infty)$.
Case 2: $k < x$.
Since $x, k \in \mathbb{Z}$, we have $k + 1 \leq x$ by corollary 1.2.4.
Thus, $x \in [k+1, \infty)$, and so $x \in (-\infty, k] \cup [k+1, \infty)$.
Therefore, $\forall k \in \mathbb{Z}, x \in (-\infty, k] \cup [k+1, \infty)$. Thus, $x \in \bigcap_{k\in\mathbb{Z}}(-\infty, k] \cup [k+1, \infty)$.
Therefore, $\mathbb{Z} \subseteq \bigcap_{k\in\mathbb{Z}}(-\infty, k] \cup [k+1, \infty)$, and hence $\bigcap_{k\in\mathbb{Z}}(-\infty, k] \cup [k+1, \infty) = \mathbb{Z}$. □

Let U and V be sets. Let $\mathcal{A} = \{A_y \mid y \in V\}$ be a family of subsets of U, indexed by V. For each $x \in U$, let $B_x = \{y \in V \mid x \in A_y\}$, and let $\mathcal{B} = \{B_x \mid x \in U\}$.

73. Prove $\forall x \in U, x \in \bigcap_{y \in V} A_y$ if and only if $B_x = V$.

Proof.
Let $x \in U$.
Assume $x \in \bigcap_{y\in V} A_y$. i.e. $\forall y \in V, x \in A_y$.
Now, $B_x \subseteq V$, since V is the universe of discourse of the set B_x.
Let $y \in V$.
Then $x \in A_y$, since $x \in \bigcap_{y\in V} A_y$ and $y \in V$.
Therefore, $y \in B_x$.
Therefore, $V \subseteq B_x$, and hence $B_x = V$.
Therefore, if $x \in \bigcap_{y\in V} A_y$, then $B_x = V$.
Conversely, assume $B_x = V$.
Let $y \in V$.
Since $y \in V$ and $B_x = V$, we have $y \in B_x$. This means $x \in A_y$.
Therefore, $\forall y \in V, x \in A_y$. This means $x \in \bigcap_{y\in V} A_y$.
Therefore, if $B_x = V$, then $x \in \bigcap_{y\in V} A_y$.
Therefore, $\forall x \in U, x \in \bigcap_{y\in V} A_y$ if and only if $B_x = V$. □

75. Prove $\forall y \in V, y \in \bigcup_{x \in U} B_x$ if and only if $A_y \neq \emptyset$.

Proof.
Let $y \in V$.
Assume $y \in \bigcup_{x\in U} B_x$.
This means $\exists x \in U, y \in B_x$. Choose such an x.
Since $y \in B_x$, we have $x \in A_y$. Therefore, $A_y \neq \emptyset$.
Therefore, if $y \in \bigcup_{x\in U} B_x$, then $A_y \neq \emptyset$.
Conversely, assume $A_y \neq \emptyset$.
Since $A_y \neq \emptyset$, we have choose an element $x \in A_y$.
Since $x \in A_y$, we have $y \in B_x$.
Therefore, $\exists x \in U, y \in B_x$. This means $y \in \bigcup_{x\in U} B_x$.
Therefore, if $A_y \neq \emptyset$, then $y \in \bigcup_{x\in U} B_x$.
Therefore, $\forall y \in V, y \in \bigcup_{x\in U} B_x$ if and only if $A_y \neq \emptyset$. □

Prove the following propositions about families of inductive sets.

77. Let $\mathcal{A} = \{A \in \mathcal{P}(\mathbb{R}) \mid 1 \in A \text{ and } A \text{ is inductive}\}$.

(a) $\bigcap_{A\in\mathcal{A}} A$ is inductive.

Proof.
Let $x \in \mathbb{R}$ and assume $x \in \bigcap_{A\in\mathcal{A}} A$.
Let $A \in \mathcal{A}$.
Since $A \in \mathcal{A}$ and $x \in \bigcap_{A\in\mathcal{A}} A$, we have $x \in A$. Since $A \in \mathcal{A}$, A is inductive, and so $x + 1 \in A$.
Therefore, $\forall A \in \mathcal{A}$, $x + 1 \in A$. This means $x + 1 \in \bigcap_{A\in\mathcal{A}} A$.
Therefore, $\forall x \in \mathbb{R}$, if $x \in \bigcap_{A\in\mathcal{A}} A$, then $x + 1 \in \bigcap_{A\in\mathcal{A}} A$. Thus, $\bigcap_{A\in\mathcal{A}} A$ is inductive. □

(c) $\bigcap_{A\in\mathcal{A}} A = \mathbb{N}$.

Proof.
Let $x \in \bigcap_{A\in\mathcal{A}} A$. This means $\forall A \in \mathcal{A}$, $x \in A$.
Since $1 \in \mathbb{N}$ and $\mathbb{N}$ is inductive, we have $\mathbb{N} \in \mathcal{A}$. Since $\mathbb{N} \in \mathcal{A}$ and $x \in \bigcap_{A\in\mathcal{A}} A$, we have $x \in \mathbb{N}$.
Therefore, $\bigcap_{A\in\mathcal{A}} A \subseteq \mathbb{N}$.
Conversely, let $x \in \mathbb{N}$.
Let $A \in \mathcal{A}$. This means $1 \in A$ and A is inductive.
Since A is inductive, we have $\forall n \in \mathbb{N}$, if $n \in A$, then $n + 1 \in A$.
By the PMI, we have $\mathbb{N} \subseteq A$; hence $x \in A$.
Therefore, $\forall A \in \mathcal{A}$, $x \in A$. This means $x \in \bigcap_{A\in\mathcal{A}} A$.
Therefore, $\mathbb{N} \subseteq \bigcap_{A\in\mathcal{A}} A$. Thus, $\bigcap_{A\in\mathcal{A}} A = \mathbb{N}$. □

78. Let $a \in \mathbb{R}$. Let $\mathcal{M} = \{A \in \mathcal{P}(\mathbb{R}) \mid a \in A \text{ and } A \text{ is inductive}\}$.

(a) $\mathcal{M}$ is non-empty.

Proof.
Since $a \in \mathbb{R}$ and $\mathbb{R}$ is inductive, we have $\mathbb{R} \in \mathcal{M}$. Thus, $\mathcal{M} \neq \emptyset$. □

(c) $\bigcap_{A\in\mathcal{M}} A$ is the smallest inductive set containing a (See definition 1.2.9).

Proof.
Letting $A \in \mathcal{M}$, we have $a \in A$. Therefore, $\forall A \in \mathcal{M}$, $a \in A$. Thus, $a \in \bigcap_{A\in\mathcal{M}} A$.
Next, let $x \in \mathbb{R}$ and assume $x \in \bigcap_{A\in\mathcal{M}} A$.
Let $A \in \mathcal{M}$. i.e. $a \in A$ and A is inductive.
Now, $x \in A$, since $x \in \bigcap_{A\in\mathcal{M}} A$. Since A is inductive, we then have $x + 1 \in A$.
Therefore, $\forall A \in \mathcal{M}$, $x + 1 \in A$. This means $x + 1 \in \bigcap_{A\in\mathcal{M}} A$.
Therefore, $\forall x \in \mathbb{R}$, if $x \in \bigcap_{A\in\mathcal{M}} A$, then $x + 1 \in \bigcap_{A\in\mathcal{M}} A$. Thus, $\bigcap_{A\in\mathcal{M}} A$ is inductive.
We now have that $\bigcap_{A\in\mathcal{M}} A$ is an inductive set containing a.
Next, let $B \subseteq \mathbb{R}$ with $a \in B$ and B inductive.
Let $x \in \bigcap_{A\in\mathcal{M}} A$.
Since $a \in B$ and B is inductive, we have $B \in \mathcal{M}$. Since $x \in \bigcap_{A\in\mathcal{M}} A$, we then have $x \in B$.
Therefore, $\bigcap_{A\in\mathcal{M}} A \subseteq B$.
Therefore, if $B \subseteq \mathbb{R}$ with $a \in B$ and B inductive, then $\bigcap_{A\in\mathcal{M}} A \subseteq B$.
Thus, $\bigcap_{A\in\mathcal{M}} A$ is the smallest inductive set containing a according to definition 1.2.9. □

Chapter 3

Relations

3.1 Equivalence Relations

Exercises 3.1.

Let A and B be sets in a universe of discourse U, and let X and Y be sets in a universe of discourse V. Let S and T be subsets of $U \times V$. Prove the following propositions.

1. If $A \subseteq B$, then $A \times X \subseteq B \times X$.

Proof.
Assume $A \subseteq B$.
  Let $(x, y) \in A \times X$. This means $x \in A$ and $y \in X$.
    Since $x \in A$ and $A \subseteq B$, we have $x \in B$.
    Now, $x \in B$ and $y \in X$, which means $(x, y) \in B \times X$.
  Therefore, $A \times X \subseteq B \times X$.
Therefore, if $A \subseteq B$, then $A \times X \subseteq B \times X$. □

3. $(A \times X) \cap (B \times Y) = (A \cap B) \times (X \cap Y)$.

Proof.
Let $(x, y) \in (A \times X) \cap (B \times Y)$.
  Then $(x, y) \in A \times X$ and $(x, y) \in B \times Y$.
  Since $(x, y) \in A \times X$, $x \in A$ and $y \in X$. Since $(x, y) \in B \times Y$, $x \in B$ and $y \in Y$.
  Since $x \in A$ and $x \in B$, $x \in A \cap B$. Since $y \in X$ and $y \in Y$, $y \in X \cap Y$.
  Now, $x \in A \cap B$ and $y \in X \cap Y$, which means $(x, y) \in (A \cap B) \times (X \cap Y)$.
Therefore, $(A \times X) \cap (B \times Y) \subseteq (A \cap B) \times (X \cap Y)$.
Conversely, let $x \in (A \cap B) \times (X \cap Y)$.
  Then $x \in A \cap B$ and $y \in X \cap Y$. This means $x \in A$, $x \in B$, $y \in X$, and $y \in Y$.
  Since $x \in A$ and $y \in X$, $(x, y) \in A \times X$. Since $x \in B$ and $y \in Y$, $(x, y) \in B \times Y$.
  Therefore, $(x, y) \in (A \times X) \cap (B \times Y)$.
Therefore, $(A \cap B) \times (X \cap Y) \subseteq (A \times X) \cap (B \times Y)$. Thus, $(A \times X) \cap (B \times Y) = (A \cap B) \times (X \cap Y)$. □

5. If $S \subseteq A \times X$ and $T \subseteq B \times Y$, then $S \cap T \subseteq (A \cap B) \times (X \cap Y)$.

Proof.
Assume $S \subseteq A \times X$ and $T \subseteq B \times Y$.
Let $(x, y) \in S \cap T$. i.e. $(x, y) \in S$ and $(x, y) \in T$.
Then $(x, y) \in A \times X$ and $(x, y) \in B \times Y$; hence $x \in A$, $y \in X$, $x \in B$ and $y \in Y$.
Since $x \in A$ and $x \in B$, $x \in A \cap B$. Since $y \in X$ and $y \in Y$, $y \in X \cap Y$. Thus, $(x, y) \in (A \cap B) \times (X \cap Y)$.
Therefore, $S \cap T \subseteq (A \cap B) \times (X \cap Y)$.
Therefore, if $S \subseteq A \times X$ and $T \subseteq B \times Y$, then $S \cap T \subseteq (A \cap B) \times (X \cap Y)$. $\square$

7. $(A \setminus B) \times X = (A \times X) \setminus (B \times X)$.

Proof.
Let $(x, y) \in (A \setminus B) \times X$.
Then $x \in A \setminus B$ and $y \in X$. i.e. $x \in A$ and $x \notin B$, and $y \in X$.
Since $x \in A$ and $y \in X$, $(x, y) \in A \times X$. Since $x \notin B$, $(x, y) \notin B \times X$. Thus, $(x, y) \in (A \times X) \setminus (B \times X)$.
Therefore, $(A \setminus B) \times X \subseteq (A \times X) \setminus (B \times X)$.
Conversely, let $(x, y) \in (A \times X) \setminus (B \times X)$.
Then $(x, y) \in A \times X$ and $(x, y) \notin B \times X$. i.e. $x \in A$ and $y \in X$, and either $x \notin B$ or $y \notin X$.
Since $y \in X$, it must be the case that $x \notin B$. Now, $x \in A$ and $x \notin B$, so $x \in A \setminus B$.
Since $x \in A \setminus B$ and $y \in X$, we have $(x, y) \in (A \setminus B) \times X$.
Therefore, $(A \times X) \setminus (B \times X) \subseteq (A \setminus B) \times X$. Thus, $(A \setminus B) \times X = (A \times X) \setminus (B \times X)$. $\square$

9. $(A \setminus B) \times (X \setminus Y) \subseteq (A \times X) \setminus (B \times Y)$.

Proof.
Let $(x, y) \in (A \setminus B) \times (X \setminus Y)$.
Then $x \in A \setminus B$, and $y \in X \setminus Y$. This means $x \in A$, $x \notin B$, $y \in X$, and $y \notin Y$.
Since $x \in A$ and $y \in X$, we have $(x, y) \in A \times X$. Since $x \notin B$, we have $(x, y) \notin B \times Y$.
Therefore, $(x, y) \in (A \times X) \setminus (B \times Y)$.
Therefore, $(A \setminus B) \times (X \setminus Y) \subseteq (A \times X) \setminus (B \times Y)$. $\square$

11. $(A \times X)^c = (U \times X^c) \cup (A^c \times V)$.

Proof.
Let $(x, y) \in U \times V$ and assume $x \in (A \times X)^c$. i.e. $(x, y) \notin A \times X$. This means $x \notin A$ or $y \notin X$.
Case 1: $x \notin A$. i.e $x \in A^c$.
Since $y \in V$, we have $(x, y) \in A^c \times V$. Thus, $(x, y) \in (U \times X^c) \cup (A^c \times V)$.
Case 2: $y \notin X$. i.e. $y \in X^c$.
Since $x \in U$, we have $(x, y) \in U \times X^c$, and so $(x, y) \in (U \times X^c) \cup (A^c \times V)$.
Therefore, $(A \times X)^c \subseteq (U \times X^c) \cup (A^c \times V)$.
Conversely, let $(x, y) \in (U \times X^c) \cup (A^c \times V)$. This means $(x, y) \in U \times X^c$ or $(x, y) \in A^c \times V$.
Case 1: $(x, y) \in U \times X^c$. i.e. $x \in U$ and $y \in X^c$.
Since $y \in X^c$, $y \notin X$, and so $(x, y) \notin A \times X$. Thus, $(x, y) \in (A \times X)^c$.
Case 2: $(x, y) \in A^c \times V$. i.e. $x \in A^c$ and $y \in V$.
Since $x \in A^c$, $x \notin A$, which implies $(x, y) \notin A \times X$. Thus, $(x, y) \in (A \times X)^c$.
Therefore, $(U \times X^c) \cup (A^c \times V) \subseteq (A \times X)^c$. Thus, $(A \times X)^c = (U \times X^c) \cup (A^c \times V)$. $\square$

Prove that each of the following are equivalence relations. Describe the equivalence classes.

13. The relation R on the set $\mathbb{R} \setminus \{0\}$ given by: $\forall x, y \in \mathbb{R} \setminus \{0\}$, xRy if and only if $xy > 0$.

Proof.
Let $x \in \mathbb{R} \setminus \{0\}$.
This means $x \in \mathbb{R}$ and $x \neq 0$.
Now, $0 \leq x^2$, and since $x \neq 0$, $x^2 \neq 0$. Therefore, $0 < x^2$, which means xRx.
Therefore, $\forall x \in \mathbb{R} \setminus \{0\}$, xRx. Thus, R is reflexive.
Let $x, y \in \mathbb{R} \setminus \{0\}$.
Assume xRy.
Then $xy > 0$, so $yx > 0$, which means yRx.
Therefore, if xRy, then yRx.
Therefore, R is symmetric.
Let $x, y, z \in \mathbb{R} \setminus \{0\}$.
Assume xRy and yRz.
Then $xy > 0$ and $yz > 0$. We then have $(xy)(yz) > 0$, so $xzy^2 > 0$.
Since $y \neq 0$, we have $0 < y^2$, so $xzy^2(y^2)^{-1} > 0(y^2)^{-1}$; hence $xz > 0$. Therefore, xRz.
Therefore, if xRy and yRz, then xRz.
Therefore, R is transitive. Thus, R is an equivalence relation. □

There are two distinct equivalence classes: $(-\infty, 0)$, and $(0, \infty)$.

15. The relation $\equiv_{\mathbb{Q}}$ on the set $\mathbb{R}$ given by: $\forall x, y \in \mathbb{R}$, $x \equiv_{\mathbb{Q}} y$ if and only if $x - y \in \mathbb{Q}$.

Proof.
Let $x \in \mathbb{R}$.
$x - x = 0 \in \mathbb{Q}$. Therefore, $x \equiv_{\mathbb{Q}} x$.
Therefore, $\equiv_{\mathbb{Q}}$ is reflexive.
Let $x, y \in \mathbb{R}$.
Assume $x \equiv_{\mathbb{Q}} y$. That is, $x - y \in \mathbb{Q}$.
We then have $-(x - y) \in \mathbb{Q}$, which means $y - x \in \mathbb{Q}$. Thus, $y \equiv_{\mathbb{Q}} x$.
Therefore, if $x \equiv_{\mathbb{Q}} y$ then $y \equiv_{\mathbb{Q}} x$.
Therefore, $\equiv_{\mathbb{Q}}$ is symmetric.
Let $x, y, z \in \mathbb{R}$.
Assume $x \equiv_{\mathbb{Q}} y$ and $y \equiv_{\mathbb{Q}} z$. i.e. $x - y \in \mathbb{Q}$ and $y - z \in \mathbb{Q}$.
We then have $(x - y) + (y - z) \in \mathbb{Q}$, and so $x - z \in \mathbb{Q}$. Therefore, $x \equiv_{\mathbb{Q}} z$.
Therefore, if $x \equiv_{\mathbb{Q}} y$ and $y \equiv_{\mathbb{Q}} z$ then $x \equiv_{\mathbb{Q}} z$.
Therefore, $\equiv_{\mathbb{Q}}$ is transitive. Since $\equiv_{\mathbb{Q}}$ is reflexive, symmetric, and transitive, $\equiv_{\mathbb{Q}}$ is an equivalence relation. □

For each $x \in \mathbb{R}$, $[x]_R = \{x + q \mid q \in \mathbb{Q}\}$. This set is often denoted $x + \mathbb{Q}$.

