Number Theory

2018A7PS0193P

CHAPTER 1

Fundamentals

1. Notation

For the rest of this course, the following notation will be followed:

(1) $\mathbb{N}$ is the set of natural numbers

(2) $\mathbb{Z}$ is the set of integers

(3) $\mathbb{W}$ is the set of whole numbers, i.e. $\mathbb{W} = \mathbb{N} \cup \{0\}$

2. Induction

Often in number theory, we use inductive proofs to prove our arguments. Induction consists of the following steps:

(1) Define an induction hypothesis $P(k)$

(2) Verify it works for some base case $k = b$. It is possible multiple base cases need to be verified.

(3) Assuming $P(k)$ is true, show that it implies that $P(k+1)$ is true

Remember that $P(k)$ is a statement, not a function. You cannot multiply it by some constant or perform any operations on it.

In weak induction (like in the steps given above), we only assume that $P(k)$ is true. However in strong induction, we assume that $P(i)$ is true $\forall i \in [b, k]$, and use this to prove that $P(k+1)$ is true.

Exercise. Prove that the principle of strong induction is true given that the principle of weak induction is true.

Solution. Let us assume that $P(1), ..., P(b)$ is true. If $P(1), ..., P(k)$ are true for some $k \geq b$, then $P(k+1)$ is true. Then, we must show that $P(n)$ is true for all $n \geq 1$.

Let $Q(n)$ be the statement that $P(1), ...P(n)$ are true. Of course, in the base case, $Q(1)$ is true. Let $Q(k)$ be true, where $K \geq 1$. This means that $P(1), ...P(k)$ is true, so $P(k+1)$ must be true. Hence, $Q(k+1)$ is true.

So, by Weak induction, $Q(n)$ is true $\forall n \geq 1$, which implies that $P(n)$ is true $\forall n \geq 1$. ■

3. Well Ordering Principle

Theorem 1.1 (Well Ordering Principle). Every non empty set of non-negative integers has a least element.

This is not true about negative integers - consider the case of infinite sets, like the set of all integers. There is no well defined least element.

Lemma 1.2. The well ordering principle is equivalent to the principle of mathematical induction.

Proof. First, let us prove that WOP $\Rightarrow$ PMI. Let $P(n)$ be a statement that depends on $n \in \mathbb{N}$. Suppose that:

- $P(1)$ is true
- $P(k)$ is true implies $P(k+1)$ is true for all $k \in N$.

We have to show that $P(n)$ is true for all $n \in N$. Let :

$$S = \{n \in \mathbb{N} : P(n) \text{ is true}\}$$

This means we must show that $S = \mathbb{N}$. Let $T := \mathbb{N}\backslash S$, i.e. T is the complement. Let as assume that $S \neq \mathbb{N}$.

By WOP, T has a least element, say m. Note that $m \geq 2$ since $1 \in S$. Then, $m-1 \notin T$ and $m-1 \in S$. As such, $P(m-1)$ must be true! However, by our initial assumptions, that would mean $P(m)$ is true as well, so $m \in S$. This creates a contradiction, since $m \in T$. Hence, $S = \mathbb{N}$.

Now, let us prove that PMI $\Rightarrow$ WOP.

Consider the statement $P(n)$ that every non empty set of non-negative integers of size n has a least element. It is clear that the base case $P(1)$ is true. Now, let us assume that $P(k)$ is true - what can we say about $P(k+1)$. When we insert an element, we have two cases:

(1) The inserted element is less than the least element. In this case, there is a new least element, and $P(k+1)$ is true.

(2) The inserted element is not less than the least element. In this case, the least element is the same, and $P(k+1)$ is true.

Hence, by PMI, we can say that $P(n)$ is true $\forall n \in N$, i.e., WOP is true.

Since PMI $\Rightarrow$ WOP and WOP $\Rightarrow$ PMI, PMI $\Leftrightarrow$ WOP. □

4. Binomial Theorem

Theorem 1.3 (Binomial Theorem). Let $x, y \in \mathbb{C}$ and let $n \in \mathbb{N}$, then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Corollary 1.3.1.

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Lemma 1.4 (Pascal's Identity).

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Lemma 1.5.

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} = F_n$$

5. Pigeonhole Principle

Theorem 1.6. If n items are put into m containers, with $n > m$, then at least one container must contain more than one item.

