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# Applied Stochastic Processes

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## CHAPTER 1

### Fundamentals

#### 1. Stochastic Processes

DEFINITION 1. A stochastic process is a probability model that describes the evolution of a system evolving randomly in time.

DEFINITION 2. A random variable is a mapping  $X : \Omega \rightarrow \mathbb{R}$  that assigns a real number  $X(\omega)$  to each outcome  $\omega \in \Omega$ , where  $\Omega$  is the sample space.

A stochastic process can be given by a collection of random variables  $\{X(t), t \in T\}$ , where  $T$  is called the **index set**. If  $T$  is countable (observed at discrete times), we get a **discrete time stochastic process**. On the other hand, if  $T$  is uncountable (observed continuously), then we get a **continuous time stochastic process**.

DEFINITION 3. The state space of a stochastic process is defined as the set of all possible values that the random variables  $X(t)$  can assume.

#### 2. Elementary Probability

For a recap of elementary probability, refer to the notes from Applied Statistical Methods.

#### 3. Transformation of Random Variables

LEMMA 3.1. Let  $X$  have a continuous, strictly increasing CDF  $F$ . Let  $U \sim \text{Uniform}(0,1)$ . If  $Y = F^{-1}(U)$ , then  $Y$  also has the CDF  $F$ .

The lemma above allows us to transform  $U$  to any other random variable, as long as it has a continuous and strictly increasing CDF. Say we had an algorithm to define a uniform random variable in the range  $(0,1)$ , now we have a way to generate random variables from a different distribution.

#### 4. Moment Generating Functions

DEFINITION 4. The moment generating function  $\phi(t)$  of the random variable  $X$  is defined for all values  $t$  as  $\phi(t) = E[e^{tx}]$ .

The moment generating functions of some oft-used distributions are as follows:

- Moment generating function of Binomial( $n, p$ ) is:

$$\phi(t) = (pe^t + 1 - p)^n$$

- Moment generating function of Poisson( $\lambda$ ) is:

$$\phi(t) = e^{\lambda(e^t - 1)}$$

- Moment generating function of Exponential( $\lambda$ ) is:

$$\phi(t) = \frac{\lambda}{\lambda - t}$$

- TODO more

THEOREM 4.1. The moment generating function of the sum of independent random variables is the product of the individual moment generating functions.

So, this means that  $\phi_{X+Y}(t) = \phi_X(t) \cdot \phi_Y(t)$ , as long as  $X \perp Y$  (this notation means that they are independent).

#### 5. Conditional distributions

The conditional probability distribution of  $Y$  given the occurrence of the value  $x$  of  $X$  is given by:

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

where  $f_{X,Y}(x,y)$  is the joint distribution and  $f_X(x)$  is the marginal density of  $X$ .

The conditional expectation of  $X$  given  $Y$  is:

$$E(X|Y = y) = \int_{-\infty}^{\infty} x f_{X|Y}(x, y) dx$$

### 6. Markov's and Chebyshev's Inequality

**THEOREM 6.1** (Markov's Inequality). Let  $X$  be a non-negative random variable and suppose that  $E(X)$  exists. For any  $t > 0$ ,

$$P(X > t) \leq \frac{E(X)}{t}$$

**THEOREM 6.2** (Chebyshev's Inequality). Let  $\mu = E(X)$  and  $\sigma^2 = V(X)$ . Then

$$P(|X - \mu| \geq t) \leq \frac{\sigma^2}{t^2}$$

$$P(|Z| \geq k) \leq \frac{1}{k^2}$$

where  $Z = (X - \mu)/\sigma$  and  $t > 0$ .

Chebyshev's Inequality is a more general version of Markov's Inequality, applicable for any random variable  $X$ .

### 7. Convergence of Random Variables

Let  $X_1, X_2, \dots$  be a sequence of random variables and let  $X$  be another random variable. Let  $F_n$  be the CDF of  $X_n$  and  $F$  be the CDF of  $X$ .

We say that  $X_n$  converges to  $X$  in probability, denoted by  $X_n \xrightarrow{P} X$ , if for every  $\epsilon > 0$ ,

$$P(|X_n - X| > \epsilon) \rightarrow 0$$

as  $n \rightarrow \infty$ .

We say that  $X_n$  converges to  $X$  in distribution, denoted by  $X_n \rightsquigarrow X$ , if

$$\lim_{n \rightarrow \infty} F_n(t) = F(t)$$

for all  $t$  for which  $F$  is continuous.

**LEMMA 7.1.** If  $X_n \xrightarrow{P} X$ , then  $X_n \rightsquigarrow X$ .

The above lemma is provided without proof, as it is beyond the scope of the course.

**EXAMPLE.** Let  $X_n \sim N(0, \frac{1}{n})$ . We claim that this series converges to

$$F_X(n) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

in distribution. Let  $t > 0$ , and define the standard normal variable  $Z_n = \sqrt{n}X_n$ , so  $Z_n \sim N(0, 1)$ . So,

$$\begin{aligned} F_{X_n}(t) &= P(X_n \leq t) \\ &= P(Z_n \leq \sqrt{nt}) \\ &= \int_{-\infty}^{\sqrt{nt}} f(x)dx \end{aligned}$$

where  $f(x)$  is the PDF of  $Z_n$ .

It is clear that as  $n \rightarrow \infty$ , we get the following distribution:

$$F_{X_n}(t) = \begin{cases} 0 & t < 0 \\ 0.5 & t = 0 \\ 1 & t > 0 \end{cases}$$

So,  $F_{X_n} \rightsquigarrow F_X \forall t \in \mathbb{R} - \{0\}$ .



## CHAPTER 2

# Markov Chains

### 1. Introduction

DEFINITION 5. A stochastic process with a finite number of state spaces  $S = \{0, 1, \dots, N\}$  and a countable index state  $T = \{t_0, t_1, t_2, \dots\}$  is a Markov chain if

$$\begin{aligned} P(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \dots, X_0 = i_0) \\ = P(X_{n+1} = j | X_n = i) \\ = P_{ij} \end{aligned}$$

This means that the probability of a transition from state  $i$  to a state  $j$  is completely dependent on the starting and the ending state. The resulting transition matrix is called the **one step transition matrix**, denoted by  $\mathbf{P}$ .

LEMMA 1.1.

$$\sum_{j=0}^{\infty} P_{ij} = 1 \forall i \in S$$