# Double Pendulum Lagrangian Problem

The double pendulum is a classical problem in physics that illustrates complex dynamical behavior, including chaos. It consists of two point masses attached by rigid, massless rods, swinging under the influence of gravity. The system is a quintessential example used to study nonlinear dynamics and is often analyzed using Lagrangian mechanics.

## 1 System Description

- **Masses**: Two point masses, $m_1$ and $m_2$ (for simplicity, we can set $m_1 = m_2 = m$).
- **Rod Lengths**: Two rigid, massless rods of lengths $l_1$ and $l_2$.
- **Angles**: The angles between the rods and the vertical are $\theta_1$ and $\theta_2$, respectively.
- **Coordinates**: The positions of the masses are determined by $\theta_1$ and $\theta_2$.

## 2 Objective

To derive the equations of motion for the double pendulum using the Lagrangian formulation.

## 3 Lagrangian Mechanics Overview

The Lagrangian $L$ is defined as the difference between the kinetic energy $T$ and the potential energy $V$:

$$L = T - V$$

The equations of motion are derived using the Euler-Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) - \frac{\partial L}{\partial \theta_i} = 0 \quad \text{for} \quad i = 1, 2$$

# 4 Calculating the Kinetic Energy $T$

## 4.1 Position Coordinates

For mass $m_1$:

$$x_1 = l_1 \sin\theta_1, \quad y_1 = -l_1 \cos\theta_1$$

For mass $m_2$:

$$x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2, \quad y_2 = -l_1 \cos\theta_1 - l_2 \cos\theta_2$$

## 4.2 Velocity Components

Velocity of $m_1$:

$$v_{1x} = l_1\dot{\theta}_1 \cos\theta_1, \quad v_{1y} = l_1\dot{\theta}_1 \sin\theta_1$$

Velocity of $m_2$:

$$v_{2x} = v_{1x} + l_2\dot{\theta}_2 \cos\theta_2, \quad v_{2y} = v_{1y} + l_2\dot{\theta}_2 \sin\theta_2$$

## 4.3 Expressing Kinetic Energy

Kinetic energy of $m_1$:

$$T_1 = \frac{1}{2}m\left(v_{1x}^2 + v_{1y}^2\right) = \frac{1}{2}ml_1^2\dot{\theta}_1^2$$

Kinetic energy of $m_2$:

$$T_2 = \frac{1}{2}m\left(v_{2x}^2 + v_{2y}^2\right)$$

Expanding $T_2$:

$$T_2 = \frac{1}{2}m\left[\left(l_1\dot{\theta}_1 \cos\theta_1 + l_2\dot{\theta}_2 \cos\theta_2\right)^2 + \left(l_1\dot{\theta}_1 \sin\theta_1 + l_2\dot{\theta}_2 \sin\theta_2\right)^2\right]$$

Simplifying $T_2$:

$$T_2 = \frac{1}{2}m\left[l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)\right]$$

## 4.4 Total Kinetic Energy

$$T = T_1 + T_2 = \frac{1}{2}m\left[2l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)\right]$$

# 5 Calculating the Potential Energy $V$

Potential Energy of $m_1$:

$$V_1 = mgy_1 = -mgl_1 \cos\theta_1$$

Potential Energy of $m_2$:

$$V_2 = mgy_2 = -mg\left(l_1 \cos\theta_1 + l_2 \cos\theta_2\right)$$

Total Potential Energy:

$$V = V_1 + V_2 = -mg\left(2l_1 \cos\theta_1 + l_2 \cos\theta_2\right)$$

# 6 Formulating the Lagrangian $L$

$$L = T - V = \frac{1}{2}m\left[2l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)\right] + mg\left(2l_1\cos\theta_1 + l_2\cos\theta_2\right)$$

# 7 Deriving the Equations of Motion

Apply the Euler-Lagrange equations to $\theta_1$ and $\theta_2$:

## 7.1 For $\theta_1$

:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} = 0$$

## 7.2 For $\theta_2$

:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} = 0$$

# 8 Resulting Equations of Motion

After performing the differentiation and simplifications (which involve considerable algebra), the equations of motion are:

## 8.1 First Equation

$$(2m)l_1\ddot{\theta}_1 + ml_2\ddot{\theta}_2\cos(\theta_1 - \theta_2) + ml_2\dot{\theta}_2^2\sin(\theta_1 - \theta_2) + (2m)g\sin\theta_1 = 0$$

### 8.2 Second Equation

$$ml_2\ddot{\theta}_2 + ml_1\ddot{\theta}_1\cos(\theta_1 - \theta_2) - ml_1\dot{\theta}_1^2\sin(\theta_1 - \theta_2) + mg\sin\theta_2 = 0$$

## 9 Simplifying (Assuming Equal Masses and Lengths)

If $m_1 = m_2 = m$ and $l_1 = l_2 = l$:

### 9.1 First Equation

$$2l\ddot{\theta}_1 + l\ddot{\theta}_2\cos(\theta_1 - \theta_2) + l\dot{\theta}_2^2\sin(\theta_1 - \theta_2) + 2g\sin\theta_1 = 0$$

### 9.2 Second Equation

$$l\ddot{\theta}_2 + l\ddot{\theta}_1\cos(\theta_1 - \theta_2) - l\dot{\theta}_1^2\sin(\theta_1 - \theta_2) + g\sin\theta_2 = 0$$

## 10 Interpretation

- **Nonlinearity**: The equations are highly nonlinear due to the trigonometric functions and the product of angular velocities.
- **Coupling**: The motion of each pendulum mass affects the other, indicating a coupled system.
- **Complex Dynamics**: Small differences in initial conditions can lead to significantly different outcomes, a hallmark of chaotic systems.

## 11 Applications

- **Chaos Theory**: The double pendulum is a standard example used to study chaotic motion.
- **Engineering**: Understanding the dynamics helps in designing structures that can withstand oscillatory motions.
- **Education**: Serves as a teaching tool for advanced classical mechanics and differential equations.

## 12 Conclusion

The double pendulum Lagrangian problem showcases the power of Lagrangian mechanics in deriving equations of motion for complex systems. While the derivation involves intricate calculations, the resulting equations provide deep insights into the dynamic behavior of coupled nonlinear systems.

## 13 References

- **Classical Mechanics Textbooks**: For a detailed derivation and analysis.
- **Dynamics Simulations**: Interactive simulations can help visualize the motion described by the equations.