CHAPTER 2

Division

1. Division Algorithm

THEOREM 2.1. Let $a, b \in \mathbb{Z}$ with $b > 0$. Then, there exist unique integers q and r such that $a = bq + r$, $r \in [0, b)$.

PROOF. Let $S = \{a - bn : n \in Z, a - bn \geq 0\}$. This set is always non-empty:

- If $a \geq 0$, then $a \in S$
- If $a < 0$, then if $n = a$, we have $a - ab \in S$ since $b \geq 1$.

By WOP, S has a least element, say r. So, there exists $q \in Z$ such that $r = a - bq$. Since $r \in S$, we have $r \geq 0$.

Suppose $r \geq b$. Then:

$$\begin{gathered} a - b(q+1) = a - bq - b = r - b \geq 0 \\ \Rightarrow a - b(q+1) \in S \\ \Rightarrow r - b \in S \end{gathered}$$

However, $r - b < r$, and r is the least element! This gives us a contradiction. So, $r < b$.

As such, we have proved the existence of this solution. Now we must prove it's uniqueness.

Suppose there exists p, r, q', r', such that:

$$a = bq + r, 0 \leq r < b$$

$$a = bq' + r', 0 \leq r' < b$$

Assume WLOG $q \geq q'$. Now,

$$r' - r = b(q - q')$$

If $q > q'$, then $r' - r \geq b$. However, $r' - r < b$. So, this is a contradiction, and $q' = q$. The solution must be unique.

□

Definition 1. If $a, b \in \mathbb{Z}$, we say that a divides b if $b = ak$ for some $k \in \mathbb{Z}$. This is denoted by $a|b$

Some properties of division are:

- If $a|b$, then $\pm a| \pm b$
- If $a|b$ and $b|c$ then $a|c$ (Transitivity)
- If $a|b$ and $a|c$ then $a|bx + cy$ (Linear Combination)
- If $a|b$ and $b \neq 0$, then $|a| \leq |b|$ (Bounds by divisibility)
- $a|b$ and $b|a$, then $b = \pm a$.

Exercise. Prove that $x^a - 1|x^b - 1 \Leftrightarrow a|b$.

Solution. First, let us prove that if $a|b$, then $x^a - 1|x^b - 1$. Let $b = qa$. Then,

$$x^b - 1 = (x^a)^q - 1^q = (x^a - 1)((x^a)^{q-1} + \cdots + x^a + 1)$$

So, $x^b - 1|x^a - 1$. Now to prove the converse. Let $b = aq + r$. Assume $a \nmid b$, then $0 < r < a$. Then,

$$x^b - 1 = x^b - x^r + x^r - 1 = x^r(x^{aq} - 1) + x^r - 1$$

$x^a - 1|x^{qa} - 1$, so $x^r - 1$ is the remainder. Since $r < a$, $x^a - 1 \nmid x^r - 1$. This would mean that $x^a - 1 \nmid x^b - 1$, which is a contradiction. So, $r = 0$. Hence proved. ■

2. Base b representations

Theorem 2.2. Let $b \in \mathbb{N}$ with $b \geq 2$. Then every positive integer can be expressed uniquely as

$$N = a_k b^k + ... + a_1 b + a_0$$

where $k \geq 0, a_k \neq 0$ and $0 \leq a_i < b$ for $i = 0, ...k$. This is denoted by $N = (a_k, ...a_1 a_0)_b$

Proof. By the division algorithm, there exist unique integers q_0 and a_0 such that:

$$N = q_0 b + a_0, a_0 \in [0, b)$$

Note that $q_0 < N$. If $q_0 \neq 0$ we apply the division algorithm again to find unique integers q_1 and a_q such that:

$$q_0 = q_1 b + a_1, a_1 \in [0, b)$$

Then,

$$N = (q_1 b + a_1)b + a_0 = q_1 b^2 + a_1 b + a_0$$

We continue till we get a quotient $q_k = 0$. This will terminate since $q_k < ... < q_2 < q_1 < q_0 < N$, forming a decreasing sequence of non-negative integers and eventually reaching zero. From this, we get:

$$N = a_k b^k + ... + a_1 b + a_0$$

Hence, the solution always exists.