17. The relation R on the set $\mathbb{Z}$ given by: $\forall x, y \in \mathbb{Z}$, xRy if and only if $x^2 = y^2$.

Proof.
Let $x \in \mathbb{Z}$.
Since $x^2 = x^2$, we have xRx
Therefore, $\forall x \in \mathbb{Z}$, xRx. That is, R is reflexive.
Let $x, y \in \mathbb{Z}$ and assume xRy.
This means $x^2 = y^2$, which can also be written as $y^2 = x^2$. Thus, yRx.
Therefore, $\forall x, y \in \mathbb{Z}$, if xRy, then yRx. That is, R is symmetric.
Let $x, y, z \in \mathbb{Z}$ and assume xRy and yRz.
Then $x^2 = y^2$ and $y^2 = z^2$. It follows that $x^2 = z^2$, and so xRz.
Therefore, $\forall x, y, z \in \mathbb{Z}$, if xRy and yRz, then xRz. This means R is transitive and is thus an equivalence relation. □

The equivalence classes are: $[0]_R = \{0\}$, and for each $x \in \mathbb{Z}$ with $x \neq 0$, $[x]_R = \{-x, x\}$.

19. The relation R on the set $\mathbb{Z}$ given by: $\forall x, y \in \mathbb{Z}$, xRy if and only if $\exists n \in \mathbb{Z}$, $x = 2^n y$.

Proof.
Let $x \in \mathbb{Z}$.
Since $x = 1x = 2^0 x$, we have xRx.
Therefore, $\forall x \in \mathbb{Z}$, xRx. Thus, R is reflexive.
Let $x, y \in \mathbb{Z}$ and assume xRy.
Choose $n \in \mathbb{Z}$ with $x = 2^n y$. Then $2^{-n} x = y$.
Putting $k = -n$ gives us $\exists k \in \mathbb{Z}$, $y = 2^k x$. Thus, yRx.
Therefore, $\forall x, y \in \mathbb{Z}$, if xRy, then yRx. This means R is symmetric.
Let $x, y, z \in \mathbb{Z}$ and assume xRy and yRz.
Choose $k, m \in \mathbb{Z}$ with $x = 2^k y$ and $y = 2^m z$. Putting $n = k + m$ gives us $x = 2^k 2^m z = 2^{k+m} z = 2^n z$.
Therefore, $\exists n \in \mathbb{Z}$, $x = 2^n z$, so xRz.
Therefore, $\forall x, y, z \in \mathbb{Z}$, if xRy and yRz, then xRz. Then R is transitive. Thus, R is an equivalence relation. □

The equivalence classes are: $[0]_R = \{0\}$, and for each odd number k, $[k]_R = \{x \in \mathbb{Z} \mid \exists n \in \mathbb{Z}, x = 2^n k\}$.

21. The relation R on the set $\mathbb{R}^2$ given by: $\forall (x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$, $(x_1, x_2)R(y_1, y_2)$ if and only if $y_2 + 2x_1 = x_2 + 2y_1$.

Proof.
Let $(x_1, x_2) \in \mathbb{R}^2$.
Since $x_2 + 2x_1 = x_2 + 2x_1$, we have $(x_1, x_2)R(x_1, x_2)$.
Therefore, $\forall (x_1, x_2) \in \mathbb{R}^2$, $(x_1, x_2)R(x_1, x_2)$. Thus, R is reflexive.
Let $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$ and assume $(x_1, x_2)R(y_1, y_2)$.
Then $y_2 + 2x_1 = x_2 + 2y_1$. This can be written as $x_2 + 2y_1 = y_2 + 2x_1$, which means $(y_1, y_2)R(x_1, x_2)$.
Therefore, $\forall (x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$, if $(x_1, x_2)R(y_1, y_2)$, then $(y_1, y_2)R(x_1, x_2)$. Thus, R is symmetric.
Let $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in \mathbb{R}^3$ and assume $(x_1, x_2)R(y_1, y_2)$ and $(y_1, y_2)R(z_1, z_2)$.
Then $y_2 + 2x_1 = x_2 + 2y_1$ and $z_2 + 2y_1 = y_2 + 2z_1$, and so $y_2 + 2x_1 - (y_2 + 2z_1) = x_2 + 2y_1 - (z_2 + 2y_1)$.
Now, $2x_1 - 2z_1 = x_2 - z_2$, and so $z_2 + 2x_1 = x_2 + 2z_1$. Thus, $(x_1, x_2)R(z_1, z_2)$.
Therefore, R is transitive. Thus, R is an equivalence relation. □

The equivalence class of a point (a, b) is the line with slope $m = 2$ passing through the point (a, b).

23. The relation R on the set $\mathbb{R}^3$ given by $\forall(x_1, x_2, x_3), (y_1, y_2, y_3) \in \mathbb{R}^3$, $(x_1, x_2, x_3)R(y_1, y_2, y_3)$ if and only if $\exists t \in \mathbb{R} \setminus \{0\}$, $(x_1, x_2, x_3) = (ty_1, ty_2, ty_3)$.

Proof.
Let $(x_1, x_2, x_3) \in \mathbb{R}^3$.
Putting $t = 1$ gives us $(x_1, x_2, x_3) = (tx_1, tx_2, tx_3)$; hence $(x_1, x_2, x_3)R(x_1, x_2, x_3)$.
Therefore, R is reflexive.
Let $(x_1, x_2, x_3), (y_1, y_2, y_3) \in \mathbb{R}^3$ and assume $(x_1, x_2, x_3)R(y_1, y_2, y_3)$.
Choose $t \in \mathbb{R} \setminus \{0\}$ with $(x_1, x_2, x_3) = (ty_1, ty_2, ty_3)$, and Put $s = \frac{1}{t}$.
Since $x_1 = ty_1$, $x_2 = ty_2$ and $x_3 = ty_3$, we have $y_1 = sx_1$, $y_2 = sx_2$ and $y_3 = sx_3$.
Thus, $(y_1, y_2, y_3) = (sx_1, sx_2, sx_3)$.
Therefore, $\exists s \in \mathbb{R} \setminus \{0\}$, $(y_1, y_2, y_3) = (sx_1, sx_2, sx_3)$. In other words, $(y_1, y_2, y_3)R(x_1, x_2, x_3)$.
Therefore, R is symmetric.
Let $(x_1, x_2, x_3), (y_1, y_2, y_3), (z_1, z_2, z_3) \in \mathbb{R}^3$ and assume $(x_1, x_2, x_3)R(y_1, y_2, y_3)$ and $(y_1, y_2, y_3)R(z_1, z_2, z_3)$.
Choose $a, b \in \mathbb{R} \setminus \{0\}$ with $(x_1, x_2, x_3) = (ay_1, ay_2, ay_3)$ and $(y_1, y_2, y_3) = (bz_1, bz_2, bz_3)$.
Then $(x_1, x_2, x_3) = (abz_1, abz_2, abz_3)$. Putting $c = ab$ then gives us $(x_1, x_2, x_3) = (cz_1, cz_2, cz_3)$.
Since $c \neq 0$, this means $(x_1, x_2, x_3)R(z_1, z_2, z_3)$.
Therefore, R is transitive. Thus, R is an equivalence relation. □

A complete list of equivalence classes is: $[\overrightarrow{0}]_R = \{\overrightarrow{0}\}$, and for $\overrightarrow{u} \in \mathbb{R}^3$ with $\|\overrightarrow{u}\| = 1$, $[\overrightarrow{u}]_R$ is the line through the origin in the direction of $\overrightarrow{u}$.

Prove the following propositions. These are analogous to theorem 3.1.13.

25. For the relation $\equiv_{\mathbb{Q}}$ defined in question 15, prove $\forall a, b, x, y \in \mathbb{R}$, if $x \in [a]_{\equiv_{\mathbb{Q}}}$ and $y \in [b]_{\equiv_{\mathbb{Q}}}$, then $x + y \in [a + b]_{\equiv_{\mathbb{Q}}}$.

Proof.
Let $a, b, x, y \in \mathbb{R}$.
Assume $x \in [a]_{\equiv_{\mathbb{Q}}}$ and $y \in [b]_{\equiv_{\mathbb{Q}}}$.
Therefore, $x \equiv_{\mathbb{Q}} a$ and $y \equiv_{\mathbb{Q}} b$, which means $x - a \in \mathbb{Q}$ and $y - b \in \mathbb{Q}$.
Then $(x - a) + (y - b) \in \mathbb{Q}$, which gives us $(x + y) - (a + b) \in \mathbb{Q}$, and hence $(x + y) \equiv_{\mathbb{Q}} (a + b)$.
Therefore, $x + y \in [a + b]_{\equiv_{\mathbb{Q}}}$.
Therefore, if $x \in [a]_{\equiv_{\mathbb{Q}}}$ and $y \in [b]_{\equiv_{\mathbb{Q}}}$, then $x + y \in [a + b]_{\equiv_{\mathbb{Q}}}$.
Therefore, $\forall a, b, x, y \in \mathbb{R}$, if $x \in [a]_{\equiv_{\mathbb{Q}}}$ and $y \in [b]_{\equiv_{\mathbb{Q}}}$, then $x + y \in [a + b]_{\equiv_{\mathbb{Q}}}$. □

Prove the following propositions about partitions.

27. Let $A, B \subseteq \mathbb{Z}$ with $A \neq B$. Let $\mathcal{P} = \{A, B\}$. If $\mathcal{P}$ is a partition of $\mathbb{Z}$, then $B \neq \mathbb{Z}$.

Proof.
Assume $\mathcal{P}$ is a partition of $\mathbb{Z}$ and $B = \mathbb{Z}$.
Since $\mathcal{P}$ is a partition, we have $A \neq \emptyset$. Therefore, we can choose $x \in \mathbb{Z}$ with $x \in A$.
Since $B = \mathbb{Z}$, we have $x \in B$. Therefore, $x \in A \cap B$.
However, since $\mathcal{P}$ is a partition, we have $A \cap B = \emptyset$. This is a contradiction.
Therefore, if $\mathcal{P}$ is a partition of $\mathbb{Z}$, then $B \neq \mathbb{Z}$. □

29. For each $n \in \mathbb{N}$, let $S_n = \{x \in \mathbb{R} \mid n - 1 \leq x^2 < n\}$. Then $\mathcal{A} = \{S_n \mid n \in \mathbb{N}\}$ is a partition of $\mathbb{R}$.

Proof.
Let $n \in \mathbb{N}$.
Putting $x = \sqrt{n-1}$ gives us $x^2 = n - 1$, and so $n - 1 \leq x^2 < n$. Thus, $x \in S_n$, which proves $S_n \neq \emptyset$.
Therefore, $\forall n \in \mathbb{N}$, $S_n \neq \emptyset$.
Next, let $m, n \in \mathbb{N}$ and assume $S_n \cap S_m \neq \emptyset$.
Choosing $x \in S_n \cap S_m$, we have $n - 1 \leq x^2 < n$ and $m - 1 \leq x^2 < m$.
Then, $n - 1 < m$; hence $n \leq m$. Similarly, $m - 1 < n$; hence $m \leq n$. This proves $n = m$, and so $S_n = S_m$.
Therefore, if $S_n \cap S_m \neq \emptyset$, then $S_n = S_m$.
Therefore, $\forall m, n \in \mathbb{N}$, if $S_n \neq S_m$, then $S_n \cap S_m = \emptyset$.
Finally, let $x \in \mathbb{R}$.
Let $B = \{k \in \mathbb{N} \mid x^2 < k\}$. By the Archimedean property, $B \neq \emptyset$.
According to the well-ordering property, we can choose $n \in \mathbb{N}$ to be the smallest element of B. Then $x^2 < n$.
Case 1: $n = 1$.
Since $0 \leq x^2$, we have $n - 1 \leq x^2$. We now have $x \in S_n$.
Case 2: $1 < n$.
In this case, $n - 1 \in \mathbb{N}$, and since $n - 1 \notin B$, we must have $n - 1 \leq x^2$. Therefore, $x \in S_n$.
Therefore, $\exists n \in \mathbb{N}$, $x \in S_n$. This means $x \in \bigcup_{n\in\mathbb{N}} S_n$.
Therefore, $\mathbb{R} \subseteq \bigcup_{n\in\mathbb{N}} S_n$ and since $\mathbb{R}$ is the universe of discourse, $\bigcup_{n\in\mathbb{N}} S_n = \mathbb{R}$.
Thus, $\mathcal{A}$ is a partition of $\mathbb{R}$. □

31. For each $y \in \mathbb{R}$, let $A_y = \{x \in \mathbb{R} \mid y = x^2\}$. Then $\mathcal{A} = \{A_y \mid y \in [0, \infty)\}$ is a partition of $\mathbb{R}$.

Proof.
Let $y \in [0, \infty)$.
Since $y \in [0, \infty)$, we can put $x = \sqrt{y}$. Then $x^2 = y$, which means $x \in A_y$; hence $A_y \neq \emptyset$.
Therefore, $\forall y \in [0, \infty)$, $A_y \neq \emptyset$.
Let $y, z \in [0, \infty)$ and assume $A_y \cap A_z \neq \emptyset$.
Choose $x \in \mathbb{R}$ with $x \in A_y \cap A_z$. Then $y = x^2$ and $z = x^2$; hence $y = z$. Therefore, $A_y = A_z$.
Therefore, if $A_y \cap A_z \neq \emptyset$ then $A_y = A_z$.
Therefore, $\forall y, z \in [0, \infty)$, if $A_y \neq A_z$ then $A_y \cap A_z = \emptyset$.
Let $x \in \mathbb{R}$.
Choosing $y = x^2$ gives us $y \in [0, \infty)$ and $x \in A_y$.
Therefore, $\exists y \in [0, \infty)$, $x \in A_y$. This means $x \in \bigcup_{y\in[0,\infty)} A_y$.
Therefore, $\mathbb{R} \subseteq \bigcup_{y\in[0,\infty)} A_y$. Since $\mathbb{R}$ is the universe of discourse, we then have $\bigcup_{y\in[0,\infty)} A_y = \mathbb{R}$.
We have thus shown that $\mathcal{A}$ is a partition of $\mathbb{R}$. □

33. The family of sets $\mathcal{A} = \{\langle 3\rangle, 1 + \langle 3\rangle, 2 + \langle 3\rangle\}$ is a partition of $\mathbb{Z}$.

Proof.
Let $S \in \mathcal{A}$. i.e. $S = \langle 3\rangle$ or $S = 1 + \langle 3\rangle$ or $S = 2 + \langle 3\rangle$.
Since $0 \in \langle 3\rangle$, $1 \in 1 + \langle 3\rangle$, and $2 \in 2 + \langle 3\rangle$, we have in every case $S \neq \emptyset$.
Therefore, $\forall S \in \mathcal{A}, S \neq \emptyset$.
Let $S, T \in \mathcal{A}$, with $S \neq T$, and suppose $S \cap T \neq \emptyset$. Choose $x \in \mathbb{Z}$ with $x \in S \cap T$.
Consider three cases: ($S = \langle 3\rangle$ and $T = 1+\langle 3\rangle$) or ($S = \langle 3\rangle$ and $T = 2+\langle 3\rangle$) or ($S = 1+\langle 3\rangle$ and $T = 2+\langle 3\rangle$).
Case 1: $x \in \langle 3\rangle \cap (1 + \langle 3\rangle)$.
Choose $s, t \in \mathbb{Z}$ with $x = 3t$ and $x = 3s + 1$. Then $1 = 3(t - s)$, so 3 divides 1, which is a contradiction.
Case 2: $x \in \langle 3\rangle \cap (2 + \langle 3\rangle)$.
Choose $s, t \in \mathbb{Z}$ with $x = 3t$ and $x = 3s + 2$. Then $2 = 3(t - s)$, so 3 divides 2, which is a contradiction.
Case 1: $x \in (1 + \langle 3\rangle) \cap (2 + \langle 3\rangle)$.
Choose $s, t \in \mathbb{Z}$, $x = 3t + 1$ and $x = 3s + 2$. Then $1 = 3(t - s)$, so 3 divides 1, which is a contradiction.
Therefore, $\forall S, T \in \mathcal{A}$, if $S \neq T$ then $S \cap T = \emptyset$.
Let $x \in \mathbb{Z}$.
By the division algorithm, choose $q, r \in \mathbb{Z}$ with $x = 3q + r$ and $0 \leq r < 3$. Then, $r = 0$ or $r = 1$ or $r = 2$.
If $r = 0$, then $x = 3q$, in which case $x \in \langle 3\rangle$.
Likewise, if $r = 1$, then $x \in 1 + \langle 3\rangle$, and if $r = 2$, then $x \in 2 + \langle 3\rangle$.
In every case, we have $\exists S \in \mathcal{A}, x \in S$. Therefore, $x \in \bigcup_{S\in\mathcal{A}} S$.
Therefore, $\mathbb{Z} \subseteq \bigcup_{S\in\mathcal{A}} S$ and since $\mathbb{Z}$ is the universe of discourse, $\bigcup_{S\in\mathcal{A}} S = \mathbb{Z}$.
Thus, $\mathcal{A}$ is a partition of $\mathbb{Z}$. □

35. $\mathcal{A} = \{\langle 4\rangle, 2 + \langle 4\rangle\}$ is a partition of $\langle 2\rangle$.

