DEPARTMENT OF MATHEMATICS AND STATISTICS
MATH 1020 Mathematical Foundations
Practice Exam Questions

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1. Write the symbolic form of the proposition "At least one subset of the real numbers has a smallest element."

   Solution.
   $\exists S \in \mathcal{P}(\mathbb{R})$, $\exists a \in S$, $\forall x \in S$, $a \leq x$. □

2. Write the symbolic form of the proposition "There is no largest natural number."

   Solution.
   $\forall x \in \mathbb{N}$, $\exists y \in \mathbb{N}$, $x < y$. □

3. Write the symbolic form of the proposition "Between any two distinct real numbers there is a rational number."

   Solution.
   $\forall x, y \in \mathbb{R}$, if $x \neq y$, then $\exists z \in \mathbb{Q}$, $x < z < y$ or $y < z < x$. □

4. Write the symbolic form of the proposition "The cube of every even integer is even."

   Solution.
   $\forall x \in \mathbb{Z}$, if $\exists a \in \mathbb{Z}$, $x = 2a$, then $\exists b \in \mathbb{Z}$, $x^3 = 2b$. □

5. Write the symbolic form of the proposition "The sum of a rational number and an irrational number is always irrational."

   Solution.
   $\forall x, y \in \mathbb{R}$, if x is rational and y is irrational, then $x + y$ is irrational. □

6. Write the negation of the proposition "The sum of a rational number and an irrational number is always irrational."

   Solution.
   $\exists x, y \in \mathbb{R}$, x is rational and y is irrational and $x + y$ is rational. □

7. Write the negation of the proposition $\forall a, b \in \mathbb{R}$, if $\forall x \in (a, \infty)$, $b \leq x$, then $b \leq a$.

   Solution.
   $\exists a, b \in \mathbb{R}$, $\forall x \in (a, \infty)$, $b \leq x$, and $a < b$. □

8. Write the negation of the proposition $\forall a \in \mathbb{R}$, if ($\forall x \in \mathbb{R}$, if $x > 0$, then $a \leq x$), then $a \leq 0$.

Solution.
$\exists a \in \mathbb{R}$, ($\forall x \in \mathbb{R}$, if $x > 0$, then $a \leq x$), and $0 < a$. □

9. Write the negation of the proposition $\forall y \in \mathbb{R}, \exists x \in \mathbb{R}, x > y^2 + 1$.

Solution.
$\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, x \leq y^2 + 1$. □

10. Write the negation of the proposition $\forall x \in \mathbb{N}$, if $\exists n \in \mathbb{N}, x^2 = n^3$, then $\exists m \in \mathbb{N}, x = m^3$.

Solution.
$\exists x \in \mathbb{N}, \exists n \in \mathbb{N}, x^2 = n^3$ and $\forall m \in \mathbb{N}, x \neq m^3$. □

11. Let $a \in \mathbb{R}$. Write the *contrapositive* form of the following implication:
if $\exists x \in \mathbb{R}, a + x = x$, then $\forall y \in \mathbb{R}, a + y = y$.

Solution.
if $\exists y \in \mathbb{R}, a + y \neq y$, then $\forall x \in \mathbb{R}, a + x \neq x$. □

12. Prove, using a proof by **contraposition**: $\forall a \in \mathbb{R}$, if ($\forall x \in \mathbb{R}$, if $x > 0$, then $a \leq x$), then $a \leq 0$.

Proof.
Let $a \in \mathbb{R}$.
Assume $0 < a$.
Put $x = \frac{a}{2}$.
Since $0 < a$, we have $0 < \frac{a}{2}$; hence $x > 0$.
Also, since $1 < 2$ and $0 < a$, we have $a < 2a$; hence $\frac{a}{2} < a$.
We now have $x > 0$ and $x < a$.
Therefore, $\exists x \in \mathbb{R}, x > 0$ and $x < a$.
Therefore, if $0 < a$, then $\exists x \in \mathbb{R}, x > 0$ and $x < a$.
Therefore, if ($\forall x \in \mathbb{R}$, if $x > 0$, then $a \leq x$), then $a \leq 0$.
Therefore, $\forall a \in \mathbb{R}$, if ($\forall x \in \mathbb{R}$, if $x > 0$, then $a \leq x$), then $a \leq 0$. □

13. Prove, using only the axioms of the real numbers, $\forall x, y, z \in \mathbb{R}$, if $x \le y$ and $y < z$, then $x < z$.

> *Proof.*
> Let $x, y, z \in \mathbb{R}$.
> Assume $x \le y$ and $y < z$.
> Since $x \le y$ means $x < y$ or $x = y$, we consider two cases.
> Case 1: $x < y$.
> In this case, since $x < y$ and $y < z$, we have $x < z$.
> Case 2: $x = y$.
> In this case, since $y < z$, we have $x < z$.
> Therefore, if $x \le y$ and $y < z$, then $x < z$.
> Therefore, $\forall x, y, z \in \mathbb{R}$, if $x \le y$ and $y < z$, then $x < z$. $\square$