Suppose N has two distinct expansions. We can write it as:

$$\begin{aligned} N &= a_k b^k + ... + a_1 b + a_0 \\ &= c_k b^k + ... + c_1 b + c_0 \end{aligned}$$

where $0 \leq a_i, c_j < b$ for all i, j. Let $d_i = a_i - c_i$. Then, $\sum_{i=0}^{k} d_i b^i = 0$. The d_i cannot all be zero as the two expansions are assumed distinct. Let j be the least integer, $0 \leq j \leq k$, such that $d_j \neq 0$. Then, $\sum_{i=j}^{k} d_i b^i = 0$. Dividing by b^j, we find that $\sum_{i=j}^{k} d_i b^{i-j} = 0$. Thus,

$$d_j + b \left(\sum_{i=j+1}^{k} d_i b^{i-j-1} \right) = 0$$

This implies that the $b|d_j$ and since $d_j \neq 0$, we get that $b = |b| \leq |d_j|$. However, $|d_j| < b$. Hence, we have a contradiction, and the two expansions cannot be distinct. Hence, the solution is also always unique. □

Lemma 2.3. If $N = (a_k ... a_1 a_0)_b$, then:

$$\begin{aligned} bN &= (a_k ... a_1 a_0 0)_b \\ \left\lfloor \frac{N}{b} \right\rfloor &= (a_k ... a_1)_b \end{aligned}$$

This is a trivial result, which can be thought of as a left or right bitwise shift.

Lemma 2.4 (Particular case of Legendre's formula). Let $n \in \mathbb{N}$ and let e denote the highest power of 2 dividing $n!$. Then

$$e = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{2^k} \right\rfloor$$

This is always a finite sum. This can alternatively expressed as, if $n = (a_k ... a_1 a_0)_2$, then:

$$e = n - (a_k + ... + a_1 + a_0)$$

Proof. It is clear that e is the sum of the no. of positive multiples of 2^i which are $\leq n$, for all i. So, this can be calculated by:

$$e = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{2^k} \right\rfloor$$

□

Thus, if r denotes the number of ones in the binary expansion of n, then 2^{n-r} is the highest power of 2 dividing $n!$. Further,

- $2^n \nmid n!$ for $n \in \mathbb{N}$
- $2^{n-1} | n!$ if and only if n is a power of 2.

CHAPTER 3

Properties of Numbers

1. Prime and Composite Numbers

DEFINITION 2. A positive integer $p > 1$ is called prime if its only positive divisors are 1 and p. A positive integer which is not prime is called composite.

The number 1 is neither prime nor composite.

LEMMA 3.1. Every integer $n \geq 2$ has a prime factor.

PROOF. Let $P(n)$ be the statement that n has a prime factor. Then $P(2)$ is true, since 2 is a prime factor of 2. Let $k \geq 2$. Assume $P(2)...P(k)$ are true.

If $k+1$ is prime, then $k+1$ is a prime factor of itself. So $P(k+1)$ is true.

If $k+1$ is composite, then there exists $d \in [2, k]$ such that $d|k+1$. By the induction hypothesis, d has a prime factor p. Since $p|d$ and $d|k+1$, $p|k+1$. So p is a prime factor of $k+1$, and $P(k+1)$ is true. By PSI, $P(n)$ is true for all $n \geq 2$. □

THEOREM 3.2 (Euclid). There are infinitely many primes.

PROOF. Suppose there are finitely many primes $p_1, ..., p_k$. Let

$$N = p_1 ... p_k + 1$$

Since $N \geq 2$, it must have a prime factor. Hence, there exists $i \in [1, k]$ such that $p_i|N$. Since $p_i|N$ and $p_i|p_1p_2...p_k$, we get that $p_i|N - p_1p_2...p_k$, i.e., $p_i|1$. However, $p_i \geq 2$, which gives us a contradiction. So, there must be infinitely many primes. □

EXERCISE. For $n \geq 1$, let p_n be the nth prime. Prove that

$$p_n \leq 2^{2^{n-1}}$$

SOLUTION. Let $P(n)$ be the statement that $p_n \leq 2^{2^{n-1}}$. It is clear that this is true for the base case $P(1)$. Let us assume that $P(1), ..., P(k)$ is true for $k \geq 1$. We observed in

Euclid's proof that $p_1...p_k + 1$ is not divisible by any of $p_1...p_k$. Hence if p_i denotes a prime factor of $p_1...p_k + 1$, then $i \geq k + 1$.

$$p_{k+1} \leq p_i \leq p_1...p_k + 1$$

Using the inductive hypothesis, we find that

$$\begin{aligned} p_{k+1} &\leq p_1...p_k + 1 \leq 2.2^2.2^{2^2}...2^{2^{k-1}} + 1 \\ &= 2^{\sum_{j=0}^{k-1} 2^j} + 1 = 2^{2^k - 1} + 1 \leq 2^{2^k} \end{aligned}$$

So, $P(k + 1)$ is true. So, by PSI, the result has been proven. ■

Definition 3. The product of the first n prime numbers is called the n^{th} primorial and is denoted by $p_n\#$.

