Applied Stochastic Processes

2018A7PS0193P

January 25, 2021

1 Introduction

Definition 1.1. *A stochastic process is a probability model that describes the evolution of a system evolving randomly in time.*

Definition 1.2. *A random variable is a mapping* $X : \Omega \to \mathbb{R}$ *that assigns a real number* $X(\omega)$ *to each outcome* $\omega \in \Omega$*, where* Ω *is the sample space.*

A stochastic process can be given by a collection of random variables $\{X(t), t \in T\}$, where T is called the **index set**. If T is countable (observed at discrete times), we get a **discrete time stochastic process**. On the other hand, if T is uncountable (observed continuously), then we get a **continuous time stochastic process**.

Definition 1.3. *The state space of a stochastic process is defined as the set of all possible values that the random variables* $X(t)$ *can assume.*

1.1 Elementary Probability

For a recap of elementary probability, refer to the notes from Applied Statistical Methods.

1.2 Transformation of Random Variables

Lemma 1.1. *Let* X *have a continuous, strictly increasing CDF* F*. Let* $U \sim$ *Uniform(0,1). If* $Y = F^{-1}(U)$*, then* Y *also has the CDF* F*.*

The lemma above allows us to transform U to any other random variable, as long as it has a continuous and strictly increasing CDF. Say we had an algorithm to define a uniform random variable in the range (0,1), now we have a way to generate random variables from a different distribution.

1.3 Moment Generating Functions

Definition 1.4. *The moment generating function $\phi(t)$ of the random variable X is defined for all values t as $\phi(t) = E[e^{tx}]$.*

The moment generating functions of some oft-used distributions are as follows:

- Moment generating function of Binomial(n, p) is:

$$\phi(t) = (pe^t + 1 - p)^n$$

- Moment generating function of Poisson(λ) is:

$$\phi(t) = e^{\lambda(e^t - 1)}$$

- Moment generating function of Exponential(λ) is:

$$\phi(t) = \frac{\lambda}{\lambda - t}$$

- TODO more

Theorem 1.2. *The moment generating function of the sum of independent random variables is the product of the individual moment generating functions.*

So, this means that $\phi_{X+Y}(t) = \phi_X(t) \cdot \phi_Y(t)$, as long as $X \perp Y$ (this notation means that they are independent).

1.4 Conditional distributions

The conditional probability distribution of Y given the occurrence of the value x of X is given by:

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x, y)}{f_X(x)}$$

where $f_{X,Y}(x, y)$ is the joint distribution and $f_X(x)$ is the marginal density of X.

The conditional expectation of X given Y is:

$$E(X|Y = y) = \int_{-\infty}^{\infty} x f_{X|Y}(x, y) dx$$

1.5 Markov's and Chebyshev's Inequality

Theorem 1.3 (Markov's Inequality)**.** *Let X be a non-negative random variable and suppose that $E(X)$ exists. For any $t > 0$,*

$$P(X > t) \leq \frac{E(X)}{t}$$

Theorem 1.4 (Chebyshev's Inequality)**.** *Let $\mu = E(X)$ and $\sigma^2 = V(X)$. Then*

$$P(|X - \mu| \geq t) \leq \frac{\sigma^2}{t^2}$$

$$P(|Z| \geq k) \leq \frac{1}{k^2}$$

where $Z = (X - \mu)/\sigma$ and $t > 0$.

Chebyshev's Inequality is a more general version of Markov's Inequality, applicable for any random variable X.

2 Convergence of Random Variables

Let $X_1, X_2, ...$ be a sequence of random variables and let X be another random variable. Let F_n be the CDF of X_n and F be the CDF of X.

We say that X_n converges to X in probability, denoted by $X_n \xrightarrow{P} X$, if for every $\epsilon > 0$,

$$P(|X_n - X| > \epsilon) \to 0$$

as $n \to \infty$.

We say that X_n converges to X in distribution, denoted by $X_n \rightsquigarrow X$, if

$$\lim_{n \to \infty} F_n(t) = F(t)$$

for all t for which F is continuous.

Lemma 2.1. *If $X_n \xrightarrow{P} X$, then $X_n \rightsquigarrow X$.*

The above lemma is provided without proof, as it is beyond the scope of the course.

Example:

Let $X_n \sim N(0, \frac{1}{n})$. We claim that this series converges to

$$F_X(n) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

in distribution. Let $t > 0$, and define the standard normal variable $Z_n = \sqrt{n}X_n$, so $Z_n \sim N(0,1)$. So,

$$\begin{aligned} F_{X_n}(t) &= P(X_n \leq t) \\ &= P(Z_n \leq \sqrt{n}t) \\ &= \int_{-\infty}^{\sqrt{n}t} f(x)dx \end{aligned}$$

where $f(x)$ is the PDF of Z_n.

It is clear that as $n \to \infty$, we get the following distribution:

$$F_{X_n}(t) = \begin{cases} 0 & t < 0 \\ 0.5 & t = 0 \\ 1 & t > 0 \end{cases}$$

So, $F_{X_n} \rightsquigarrow F_X \forall t \in \mathbb{R} - \{0\}$.