14. Prove $\forall a \in \mathbb{R}$, if $\forall x \in \mathbb{R}$, $ax \le x$, then $a = 1$.

> *Proof.*
> Let $a \in \mathbb{R}$.
> Assume $\forall x \in \mathbb{R}$, $ax \le x$.
> Since $1 \in \mathbb{R}$, we have $a(1) \le 1$; hence $a \le 1$.
> Since $-1 \in \mathbb{R}$, we have $a(-1) \le -1$; hence $1 \le a$.
> Now, since $a \le 1$ and $1 \le a$, we have $a = 1$.
> Therefore, if $\forall x \in \mathbb{R}$, $ax \le x$, then $a = 1$.
> Therefore, $\forall a \in \mathbb{R}$, if $\forall x \in \mathbb{R}$, $ax \le x$, then $a = 1$. $\square$

15. Prove $\forall a \in \mathbb{R}$, if $\forall x \in \mathbb{R}$, $ax \le 0$, then $a = 0$.

> *Proof.*
> Let $a \in \mathbb{R}$.
> Assume $\forall x \in \mathbb{R}$, $ax \le 0$.
> Since $1 \in \mathbb{R}$, we have $a(1) \le 0$; hence $a \le 0$.
> Since $-1 \in \mathbb{R}$, we have $a(-1) \le 0$; hence $0 \le a$.
> Since $a \le 0$ and $0 \le a$, we have $a = 0$.
> Therefore, if $\forall x \in \mathbb{R}$, $ax \le 0$, then $a = 0$.
> Therefore, $\forall a \in \mathbb{R}$, if $\forall x \in \mathbb{R}$, $ax \le 0$, then $a = 0$. $\square$

16. Prove $\forall x \in \mathbb{Z}$, if 10 divides x and 4 divides x, then 20 divides x.

> *Proof.*
> Let $x \in \mathbb{Z}$.
> Assume 10 divides x and 4 divides x.
> Choose $a, b \in \mathbb{Z}$ with $x = 10a$ and $x = 4b$.
> Put $c = b - 2a$.
> $x = 5x - 4x = 5(4b) - 4(10a) = 20b - 40a = 20(b - 2a) = 20c$.
> Therefore, $\exists c \in \mathbb{Z}$, $x = 20c$.
> This means 20 divides x.
> Therefore, if 10 divides x and 4 divides x, then 20 divides x.
> Therefore, $\forall x \in \mathbb{Z}$, if 10 divides x and 4 divides x, then 20 divides x. $\square$

17. Using a proof by contradiction, prove $\forall x, y \in \mathbb{Z}$, if x is odd and xy is even, then y is even.

Proof.
Let $x, y \in \mathbb{Z}$.
Assume x is odd and xy is even and y is not even.
Since y is not even, y is odd.
Choose $a, b, c \in \mathbb{Z}$ with $x = 2a + 1$ and $xy = 2b$ and $y = 2c + 1$.
Then $(2a + 1)(2c + 1) = 2b$; $4ac + 2a + 2c + 1 = 2b$; $1 = 2(b - 2ac - a - c)$.
Since $0 < 1$ and $0 < 2$, we must have $0 < b - 2ac - a - c$. It follows that $1 \leq b - 2ac - a - c$.
We then have $2 \leq 2(b - 2ac - a - c)$; hence $2 \leq 1$. This is a contradiction, since $1 < 2$.
Therefore, if x is odd and xy is even, then y is even.
Therefore, $\forall x, y \in \mathbb{Z}$, if x is odd and xy is even, then y is even. $\square$

18. Prove $\forall x, y \in \mathbb{R}$, if $\min(x, y) < |x - y|$, then $2\min(x, y) < \max(x, y)$.

Proof.
Let $x, y \in \mathbb{R}$.
Assume $\min(x, y) < |x - y|$.
Case 1: $x < y$.
In this case, $\min(x, y) = x$ and $\max(x, y) = y$.
Also, in this case $x - y < 0$, so $|x - y| = y - x$.
By assumption, we then have $x < y - x$, which gives us $2x < y$. Thus, $2\min(x, y) < \max(x, y)$.
Case 2: $y < x$.
In this case, $\min(x, y) = y$ and $\max(x, y) = s$.
Also, in this case $0 < x - y$, so $|x - y| = x - y$.
Our assumption then becomes $y < x - y$, which implies $2y < x$. That is, $2\min(x, y) < \max(x, y)$.
Case 3: $x = y$.
In this case, $\min(x, y) = x$ and $\max(x, y) = x$.
Also, $|x - y| = |0| = 0$.
Our assumption then gives us $x < 0$.
Adding x to both sides gives $2x < x$, which means $2\min(x, y) < \max(x, y)$.
Therefore, if $\min(x, y) < |x - y|$, then $2\min(x, y) < \max(x, y)$.
Therefore, $\forall x, y \in \mathbb{R}$, if $\min(x, y) < |x - y|$, then $2\min(x, y) < \max(x, y)$. $\square$

19. Prove $\forall x, y \in \mathbb{R}$, $\min(x, y) = \frac{1}{2}(x + y - |x - y|)$.

Proof.
Let $x, y \in \mathbb{R}$.
Case 1: $x \leq y$.
In this case, $\min(x, y) = x$.
Also, in this case $x - y \leq 0$, so $|x - y| = y - x$.
Now, $\min(x, y) = x = \frac{1}{2}(x + x) = \frac{1}{2}(x + y - (y - x)) = \frac{1}{2}(x + y - |x - y|)$.
Case 2: $y < x$.
In this case, $\min(x, y) = y$.
Also, in this case $0 < x - y$, so $|x - y| = x - y$.
$\min(x, y) = y = \frac{1}{2}(y + y) = \frac{1}{2}(x + y - (x - y)) = \frac{1}{2}(x + y - |x - y|)$.
Therefore, $\min(x, y) = \frac{1}{2}(x + y - |x - y|)$.
Therefore, $\forall x, y \in \mathbb{R}$, $\min(x, y) = \frac{1}{2}(x + y - |x - y|)$. $\square$