Definition 4. Euclid numbers are integers of the form $E_n = p_n\# + 1$.

All Euclid numbers are not primes - E_6 is not a prime!

Theorem 3.3. Every composite number n has a prime factor $\leq \lfloor\sqrt{n}\rfloor$

Proof. Since n is composite, there exists integers $k, l \in (1, n)$ such that

$$n = kl$$

If $k > \sqrt{n}$ and $l > \sqrt{n}$ then $kl > n$, which is false. So, one of them must be less than or equal to $\sqrt{n}$. □

So, if $n > 1$ has no prime factors $\leq \lfloor\sqrt{n}\rfloor$, then n is prime. We can use this as a test of primality.

It is faster to do this using the Sieve of Eratosthenes. Using this, we can test primality of the first n integers in $O(n \log\log n)$ instead of $O(n\sqrt{n})$. This is a pretty well known algorithm so it's left to the reader to see it on cp-algorithms.

Theorem 3.4. There is no non-constant polynomial $f(x)$ with integer coefficients such that $f(n)$ is prime for all integer n.

Proof. Suppose such a polynomial $f(x)$ exists:

$$f(x) = a_k x^k + ... + a_1 x + a_0, k \geq 1, a_k \neq 0$$

Let $b \in \mathbb{Z}$. Then $f(b)$ is a prime number, say p. Let $t \in \mathbb{Z}$. We have:

$$\begin{aligned} f(b+tp) &= a_k(b+tp)^k + ... + a_1(b+tp) + a_0 \\ &= (a_k b^k + ...a_1 b + a_0) + p \cdot g(t) \\ &= f(b) + p \cdot g(t) = p(1+g(t)) \end{aligned}$$

where $g(t)$ is a polynomial in t. Since $p|f(b+tp)$ and it must be prime, so $f(b+tp) = p$. This implies that f assumes the value p infinitely many times. This is a contradiction, since a polynomial of degree k cannot assume the same value ¿ k times. □

2. Prime Counting function

Let x be a positive real number. We define :

$$\pi(x) = \sum_{p \leq x} 1$$

where p denotes a prime. So $\pi(x)$ counts the number of primes $\leq x$. This is called the prime counting function.

THEOREM 3.5 (Prime Number Theorem).

$$\lim_{x \to \infty} \frac{\pi(x)}{x/\log x} = 1$$

This essentially states that $\pi(x) \sim \frac{x}{\log x}$. The proof is too complicated to be covered in this course.

3. Gaps between Primes

The following lemma states that we can find a gap between primes of any arbitrary length.

LEMMA 3.6. For every $n \in \mathbb{N}$, there are n consecutive integers that are all composite.

PROOF. Consider the numbers:

$$(n+1)! + 2, (n+1)! + 3, ..., (n+1)! + (n+1)$$

It is clear that for $n \geq 1$, $2|(n+1)! + 2$. However, $(n+1)! + 2 \neq 2$. So, $(n+1)! + 2$ cannot be prime, and must be composite. We can extend this to each of the given numbers, and prove that they are all composite. □

DEFINITION 5. A pair (p, q) of primes with $p < q$ is called a twin prime pair if $q - p = 2$.

It is unknown how many twin primes exist. It is conjectured that there are infinitely many twin primes, but this has not yet been proved.

THEOREM 3.7 (Bertrand's Postulate). For every integer $n \geq 2$, there is always at least one prime between n and $2n$.

This was verified by Bertrand but proved by Chebyshev. It is sometimes called Chebyshev's theorem. The proof of this result goes beyond the scope of this course.

REMARK. Do not use this result unless mentioned that we can, in the exam .