Proof.
Let $S \in \mathcal{A}$. i.e. $S = \langle 4\rangle$ or $S = 2 + \langle 4\rangle$.
Since $0 \in \langle 4\rangle$, $2 \in 2 + \langle 4\rangle$, we have in both cases $S \neq \emptyset$.
Therefore, $\forall S \in \mathcal{A}, S \neq \emptyset$.
Let $S, T \in \mathcal{A}$, with $S \neq T$, and suppose $S \cap T \neq \emptyset$. Choose $x \in \mathbb{Z}$ with $x \in S \cap T$.
Without loss of generality, assume $S = \langle 4\rangle$, in which case $T = 2 + \langle 4\rangle$.
Choose $s, t \in \mathbb{Z}$ with $x = 4t$ and $x = 2 + 4s$. Then $2 = 4(t - s)$, and so 4 divides 2, which is a contradiction.
Therefore, $\forall S, T \in \mathcal{A}$, if $S \neq T$ then $S \cap T = \emptyset$.
Let $x \in \langle 2\rangle$. Choose $a \in \mathbb{Z}$ with $x = 2a$.
Case 1: a is even.
Choose $m \in \mathbb{Z}$ with $a = 2m$. Then $x = 2(2m) = 4m$; hence $x \in \langle 4\rangle$.
Case 2: a is odd.
Choose $k \in \mathbb{Z}$ with $a = 2k + 1$. Then $x = 2(2k + 1) = 4k + 2$; hence $x \in 2 + \langle 4\rangle$.
In noth cases, we have $\exists S \in \mathcal{A}, x \in S$. Therefore, $x \in \bigcup_{S\in\mathcal{A}} S$.
Therefore, $\langle 2\rangle \subseteq \bigcup_{S\in\mathcal{A}} S$.
Conversely, let $x \in \bigcup_{S\in\mathcal{A}} S$. Choose $S \in \mathcal{A}$ with $x \in S$.
Case 1: $S = \langle 4\rangle$.
Choose $b \in \mathbb{Z}$ with $x = 4b$, and put $q = 2b$. Then $x = 2q$; hence $x \in \langle 2\rangle$.
Case 2: $S = 2 + \langle 4\rangle$.
Choose $c \in \mathbb{Z}$ with $x = 2 + 4c$, and put $w = 1 + 2c$. Then $x = 2w$; hence $x \in \langle 2\rangle$.
Therefore, $\bigcup_{S\in\mathcal{A}} S \subseteq \langle 2\rangle$, and hence $\bigcup_{S\in\mathcal{A}} S = \langle 2\rangle$.
Thus, $\mathcal{A}$ is a partition of $\langle 2\rangle$. □

37. For each $n \in \mathbb{Z}_{\geq 0}$, let $A_n = 2^n + \langle 2^{n+1} \rangle$. Then $\mathcal{A} = \{A_n \mid n \in \mathbb{Z}_{\geq 0}\}$ is a partition of $\mathbb{Z} \setminus \{0\}$.

Proof.
Let $n \in \mathbb{Z}_{\geq 0}$.
Since $2^n = 2^n + 2^{n+1}(0)$, we have $2^n \in A_n$. Thus, $A_n \neq \emptyset$.
Therefore, $\forall n \in \mathbb{Z}_{\geq 0}$, $A_n \neq \emptyset$.
Let $m, n \in \mathbb{Z}_{\geq 0}$ with $A_m \neq A_n$, and suppose $A_m \cap A_n \neq \emptyset$. WLOG, assume $m < n$.
Choose $x \in A_m \cap A_n$, and choose $s, t \in \mathbb{Z}$ with $x = 2^m + 2^{m+1}s$ and $x = 2^n + 2^{n+1}t$.
Then $2^m - 2^n = 2^{n+1}t - 2^{m+1}s$, which gives us $2^m(1 - 2^{n-m}) = 2^{m+1}(2^{n-m}t - s)$; hence $1 - 2^{n-m} = 2(2^{n-m}t - s)$.
Thus, $1 - 2^{n-m}$ is even, but since $m < n$, we can put $a = -2^{n-m-1}$, giving us $1 - 2^{n-m} = 1 + 2a$, which is odd.
This is a contradiction.
Therefore, $\forall m, n \in \mathbb{Z}_{\geq 0}$, if $A_m \neq A_n$, then $A_m \cap A_n = \emptyset$.
Let $x \in \mathbb{Z} \setminus \{0\}$ and let $S = \{n \in \mathbb{Z}_{\geq 0} \mid 2^n \text{ divides } x\}$.
Since $2^0 = 1$ and 1 divides x, we have $1 \in S$, and so $S \neq \emptyset$.
Also, if 2^n divides x, then since $x \neq 0$, $2^n \leq |x|$.
Thus, if $n \in S$, then $n \leq 2^n \leq |x|$, which means S is bounded above by $|x|$.
By the well-ordering property, choose n to be the largest element of S.
Then 2^n divides x. Choose $q \in \mathbb{Z}$ with $x = 2^n q$.
Suppose, looking for a contradiction, that q is even.
Choose $w \in \mathbb{Z}$ with $q = 2w$. Then $x = 2^n(2w) = 2^{n+1}2$; hence 2^{n+1} divides x.
This is a contradiction, since $n + 1 \notin S$.
Therefore, q is odd. We can therefore choose $c \in \mathbb{Z}$ with $q = 2c + 1$.
Now, $x = 2^n(2c + 1) = 2^n + 2^{n+1}c$; hence $x \in A_n$.
Therefore, $\exists n \in \mathbb{Z}_{\geq 0}$, $x \in A_n$. Thus, $x \in \bigcup_{n \in \mathbb{Z}_{\geq 0}} A_n$.
Therefore, $\mathbb{Z}_{\geq 0} \subseteq \bigcup_{n \in \mathbb{Z}_{\geq 0}} A_n$.
Conversely, let $x \in \bigcup_{n \in \mathbb{Z}_{\geq 0}} A_n$, and suppose $x \notin \mathbb{Z} \setminus \{0\}$. Since $x \in \mathbb{Z}$, this means $x = 0$.
Choose $k \in \mathbb{Z}$ with $x = 2^n + 2^{n+1}k$. i.e. $0 = 2^n + 2^{n+1}k$. Thus, $0 = 1 + 2k$.
This means 0 is odd, which is a contradiction, because 0 is even.
Therefore, $\bigcup_{n \in \mathbb{Z}_{\geq 0}} A_n \subseteq \mathbb{Z} \setminus \{0\}$. Thus, $\bigcup_{n \in \mathbb{Z}_{\geq 0}} A_n = \mathbb{Z} \setminus \{0\}$.
Therefore, $\mathcal{A}$ is a partition of $\mathbb{Z} \setminus \{0\}$. □

The following exercises investigate multiplication of congruence classes.

39. Write a complete multiplication table for the family of equivalence classes $\mathbb{Z}_5$.

Solution.

	$[0]_5$	$[1]_5$	$[2]_5$	$[3]_5$	$[4]_5$
$[0]_5$	$[0]_5$	$[0]_5$	$[0]_5$	$[0]_5$	$[0]_5$
$[1]_5$	$[0]_5$	$[1]_5$	$[2]_5$	$[3]_5$	$[4]_5$
$[2]_5$	$[0]_5$	$[2]_5$	$[4]_5$	$[1]_5$	$[3]_5$
$[3]_5$	$[0]_5$	$[3]_5$	$[1]_5$	$[4]_5$	$[2]_5$
$[4]_5$	$[0]_5$	$[4]_5$	$[3]_5$	$[2]_5$	$[1]_5$

□

41. Write a complete multiplication table for the family of equivalence classes $\mathbb{Z}_6$.

Solution.

	$[0]_6$	$[1]_6$	$[2]_6$	$[3]_6$	$[4]_6$	$[5]_6$
$[0]_6$	$[0]_6$	$[0]_6$	$[0]_6$	$[0]_6$	$[0]_6$	$[0]_6$
$[1]_6$	$[0]_6$	$[1]_6$	$[2]_6$	$[3]_6$	$[4]_6$	$[5]_6$
$[2]_6$	$[0]_6$	$[2]_6$	$[4]_6$	$[0]_6$	$[2]_6$	$[4]_6$
$[3]_6$	$[0]_6$	$[3]_6$	$[0]_6$	$[3]_6$	$[0]_6$	$[3]_6$
$[4]_6$	$[0]_6$	$[4]_6$	$[2]_6$	$[0]_6$	$[4]_6$	$[2]_6$
$[5]_6$	$[0]_6$	$[5]_6$	$[4]_6$	$[3]_6$	$[2]_6$	$[1]_6$

□

43. Prove your answer to exercise 42.

Proof. We will prove $\forall x, y \in \mathbb{Z}$, if $[x]_5[y]_5 = [0]_5$, then $[x]_5 = [0]_5$ or $[y]_5 = [0]_5$.
Let $x, y \in \mathbb{Z}$ and assume $[x]_5[y]_5 = [0]_5$.
This means $[xy]_5 = [0]_5$, and so $xy \equiv_5 0$. That is, $xy \in \langle 5 \rangle$.
By Euclid's Lemma (exercise 23), since 5 is prime, we have $x \in \langle 5 \rangle$ or $y \in \langle 5 \rangle$.
Therefore, $x \equiv_5 0$ or $y \equiv_5 0$. This means $[x]_5 = [0]_5$ or $[y]_5 = [0]_5$.
Therefore, $\forall x, y \in \mathbb{Z}$, if $[x]_5[y]_5 = [0]_5$, then $[x]_5 = [0]_5$ or $[y]_5 = [0]_5$. □

45. Prove your answer to exercise 44.

Proof. We will prove $\exists x, y \in \mathbb{Z}$, $[x]_6[y]_6 = [0]_6$, but $[x]_6 \neq [0]_6$ and $[y]_6 \neq [0]_6$.
Put $x = 2$ and $y = 3$.
Then $[x]_6[y]_6 = [2]_6[3]_6 = [6]_6 = [0]_6$,
but $[x]_6 \neq [0]_6$ and $[y]_6 \neq [0]_6$, since $2 \notin \langle 6 \rangle$ and $3 \notin \langle 6 \rangle$.
Therefore, $\exists x, y \in \mathbb{Z}$, $[x]_6[y]_6 = [0]_6$, but $[x]_6 \neq [0]_6$ and $[y]_6 \neq [0]_6$. □

47. Prove your answer to exercise 46.

Proof. We will prove $\forall x \in \mathbb{Z}$, if $[x]_5 \neq [0]_5$, then $\exists y \in \mathbb{Z}$, $[x]_5[y]_5 = [1]_5$.
Let $x \in \mathbb{Z}$ and assume $[x]_5 \neq [0]_5$.
Since 5 is prime, either $\gcd(x, 5) = 1$ or $\gcd(x, 5) = 5$.
Suppose $\gcd(x, 5) = 5$.
Then 5 divides x, which means $x \in \langle 5 \rangle$. This gives us $x \equiv_5 0$, and so $[x]_5 = [0]_5$. This is a contradiction.
Therefore, $\gcd(x, 5) \neq 5$, which means it must be the case that $\gcd(x, 5) = 1$.
We can therefore choose $y, z \in \mathbb{Z}$ with $xy + 5z = 1$.
Then $1 - xy = 5z$, and so $1 \equiv_5 xy$. This means $[xy]_5 = [1]_5$, and hence $[x]_5[y]_5 = [1]_5$.
Therefore, $\exists y \in \mathbb{Z}$, $[x]_5[y]_5 = [1]_5$.
Therefore, $\forall x \in \mathbb{Z}$, if $[x]_5 \neq [0]_5$, then $\exists y \in \mathbb{Z}$, $[x]_5[y]_5 = [1]_5$. □

49. Prove your answer to exercise 48.

Proof.
We will prove $\exists x \in \mathbb{Z}$, $[x]_6 \neq [0]_6$ and $\forall y \in \mathbb{Z}$, $[x]_6[y]_6 \neq [1]_6$.
Put $x = 3$. Then $[x]_6 \neq [0]_6$, since $3 \notin \langle 6 \rangle$.
Let $y \in \mathbb{Z}$ and suppose $[x]_6[y]_6 = [1]_6$.
Then $[3y]_6 = [1]_6$, which means $3y \equiv_6 1$. Choose $t \in \mathbb{Z}$ with $3y - 1 = 6t$.
We then have $1 = 3y - 6t$, which can be written as $1 = 3(y - 2t)$.
This implies that 3 divides 1, which is a contradiction since $1 < 3$.
Therefore, $\forall y \in \mathbb{Z}$, $[x]_6[y]_6 \neq [1]_6$.
Therefore, $\exists x \in \mathbb{Z}$, $[x]_6 \neq [0]_6$ and $\forall y \in mathbbZ$, $[x]_6[y]_6 \neq [1]_6$. □

51. Use the statement in exercise 46 to prove $\forall x, y, z \in \mathbb{Z}$, if $[x]_5[z]_5 = [y]_5[z]_5$ and $[z]_5 \neq [0]_5$, then $[x]_5 = [y]_5$.

Proof.
Let $x, y, z \in \mathbb{Z}$ and assume $[x]_5[z]_5 = [y]_5[z]_5$ and $[z]_5 \neq [0]_5$.
Since $[z]_5 \neq [0]_5$, by exercise 46 we can choose $k \in \mathbb{Z}$ with $[z]_5[k]_5 = [1]_5$.
We then have $[x]_5[z]_5[k]_5 = [y]_5[z]_5[k]_5$. Thus, $[x]_5[1]_5 = [y]_5[1]_5$, which gives us $[x]_5 = [y]_5$.
Therefore, $\forall x, y, z \in \mathbb{Z}$, if $[x]_5[z]_5 = [y]_5[z]_5$ and $[z]_5 \neq [0]_5$, then $[x]_5 = [y]_5$. □

Prove the following propositions about congruence classes.

53. Let $n \in \mathbb{N}$. $\forall x, y, z \in \mathbb{Z}$, $x - y \in [z]_n$ if and only if $x \in [y+z]_n$.

Proof.
Let $n \in \mathbb{N}$ and let $x, y, z \in \mathbb{Z}$.
Assume $x - y \in [z]_n$. Then $x - y \equiv_n z$, which means $x - y - z \in \langle n \rangle$.
This can be written as $x - (y + z) \in \langle n \rangle$, and so $x \equiv_n y + z$. Thus, $x \in [y + z]_n$.
Therefore, if $x - y \in [z]_n$, then $x \in [y + z]_n$.
Conversely, assume $x \in [y + z]_n$.
This means $x \equiv_n y + z$, which in turn means $x - (y + z) \in \langle n \rangle$.
This can be written as $(x - y) - z \in \langle n \rangle$, and so $x - y \equiv_n z$. Thus, $x - y \in [z]_n$.
Therefore, if $x \in [y + z]_n$, then $x - y \in [z]_n$.
Therefore, $\forall x, y, z \in \mathbb{Z}$, $x - y \in [z]_n$ if and only if $x \in [y + z]_n$. □

55. $\langle 2 \rangle \setminus \langle 4 \rangle = 2 + \langle 4 \rangle$.

Proof.
Let $x \in \langle 2 \rangle \setminus \langle 4 \rangle$. i.e. $x \in \langle 2 \rangle$ and $x \notin \langle 4 \rangle$.
Choose $a \in \mathbb{Z}$ with $x = 2a$.
Suppose, looking for a contradiction, that a is even. Choose $b \in \mathbb{Z}$ with $a = 2b$.
Then $x = 2(2b) = 4b$, which means $x \in \langle 4 \rangle$. This is a contradiction.
Therefore, a is odd. Choose $c \in \mathbb{Z}$ with $a = 2c + 1$.
Then $x = 2(2c + 1) = 2 + 4c$, and so $x \in 2 + \langle 4 \rangle$.
Therefore, $\langle 2 \rangle \setminus \langle 4 \rangle \subseteq 2 + \langle 4 \rangle$.
Conversely, let $x \in 2 + \langle 4 \rangle$. Choose $k \in \mathbb{Z}$ with $x = 2 + 4k$.
Putting $m = 1 + 2k$ gives us $x = 2 + 4k = 2(1 + 2k) = 2m$; hence $x \in \langle 2 \rangle$.
Suppose, looking for a contradiction, that $x \in \langle 4 \rangle$. Choose $n \in \mathbb{Z}$ with $x = 4n$.
Now, $2 + 4k = 4n$, which means $1 = 2(n - k)$. This shows 1 is even, which is a contradiction.
Therefore, $x \notin \langle 4 \rangle$, which means $x \in \langle 2 \rangle \setminus \langle 4 \rangle$.
Therefore, $2 + \langle 4 \rangle \subseteq \langle 2 \rangle \setminus \langle 4 \rangle$. Thus, $\langle 2 \rangle \setminus \langle 4 \rangle = 2 + \langle 4 \rangle$. □

57. $\langle 3 \rangle \setminus (3 + \langle 6 \rangle) = \langle 6 \rangle$.

Proof.
Let $x \in \langle 3 \rangle \setminus (3 + \langle 6 \rangle)$. i.e. $x \in \langle 3 \rangle$ and $x \notin 3 + \langle 6 \rangle$.
Choose $a \in \mathbb{Z}$ with $x = 3a$.
Suppose, looking for a contradiction, that a is odd. Choose $b \in \mathbb{Z}$ with $a = 2b + 1$.
Then $x = 3(2b + 1) = 3 + 6b$, which means $x \in 3 + \langle 6 \rangle$. This is a contradiction.
Therefore, a is even. Choosing $c \in \mathbb{Z}$ with $a = 2c$ gives us $x = 3(2c) = 6c$; hence $x \in \langle 6 \rangle$.
Therefore, $\langle 3 \rangle \setminus (3 + \langle 6 \rangle) \subseteq \langle 6 \rangle$.
Conversely, let $x \in \langle 6 \rangle$. Choose $k \in \mathbb{Z}$ with $x = 6k$.
Putting $m = 2k$ gives us $x = 6k = 3(2k) = 3m$; hence $x \in \langle 3 \rangle$.
Suppose, looking for a contradiction, that $x \in 3 + \langle 6 \rangle$. Choose $n \in \mathbb{Z}$ with $x = 3 + 6n$.
Now, $6k = 3 + 6n$, which means $1 = 2(k - n)$. This shows 1 is even, which is a contradiction.
Therefore, $x \notin 3 + langle 6 \rangle$, which means $x \in \langle 3 \rangle \setminus (3 + \langle 6 \rangle)$.
Therefore, $\langle 6 \rangle \subseteq \langle 3 \rangle \setminus (3 + \langle 6 \rangle)$. Thus, $\langle 3 \rangle \setminus (3 + \langle 6 \rangle) = \langle 6 \rangle$. □

59. Let $n \in \mathbb{N}$. $\forall a, x \in \mathbb{Z}$, if $a \in [x]_n$, then $-a \in [-x]_n$.

Proof.
Let $n \in \mathbb{N}$ and let $a, x \in \mathbb{Z}$. Assume $a \in [x]_n$.
Then $a \equiv_n x$, which means $a - x \in \langle n \rangle$. Choose $q \in \mathbb{Z}$ with $a - x = nq$.
Putting $p = -q$, we have $-a - (-x) = np$, which shows $-a - (-x) \in \langle n \rangle$.
Thus, $-a \equiv_n -x$, and so $-a \in [-x]_n$.
Therefore, $\forall a, x \in \mathbb{Z}$, if $a \in [x]_n$, then $-a \in [-x]_n$. □

61. $\forall n \in \mathbb{N}$, if $n \neq 1$ and n is not prime, then $\exists x, y \in \mathbb{Z}$, $[x]_n[y]_n = [0]_n$ but $[x]_n \neq [0]_n$ and $[y]_n \neq [0]_n$.