20. Prove $\forall a, x, y \in \mathbb{R}$, if $a < \max(x, y)$, then $a < x$ or $a < y$.

Proof.
Let $x, y, a \in \mathbb{R}$.
Assume $a < \max(x, y)$.
Case 1: $x \geq y$.
In this case, $\max(x, y) = x$, and hence $a < x$.
Therefore, $a < x$ or $a < y$.
Case 2: $x < y$.
In this case, $\max(x, y) = y$, so $a < y$.
Again, $a < x$ or $a < y$ is true.
Therefore, if $a < \max(x, y)$, then $a < x$ or $a < y$.
Therefore, $\forall a, x, y \in \mathbb{R}$, if $a < \max(x, y)$, then $a < x$ or $a < y$. $\square$

21. Prove $\forall x, y \in \mathcal{B}$, if $x \wedge y = F$, then $\neg x \vee y = \neg x$.

Proof.
Let $x, y \in \mathbb{B}$.
Assume $x \wedge y = F$.
Then $\neg x \vee (x \wedge y) = \neg x \vee F$.
This gives us $(\neg x \vee x) \wedge (\neg x \vee y) = \neg x$. So, $T \wedge (\neg x \vee y) = \neg x$.
Thus, $\neg x \vee y = \neg x$.
Therefore, if $x \wedge y = F$, then $\neg x \vee y = \neg x$.
Q.E.D. $\square$

22. Prove $\forall x, y, z \in \mathcal{B}$, if $x \Rightarrow y$ and $\neg z \Rightarrow x$, then $y \vee z = T$.

Proof.
Let $x, y, z \in \mathbb{B}$.
Assume $x \Rightarrow y$ and $\neg z \Rightarrow x$.
Then $\neg z \Rightarrow y$ by transitivity. This means $\neg z \wedge y = \neg z$. Then $(\neg z \wedge y) \vee z = \neg z \vee z$.
Expanding, we have $(\neg z \vee z) \wedge (y \vee z) = T$, and so $T \wedge (y \vee z) = T$.
Thus, $y \vee z = T$.
Therefore, if $x \Rightarrow y$ and $\neg z \Rightarrow x$, then $y \vee z = T$.
Q.E.D. $\square$

Alternate Proof.
Let $x, y, z \in \mathbb{B}$.
Assume $x \Rightarrow y$ and $\neg z \Rightarrow x$.
Since $\neg z \Rightarrow x$, we have $\neg x \Rightarrow z$ by contraposition.
Now, $x \wedge y = x$ and $\neg x \wedge z = \neg x$.
Then $(x \wedge y) \vee (\neg x \wedge z) = x \vee \neg x$. Thus, $(x \wedge y) \vee (\neg x \wedge z) = T$.
This gives us $y \vee (x \wedge y) \vee (\neg x \wedge z) = y \vee T$.
By absorption, we then have $y \vee (\neg x \wedge z) = T$.
Now, $y \vee (\neg x \wedge z) \vee z = T \vee z$.
By absorption again, we have $y \vee z = T$.
Therefore, if $x \Rightarrow y$ and $\neg z \Rightarrow x$, then $y \vee z = T$.
Q.E.D. $\square$

23. Using a proof by contradiction and the well-ordering property, prove $\forall a \in \mathbb{R}$, if $1 \le a$, then $\forall n \in \mathbb{N}$, $a \le a^n$.

Proof.
Let $a \in \mathbb{Z}$.
Assume $1 \le a$ and $\exists n \in \mathbb{N}$, $a^n < a$.
Let $n \in \mathbb{N}$ be smallest for which $a^n < a$.
Since $a^1 = a$, but $a^n < a$, we know $n \ne 1$.
Then $n - 1 \in \mathbb{N}$, which means $a \le a^{n-1}$.
Now, $a^n < a$ and $a \le a^{n-1}$ which gives us $a^n < a^{n-1}$ by transitivity.
This means $a^{n-1}a < a^{n-1}$.
Note that since $0 < 1 \le a$, we have $0 < a$ and so $0 < a^{n-1}$.
This allows us to divide both sides of the inequality $a^{n-1}a < a^{n-1}$ by a^{n-1}.
Doing so gives us $a < 1$, which is a contradiciton, since $1 \le a$.
Therefore, if $1 \le a$, then $\forall n \in \mathbb{N}$, $a \le a^n$.
Therefore, $\forall a \in \mathbb{R}$, if $1 \le a$, then $\forall n \in \mathbb{N}$, $a \le a^n$. □

24. Using the principle of mathematical induction, prove $\forall a \in \mathbb{R}$, if $1 \le a$, then $\forall n \in \mathbb{N}$, $a \le a^n$.