EXERCISE. Using Bertrand's postulate, prove that for $n \geq 2$:

$$p_n < 2^n$$

SOLUTION. Consider the statement $P(n)$ that $p_n < 2^n$. This is true for the base case that $P(2)$. Now assume that $P(k)$ is true. This means:

$$p_k < 2^k$$

From Bertrand's postulate,

$$k < p_k < 2k$$

$$k + 1 \leq p_k$$

We also know from Bertrand's postulate that:

$$k + 1 < p_{k+1} < 2(k + 1)$$

$$p_{k+1} < 2p_k$$

So, from the induction hypothesis,

$$p_{k+1} < 2.2^k$$

$$p_{k+1} < 2^{k+1}$$

Hence, $P(k) \Rightarrow P(k + 1)$. From PMI, $P(n)$ is true $\forall n \geq 2$. ■

EXERCISE. Prove that if $2^m + 1$ is prime, then $m = 2^n$ for some n.

SOLUTION. Here, we use the following lemma - if k is odd, then $x^k + 1$ is divisible by $x + 1$. Suppose that m has an odd factor k. Then, we can express m as kp. So,

$$2^{kp} + 1 = (2^p)^k + 1$$

From our lemma, this is divisible by $2^p + 1$ which is a number other than 1 and itself. This means $2^m + 1$ cannot be prime if it has an odd factor, and hence m must be a power of 2. ■

4. Fermat Numbers

DEFINITION 6. Fermat numbers are f_n such that:

$$f_n = 2^{2^n} + 1$$

LEMMA 3.8 (Recursive definition).

$$f_n = f_{n-1}^2 - 2f_{n-1} + 2$$

This result is obvious from expanding the RHS, so the proof is not given here.

EXERCISE. Prove that f_n, $n \geq 2$, all end in 7.

SOLUTION. Let $P(n)$ be the statement that f_n ends in 7. This is true for our base case $P(2)$. Let us assume that $P(k-1)$ is true, i.e. $f_k \mod 10 = 7$. So, by the recursive definition:

$$\begin{aligned} f_k &= f_{k-1}^2 - 2f_{k-1} + 2 \mod 10 \\ &= 7^2 - 2 * 7 + 2 \mod 10 \\ &= 7 \mod 10 \end{aligned}$$

So, by PMI, $P(n)$ is true for all $n \geq 2$. ■

LEMMA 3.9 (Duncan's Identity).

$$f_0 f_1 ... f_{n-1} = f_n - 2$$

PROOF. Let $P(n)$ be the statement that this is true for f_n. This is clearly true for the base case $P(1)$. Let us assume $P(k)$ is true, i.e.

$$f_0 f_1 ... f_{k-1} = f_k - 2$$

$$f_0 f_1 ... f_k = f_k(f_k - 2) = f_k^2 - 2f_k = f_{k+1} - 2$$

The above result comes from the recursive definition. Since $P(k+1)$ follows from $P(k)$, by PMI, Duncan's identity is true. □

THEOREM 3.10. Every prime factor of f_n, $n \geq 2$, is of the form $k \cdot 2^{n+2} + 1$.

PROOF. To be discussed later in the course. □

This theorem can be helpful to quickly find the primality of f_n. For instance, f_4 is prime - we can see this by checking all the numbers of the form 2^6k+1 which are less than $\sqrt{f_4}$. This cuts down the search space and makes primality checking faster.

5. Fibonacci Numbers

Definition 7. Fibonacci numbers are numbers of the form:

$$F_n = F_{n-1} + F_{n-2}$$

where $F_1 = 1, F_2 = 1$

Lemma 3.11.

$$\sum_{i=1}^{k} F_i = F_{k+2} - 1$$

Lemma 3.12 (Cassini's Formula).

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n, n \geq 2$$

The proofs of lemmas 3.11 and 3.12 come directly from induction, so I am not discussing the proof here.

Lemma 3.13.

$$F_{n+m} = F_mF_{n+1} = F_mF_{n+1} + F_{m-1}F_n, m \geq 2, n \geq 1$$

Proof. This is a non-trivial case of induction. Let us fix $m \in \mathbb{N}$, and do induction on n. For $n = 1$, the RHS is:

$$F_{m-1}F_1 + F_mF_2 = F_{m-1} + F_m = F_{m+1}$$

This is also true for $n = 2$. Assume that the result is true for $k = 3, 4, ..., n$. We want to show that the result is true for $k = n+1$. For $k = n-1$,

$$F_{m+n-1} = F_{m-1}F_{n-1} + F_mF_n$$

For $k = n$,

$$F_{m+n} = F_{m-1}F_n + F_mF_{n+1}$$

Add both sides, we get:

$$F_{m+n-1} + F_{m+n} = F_{m+n+1} = F_{m-1}F_{n+1} + F_mF_{n+2}$$

Hence Proved. □

6. Lucas Numbers

Definition 8. Lucas numbers are numbers L_n such that:

$$L_n = L_{n-1} + L_{n-2}$$

where $\mathrm{L}_1 = 1, L_2 = 3$.