Proof.
Let $n \in \mathbb{N}$ and assume $n \neq 1$ and n is not prime.
This means $\exists x, y \in \mathbb{N}$, $n = xy$ and $x \neq 1$ and $y \neq 1$. Choose such x and y.
Suppose $[x]_n = [0]_n$.
Then $x \equiv_n 0$, which means $x \in \langle n \rangle$. Thus, n divides x, and so $n \leq x$.
However, $n = xy$, which means x divides n, and so $x \leq n$.
Therefore, $x = n$, which means $x = xy$. Thus, $y = 1$. This is a contradiction.
Therefore, $[x]_n \neq [0]_n$. Similarly, $[y]_n \neq [0]_n$.
Further, $[x]_n[y]_n = [xy]_n = [n]_n = [0]_n$.
Therefore, $\exists x, y \in \mathbb{Z}$, $[x]_n[y]_n = [0]_n$ but $[x]_n \neq [0]_n$ and $[y]_n \neq [0]_n$.
Therefore, $\forall n \in \mathbb{N}$, if $n \neq 1$ and n is not prime, then $\exists x, y \in \mathbb{Z}$, $[x]_n[y]_n = [0]_n$ but $[x]_n \neq [0]_n$ and $[y]_n \neq [0]_n$. □

3.2 Order Relations

Exercises 3.2.

For each of the following, prove that the relation is a partial ordering. If the relation is also a total ordering, prove it. Otherwise, prove that it is not a total ordering.

1. The relation $\preccurlyeq_{\mathbb{Z}}$ on $\mathbb{Q}$ given by: $\forall x, y \in \mathbb{Q}$, $x \preccurlyeq y$ if and only if $y - x \in \mathbb{Z}_{\geq 0}$.

Proof.
Let $x \in \mathbb{Q}$.
 $x - x = 0 \in \mathbb{Z}_{\geq 0}$, which means $x \preccurlyeq x$.
Therefore, $\forall x \in \mathbb{Q}$, $x \preccurlyeq x$. Thus, $\preccurlyeq$ is reflexive.
Next, let $x, y \in \mathbb{Q}$ and assume $x \preccurlyeq y$ and $y \preccurlyeq x$. This means $y - x \in \mathbb{Z}_{\geq 0}$ and $x - y \in \mathbb{Z}_{\geq 0}$.
 Then $0 \leq y - x$ and $0 \leq x - y$. Since $0 \leq x - y$, we have $y - x \leq 0$. We now have $y - x = 0$, and so $x = y$.
Therefore, $\forall x, y \in \mathbb{Q}$, if $x \preccurlyeq y$ and $y \preccurlyeq x$, then $x = y$. That is, $\preccurlyeq$ is antisymmetric.
Finally, let $x, y, z \in \mathbb{Q}$ and assume $x \preccurlyeq y$ and $y \preccurlyeq z$.
 Then $y - x \in \mathbb{Z}_{\geq 0}$ and $z - y \in \mathbb{Z}_{\geq 0}$.
 Therefore, $(y - x) + (z - y) \in \mathbb{Z}_{\geq 0}$, which gives us $z - x \in \mathbb{Z}_{\geq 0}$. Thus, $x \preccurlyeq z$.
Therefore, $\forall x, y, z \in \mathbb{Q}$, if $x \preccurlyeq y$ and $y \preccurlyeq z$, then $x \preccurlyeq z$. This means $\preccurlyeq$ is transitive.
Therefore, $\preccurlyeq$ is a partial ordering.

$\preccurlyeq$ is not a total ordering. Indeed, putting $x = 1$ and $y = \frac{1}{2}$, we have $x - y = \frac{1}{2} \notin \mathbb{Z}_{\geq 0}$ and $y - x = -\frac{1}{2} \notin \mathbb{Z}_{\geq 0}$.
For this choice of x and y, we have $x \npreceq y$ and $y \npreceq x$. Thus, $\exists x, y \in \mathbb{Q}$, $x \npreceq y$ and $y \npreceq x$.
Therefore, $\preccurlyeq$ is not a total ordering. □

3. The relation $\preccurlyeq$ on $\mathbb{Z}$ given by: $\forall x, y \in \mathbb{Z}$, $x \preccurlyeq y$ if and only if $\exists a \in \mathbb{Z}_{\geq 0}$, $y = x + 3a$.

Proof.
Let $x \in \mathbb{Z}$.
 $x = x + 3(0)$, and so $\exists a \in \mathbb{Z}_{\geq 0}$, $x = x + 3a$. Thus, $x \preccurlyeq x$.
Therefore, $\forall x \in \mathbb{Z}$, $x \preccurlyeq x$. This means $\preccurlyeq$ is reflexive.
Next, let $x, y \in \mathbb{Z}$ and assume $x \preccurlyeq y$ and $y \preccurlyeq x$.
 We then have $\exists a \in \mathbb{Z}_{\geq 0}$, $y = x + 3a$ and $\exists b \in \mathbb{Z}_{\geq 0}$, $x = y + 3b$. Choose such $a, b \in \mathbb{Z}_{\geq 0}$.
 Then $x = x + 3a + 3b$, and so $3(a + b) = 0$. It follows that $a = -b$.
 Since $0 \leq a$, we have $0 \leq -b$; hence $b \leq 0$. But, $0 \leq b$, which gives us $b = 0$. Now, $x = y + 3b = y + 3(0) = y$.
Therefore, $\forall x, y \in \mathbb{Z}$, if $x \preccurlyeq y$ and $y \preccurlyeq x$, then $x = y$. Thus, $\preccurlyeq$ is antisymmetric.
Finally, let $x, y, z \in \mathbb{Z}$ and assume $x \preccurlyeq y$ and $y \preccurlyeq z$.
 Then $\exists s \in \mathbb{Z}_{\geq 0}$, $y = x + 3s$ and $\exists t \in \mathbb{Z}_{\geq 0}$, $z = y + 3t$. Choose such $s, t \in \mathbb{Z}_{\geq 0}$.
 Putting $u = s + t$ gives us $z = y + 3t = x + 3s + 3t = x + 3u$, which proves $x \preccurlyeq z$.
Therefore, $\forall x, y, z \in \mathbb{Z}$, if $x \preccurlyeq y$ and $y \preccurlyeq z$, then $x \preccurlyeq z$. This means $\preccurlyeq$ is transitive. Thus, $\preccurlyeq$ is a partial ordering.

$\preccurlyeq$ is not a total ordering. To see this, put $x = 0$ and $y = 1$.
 Suppose $x \preccurlyeq y$. Then $\exists a \in \mathbb{Z}_{\geq 0}$, $1 = 0 + 3a$.
  Choosing such an $a \in \mathbb{Z}_{\geq 0}$ we have $a \neq 0$, since $1 \neq 0 + 3(0)$, and so $1 \leq a$.
  Then $3 \leq 3a$, which gives us $3 \leq 1$. This is a contradiction, and so $x \npreceq y$.
 Likewise, suppose $y \preccurlyeq x$ and accordingly, choose $b \in \mathbb{Z}_{\geq 0}$ with $0 = 1 + 3a$.
  Since $0 \leq a$, we have $1 \leq 1 + 3a$. This gives us $1 \leq 0$, which is a contradiction. Hence, $y \npreceq x$.
Therefore, $\exists x, y \in \mathbb{Z}$, $x \npreceq y$ and $y \npreceq x$. Therefore, $\preccurlyeq$ is not a total ordering. □

5. The relation $\preccurlyeq$ on $\mathbb{R}^2$ given by: $\forall (x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$, $(x_1, x_2) \preccurlyeq (y_1, y_2)$ if and only if $x_1 \leq y_1$ and $x_2 \leq y_2$.

Proof.
Let $(x_1, x_2) \in \mathbb{R}^2$.
Since $x_1 = x_1$ and $x_2 = x_2$, we have $x_1 \leq x_1$ and $x_2 \leq x_2$. Thus, $(x_1, x_2) \preccurlyeq (x_1, x_2)$.
Therefore, $\forall (x_1, x_2) \in \mathbb{R}^2$, $(x_1, x_2) \preccurlyeq (x_1, x_2)$. This means $\preccurlyeq$ is reflexive.
Next, let $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$, and assume $(x_1, x_2) \preccurlyeq (y_1, y_2)$ and $(y_1, y_2) \preccurlyeq (x_1, x_2)$.
Then $x_1 \leq y_1$, $x_2 \leq y_2$, $y_1 \leq x_1$, and $y_2 \leq x_2$.
Since $x_1 \leq y_1$ and $y_1 \leq x_1$, we have $x_1 = y_1$.
Similarly, since $x_2 \leq y_2$ and $y_2 \leq x_2$, we have $x_2 = y_2$. Therefore, $(x_1, x_2) = (y_1, y_2)$.
Therefore, $\forall (x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$, if $(x_1, x_2) \preccurlyeq (y_1, y_2)$ and $(y_1, y_2) \preccurlyeq (x_1, x_2)$, then $(x_1, x_2) = (y_1, y_2)$.
Thus, $\preccurlyeq$ is antisymmetric.
Finally, let $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in \mathbb{R}^2$, and assume $(x_1, x_2) \preccurlyeq (y_1, y_2)$ and $(y_1, y_2) \preccurlyeq (z_1, z_2)$.
Then $x_1 \leq y_1$, $x_2 \leq y_2$, $y_1 \leq z_1$, and $y_2 \leq z_2$.
Since $x_1 \leq y_1$ and $y_1 \leq z_1$, we have $x_1 \leq z_1$. Likewise, since $x_2 \leq y_2$ and $y_2 \leq z_2$, we have $x_2 \leq z_2$.
Now, $x_1 \leq z_1$ and $x_2 \leq z_2$, which means $(x_1, x_2) \preccurlyeq (z_1, z_2)$.
Therefore, $\forall (x_1, x_2), (y_1, y_2), (z_1, z_2) \in \mathbb{R}^2$, if $(x_1, x_2) \preccurlyeq (y_1, y_2)$ and $(y_1, y_2) \preccurlyeq (z_1, z_2)$, then $(x_1, x_2) \preccurlyeq (z_1, z_2)$.
Therefore, $\preccurlyeq$ is transitive. Thus, $\preccurlyeq$ is a partial ordering.

$\preccurlyeq$ is not a total ordering.
Indeed, putting $(x_1, x_2) = (0, 1)$ and $(y_1, y_2) = (1, 0)$, we have $x_2 \nleq y_2$, which implies $(x_1, x_2) \not\preccurlyeq (y_1, y_2)$, and $y_1 \nleq x_1$, which implies $(y_1, y_2) \not\preccurlyeq (x_1, x_2)$. Therefore, $\preccurlyeq$ is not a total ordering. □

7. The relation $\preccurlyeq$ on $\mathbb{R} \times (\mathbb{R} \setminus \{0\})$ given by: $\forall (x_1, x_2), (y_1, y_2) \in \mathbb{R} \times (\mathbb{R} \setminus \{0\})$, $(x_1, x_2) \preccurlyeq (y_1, y_2)$ if and only if $x_1 y_2 = x_2 y_1$ and $x_2 \leq y_2$.

Proof.
Let $(x_1, x_2) \in \mathbb{R} \times (\mathbb{R} \setminus \{0\})$.
Since $x_1 x_2 = x_2 x_1$ and $x_2 \leq x_2$, we have $(x_1, x_2) \preccurlyeq (x_1, x_2)$.
Therefore, $\forall (x_1, x_2) \in \mathbb{R} \times (\mathbb{R} \setminus \{0\})$, $(x_1, x_2) \preccurlyeq (x_1, x_2)$. This means $\preccurlyeq$ is reflexive.
Next, let $(x_1, x_2), (y_1, y_2) \in \mathbb{R} \times (\mathbb{R} \setminus \{0\})$, and assume $(x_1, x_2) \preccurlyeq (y_1, y_2)$ and $(y_1, y_2) \preccurlyeq (x_1, x_2)$.
Then $x_1 y_2 = x_2 y_1$, $x_2 \leq y_2$, $y_1 x_2 = y_2 x_1$, and $y_2 \leq x_2$. Since $x_2 \leq y_2$ and $y_2 \leq x_2$, we have $x_2 = y_2$.
Now, since $x_2 = y_2$, $x_1 y_2 = x_2 y_1$ can be written as $x_1 x_2 = y_1 x_2$.
Since $x_2 \neq 0$, this means $x_1 = y_1$. We now have $(x_1, x_2) = (y_1, y_2)$.
Therefore, $\forall (x_1, x_2), (y_1, y_2) \in \mathbb{R} \times (\mathbb{R} \setminus \{0\})$, if $(x_1, x_2) \preccurlyeq (y_1, y_2)$ and $(y_1, y_2) \preccurlyeq (x_1, x_2)$, then $(x_1, x_2) = (y_1, y_2)$.
Thus, $\preccurlyeq$ is antisymmetric.
Finally, let $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in \mathbb{R} \times (\mathbb{R} \setminus \{0\})$, and assume $(x_1, x_2) \preccurlyeq (y_1, y_2)$ and $(y_1, y_2) \preccurlyeq (z_1, z_2)$.
Then $x_1 y_2 = x_2 y_1$, $x_2 \leq y_2$, $y_1 z_2 = y_2 z_1$, and $y_2 \leq z_2$. Since $x_2 \leq y_2$ and $y_2 \leq z_2$, we have $x_2 \leq z_2$.
Also, since $x_1 y_2 = x_2 y_1$ and $y_1 z_2 = y_2 z_1$, we have $x_1 y_2 z_2 = x_2 y_1 z_2$ and $x_2 y_1 z_2 = x_2 y_2 z_1$.
Thus, $x_1 y_2 z_2 = x_2 y_2 z_1$. Since $y_2 \neq 0$, this gives us $x_1 z_2 x_2 z_1$. We now have $(x_1, x_2) \preccurlyeq (z_1, z_2)$.
Therefore, $\preccurlyeq$ is transitive. Thus, $\preccurlyeq$ is a partial ordering.

$\preccurlyeq$ is not a total ordering.
Indeed, putting $(x_1, x_2) = (1, 1)$ and $(y_1, y_2) = (0, 1)$, we have $x_1 y_2 = 1$ and $x_2 y_1 = 0$, and so $x_1 y_2 \neq x_2 y_1$. This implies $(x_1, x_2) \not\preccurlyeq (y_1, y_2)$ and $(y_1, y_2) \not\preccurlyeq (x_1, x_2)$. Therefore, $\preccurlyeq$ is not a total ordering. □

Prove the following propositions about extensions of the divides relation.

9. The relation $|$ on $\mathbb{Z}$, given by: $\forall x, y \in \mathbb{Z}$, $x|y$ if and only if $\exists q \in \mathbb{Z}$, $y = xq$, is not a partial ordering.

Proof.
Put $x = 1$ and $y = -1$.
Put $q = -1$.
Then $y = -1 = 1(-1) = xq$.
Therefore, $\exists q \in \mathbb{Z}$, $y = xq$. Thus, $x|y$.
For this same choice of $q = -1$, we also have $x = 1 = (-1)(-1) = yq$.
Therefore, $\exists q \in \mathbb{Z}$, $x = yq$. Thus, $y|x$.
We now have $x|y$ and $y|x$, but $x \neq y$.
Therefore, $\exists x, y \in \mathbb{Z}$, $x|y$ and $y|x$, and $x \neq y$. This means $|$ is not antisymmetric.
Therefore, $|$ is not a partial ordering. □

Prove the following propositions where the ordering is the usual ordering $\leq$ on $\mathbb{R}$.

11. The set $A = \{x \in \mathbb{Q} \mid x^2 < 2\}$ has both an upper bound and a lower bound in $\mathbb{Q}$.

Proof.
Put $k = \frac{3}{2}$.
Let $x \in A$. This means $x^2 < 2$.
Suppose $k < |x|$. i.e. $\frac{3}{2} < |x|$.
Since $\frac{3}{2} < |x|$, we have $\frac{9}{4} < |x|^2$, and so $\frac{9}{4} < x^2$.
By transitivity, $\frac{9}{4} < 2$, which gives us $9 < 8$. This is a contradiction.
Therefore, $|x| < \frac{3}{2}$, which implies $-\frac{3}{2} < x < \frac{3}{2}$.
Therefore, $\forall x \in A$, $-k < x$ and $x < k$. i.e. $-k$ is a lower bound of A and k is an upper bound of A.
Therefore, A has both an upper and a lower bound in $\mathbb{Q}$. □

13. $\forall a, b \in \mathbb{R}$, if $a < b$, then $\inf(a, b) = a$.