Proof.
Let $a \in \mathbb{R}$.
Assume $1 \le a$
Let $A = \{n \in \mathbb{N} \mid a \le a^n\}$.
Since $a = a^1$, we have $a \le a^1$. Thus, $1 \in A$.
Let $n \in \mathbb{N}$, and assume $n \in A$.
Then $a \le a^n$.
Since $0 < 1$ and $1 \le a$ and $a \le a^n$, we have $0 \le a^n$.
Now, since $1 \le a$ and $0 \le a^n$, we have $a^n(1) \le a^n(a)$. That is, $a^n \le a^{n+1}$.
Finally, since $a \le a^n$ and $a^n \le a^{n+1}$, we have $a \le a^{n+1}$.
Thus, $n + 1 \in A$.
Therefore, $\forall n \in \mathbb{N}$, if $n \in A$, then $n + 1 \in A$.
By the Principle of Mathematical Induction $\mathbb{N} \subseteq A$.
Therefore, $\forall n \in \mathbb{N}$, $a \le a^n$.
Therefore, if $1 \le a$, then $\forall n \in \mathbb{N}$, $a \le a^n$.
Therefore, $\forall a \in \mathbb{R}$, if $1 \le a$, then $\forall n \in \mathbb{N}$, $a \le a^n$. □

25. Using a proof by contradiction and the well-ordering property, prove $\forall x \in \mathbb{Z}$, if x is odd then $\forall n \in \mathbb{N}$, x^n is odd.

Proof.
Let $x \in \mathbb{Z}$.
Assume x is odd and $\exists n \in \mathbb{N}$, x^n is even.
Choose $a \in \mathbb{Z}$ with $x = 2a + 1$.
Let $n \in \mathbb{N}$ be smallest for which x^n is even.
Choose $b \in \mathbb{Z}$ with $x^n = 2b$.
Since x^1 is odd, but x^n is even, we know $n \neq 1$.
Then $n - 1 \in \mathbb{N}$, which means x^{n-1} is odd.
Now, $x^n = x^{n-1}x$, which gives us $2b = x^{n-1}(2a + 1)$.
Rearranging this, we have $x^{n-1} = 2b - 2ax^{n-1}$.
Letting $c = b - ax^{n-1}$, we then have $x^{n-1} = 2c$.
This proves x^{n-1} is even, which is a contradiction, since x^{n-1} is odd.
Therefore, if x is odd, then $\forall n \in \mathbb{N}$, x^n is odd.
Therefore, $\forall x \in \mathbb{Z}$, if x is odd, then $\forall n \in \mathbb{N}$, x^n is odd. $\square$

26. Using the principle of mathematical induction, prove $\forall x \in \mathbb{Z}$, if x is odd then $\forall n \in \mathbb{N}$, x^n is odd.

Proof.
Let $x \in \mathbb{Z}$.
Assume x is odd.
Choose $a \in \mathbb{Z}$ with $x = 2a + 1$.
Let $A = \{n \in \mathbb{N} \mid x^n \text{ is odd}\}$.
Since $x^1 = x$ and x is odd, we have $1 \in A$.
Let $n \in \mathbb{N}$.
Assume $n \in A$.
Hence x^n is odd.
Choose $b \in \mathbb{Z}$ with $x^n = 2b + 1$.
Then $x^{n+1} = x^n x = (2b + 1)(2a + 1) = 4ab + 2a + 2b + 1$
Putting $c = 2ab + a + b$ then gives us $x^{n+1} = 2c + 1$.
Therefore, x^{n+1} is odd.
Hence $n + 1 \in A$.
Therefore, if $n \in A$ then $n + 1 \in A$.
Therefore, by the Principle of Mathematical Induction, $\mathbb{N} \subseteq A$.
Therefore, $\forall n \in \mathbb{N}$, x^n is odd.
Therefore, if x is odd, then $\forall n \in \mathbb{N}$, x^n is odd.
Therefore, $\forall x \in \mathbb{Z}$, if x is odd, then $\forall n \in \mathbb{N}$, x^n is odd. $\square$

27. Prove $\forall x \in \mathbb{N}$, 7 divides $8^x - 1$.

Proof.
Let $A = \{x \in \mathbb{N} \mid 7 \text{ divides } 8^x - 1\}$.
Letting $q = 1$ gives us $8^1 - 1 = 7 = 7q$.
Therefore, $\exists q \in \mathbb{Z}$, $8^1 - 1 = 7q$. That is, 7 divides $8^1 - 1$, and hence $1 \in A$.
Let $n \in A$. Then 7 divides $8^n - 1$.
Accordingly, let $p \in \mathbb{Z}$ with $8^n - 1 = 7p$.
Let $k = 8^n + p$

$$\begin{aligned} 8^{n+1} - 1 &= 8(8^n) - 1 \\ &= (7+1)(8^n) - 1 \\ &= 7(8^n) + 8^n - 1 \\ &= 7(8^n) + 7p \\ &= 7(8^n + p) \\ &= 5k \end{aligned}$$

Therefore, $\exists k \in \mathbb{Z}$, $8^{n+1} - 1 = 7k$, and so $n + 1 \in A$.
Therefore, $\forall n \in \mathbb{N}$, if $n \in A$, then $n + 1 \in A$.
By the Principle of Mathematical Induction $\mathbb{N} \subseteq A$.
Therefore, $\forall x \in \mathbb{N}$, 7 divides $8^x - 1$. $\square$

28. Prove $\forall a, r \in \mathbb{R}$, if $r \neq 1$ and $r \neq 0$, then $\forall n \in \mathbb{N}$, $\displaystyle\sum_{k=0}^{n} ar^k = \frac{a(1 - r^{n+1})}{1 - r}$.