Theorem 3.14 (Binet's formulas). Let $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

$$L_n = \alpha^n + \beta^n$$

Proof. TODO □

CHAPTER 4

Greatest Common Divisor and Least Common Multiple

1. Greatest Common Divisor

Definition 9. Let $a, b \in \mathbb{Z}$, not both zero. The greatest common divisor of a and b is the positive integer d such that:

- $d|a$ and $d|b$
- If c is a positive integer such that $c|a$ and $c|b$, then $c \leq d$.

This is generally denoted by (a, b).

The GCD of two non-zero numbers always exists and is unique. Observe that $(a, b) = (a, -b) = (-a, b) = (-a, -b)$. If $a \neq 0$, then $(a, 0) = |a|$.

Definition 10. $a, b \in \mathbb{Z}$ are relatively prime (coprime) if $(a, b) = 1$.

Exercise. Prove that $(F_n, F_{n+1}) = 1$ for $n \geq 1$.

Solution. From Cassini's formula, we know that:

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n$$

Let $d = (F_n, F_{n+1})$. Since $d|F_n$ and $d|F_{n+1}$, we have $d|F_{n-1}F_{n+1} - F_n^2$. So $d|(-1)^n$. This means that $|d| \leq |(-1)^n|$. Since d is a positive number, $d = 1$. ■

Exercise. Prove that $(f_m, f_n) = 1$ for distinct non-negative m, n.

Solution. Suppose $m < n$. Let $d = (f_m, f_n)$. Since $d|f_m$ and $d|f_n$, $d|f_n - (f_0 ... f_m .. f_{n-1})$. From Duncan's identity, this implies that $d|2$. Since $d > 0$,, $d = 1$ or $d = 2$. However, Fermat numbers are always odd - so $d \neq 2$. This implies that $d = 1$. ■

Theorem 4.1. Let $a, b \in \mathbb{Z}$, not both zero. Then there exist integers x_0, y_0 such that:

$$(a, b) = ax_0 + by_0$$

PROOF. Consider the set $S = \{ax + by > 0 : a, b \in \mathbb{Z}\}$. Let $d = \min S$. Suppose that d does not divide a. Then by division algorithm:

$$a = qd + r$$

$$qd = a - r$$

$$q(ax + by) = a - r$$

$$r = a(1 - qx) - bqy$$

So, r is a linear combination of a and b, and since $r > 0$, $r \in S$. From division algorithm, $r < d$, which contradicts the fact that $d = \min S$. So, by contradiction, d divides a (and by similiar argument, d divides b). We also know that any common divisor of a and b must divide d. This is obvious since if $a = uc$ and $b = vc$, then $d = ax + by = c(ux + vy)$, so $c|d$. From these two facts, it is clear that d is the GCD, and is of the form $ax + by$. □

EXERCISE. Let $a, b \in \mathbb{N}$. If $b = aq + r$, then $(a, b) = (a, r)$.

SOLUTION. Let $d = (a, b)$ and $e = (b, r)$. We need to show that $d = e$. Since $d|a$ and $d|b$, $d|a - bq = r$. So d is a common divisor of b and r. Hence $d \leq e$. Similarly, as $e|b$ and $e|r$, e divides $bq + r = q$. Thus e is a common divisor of a and b, so $e \leq d$. Thus, $e = d$. ■

EXERCISE. Let $a, b, c \in \mathbb{N}$. Prove that $(ac, bc) = c(a, b)$.

SOLUTION. Let $d = (a, b)$. Then $d|a$ and $d|b$, so $d|ca$ and $d|cb$. There exist integers x, y such that:

$$d = ax + by$$

$$cd = (ac)x + (bc)y$$

If e is a positive integer such that $e|ac$ and $e|bc$, then $e|(ac)x + (bc)y$, i.e. $e|cd$. So, cd is the GCD of ac and bc. ■

THEOREM 4.2. Let $a, v \in \mathbb{Z}$, not both zero. Then $(a, b) = 1$ if and only if there exist integers x_0, y_0 such that $ax_0 + by_0 = 1$.