Proof.
Let $a, b \in \mathbb{R}$ and assume $a < b$.
Let $x \in (a, b)$. i.e. $a < x$ and $x < b$.
In particular, $a < x$, which implies $a \leq x$.
Therefore, $\forall x \in (a, b)$, $a \leq x$.
Let $c \in \mathbb{R}$ and assume c is a lower bound of the set (a, b). Suppose $a < c$.
Since $a < b$, we have $\frac{a+b}{2} \in (a, b)$, which gives us $c \leq \frac{a+b}{2} < b$; hence $c < b$.
Since $a < c$, putting $y = \frac{a+c}{2}$ gives us $a < y$ and $y < c$. Further, and since $c < b$, we have $y < b$.
Now, since $a < y$ and $y < b$, we have $y \in (a, b)$.
Since c is a lower bound of (a, b), we then have $x \leq y$, which contradicts $y < c$.
Therefore, $\forall c \in \mathbb{R}$, if c is a lower bound of (a, b), then $c \leq a$.
Therefore, a is the greatest lower bound of the set (a, b). That is, $\inf(a, b) = a$.
Therefore, $\forall a, b \in \mathbb{R}$, if $a < b$, then $\inf(a, b) = a$. □

15. $\sup\{x \in \mathbb{R} \mid \exists n \in \mathbb{N},\ x = \frac{n-1}{n}\} = 1$.

Proof. Let $A = \{x \in \mathbb{R} \mid \exists n \in \mathbb{N},\ x = \frac{n-1}{n}\}$.
Let $x \in A$, and choose $n \in \mathbb{N}$ with $x = \frac{n-1}{n}$.
Since $n - 1 \leq n$, we have $\frac{n-1}{n} \leq 1$; hence $x \leq 1$.
Therefore, $\forall x \in A$, $x \leq 1$. This means 1 is an upper bound of A.
Next, let $a \in \mathbb{R}$ and assume a is an upper bound of the set A. Suppose $a < 1$.
Since $a < 1$, we have $0 < 1 - a$. By the Archimedean property, choose $m \in \mathbb{N}$ with $1 < m(1 - a)$.
Then $1 < m - ma$, which means $ma < m - 1$, and so $a < \frac{m-1}{m}$.
But, $\frac{m-1}{m} \in A$, and so since a is an upper bound of A, we must have $\frac{m-1}{m} \leq a$. This is a contradiction.
Therefore, $\forall a \in \mathbb{R}$, if a is an upper bound of the set A, then $1 \leq a$.
Therefore, 1 is the least upper bound of the set A. That is, $\sup A = 1$.
Therefore, $\sup\{x \in \mathbb{R} \mid \exists n \in \mathbb{N},\ x = \frac{n-1}{n}\} = 1$. □

17. $\inf\{x \in \mathbb{R} \mid \exists n \in \mathbb{N},\ x = \frac{2^n+1}{2^n}\} = 1$.

Proof. Let $A = \{x \in \mathbb{R} \mid \exists n \in \mathbb{N},\ x = \frac{2^n+1}{2^n}\}$.
Let $x \in A$, and choose $n \in \mathbb{N}$ with $x = \frac{2^n+1}{2^n}$.
Since $2^n \leq 2^n + 1$, we have $1 \leq \frac{2^n+1}{2^n}$; hence $1 \leq x$.
Therefore, $\forall x \in A$, $1 \leq x$. This means 1 is a lower bound of A.
Next, let $a \in \mathbb{R}$ and assume a is a lower bound of the set A. Suppose $1 < a$.
Since $1 < a$, we have $0 < a - 1$. By the Archimedean property, choose $m \in \mathbb{N}$ with $1 < m(a - 1)$.
Since $m \leq 2^m$ and $0 < a - 1$, we have $m(a - 1) \leq 2^m(a - 1)$. By transitivity, $1 < 2^m(a - 1)$.
This gives us $1 < 2^m a - 2^m$, and so $2^m + 1 < 2^m a$. Now, $\frac{2^m+1}{2^m} < a$.
But, $\frac{2^m+1}{2^m} \in A$, and so since a is a lower bound of A, we must have $a \leq \frac{2^m+1}{2^m}$. This is a contradiction.
Therefore, $\forall a \in \mathbb{R}$, if a is a lower bound of the set A, then $a \leq 1$.
Therefore, 1 is the greatest lower bound of the set A. That is, $\inf A = 1$.
Therefore, $\inf\{x \in \mathbb{R} \mid \exists n \in \mathbb{N},\ x = \frac{2^n+1}{2^n}\} = 1$. □

19. $\sup\{x \in \mathbb{R} \mid \exists n \in \mathbb{N},\ x = \frac{3n-2}{n}\} = 3$.

Proof. Let $A = \{x \in \mathbb{R} \mid \exists n \in \mathbb{N},\ x = \frac{3n-2}{n}\}$.
Let $x \in A$, and choose $n \in \mathbb{N}$ with $x = \frac{3n-2}{n}$.
Since $3n - 2 \leq 3n$, we have $\frac{3n-2}{n} \leq 3$; hence $x \leq 3$.
Therefore, $\forall x \in A$, $x \leq 3$. This means 3 is an upper bound of A.
Next, let $a \in \mathbb{R}$ and assume a is an upper bound of the set A. Suppose $a < 3$.
Since $a < 3$, we have $0 < 3 - a$. By the Archimedean property, choose $m \in \mathbb{N}$ with $2 < m(3 - a)$.
Then $2 < 3m - ma$, which means $ma < 3m - 2$, and so $a < \frac{3m-2}{m}$.
But, $\frac{3m-2}{m} \in A$, and so since a is an upper bound of A, we must have $\frac{3m-2}{m} \leq a$. This is a contradiction.
Therefore, $\forall a \in \mathbb{R}$, if a is an upper bound of the set A, then $3 \leq a$.
Therefore, 3 is the least upper bound of the set A. That is, $\sup A = 3$.
Therefore, $\sup\{x \in \mathbb{R} \mid \exists n \in \mathbb{N},\ x = \frac{3n-2}{n}\} = 3$. □

21. $\sup\{x \in \mathbb{R} \mid \exists a \in (-\infty, 0),\ x = 1 + a\} = 1$.

Proof. Let $A = \{x \in \mathbb{R} \mid \exists a \in (-\infty, 0),\ x = 1 + a\}$.
Let $x \in A$. Choose $a \in (-\infty, 0)$ with $x = 1 + a$.
Then $a < 0$, and so $1 + a < 1$. This implies $x \leq 1$.
Therefore, $\forall x \in A$, $x \leq 1$. This means 1 is an upper bound of the set A.
Next, let $c \in \mathbb{R}$ and assume c is an upper bound of the set A. Suppose $c < 1$.
Let $y = \frac{c+1}{2}$. This gives us $c < y$ and $y < 1$.
Put $b = y - 1$. Since $y < 1$, this means $b < 0$. Thus, $b \in (-\infty, 0)$, and also, $y = 1 + b$.
Therefore, $\exists b \in (-\infty, 0)$, $y = 1 + b$. This proves $y \in A$.
Since c is an upper bound of A, we must then have $y \leq c$, which contradicts $c < y$.
Therefore, $\forall c \in \mathbb{R}$, if c is an upper bound of A, then $1 \leq c$.
Thus, 1 is the least upper bound of A. That is, $\sup A = 1$.
Therefore, $\sup\{x \in \mathbb{R} \mid \exists a \in (-\infty, 0),\ x = 1 + a\} = 1$. □

23. $\inf\{x \in \mathbb{R} \mid \exists a \in (-\infty, 0),\ x = 1 - a\} = 1$.

Proof. Let $A = \{x \in \mathbb{R} \mid \exists a \in (-\infty, 0),\ x = 1 - a\}$.
Let $x \in A$. Choose $a \in (-\infty, 0)$ with $x = 1 - a$.
Then $a < 0$, which means $0 < -a$, and so $1 < 1 - a$. This implies $1 \leq x$.
Therefore, $\forall x \in A$, $1 \leq x$. This means 1 is a lower bound of the set A.
Next, let $c \in \mathbb{R}$ and assume c is a lower bound of the set A. Suppose $1 < c$.
Let $y = \frac{c+1}{2}$. This gives us $1 < y$ and $y < c$.
Put $b = 1 - y$. Since $1 < y$, this means $b < 0$. Thus, $b \in (-\infty, 0)$, and also, $y = 1 - b$.
Therefore, $\exists b \in (-\infty, 0)$, $y = 1 - b$. This proves $y \in A$.
Since c is a lower bound of A, we must then have $c \leq y$, which contradicts $y < c$.
Therefore, $\forall c \in \mathbb{R}$, if c is a lower bound of A, then $c \leq 1$.
Thus, 1 is the greatest lower bound of A. That is, $\inf A = 1$.
Therefore, $\inf\{x \in \mathbb{R} \mid \exists a \in (-\infty, 0),\ x = 1 - a\} = 1$. □

25. $\sup\{x \in \mathbb{R} \mid \exists a \in (-2, 2),\ x = a^2\} = 4$.

Proof. Let $A = \{x \in \mathbb{R} \mid \exists a \in (-2, 2),\ x = a^2\}$.
Let $x \in A$. Choose $a \in (-2, 2)$ with $x = a^2$.
Then $-2 < a < 2$, which by exercise 127 implies $a^2 < 4$. This gives us $x \leq 4$.
Therefore, $\forall x \in A$, $x \leq 4$. This means 4 is an upper bound of the set A.
Next, let $c \in \mathbb{R}$ and assume c is an upper bound of the set A. Suppose $c < 4$.
Since $0 \in (-2, 2)$, we have $0 = 0^2 \in A$. Since c is an upper bound of A, we then have $0 \leq c$.
Put $b = \frac{c}{4} + 1$. Since $0 \leq c < 4$, we have $0 \leq \frac{c}{4} < 1$, and so $1 \leq b < 2$.
This implies $-2 < b < 2$, and hence $b \in (-2, 2)$.
Putting $y = b^2$ then gives us $y \in A$. Since c is an upper bound of A, we have $y \leq c$.
This means $b^2 \leq c$, which can be written as $(\frac{c}{4} + 1)^2 \leq c$, and so $\frac{c^2}{16} + \frac{c}{2} + 1 \leq c$.
Rearranging, we have $c^2 - 8c + 16 \leq 0$, which means $(c - 4)^2 \leq 0$.
However, since $0 \leq (c - 4)^2$, we have $(c - 4)^2 = 0$; hence $c = 4$. This is a contradiction, since $c < 4$.
Therefore, $\forall c \in \mathbb{R}$, if c is an upper bound of A, then $4 \leq c$.
Thus, 4 is the least upper bound of A. That is, $\sup A = 4$.
Therefore, $\sup\{x \in \mathbb{R} \mid \exists a \in (-2, 2),\ x = a^2\} = 4$. □

27. $\inf\{x \in \mathbb{R} \mid \exists a \in [0,1),\ x^2 = a\} = -1$.

Proof. Let $A = \{x \in \mathbb{R} \mid \exists a \in [0,1),\ x^2 = a\}$.
Let $x \in A$ and suppose $x < -1$. Choose $a \in [0,1)$ with $x^2 = a$.
Since $x < -1$, we have $1 < -x$, and so $0 < -x$. Therefore, $(-x)(1) < (-x)(-x)$, which means $-x < x^2$.
By transitivity, $1 < x^2$, and so $1 < a$. This is a contradiction, since $a \in [0,1)$.
Therefore, $\forall x \in A$, $-1 \leq x$. This means -1 is a lower bound of the set A.
Next, let $c \in \mathbb{R}$ and assume c is a lower bound of the set A. Suppose $-1 < c$.
Since $0 \in [0,1)$ and $0^2 = 0$, we have $0 \in A$. Since c is a lower bound of A, we have $c \leq 0$.
Put $y = \frac{c-1}{2}$ and put $b = y^2$. Since $-1 < c$ and $y = \frac{c-1}{2}$, we have $-1 < y < c \leq 0$.
Then $0 \leq y^2 < 1$, and so $b \in [0,1)$. Since $y^2 = b$, this proves $y \in A$.
Since c is a lower bound of A, we must then have $c \leq y$. This contradicts $y < c$.
Therefore, $\forall c \in \mathbb{R}$, if c is a lower bound of A, then $c \leq -1$.
Thus, -1 is the greatest lower bound of A. That is, $\inf A = -1$.
Therefore, $\inf\{x \in \mathbb{R} \mid \exists a \in [0,1),\ x^2 = a\} = -1$. □

Prove the following propositions about the subset ordering.

29. For a family of sets $\mathcal{A} \subseteq \mathcal{P}(U)$, $\bigcap_{S\in\mathcal{A}} S = \inf(\mathcal{A})$ under the partial ordering $\subseteq$.

Proof. Let $\mathcal{A} \subseteq \mathcal{P}(U)$ be a family of sets.
Let $X \in \mathcal{A}$.
  Let $x \in \bigcap_{S\in\mathcal{A}} S$. This means $\forall S \in \mathcal{A}, x \in S$.
    Since $X \in \mathcal{A}$, we then have $x \in X$.
  Therefore, $\bigcap_{S\in\mathcal{A}} S \subseteq X$.
Therefore, $\forall X \in \mathcal{A}$, $\bigcap_{S\in\mathcal{A}} S \subseteq X$. Thus, $\bigcap_{S\in\mathcal{A}} S$ is a lower bound of the set $\mathcal{A}$.
Next, assume C is a lower bound of $\mathcal{A}$.
  Let $x \in C$.
    Let $S \in \mathcal{A}$.
      Since C is a lower bound of $\mathcal{A}$, we have $C \subseteq S$, and so $x \in S$.
    Therefore, $\forall S \in \mathcal{A}, x \in S$. This means $x \in \bigcap_{S\in\mathcal{A}} S$.
  Therefore, $C \subseteq \bigcap_{S\in\mathcal{A}} S$.
Therefore, if C is a lower bound of $\mathcal{A}$, then $C \subseteq \bigcap_{S\in\mathcal{A}} S$.
Thus, $\bigcap_{S\in\mathcal{A}} S$ is the greatest lower bound of $\mathcal{A}$. In other words, $\bigcap_{S\in\mathcal{A}} S = \inf(\mathcal{A})$. □

31. Under the partial ordering $\subseteq$ on the family of sets $\mathcal{I} = \{\langle n\rangle \mid n \in \mathbb{Z}\}$, $\forall a, b \in \mathbb{Z}$, $\langle a\rangle + \langle b\rangle = \sup\{\langle a\rangle, \langle b\rangle\}$.

Proof.
Let $a, b \in \mathbb{Z}$, and let $\mathcal{A} = \{\langle a\rangle, \langle b\rangle\}$.
Note that by Proposition 2.3.11, $\langle a\rangle + \langle b\rangle \in \mathcal{I}$.
  Let $S \in \mathcal{A}$. i.e. $S = \langle a\rangle$ or $S = \langle b\rangle$.
    Let $x \in S$.
      Case 1: $S = \langle a\rangle$.
        Choose $m \in \mathbb{Z}$ with $x = am$, and put $n = 0$. Then $x = am + bn$; hence $x \in \langle a\rangle + \langle b\rangle$.
      Case 2: $S = \langle b\rangle$.
        Choose $t \in \mathbb{Z}$ with $x = bt$, and put $s = 0$. Then $x = as + bt$; hence $x \in \langle a\rangle + \langle b\rangle$.
    Therefore, $S \subseteq \langle a\rangle + \langle b\rangle$.
  Therefore, $\forall S \in \mathcal{A}$, $S \subseteq \langle a\rangle + \langle b\rangle$. This means $\langle a\rangle + \langle b\rangle$ is an upper bound of $\mathcal{A}$.
  Next, let $\langle c\rangle \in \mathcal{I}$ and assume $\langle c\rangle$ is an upper bound of the set $\mathcal{A}$.
    Since $\langle a\rangle \in \mathcal{A}$ and $\langle b\rangle \in \mathcal{A}$, we then have $\langle a\rangle \subseteq \langle c\rangle$ and $\langle b\rangle \subseteq \langle c\rangle$.
    Let $x \in \langle a\rangle + \langle b\rangle$, and choose $p, q \in \mathbb{Z}$ with $x = ap + bq$.
      Since $ap \in \langle a\rangle$ and $\langle a\rangle \subseteq \langle c\rangle$, we have $ap \in \langle c\rangle$. Choose $u \in \mathbb{Z}$ with $ap = cu$.
      Since $bq \in \langle b\rangle$ and $\langle b\rangle \subseteq \langle c\rangle$, we have $bq \in \langle c\rangle$. Choose $v \in \mathbb{Z}$ with $bq = cv$.
      Now, $x = ap + bq = cu + cv = c(u + v)$. Thus, $x \in langlec\rangle$.
    Therefore, $\langle a\rangle + \langle b\rangle \subseteq \langle c\rangle$.
  Therefore, $\forall\langle c\rangle \in \mathcal{I}$, if $\langle c\rangle$ is an upper bound of $\mathcal{A}$, then $\langle a\rangle + \langle b\rangle \subseteq \langle c\rangle$.
  Therefore, $\langle a\rangle + \langle b\rangle$ is the least upper bound of the set $\mathcal{A}$.
Therefore, $\forall a, b \in \mathbb{Z}$, $\langle a\rangle + \langle b\rangle = \sup\{\langle a\rangle, \langle b\rangle\}$. □

Let $\preccurlyeq$ be a partial ordering on a set U for which $\sup\{x, y\}$ and $\inf\{x, y\}$ exist for all $x, y \in U$. Prove the following propositions.

33. (Idempotence) $\forall x \in U$, $\sup\{x, x\} = x$.

Proof.
Let $x \in U$.
Let $a \in \{x, x\}$. i.e. $a = x$.
Since $\preccurlyeq$ is reflexive, we have $x \preccurlyeq x$; hence $a \preccurlyeq x$.
Therefore, $\forall a \in \{x, x\}$, $a \preccurlyeq x$. Thus, x is an upper bound of the set $\{x, x\}$.
Next, let $c \in U$ and assume c is an upper bound of the set $\{x, x\}$.
Since $x \in \{x, x\}$, we then have $x \preccurlyeq c$.
Therefore, $\forall c \in U$, if c is an upper bound of $\{x, x\}$, then $x \preccurlyeq c$.
Therefore, x is the least upper bound of the set $\{x, x\}$. This means $\sup\{x, x\} = x$.
Therefore, $\forall x \in U$, $\sup\{x, x\} = x$. □

35. (Associativity) $\forall x, y, z \in U$, $\inf\{\inf\{x, y\}, z\} = \inf\{x, \inf\{y, z\}\}$.