Proof.
Let $a, r \in \mathbb{R}$, and assume $r \neq 1$ and $r \neq 0$.
Let $A = \{n \in \mathbb{N} \mid \sum_{k=0}^{n} ar^k = \frac{a(1-r^{n+1})}{1-r}\}$.
$\sum_{k=0}^{1} ar^k = ar^0 + ar^1 = a(1+r) = \frac{a(1+r)(1-r)}{1-r} = \frac{a(1-r^2)}{1-r}$. Thus, $1 \in A$.
Let $n \in A$. Then $\sum_{k=0}^{n} ar^k = \frac{a(1-r^{n+1})}{1-r}$.

$$\sum_{k=0}^{n+1} ar^k = \sum_{k=0}^{n} ar^k + ar^{n+1} = \frac{a(1 - r^{n+1})}{1 - r} + ar^{n+1} = \frac{a(1 - r^{n+1} + r^{n+1}(1 - r))}{1 - r} = \frac{a(1 - r^{n+2})}{1 - r}$$

Thus, $n + 1 \in A$.
Therefore, A is inductive. By the PMI, $\forall n \in \mathbb{N}$, $n \in A$.
Therefore, $\forall n \in \mathbb{N}$, $\sum_{k=0}^{n} ar^k = \frac{a(1-r^{n+1})}{1-r}$.
Therefore, $\forall a, r \in \mathbb{R}$, if $r \neq 1$ and $r \neq 0$, then $\forall n \in \mathbb{N}$, $\sum_{k=0}^{n} ar^k = \frac{a(1-r^{n+1})}{1-r}$. $\square$

29. Let $A \in \mathcal{P}(U)$. Prove if $U \setminus A \subseteq A$, then $A = U$.

Proof.
Assume $U \setminus A \subseteq A$ and assume $A \neq U$.
Since $A \subseteq U$, it must be the case that $U \nsubseteq A$.
That is, $\exists x \in U$, $x \notin A$. Choose such an x.
Then $x \in U$ and $x \notin A$, which means $x \in U \setminus A$.
Since $U \setminus A \subseteq A$, we then have $x \in A$.
We now have the contradiction $x \in A$ and $x \notin A$.
Therefore, if $U \setminus A \subseteq A$, then $A = U$. □

30. Let A, B, C, and D be sets. Prove if $A \subseteq C$ and $B \subseteq D$, then $A \cup B \subseteq C \cup D$.

Proof.
Assume $A \subseteq C$ and $B \subseteq D$.
Let $x \in A \cup B$.
Then $x \in A$ or $x \in B$.
Case 1: $x \in A$.
Since $A \subseteq C$, we then have $x \in C$.
This proves $x \in C$ or $x \in D$, which means $x \in C \cup D$.
Case 1: $x \in B$.
Since $B \subseteq D$, we then have $x \in D$.
This proves again $x \in C$ or $x \in D$, so $x \in C \cup D$.
Therefore, $\forall x \in A \cup B$, then $x \in C \cup D$.
Thus, $A \cup B \subseteq C \cup D$.
Therefore, if $A \subseteq C$ and $B \subseteq D$, then $A \cup B \subseteq C \cup D$. □

Alternate Proof.
Assume $A \subseteq C$ and $B \subseteq D$.
Since $A \subseteq C$, we have $A \cup B \subseteq C \cup B$.
Since $B \subseteq D$, we have $C \cup B \subseteq C \cup D$.
By transitivity, we then have $A \cup B \subseteq C \cup D$.
Therefore, if $A \subseteq C$ and $B \subseteq D$, then $A \cup B \subseteq C \cup D$. □

31. Let A, B, and C be sets. Prove if $A \cap B = A \cap C$ and $A \subseteq B \cup C$, then $A \subseteq B \cap C$.

Proof.
Assume $A \cap B = A \cap C$ and $A \subseteq B \cup C$.
Let $x \in A$.
Then $x \in B \cup C$ (since $A \subseteq B \cup C$).
Case 1: $x \in B$.
In this case, we have $x \in A$ and $x \in B$, which means $x \in A \cap B$.
Then $x \in A \cap C$ (since $A \cap B = A \cap C$).
This means $x \in C$ as well.
Now, $x \in B$ and $x \in C$, which means $x \in B \cap C$.
Case 2: $x \in C$.
In this case, $x \in A$ and $x \in C$, and so $x \in A \cap C$.
Then $x \in A \cap B$ (since $A \cap B = A \cap C$).
In particular $x \in B$.
We now have $x \in B$ and $x \in C$, which means $x \in B \cap C$.
Therefore, $\forall x \in A$, $x \in B \cap C$. In other words, $A \subseteq B \cap C$.
Therefore, if $A \cap B = A \cap C$ and $A \subseteq B \cup C$, then $A \subseteq B \cap C$. □