PROOF. If $(a, b) = 1$, there must exist x_0, y_0 such that $ax_0 + by_0 = 1$, from Theorem 4.1. Conversely, suppose there exists $x_0, y_0 \in \mathbb{Z}$, such that

$$ax_0 + by_0 = 1$$

Let $d = (a, b)$. Then $d|ax_0 + by_0$, which means that $d|1|$. Since $d \in \mathbb{N}$, $d = 1$. □

Corollary 4.2.1. Let $d = (a, b)$. Then, $(\frac{a}{d}, \frac{b}{d}) = 1$

Corollary 4.2.2. If $(a, b) = 1$, and a and b both divide c, then $ab|c$.

Theorem 4.3 (Euclid's Lemma). If $a|bc$ and $(a, b) = 1$, then $a|c$.

Proof.

$$ax + by = 1$$

$$acx + bcy = c$$

Since $a|acx$ and $a|bcy$, so $a|c$. □

Exercise. Let $m, n \in \mathbb{N}, m > 2$. If $F_m|F_n$, prove that $m|n$.

Proof. We know that:

$$F_n = F_{n-m}F_{m-1} + F_{n-m+1}F_m$$

Since $F_m|F_n$ and $F_m|F_{n-m+1}F_m$, we get $F_m|F_n - F_{n-m+1}F_m$ and hence $F_m|F_{n-m}F_{m-1}$. But, we also know that $(F_m, F_{m-1}) = 1$. By Euclid's Lemma, $F_m|F_{n-m}$.

From the division algorithm, let $n = mq + r$. Suppose $r > 0$. From our previous result, $F_m|F_{n-m}$, so $F_m|F_{n-2m}...F_m|F_r$. This means that $F_m \leq F_r$. But $r < m$, so $F_r < F_m$. This is a contradiction. So $r = 0$. Hence proved. □

Definition 11. Let $n \geq 2$ and let $a_1, ...a_n \in \mathbb{Z}$, not all zero. The GCD of $a_1, ..., a_n$ is the largest positive integer that divides each a_i. This is denoted by $a_1, ..., a_n$.

This has the following properties:

- $(a_1, a_2...a_n)$ is the least positive integer that is a linear combination of $a_1...a_n$.
- $(a_1, ..., a_n) = ((a_1, ..., a_{n-1}), a_n)$
- If $d|a_1...a_n$ and $(d, a_i) = 1$ for all $i \in [1, n-1]$, then $d|a_n$.
- If $a_1, ..., a_n$ are pairwise relatively prime, then $(a_1, ..., a_n) = 1$.

2. Euclidean Algorithm

We are given $a, b \in \mathbb{Z}$, not both zero, and want to compute (a, b). The algorithm to do this works as follows:

(1) If a or b are negative, replace with their absolute value.

(2) If $a > b$, then swap a and b.

(3) If $a = 0$, then $(a, b) = b$.

(4) If $a > 0$, write $b = aq + r$. Then,

$$(a, b) = (r, a)$$

Go to step 3 with $a = r$ and $b = a$ respectively.

This is called the Euclidean Algorithm.

To express (a, b) as a linear combination of a and b where $0 \leq a \leq b$, we create a table with four columns with headings x, y, r, q. We denote the rows as $R_{-1}, R_0, R_1, ...R_{i-1}$ and the entries in R_i as x_i, y_i, r_i, q_i. Suppose we have filled R_{i-1} for some $i \geq 1$. To fill R_i, we first compute q_i, which is the quotient obtained on dividing r_{i-2} by r_{i-1}. Next, $R_i = R_{i-2} - q_i R_{i-1}$. This is known as the Extended Euclidean Algorithm.

Theorem 4.4 (Lame's theorem). Let $b \geq a \geq 2$. The number of divisions required to compute (a, b) by the Euclidean algorithm is at most 5 times the number of decimal digits in a.

Proof. Suppose that a contains k decimal digits and takes n divisions to compute (a, b). We need to show that $n \leq 5k$. Let $r_0 = b, r_1 = a$. Applying Division algorithm repeatedly, we have:

$$r_0 = r_1 q_1 + r_2, 0 < r_2 < r_1$$

$$r_1 = r_2 q_2 + r_3, 0 < r_3 < r_2$$

$$\cdots$$

$$r_{n-1} = r_n \cdot q_n + 0$$

We can prove that $q_i \geq 1$ for $1 \leq i \leq n - 1$ and $q_n \geq 2$. This is because if q_n is 1, then $r_{n-1} = r_n$, which is a contradiction. If $q_i = 0$, then $r_{i-1} = r_{i+1}$, which is also a contradiction.