Proof.
Let $x, y, z \in U$.
Let $a \in \{x, \inf\{y, z\}\}$. i.e. $a = x$ or $a = \inf\{y, z\}$.
Case 1: $a = x$.
Since $\inf\{x, y\} \in \{\inf\{x, y\}, z\}$, we have $\inf\{\inf\{x, y\}, z\} \preccurlyeq \inf\{x, y\}$.
Since $x \in \{x, y\}$, we have $\inf\{x, y\} \preccurlyeq x$.
By transitivity, $\inf\{\inf\{x, y\}, z\} \preccurlyeq x$. Thus, $\inf\{\inf\{x, y\}, z\} \preccurlyeq a$.
Case 2: $a = \inf\{y, z\}$.
Since $y \in \{x, y\}$, we have $\inf\{x, y\} \preccurlyeq y$.
Since $\inf\{\inf\{x, y\}, z\} \preccurlyeq \inf\{x, y\}$, we then have $\inf\{\inf\{x, y\}, z\} \preccurlyeq y$.
Also, since $z \in \{\inf\{x, y\}, z\}$, we have $\inf\{\inf\{x, y\}, z\} \preccurlyeq z$.
This proves $\inf\{\inf\{x, y\}, z\}$ is a lower bound of the set $\{y, z\}$,
which gives us $\inf\{\inf\{x, y\}, z\} \preccurlyeq \inf\{y, z\}$. Thus, $\inf\{\inf\{x, y\}, z\} \preccurlyeq a$.
Therefore, $\forall a \in \{x, \inf\{y, z\}\}$, $\inf\{\inf\{x, y\}, z\} \preccurlyeq a$.
Thus, $\inf\{\inf\{x, y\}, z\}$ is a lower bound of the set $\{x, \inf\{y, z\}\}$, and so $\inf\{\inf\{x, y\}, z\} \preccurlyeq \inf\{x, \inf\{y, z\}\}$.
Next, let $a \in \{\inf\{x, y\}, z\}$.
Case 1: $a = \inf\{x, y\}$.
Since $\inf\{y, z\} \in \{x, \inf\{y, z\}\}$, we have $\inf\{x, \inf\{y, z\}\} \preccurlyeq \inf\{y, z\}$.
Since $y \in \{y, z\}$, we have $\inf\{y, z\} \preccurlyeq y$. Thus, $\inf\{x, \inf\{y, z\}\} \preccurlyeq y$ by transitivity.
Since $x \in \{x, \inf\{y, z\}\}$, we have $\inf\{x, \inf\{y, z\}\} \preccurlyeq x$.
This proves $\inf\{x, \inf\{y, z\}\}$ is a lower bound of the set $\{x, y\}$. We then have $\inf\{x, \inf\{y, z\}\} \preccurlyeq \inf\{x, y\}$.
Thus, $\inf\{x, \inf\{y, z\}\} \preccurlyeq a$.
Case 2: $a = z$.
Since $z \in \{y, z\}$, we have $\inf\{y, z\} \preccurlyeq z$. But, we also have $\inf\{x, \inf\{y, z\}\} \preccurlyeq \inf\{y, z\}$.
Therefore, $\inf\{x, \inf\{y, z\}\} \preccurlyeq z$ by transitivity. Thus, $\inf\{x, \inf\{y, z\}\} \preccurlyeq a$.
Therefore, $\forall a \in \{\inf\{x, y\}, z\}$, $\inf\{x, \inf\{y, z\}\} \preccurlyeq a$.
This means $\inf\{x, \inf\{y, z\}\}$ is a lower bound of $\{\inf\{x, y\}, z\}$, and so $\inf\{x, \inf\{y, z\}\} \preccurlyeq \{\inf\{x, y\}, z\}$.
Now, $\inf\{\inf\{x, y\}, z\} \preccurlyeq \inf\{x, \inf\{y, z\}\}$ and $\inf\{x, \inf\{y, z\}\} \preccurlyeq \{\inf\{x, y\}, z\}$.
By antisymmetry, $\inf\{\inf\{x, y\}, z\} = \inf\{x, \inf\{y, z\}\}$.
Therefore, $\forall x, y, z \in U$, $\inf\{\inf\{x, y\}, z\} = \inf\{x, \inf\{y, z\}\}$. □

37. (Absorption) $\forall x, y \in U$, $\sup\{x, \inf\{x, y\}\} = x$.

Proof.
Let $x, y \in U$.
  Let $a \in \{x, \inf\{x, y\}\}$.
    Case 1: $a = x$.
      Since $\preccurlyeq$ is reflexive, we have $x \preccurlyeq x$; hence $a \preccurlyeq x$.
    Case 2: $a = \inf\{x, y\}$.
      Since $x \in \{x, y\}$, we have $a \preccurlyeq x$.
  Therefore, $\forall a \in \{x, \inf\{x, y\}\}$, $a \preccurlyeq x$. Thus, x is an upper bound of the set $\{x, \inf\{x, y\}\}$.
  Let $c \in U$ and assume c is an upper bound of the set $\{x, \inf\{x, y\}\}$.
    Since $x \in \{x, \inf\{x, y\}\}$, we have $x \preccurlyeq c$.
  Therefore, $\forall c \in U$, if c is an upper bound of $\{x, \inf\{x, y\}\}$, then $x \preccurlyeq c$.
  Therefore, x is the least upper bound of the set $\{x, \inf\{x, y\}\}$. This means $\sup\{x, \inf\{x, y\}\} = x$.
Therefore, $\forall x, y \in U$, $\sup\{x, \inf\{x, y\}\} = x$. □

39. (Identity) If b is the greatest element of U, then $\forall x \in U$, $\inf\{x, b\} = x$.

Proof.
Assume b is the greatest element of U.
  Let $x \in U$.
    Since $\preccurlyeq$ is reflexive, $x \preccurlyeq x$.
    Since b is the greatest element of U, $x \preccurlyeq b$.
    Since $x \preccurlyeq x$ and $x \preccurlyeq b$, we have that x is a lower bound of the set $\{x, b\}$.
    Let $c \in U$ and assume c is a lower bound of the set $\{x, b\}$.
      Then, since $x \in \{x, b\}$, we have $c \preccurlyeq x$.
    Therefore, $\forall c \in U$, if c is a lower bound of $\{x, b\}$, then $c \preccurlyeq x$.
    Thus, x is the greatest lower bound of the set $\{x, b\}$.
  Therefore, $\forall x \in U$, $\inf\{x, b\} = x$.
Therefore, if b is the greatest element of U, then $\forall x \in U$, $\inf\{x, b\} = x$. □

41. (Annihilator) If b is the greatest element of U, then $\forall x \in U$, $\sup\{x, b\} = b$.

Proof.
Assume b is the greatest element of U.
  Let $x \in U$.
    Since b is the greatest element of U, $x \preccurlyeq b$.
    Since $\preccurlyeq$ is reflexive, $b \preccurlyeq b$.
    Since $x \preccurlyeq b$ and $b \preccurlyeq b$, we have that b is an upper bound of the set $\{x, b\}$.
    Let $c \in U$ and assume c is an upper bound of the set $\{x, b\}$.
      Then, since $b \in \{x, b\}$, we have $b \preccurlyeq c$.
    Therefore, $\forall c \in U$, if c is an upper bound of $\{x, b\}$, then $b \preccurlyeq c$.
    Thus, b is the least upper bound of the set $\{x, b\}$.
  Therefore, $\forall x \in U$, $\sup\{x, b\} = b$.
Therefore, if b is the greatest element of U, then $\forall x \in U$, $\sup\{x, b\} = b$. □

43. $\forall a, x, y \in U$, if $x \preccurlyeq y$, then $\inf\{a, x\} \preccurlyeq \inf\{a, y\}$.

Proof.
Let $a, x, y \in U$, and assume $x \preccurlyeq y$.
Since $a \in \{a, x\}$, we have $\inf\{a, x\} \preccurlyeq a$.
Since $x \in \{a, x\}$, we have $\inf\{a, x\} \preccurlyeq x$, and since $x \preccurlyeq y$, this implies $\inf\{a, x\} \preccurlyeq y$.
Now, since $\inf\{a, x\} \preccurlyeq a$ and $\inf\{a, x\} \preccurlyeq y$, $\inf\{a, x\}$ is a lower bound of the set $\{a, y\}$.
Therefore, $\inf\{a, x\} \preccurlyeq \inf\{a, y\}$.
Therefore, $\forall a, x, y \in U$, if $x \preccurlyeq y$, then $\inf\{a, x\} \preccurlyeq \inf\{a, y\}$. □

45. If $\preccurlyeq$ is a total ordering, then $\forall x, y \in U$, $\sup\{x, y\} = x$ or $\sup\{x, y\} = y$.

Proof.
Assume $\preccurlyeq$ is a total ordering, and let $x, y \in U$.
Since $\preccurlyeq$ is a total ordering, we have $x \preccurlyeq y$ or $y \preccurlyeq x$.
Case 1: $x \preccurlyeq y$.
Since $\preccurlyeq$ is reflexive, $y \preccurlyeq y$. Since $x \preccurlyeq y$ and $y \preccurlyeq y$, we have that y is an upper bound of the set $\{x, y\}$.
Let $c \in U$ and assume c is an upper bound of the set $\{x, y\}$.
Since $y \in \{x, y\}$, we then have $y \preccurlyeq c$.
Therefore, $\forall c \in U$, if c is an upper bound of $\{x, y\}$, then $y \preccurlyeq c$.
This proves y is the least upper bound of the set $\{x, y\}$. This means $\sup\{x, y\} = y$.
Case 2: $y \preccurlyeq x$.
Since $\preccurlyeq$ is reflexive, $x \preccurlyeq x$. Since $x \preccurlyeq x$ and $y \preccurlyeq x$, we have that x is an upper bound of the set $\{x, y\}$.
Let $c \in U$ and assume c is an upper bound of the set $\{x, y\}$.
Since $x \in \{x, y\}$, we then have $x \preccurlyeq c$.
Therefore, $\forall c \in U$, if c is an upper bound of $\{x, y\}$, then $x \preccurlyeq c$.
Thus, x is the least upper bound of the set $\{x, y\}$. That is, $\sup\{x, y\} = x$.
Therefore, $\sup\{x, y\} = x$ or $\sup\{x, y\} = y$.
Therefore, if $\preccurlyeq$ is a total ordering, then $\forall x, y \in U$, $\sup\{x, y\} = x$ or $\sup\{x, y\} = y$. □

Let $\preccurlyeq$ be a partial ordering on a set U, and let A and B be subsets of U for which $\sup A$, $\inf A$, $\sup B$, and $\inf B$ exist. Prove the following propositions.

47. $\forall a, b \in U$, if a and b are both greatest elements of A, then $a = b$.

Proof.
Let $a, b \in U$, and assume a and b are both greatest elements of A.
This means ($a \in A$ and $\forall x \in A$, $x \preccurlyeq a$) and ($b \in A$ and $\forall x \in A$, $x \preccurlyeq b$).
Since $a \in A$ and b is a greatest element of A, we have $a \preccurlyeq b$.
Likewise, since $b \in A$ and a is a greatest element of A, we have $b \preccurlyeq a$.
Now, $a \preccurlyeq b$ and $b \preccurlyeq a$. By antisymmetry, we then have $a = b$.
Therefore, $\forall a, b \in U$, if a and b are both greatest elements of A, then $a = b$. □

49. $\forall a \in U$, if $a \in A$ and $a = \sup A$, then a is the greatest element of A.

Proof.
Let $a \in U$, and assume $a \in A$ and $a = \sup A$.
Since $a = \sup A$, we have that a is an upper bound of A. This means $\forall x \in A$, $x \preccurlyeq a$.
Now, $a \in A$ and $\forall x \in A$, $x \preccurlyeq a$, which means a is the greatest element of A. Therefore, $\forall a \in U$, if $a \in A$ and $a = \sup A$, then a is the greatest element of A. □

51. $\forall a \in U$, if a is the least element of A, then $a = \inf A$.

Proof.
Let $a \in U$, and assume a is the least element of A.
This means $a \in A$ and $\forall x \in A$, $a \preccurlyeq x$.
Since $\forall x \in A$, $a \preccurlyeq x$, we have that a is a lower bound of A.
Let $c \in U$ and assume c is a lower bound of A.
Since $a \in A$, we then have $c \preccurlyeq a$.
Therefore, $\forall c \in U$, if c is a lower bound of A, then $c \preccurlyeq a$.
This proves a is the greatest lower bound of A. That is, $a = \inf A$.
Therefore, $\forall a \in U$, if a is the least element of A, then $a = \inf A$. □

53. If $\inf A \notin A$, then A does not have a least element.

Proof.
Suppose $\inf A \notin A$ and A has a least element.
Choose $a \in U$ to be the least element of A.
Then $a \in A$, and by exercise 51, we have $a = \inf A$. This is a contradiction, since $\inf A \notin A$.
Therefore, if $\inf A \notin A$, then A does not have a least element. □

55. If $A \subseteq B$, then $\sup A \preccurlyeq \sup B$.

Proof.
Assume $A \subseteq B$.
Let $x \in A$.
Since $A \subseteq B$, we then have $x \in B$. Since $\sup B$ is an upper bound of B, we then have $x \preccurlyeq \sup B$.
Therefore, $\forall x \in A$, $x \preccurlyeq \sup B$. This means $\sup B$ is an upper bound of A.
Since $\sup A$ is the least upper bound of A, we then have $\sup A \preccurlyeq \sup B$.
Therefore, if $A \subseteq B$, then $\sup A \preccurlyeq \sup B$. □

57. If $A \neq \emptyset$, then $\inf A \preccurlyeq \sup A$.

Proof.
Assume $A \neq \emptyset$. Then we can choose an element $x \in A$.
Since $x \in A$ and $\inf A$ is a lower bound of A, we have $\inf A \preccurlyeq x$.
Likewise, since $\sup A$ is an upper bound of A, we have $x \preccurlyeq \sup A$.
Now, $\inf A \preccurlyeq x$ and $x \preccurlyeq \sup A$, and so by transitivity, $\inf A \preccurlyeq \sup A$.
Therefore, if $A \neq \emptyset$, then $\inf A \preccurlyeq \sup A$. □

59. $\forall a \in U$, if $\preccurlyeq$ is a total ordering and a is maximal in A, then a is the greatest element of A.

Proof.
Let $a \in U$, and assume $\preccurlyeq$ is a total ordering and a is maximal in A.
This means $a \in A$ and $\forall x \in A$, if $a \preccurlyeq x$, then $a = x$.
Let $x \in A$.
Since $\preccurlyeq$ is a total ordering, we have $a \preccurlyeq x$ or $x \preccurlyeq a$.
However, in the case where $a \preccurlyeq x$, since a is maximal, we have $a = x$; hence $x \preccurlyeq a$ by reflexivity.
Therefore, if either case, we have $x \preccurlyeq a$.
Therefore, $\forall x \in A$, $x \preccurlyeq a$.
Now, $a \in A$ and $\forall x \in A$, $x \preccurlyeq a$, which means a is the greatest element of A.
Therefore, $\forall a \in U$, if $\preccurlyeq$ is a total ordering and a is maximal in A, then a is the greatest element of A. □

61. If $\forall x \in B$, $\forall y \in A$, $x \preccurlyeq y$, then $\sup B \preccurlyeq \inf A$.

Proof.
Assume $\forall x \in B$, $\forall y \in A$, $x \preccurlyeq y$.
Let $x \in B$.
Let $y \in A$.
Since by assumption, we have $\forall x \in B$, $\forall y \in A$, $x \preccurlyeq y$, this gives us $x \preccurlyeq y$.
Therefore, $\forall y \in A$, $x \preccurlyeq y$. Thus, x is a lower bound of the set A.
Since $\inf A$ is the greatest lower bound of A, we then have $x \preccurlyeq \inf A$.
Therefore, $\forall x \in B$, $x \preccurlyeq \inf A$. Thus, $\inf A$ is an upper bound of the set B.
Since $\sup B$ is the least upper bound of the set B, we then have $\sup B \preccurlyeq \inf A$.
Therefore, if $\forall x \in B$, $\forall y \in A$, $x \preccurlyeq y$, then $\sup B \preccurlyeq \inf A$. □

For the partial ordering $\preccurlyeq_{\mathbb{Z}}$ on the set $\mathbb{Q}$, defined in exercise 1, prove the following propositions.

63. $\frac{1}{2}$ is a maximal element of the set $A = \{x \in \mathbb{Q} \mid 0 \leq x \leq 1\}$.

Proof.
Since $0 \leq \frac{1}{2} \leq 1$, we have $\frac{1}{2} \in A$.
Let $x \in A$, and assume $\frac{1}{2} \preccurlyeq_{\mathbb{Z}} x$.
This means $x - \frac{1}{2} \in \mathbb{Z}_{\geq 0}$.
Since $x \in A$, we have $0 \leq x$ and $x \leq 1$. Since $x \leq 1$, we have $x - \frac{1}{2} \leq \frac{1}{2}$, and so $x - \frac{1}{2} < 1$.
Since $x - \frac{1}{2} \in \mathbb{Z}_{\geq 0}$, we have $x - \frac{1}{2} \leq 0$ by Theorem 1.2.3.
However, since $x - \frac{1}{2} \in \mathbb{Z}_{\geq 0}$, we also have $0 \leq x - \frac{1}{2}$. This gives us $x - \frac{1}{2} = 0$; hence $x = \frac{1}{2}$.
Therefore, $\forall x \in A$, if $\frac{1}{2} \preccurlyeq_{\mathbb{Z}} x$, then $x = \frac{1}{2}$.
Now, $\frac{1}{2} \in A$ and $\forall x \in A$, if $\frac{1}{2} \preccurlyeq_{\mathbb{Z}} x$, then $x = \frac{1}{2}$.
Therefore, $\frac{1}{2}$ is maximal in the set A. □

Let U be a non-empty set with at least two elements, and let $\mathcal{A} = \mathcal{P}(U) \setminus \{\emptyset, U\}$. That is, $\mathcal{A}$ is the family of non-empty, non-trivial subsets of U. Prove the following propositions for the partial ordering $\subseteq$.

65. $\forall a \in U$, $\{a\}$ is a minimal element of $\mathcal{A}$.