Alternate Proof.
Assume $A \cap B = A \cap C$ and $A \subseteq B \cup C$.
Then $(A \cap B) \cap C = (A \cap C) \cap C = A \cap C$.
This means $A \cap B = A \cap C = A \cap B \cap C$.
Now, since $A \subseteq B \cup C$, we have $A \cap A \subseteq A \cap (B \cup C)$, and so $A \subseteq (A \cap B) \cup (B \cup C)$,
Since $A \cap B = A \cap C = A \cap B \cap C$, this gives us $A \subseteq (A \cap B \cap C) \cup (A \cap B \cap C)$.
Thus, $A \subseteq A \cap B \cap C$.
Finally, since $A \cap B \cap C \subseteq B \cap C$ (by consistency), we have $A \subseteq B \cap C$.
Therefore,if $A \cap B = A \cap C$ and $A \subseteq B \cup C$, then $A \subseteq B \cap C$. □

32. Prove $\forall a, b \in \mathbb{R}$, if $a < b$, then $(-\infty, b) \setminus (-\infty, a) = [a, b)$.

Proof.
Let $a, b \in \mathbb{R}$.
Assume $a < b$.
Let $x \in \mathbb{R}$ and assume $x \in (-\infty, b) \setminus (-infty, a)$.
Then $x \in (-\infty, b)$, so $x < b$, and $x \notin (-\infty, a)$, so $a \leq x$.
Since $a \leq x$ and $x < b$, we have $x \in [a, b)$.
Therefore, $\forall x \in \mathbb{R}$, if $x \in (-\infty, b) \setminus (-infty, a)$, then $x \in [a, b)$.
Thus, $(-\infty, b) \setminus (-\infty, a) \subseteq [a, b)$.
Conversely, let $x \in \mathbb{R}$ and assume $x \in [a, b)$.
Then $a \leq x$ and $x < b$.
Since $a \leq x$, we have $x \notin (-\infty, a)$.
Since $x < b$, we have $x \in (-\infty, b)$.
Now, $x \in (-\infty, b)$ and $x \notin (-\infty, a)$, so $x \in (-\infty, b) \setminus (-\infty, a)$.
Therefore, $[a, b) \subseteq (-\infty, b) \setminus (-\infty, a)$.
Therefore, $(-\infty, b) \setminus (-\infty, a) = [a, b)$.
Therefore, if $a < b$, then $(-\infty, b) \setminus (-\infty, a) = [a, b)$.
Therefore, $\forall a, b \in \mathbb{R}$, if $a < b$, then $(-\infty, b) \setminus (-\infty, a) = [a, b)$. □

33. Prove $(-\infty, 3] \setminus (1, 2) = (-\infty, 1] \cup [2, 3]$.

Proof.
Let $x \in \mathbb{R}$ and assume $x \in (-\infty, 3] \setminus (1, 2)$.
Then $x \in (-\infty, 3]$, so $x \leq 3$, and $x \notin (1, 2)$, so either $x \leq 1$ or $2 \leq x$.
Case 1: $x \leq 1$.
In this case, we have $x \in (-\infty, 1]$, so $x \in (-\infty, 1] \cup [2, 3]$.
Case 2: $2 \leq x$.
Now, $2 \leq x$ and $x \leq 3$, so $x \in [2, 3]$.
This implies that $x \in (-\infty, 1] \cup [2, 3]$.
Therefore, $(-\infty, 3] \setminus (1, 2) \subseteq (-\infty, 1] \cup [2, 3]$.
Conversely, let $x \in \mathbb{R}$ and assume $x \in (-\infty, 1] \cup [2, 3]$.
Then $x \in (-\infty, 1]$ or $x \in [2, 3]$.
Case 1: $x \in (-\infty, 1]$.
In this case, $x \leq 1$, so $x \notin (1, 2)$.
Also, since $x \leq 1$ and $1 \leq 3$, we have $x \leq 3$, so $x \in (-\infty, 3]$.
Now, $x \in (-\infty, 3]$ and $x \notin (1, 2)$, so $x \in (-\infty, 3] \setminus (1, 2)$.
Case 2: $x \in [2, 3]$.
In this case, $2 \leq x$, so $x \notin (1, 2)$, and $x \leq 3$, so $x \in (-\infty, 3]$.
Again, we have $x \in (-\infty, 3]$ and $x \notin (1, 2)$, so $x \in (-\infty, 3] \setminus (1, 2)$.
Therefore, $(-\infty, 1] \cup [2, 3] \subseteq (-\infty, 3] \setminus (1, 2)$.
Therefore, $(-\infty, 3] \setminus (1, 2) = (-\infty, 1] \cup [2, 3]$. $\square$