We claim that $r_{n-i} \geq F_{i+2}$ for $1 \leq i \leq n-1$. This is true for the base case, where $r_{n-1} \geq F_3 = 2$.

$$\begin{aligned} r_{n-j} &= r_{n-(j-1)} q_{n-(j-1)} + r_{n-(j-2)} \\ &\geq r_{n-(j-1)} + r_{n-(j-2)} \\ &\geq F_{j+1} + F_j = F_j + 2 \end{aligned}$$

In particular, $a = r_1 \geq F_{n+1}$. Now $10^k > a \geq F_{n+1}\alpha^{n-1}$, where $\alpha = \frac{1+\sqrt{5}}{2}$ and $n \geq 3$. Taking logarithms and using the fact that $\log \alpha > 1/5$, we get that $n \leq 5k$. □

Exercise. Prove that for all $a, m, n \in \mathbb{N}$,

$$(a^n - 1, a^m - 1) = a^{(n,m)} - 1$$

Solution. Let $f_n = a^n - 1$. Our goal is to prove that $(f_n, f_m) = f_{(n,m)}$.

$$\begin{aligned} f_n &= a^n - 1 \\ &= a^{n-m}(a^m - 1) + a^{n-m} - 1 \\ &= f_{n-m} + kf_m \end{aligned}$$

So now, when performing Euclid's algorithm to find the GCD, $(f_n, f_m) = (f_{n-m}, f_m)$. This occurs recursively, and as we can see it is also performing Euclid's algorithm in the subscript. Hence, $(f_n, f_m) = f_{(n,m)}$. ■

3. Least Common Multiple

Definition 12. Let $a, b \in \mathbb{N}$. We say that a positive integer l is the least common multiple of a and b if

- a and b both divide l
- c is a positive integer divisible by both a and b, then $l \leq c$.

It is generally denoted by $[a, b]$.

Exercise. Prove that $(a, b)[a, b] = ab$.

Solution. Let $d = (a, b)$ and let $k = ab/d$. Obviously, $k \in \mathbb{N}$. We have to show that $k = [a, b]$. We first prove that a and b both divide k. Write

$$a = da_1, b = db_1, a_1, b_1 \in \mathbb{Z}$$

Then, $k = \frac{ab}{d} = ab_1 = ba_1$. Hence a and b both divide k. Suppose c is a positive integer divisible by both a and b. Write

$$c = aa', c = bb', a', b' \in \mathbb{Z}$$

Since $d = (a, b)$, there exists integers x and y such that:

$$d = ax + by$$

Then

$$\frac{c}{k} = \frac{cd}{ab} = \frac{c(ax + by)}{ab} = \frac{c}{b}x + \frac{c}{a}y = b'x + a'y \in \mathbb{Z}$$

Hence k divides c. Since $c \neq 0$, this implies $k \leq c$. Therefore, by definition, $k = [a, b]$ and $(a, b)[a, b] = ab$. ■

4. Fundamental Theorem of Arithmetic

Theorem 4.5 (Fundamental Theorem of Arithmetic). Every integer $n \geq 2$ can be written as a product of primes and the factorization into primes is unique up to the order of the factors.

Proof. Let $P(n)$ be the statement that n can be written as a product of primes. $P(2)$ is true, since it is a prime. Let $k \geq 2$ and $P(2), \cdots, P(k)$ is true. If $k+1$ is prime, then $P(k+1)$ is immediately true. If it is composite, then $k+1$ is of the form $a \cdot b$ where $2 \leq a, b \geq k$. By induction hypothesis, since $P(a)$ and $P(b)$ is true, $P(k+1)$ is also true.

Suppose that

$$n = p_1 p_2 .. p_r = q_1 ... q_s$$

Assume $r \leq s$. Since if $p_1 | q_1 ... q_s$, $p_1 = q_i$ for some $i, 1 \leq i \leq s$. We can cancel this out and keep doing this till we get

$$1 = q_{i_1} q_{i_2} q_{i_{s-r}}$$

However, q_i is a prime. This is a contradiction unless $s = r$. Hence uniqueness is proved. □

The largest power x such that $p^x | a$ for a prime p is called the multiplicity of p in a.