Proof.
Let $a \in U$.
Since U has at least two elements, we have $\{a\} \neq U$. Also, since $a \in \{a\}$, we have $\{a\} \neq \emptyset$.
Thus, $\{a\} \notin \{\emptyset, U\}$, which proves $\{a\} \in \mathcal{A}$.
Let $S \in \mathcal{A}$ and assume $S \subseteq \{a\}$.
Since $S \in \mathcal{A}$, we have that $S \neq \emptyset$. Choose an element $x \in S$.
Then $x \in \{a\}$, since $S \subseteq \{a\}$, which gives us $x = a$.
Therefore, $a \in S$, which proves $\{a\} \subseteq S$. Since we also have $S \subseteq \{a\}$, this means $S = \{a\}$.
Therefore, $\forall S \in \mathcal{A}$, if $S \subseteq \{a\}$, then $S = \{a\}$. Thus, $\{a\}$ is minimal in $\mathcal{A}$.
Therefore, $\forall a \in U$, $\{a\}$ is a minimal element of $\mathcal{A}$. □

67. $\forall S \in \mathcal{A}$, if S is a maximal element of $\mathcal{A}$, then $\exists a \in U$, $S = U \setminus \{a\}$.

Proof.
Let $S \in \mathcal{A}$, and assume S is maximal element of $\mathcal{A}$.
Since $S \in \mathcal{A}$, we have $S \neq U$. Choose an element $a \in U$ with $a \notin S$.
Let $x \in S$.
Since $S \in \mathcal{P}(U)$, we have $S \subseteq U$, which gives us $x \in U$.
Since $a \notin S$, we have $x \neq a$, which means $x \notin \{a\}$. Thus, $x \in U \setminus \{a\}$.
Therefore, $S \subseteq U \setminus \{a\}$.
Since S is maximal in $\mathcal{A}$, this implies $S = U \setminus \{a\}$.
Therefore, $\exists a \in U$, $S = U \setminus \{a\}$.
Therefore, $\forall S \in \mathcal{A}$, if S is a maximal element of $\mathcal{A}$, then $\exists a \in U$, $S = U \setminus \{a\}$. □

3.3 Functions

Exercises 3.3.

For each of the following, prove that the relation f is a function. State the domain and codomain of f.

1. $f = \{(x, n) \in (0, 1] \times \mathbb{N} \mid \frac{1}{n+1} < x \leq \frac{1}{n}\}$.

Proof.
Let $x \in (0, 1]$.
Let $S = \{n \in \mathbb{N} \mid x \leq \frac{1}{n}\}$.
Since $x \in (0, 1]$, we have $x \leq 1$. This means $x \leq \frac{1}{1}$, which tells us $1 \in S$. Thus, $S \neq \emptyset$.
By the Archimedean property, choose $k \in \mathbb{N}$ with $\frac{1}{k} < x$.
Then for any $m \in S$, we have $\frac{1}{k} < x \leq \frac{1}{m}$, which means $m < k$. Thus, S is bounded above by k.
Applying the well-ordering property, choose $n \in \mathbb{N}$ to be the largest element of S.
Since $n \in S$, we have $x \leq \frac{1}{n}$.
Since $n + 1 \notin S$, we have $\frac{1}{n+1} < x$.
Therefore, $\frac{1}{n+1} < x \leq \frac{1}{n}$; hence $(x, n) \in f$.
Therefore, $\exists n \in \mathbb{N}, (x, n) \in f$.
Therefore, $\forall x \in (0, 1], \exists n \in \mathbb{N}, (x, n) \in f$.
Next, let $x \in (0, 1]$, let $n_1, n_2 \in \mathbb{N}$, and assume $(x, n_1) \in f$ and $(x, n_2) \in f$.
Then $\frac{1}{n_1+1} < x \leq \frac{1}{n_1}$ and $\frac{1}{n_2+1} < x \leq \frac{1}{n_2}$.
Now, $\frac{1}{n_1+1} < \frac{1}{n_2}$ and $\frac{1}{n_2+1} < \frac{1}{n_1}$ by transitivity.
Therefore, $n_2 < n_1 + 1$ and $n_1 < n_2 + 1$.
This implies $n_2 \leq n_1$ and $n_1 \leq n_2$; hence $n_1 = n_2$.
Therefore, $\forall x \in [0, 1), \forall n_1, n_2 \in \mathbb{N}$, if $(x, n_1) \in f$ and $(x, n_2) \in f$, then $n_1 = n_2$.
Therefore, f is a function. □

3. Let $S = \{x \in \mathbb{Z} \mid 0 \leq x < 5\}$.
$f = \{(x, r) \in \mathbb{N} \times S \mid \exists q \in \mathbb{Z},\ x = 5q + r\}$.

Proof.
Let $x \in \mathbb{N}$.
Applying the division algorithm, choose $q, r \in \mathbb{Z}$ with $x = 5q + r$ and $0 \leq r < 5$.
Since $0 \leq r < 5$, we have $r \in S$ and hence $(x, r) \in f$.
Therefore, $\exists r \in S, (x, r) \in f$.
Therefore, $\forall x \in \mathbb{N}, \exists r \in S, (x, r) \in f$.
Next, let $x \in \mathbb{N}$, let $r_1, r_2 \in S$, and assume $(x, r_1) \in f$ and $(x, r_2) \in f$.
Choose $q_1, q_2 \in \mathbb{Z}$ with $x = 5q_1 + r_1$ and $x = 5q_2 + r_2$. Then $5q_1 + r_1 = 5q_2 + r_2$.
Now, $5(q_1 - q_2) = r_1 - r_2$, which means 5 divides $r_1 - r_2$. However, since $r_1, r_2 \in S$, we have $|r_1 - r_2| < 5$.
It must therefore be the case that $r_1 - r_2 = 0$. Thus, $r_1 = r_2$.
Therefore, $\forall x \in \mathbb{N}, \forall r_1, r_2 \in S$, if $(x, r_1) \in f$ and $(x, r_2) \in f$, then $r_1 = r_2$.
Therefore, f is a function. □

For each of the following, find expressions for the functions $g \circ f$ and $f \circ g$.

5. $f : \mathbb{Z} \to \mathbb{Z}$, given by $\forall x \in \mathbb{Z},\ f(x) = 2x + 1$.
   $g : \mathbb{Z} \to \mathbb{Z}$, given by $\forall x \in \mathbb{Z},\ g(x) = \begin{cases} x+1 & \text{if } x \text{ is even} \\ x-1 & \text{if } x \text{ is odd} \end{cases}.$

   Solution.

   $$\forall x \in \mathbb{Z},\ g \circ f(x) = 2x.$$
   $$\forall x \in \mathbb{Z},\ f \circ g(x) = \begin{cases} 2x+3 & \text{if } x \text{ is even} \\ 2x-1 & \text{if } x \text{ is odd} \end{cases}.$$

   □

Let A, B, and C be sets, and let $f : A \to B$ and $g : B \to C$ be functions. Prove the following.

7. If $\alpha : B \to A$ and $\beta : B \to A$ are functions for which $\alpha \circ f = i_A$ and $f \circ \beta = i_B$, then $\alpha = \beta$. (Thus, the inverse of a function is unique).

   Proof.
   Let $x \in B$.
   Then $\alpha(x) = \alpha(i_B(x)) = \alpha(f \circ \beta(x)) = \alpha(f(\beta(x))) = \alpha \circ f(\beta(x)) = i_A(\beta(x)) = \beta(x)$.
   Therefore, $\forall x \in B$, $\alpha(x) = \beta(x)$.
   Thus, $\alpha = \beta$. □

9. Suppose $A = B$, so that $f : A \to A$. If $f \circ f$ is invertible, then f is invertible.

   Proof.
   Assume $f \circ f$ is invertible, and let h be the inverse of $f \circ f$.
   Then $h \circ f \circ f = i_A$ and $f \circ f \circ h = i_A$.
   Letting $\alpha = h \circ f$ and $\beta = f \circ h$, we have $\alpha \circ f = i_A$ and $f \circ \beta = i_A$.
   From 7, we have $\alpha = \beta$.
   Therefore, $\alpha \circ f = i_A$ and $f \circ \alpha = i_A$; hence f is invertible with $\alpha = f^{-1}$.
   Therefore, if $f \circ f$ is invertible, then f is invertible. □

11. If $g \circ f$ is invertible and g is invertible, then f is invertible.

    Proof.
    Assume $g \circ f$ is invertible and g is invertible.
    Let $h : C \to A$ be the inverse of $g \circ f$. i.e. $h \circ g \circ f = i_A$ and $g \circ f \circ h = i_C$.
    We claim that $h \circ g = f^{-1}$.
    Indeed, we have $(h \circ g) \circ f = i_A$.
    Also, $f \circ (h \circ g) = g^{-1} \circ g \circ f \circ h \circ g = g^{-1} \circ i_C \circ g = g^{-1} \circ g = i_B$.
    Therefore, f is invertible with $h \circ g = f^{-1}$.
    Therefore, if $g \circ f$ is invertible and g is invertible, then f is invertible. □

For each of the following, prove the function is bijective.

13. $f : \mathbb{R} \to \mathbb{R}$, given by $\forall x \in \mathbb{R}$, $f(x) = \dfrac{5x-7}{2}$.

Proof.
Let $x_1, x_2 \in \mathbb{R}$ and assume $f(x_1) = f(x_2)$.
This means $\frac{5x_1-7}{2} = \frac{5x_2-7}{2}$; hence $5x_1 - 7 = 5x_2 - 7$; $5x_1 = 5x_2$; $x_1 = x_2$.
Therefore, $\forall x_1, x_2 \in \mathbb{R}$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.
Thus, f is injective.
Next, let $y \in \mathbb{R}$.
Put $x = \frac{2y+7}{5}$.
Then $f(x) = \frac{5x-7}{2} = \frac{(2y+7)-7}{2} = y$.
Therefore, $\exists x \in \mathbb{R}$, $f(x) = y$.
Therefore, $\forall y \in \mathbb{R}, \exists x \in \mathbb{R}, f(x) = y$.
Thus, f is surjective. We now have that f is injective and surjective, which means f is bijective. □

15. $f : \mathbb{R}^2 \to \mathbb{R}^2$, given by $\forall (x_1, x_2) \in \mathbb{R}^2,\ f(x_1, x_2) = (x_1 + x_2, x_1 - x_2)$.

Proof.
Let $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$ and assume $f(x_1, x_2) = f(y_1, y_2)$.
Then $(x_1 + x_2, x_1 - x_2) = (y_1 + y_2, y_1 - y_2)$. This means $x_1 + x_2 = y_1 + y_2$ and $x_1 - x_2 = y_1 - y_2$.
Adding these two equations gives us $2x_1 = 2y_1$; hence $x_1 = y_1$.
Similarly, by subtracting the equations, we have $2x_2 = 2y_2$; hence $x_2 = y_2$.
Therefore, $x_1 = y_1$ and $x_2 = y_2$; hence $(x_1, x_2) = (y_1, y_2)$.
Therefore, $\forall (x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$, if $f(x_1, x_2) = f(y_1, y_2)$, then $(x_1, x_2) = (y_1, y_2)$.
Thus, f is injective.
Next, let $(y_1, y_2) \in \mathbb{R}^2$.
Put $x_1 = \frac{y_1+y_2}{2}$ and put $x_2 = \frac{y_1-y_2}{2}$.
Then $f(x_1, x_2) = (x_1 + x_2, x_1 - x_2) = (\frac{y_1+y_2}{2} + \frac{y_1-y_2}{2}, \frac{y_1+y_2}{2} - \frac{y_1-y_2}{2}) = (y_1, y_2)$.
Therefore, $\forall (y_1, y_2) \in \mathbb{R}^2, \exists (x_1, x_2) \in \mathbb{R}^2, f(x_1, x_2) = (y_1, y_2)$. □

Let A be a set, and let $f : A \to A$ be a function. Prove the following.

17. If $f \circ f$ is injective, then f is injective.

Proof.
Assume $f \circ f$ is injective.
Let $x_1, x_2 \in A$, and assume $f(x_1) = f(x_2)$.
Then $f(f(x_1)) = f(f(x_2))$. In other words, $f \circ f(x_1) = f \circ f(x_2)$.
Since $f \circ f$ is injective, we have $x_1 = x_2$.
Therefore, $\forall x_1, x_2 \in A$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.
Thus, f is injective.
Therefore, if $f \circ f$ is injective, then f is injective. □

19. If f is surjective, then $f \circ f$ is surjective.

Proof.
Assume f is surjective.
Let $z \in A$.
Since f is surjective, we can choose $y \in A$ with $f(y) = z$.
Again, since f is surjective, we can choose $x \in A$ with $f(x) = y$.
Now, $f \circ f(x) = f(f(x)) = f(y) = z$.
Therefore, $\forall z \in A, \exists x \in A, f \circ f(x) = z$.
Thus, $f \circ f$ is surjective.
Therefore, if f is surjective, then $f \circ f$ is surjective. □

21. If f is injective, then $\forall n \in \mathbb{N}$, f^n is injective.

Proof.
Assume f is injective.
Let $S = \{n \in \mathbb{N} \mid f^n \text{ is injective}\}$.
Since f is injective, we have that f^1 is injective; hence $1 \in S$.
Let $n \in \mathbb{N}$ and assume $n \in S$.
Then f^n is injective.
Let $x_1, x_2 \in A$ and assume $f^{n+1}(x_1) = f^{n+1}(x_2)$.
This means $f^n \circ f(x_1) = f^n \circ f(x_2)$. In other words, $f^n(f(x_1)) = f^n(f(x_2))$.
Since f^n is injective, this implies that $f(x_1) = f(x_2)$,
and since f is injective, we then have $x_1 = x_2$.
Therefore, $\forall x_1, x_2 \in A$, if $f^{n+1}(x_1) = f^{n+1}(x_2)$, then $x_1 = x_2$.
Thus, f^{n+1} is injective, which means $n + 1 \in S$.
Therefore, $\forall n \in \mathbb{N}$, if $n \in S$, then $n + 1 \in S$.
By the PMI, $\mathbb{N} \subseteq S$. Thus, $\forall n \in \mathbb{N}$, f^n is injective.
Therefore, if f is injective, then $\forall n \in \mathbb{N}$, f^n is injective. □

23. If $\exists n \in \mathbb{N}$, f^n is surjective, then f is surjective.

Proof.
Assume $\exists n \in \mathbb{N}$, f^n is surjective. Choose such an n.
In the case where $n = 1$, we have that f^1 is surjective, which means f is surjective.
In the case where $n \geq 2$, we begin by letting $y \in A$.
Since f^n is surjective, we can choose $z \in A$ with $f^n(z) = y$.
Put $x = f^{n-1}(z)$.
Then $f(x) = f(f^{n-1}(z)) = f^n(z) = y$.
Therefore, $\exists x \in A, f(x) = y$.
Therefore, f is surjective.
Therefore, if $\exists n \in \mathbb{N}$, f^n is surjective, then f is surjective. □

Let A, B, and C be sets, and let $f : A \to B$ and $g : B \to C$ be functions. Prove the following.

25. If $g \circ f$ is injective, then f is injective.

Proof.
Assume $g \circ f$ is injective.
Let $x_1, x_2 \in A$, and assume $f(x_1) = f(x_2)$.
Then $g(f(x_1)) = g(f(x_2))$. In other words, $g \circ f(x_1) = g \circ f(x_2)$.
Since $g \circ f$ is injective, we have $x_1 = x_2$.
Therefore, $\forall x_1, x_2 \in A$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.
Thus, f is injective.
Therefore, if $g \circ f$ is injective, then f is injective. □

27. If f and g are surjective, then $g \circ f$ is surjective.

Proof.
Assume f and g are surjective.
Let $z \in C$.
Since g is surjective, we can choose $y \in B$ with $g(y) = z$.
Again, since f is surjective, we can choose $x \in A$ with $f(x) = y$.
Now, $g \circ f(x) = g(f(x)) = g(y) = z$.
Therefore, $\forall z \in C, \exists x \in A, g \circ f(x) = z$.
Thus, $g \circ f$ is surjective.
Therefore, if f and g are surjective, then $g \circ f$ is surjective. □

29. If $g \circ f$ is injective and f is surjective, then g is injective.

Proof.
Assume $g \circ f$ is injective and f is surjective.
Let $y_1, y_2 \in B$, and assume $g(y_1) = g(y_2)$.
Since f is surjective, we can choose $x_1, x_2 \in A$ with $f(x_1) = y_1$ and $f(x_2) = y_2$.
Since $g(y_1) = g(y_2)$, we have $g(f(x_1)) = g(f(x_2))$; hence $g \circ f(x_1) = g \circ f(x_2)$.
Since $g \circ f$ is injective, this implies that $x_1 = x_2$.
We then have $f(x_1) = f(x_2)$, which means $y_1 = y_2$.
Therefore, $\forall y_1, y_2 \in B$, if $g(y_1) = g(y_2)$, then $y_1 = y_2$.
Thus, g is injective.
Therefore, if $g \circ f$ is injective and f is surjective, then g is injective. □

For the given function f and set S, find $f(S)$. Prove your result.

31. $f : \mathbb{R} \to \mathbb{R}$, given by $\forall x \in \mathbb{R}$, $f(x) = 2x + 1$.

    (a) $S = [-1, 1)$.

Proof. For $S = [-1, 1)$, we claim $f(S) = [-1, 3)$.
Let $y \in f(S)$.
Then $\exists x \in S$, $y = f(x)$. Choose such an x.
Since $x \in S$, we have $-1 \leq x < 1$. Moreover, since $y = f(x)$, we have $y = 2x + 1$.
Since $-1 \leq x$, we have $-2 \leq 2x$, and so $-1 \leq 2x + 1$. Thus, $-1 \leq y$.
Since $x < 1$, we have $2x < 2$, and so $2x + 1 < 3$. Thus, $y < 3$.
Therefore, $y \in [-1, 3)$.
Therefore, $f(S) \subseteq [-1, 3)$.
Next, let $y \in [-1, 3)$. This means $-1 \leq y < 3$.
Put $x = \frac{y-1}{2}$.
Then $f(x) = 2\left(\frac{y-1}{2}\right) + 1 = (y - 1) + 1 = y$. Thus, $y = f(x)$.
Moreover, since $-1 \leq y$, we have $-2 \leq y - 1$, and so $-1 \leq \frac{y-1}{2}$. Thus, $-1 \leq x$.
Likewise, since $y < 3$, we have $y - 1 < 2$, and so $\frac{y-1}{2} < 1$. Thus, $x < 1$.
Therefore, $x \in [-1, 1)$ and so $x \in S$.
Therefore, $\exists x \in S$, $y = f(x)$. This means $y \in f(S)$.
Therefore, $[-1, 3) \subseteq f(S)$, and so $f(S) = [-1, 3)$. □

32. $f : \mathbb{R} \to \mathbb{R}$, given by $\forall x \in \mathbb{R}$, $f(x) = 1 - 2x$.