34. Let $A = (-7, 1)$ and $B = [-2, 17)$. Find $A \cup B$. Prove your result.

Solution.
$A \cup B = (-7, 17)$. $\square$

Proof.
Let $x \in \mathbb{R}$ and assume $x \in A \cup B$.
Then $x \in A$ or $x \in B$.
Case 1: $x \in A$.
In this case, we have $x \in (-7, 1)$, so $-7 < x$ and $x < 1$.
Since $x < 1$ and $1 < 17$, we have $x < 17$.
Now, $-7 < x$ and $x < 17$, so $x \in (-7, 17)$.
Case 2: $x \in B$.
Then $x \in [-2, 17)$, so $2 \leq x$ and $x < 17$.
Since $-7 < 2$ and $2 \leq x$, we have $-7 < x$.
Again, $-7 < x$ and $x < 17$, so $x \in (-7, 17)$.
Therefore, $A \cup B \subseteq (-7, 17)$.
Conversely, let $x \in \mathbb{R}$ and assume $x \in (-7, 17)$.
Then $-7 < x$ and $x < 17$. We consider two cases: $x < 1$ and $1 \leq x$.
Case 1: $x < 1$.
In this case, $-7 < x$ and $x < 1$, so $x \in A$. Thus, $x \in A \cup B$.
Case 2: $1 \leq x$.
Since $-2 \leq 1$ and $1 \leq x$, we have $-2 \leq x$.
Now, $-2 \leq x$ and $x < 17$, so $x \in B$. Thus, $x \in A \cup B$.
Therefore, $(-7, 17) \subseteq A \cup B$.
Therefore, $A \cup B = (-7, 17)$. $\square$

35. Prove $\langle 5\rangle \cap \langle 4\rangle = \langle 20\rangle$.

Proof.
Let $x \in \mathbb{Z}$ and assume $x \in \langle 5\rangle \cap \langle 4\rangle$.
Then $x \in \langle 5\rangle$ and $x \in \langle 4\rangle$.
Choose $a, b \in \mathbb{Z}$ with $x = 5a$ and $x = 4b$.
Let $c = b - a$.
$x = 5x - 4x = 5(4b) - 4(5a) = 20b - 20a = 20c$.
Therefore, $\exists c \in \mathbb{Z}$, $x = 20c$. Thus, $x \in \langle 20\rangle$.
Therefore, $\langle 5\rangle \cap \langle 4\rangle \subseteq \langle 20\rangle$
Conversely, let $x \in \mathbb{Z}$ and assume $x \in \langle 20\rangle$.
Accordingly, choose $q \in \mathbb{Z}$ with $x = 20q$.
Let $s = 4q$.
Then $x = 20q = 5(4q) = 5s$.
Therefore, $\exists s \in \mathbb{Z}$, $x = 5s$, which means $x \in \langle 5\rangle$.
Let $t = 5q$.
$x = 20q = 4(5q) = 4t$.
Therefore, $\exists t \in \mathbb{Z}$, $x = 4t$. Thus, $x \in \langle 4\rangle$.
We now have $x \in \langle 5\rangle$ and $x \in \langle 4\rangle$, which means $x \in \langle 5\rangle \cap \langle 4\rangle$.
Therefore, $\langle 20\rangle \subseteq \langle 5\rangle \cap \langle 4\rangle$.
Therefore, $\langle 5\rangle \cap \langle 4\rangle = \langle 20\rangle$. $\square$

36. Prove $\displaystyle\bigcup_{n\in\mathbb{N}} [1, 3n) = [1, \infty)$.

Proof.
Let $x \in \mathbb{R}$ and assume $x \in \bigcup_{n\in\mathbb{N}} [1, 3n)$.
This means $\exists n \in \mathbb{N}$, $x \in [1, 3n)$. Choose such an n.
Since $x \in [1, 3n)$, we have $1 \leq x$ and $x < 3n$.
Since $1 \leq x$, we have $x \in [1, \infty)$.
Therefore, $\bigcup_{n\in\mathbb{N}} [1, 3n) \subseteq [1, \infty)$.
Conversely, let $x \in \mathbb{R}$ and assume $x \in [1, \infty)$.
Then $1 \leq x$.
By the Archimedean property, since $0 < 3$, $\exists n \in \mathbb{N}$, $x < 3n$. Choose such an n.
Now, $1 \leq x$ and $x < 3n$, so $x \in [1, 3n)$.
Therefore, $\exists n \in \mathbb{N}$, $x \in [1, 3n)$.
This means $x \in \bigcup_{n\in\mathbb{N}} [1, 3n)$.
Therefore, $[1, \infty) \subseteq \bigcup_{n\in\mathbb{N}} [1, 3n)$.
Therefore, $\bigcup_{n\in\mathbb{N}} [1, 3n) = [1, \infty)$. $\square$

37. Prove $\bigcap\limits_{n\in\mathbb{N}}[1, n+1) = [1, 2)$.

Proof.
Let $x \in \mathbb{R}$ and assume $x \in \bigcap_{n\in\mathbb{N}}[1, n+1)$.
This means $\forall n \in \mathbb{N}$, $x \in [1, n+1)$.
Since $1 \in \mathbb{N}$, we have $x \in [1, 1+1)$, which means $x \in [1, 2)$.
Therefore, $\bigcap_{n\in\mathbb{N}}[1, n+1) \subseteq [1, 2)$.
Conversely, let $x \in \mathbb{R}$ and assume $x \in [1, 2)$.
This means $1 \leq x$ and $x < 2$.
Let $n \in \mathbb{N}$.
Then $1 \leq n$, so $2 \leq n+1$.
Since $x < 2$ and $2 \leq n+1$, we have $x < n+1$.
Now, $1 \leq x$ and $x < n+1$, so $x \in [1, n+1)$.
Therefore, $\forall n \in \mathbb{N}$, $x \in [1, n+1)$.
This means $x \in \bigcap_{n\in\mathbb{N}}[1, n+1)$.
Therefore, $[1, 2) \subseteq \bigcap_{n\in\mathbb{N}}[1, n+1)$.
Therefore, $\bigcap_{n\in\mathbb{N}}[1, n+1) = [1, 2)$. $\square$

38. For $n \in \mathbb{N}$, let $\equiv_n^2$ be the relation on $\mathbb{Z}$ given by: $\forall x, y \in \mathbb{Z}$, $x \equiv_n^2 y$ means $(x-y)(x+y) \in \langle n \rangle$. Prove $\equiv_n^2$ is an equivalence relation.

Proof.
Let $x \in \mathbb{Z}$.
Then $(x-x)(x+x) = 0(2x) = 0 = 0n$. Thus, $(x-x)(x+x) \in \langle n \rangle$.
Therefore, $x \equiv_n^2 x$.
Therefore, $\equiv_n^2$ is reflexive.
Let $x, y \in \mathbb{Z}$.
Assume $x \equiv_n^2 y$. Then $(x-y)(x+y) \in \langle n \rangle$.
Let $q \in \mathbb{Z}$ with $(x-y)(x+y) = nq$.
Then $(y-x)(y+x) = -(x-y)(x+y) = -nq = n(-q)$.
Thus, $(y-x)(y+x) \in \langle n \rangle$, meaning $y \equiv_n^2 x$.
Therefore, if $x \equiv_n^2 y$, then $y \equiv_n^2 x$.
Therefore, $\equiv_n^2$ is symmetric.
Let $x, y, z \in \mathbb{Z}$.
Assume $x \equiv_n^2 y$ and $y \equiv_n^2 z$. Then $(x-y)(x+y) \in \langle n \rangle$ and $(y-z)(y+z) \in \langle n \rangle$.
Let $a, b \in \mathbb{Z}$ with $(x-y)(x+y) = an$ and $(y-z)(y+z) = bn$.
Then $x^2 - y^2 = an$ and $y^2 - z^2 = bn$.
Adding these equations gives $x^2 - z^2 = an + bn = (a+b)n$.
Thus, $(x-z)(x+z) \in \langle n \rangle$, and so $x \equiv_n^2 z$.
Therefore, if $x \equiv_n^2 y$ and $y \equiv_n^2 z$, then $x \equiv_n^2 z$.
Therefore, $\equiv_n^2$ is transitive.
Q.E.D. $\square$

39. Let R be the relation $R = \{(x, y) \in \mathbb{R}^2 \mid 2x \leq 3y\}$. Prove R is not reflexive, R is not symmetric, R is not antisymmetric, and R is not transitive.

Proof.
Let $x = -1$.
Then $3x = -3$ and $2x = -2$. Since $-3 < -2$, we have $3x < 2x$, which means $x \not R x$.
Therefore, $\exists x \in \mathbb{R}$, $x \not R x$. This means R is not reflexive.
Next, let $x = 1$ and $y = 2$.
Then $2x = 2 \leq 6 = 3y$, and so xRy. However, $2y = 4$ and $3x = 3$, so $3x < 2y$. Thus, $y \not R x$.
Therefore, $\exists x, y \in \mathbb{R}$, xRy, but $y \not R x$. This means R is not symmetric.
Next, let $x = 3$ and $y = 4$.
Then $2x = 6 \leq 12 = 3y$, so xRy. Also, $2y = 8 \leq 9 = 3x$, so yRx.
However, $x \neq y$.
Therefore, $\exists x, y \in \mathbb{R}$, xRy and yRx, but $x \neq y$. This means R is not antisymmetric.
Finally, let $x = 5$, $y = 4$, and $z = 3$.
$2x = 10 \leq 12 = 3y$, so xRy. Also, $2y = 8 \leq 9 = 3z$, so yRz.
However, $2x = 10$ and $3z = 9$, so $3z < 2x$. Thus, $x \not R z$.
Therefore, $\exists x, y, z \in \mathbb{R}$, xRy and yRz, but $x \not R z$. This means R is not transitive. □

40. Let R be the relation $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid \exists n \in \mathbb{Z}_{\geq 0},\ y = 2^n x\}$. Prove R is antisymmetric.

Proof.
Let $x, y \in \mathbb{R}$.
Assume xRy and yRx.
This means $\exists n \in \mathbb{Z}_{\geq 0}$, $y = 2^n x$ and $\exists n \in \mathbb{Z}_{\geq 0}$, $x = 2^n y$.
Choose $n \in \mathbb{Z}_{\geq 0}$ with $y = 2^n x$, and choose $m \in \mathbb{Z}_{\geq 0}$ with $x = 2^m y$.
Then $x = 2^{m+n} x$; hence either $x = 0$ or $1 = 2^{m+n}$.
Case 1: $x = 0$.
In this case, $y = 2^n x = 2^n(0) = 0$; thus $y = x$.
Case 2: $1 = 2^{m+n}$.
Since $m + n < 2^{m+n}$ (Proposition 1.2.20), we have $m + n < 1$; hence $m + n \leq 0$.
Since $0 \leq n$, we have $m \leq m + n \leq 0$, and so $m \leq 0$. Since we also have $0 \leq m$, this means $m = 0$.
Now, $x = 2^m y = 2^0 y = (1)y = y$.
Therefore, if xRy and yRx, then $x = y$.
Therefore, $\forall x, y \in \mathbb{R}$, if xRy and yRx, then $x = y$.
Thus, R is antisymmetric. □