    (a) $S = (-3, 5]$.

Proof. For $S = (-3, 5]$, we claim $f(S) = [-9, 7)$.
Let $y \in f(S)$.
Then $\exists x \in S$, $y = f(x)$. Choose such an x.
Since $x \in S$, we know $-3 < x \leq 5$. Also, since $y = f(x)$, we have $y = 1 - 2x$.
Since $-3 < x$, we have $-2x < 6$; hence $1 - 2x < 7$. Thus, $y < 7$.
Since $x \leq 5$, we have $-10 \leq -2x$, and so $-9 \leq 1 - 2x$. Thus, $-9 \leq y$.
Therefore, $y \in [-9, 7)$.
Therefore, $f(S) \subseteq [-9, 7)$.
Next, let $y \in [-9, 7)$. That is, $-9 \leq y < 7$.
Put $x = \frac{1-y}{2}$.
Then $f(x) = 1 - 2\left(\frac{1-y}{2}\right) = 1 - (1 - y) = y$. This proves $y = f(x)$.
Further, since $-9 \leq y$, we have $-y \leq 9$; $1 - y \leq 10$; hence $\frac{1-y}{2} \leq 5$. Thus, $x \leq 5$.
Likewise, since $y < 7$, we have $-7 < -y$; $-6 < 1 - y$; and so $-3 < \frac{1-y}{2}$. Thus, $-3 < x$.
Therefore, $x \in (-3, 5] = S$.
Therefore, $\exists x \in S$, $y = f(x)$. Thus, $y \in f(S)$.
Therefore, $[-9, 7) \subseteq f(S)$, which completes the proof of $f(S) = [-9, 7)$. □

33. $f : \mathbb{R} \to \mathbb{R}$, given by $\forall x \in \mathbb{R}$, $f(x) = x^2$.

(a) $S = [-1, 1)$.

Proof. For $S = [-1, 1)$, we claim $f(S) = [0, 1]$.
Let $y \in f(S)$.
Then $\exists x \in S$, $y = f(x)$. Choosing such an $x \in S$, we have $-1 \leq x < 1$.
Since $y = f(x)$, we have $y = x^2$, and since $0 \leq x^2$, we have $0 \leq y$.
Case 1: $x < 0$.
In this case, since $-1 \leq x$, we have $x^2 \leq -x$ and $-x \leq 1$. Thus, $x^2 \leq 1$.
Case 2: $0 \leq x$.
In this case, since $x < 1$ and $0 \leq x$, we have $x^2 \leq x$. By transitivity, $x^2 \leq 1$.
In both cases, $x^2 \leq 1$, which proves $y \leq 1$. Thus, $y \in [0, 1]$.
Therefore, $f(S) \subseteq [0, 1]$.
Next, let $y \in [0, 1]$. That is, $0 \leq y \leq 1$.
Put $x = -\sqrt{y}$. Then $f(x) = (-\sqrt{y})^2 = y$.
Also, $x \leq 0$, and since $0 < 1$, we have $x < 1$.
Next, suppose $x < -1$ (looking for a contradiction).
Then $x < 0$, and so $-x < x^2$.
Now, since $x < -1$, we have $1 < -x$. We then have $1 < x^2$, which gives us $1 < y$.
This is a contradiction, since $y \leq 1$.
Therefore, $-1 \leq x$. This proves $x \in [-1, 1)$ and so $x \in S$.
Therefore, $\exists x \in S$, $y = f(x)$. Thus, $y \in f(S)$.
Therefore, $[0, 1] \subseteq f(S)$, and so $f(S) = [0, 1]$. □

34. $f : \mathbb{R} \to \mathbb{R}$, given by $\forall x \in \mathbb{R}$, $f(x) = 1 - x^2$.

(a) $S = (-3, 5]$.

Proof. □

35. $f : \mathbb{Z}^2 \to \mathbb{Z}$, given by $\forall x, y \in \mathbb{Z}$, $f(x, y) = x + y$.

(a) $S = \mathbb{E}^2$.

Proof. □

36. $f : \mathbb{Z}^3 \to \mathbb{Z}^3$, given by $\forall (x_1, x_2, x_3) \in \mathbb{Z}^3$, $f(x_1, x_2, x_3) = (x_1 + x_2 + x_3, x_2 + x_3, x_3)$.

(a) $S = \mathbb{O}^3$.

Proof. □

37. $f : \mathbb{Z} \to \mathbb{Z}$, given by $\forall x \in \mathbb{Z}$, $f(x) = \begin{cases} x - 1 & \text{if } x \text{ is even} \\ x + 1 & \text{if } x \text{ is odd} \end{cases}$.

(a) $S = \langle 3 \rangle$.

Proof. □

38. $f : \mathbb{Z} \to \mathbb{Z}$, given by $\forall x \in \mathbb{Z},\ f(x) = \begin{cases} 2x - 2 & \text{if } x \text{ is even} \\ 2x & \text{if } x \text{ is odd} \end{cases}$.

    (a) $S = \mathbb{Z}$.

    Proof. □

39. $f : \mathbb{Z} \times \mathbb{N} \to \mathbb{Q}$, given by $\forall (m, n) \in \mathbb{Z} \times \mathbb{N},\ f(m, n) = \dfrac{m}{n}$.

    (a) $S = \mathbb{E} \times (\mathbb{E} \cap \mathbb{N})$.

    Proof. □

40. $f : \mathbb{Z} \to \mathbb{Z}_6$, given by $\forall x \in \mathbb{Z}$, $f(x) = [x]_6$.

    (a) $S = \langle 3 \rangle$.

    Proof. □

For the given function f and set V, find $f^{-1}(V)$. Prove your result.

41. $f : \mathbb{R} \to \mathbb{R}$, given by $\forall x \in \mathbb{R}$, $f(x) = 2x + 1$.

    (a) $V = (0, 3]$.

    Proof. □

42. $f : \mathbb{R} \to \mathbb{R}$, given by $\forall x \in \mathbb{R}$, $f(x) = 1 - 2x$.

    (a) $V = (0, 3]$.

    Proof. □

43. $f : \mathbb{R} \to \mathbb{R}$, given by $\forall x \in \mathbb{R}$, $f(x) = x^2$.

    (a) $V = (0, 9]$.

    Proof. □

44. $f : \mathbb{R} \to \mathbb{R}$, given by $\forall x \in \mathbb{R}$, $f(x) = 1 - x^2$.

    (a) $V = (1, 2]$.

    Proof. □

45. $f : \mathbb{Z}^2 \to \mathbb{Z}$, given by $\forall x, y \in \mathbb{Z}$, $f(x, y) = x + y$.

    (a) $V = \mathbb{E}$.

    Proof. □

46. $f : \mathbb{Z} \to \mathbb{Z}$, given by $\forall x \in \mathbb{Z}$, $f(x) = \begin{cases} x - 1 & \text{if } x \text{ is even} \\ x + 1 & \text{if } x \text{ is odd} \end{cases}$.

    (a) $V = \langle 4 \rangle$.

    Proof. □

47. $f : \mathbb{Z} \to \mathbb{Z}$, given by $\forall x \in \mathbb{Z}$, $f(x) = \begin{cases} 2x - 2 & \text{if } x \text{ is even} \\ 2x & \text{if } x \text{ is odd} \end{cases}$.

    (a) $V = \{6, 12\}$.

    Proof. □

48. $f : \mathbb{Z} \to \mathbb{Z}_{15}$, given by $\forall x \in \mathbb{Z}$, $f(x) = [x]_{15}$.

    (a) $V = \{[0]_{15}\}$.

    Proof. □

Let $f : A \to B$ be a function. Let S and T be subsets of A, and let V and W be subsets of B. Prove the following.

49. $f(S \cup T) \subseteq f(S) \cup f(T)$.

Proof.
Let $y \in f(S \cup T)$.
Then we can choose $x \in S \cup T$ with $y = f(x)$.
For this x, we have either $x \in S$ or $x \in T$.
Case 1: $x \in S$.
Then $\exists x \in S, y = f(x)$; hence $y \in f(S)$.
Therefore, $y \in f(S) \cup f(T)$.
Case 2: $x \in T$.
Then $\exists x \in T, y = f(x)$, and so $y \in f(T)$.
Therefore, $y \in f(S) \cup f(T)$.
Therefore, in both cases, $y \in f(S) \cup f(T)$.
Thus, $f(S \cup T) \subseteq f(S) \cup f(T)$. □

51. $f(S \cap T) \subseteq f(S) \cap f(T)$. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ and subsets S and T of $\mathbb{R}$ for which $f(S) \cap f(T) \nsubseteq f(S \cap T)$.

Proof.
Let $y \in f(S \cap T)$.
Then we can choose $x \in S \cap T$ with $y = f(x)$.
For this x, we have both $x \in S$ and $x \in T$.
Since $x \in S$ and $y = f(x)$, we have $y \in f(S)$.
Likewise, since $x \in T$ and $y = f(x)$, we have $y \in f(T)$.
Thus, $y \in f(S) \cap f(T)$.
Therefore, $f(S \cap T) \subseteq f(S) \cap f(T)$.

The converse is not generally true. For example, for the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$, we have $1 \in f((-\infty, 0))$, since $1 = f(-1)$ and $-1 \in (-\infty, 0)$. Also, $1 \in f((0, \infty))$, since $1 = f(1)$ and $1 \in (0, \infty)$. This proves $1 \in f((-\infty, 0)) \cap f((0, \infty))$. However, $1 \notin f((-\infty, 0) \cap (0, \infty))$, since $(-\infty, 0) \cap (0, \infty) = \emptyset$. Thus, $f((-\infty, 0)) \cap f((0, \infty)) \nsubseteq f((-\infty, 0) \cap (0, \infty))$. □

53. If f is injective, then $f(S \setminus T) \subseteq f(S) \setminus f(T)$.

Proof.
Assume f is injective.
Let $y \in f(S \setminus T)$ and suppose $y \notin f(S) \setminus f(T)$.
Since $y \in f(S \setminus T)$, we can choose $x \in S \setminus T$ with $y = f(x)$.
For the chosen x, since $x \in S$ and $y = f(x)$, we have $y \in f(S)$.
However, since $y \notin f(S) \setminus f(T)$, we must then have $y \in f(T)$.
This means we can choose $t \in T$ with $y = f(t)$.
Now, since $y = f(x)$ and $y = f(t)$, we have $f(x) = f(t)$; hence $x = t$ by injectivity.
This is troublesome, because since $x = t$ and $t \in T$, we have $x \in T$. But, since $x \in S \setminus T$, $x \notin T$.
This is a contradiction.
Therefore, $f(S \setminus T) \subseteq f(S) \setminus f(T)$.
Therefore, if f is injective, then $f(S \setminus T) \subseteq f(S) \setminus f(T)$. □

55. If f is injective, then $f(S^c) \subseteq (f(S))^c$.

Proof.
Assume f is injective.
Let $y \in f(S^c)$ and suppose $y \notin (f(S))^c$. i.e., $y \in f(S)$.
Since $y \in f(S^c)$, we can choose $x \in S^c$ with $y = f(x)$.
Likewise, since $y \in f(S)$, we can choose $a \in S$ with $y = f(a)$.
Now, since $f(x) = f(a)$, we have $x = a$ by injectivity.
However, since $a \in S$, we then have $x \in S$, which contradicts $x \in S^c$.
Therefore, $f(S^c) \subseteq (f(S))^c$.
Therefore, if f is injective, then $f(S^c) \subseteq (f(S))^c$. □

57. $f^{-1}(V \cup W) = f^{-1}(V) \cup f^{-1}(W)$.

Proof.
Let $x \in f^{-1}(V \cup W)$.
Then $f(x) \in V \cup W$. This means $f(x) \in V$ or $f(x) \in W$.
In the case where $f(x) \in V$, we have $x \in f^{-1}(V)$.
In the case where $f(x) \in W$, we have $x \in f^{-1}(W)$.
Therefore, in either case, we have $x \in f^{-1}(V) \cup f^{-1}(W)$.
Thus, $f^{-1}(V \cup W) \subseteq f^{-1}(V) \cup f^{-1}(W)$.
Conversely, let $x \in f^{-1}(V) \cup f^{-1}(W)$. i.e., $x \in f^{-1}(V)$ or $x \in f^{-1}(W)$.
In the case where $x \in f^{-1}(V)$, we have $f(x) \in V$, and so $f(x) \in V \cup W$. Thus, $x \in f^{-1}(V \cup W)$.
In the case where $x \in f^{-1}(W)$, we have $f(x) \in W$, and so $f(x) \in V \cup W$. Again, $x \in f^{-1}(V \cup W)$.
Since in both cases, $x \in f^{-1}(V \cup W)$, this proves $f^{-1}(V) \cup f^{-1}(W) \subseteq f^{-1}(V \cup W)$.
Thus, $f^{-1}(V \cup W) = f^{-1}(V) \cup f^{-1}(W)$. □

59. $f^{-1}(V \setminus W) = f^{-1}(V) \setminus f^{-1}(W)$.

Proof.
Let $x \in f^{-1}(V \setminus W)$.
This means $f(x) \in V \setminus W$.
Since $f(x) \in V$, we have $x \in f^{-1}(V)$.
Since $f(x) \notin W$, we have $x \notin f^{-1}(W)$.
This gives us $x \in f^{-1}(V) \setminus f^{-1}(W)$.
Therefore, $f^{-1}(V \setminus W) \subseteq f^{-1}(V) \setminus f^{-1}(W)$.
Conversely, let $x \in f^{-1}(V) \setminus f^{-1}(W)$.
Since $x \in f^{-1}(V)$, $f(x) \in V$.
Since $x \notin f^{-1}(W)$, $f(x) \notin W$.
We then have $f(x) \in V \setminus W$, and so $x \in f^{-1}(V \setminus W)$.
Therefore, $f^{-1}(V) \setminus f^{-1}(W) \subseteq f^{-1}(V \setminus W)$.
Thus, $f^{-1}(V \setminus W) = f^{-1}(V) \setminus f^{-1}(W)$. □

61. If $\forall E \in \mathcal{P}(B)$, $E \subseteq f(f^{-1}(E))$, then f is surjective.

Proof.
Assume $\forall E \in \mathcal{P}(B)$, $E \subseteq f(f^{-1}(E))$.
Let $y \in B$.
Since $B \in \mathcal{P}(B)$, we have $B \subseteq f(f^{-1}(B))$.
Since $y \in B$, we have $y \in f(f^{-1}(B))$.
This means there is an $x \in f^{-1}(B)$ with $y = f(x)$.
This means there is an $x \in A$ with $y = f(x)$.
Therefore, $\forall y \in B, \exists x \in A, y = f(x)$. In other words, f is surjective.
Therefore, if $\forall E \in \mathcal{P}(B)$, $E \subseteq f(f^{-1}(E))$, then f is surjective. □

63. If f is bijective, then $f(S) = V$ if and only if $f^{-1}(V) = S$.

Proof.
Assume f is bijective.
Assume $f(S) = V$.
Let $x \in f^{-1}(V)$. This means $f(x) \in V$.
Since $f(S) = V$, we then have $f(x) \in f(S)$.
This means we can choose $a \in S$ with $f(x) = f(a)$.
Since f is injective, we have $x = a$, and so $x \in S$.
Thus, $f^{-1}(V) \subseteq S$.
Next, let $x \in S$.
Then $f(x) \in f(S)$, and so $f(x) \in V$ (since $f(S) = V$). Thus, $x \in f^{-1}(V)$.
Therefore, $S \subseteq f^{-1}(V)$, and hence $f^{-1}(V) = S$.
Therefore, if $f(S) = V$, then $f^{-1}(V) = S$.
Conversely, assume $f^{-1}(V) = S$.
Let $y \in f(S)$. i.e., we can choose $x \in S$ with $y = f(x)$.
For such x, we have $x \in f^{-1}(V)$, since $f^{-1}(V) = S$. Thus, $f(x) \in V$.
This gives us $y \in V$, which proves $f(S) \subseteq V$.
Next, let $y \in V$.
Since f is surjective, we can choose $x \in A$ with $y = f(x)$.
Since $y \in V$, for or chosen x we have $f(x) \in V$. Thus, $x \in f^{-1}(V)$.
This gives us $x \in S$, since $f^{-1}(V) = S$. Now, since $x \in S$ and $y = f(x)$, we have $y \in f(S)$.
Therefore, $V \subseteq f(S)$, which completes the proof that $f(S) = V$.
Therefore, if $f^{-1}(V) = S$, then $f(S) = V$.
We have now shown, under the assumption that f is bijective, that $f(S) = V$ if and only if $f^{-1}(V) = S$.
Therefore, if f is bijective, then $f(S) = V$ if and only if $f^{-1}(V) = S$. □

The following exercises combine the topics in this section with those in the section 3.1.

65. Let $f : A \to B$ be a function. Let R be the relation on A given by: For all $a, b \in A$, aRb if and only if $f(a) = f(b)$. Prove R is an equivalence relation.

Proof. $\square$

67. Let $f : A \to B$ be a function. Let R be the relation on A given by: For all $a, b \in A$, aRb if and only if $f(a) = f(b)$. Prove the relation
$$g = \{([x]_R, f(x)) \in A/R \times B \mid x \in A\}$$
is an injective function.

Proof. $\square$

69. Let $f : A \to B$ be a function, and let $\mathcal{P}$ be a partition of A. Let $\mathcal{Q} = \{f(S) \mid S \in \mathcal{P}\}$. Prove if f is bijective, then $\mathcal{Q}$ is a partition of B.

Proof. $\square$